# PERMUTATIONS AS GENOME REARRANGEMENTS

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Remark 0.1. *RECALL: inversion*

## 1. Block Interchanges

### 1.1. The Cycle Graph and Its Relation to *bid*.

Definition 1.1. *A Block Interchange is an operation that interchanges two blocks of consecutive entries without rearranging said entries.*

Example 1.1. $|34|17|562| \to |562|17|34|$

Definition 1.2. *Let the Block Interchange Distance between two n-permutation $p, q$, denoted as $bid(p,q)$, be the smallest number of block interchanges required to transform $p$ into $q$. Let $bid(p) := bid(p, id)$ be the number of interchanges required to sort $p$.*

Remark 1.1. *$bid(p,q)$ is a metric and is left-invariant, just as we have seen for $btd(p,q)$.*

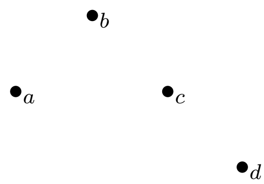
Definition 1.3. *Let $p = p_1 p_2 \cdots p_n$ be an n-permutation. The Cycle Graph $G(p)$ is a graph of coloured directed edges on the vertex set $\{0, 1, \cdots, n, n+1\}$ constructed as follows:*

*(1) For every $i$ with $1 \leq i \leq n+1$: Add a black edge from $p_i$ to $p_{i-1}$*
*(2) For every $i$ with $0 \leq i \leq n$: Add a red edge from $i$ to $i+1$*

*with $p_0 = 0$ and $p_{n+1} = n+1$.*

Remark 1.2. *The Cycle Graph has $2n+2$ edges.*

Example 1.2. *$p$ = 123*




*$p$ = 4213? $p$ = 4312 draw cycle graph count cycles find minimal transpositions to id and show how it works*

Lemma 1.1. *The cycle graph $G(p)$ has a unique decomposition into edge-disjoint directed cycles in which the colors of the edges alternate.*

*Proof.* It is easy to see that a path of alternating colours cannot branch since by construction there is always only one edge per color directed away from every node. We begin at the 0 node and complete a cycle (this must occur since $G(p)$ is finite). Once we have done so, we remove the edges of this cycle from $G(p)$ and iterate. Notice that in removing full cycles the structure is maintained as we only remove disjointly coloured pairs of edges from every node. The cycles are then edge-disjoint by construction. □

Definition 1.4. *Let $c(G(p))$ be the number of alternating directed cycles in the decomposition of $G(p)$. To avoid confusion, let $c(\Gamma(p))$ denote the number of cycles in the traditional sense of a permutation. From now on the cycles of $G(p)$ will refer to the alternating cycles.*

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Remark 1.3. *The cycles are only edge-disjoint not vertex-disjoint.*

Remark 1.4. *If there is an alternating path from one node to another, both nodes are in the same cycle.*

Remark 1.5. *The identity permutation has $n+1$ cycles. In fact, it is the only permutation with this many cycles.*

Before we can find a formula for $bid(p)$ we shall examine how single block interchanges can influence the block interchange distance of a permutation in the following two lemmas.

Definition 1.5. *Canonical Block Interchange of $p$.*

Lemma 1.2. *9.10 Let $p \in S_n$ with $p \neq id$. Then there exists a block interchange which increases $c(G(p))$ by two.*

*Proof.* 3 cases □

Lemma 1.3. *9.11 A block interchange cannot increase $c(G(p))$ by more than two.*

*Proof.* Proof Sketch Only □

Theorem 1.1. *9.9 Let $p \in S_n$. Then $bid(p) = \frac{n+1-c(G(p))}{2}$*

*Proof.* By the previous two lemmata we have for any $p \in S_n$ that

$$\frac{n+1-c(G(p))}{2} \leq bid(p) \leq \frac{n+1-c(G(p))}{2}$$

since we require this amount of block interchanges to attain $n+1$ cycles in $p$, thus culminating our desired equality. □

Remark 1.6. *We have discovered that we need at most $\lfloor \frac{n}{2} \rfloor$ block interchanges to sort an $n$-permutation. Furthermore $bid(p) \in \mathbb{N}$ implies that $(n+1)$ and $c(G(p))$ must either both be odd or even.*

### 1.2. Average Number of Block Interchanges Required to Sort $p$.

Definition 1.6. *Define the Hultman Number as*

$$S_H(n,k) := |\{\pi \in S_n : c(G(p)) = k\}|$$

Example 1.3. *$S_H(3,4) = 1$ and $S_H(3,2) = 5$. Draw sample graphs quickly? Maybe at least two. Continue example after 9.14 et al.*

Theorem 1.2. *9.14 and Corollary? Proof Sketch?*

Remark 1.7. *The previous theorem illustrates the connection between $c(G(p))$ and $c(\Gamma(p))$. It allows us to translate ?????????? Skip all the other stuff, useless?*

Theorem 1.3. *9.20? 9.21, 9.22? just write theorems and quickly talk about at most.*

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THEOREM 1.4. *9.23. The average number of block interchanges needed to sort an n-permutation is*

$$\frac{1}{2}(n - \frac{1}{\left\lfloor \frac{n+2}{2} \right\rfloor} - \sum_{i=2}^{n} \frac{1}{i})$$