A CONVEXITY THEOREM FOR CERTAIN GROUPS OF TRANSFORMATIONS

By

I. I. Hirschman, Jr.[1]

in Saint Louis, Mo., U.S.A.

1. If $f(x) \in L^p(0, 2\pi)$ and if $c_n = (2\pi)^{-1} \int_0^{2\pi} f(x) e^{-inx} dx$ then we write $f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}$. Let $\{\lambda_n\}_{n=-\infty}^{\infty}$ be a set of real constants and consider the group of transformations

$$\Lambda(\sigma) f(x) \sim \sum_{-\infty}^{\infty} e^{\lambda_n \sigma} c_n e^{inx} \qquad (-\infty < \sigma < \infty).$$

The object of this paper is to establish the following "convexity" theorem.

Theorem 1. *Suppose that for every function $g \in L^p(0, 2\pi)$ and $h \in L^q(0, 2\pi)$ we have*

$$\| \Lambda(i\tau) g \|_p \leqq A(\tau) \| g \|_p, \quad \| \Lambda(i\tau) h \|_q \leqq A(\tau) \| h \|_q,$$

where for every $a > 0$

$$\log A(\tau) = O(e^{a|\tau|}) \qquad \tau \to \pm \infty.$$

Then if $0 < \Theta < 1$, and if $\sigma = \sigma_1 \Theta$, $\frac{1}{n} = (1-\Theta)\frac{1}{p} + \Theta \frac{1}{q}$, we have

$$\| \Lambda(\sigma) f \|_n \leqq B \| f \|_p^{1-\theta} \| \Lambda(\sigma_1) f \|_q^{\theta},$$

where B depends upon $\{\lambda_n\}$, σ_1, p, q, and Θ, but not upon f.

2. We need some results from the theory of harmonic majoration. Let us define

$$\omega(x, y) = \frac{\frac{1}{2} \tan \frac{\pi x}{2}}{\left[\tan^2 \frac{\pi x}{2} + \tanh^2 \frac{\pi y}{2}\right] \cosh^2 \frac{\pi y}{2}}. \tag{1}$$

---

1. John Simon Guggenheim Memorial Fellow. Work supported in part by the U.S. Office of Ordnance Research under Contract DA-23-072-ORD-392.

If $a_1(t)$, $a_2(t)$ are such that

$$\int_{-\infty}^{\infty} |a_i(t)| e^{-\pi|t|} dt < \infty \qquad (i = 1, 2),$$

then the formula

$$(2) \qquad u(x, y) = \int_{-\infty}^{\infty} \omega(x, y-t) a_1(t) dt + \int_{-\infty}^{\infty} \omega(1-x, y-t) a_2(t) dt$$

defines $u(x, y)$ as a harmonic function in the strip $0 < x < 1$, $-\infty < y < \infty$, and if $a_1(y)$ or $a_2(y)$ is continuous at y_0 then

$u(x, y) \to a_1(y_0)$ as $x \to 0+$, $y \to y_0$ or $u(x, y) \to a_2(y_0)$ as $x \to 1-$, $y \to y_0$.

This formula is entirely analogous to the Poisson representation of a function harmonic in the unit circle in terms of its boundary values.

We recall the principle of harmonic majoration. Let D be a domain with compact closure $\bar{D}$. If $f(z)$ is analytic in D and $u(z)$ is harmonic in D and if for every $z_0 \in \bar{D} - D$

$$\overline{\lim_{z \to z_0}} \log |f(z)| - u(z) \leqq 0,$$

then for $z \in D$

$$\log |f(z)| \leqq u(z).$$

See [3; pp. 37—45].

