PERMUTATIONS AS GENOME REARRANGEMENTS

SIMON GRÜNING

(Block Interchanges)

(Simon Grüning) University of Zurich, Rämistrasse 71, 8006 Zurich
E-mail address: simon.gruening@uzh.ch.

Simon Grüning
simon.gruening@uzh.ch

1. The Cycle Graph and Block Interchange Distance

Remark 1.1. *Recall an inversion in* $p = p_1 p_2 \cdots p_n$ *is such that* $(i, j \in [n] : p_i > p_j \wedge i < j)$.

Definition 1.1. *A* ***Block Interchange*** *is an operation that interchanges two blocks of consecutive entries without rearranging said entries.*

Example 1.1. $|34|17|562| \to |562|17|34|$

Definition 1.2. *Let the* ***Block Interchange Distance*** *between two n-permutation* p, q*, denoted as* $bid(p, q)$*, be the smallest number of block interchanges required to transform* p *into* q*. Let* $bid(p) := bid(p, id)$ *be the number of interchanges required to sort* p*.*

Remark 1.2. $bid(p, q)$ *is a metric and is left-invariant, just as we have seen for* $btd(p, q)$*.*

Definition 1.3. *Let* $p = p_1 p_2 \cdots p_n$ *be an n-permutation. The Cycle Graph* $G(p)$ *is a graph of coloured directed edges on the vertex set* $\{0, 1, \cdots, n, n+1\}$ *constructed as follows:*

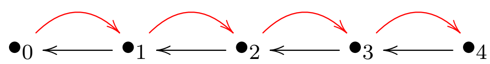
(1) *For every* i *with* $1 \leq i \leq n+1$*: Add a black edge from* p_i *to* p_{i-1}
(2) *For every* i *with* $0 \leq i \leq n$*: Add a red edge from* i *to* $i+1$

with $p_0 = 0$ *and* $p_{n+1} = n+1$*.*

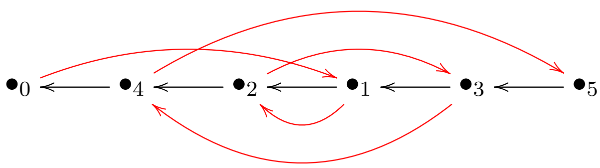
Remark 1.3. *The Cycle Graph has* $2n+2$ *edges for any n-permutation.*

Example 1.2.

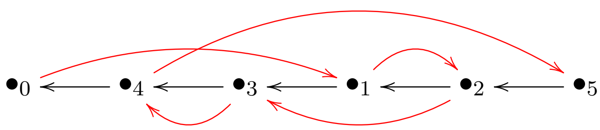
$p_1 = 123 \implies c(G(p_1)) = 4$



$p_2 = 4213 \implies c(G(p_2)) = 1$



$p_3 = 4312 \implies c(G(p_3)) = 3$



For the final permutation p_3 *notice that* $bid(p_3) = 1 = \frac{4+1-3}{2}$*, something which will become relevant in a later theorem.*

Lemma 1.1. *The cycle graph* $G(p)$ *has a unique decomposition into edge-disjoint directed cycles in which the colors of the edges alternate.*

Simon Grüning
simon.gruening@uzh.ch

Proof. It is easy to see that a path of alternating colours cannot branch since by construction there is always only one edge per color directed away from every node. We begin at the 0-node and complete a cycle (this must occur since $G(p)$ is finite). Once we have done so, we remove the edges of this cycle from $G(p)$ and iterate. Notice that in removing full cycles the structure is maintained as we only remove disjointly coloured pairs of edges from every node. The cycles are then edge-disjoint by construction. □

Definition 1.4. *Let $c(G(p))$ be the* ***number of alternating directed cycles*** *in the decomposition of $G(p)$. To avoid confusion, let $c(\Gamma(p))$ denote the number of cycles in the traditional sense of a permutation. From now on the cycles of $G(p)$ will refer to the alternating cycles.*

Remark 1.4.

