CLASSICAL FOURIER ANALYSIS: INTERPOLATION OF L^p SPACES

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Definition. *Let (X, μ) and (Y, ν) be measure spaces. Further let T be an operator defined on a linear space of complex-valued measurable functions on X and taking values in the set of all complex-valued, finite almost everywhere, measurable functions on Y. Then T is called* linear *if for all functions f and g in the domain of T and all $z \in \mathbb{C}$ holds*

$$T(f+g) = T(f) + T(g) \qquad T(zf) = zT(f) \tag{1}$$

and quasi-linear *if*

$$|T(f+g)| \leqslant K(|T(f)| + |T(g)|) \qquad |T(zf)| = |z|\,|T(f)| \tag{2}$$

holds for some real constant $K > 0$. If $K = 1$, T is called sublinear.

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A complex-valued function f is said to be *holomorphic* in $\Omega \subseteq \mathbb{C}$ open, if $f'(z)$ exists for any $z \in \Omega$.

Lemma. (Hadamard's three lines lemma) *Let F be a holomorphic function in the strip $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$, continuous and bounded on $\overline{S}$, such that $|F(z)| \leqslant B_0$ when $\operatorname{Re} z = 0$ and $|F(z)| \leqslant B_1$ when $\operatorname{Re} z = 1$, for some $0 < B_0, B_1 < \infty$. Then $|F(z)| \leqslant B_0^{1-\theta} B_1^{\theta}$ when $\operatorname{Re} z = \theta$, for any $0 \leqslant \theta \leqslant 1$.*

Proof. For $z \in \overline{S}$ define

$$G(z) := \frac{F(z)}{B_0^{1-z} B_1^z} \qquad G_n(z) := G(z) e^{\left(z^2-1\right)/n}, \ n \in \mathbb{N}_{>0}$$

$G(z)$ and $G_n(z)$ are holomorphic in S by

$$G'(z) = \frac{F'(z) - F(z)\log\left(B_1/B_0\right)}{B_0^{1-z} B_1^z} \qquad G_n'(z) = G'(z) e^{\left(z^2-1\right)/n} + \frac{2}{n} z G_n(z)$$

and $e^z \neq 0$ for every $z \in \mathbb{C}$. Further, we have

$$\left|B_0^{1-z} B_1^z\right| = \left(B_0^{1-z} B_0^{1-\overline{z}} B_1^z B_1^{\overline{z}}\right)^{1/2} = B_0^{1-\operatorname{Re} z} B_1^{\operatorname{Re} z}$$

Consider $0 \leqslant \operatorname{Re} z \leqslant 1$ and $B_0 \geqslant 1$. Then $B_0^{1-\operatorname{Re} z} \geqslant 1$ and $B_0^{1-\operatorname{Re} z} \geqslant B_0$ in the case $B_0 < 1$. Similarly, $B_1^{\operatorname{Re} z} \geqslant 1$ if $B_1 \geqslant 1$ and $B_1^{\operatorname{Re} z} \leqslant B_1$ if $B_1 < 1$. Hence

$$\left|B_0^{1-z} B_1^z\right| \geqslant \min\{1, B_0\} \min\{1, B_1\} > 0 \tag{3}$$

for all $z \in \overline{S}$. Since F is bounded on $\overline{S}$, we have $|F(z)| \leqslant L$ for some $L > 0$ and all $z \in \overline{S}$. Thus by (3)

$$|G(z)| = \frac{|F(z)|}{\left|B_0^{1-z} B_1^z\right|} \leqslant \frac{L}{\min\{1, B_0\} \min\{1, B_1\}} =: M$$

for every $z \in \overline{S}$. Fix $n \in \mathbb{N}_{>0}$ and write $z := x + iy \in \overline{S}$. Then

$$|G_n(z)| \leqslant M \left(e^{\left(x^2+2ixy-y^2-1\right)/n} e^{\left(x^2-2ixy-y^2-1\right)/n}\right)^{1/2} = M e^{-y^2/n} e^{\left(x^2-1\right)/n} \leqslant M e^{-y^2/n}$$