Lemma 1. *Let $f(z)$ be analytic in the strip*

$$0 \leqq x \leqq 1, \ -\infty < y < \infty$$

and let

$$\log M(r) = O(e^{a|r|}) \quad r \to \pm\infty \quad (a < \pi)$$

where $M(r) = $ l. u. b. $|f(z)|$ for $0 \leqq x \leqq 1$, $y = r$. If $\log |f(iy)| \leqq \alpha_1(y)$, $\log |f(1 + iy)| \leqq \alpha_2(y)$ for $-\infty < y < \infty$, then

$$\log |f(\Theta)| \leqq \int_{-\infty}^{\infty} \omega(\Theta, y) \alpha_1(y) dy + \int_{-\infty}^{\infty} \omega(1-\Theta, y) \alpha_2(y) dy.$$

It is assumed that $\alpha_1(y)$ and $\alpha_2(y)$ are continuous. Let D_T be the rectangle $0 < x < 1$, $-T < y < T$. Choose a', $a < a' < \pi$, and $\varepsilon > 0$ and consider

$$\log|f(z)| - \int_{-T}^{T} \omega(x, y-t)\,\alpha_1(t)\,dt - \int_{-T}^{T} \omega(1-x, y-t)\,\alpha_2(t)\,dt$$
$$- \varepsilon \cosh a'y \cos a'\left(x - \frac{1}{2}\right).$$

If T is sufficiently large then this will be negative on the boundary of D_T, and therefore, by the principle of harmonic majoration, negative in D_T. Letting T increase without limit and using Fatou's lemma we find that

$$\log|f(z)| \leqq \int_{-\infty}^{\infty} \omega(x, y-t)\,\alpha_1(t)\,dt + \int_{-\infty}^{\infty} \omega(1-x, y-t)\,\alpha_2(t)\,dt$$
$$+ \varepsilon \cosh a'y \cos a'\left(x - \frac{1}{2}\right).$$

Since ε is arbitrary

$$\log|f(z)| \leqq \int_{-\infty}^{\infty} \omega(x, y-t)\,\alpha_1(t)\,dt + \int_{-\infty}^{\infty} \omega(1-x, y-t)\,\alpha_2(t)\,dt.$$

If we set $z = \Theta$ we obtain our desired result.

We now proceed to the demonstration of Theorem 1. It is to be noted that $\|\Lambda(i\tau)g\|_2 \leqq \|g\|_2$ for every $g \in L^2$. Thus our assumption concerning $\Lambda(i\tau)$ is "reasonable". However it is not satisfied in general.

Theorem 1 is essentially a variant of the Riesz-Thorin convexity theorem and its proof proceeds along almost identical lines. See [4] and [5].

Let $g(x)$, $0 \leqq x \leqq 2\pi$, be a step function constant on the intervals I_N, $N = 1, 2, \ldots, M$, so that

$$g(x) = \exp(a_N + ib_N) \qquad x \in I_N.$$

We suppose $\|g(x)\|_{n'} \leqq 1$ where $\frac{1}{n'} + \frac{1}{n} = 1$. Let $w = u + iv$ be a complex variable. We set

$$g(x, w) = \exp\left[a_N\left\{(1-\bar{w})\frac{n'}{p'} + \bar{w}\frac{n'}{q'}\right\} + ib_N\right] \qquad x \in I_N.$$

Here $\frac{1}{q'} + \frac{1}{q} = 1$, $\frac{1}{p'} + \frac{1}{p} = 1$. Note that

$$g(x, \Theta) \equiv g(x),$$

and that

(3) $\qquad \| g(x\,,\,0+iv)\|_{p'} \leqq 1\,,\ \| g(x\,,\,1+iv)\|_{q'} \leqq 1 \qquad (-\infty < v < \infty)\,.$

Let

$$[\Lambda(s) f(x)]_\mu = \sum_{-\mu}^{\mu} \left(1 - \frac{|\nu|}{\mu+1}\right) e^{\lambda_\nu s} c_\nu e^{i\nu x}$$

be the Fejér sum of order μ of the formal Fourier series $\Lambda(s) f(x)$. We define

$$T_\mu(w) = \int_0^{2\pi} [\Lambda(\sigma_1 w) f(x)]_\mu \overline{g(x\,,\,w)}\, dx\,.$$

Note that

$$T_\mu(\Theta) = \int_0^{2\pi} [\Lambda(\sigma) f(x)]_\mu \overline{g(x)}\, dx\,. \tag{4}$$