(1) The cycles are only edge-disjoint not necessarily vertex-disjoint.
(2) If there is an alternating path from one node to another, both nodes are in the same cycle.
(3) The identity permutation has $n+1$ cycles. In fact, it is the only permutation with this many cycles.

Before we can find a formula for $bid(p)$ we shall examine how single block interchanges can influence the block interchange distance of a permutation in the following two lemmas.

Definition 1.5. *Let $p = p_1p_2\cdots p_n$ with $p \neq id$, or in other words, p contains at least one inversion. Select the inversion in p $(i, j \in [n] : p_i > p_j \wedge i < j)$ such that p_j is minimal and if necessary, as a second criterion that p_i is maximal.*

Let $x := p_j, y := p_i$ to ease our eyes. By our selection criterion we find that $x-1$ occurs before y in p, whilst $y+1$ occurs after x. (In the case of $x = 1$ simply let $x-1 = 0$)

$$p = \cdots(x-1)|\cdots y|\cdots|x\cdots|(y+1)\cdots$$

The ***Canonical Block Interchange*** *is defined as the interchange transforming p into*

$$p' = \cdots(x-1)|x\cdots|\cdots|\cdots y|(y+1)\cdots$$

Lemma 1.2. *(9.10) Let $p \in S_n$ with $p \neq id$. Then there exists a block interchange which increases $c(G(p))$ by 2.*

Proof. We shall prove that the canonical block interchange suffices, or symbolically:

$$c(G(p')) = c(G(p)) + 2.$$

Notice first that any block interchange does not affect the red edges in $G(p)$. The canonical interchange changes 3 black edges when x and y are adjacent, and 4 edges otherwise. We shall perform a proof by cases.

(1) <u>x, y adjacent:</u> The cycle decomposition of $G(p)$ will contain an alternating path through the following nodes in respective order:

$$a \to (x-1) \to x \to y \to (y+1) \to b$$

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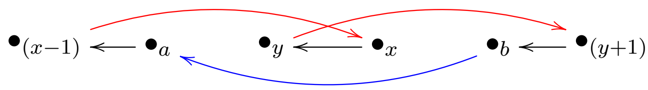
where a is the entry in p directly to the right of $(x-1)$ and b the one to the left of $(y+1)$.

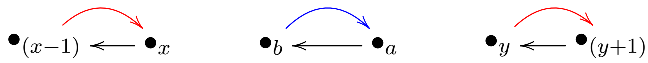
$$p = \cdots(x-1)|a\cdots y|\cdots|x\cdots b|(y+1)\cdots$$

Since a path exists, we find that the entries are part of the same cycle C. After performing the canonical interchange, C is split into 3 cycles. In the black edges we now have $x \to (x-1)$, $(y+1) \to y$, and $a \to b$ while the red have been untouched as seen in the following illustration:

p



$\rightsquigarrow p'$



Notice the arrow in blue is not necessarily a singular edge, however it is a well defined alternating path.

(2) <u>x, y not adjacent:</u> We must introduce another notation in this case. Let a, b, c, d be such that they are adjacent to $(x-1), y, x, (y+1)$ respectively pair-wise. Since they are neighbours we will find the alternating paths

$$P_a = a \to (x-1) \to x \to c$$

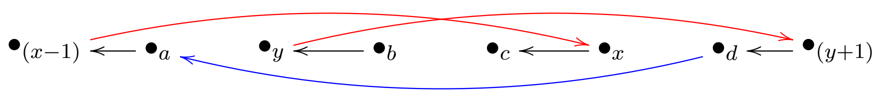
and

$$P_b = b \to y \to y+1 \to d$$

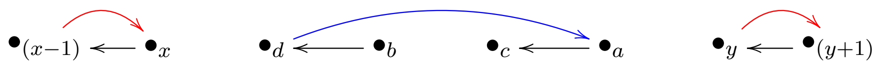
We must make a further division of cases at this point.