for $0 \leqslant x \leqslant 1$. Thus

$$\lim_{y \to \pm\infty} \sup\{|G_n(z)| : 0 \leqslant x \leqslant 1\} = 0$$

by the pinching-principle. Hence there exist $C_0, C_1 \in \mathbb{R}$, such that

$$\sup\{|G_n(z)| : 0 \leqslant x \leqslant 1\} \leqslant 1$$

when $y > C_0$ or $y < C_1$. Letting

$$C(n) := \max\{|C_0| + 1, |C_1| + 1\}$$

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we conclude $|G_n(z)| \leqslant 1$ for all $0 \leqslant x \leqslant 1$ when $|y| \geqslant C(n)$. Now consider the rectangle $R := (0,1) \times (-C(n), C(n))$. We have $|G_n(z)| \leqslant 1$ on the lines $[0,1] \times \{\pm C(n)\}$. By

$$|G_n(iy)| = \frac{|F(iy)|}{\left|B_0^{1-iy} B_1^{iy}\right|} e^{-(y^2+1)/n} \leqslant 1 \qquad |G_n(1+iy)| = \frac{|F(1+iy)|}{\left|B_0^{-iy} B_1^{1+iy}\right|} e^{-y^2/n} \leqslant 1$$

we have $|G_n(z)| \leqslant 1$ on the lines $\{0\} \times [-C(n), C(n)]$, $\{1\} \times [-C(n), C(n)]$. Thus $|G_n(z)| \leqslant 1$ on ∂R. Since $|G_n(z)|$ is continuous on $\overline{R}$, holomorphic in R and R is a bounded region, the maximum modulus theorem implies

$$|G_n(z)| \leqslant \sup\{|G_n(z)| : z \in \partial R\} \leqslant 1$$

for every $z \in R$. Therefore $|G_n(z)| \leqslant 1$ on $\overline{R}$ and so $|G_n(z)| \leqslant 1$ on $\overline{S}$. Since inequalities are preserved by limits and the modulus is a continuous function, we have that $|G(z)| = \lim_{n\to\infty} |G_n(z)| \leqslant 1$ for $z \in \overline{S}$. We conclude by

$$|F(\theta + it)| = |G(\theta + it)| \left|B_0^{1-\theta-it} B_1^{\theta+it}\right| \leqslant B_0^{1-\theta} B_1^{\theta}$$

whenever $0 \leqslant \theta \leqslant 1$, $t \in \mathbb{R}$. $\square$

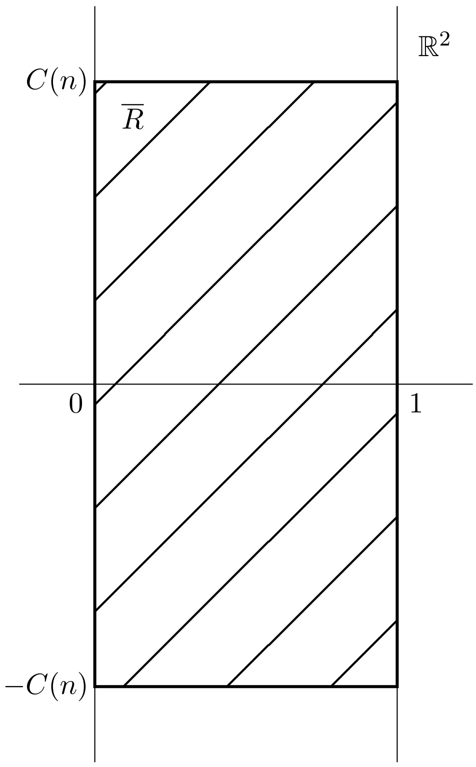

FIGURE 1. Sketch of the rectangle $\overline{R}$.

