

# CLASSICAL FOURIER ANALYSIS: INTERPOLATION ON $L^p$ SPACES

**Abstract.** In this written seminar work I will basically follow [Gra14, pp. 33–48]. I will review three basic but important theorems on interpolation of operators on  $L^p$  spaces, namely the *Marcinkiewicz Interpolation Theorem*, the *Riesz-Thorin Interpolation Theorem* and finally an important extension of the Riesz-Thorin Interpolation Theorem to analytic families of operators (the so-called *Stein's theorem on interpolation of analytic families of operators*). We are mainly concerned with the notion of linear operators as well as slight generalizations of them.

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## 1. Introduction and Basic Definitions.

**1.1. Linear Operators.** First we need to have a precise and suitable idea of *linear operators* in the generalized setting of measure spaces.

DEFINITION 1.1. Let  $(X, \mathcal{A}, \mu)$  and  $(Y, \mathcal{B}, \nu)$  be measure spaces. Further let  $T$  be an operator defined on a linear space of complex-valued measurable functions on  $X$  and taking values in the set of all complex-valued, finite almost everywhere, measurable functions on  $Y$ . Then  $T$  is called *linear* if for all functions  $f$  and  $g$  in the domain of  $T$  and all  $z \in \mathbb{C}$  holds

$$(1) \quad T(f + g) = T(f) + T(g) \quad T(zf) = zT(f)$$

and quasi-linear if

$$(2) \quad |T(f + g)| \leq K(|T(f)| + |T(g)|) \quad |T(zf)| = |z||T(f)|$$

holds for some real constant  $K > 0$ . If  $K = 1$ ,  $T$  is called *sublinear*.

REMARK 1.1. *For simplicity I will omit in most of what follows that a complex-valued function assumes values in the measured space  $(\mathbb{C}, \mathfrak{B}^2)$  where for  $p \in \mathbb{N}_{>0}$  we define  $\mathfrak{B}^p := \mathfrak{B}(\mathbb{R}^p) = \sigma(\mathfrak{O}^p) := \sigma(\{U \subseteq \mathbb{R}^p : U \text{ open}\})$ , the  $\sigma$ -algebra of the Borel sets of  $\mathbb{R}^p$  (the notation  $\sigma(\mathfrak{C})$  for  $\mathfrak{C} \subseteq \mathcal{P}(X)$  of any set  $X$  denotes the  $\sigma$ -algebra generated by  $\mathfrak{C}$ ). For more details see [Els11, pp. 16–19].*

## 2. The Real Method.

**2.1. The Marcinkiewicz Interpolation Theorem.** The name originates from the real variables technique used for proving the theorem.

**THEOREM 2.1.** (The Marcinkiewicz Interpolation Theorem) *Let  $(X, \mathcal{A}, \mu)$  be a  $\sigma$ -finite measure space,  $(Y, \mathcal{B}, \nu)$  another measure space and  $0 < p_0 < p_1 \leq +\infty$ . Further let  $T$  be a sublinear operator defined on*

$$L^{p_0}(X, \mathcal{A}, \mu) + L^{p_1}(X, \mathcal{A}, \mu) := \{f_0 + f_1 : f_0 \in L^{p_0}(X, \mathcal{A}, \mu), f_1 \in L^{p_1}(X, \mathcal{A}, \mu)\}$$

*and taking values in the space of measurable functions on  $Y$ . Assume that there exist  $A_0, A_1 < +\infty$  such that*

$$(3) \quad \forall f \in L^{p_0}(X, \mathcal{A}, \mu), \|T(f)\|_{L^{p_0, \infty}(Y, \mathcal{B}, \nu)} \leq A_0 \|f\|_{L^{p_0}(X, \mathcal{A}, \mu)}$$

$$(4) \quad \forall f \in L^{p_1}(X, \mathcal{A}, \mu), \|T(f)\|_{L^{p_1, \infty}(Y, \mathcal{B}, \nu)} \leq A_1 \|f\|_{L^{p_1}(X, \mathcal{A}, \mu)}$$

*Then for all  $p_0 < p < p_1$  and for all  $f \in L^p(X, \mathcal{A}, \mu)$  we have the estimate*

$$(5) \quad \|T(f)\|_{L^p(Y, \mathcal{B}, \nu)} \leq A \|f\|_{L^p(X, \mathcal{A}, \mu)}$$

*where*

$$(6) \quad A := 2 \left( \frac{p}{p - p_0} + \frac{p}{p_1 - p} \right)^{1/p} A_0^{\frac{\frac{1}{p} - \frac{1}{p_1}}{\frac{1}{p_0} - \frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0} - \frac{1}{p}}{\frac{1}{p_0} - \frac{1}{p_1}}}$$

*Proof.* The proof is subdivided into two main parts, which are further subdivided. In detail, we have the following partitioning:

- (i.)  $p_1 < +\infty$ .
  - a. Split  $f$  using cut-off functions.
  - b. Estimate the distribution function  $d_{T(f)}$ .
  - c. Estimate  $\|T(f)\|_{L^p(Y, \mathcal{B}, \nu)}^p$ .
- (ii.)  $p_1 = +\infty$ .
  - a. Show that  $\mu(\{|T(f_1)| > \alpha/2\}) = 0$ .
  - b. Estimate the distribution function  $d_{T(f_0)}$ .
  - c. Estimate  $\|T(f)\|_{L^p(Y, \mathcal{B}, \nu)}^p$ .
- (i.) a. Let us first consider the case  $p_1 < +\infty$ . Fix  $f \in L^p(X, \mathcal{A}, \mu)$ ,  $\alpha > 0$  and  $\delta > 0$  ( $\delta$  will be determined later). We split  $f$  using so-called *cut-off* functions, by stipulating  $f \equiv f_0(\cdot; \alpha, \delta) + f_1(\cdot; \alpha, \delta)$ , where  $f_0(\cdot; \alpha, \delta)$  is the *unbounded part* of  $f$  and  $f_1(\cdot; \alpha, \delta)$  is the *bounded part* of  $f$ , defined by

$$(7) \quad \begin{aligned} f_0(x; \alpha, \delta) &:= \begin{cases} f(x), & |f(x)| > \delta\alpha, \\ 0, & |f(x)| \leq \delta\alpha. \end{cases} \\ f_1(x; \alpha, \delta) &:= \begin{cases} f(x), & |f(x)| \leq \delta\alpha, \\ 0, & |f(x)| > \delta\alpha. \end{cases} \end{aligned}$$

for  $x \in X$ . To facilitate reading I will omit the dependency of  $f_0(\cdot; \alpha, \delta)$  and  $f_1(\cdot; \alpha, \delta)$  upon the parameters  $\alpha$  and  $\delta$  and simply use  $f_0, f_1$  respectively. Since  $p_0 < p$  we have

$$\begin{aligned}
 \|f_0\|_{L^{p_0}(X, \mathcal{A}, \mu)}^{p_0} &= \int_X |f_0|^{p_0} d\mu \\
 &= \int_X |f|^{p_0} \cdot \chi_{\{|f| > \delta\alpha\}} d\mu \\
 &\stackrel{(\dagger)}{=} \int_{\{|f| > \delta\alpha\}} |f|^{p_0} d\mu \\
 &= \int_{\{|f| > \delta\alpha\}} |f|^p |f|^{p_0-p} d\mu \\
 &= \int_{\{|f| > \delta\alpha\}} \frac{|f|^p}{|f|^{p-p_0}} d\mu \\
 &\leq \frac{1}{(\delta\alpha)^{p-p_0}} \int_{\{|f| > \delta\alpha\}} |f|^p d\mu \\
 &= (\delta\alpha)^{p_0-p} \int_X |f|^p \cdot \chi_{\{|f| > \delta\alpha\}} d\mu \\
 &\leq (\delta\alpha)^{p_0-p} \int_X |f|^p d\mu \\
 &= (\delta\alpha)^{p_0-p} \|f\|_{L^p(X, \mathcal{A}, \mu)}^p
 \end{aligned}
 \tag{8}$$

Since  $f \in L^p(X, \mathcal{A}, \mu)$  and thus  $\|f\|_{L^p(X, \mathcal{A}, \mu)} < +\infty$ , we have by estimate (8)  $f_0 \in L^{p_0}(X, \mathcal{A}, \mu)$ . Analogously we get  $\|f_1\|_{L^{p_1}(X, \mathcal{A}, \mu)}^{p_1} \leq (\delta\alpha)^{p_1-p} \|f\|_{L^p(X, \mathcal{A}, \mu)}^p$  and so  $f_1 \in L^{p_1}(X, \mathcal{A}, \mu)$ . Therefore  $f \equiv f_0 + f_1 \in L^{p_0}(X, \mathcal{A}, \mu) + L^{p_1}(X, \mathcal{A}, \mu)$ .