We have

$$T_\mu(w) =$$
$$\sum_{\nu=-\mu}^{\mu} \sum_{N=1}^{M} c_\nu \left(1 - \frac{|\nu|}{\mu+1}\right) \exp\left[\lambda_\nu \sigma_1 w + a_N\left\{(1-w)\frac{n'}{p'} + \frac{n'}{q'}\right\} - i b_N\right] \int_{I_N} e^{i\nu x}\, dx\,.$$

It is evident from this that $T_\mu(w)$ is an entire function bounded in every vertical strip. We have

$$T_\mu(0 + iv) = \int_0^{2\pi} [\Lambda(i\sigma_1 v) f(x)]_\mu \overline{g(x\,,\,iv)}\, dx\,.$$

Using Hölder's inequality and (3) we find that

$$|T_\mu(0 + iv)| \leqq A(\sigma_1 v) \| f \|_p\,. \tag{5}$$

In exactly the same way we find that

$$\begin{aligned} |T_\mu(1 + iv)| &= \left| \int_0^{2\pi} [\Lambda(\sigma_1 + i\sigma_1 v) f(x)]_\mu \overline{g(x\,,\,1+iv)}\, dx \right| \\ &= \left| \int_0^{2\pi} [\Lambda(i\sigma_1 v) \{\Lambda(\sigma_1) f(x)\}]_\mu \overline{g(x\,,\,1+iv)}\, dx \right| \\ &\leqq A(\sigma_1 v) \| \Lambda(\sigma_1) f \|_q\,. \end{aligned} \tag{6}$$

By Lemma 1 we have

$$\log |T_\mu(\Theta)| \leqq \int_{-\infty}^{\infty} \log [A(\sigma_1 v) \| f \|_p] \, \omega(\Theta, v) \, dv$$

$$+ \int_{-\infty}^{\infty} \log [A(\sigma_1 v) \| \Lambda(\sigma_1) f \|_q] \, \omega(1-\Theta, v) \, dv .$$

Using the relations

$$\int_{-\infty}^{\infty} \omega(\Theta, v) \, dv = (1-\Theta), \quad \int_{-\infty}^{\infty} \omega(1-\Theta, v) \, dv = \Theta ,$$

and using (5) and (6) we obtain

$$\left| \int_0^1 [\Lambda(\sigma) f(x)]_\mu \overline{g(x)} \, dx \right| \leqq B \| f \|_p^{1-\theta} \| \Lambda(\sigma_1) f \|_q^{\theta} , \tag{7}$$

where

$$\log B = \int_{-\infty}^{\infty} \log A(\sigma_1 v) [\omega(\Theta, v) + \omega(1-\Theta, v)] \, dv .$$

Since (7) holds for every step function $g(x)$ with $\| g(x) \|_{n'} \leqq 1$ it implies that

$$\| [\Lambda(\sigma) f(x)]_\mu \|_n \leqq B \| f \|_p^{1-\theta} \| \Lambda(\sigma_1) f \|_q^{\theta} . \tag{8}$$

Finally since this is true for every μ it implies, if $n > 1$, that $\Lambda(\sigma) f(x)$ is the Fourier series of a function in $L^n(0, 2\pi)$ and that

$$\| \Lambda(\sigma) f(x) \|_n \leqq B \| f \|_p^{1-\theta} \| \Lambda(\sigma_1) f \|_q^{\theta} ;$$

if $n = 1$, in which case $p = q = 1$, we must verify in addition to (8), that

$$\| \Lambda(\sigma) [f_{\mu_1}(x) - f_{\mu_2}] \|_1 \leqq B \| f_{\mu_1} - f_{\mu_2} \|_1^{1-\theta} \| \Lambda(\sigma_1) [f_{\mu_1} - f_{\mu_2}] \|_1^{\theta} , \tag{9}$$

where

$$f_\mu \sim \sum_{-\mu}^{\mu} \left(1 - \frac{|\nu|}{\mu+1}\right) c_\nu e^{i\nu x} .$$

This is done in exactly the same way. Inequality (9) implies that

$$\lim_{\mu_1, \mu_2 \to \infty} \| \Lambda(\sigma) [f_{\mu_1}(x) - f_{\mu_2}(x)] \|_1 = 0$$

and thus that $\Lambda(\sigma) f(x)$ is the Fourier series of a function in $L^1(0, 2\pi)$, etc.. See [6; pp. 78—88].