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Theorem. (Riesz-Thorin Interpolation Theorem) *Let (X, μ) be a measure space, (Y, ν) a semifinite measure space and T be a linear operator defined on Σ_X and taking values in the set of measurable functions on Y. Let $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$ and assume that*

$$\|T(f)\|_{L^{q_0}} \leqslant M_0 \|f\|_{L^{p_0}} \qquad \|T(f)\|_{L^{q_1}} \leqslant M_1 \|f\|_{L^{p_1}} \tag{4}$$

for all $f \in \Sigma_X$ and $M_0, M_1 < \infty$. Then for all $0 < \theta < 1$ we have

$$\|T(f)\|_{L^q} \leqslant M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p} \tag{5}$$

for all $f \in \Sigma_X$, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \tag{6}$$

Proof. Fix

$$f :\equiv \sum_{j=1}^{n} a_j e^{i\alpha_j} \chi_{A_j} \in \Sigma_X \qquad g :\equiv \sum_{k=1}^{m} b_k e^{i\beta_k} \chi_{B_k} \in \Sigma_Y$$

where $a_j, b_k > 0$ and $\alpha_j, \beta_k \in \mathbb{R}$ for every $j = 1, \dots, n$, $k = 1, \dots, m$. Define

$$P(z) := \frac{p}{p_0}(1-z) + \frac{p}{p_1} z \qquad Q(z) := \frac{q'}{q_0'}(1-z) + \frac{q'}{q_1'} z$$

for $z \in \overline{S}$ (if $p, q' = \infty$ then also $p_0, p_1, q_0', q_1' = \infty$ and hence P, Q are well defined). Further let

$$f_z :\equiv \sum_{j=1}^{n} a_j^{P(z)} e^{i\alpha_j} \chi_{A_j} \qquad g_z :\equiv \sum_{k=1}^{m} b_k^{Q(z)} e^{i\beta_k} \chi_{B_k} \tag{7}$$

and

$$F(z) := \int_Y T(f_z)(y) g_z(y) d\nu(y) \tag{8}$$

By the linearity of the operator T we have

$$F(z) = \sum_{j=1}^{n} \sum_{k=1}^{m} a_j^{P(z)} b_k^{Q(z)} e^{i\alpha_j} e^{i\beta_k} \int_Y T(\chi_{A_j})(y) \chi_{B_k}(y) d\nu(y)$$

and by Hölder's inequality

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$$
\begin{aligned}
\left|\int_Y T(\chi_{A_j})(y)\chi_{B_k}(y)d\nu(y)\right| &\leqslant \int_Y \left|T(\chi_{A_j})(y)\chi_{B_k}(y)\right| d\nu(y) \\
&= \left\|T(\chi_{A_j})\chi_{B_k}\right\|_{L^1} \\
&\leqslant \left\|T(\chi_{A_j})\right\|_{L^{q_0}} \left\|\chi_{B_k}\right\|_{L^{q_0'}} \\
&\leqslant M_0 \left\|\chi_{A_j}\right\|_{L^{p_0}} \left\|\chi_{B_k}\right\|_{L^{q_0'}} \\
&\overset{p_0,q_0'\neq\infty}{=} M_0 \mu\left(A_j\right)^{1/p_0} \nu\left(B_k\right)^{1/q_0'} \\
&< \infty
\end{aligned}
$$

for each $j = 1, \dots, n$, $k = 1, \dots, m$. In the case where either $p_0 = \infty$ or $q_0' = \infty$, consider that $\left\|\chi_{A_j}\right\|_{L^\infty}, \left\|\chi_{B_k}\right\|_{L^\infty} \leqslant 1$. Thus the function F is well-defined on $\overline{S}$. Let $t \in \mathbb{R}$. For $p, p_0 \neq \infty$

$$
\begin{aligned}
\|f_{it}\|_{L^{p_0}} &= \left(\sum_{j=1}^n \int_X |f_{it}|^{p_0}\, d\mu + \int_{X\setminus\bigcup_{j=1}^n A_j} |f_{it}|^{p_0}\, d\mu\right)^{1/p_0} \\
&= \left(\sum_{j=1}^n \left|a_j^{P(it)} e^{i\alpha_j}\right|^{p_0} \int_X \chi_{A_j}\, d\mu\right)^{1/p_0} \\
&= \left(\sum_{j=1}^n a_j^{p_0 \operatorname{Re} P(it)} \mu\left(A_j\right)\right)^{1/p_0} \\
&= \left(\sum_{j=1}^n a_j^{p} \mu\left(A_j\right)\right)^{p/(p_0 p)} \\
&= \|f\|_{L^p}^{p/p_0}
\end{aligned}
$$