*Proof of the equality (†).* We have to prove that  $\{|f| > \delta\alpha\} \in \mathcal{A}^1$ . Since  $f$  is complex-valued, we may write  $f \equiv \operatorname{Re} f + i \operatorname{Im} f$  and thus  $|f|^2 \equiv \operatorname{Re}^2 f + \operatorname{Im}^2 f$ . Since  $f$  is measurable by hypothesis this implies that  $\operatorname{Re} f$  and  $\operatorname{Im} f$  are measurable<sup>2</sup>. Further for measurable real-valued functions  $f, g : (X, \mathcal{A}) \rightarrow (\overline{\mathbb{R}}, \overline{\mathfrak{B}})$ <sup>3</sup> the functions  $f + g$  and  $f \cdot g$  are measurable<sup>4</sup> and thus  $|f|^2$  is measurable. Hence  $\{\operatorname{Re}^2 f + \operatorname{Im}^2 f > \lambda\} \in \mathcal{A}^5$  for any  $\lambda \in \mathbb{R}$ . So especially for  $\lambda := (\delta\alpha)^2$  we have  $\{|f| > \delta\alpha\} \in \mathcal{A}^6$ . In a similar manner it can also be proven that  $\{|f| \leq \delta\alpha\} \in \mathcal{A}$ . Let us prove a useful

LEMMA 2.1. *Let  $A \in \mathcal{P}(X)$  and  $\chi_A : (X, \mathcal{A}) \rightarrow (\mathbb{C}, \mathfrak{B}^2)$  be the characteristic function of the set  $A$ . Then  $\chi_A$  is measurable if and only if  $A$  is measurable.*

*Proof.* Assume  $\chi_A$  is measurable. Then  $\operatorname{Re} \chi_A$  and  $\operatorname{Im} \chi_A$  are measurable. Especially for  $0 < \lambda < 1$  we have that  $\{\operatorname{Re} \chi_A > \lambda\} = A \in \mathcal{A}$ . Conversely, assume  $A$  is measurable.

<sup>1</sup> For  $Y \in \mathcal{A}$  the  $\mu$ -integral of  $f : X \rightarrow \mathbb{C}$  over  $Y$  is defined to be  $\int_Y f d\mu := \int_X f \cdot \chi_Y d\mu$ . For more details see [Els11, pp. 135–136].

<sup>2</sup> For a proof see [Els11, p. 106].

<sup>3</sup>  $\overline{\mathfrak{B}} := \sigma(\mathbb{R})$  and  $\overline{\mathfrak{B}} = \{B \cup E : B \in \mathfrak{B}, E \subseteq \{\pm\infty\}\}$ .

<sup>4</sup> For a proof see [Els11, p. 107].

<sup>5</sup> For a proof see [Els11, pp. 105–106].

<sup>6</sup> This follows from the fact that  $x < y$  if and only if  $x^n < y^n$  for  $n \in \mathbb{N}_{>0}$  and some real numbers  $x, y > 0$ .

For  $\lambda \in \mathbb{R}_{<0}$  we have  $\{\operatorname{Re}\chi_A > \lambda\} = X \in \mathcal{A}$ ,  $\lambda \in [0, 1[$ ,  $\{\operatorname{Re}\chi_A > \lambda\} = A \in \mathcal{A}$  and  $\{\operatorname{Re}\chi_A > \lambda\} = \emptyset \in \mathcal{A}$  for  $\lambda \in \mathbb{R}_{\geq 1}$ . Since  $\operatorname{Im}\chi_A \equiv 0$  we have  $\{\operatorname{Im}\chi_A > \lambda\} = X \in \mathcal{A}$  if  $\lambda \in \mathbb{R}_{<0}$  and  $\{\operatorname{Im}\chi_A > \lambda\} = \emptyset \in \mathcal{A}$  if  $\lambda \in \mathbb{R}_{\geq 0}$ .  $\square$

By Lemma 2.1 and the fact that  $f \cdot g$  is measurable for two measurable functions  $f, g : (X, \mathcal{A}) \rightarrow (\mathbb{C}, \mathfrak{B}^2)$ <sup>7</sup>,  $f_0$  and  $f_1$  are measurable since  $f_0 \equiv f \cdot \chi_{\{|f|>\delta\alpha\}}$  and  $f_1 \equiv f \cdot \chi_{\{|f|\leq\delta\alpha\}}$ .

One subtlety is left to clear: the  $\mu$ -integrability of either  $|f_1|^{p_0}$  or  $|f_1|^{p_1}$  requires that  $|f_0|^{p_0}$  and  $|f_1|^{p_1}$  are measurable functions. By the fact that any continuous map  $g : (X, d_X) \rightarrow (Y, d_Y)$  between metric spaces is Borel-measurable (see [Els11, p. 86]) and that the composition of measurable functions is again measurable (see [Els11, p. 87]), the measurability of either  $f_0$  or  $f_1$  follows by  $|f_0|^{p_0} \equiv \cdot^{p_0} \circ |f \cdot \chi_{\{|f|>\delta\alpha\}}|$  and  $|f_1|^{p_1} \equiv \cdot^{p_1} \circ |f \cdot \chi_{\{|f|\leq\delta\alpha\}}|$  by stipulating  $\cdot^p : (\mathbb{R}_{\geq 0}, |\cdot|) \rightarrow (\mathbb{C}, |\cdot|)$ ,  $x^p := \exp(p \log(x))$  for  $p > 0$  and  $x \in \mathbb{R}_{>0}$  and  $x^p := 0$  if  $x = 0$ .

- b. Since  $T$  is a sublinear operator we have  $|T(f)| = |T(f_0 + f_1)| \leq |T(f_0)| + |T(f_1)|$ . Thus for any  $y \in Y$  with  $|T(f)(y)| > \alpha$  we therefore have either  $|T(f_0)(y)| > \frac{\alpha}{2}$  or  $|T(f_1)(y)| > \frac{\alpha}{2}$ <sup>8</sup>. Hence

$$\{|T(f)| > \alpha\} \subseteq \{|T(f_0)| > \alpha/2\} \cup \{|T(f_1)| > \alpha/2\}$$

and so by the monotonicity<sup>9</sup> and subadditivity<sup>10</sup> property of the measure  $\mu : \mathcal{A} \rightarrow \overline{\mathbb{R}}$  we have

$$\begin{aligned} d_{T(f)}(\alpha) &= \mu(\{|T(f)| > \alpha\}) \\ &\leq \mu(\{|T(f_0)| > \alpha/2\} \cup \{|T(f_1)| > \alpha/2\}) \\ &\leq \mu(\{|T(f_0)| > \alpha/2\}) + \mu(\{|T(f_1)| > \alpha/2\}) \\ &= d_{T(f_0)}(\alpha/2) + d_{T(f_1)}(\alpha/2) \end{aligned} \tag{9}$$

Now by the hypothesis  $\|T(f)\|_{L^{p_0,\infty}(Y,B,\nu)} \leq A_0 \|f\|_{L^{p_0}(X,\mathcal{A},\mu)}$  we have for  $d_{T(f_0)}(\alpha/2)$  the estimate

<sup>7</sup>Els11, p. 107.

<sup>8</sup>Without loss of generality assume  $|T(f_0)(y)| \leq |T(f_1)(y)|$ . Then we have  $\alpha < |T(f)(y)| \leq |T(f_0)(y)| + |T(f_1)(y)| \leq 2|T(f_1)(y)|$  (this is possible since  $\mathbb{R}$  is an ordered field).

<sup>9</sup> $A, B \in \mathcal{A}$  with  $A \subseteq B$  implies  $\mu(A) \leq \mu(B)$ . This follows immediately from the observation that  $B = (B \setminus A) \cup A$  and  $(B \setminus A) \cap A = \emptyset$  implies by the  $\sigma$ -additivity of  $\mathcal{A}$  directly the inequality  $\mu(A) \leq \mu(B \setminus A) + \mu(A) = \mu(B)$  since  $B \setminus A$  is measurable and  $\mu(B \setminus A) \geq 0$ .

