$$\log|F(h(z))| \le \frac{1}{2\pi}\int_{-\pi}^{+\pi} \log|F(h(Re^{i\varphi}))| \frac{R^2-\rho^2}{|Re^{i\varphi}-\rho e^{i\theta}|^2}\,d\varphi \tag{1.3.36}$$

when $z=\rho e^{i\theta}$ and $|z|=\rho<R$. We observe that for $R<|\zeta|=1$ the hypothesis on F imply that

$$\begin{aligned}
\log|F(h(Re^{i\varphi}))| &\le Ae^{\tau_0\left|\operatorname{Im}\frac{1}{\pi i}\log\left(i\frac{1+R\zeta}{1-R\zeta}\right)\right|}\\
&\le Ae^{\frac{\tau_0}{\pi}\left|\log\left|\frac{1+R\zeta}{1-R\zeta}\right|\right|}\\
&\le Ae^{\frac{\tau_0}{\pi}\left|\log\left|\frac{1+R\zeta}{1-R\zeta}\right|\right|}\\
&\le A2^{\frac{\tau_0}{\pi}}\left[|1+R\zeta|^{-\frac{\tau_0}{\pi}}+|1-R\zeta|^{-\frac{\tau_0}{\pi}}\right].
\end{aligned}$$

Now $|1\pm Re^{i\varphi}|^2=(1\pm R\cos\varphi)^2+R^2\sin^2\varphi\ge\frac14\sin^2\varphi$, since if $R\le 1/2$ the first term is at least $1/4$ while if $R>1/2$ the second term in the sum is at least $\frac14\sin^2\varphi$. Hence $|1\pm Re^{i\varphi}|\ge\frac12|\sin\varphi|$ and from this it follows that

$$\log|F(h(Re^{i\varphi}))|\le C|\sin\varphi|^{-\frac{\tau_0}{\pi}}.$$

Now $|\sin\varphi|^{-\frac{\tau_0}{\pi}}$ is integrable over $[-\pi,\pi]$, in view of the assumption $\tau_0<\pi$. Moreover, the bound $\frac{R^2-\rho^2}{|Re^{i\varphi}-\rho e^{i\theta}|^2}\le\frac{4}{1-\rho}$ holds for $1>R>\frac12(\rho+1)$.

We will now use the following consequence of Fatou's lemma: suppose that $F_R\le G$, where $G\ge 0$ is integrable. Then $\limsup_{R\to\infty}\int F_R\,d\varphi\le\int\limsup_{R\to\infty}F_R\,d\varphi$. Letting $R\uparrow 1$ in (1.3.36) and using this version of Fatou's lemma we obtain

$$\log|F(h(\rho e^{i\theta}))|\le\frac{1}{2\pi}\int_{-\pi}^{+\pi}\log|F(h(e^{i\varphi}))|\frac{1-\rho^2}{1-2\rho\cos(\theta-\varphi)+\rho^2}\,d\varphi. \tag{1.3.37}$$

Setting $x=h(\rho e^{i\theta})$, we obtain that

$$\rho e^{i\theta}=h^{-1}(x)=\frac{e^{\pi ix}-i}{e^{\pi ix}+i}=-i\frac{\cos(\pi x)}{1+\sin(\pi x)}=\left(\frac{\cos(\pi x)}{1+\sin(\pi x)}\right)e^{-i(\pi/2)},$$

from which it follows that $\rho=(\cos(\pi x))/(1+\sin(\pi x))$ and $\theta=-\pi/2$ when $0<x\le\frac12$, while $\rho=-(\cos(\pi x))/(1+\sin(\pi x))$ and $\theta=\pi/2$ when $\frac12\le x<1$. In either case we easily deduce that

$$\frac{1-\rho^2}{1-2\rho\cos(\theta-\varphi)+\rho^2}=\frac{\sin(\pi x)}{1+\cos(\pi x)\sin(\varphi)}.$$

Using this we write (1.3.37) as

$$\log|F(x)|\le\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin(\pi x)}{1+\cos(\pi x)\sin(\varphi)}\log|F(h(e^{i\varphi}))|\,d\varphi. \tag{1.3.38}$$