holds. Let $p_0 = \infty$, $p \neq \infty$. Then either $\|f_{it}\|_{L^\infty} = 0$ or $\|f_{it}\|_{L^\infty} = 1$. In the former case $f \equiv 0$ μ-a.e which implies $\mu\left(A_j\right) = 0$ for any $j = 1, \dots, n$ and thus $\|f_{it}\|_{L^\infty} = 0$ and in the latter case $\|f_{it}\|_{L^\infty} = 1$ by the simple observation that $\left|a_j^{P(it)}\right| = a_j^{p/p_0} = 1$ and that there exists some index j, such that $\mu\left(A_j\right) \neq 0$. If $p = \infty$, observe that $P(z) = 1$ and thus $\|f_{it}\|_{L^\infty} = \|f\|_{L^\infty}$. By the same considerations we see that $\|g_{it}\|_{L^{q_0'}} = \|g\|_{L^{q'}}^{q'/q_0'}$ any legitime q_0, q. Hence

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$$
\begin{aligned}
|F(it)| &\leqslant \int_Y |T(f_{it})(y)g_{it}(y)|\,d\nu(y) \\
&= \|T(f_{it})g_{it}\|_{L^1} \\
&\leqslant \|T(f_{it})\|_{L^{q_0}} \|g_{it}\|_{L^{q_0'}} \\
&\leqslant M_0 \|f_{it}\|_{L^{p_0}} \|g_{it}\|_{L^{q_0'}} \\
&= M_0 \|f\|_{L^p}^{p/p_0} \|g\|_{L^{q'}}^{q'/q_0'} \\
&< \infty
\end{aligned}
$$

by Hölder's inequality. In an analogous manner s we can estimate

$$
\|f_{1+it}\|_{L^{p_1}} = \|f\|_{L^p}^{p/p_1} \qquad \|g_{1+it}\|_{L^{q_1'}} = \|g\|_{L^{q'}}^{q'/q_1'}
$$

and thus

$$
|F(1+it)| \leqslant M_1 \|f\|_{L^p}^{p/p_1} \|g\|_{L^{q'}}^{q'/q_1'}
$$

Further

$$
\begin{aligned}
|F(z)| &\leqslant \int_Y |T(f_z)(y)g_z(y)|\,d\nu(y) = \|T(f_z)g_z\|_{L^1} \leqslant \|T(f_z)\|_{L^{q_0}} \|g_z\|_{L^{q_0'}} \\
&\leqslant M_0 \|f_z\|_{L^{p_0}} \|g_z\|_{L^{q_0'}} \overset{p_0,q_0'\neq\infty}{=} M_0 \left(\int_X |f_z|^{p_0}\,d\mu\right)^{1/p_0} \left(\int_Y |g_z|^{q_0'}\,d\nu\right)^{1/q_0'} \\
&= M_0 \left(\sum_{j=1}^n a_j^{p_0 \operatorname{Re} P(z)} \mu(A_j)\right)^{1/p_0} \left(\sum_{k=1}^m b_k^{q_0' \operatorname{Re} Q(z)} \nu(B_k)\right)^{1/q_0'} \\
&= M_0 \left(\sum_{j=1}^n a_j^{p(1-\operatorname{Re} z)+(pp_0 \operatorname{Re} z)/p_1} \mu(A_j)\right)^{1/p_0} \left(\sum_{k=1}^m b_k^{q'(1-\operatorname{Re} z)+\left(q'q_0' \operatorname{Re} z\right)/q_1'} \nu(B_k)\right)^{1/q_0'} \\
&\leqslant M_0 \left(\sum_{j=1}^n a_j^{p+(pp_0)/p_1} \mu(A_j)\right)^{1/p_0} \left(\sum_{k=1}^m b_k^{q'+\left(q'q_0'\right)/q_1'} \nu(B_k)\right)^{1/q_0'} \\
&= M_0 \|f\|_{L^{p+(pp_0)/p_1}}^{p/p_0+p/p_1} \|g\|_{L^{q'+\left(q'q_0'\right)/q_1'}}^{q'/q_0'+q'/q_1'} =: C(f,g)
\end{aligned}
$$