<sup>10</sup> $(A_\iota)_{\iota \in I} \in \mathcal{A}^I$  where  $|I| < \aleph_0$  implies  $\mu\left(\bigcup_{\iota \in I} A_\iota\right) \leq \sum_{\iota \in I} \mu(A_\iota)$ . A proof can be found in [Els11, p. 31].

$$\begin{aligned}
 d_{T(f_0)}(\alpha/2) &= \left(\frac{\alpha/2}{\alpha/2}\right)^{p_0} d_{T(f_0)}(\alpha/2) \\
 &\leq \left(\frac{1}{\alpha/2}\right)^{p_0} \left[ \sup \left\{ \gamma d_{T(f_0)}(\gamma)^{1/p_0} : \gamma \in \mathbb{R}_{>0} \right\} \right]^{p_0} \\
 &= \left(\frac{1}{\alpha/2}\right)^{p_0} \|T(f_0)\|_{L^{p_0,\infty}(Y,\mathcal{B},\nu)}^{p_0} \\
 &\leq \left(\frac{A_0}{\alpha/2}\right)^{p_0} \|f_0\|_{L^{p_0}(X,\mathcal{A},\mu)}^{p_0}
 \end{aligned}
 \tag{10}$$

Analogously we get by the second hypothesis  $\|T(f)\|_{L^{p_1,\infty}(Y,\mathcal{B},\nu)} \leq A_1 \|f\|_{L^{p_1}(X,\mathcal{A},\mu)}$  an estimate for  $d_{T(f_1)}(\alpha/2)$  of the form  $d_{T(f_1)}(\alpha/2) \leq \left(\frac{A_1}{\alpha/2}\right)^{p_1} \|f_1\|_{L^{p_1}(X,\mathcal{A},\mu)}^{p_1}$ . Combining estimates (9), (10) and (b.) and using the definitions of  $f_0, f_1$  we arrive at

$$d_{T(f)}(\alpha) \leq \left(\frac{A_0}{\alpha/2}\right)^{p_0} \int_{\{|f|>\delta\alpha\}} |f|^{p_0} d\mu + \left(\frac{A_1}{\alpha/2}\right)^{p_1} \int_{\{|f|\leq\delta\alpha\}} |f|^{p_1} d\mu
 \tag{11}$$

c. By

$$\int_0^{\frac{1}{\delta}|f|} \alpha^{p-p_0-1} d\lambda = \begin{cases} \frac{1}{p-p_0} \frac{1}{\delta^{p-p_0}} |f|^{p-p_0}, & p \geq p_0 + 1 \\ \lim_{\omega \rightarrow 0^+} \int_{\omega}^{\frac{1}{\delta}|f|} \alpha^{p-p_0-1} d\lambda \\ = \lim_{\omega \rightarrow 0^+} \left[ \frac{1}{p-p_0} \alpha^{p-p_0} \left| \frac{1}{\delta} |f| \right| \right] \\ = \frac{1}{p-p_0} \left[ \frac{1}{\delta^{p-p_0}} |f|^{p-p_0} - \lim_{\omega \rightarrow 0^+} \omega^{p-p_0} \right] \\ = \frac{1}{p-p_0} \frac{1}{\delta^{p-p_0}} |f|^{p-p_0}, & p_0 < p < p_0 + 1 \end{cases}
 \tag{12}$$

and

$$\begin{aligned}
 \int_{\frac{1}{\delta}|f|}^{+\infty} \alpha^{p-p_1-1} d\lambda &= \lim_{\omega \rightarrow +\infty} \left[ \frac{1}{p-p_1} \alpha^{p-p_1} \right]_{\frac{1}{\delta}|f|}^{\omega} \\
 &= \frac{1}{p-p_1} \left[ \lim_{\omega \rightarrow +\infty} \omega^{p-p_1} - \frac{1}{\delta^{p-p_1}} |f|^{p-p_1} \right] \\
 &= \frac{1}{p_1-p} \frac{1}{\delta^{p-p_1}} |f|^{p-p_1}
 \end{aligned}
 \tag{13}$$

and the representation  $\|f\|_{L^p(X,\mathcal{A},\mu)}^p = p \int_0^{+\infty} \alpha^{p-1} d_f(\alpha) d\lambda$  for  $0 < p < +\infty$  we get

$$\begin{aligned}
& \|T(f)\|_{L^p(Y, \mathcal{B}, \nu)}^p \\
&= p \int_0^{+\infty} \alpha^{p-1} d_{T(f)} \lambda \\
&\leq p(2A_0)^{p_0} \int_0^{+\infty} \alpha^{p-p_0-1} \int_{\{|f|>\delta\alpha\}} |f|^{p_0} d\mu d\lambda + p(2A_1)^{p_1} \int_0^{+\infty} \alpha^{p-p_1-1} \int_{\{|f|\leq\delta\alpha\}} |f|^{p_1} d\mu d\lambda \\
&\stackrel{(\dagger)}{=} p(2A_0)^{p_0} \int_{\{|f|>0\}} |f|^{p_0} \int_0^{\frac{1}{\delta}|f|} \alpha^{p-p_0-1} d\lambda d\mu + p(2A_0)^{p_0} \int_{\{|f|=0\}} |f|^{p_0} \int_0^{\frac{1}{\delta}|f|} \alpha^{p-p_0-1} d\lambda d\mu \\
(14) \quad &+ p(2A_1)^{p_1} \int_X |f|^{p_1} \int_{\frac{1}{\delta}|f|}^{+\infty} \alpha^{p-p_1-1} d\lambda d\mu \\
&= p(2A_0)^{p_0} \int_X |f|^{p_0} \int_0^{\frac{1}{\delta}|f|} \alpha^{p-p_0-1} d\lambda d\mu + p(2A_1)^{p_1} \int_X |f|^{p_1} \int_{\frac{1}{\delta}|f|}^{+\infty} \alpha^{p-p_1-1} d\lambda d\mu \\
&= \frac{p(2A_0)^{p_0}}{p-p_0} \frac{1}{\delta^{p-p_0}} \int_X |f|^{p_0} |f|^{p-p_0} d\mu + \frac{p(2A_1)^{p_1}}{p_1-p} \frac{1}{\delta^{p-p_1}} \int_X |f|^{p_1} |f|^{p-p_1} d\mu \\
&= p \left( \frac{(2A_0)^{p_0}}{p-p_0} \frac{1}{\delta^{p-p_0}} + \frac{(2A_1)^{p_1}}{p_1-p} \delta^{p_1-p} \right) \|f\|_{L^p(X, \mathcal{A}, \mu)}^p
\end{aligned}$$

We pick  $\delta > 0$  such that  $(2A_0)^{p_0} \frac{1}{\delta^{p-p_0}} = (2A_1)^{p_1} \delta^{p_1-p}$ . Solving for  $\delta$  yields

$$(15) \quad \delta = \frac{1}{2} \frac{A_0^{\frac{p_0}{p_1-p_0}}}{A_1^{\frac{p_1}{p_1-p_0}}}$$

Substituting this in estimate (14) leads to

$$\begin{aligned}
(16) \quad \|T(f)\|_{L^p(Y, \mathcal{B}, \nu)}^p &\leq p \left( \frac{(2A_0)^{p_0}}{p-p_0} \frac{2^{p-p_0} A_1^{\frac{p_1(p-p_0)}{p_1-p_0}}}{A_0^{\frac{p_0(p-p_0)}{p_1-p_0}}} + \frac{(2A_1)^{p_1}}{p_1-p} \frac{A_0^{\frac{p_0(p_1-p)}{p_1-p_0}}}{2^{p_1-p} A_1^{\frac{p_1(p_1-p)}{p_1-p_0}}} \right) \|f\|_{L^p(X, \mathcal{A}, \mu)}^p \\
&= 2^p p \left( \frac{A_0^{\frac{p_0(p_1-p)}{p_1-p_0}} A_1^{\frac{p_1(p-p_0)}{p_1-p_0}}}{p-p_0} + \frac{A_0^{\frac{p_0(p_1-p)}{p_1-p_0}} A_1^{\frac{p_1(p-p_0)}{p_1-p_0}}}{p_1-p} \right) \|f\|_{L^p(X, \mathcal{A}, \mu)}^p
\end{aligned}$$