3. Let $f(z)$, $z=\rho e^{i\varphi}$ be analytic for $\rho_1 \leqq \rho \leqq \rho_2$. We set

$$\mathbf{M}_r(f, \rho) = \left[\frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{i\varphi})|^r d\varphi\right]^{1/r}.$$

Theorem 2. *If $f(z)$, $z=\rho e^{i\varphi}$ is analytic and single valued for $\rho_1 \leqq \rho \leqq \rho_2$ and if*

$$\log \rho_0 = (1-\Theta)\log\rho_1 + \Theta\log\rho_2,$$

$$\frac{1}{n} = (1-\Theta)\frac{1}{p} + \Theta\frac{1}{q}, \quad \begin{array}{l}(0<\Theta<1)\\ (1\leqq p,\ q\leqq\infty)\end{array}$$

then

$$\mathbf{M}_n(f, \rho_0) \leqq \mathbf{M}_p(f, \rho_1)^{1-\theta}\mathbf{M}_q(f, \rho_2)^{\theta}.$$

Let $\lambda_n = n$ $(-\infty<n<\infty)$. We have

$$\Lambda(i\tau)f(x) \sim \sum_{-\infty}^{\infty} c_n e^{in\tau} e^{inx} \sim f(x+\tau).$$

Thus

$$\|\Lambda(i\tau)f(x)\|_r \leqq \|f(x)\|_r \qquad 1\leqq r\leqq\infty.$$

Applying Theorem 1 and setting $\log\rho=\sigma$ we obtain the above result.

This result is known for $n=p=q$. Prof. A. Beurling tells me that he has long been in possession of a proof of the general case, although he has never published it.

If $f(x)\in L^r(0, 2\pi)$ and if $c_0=0$ then the fractional integral of $f(x)$ of order σ is given by the formula

$$f_\sigma(x) = \sum_{-\infty}^{\infty}{}' \frac{c^n}{(in)^\sigma} e^{inx}.$$

Here $(in)^\sigma$ is taken as $|n|^\sigma \exp(i\pi\sigma \operatorname{sgn} n/2)$.

Theorem 3. *If $1<p$, $q<\infty$ and if*

$$\sigma_1 = \sigma\Theta, \quad \frac{1}{n} = (1-\Theta)\frac{1}{p} + \Theta\frac{1}{q}, \qquad (0<\Theta<1)$$

then

$$\|f_\sigma(x)\|_n \leqq A\|f(x)\|_p^{1-\theta}\|f_{\sigma_1}(x)\|_q^{\theta},$$

where A depends only upon p, q, σ_1 and Θ but not upon f.

We need the following result. Let $\{\mu_n\}_{-\infty}^{\infty}$ be a sequence of complex constants, and consider the transformation M defined by

$$Mf(x) \sim \sum_{-\infty}^{\infty} \mu_n c_n e^{inx}.$$

Marcinkiewicz [2] has shown that if

$$(1) \qquad |\mu_n| \leqq A \quad n = 0, \pm 1, \pm 2, \ldots,$$

$$\sum_{2^m}^{2^{m+1}} |\mu_n - \mu_{n+1}| \leqq A, \quad \sum_{-2^{m+1}}^{-2^m} |\mu_n - \mu_{n+1}| \leqq A \qquad m = 0, 1, \ldots,$$

then

$$\| Mf(x) \|_r \leqq AB \| f(x) \|_r \qquad 1 < r < \infty,$$

where B is a constant which depends only upon r.