by Hölder's inequality and in the edge cases

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$$
\begin{aligned}
p_0 = \infty, q_0' \neq \infty : &\qquad C(f,g) := M_0 \max_{j=1,\dots,n} a_j^{p/p_1} \|g\|_{L^{q'+(q'q_0')/q_1'}}^{q'/q_0'+q'/q_1'} \\
p_0 \neq \infty, q_0' = \infty : &\qquad C(f,g) := M_0 \|f\|_{L^{p+(pp_0)/p_1}}^{p/p_0+p/p_1} \max_{k=1,\dots,m} b_k^{q'/q_1'} \\
p_0 = \infty, q_0' = \infty : &\qquad C(f,g) := M_0 \max_{j=1,\dots,n} a_j^{p/p_1} \max_{k=1,\dots,m} b_k^{q'/q_1'}
\end{aligned}
$$

Hence F is bounded on $\overline{S}$. By

$$
F'(z) = \sum_{j=1}^{n}\sum_{k=1}^{m} a_j^{P(z)} \log(a_j)\left(\frac{p}{p_1} - \frac{p}{p_0}\right) b_k^{Q(z)} \log(b_j)\left(\frac{q'}{q_1'} - \frac{q'}{q_0'}\right) e^{i\alpha_j} e^{i\beta_k} \int_Y T(\chi_{A_j})(y)\chi_{B_k}(y)d\nu(y)
$$

it is immediate, that F is an entire function and thus holomorphic in S and continuous on $\overline{S}$. Therefore Hadamard's three lines lemma yields

$$
\begin{aligned}
|F(z)| &\leqslant \left(M_0 \|f\|_{L^p}^{p/p_0} \|g\|_{L^{q'}}^{q'/q_0'}\right)^{1-\theta} \left(M_1 \|f\|_{L^p}^{p/p_1} \|g\|_{L^{q'}}^{q'/q_1'}\right)^{\theta} \\
&= M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p} \|g\|_{L^{q'}}
\end{aligned}
$$

for $\operatorname{Re} z = \theta$. By $P(\theta) = Q(\theta) = 1$ and

$$
\begin{aligned}
M_q(T(f)) &= \sup\left\{\left|\int_Y T(f) g d\nu\right| : g \in \Sigma_Y, \|g\|_{L^{q'}} = 1\right\} \\
&= \sup\left\{|F(\theta)| : g \in \Sigma_Y, \|g\|_{L^{q'}} = 1\right\} \\
&\leqslant M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p} \\
&< \infty
\end{aligned}
$$

we conclude $\|T(f)\|_{L^q} = M_q(T(f))$ for any $f \in \Sigma_X$ by observing, that $T(f)g \in L^1$ for any $g \in \Sigma_Y$ by either one of the hypotheses on the linear operator T and the semifiniteness of ν. $\square$

Lemma 1.1. (Hadamard's three lines lemma, extension) *Let F be a holomorphic function in the strip $S := \{z \in \mathbb{C} : 0 < \mathrm{Re} z < 1\}$ and continuous on $\overline{S}$, such that for some $0 < A < \infty$ and $\tau_0 \in (0, \pi)$ we have $\log|F(z)| \leqslant Ae^{\tau_0 |\operatorname{Im} z|}$ for every $z \in \overline{S}$. Then*

$$
|F(z)| \leqslant \exp\left(\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log|F(it+iy)|}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log|F(1+it+iy)|}{\cosh(\pi t) + \cos(\pi x)}\right] d\lambda(t)\right)
$$

whenever $z := x + iy \in S$.