And taking the  $p$ -th power further

$$\begin{aligned}
 \|T(f)\|_{L^p(Y, \mathcal{B}, \nu)} &\leq 2 \left( \frac{p}{p-p_0} + \frac{p}{p_1-p} \right)^{1/p} A_0^{\frac{p_0(p_1-p)}{p(p_1-p_0)}} A_1^{\frac{p_1(p-p_0)}{p(p_1-p_0)}} \|f\|_{L^p(X, \mathcal{A}, \mu)} \\
 &= 2 \left( \frac{p}{p-p_0} + \frac{p}{p_1-p} \right)^{1/p} A_0^{\frac{p_0(p_1-p)}{p(p_1-p_0)} \frac{p_1}{p_1}} A_1^{\frac{p_1(p-p_0)}{p(p_1-p_0)} \frac{p_0}{p_0}} \|f\|_{L^p(X, \mathcal{A}, \mu)} \\
 &= 2 \left( \frac{p}{p-p_0} + \frac{p}{p_1-p} \right)^{1/p} A_0^{\frac{p_1-p}{p p_1} \frac{p_1-p_0}{p_0 p_1}} A_1^{\frac{p-p_0}{p_0 p} \frac{p_1-p_0}{p_0 p_1}} \|f\|_{L^p(X, \mathcal{A}, \mu)} \\
 &= 2 \left( \frac{p}{p-p_0} + \frac{p}{p_1-p} \right)^{1/p} A_0^{\frac{\frac{1}{p} - \frac{1}{p_1}}{\frac{1}{p_0} - \frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0} - \frac{1}{p}}{\frac{1}{p_0} - \frac{1}{p_1}}} \|f\|_{L^p(X, \mathcal{A}, \mu)}
 \end{aligned}
 \tag{17}$$

*Proof of the equality (‡).* The set  $\{|f| = 0\}$  is clearly measurable since by the measurability of  $f$  the sets  $\{|f| \geq 0\}$  and  $\{|f| \leq 0\}$  are measurable and thus by the property of a  $\sigma$ -algebra we have  $\{|f| = 0\} = \{|f| \geq 0\} \cap \{|f| \leq 0\} \in \mathcal{A}$  (or by the property of a  $\sigma$ -algebra we have  $\{|f| = 0\}^c = \{|f| > 0\} \in \mathcal{A}$ ).

- (ii.) **a.** Now consider the case  $p_1 = +\infty$ . We again use the cut-off functions defined in 7 to write  $f \equiv f_0 + f_1 \in L^{p_0}(X, \mathcal{A}, \mu) + L^{p_1}(X, \mathcal{A}, \mu)$ . Since  $\{|f_1| > \delta\alpha\} = \emptyset$ , we have

$$\begin{aligned}
 \|T(f_1)\|_{L^\infty(Y, \mathcal{B}, \nu)} &\leq A_1 \|f_1\|_{L^\infty(X, \mathcal{A}, \mu)} \\
 &= A_1 \inf \{B \in \mathbb{R}_{>0} : \mu(\{|f_1| > B\}) = 0\} \\
 &\leq A_1 \delta\alpha \\
 &= \frac{\alpha}{2}
 \end{aligned}
 \tag{18}$$

Provided we stipulate  $\delta := \frac{1}{2A_1}$ . Therefore the set  $\{|T(f_1)| > \alpha/2\}$  has measure zero (this is immediate since  $\|T(f_1)\|_{L^\infty(Y, \mathcal{B}, \nu)} = \inf \{B \in \mathbb{R}_{>0} : \mu(\{|T(f_1)| > B\}) = 0\} \leq \alpha/2$  and any subset of a set with measure zero has itself measure zero<sup>11</sup>). Thus similar to part **b.** of **(i.)** we get  $d_{T(f)}(\alpha) \leq d_{T(f_0)}(\alpha/2)$ .

- b.** By the hypothesis  $\|T(f)\|_{L^{p_0, \infty}(Y, \mathcal{B}, \nu)} \leq A_0 \|f\|_{L^{p_0}(X, \mathcal{A}, \mu)}$  for all  $f \in L^{p_0}(X, \mathcal{A}, \mu)$  we have again the estimate  $d_{T(f_0)}(\alpha/2) \leq \left(\frac{A_0}{\alpha/2}\right)^{p_0} \int_{\{2A_1|f| > \alpha\}} |f|^{p_0} d\mu$ .
- c.** Thus by **a.** and **b.**

<sup>11</sup>Let  $A, B \in \mathcal{A}$  with  $A \subseteq B$  and  $\mu(B) = 0$ . Then  $0 \leq \mu(A) \leq \mu(B) = 0$ .



$$\begin{aligned}
 \|T(f)\|_{L^p(Y, \mathcal{B}, \nu)}^p &= p \int_0^{+\infty} \alpha^{p-1} d_{T(f)} d\lambda \\
 &\leq p(2A_0)^{p_0} \int_0^{+\infty} \alpha^{p-p_0-1} \int_{\{2A_1|f|>\alpha\}} |f|^{p_0} d\mu d\lambda \\
 &= p(2A_0)^{p_0} \int_X |f|^{p_0} \int_0^{2A_1|f|} \alpha^{p-p_0-1} d\lambda d\mu \\
 &= \frac{2^p p A_0^{p_0} A_1^{p-p_0}}{p-p_0} \int_X |f|^p d\mu \\
 &= \frac{2^p p A_0^{p_0} A_1^{p-p_0}}{p-p_0} \|f\|_{L^p(X, \mathcal{A}, \mu)}^p
 \end{aligned}
 \tag{19}$$

That the constant found in (19) really agrees with the one stated in the theorem, can be seen by passing the constant (6) to the limit  $p_1 \rightarrow +\infty$ . We get

$$\begin{aligned}
 \lim_{p_1 \rightarrow +\infty} A &= \lim_{p_1 \rightarrow +\infty} \left[ 2 \left( \frac{p}{p-p_0} + \frac{p}{p_1-p} \right)^{1/p} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}} \right] \\
 &= 2 \exp \left[ \frac{1}{p} \log \left( \frac{p}{p-p_0} + \lim_{p_1 \rightarrow +\infty} \frac{1}{p_1} \frac{p}{1-p \lim_{p_1 \rightarrow +\infty} \frac{1}{p_1}} \right) \right] \\
 &\quad \cdot \lim_{p_1 \rightarrow +\infty} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} \cdot \lim_{p_1 \rightarrow +\infty} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}} \\
 &= 2 \left( \frac{p}{p-p_0} \right)^{1/p} \exp \left[ \frac{\frac{1}{p} - \lim_{p_1 \rightarrow +\infty} \frac{1}{p_1}}{\frac{1}{p_0} - \lim_{p_1 \rightarrow +\infty} \frac{1}{p_1}} \log(A_0) \right] \exp \left[ \frac{\frac{1}{p_0} - \frac{1}{p}}{\frac{1}{p_0} - \lim_{p_1 \rightarrow +\infty} \frac{1}{p_1}} \log(A_1) \right] \\
 &= 2 \left( \frac{p}{p-p_0} \right)^{1/p} A_0^{\frac{p_0}{p}} A_1^{1-\frac{p_0}{p}}
 \end{aligned}
 \tag{20}$$

Taking the  $p$ -th power in the estimate (19) finally yields the desired result. □

**3. The Complex Method.** This theorem will unfortunately only be applicable to linear operators but will yield a more natural bound of the operator on the intermediate space. The proof will make strong use of complex variables technique. A major tool will be an application of the maximum modulus principle, known as *Hadamard's three lines lemma*.

### 3.1. Hadamard's Three Lines Lemma.

LEMMA 3.1. *Hadamard's three lines lemma*) Let  $F$  be an analytic function on the strip  $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ , continuous and bounded on  $\overline{S}$ , such that  $|F(z)| \leq B_0$  when  $\operatorname{Re} z = 0$  and  $|F(z)| \leq B_1$  when  $\operatorname{Re} z = 1$ , for some  $0 < B_0, B_1 < \infty$ . Then  $|F(z)| \leq B_0^{1-\theta} B_1^\theta$  when  $\operatorname{Re} z = \theta$ , for any  $0 \leq \theta \leq 1$ .

