PERMUTATIONS AS GENOME REARRANGEMENTS

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Remark 0.1. *RECALL: inversion*

1. Block Interchanges

1.1. The Cycle Graph and Its Relation to bid.

Definition 1.1. *Block Interchange*

Definition 1.2. *Block Interchange Distance* $bid(p, q)$ $bid(p) := bid(p, id)$

Remark 1.1. $bid(p, q)$ *is a metric and is left-invariant, just as we have seen for* $btd(p, q)$.

Definition 1.3. *Cycle Graph* $G(p)$

Lemma 1.1. *The cycle graph* $G(p)$ *has a unique decomposition into edge-disjoint directed cycles in which the colors of the edges alternate.*

Proof. It is easy to see that a path of alternating colours cannot branch since by construction there is always only one edge per color directed away from every node. We begin at the 0 node and complete a cycle (this must occur since $G(p)$ is finite). Once we have done so, we remove the edges of this cycle from $G(p)$ and iterate. Notice that in removing full cycles the structure is maintained as we only remove disjointly coloured pairs of edges from every node. The cycles are then edge-disjoint by construction. □

Definition 1.4. $c(G(p))$, $c(\Gamma(p))$ *From now on the cycles of* $G(p)$ *will refer to the alternating cycles, not the traditional cycles of a permutation we have seen previously.*

Remark 1.2. *The cycles are only edge-disjoint not vertex-disjoint.*

Remark 1.3. *If there is an alternating path from one node to another, both nodes are in the same cycle.*

Example 1.1. $p = 123$ $p = 4213?$ $p = 4312$ *draw cycle graph count cycles find minimal transpositions to id and show how it works*

Remark 1.4. *The identity permutation has* $n+1$ *cycles. In fact, it is the only permutation with this many cycles.*

Before we can find a formula for $bid(p)$ we shall examine how single block interchanges can influence the block interchange distance of a permutation in the following two lemmas.

Definition 1.5. *Canonical Block Interchange of* p.

Lemma 1.2. *9.10 Let* $p \in S_n$ *with* $p \neq id$. *Then there exists a block interchange which increases* $c(G(p))$ *by two.*

Proof. 3 cases □

Lemma 1.3. *9.11 A block interchange cannot increase* $c(G(p))$ *by more than two.*

Proof. Proof Sketch Only □

Theorem 1.1. *9.9 Let* $p \in S_n$. *Then* $bid(p) = \frac{n+1-c(G(p))}{2}$

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Proof. By the previous two lemmata we have for any $p \in S_n$ that

$$\frac{n+1-c(G(p))}{2} \leq bid(p) \leq \frac{n+1-c(G(p))}{2}$$

since we require this amount of block interchanges to attain $n+1$ cycles in p, thus culminating our desired equality. □

REMARK 1.5. *We have discovered that we need at most* $\lfloor \frac{n}{2} \rfloor$ *block interchanges to sort an* n*-permutation. Furthermore* $bid(p) \in \mathbb{N}$ *implies that* $(n+1)$ *and* $c(G(p))$ *must either both be odd or even.*

1.2. **Average Number of Block Interchanges Required to Sort** p.

DEFINITION 1.6. *Define the Hultman Number as*

$$S_H(n,k) := |\{\pi \in S_n : c(G(p)) = k\}|$$

EXAMPLE 1.2. $S_H(3,4) = 1$ *and* $S_H(3,2) = 5$*. Draw sample graphs quickly? Maybe at least two. Continue example after 9.14 et al.*

THEOREM 1.2. *9.14 and Corollary? Proof Sketch?*

REMARK 1.6. *The previous theorem illustrates the connection between* $c(G(p))$ *and* $c(\Gamma(p))$*. It allows us to translate ?????????? Skip all the other stuff, useless?*

THEOREM 1.3. *9.20? 9.21, 9.22? just write theorems and quickly talk about at most.*

THEOREM 1.4. *9.23. The average number of block interchanges needed to sort an* n*-permutation is*

$$\frac{1}{2}(n - \frac{1}{\lfloor \frac{n+2}{2} \rfloor} - \sum_{i=2}^{n} \frac{1}{i})$$