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Definition 1.1. (Analytic family, admissible growth) *Let* (X, μ) *be a measure space,* (Y, ν) *be a semifinite measure spaces and* $(T_z)_{z\in\overline{S}}$*, where* T_z *is defined* Σ_X *and taking values in the space of all measurable functions on* Y *such that*

$$\int_Y |T_z(\chi_A)\chi_B|\, d\nu \tag{9}$$

whenever $\mu(A), \nu(B) < \infty$. *The family* $(T_z)_{z\in\overline{S}}$ *is said to be* analytic *if for all* $f \in \Sigma_X$, $g \in \Sigma_Y$ *we have that*

$$z \mapsto \int_Y T_z(f) g d\nu \tag{10}$$

is analytic on S *and continuous on* $\overline{S}$. *Further, an analytic family* $(T_z)_{z\in\overline{S}}$ *is called of* admissible growth, *if there is a constant* $\tau_0 \in (0, \pi)$, *such that for all* $f \in \Sigma_X$, $g \in \Sigma_Y$ *a constant* $C(f, g)$ *exists with*

$$\log \left| \int_Y T_z(f) g d\nu \right| \leqslant C(f, g) e^{\tau_0 |\operatorname{Im} z|} \tag{11}$$

for all $z \in \overline{S}$.

Theorem 1.1. (Stein's Theorem on Interpolation of Analytic Families of Operators) *Let* $(T_z)_{z\in\overline{S}}$ *be an analytic family of admissible growth,* $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$ *and suppose that* M_0, M_1 *are positive functions on the real line such that for some* $\tau_1 \in (0, \pi)$

$$\sup\left\{e^{-\tau_1|y|} \log M_0(y) : y \in \mathbb{R}\right\} < \infty \qquad \sup\left\{e^{-\tau_1|y|} \log M_1(y) : y \in \mathbb{R}\right\} < \infty \tag{12}$$

Fix $0 < \theta < 1$ *and define*

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \tag{13}$$

Further suppose that for all $f \in \Sigma_X$ *and* $y \in \mathbb{R}$ *we have*

$$\|T_{iy}(f)\|_{L^{q_0}} \leqslant M_0(y) \|f\|_{L^{p_0}} \qquad \|T_{1+iy}(f)\|_{L^{q_1}} \leqslant M_1(y) \|f\|_{L^{p_1}} \tag{14}$$

Then for all $f \in \Sigma_X$ *we have*

$$\|T_\theta(f)\|_{L^q} \leqslant M(\theta) \|f\|_{L^p}$$

where for $0 < x < 1$

$$M(x) = \exp\left(\frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \left[\frac{\log M_0(t)}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log M_1(t)}{\cosh(\pi t) + \cos(\pi x)}\right] d\lambda(t)\right)$$

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THEOREM 1.2. (The Marcinkiewicz Interpolation Theorem) *Let* (X, μ) *be a* σ*-finite measure space,* (Y, ν) *another measure space and* $0 < p_0 < p_1 \leqslant \infty$*. Further let* T *be a sublinear operator defined on*

$$L^{p_0} + L^{p_1} := \{f_0 + f_1 : f_0 \in L^{p_0}(X, \mu), f_1 \in L^{p_1}(X, \mu)\}$$

and taking values in the space of measurable functions on Y*. Assume that there exist* $A_0, A_1 < \infty$ *such that*

$$\forall f \in L^{p_0}(X, \mu) \ \|T(f)\|_{L^{p_0,\infty}} \leqslant A_0 \|f\|_{L^{p_0}} \tag{15}$$

$$\forall f \in L^{p_1}(X, \mu) \ \|T(f)\|_{L^{p_1,\infty}} \leqslant A_1 \|f\|_{L^{p_1}} \tag{16}$$

Then for all $p_0 < p < p_1$ *and for all* $f \in L^p(X, \mu)$ *we have the estimate*

$$\|T(f)\|_{L^p} \leqslant A \|f\|_{L^p} \tag{17}$$

where

$$A := 2 \left(\frac{p}{p - p_0} + \frac{p}{p_1 - p} \right)^{1/p} A_0^{\frac{\frac{1}{p} - \frac{1}{p_1}}{\frac{1}{p_0} - \frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0} - \frac{1}{p}}{\frac{1}{p_0} - \frac{1}{p_1}}} \tag{18}$$