Let $\lambda_n^{(1)} = 0$ for $-\infty < n \leqq 0$ and let $\lambda_n^{(1)} = -\log|n|$ for $1 \leqq n < \infty$. Similarly let $\lambda_n^{(2)} = 0$ for $0 \leqq n < \infty$ and let $\lambda_n^{(2)} = -\log|n|$ for $-\infty < n \leqq -1$. It follows from the result above that if $1 < r < \infty$ then

$$\| \Lambda^{(1)}(i\tau) f(x) \|_r \leqq A(\tau) \| f(x) \|_r, \quad \| \Lambda^{(2)}(i\tau) f(x) \|_r \leqq A(\tau) \| f(x) \|_r,$$

where

$$A(\tau) = O(|\tau|) \qquad \tau \to \pm \infty.$$

Thus Theorem 1 is applicable to $\Lambda^{(1)}(\sigma)$ and $\Lambda^{(2)}(\sigma)$.

We may assume that $\| f(x) \|_p$ and $\| f_{\sigma_1}(x) \|_q$ are finite. By Riesz's theorem on conjugate functions if

$$f^{(1)}(x) = \sum_{1}^{\infty} c_n e^{inx}, \quad f^{(2)}(x) = \sum_{-\infty}^{-1} c_n e^{inx},$$

then

$$\| f^{(i)}(x) \|_p \leqq A \| f(x) \|_p \quad i = 1, 2,$$
$$\| f_{\sigma_1}^{(i)}(x) \|_q \leqq A \| f_{\sigma_1}(x) \|_q \quad i = 1, 2,$$

where A is a constant depending only on p and q. We have

$$f_\alpha^{(1)}(x) = e^{-i\pi\alpha/2} \Lambda^{(1)}(\alpha) f^{(1)}(x),$$
$$f_\alpha^{(2)}(x) = e^{+i\pi\alpha/2} \Lambda^{(2)}(\alpha) f^{(2)}(x).$$

Hence

$$\| f_\sigma^{(i)}(x) \|_n \leqq A \| f^{(i)}(x) \|_p^{1-\theta} \| f_{\sigma_1}^{(i)}(x) \|_q^{\theta} \quad i = 1, 2,$$

where A is (another) constant depending only on p, q, σ_1, and Θ. Since

$$\|f_\sigma(x)\|_n \leqq \|f_\sigma^{(1)}(x)\|_n + \|f_\sigma^{(2)}(x)\|_n,$$

we obtain our desired result.

Convexity theorems for $n = p = q = \infty$ have been given by Kolmogoroff and Bang, see [1].

4. We shall give here an abstract form of our principal theorem. Let (S, μ) be a measure space. We write, as is usual, $L^r(S)$ for the set of all complex valued functions for which

$$\|f\|_r = \left[\int_S |f(x)|^r \, d\mu(x)\right]^{1/r}$$

is finite. We restrict ourselves to $1 < r < \infty$. If $\frac{1}{r} + \frac{1}{r'} = 1$ then $L^r(S)$ and $L^{r'}(S)$ are conjugate Banach spaces; when $f \in L^r(S)$, $g \in L^{r'}(S)$ we set

$$(f, g) = \int_S f(x)\overline{g(x)} \, d\mu(x).$$

Let $\mathbf{A}$ be a directed set and let $\{M_\alpha\}_{\alpha \in \mathbf{A}}$ be a set of linear transformations each of which carries $L^r(S)$ into $L^r(S)$ for every r, [2] and such that:

1. $\|M_\alpha f\|_r \leqq A(r)\|f\|_r \quad \alpha \in \mathbf{A}, \ 1 < r < \infty, \ f \in L^r(S)$;
2. $\lim\limits_{\alpha \in \mathbf{A}} \|f - M_\alpha f\|_r = 0 \quad 1 < r < \infty, \ f \in L^r(S)$;
3. $M_\alpha M_\beta = M_\beta M_\alpha$;
4. $(M_\alpha f, g) = (f, M_\alpha g) \quad f \in L^r(S), \ g \in L^{r'}(S)$.