*Proof.* For  $z \in \overline{S}$  define

$$(21) \quad G(z) := \frac{F(z)}{B_0^{1-z} B_1^z} \quad \forall n \in \mathbb{N}_{>0} : G_n(z) := G(z) e^{(z^2-1)/n}$$

Obviously,  $G(z)$  and  $G_n(z)$  are analytic functions on  $S$  for  $n \in \mathbb{N}_{>0}$ <sup>12</sup>. Further, we have

$$(22) \quad |B_0^{1-z} B_1^z|^2 = |B_0^{1-z}|^2 |B_1^z|^2 \stackrel{(\dagger)}{=} B_0^{1-z} B_0^{1-\bar{z}} B_1^z B_1^{\bar{z}} = (B_0^{1-\operatorname{Re} z})^2 (B_1^{\operatorname{Re} z})^2$$

Consider  $0 \leq \operatorname{Re} z \leq 1$  and  $B_0 \geq 1$ . Then  $B_0^{1-\operatorname{Re} z} = \exp((1 - \operatorname{Re} z) \log B_0) \geq 1$  and  $B_0^{1-\operatorname{Re} z} \geq B_0$  in the case  $B_0 < 1$ . A similar estimation of  $B_1^{\operatorname{Re} z}$  leads to

$$(23) \quad |B_0^{1-z} B_1^z| \geq \min\{1, B_0\} \min\{1, B_1\}$$

for all  $z \in \overline{S}$ . By this,  $G(z)$  is bounded on  $\overline{S}$  (by the boundedness of  $F$ ). Let  $M > 0$ , such that  $|G(z)| \leq M$  for  $z \in \overline{S}$ . Fix  $n \in \mathbb{N}_{>0}$  and write  $z := x + iy \in \overline{S}$ . Since

$$(24) \quad \begin{aligned} |G_n(z)|^2 &= |G(z)|^2 |e^{(x+iy)^2-1)/n}|^2 \\ &\leq M^2 e^{(x^2+2ixy-y^2-1)/n} e^{(x^2-2ixy-y^2-1)/n} \\ &= M^2 \left(e^{-y^2/n}\right)^2 \left(e^{(x^2-1)/n}\right)^2 \\ &\leq M^2 \left(e^{-y^2/n}\right)^2 \\ &= M^2 \left(e^{-|y|^2/n}\right)^2 \end{aligned}$$

we have  $\lim_{y \rightarrow \pm\infty} \sup\{|G_n(z)| : x \in [0, 1]\} = 0$  by the pinching-principle. Hence there exists some  $C(n) \in \mathbb{R}_{>0}$ , such that  $|G_n(z)| \leq 1$  for all  $|y| \geq C(n)$  and all  $x \in [0, 1]$ . Consider the rectangle  $R := [0, 1] \times [-C(n), C(n)]$ . Now  $|G_n(z)| \leq 1$  on the lines  $[0, 1] \times \{\pm C(n)\}$  and since  $|G(z)| = |F(z)|/B_0 \leq 1$ ,  $|G(z)| = |F(z)|/B_1 \leq 1$  on the line  $\{0\} \times [-C(n), C(n)]$  and

<sup>12</sup> Recall, that a function  $f$  is called *analytic on  $U$* ,  $U \subseteq \mathbb{C}$  open, if  $f$  is analytic at every  $z_0 \in U$ , that is, there exists a power series  $\sum_{n \in \mathbb{N}} a_n (z - z_0)^n$  and some  $r > 0$ , such that the series converges absolutely for  $|z - z_0| < r$ , and such that for such  $z$ , we have  $f(z) = \sum_{n \in \mathbb{N}} a_n (z - z_0)^n$  (as defined in [Lan93, pp. 68–69]). If  $f$  and  $g$  are analytic on  $U \subseteq \mathbb{C}$ , so are  $f + g$ ,  $f \cdot g$ . Also  $f/g$  is analytic on the open subset of  $z \in U$  such that  $g(z) \neq 0$ . If  $g : U \rightarrow V$  and  $f : V \rightarrow C$  are analytic so is  $f \circ g$ . Further, if  $f(z) = \sum_{n \in \mathbb{N}} a_n z^n$  is a power series with radius of convergence  $r$ ,  $f$  is analytic on  $B_r(0)$  (for a proof see [Lan93, pp. 69–70]).

$\{1\} \times [-C(n), C(n)]$  respectively by assumption, we have  $|G_n(z)| \leq 1$  on  $\partial S$ . By the maximum modulus principle<sup>13</sup> we have  $|G_n(z)| \leq 1$  on  $R$  and thus  $|G_n(z)| \leq 1$  on  $\bar{S}$ . Since inequalities are preserved by limits and the modulus is a continuous function, we have that  $|G(z)| = \lim_{n \rightarrow \infty} |G_n(z)| \leq 1$  on  $\bar{S}$ . Taking  $z := \theta + it$ , where  $0 \leq \theta \leq 1$  and  $t \in \mathbb{R}$ , we conclude  $|F(z)| = |G(z)| |B_0^{1-z} B_1^z| \leq B_0^{1-\theta} B_1^\theta$ , which completes the proof.

*Proof of the equality (†).* For any  $\alpha \in \mathbb{R}_{>0}$  and  $z \in \mathbb{C}$  we have  $\alpha^z = \exp(z \log(\alpha))$ . Since the exponential function is convergent on the whole complex plane, for fixed  $\varepsilon > 0$  we find  $C \in \mathbb{N}$  such that  $|\sum_{k=0}^N \frac{z^k}{k!} - \exp(z)| < \varepsilon$  whenever  $N > C$ . But by the properties of the complex conjugate we get  $|\sum_{k=0}^N \frac{\bar{z}^k}{k!} - \overline{\exp(z)}| = |\sum_{k=0}^N \frac{z^k}{k!} - \exp(z)| = |\sum_{k=0}^N \frac{z^k}{k!} - \exp(z)| < \varepsilon$ . Therefore  $\overline{\exp(z)} = \sum_{k \in \mathbb{N}} \frac{\bar{z}^k}{k!} = \exp(\bar{z})$  and thus  $\overline{\alpha^z} = \alpha^{\bar{z}}$ .  $\square$

**REMARK 3.1.** *To apply the maximum modulus principle it is mandatory for  $G_n$  to be non-constant. That the constant case is obviously true can be seen as follows. Assume  $G_n(z) \equiv w \in \mathbb{C}$  for  $z \in S$ . This immediately implies  $F(z) = w B_0^{1-z} B_1^z e^{(1-z^2)/n}$ . Hence  $F(z) = F(z; n)$ . Thus the only possible case left is  $w = 0$  and so  $F \equiv 0$ . But then the lemma holds trivially.*

**3.2. The Riesz-Thorin Interpolation Theorem.** Now we are able to prove the Riesz-Thorin Interpolation theorem without an interruption.

**THEOREM 3.1. (Riesz-Thorin Interpolation Theorem)** *Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $(Y, \mathcal{B}, \nu)$  a  $\sigma$ -finite measure space and  $T$  be a linear operator defined on the set of all finitely simple functions on  $X$  and taking values in the set of measurable functions on  $Y$ . Let  $1 \leq p_0, p_1, q_0, q_1 \leq \infty$  and assume that*

$$(25) \quad \|T(f)\|_{L^{q_0}(Y, \mathcal{B}, \nu)} \leq M_0 \|f\|_{L^{p_0}(X, \mathcal{A}, \mu)} \quad \|T(f)\|_{L^{q_1}(Y, \mathcal{B}, \nu)} \leq M_1 \|f\|_{L^{p_1}(X, \mathcal{A}, \mu)}$$

*holds for all finitely simple functions  $f$  on  $X$  and  $0 < M_0, M_1 < \infty$ . Then for all  $0 < \theta < 1$  we have*

$$(26) \quad \|T(f)\|_{L^q(Y, \mathcal{B}, \nu)} \leq M_0^{1-\theta} M_1^\theta \|f\|_{L^p(X, \mathcal{A}, \mu)}$$

*for all finitely simple functions  $f$  on  $X$ , where*

$$(27) \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

:

<sup>13</sup> The theorem can be found in [Lan93, pp. 91–92]. I will reproduce it here.

**LEMMA 3.2. (Maximum Modulus Principle, global version)** *Let  $U \subseteq \mathbb{C}$  be a connected open set, and let  $f$  be an analytic function on  $U$ . If  $z_0 \in U$  is a maximum point for  $|f|$ , that is  $|f(z_0)| \geq |f(z)|$  for all  $z \in U$ , then  $f$  is constant on  $U$ .*

For our purpose the following corollary is more appropriate.