(a) <u>P_a, P_b are of one cycle:</u> Denote the cycle as C_{ab}. In this case there exists an alternating path A_{da} from d to a with the final edge being grey. See the following illustration:

p



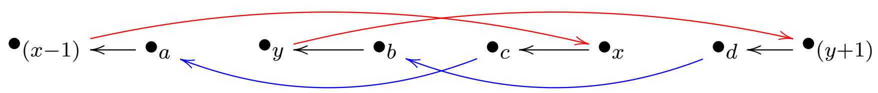
$\rightsquigarrow p'$



where the blue edge is our path A_{da}, the red edges are kept static, and once more we have split one cycle into three, all else remaining equal.

(b) P_a, P_b are of distinct cycles: Denote these cycles C_a and C_b respectively. We find that each cycle shall split into two, we shall cement this only with a short illustration as the argument is analogue.

p



$\rightsquigarrow p'$

Where the blue edges are our paths C_a and C_b exempted $(x-1) \to x$ and $y \to (y+1)$ correspondingly.

Having exhausted all cases, we conclude the proof.

$\square$

LEMMA 1.3. *(9.11) Let $p \in S_n$. Any block interchange performed on p cannot increase $c(G(p))$ by more than 2.*

Proof. We shall only provide a quick sketch of the proof which makes use of the reversibility of block interchanges to create a contradiction.

(1) Any block interchange changes at most 4 black edges of $G(p)$.
(2) For a block interchange to create more than 2 cycles, it must break one cycle into 4 smaller cycles.
(3) If this where the case, the inverse block interchange would have to turn 4 cycles into one cycle, which leads to contradiction upon closer examination. $\lightning$

$\square$

THEOREM 1.1. *(9.9) Let $p \in S_n$. Then*

$$bid(p) = \frac{n+1-c(G(p))}{2}.$$

Proof. By the previous two lemmata we have for any $p \in S_n$ that

$$\frac{n+1-c(G(p))}{2} \leq bid(p) \leq \frac{n+1-c(G(p))}{2}$$

since we require a change in $c(G(p))$ of magnitude $(n+1) - c(G(p))$ to achieve

$$c(G(\tilde{p})) = n+1 \iff \tilde{p} = id$$

and this can be achieved by half the same number of block interchanges at most and at least, thus culminating our desired equality. $\square$

REMARK 1.5. *We have discovered that we need at most $\left\lfloor \frac{n}{2} \right\rfloor$ block interchanges to sort an n-permutation. Furthermore $bid(p) \in \mathbb{N}$ implies that $(n+1)$ and $c(G(p))$ must either both be odd or even.*

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2. Average Number of Block Interchanges Required

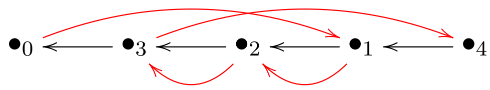
We would now like to find the average number of block interchanges required for any arbitrary n-permutation. We wish to count the cardinality of the set of n-permutations restricted to a specific number of cycles in order to calculate this average.

Definition 2.1. *Define the Hultman Number as*

$$S_H(n,k) := |\{\pi \in S_n : c(G(p)) = k\}|$$

Example 2.1. $S_H(3,4) = 1$ *and* $S_H(3,2) = 5$*. Thus all 5 non-trivial 3-permutations have 2 cycles.*

Have a glance at the graph of $p = 321$*:*



Theorem 2.1. *(9.14)*

$$S_H(n,k) = |\{(q,r) \in S_n^2 : c(\Gamma(r)) = k \wedge qr = (12\cdots n(n+1))\}|$$

The previous theorem illustrates the connection between $c(G(p))$ and $c(\Gamma(p))$. It allows us to translate our question into previously explored territory. After some complex and difficult papers alleged to require things such as symmetric functions, character theory, and non-elementary integration, we find our answer:

Theorem 2.2. *(9.23) The average number of block interchanges needed to sort an n-permutation is*

$$\frac{1}{2}\left(n - \frac{1}{\left\lfloor \frac{n+2}{2} \right\rfloor} - \sum_{i=2}^{n} \frac{1}{i}\right)$$