A transformation Λ is said to be associated with $\{M_\alpha\}_{\alpha \in \mathbf{A}}$ if it maps a subset $\mathbf{D}(\Lambda)$ of $\bigcup\limits_{1<r<\infty} L^r(S)$ into $\bigcup\limits_{1<r<\infty} L^r(S)$ and is such that:

5. $f, g \in L^r(S) \cap \mathbf{D}(\Lambda)$ implies that $af + bg \in L^r(S) \cap \mathbf{D}(\Lambda)$ and that $\Lambda(af + bg) = a\Lambda(f) + b\Lambda(g)$, a and b being complex constants;

6. $M_\alpha L^r(S) \subset \mathbf{D}(\Lambda)$, $\Lambda M_\alpha L^r(S) \subset L^r(S)$ for $a \in \mathbf{A}$, $1 < r < \infty$;

7. $M_\alpha \Lambda f = \Lambda M_\alpha f, \quad f \in \mathbf{D}(\Lambda)$;

8. $f \in L^r(S)$, $F \in L^s(S)$, and $\lim\limits_{\alpha \in \mathbf{A}} \|F - \Lambda M_\alpha f\|_s = 0$,

imply that $f \in \mathbf{D}(\Lambda)$ and $\Lambda f = F$.

2. We assume of course that if $f \in L^p(S)$ and $L^q(S)$ then $M_\alpha f$ is the same regardless of which space f is thought of as belonging to.

Lemma 2. Let Λ be associated with $\{M_\alpha\}_{\alpha\in\mathbf{A}}$ and let $f\in L^r(S)$. If

(1) $$\|\Lambda M_\alpha f\|_s \leqq M \qquad \alpha\in\mathbf{A}$$

where M is independent of α, then $f\in\mathbf{D}(\Lambda)$ and $\|\Lambda f\|_s\leqq M$.

By the theory of weak compactness (1) implies that the directed sequence $\Lambda M_\alpha f$ has at least one weak limit point F, $\|F\|_s\leqq M$, in $L^s(S)$. In particular if $g_i\in L^{s}(S)$, $i=1,\ldots,n$, and thus, β being fixed, $M_\beta g_i\in L^{s'}(S)$, $i=1,\ldots,n$, and if $\varepsilon>0$ is given then there exists $\alpha_0\in\mathbf{A}$ such that

$$|(F-\Lambda M_\alpha f\ ,\ M_\beta g_i)|\leqq\varepsilon \quad i=1,\ldots,n, \quad \alpha>\alpha_0.$$

Using assumptions 4, 7, and 3 we have

$$(F-\Lambda M_\alpha f\ ,\ M_\beta g_i)=(M_\beta F-M_\alpha\Lambda M_\beta f\ ,\ g_i).$$

Thus $M_\beta F$ is a week limit point in $L^s(S)$ of the directed sequence $M_\alpha\Lambda M_\beta f$. By assumptions 2 and 6, $M_\alpha\Lambda M_\beta f$ converges strongly to $\Lambda M_\beta f$. The strong and weak limits must coincide and thus $M_\beta F=\Lambda M_\beta f$. Appealing to assumptions 2 and 8 we see that $f\in\mathbf{D}(\Lambda)$ and that $\Lambda f=F$.

Let $\Lambda(s)$, $s=\sigma+i\tau$ be a family of mappings associated with $\{M_\alpha\}_{\alpha\in\mathbf{A}}$. We make the following assumptions:

9. $f\in\mathbf{D}(\Lambda(s_2))$, $\Lambda(s_2)f\in\mathbf{D}(\Lambda(s_1))$ implies $f\in\mathbf{D}(\Lambda(s_1+s_2))$ and $\Lambda(s_1)[\Lambda(s_2)f]=\Lambda(s_1+s_2)f$;

10. $(\Lambda(s)M_\alpha f\ ,g)$ is for every $f\in L^r(S)$, $g\in L^{r'}(S)$, and α an entire function of s bounded in every finite vertical strip;

11. $\mathbf{D}(\Lambda(i\tau))\supset L^r(S)$, $1<r<\infty$, and $\|\Lambda(i\tau)f\|_r\leqq A(r,\tau)\|f\|_r$ where $\log A(r,\tau)=O(e^{a|\tau|})$ as $\tau\to\pm\infty$, for every $a>0$.