**COROLLARY 3.1.** *Let  $U \subseteq \mathbb{C}$  be a connected open set and  $f$  be a continuous function on  $\bar{U}$ , analytic and non-constant on  $U$ . If  $z_0 \in \bar{U}$  is a maximum for  $f$ , that is  $|f(z_0)| \geq |f(z)|$  for all  $z \in \bar{U}$ , then  $z_0 \in \partial U$ .*

*Proof.* We will use the fact that the  $L^p(Y, \mathcal{B}, \nu)$  norm of a function can be obtained via duality for  $1 < p \leq \infty$  (for  $p = \infty$  the underlying space has to be  $\sigma$ -finite according to [Els11, pp. 288–289]) by

$$\|f\|_{L^p(Y, \mathcal{B}, \nu)} = \sup \left\{ \left| \int_Y f g d\nu \right| : \|g\|_{L^{p'}(Y, \mathcal{B}, \nu)} = 1 \right\}$$

with  $p' := \frac{p}{p-1}$  for  $p \in ]1, \infty[$  and  $p' := 1$  for  $p = \infty$ . Since we will also make use of

$$\|f\|_{L^p(Y, \mathcal{B}, \nu)} = \sup \left\{ \left| \int_Y f g d\nu \right| : \|g\|_{L^{p'}(Y, \mathcal{B}, \nu)} \leq 1 \right\}$$

I will prove their equivalence. If we define  $\varphi_f(g) : L^{p'}(Y, \mathcal{B}, \nu) \rightarrow \mathbb{C}$ ,  $\varphi_f(g) := \int_Y f g d\nu$ ,  $\varphi_f$  is clearly a linear functional (to be precise, a continuous linear functional by [Els11, p. 289]). Hence let  $(V, \|\cdot\|)$  and  $(W, \|\cdot\|)$  be two normed vector spaces over  $\mathbb{C}$  and  $L \in \text{Hom}_{\mathbb{C}}(V, W)$  continuous. Then we define  $v_n := (1 - \frac{1}{n})v$  for  $v \in V$  with  $\|v\| = 1$  and  $n \in \mathbb{N}_{>0}$ . We have  $\|v_n\| = 1 - \frac{1}{n} \leq 1$ . Thus  $\|L(v_n)\| \leq \sup\{\|L(v)\| : \|v\| \leq 1\}$  and so  $\lim_{n \rightarrow \infty} \|L(v_n)\| = \|L(v)\| \leq \sup\{\|L(v)\| : \|v\| \leq 1\}$ . On the other hand we have  $\|L(v)\| \leq \frac{1}{\|v\|} \|L(v)\| = \left\| L \left( \frac{v}{\|v\|} \right) \right\| \leq \sup\{\|L(v)\| : \|v\| = 1\}$  for any  $v \in V$  with  $\|v\| \leq 1$ .

Define  $\mathfrak{F} := \text{span}_{\mathbb{C}}\{\chi_E : E \in \mathcal{B}, \nu(E) < \infty\}$ , the set of all finitely simple functions on  $Y$ <sup>14</sup>. Since  $\mathfrak{F}$  is dense in  $L^p(Y, \mathcal{B}, \nu)$  for every  $0 < p < \infty$ <sup>15</sup>, we may use the corollary found in [Bou95, p. 76]

**COROLLARY 3.2.** (Principle of extension of identities) *Let  $f, g$  be two continuous mappings of a topological space  $X$  into a Hausdorff space  $Y$ . If  $f(x) = g(x)$  at all points of a dense subset of  $X$ , then  $f \equiv g$ .*

to see, that also

$$\|f\|_{L^p(Y, \mathcal{B}, \nu)} = \sup \left\{ \left| \int_Y f g d\nu \right| : g \in \mathfrak{F}, \|g\|_{L^{p'}(Y, \mathcal{B}, \nu)} \leq 1 \right\}$$

Assume  $q > 1$ . Fix  $f := \sum_{k=1}^n a_k e^{i\alpha_k} \chi_{X_k}$ , where  $n \in \mathbb{N}_{>0}, a_k > 0, \alpha_k \in [0, 2\pi[, X_i \cap X_j = \emptyset$  for  $i, j = 1, \dots, n$  and  $\mu(X_k) < \infty$  for every  $k = 1, \dots, n$ . Further let  $g := \sum_{k=1}^m b_k e^{i\beta_k} \chi_{Y_k} \in \mathfrak{F}$ , where  $m \in \mathbb{N}_{>0}, b_k > 0$  and  $\beta_k \in [0, 2\pi[$ . Define

$$P(z) := \frac{p}{p_0}(1-z) + \frac{p}{p_1}z \quad Q(z) := \frac{q'}{q'_0}(1-z) + \frac{q'}{q'_1}z$$

for  $z \in \overline{S}$  (in the case  $p = \infty$  we get also  $p_0 = p_1 = \infty$  and hence by stipulating  $\frac{\infty}{\infty} := 1$  the function  $P$  is well-defined). Further let

<sup>14</sup> This is almost trivial. Consider  $Y_1, Y_2 \in \mathcal{B}$  with  $\nu(Y_1), \nu(Y_2) < \infty$  and  $Y_1 \cap Y_2 \neq \emptyset$ . Then  $f \equiv z_1 \chi_{Y_1} + z_2 \chi_{Y_2} \in \mathfrak{F}$  for  $z_1, z_2 \in \mathbb{C}$ . We see, that  $f \equiv z_1 \chi_{Y_1 \setminus Y_2} + z_2 \chi_{Y_2 \setminus Y_1} + (z_1 + z_2) \chi_{Y_1 \cap Y_2} \in \mathfrak{F}$  where the latter function is a finitely simple one since  $\nu(Y_1 \cup Y_2) \leq \nu(Y_1) + \nu(Y_2) < \infty$  and  $Y_1 \setminus Y_2, Y_2 \setminus Y_1, Y_1 \cap Y_2 \subseteq Y_1 \cup Y_2$ .

<sup>15</sup> In [Els11, p. 242] a proof can be found, that  $\mathfrak{F}$  is dense in  $L^p$  for  $0 < p < \infty$ . Now the canonical map  $\pi : \mathcal{L}^p \rightarrow L^p/\mathcal{N}$  is continuous. Hence we may use the following lemma.

**LEMMA 3.3.** *Let  $X$  and  $Y$  be topological spaces,  $f : X \rightarrow Y$  and  $A \subseteq X$  dense in  $X$ . Then  $f(A)$  is dense in  $Y$ .*

*Proof.* By [Mun00, p. 104] we have  $Y = f(X) = \overline{f(A)} \subseteq \overline{f(\overline{A})} \subseteq Y$ . □

$$(28) \quad f_z := \sum_{k=1}^n a_k^{P(z)} e^{i\alpha_k} \chi_{X_k} \quad g_z := \sum_{k=1}^m b_k^{Q(z)} e^{i\beta_k} \chi_{Y_k}$$

and

$$(29) \quad F(z) := \int_Y T(f_z)(y) g_z(y) d\nu(y)$$

By the linearity of the operator  $T$  we have

$$(30) \quad F(z) = \sum_{j=1}^n \sum_{k=1}^m a_j^{P(z)} b_k^{Q(z)} e^{i\alpha_j} e^{i\beta_k} \int_Y T(\chi_{X_j})(y) \chi_{Y_k}(y) d\nu(y)$$

and by using Hölder's inequality <sup>16</sup>

$$(31) \quad \begin{aligned} \left| \int_Y T(\chi_{X_j})(y) \chi_{Y_k}(y) d\nu(y) \right| &\leq \int_Y |T(\chi_{X_j})(y)| \chi_{Y_k}(y) d\nu(y) \\ &= \|T(\chi_{X_j}) \chi_{Y_k}\|_{L^1(Y, \mathcal{B}, \nu)} \\ &\leq \|T(\chi_{X_j})\|_{L^{q_0}(Y, \mathcal{B}, \nu)} \|\chi_{Y_k}\|_{L^{q'_0}(Y, \mathcal{B}, \nu)} \\ &\leq M_0 \|\chi_{X_j}\|_{L^{p_0}(X, \mathcal{A}, \mu)} \|\chi_{Y_k}\|_{L^{q'_0}(Y, \mathcal{B}, \nu)} \\ &= M_0 \left( \int_X |\chi_{X_j}(x)|^{p_0} d\mu(x) \right)^{1/p_0} \left( \int_Y |\chi_{Y_k}(y)|^{q'_0} d\nu(y) \right)^{1/q'_0} \\ &= M_0 \left( \int_X \chi_{X_j}(x) d\mu(x) \right)^{1/p_0} \left( \int_Y \chi_{Y_k}(y) d\nu(y) \right)^{1/q'_0} \\ &= M_0 \mu(X_j)^{p_0} \nu(Y_k)^{q'_0} \\ &< \infty \end{aligned}$$

for  $p_0 < \infty$  and each  $j = 1, \dots, n$ ,  $k = 1, \dots, m$  we get that  $F(z)$  is analytic on  $S$ . The case  $p_0, q'_0 = \infty$  is trivial since  $\|\chi_{X_j}\|_{L^\infty(X, \mathcal{A}, \mu)}, \|\chi_{Y_k}\|_{L^\infty(Y, \mathcal{B}, \nu)} \leq 1$ . Now