Theorem 4. Under the above assumptions if $0<\Theta<1$, $1<p$, $q<\infty$, and if $\sigma=\sigma_1\Theta$, $\dfrac{1}{n}=(1-\Theta)\dfrac{1}{p}+\Theta\dfrac{1}{q}$, then

$$\|\Lambda(\sigma)f\|_n\leqq B\|f\|_p^{1-\theta}\|\Lambda(\sigma_1)f\|_q^{\theta}.$$

Let I_N, $N=1,\ldots,M$, be disjoint sets of finite measure in S and let

$$\begin{aligned} g(x) &= \exp(a_N+ib_N) && x\in I_N \\ &= 0 && x\notin \bigcup_{N=1}^{M} I_N. \end{aligned}$$

We suppose that $\|g(x)\|_{n'}=1$. Let $w=u+iv$ be a complex variable and let

$$g(x\,,\,w)=\exp\left[a_N\left\{(1-\overline{w})\,\frac{n'}{p'}+\overline{w}\,\frac{n'}{q'}\right\}+ib_N\right]\qquad x\in I_N$$

$$=0\qquad\qquad x\notin\bigcup_{N=1}^{M}I_N\,.$$

Consider

$$T_\alpha(w)=\int_S[\Lambda(\sigma_1 w)M_\alpha f]\,\overline{g(w\,,\,x)}\,d\mu(x)\,.$$

Note that

$$T_\alpha(\Theta)=\int_S[\Lambda(\sigma)M_\alpha f]\,\overline{g(x)}\,d\mu(x).$$

Now if $g_N(x)=1$ for $x\in I_N$ and 0 for $x\notin I_N$ then

$$T_\alpha(w)=\sum_{N=1}^{M}(\Lambda(\sigma_1 w)M_\alpha f\,,\,g_N)\exp\left[a_N\left\{(1-w)\,\frac{n'}{p'}+w\,\frac{n'}{q'}\right\}-ib_N\right].$$

Thus $T_\alpha(w)$ is an entire function of w bounded in every finite vertical strip. Arguing just as in the proof of Theorem 1 we find that

$$T_\alpha(\Theta)\leqq B\,\|f\|_p^{1-\theta}\,\|\Lambda(\sigma_1)f\|_q^{\theta}\,,$$

where B is independent of f. Since g is an arbitrary step function such that $\|g\|_{n'}=1$ this implies that

$$\|\Lambda(\sigma)M_\alpha f\|_n\leqq B\,\|f\|_p^{1-\theta}\,\|\Lambda(\sigma_1)f\|_q^{\theta}\,,$$

where B is independent of f. Appealing to Lemma 2 we see that $f\in\mathbf{D}(\Lambda(\sigma))$ and that $\|\Lambda(\sigma)f\|_n\leqq B\,\|f\|_p^{1-\theta}\,\|\Lambda(\sigma_1)f\|_q^{\theta}$.

Washington University and
The Institute for Advanced Study

REFERENCES

1. T. Bang, Une inégalité de Kolmogoroff et les fonctions presque-périodiques, *Danske Vid. Selsk. Math. Fys. Medd.* 19, no. 4 (1941), 28 pp.

2. J. Marcinkiewicz, Sur les multiplicateurs des séries de Fourier, *Studia Math.* vol. 8 (1939), pp. 78—91.

3. R. Nevanlinna, Eindeutige Analytische Funktionen, Berlin, 1936.

4. M. Riesz, Sur les maxima des formes billinéaires et sur les fonctionnelles linéaires, *Acta Math.* vol. 49 (1926), pp. 465—497.

5. G. Thorin, Convexity theorems generalizing those of M. Riesz and Hadamard with some applications, *Comm. Sem. Math. Univ. Lund*, vol. 9 (1948), pp. 1—58.

6. A. Zygmund, Trigonometrical Series, Warsaw—Lwow, 1936.

(Received December 10, 1952)