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<sup>16</sup>A proof can be found in [Els11, p. 223].

$$\begin{aligned}
 \|f_{it}\|_{L^{p_0}(X, \mathcal{A}, \mu)} &= \left( \sum_{k=1}^n \int_X |f_{it}|^{p_0} d\mu + \int_{X \setminus \bigcup_{k=1}^n X_k} |f_{it}|^{p_0} d\mu \right)^{1/p_0} \\
 &= \left( \sum_{k=1}^n |a_k^{P(it)} e^{i\alpha_k}|^{p_0} \int_X \chi_{X_k} d\mu \right)^{1/p_0} \\
 (32) \quad &= \left( \sum_{k=1}^n a_k^{p_0 \operatorname{Re} P(it)} \mu(X_k) \right)^{1/p_0} \\
 &= \left( \sum_{k=1}^n a_k^p \mu(X_k) \right)^{p/p_0} \\
 &= \|f\|_{L^p(X, \mathcal{A}, \mu)}^{p/p_0}
 \end{aligned}$$

for  $p_0 \neq \infty$  and  $p < \infty$ . Let us consider  $p_0 = \infty$ . Then either  $\|f_{it}\|_{L^\infty(X, \mathcal{A}, \mu)} = 0$  or  $\|f_{it}\|_{L^\infty(X, \mathcal{A}, \mu)} = 1$ . Since  $\|\cdot\|_{L^p(X, \mathcal{A}, \mu)}$  is a norm for  $1 \leq p \leq \infty$  (see [Els11, p. 231]), we have  $f = 0 + \mathcal{N}$  if  $\|f_{it}\|_{L^\infty(X, \mathcal{A}, \mu)} = 0$ . Since  $f \in \mathfrak{F}$ , we may conclude  $f \equiv \sum_{k=1}^n a_k e^{i\alpha_k} \chi_{X_k}$ , where  $\mu(X_k) = 0$  for  $k = 1, \dots, n$ . But then  $\|f_{it}\|_{L^\infty(X, \mathcal{A}, \mu)} = \inf\{B > 0 : \mu(\{|f_{it}| > B\}) = 0\} = \inf\{B > 0 : \mu(\{1 > B\}) = 0\} = 0$  since  $|a_k^{P(it)}| = \lim_{p_0 \rightarrow \infty} a_k^{p/p_0} = 1$ . In the other case we simply have  $\|f_{it}\|_{L^\infty(X, \mathcal{A}, \mu)} = 1$  since there exists at least one subset  $X_k$  such that  $\mu(X_k) \neq 0$ . Now consider  $p = \infty$ . Then  $p_0 = p_1 = \infty$ . Thus  $P(it) = 1$  and so  $f_z \equiv f$ . By the same considerations we see that  $\|g_{it}\|_{L^{q'_0}(Y, \mathcal{B}, \nu)} = \|g\|_{L^{q'}(Y, \mathcal{B}, \nu)}^{q'/q'_0}$  for  $q_0 \in [1, \infty]$  (set  $\infty' := 1$ ). Hence

$$\begin{aligned}
 |F(it)| &\leq \int_Y |T(f_{it})(y)g_{it}(y)| d\nu(y) \\
 (33) \quad &= \|T(f_{it})g_{it}\|_{L^1(Y, \mathcal{B}, \nu)} \\
 &\leq \|T(f_{it})\|_{L^{q_0}(Y, \mathcal{B}, \nu)} \|g_{it}\|_{L^{q'_0}(Y, \mathcal{B}, \nu)} \\
 &\leq M_0 \|f_{it}\|_{L^{p_0}(X, \mathcal{A}, \mu)} \|g_{it}\|_{L^{q'_0}(Y, \mathcal{B}, \nu)} \\
 &= M_0 \|f\|_{L^p(X, \mathcal{A}, \mu)}^{p/p_0} \|g\|_{L^{q'}(Y, \mathcal{B}, \nu)}^{q'/q'_0}
 \end{aligned}$$

by Hölder's inequality. By similar calculations we get

$$(34) \quad \|f_{1+it}\|_{L^{p_1}(X, \mathcal{A}, \mu)} = \|f\|_{L^p(X, \mathcal{A}, \mu)}^{p/p_1} \quad \|g_{1+it}\|_{L^{q'_1}(Y, \mathcal{B}, \nu)} = \|g\|_{L^{q'}(Y, \mathcal{B}, \nu)}^{q'/q'_1}$$

and thus

$$(35) \quad |F(1+it)| \leq M_1 \|f\|_{L^p(X, \mathcal{A}, \mu)}^{p/p_1} \|g\|_{L^{q'}(Y, \mathcal{B}, \nu)}^{q'/q'_1}$$

Since  $F$  is analytic on  $S$  and continuous on  $\overline{S}$  and further

$$\begin{aligned}
|F(z)| &\leq \int_Y |T(f_z)(y)g_z(y)|d\nu(y) \\
&= \|T(f_z)g_z\|_{L^1(Y, \mathcal{B}, \nu)} \\
&\leq \|T(f_z)\|_{L^{q_0}(Y, \mathcal{B}, \nu)} \|g_z\|_{L^{q'_0}(Y, \mathcal{B}, \nu)} \\
&\leq M_0 \|f_z\|_{L^{p_0}(X, \mathcal{A}, \mu)} \|g_z\|_{L^{q'_0}(Y, \mathcal{B}, \nu)} \\
(36) \quad &= M_0 \left( \int_X |f_z|^{p_0} d\mu \right)^{1/p_0} \left( \int_Y |g_z|^{q'_0} d\nu \right)^{1/q'_0} \\
&= M_0 \left( \sum_{j=1}^n a_j^{\operatorname{Re} P(z)} \mu(X_j) \right)^{1/p_0} \left( \sum_{k=1}^m b_k^{\operatorname{Re} Q(z)} \nu(Y_k) \right)^{1/q'_0} \\
&\leq M_0 \left( \sum_{j=1}^n a_j^{p/p_0 + p/p_1} \mu(X_j) \right)^{1/p_0} \left( \sum_{k=1}^m b_k^{q'/q'_0 + q'/q'_1} \nu(Y_k) \right)^{1/q'_0}
\end{aligned}$$

by Hölder's inequality  $F$  is bounded on  $\bar{S}$  we can apply Hadamard's three lines lemma to get

$$\begin{aligned}
(37) \quad |F(z)| &\leq \left( M_0 \|f\|_{L^p(X, \mathcal{A}, \mu)}^{p/p_0} \|g\|_{L^{q'_0}(Y, \mathcal{B}, \nu)}^{q'/q'_0} \right)^{1-\theta} \left( M_1 \|f\|_{L^p(X, \mathcal{A}, \mu)}^{p/p_1} \|g\|_{L^{q'_1}(Y, \mathcal{B}, \nu)}^{q'/q'_1} \right)^{\theta} \\
&= M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p(X, \mathcal{A}, \mu)} \|g\|_{L^{q'}(Y, \mathcal{B}, \nu)}
\end{aligned}$$

for  $\operatorname{Re} z = \theta$  where  $0 \leq \theta \leq 1$ . Further observe  $P(\theta) = Q(\theta) = 1$  and thus

$$\begin{aligned}
(38) \quad \|T(f)\|_{L^q(Y, \mathcal{B}, \nu)} &= \sup \left\{ \left| \int_Y T(f)g d\nu \right| : g \in \mathfrak{F}, \|g\|_{L^{q'}(Y, \mathcal{B}, \nu)} \leq 1 \right\} \\
&= \sup \left\{ |F(\theta)| : g \in \mathfrak{F}, \|g\|_{L^{q'}(Y, \mathcal{B}, \nu)} \leq 1 \right\} \\
&\leq M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p(X, \mathcal{A}, \mu)}
\end{aligned}$$

Now assume  $q = 1$ . Then  $q_0 = q_1 = 1$  and so  $Q(z) = 1$  which implies  $g_z \equiv g$  for every  $z \in \bar{S}$ . Assume, that  $\|g\|_{L^\infty(Y, \mathcal{B}, \nu)} \leq 1$ . Then the above proof is also valid, if we take the supremum over the simple functions, instead of finitely simple functions, since by [Coh13, p. 100] the simple functions are dense in  $L^\infty(Y, \mathcal{B}, \nu)$ .  $\square$

#### 4. Interpolation of Analytic Families of Operators.

**4.1. The Poisson Formula.** First, we have to extend Hadamard's three lines lemma appropriately (lemma 3.1). To do so, we first need some theorems and definitions of complex analysis.

**THEOREM 4.1.** (Complex Analysis Revisited) *Let  $h(e^{i\theta})$  be a continuous function on the unit circle. Then the Poisson integral*

$$\tilde{h}(z) = \int_{-\pi}^{\pi} h(e^{i\varphi}) P_r(\theta - \varphi) \frac{d\lambda(\varphi)}{2\pi} \quad z := re^{i\theta} \in \mathbb{D} := \{|z| < 1\}$$

where

$$(39) \quad P_r(\theta) := \sum_{k \in \mathbb{Z}} r^{|k|} e^{ik\theta} \quad 0 \leq r < 1, -\pi \leq \theta \leq \pi$$

denotes the Poisson kernel function, is a harmonic function on  $\mathbb{D}$  with boundary values  $h(e^{i\theta})$ , that is,  $\tilde{h}(e^{i\theta})$  tends to  $h(\zeta)$  as  $z \in \mathbb{D}$  tends to  $\zeta \in \partial\mathbb{D}$ .

*Proof.* A proof can be found in [Gam01, pp. 277–278]. □

Further we introduce the notion of a subharmonic function as found in [Gam01, p. 394].

**DEFINITION 4.1.** *Let  $D \subseteq \mathbb{C}$  be a domain (open and path-connected), and let  $u : D \rightarrow [-\infty, \infty[$  be continuous. We say that  $u(z)$  is subharmonic if for each  $z_0 \in D$ , there is  $\varepsilon > 0$  such that  $u(z)$  satisfies the mean value inequality*

$$(40) \quad u(z_0) \leq \int_0^{2\pi} u(z_0 + re^{i\theta}) \frac{d\lambda(\theta)}{2\pi} \quad 0 < r < \varepsilon$$

And the notion of a conformal mapping ([Gam01, p. 59]).

**DEFINITION 4.2.** *A smooth complex-valued function  $g(z)$  (that is,  $g(z)$  has as many derivatives as is necessary for whatever is being asserted to be true) is conformal at  $z_0$  if whenever  $\gamma_0, \gamma_1$  are two curves terminating at  $z_0$  with non-zero tangents, then the curves  $g \circ \gamma_0, g \circ \gamma_1$  have non-zero tangents at  $g(z_0)$  and the angle from  $(g \circ \gamma_0)'(z_0)$  to  $(g \circ \gamma_1)'(z_0)$  is the same as the angle from  $\gamma_0'(z_0)$  to  $\gamma_1'(z_0)$ . A conformal mapping of one domain  $D$  onto another  $V$  is a continuously differentiable function that is conformal at each point of  $D$  and that maps  $D$  one-to-one onto  $V$ .*

Now we are able to formulate the proof of the extension of Hadamard's three lines lemma.

**LEMMA 4.1.** (Hadamard's three lines lemma, extension) *Let  $F$  be an analytic function on the strip  $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$  and continuous on  $\overline{S}$ , such that for every  $z \in \overline{S}$  we have  $\log |F(z)| \leq Ae^{\tau|\operatorname{Im} z|}$  for some  $A < \infty$  and  $\tau \in [0, \pi[$ . Then*

$$|F(z)| \leq \exp \left( \frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \left[ \frac{\log |F(it + iy)|}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log |F(1 + it + iy)|}{\cosh(\pi t) + \cos(\pi x)} \right] d\lambda(t) \right)$$

whenever  $z := x + iy \in S$ .



*Proof.* Consider the function

$$(41) \quad h(z) := \frac{1}{\pi i} \operatorname{Log} \left( \frac{z+1}{iz-i} \right) = \frac{1}{\pi} \left( \operatorname{Arg} \left( \frac{1+z}{1-z} \right) - i \log \left| \frac{1+z}{1-z} \right| \right)$$

which maps  $\mathbb{D}$  onto  $]0, 1[ \times \mathbb{R}$ . □

#### 4.2. Stein's Theorem on Interpolation of Analytic Families of Operators.

**DEFINITION 4.3.** (Analytic family, admissible growth) *Let  $(X, \mu)$ ,  $(Y, \nu)$  be measure spaces and  $(T_z)_{z \in \overline{S}}$ , where  $T_z$  is defined on the space of all finitely simple functions on  $X$  and taking values in the space of all measurable functions on  $Y$  such that*

$$(42) \quad \int_Y |T_z(\chi_A) \chi_B| d\nu$$

*whenever  $\mu(A), \nu(B) < \infty$ . The family  $(T_z)_{z \in \overline{S}}$  is said to be analytic if for all  $f, g$  finitely simple we have that*

$$(43) \quad z \mapsto \int_Y T_z(f) g d\nu$$

*is analytic on  $S$  and continuous on  $\overline{S}$ . Further, an analytic family  $(T_z)_{z \in \overline{S}}$  is called of admissible growth, if there is a constant  $\tau \in [0, \pi[$ , such that for all finitely simple functions  $f, g$  a constant  $C(f, g)$  exists with*

$$(44) \quad \log \left| \int_Y T_z(f) g d\nu \right| \leq C(f, g) e^{\tau |\operatorname{Im} z|}$$

*for all  $z \in \overline{S}$ .*

**THEOREM 4.2.** (Riesz-Thorin interpolation theorem, extension) *Let  $(T_z)_{z \in \overline{S}}$  be an analytic family of admissible growth,  $1 \leq p_0, p_1, q_0, q_1 \leq \infty$  and suppose that  $M_0, M_1$  are positive functions on the real line such that for some  $\tau \in [0, \pi[$*

$$(45) \quad \sup \left\{ e^{-\tau |y|} \log M_0(y) : y \in \mathbb{R} \right\} < \infty \quad \sup \left\{ e^{-\tau |y|} \log M_1(y) : y \in \mathbb{R} \right\} < \infty$$

*Fix  $0 < \theta < 1$  and define*

$$(46) \quad \frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

*Further suppose that for all finitely simple functions  $f$  on  $X$  and  $y \in \mathbb{R}$  we have*

$$(47) \quad \|T_{iy}(y)\|_{L^{q_0}} \leq M_0(y) \|f\|_{L^{p_0}} \quad \|T_{1+iy}(y)\|_{L^{q_1}} \leq M_1(y) \|f\|_{L^{p_1}}$$

*Then for all finitely simple functions  $f$  on  $X$  we have*

$$\|T_\theta(f)\|_{L^q} \leq M(\theta) \|f\|_{L^p}$$

*where for  $0 < x < 1$*

$$M(x) = \exp \left( \frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \left[ \frac{\log M_0(t)}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log M_1(t)}{\cosh(\pi t) + \cos(\pi x)} \right] d\lambda(t) \right)$$

*Proof.* Fix  $0 < \theta < 1$  and finitely simple functions  $f, g$  on  $X, Y$  respectively with  $\|f\|_{L^p} = \|g\|_{L^{q'}} = 1$ . Define  $f_z, g_z$  as in (28) and for  $z \in \bar{S}$

$$(48) \quad F(z) := \int_Y T_z(f_z) g_z d\nu$$

Observe, that  $|a_j^{P(z)}| \leq a_j^{p/p_0+p/p_1}$  and  $|b_k^{Q(z)}| \leq b_k^{q'/q'_0+q'/q'_1}$  for  $z \in \bar{S}$ . Hence

$$(49) \quad \begin{aligned} \log |F(z)| &= \log \left| \sum_{j=1}^n \sum_{k=1}^m a_j^{P(z)} b_k^{Q(z)} e^{i\alpha_j} e^{i\beta_k} \int_Y T_z(\chi_{X_j})(y) \chi_{Y_k}(y) d\nu(y) \right| \\ &\leq \log \left( \sum_{j=1}^n \sum_{k=1}^m |a_j^{P(z)}| |b_k^{Q(z)}| \int_Y |T_z(\chi_{X_j})(y)| \chi_{Y_k}(y) d\nu(y) \right) \\ &\leq \log \left( \sum_{j=1}^n \sum_{k=1}^m a_j^{p/p_0+p/p_1} b_k^{q'/q'_0+q'/q'_1} \int_{Y_k} |T_z(\chi_{X_j})| d\nu \right) \end{aligned}$$

□

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