CLASSICAL FOURIER ANALYSIS: INTERPOLATION OF L^p SPACES

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Definition 1.1. *Let (X, μ) and (Y, ν) be measure spaces. Further let T be an operator defined on a linear space of complex-valued measurable functions on X and taking values in the set of all complex-valued, finite almost everywhere, measurable functions on Y. Then T is called* linear *if for all functions f and g in the domain of T and all $z \in \mathbb{C}$ holds*

$$T(f+g) = T(f) + T(g) \qquad T(zf) = zT(f) \tag{1}$$

and quasi-linear *if*

$$|T(f+g)| \leqslant K(|T(f)| + |T(g)|) \qquad |T(zf)| = |z|\,|T(f)| \tag{2}$$

holds for some real constant $K > 0$. If $K = 1$, T is called sublinear*.*

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Recall, that a complex-valued function f is said to be *holomorphic* in $\Omega \subseteq \mathbb{C}$ open, if $f'(z)$ exists for any $z \in \Omega$.

LEMMA 1.1. (Hadamard's three lines lemma) *Let F be a holomorphic function in the strip $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$, continuous and bounded on $\overline{S}$, such that $|F(z)| \leqslant B_0$ when $\operatorname{Re} z = 0$ and $|F(z)| \leqslant B_1$ when $\operatorname{Re} z = 1$, for some $0 < B_0, B_1 < \infty$. Then $|F(z)| \leqslant B_0^{1-\theta} B_1^{\theta}$ when $\operatorname{Re} z = \theta$, for any $0 \leqslant \theta \leqslant 1$.*

Proof. For $z \in \overline{S}$ define

$$G(z) := \frac{F(z)}{B_0^{1-z} B_1^z} \qquad G_n(z) := G(z) e^{\left(z^2-1\right)/n}, \ n \in \mathbb{N}_{>0}$$

$G(z)$ and $G_n(z)$ are holomorphic in S by

$$G'(z) = \frac{F'(z) - F(z)\log\left(B_1/B_0\right)}{B_0^{1-z} B_1^z} \qquad G_n'(z) = G'(z) e^{\left(z^2-1\right)/n} + \frac{2}{n} z G_n(z)$$

and $e^z \neq 0$ for every $z \in \mathbb{C}$. Further, we have

$$\left|B_0^{1-z} B_1^z\right| = \left(B_0^{1-z} B_0^{1-\overline{z}} B_1^z B_1^{\overline{z}}\right)^{1/2} = B_0^{1-\operatorname{Re} z} B_1^{\operatorname{Re} z}$$

Consider $0 \leqslant \operatorname{Re} z \leqslant 1$ and $B_0 \geqslant 1$. Then $B_0^{1-\operatorname{Re} z} \geqslant 1$ and $B_0^{1-\operatorname{Re} z} \geqslant B_0$ in the case $B_0 < 1$. Similarly, $B_1^{\operatorname{Re} z} \geqslant 1$ if $B_1 \geqslant 1$ and $B_1^{\operatorname{Re} z} \leqslant B_1$ if $B_1 < 1$. Hence

$$\left|B_0^{1-z} B_1^z\right| \geqslant \min\{1, B_0\} \min\{1, B_1\} > 0 \tag{3}$$

for all $z \in \overline{S}$. Since F is bounded on $\overline{S}$, we have $|F(z)| \leqslant L$ for some $L > 0$ and all $z \in \overline{S}$. Thus by (3)

$$|G(z)| = \frac{|F(z)|}{\left|B_0^{1-z} B_1^z\right|} \leqslant \frac{L}{\min\{1, B_0\} \min\{1, B_1\}} =: M$$

for every $z \in \overline{S}$. Fix $n \in \mathbb{N}_{>0}$ and write $z := x + iy \in \overline{S}$. Then

$$|G_n(z)| \leqslant M \left(e^{\left(x^2+2ixy-y^2-1\right)/n} e^{\left(x^2-2ixy-y^2-1\right)/n}\right)^{1/2} = M e^{-y^2/n} e^{\left(x^2-1\right)/n} \leqslant M e^{-y^2/n}$$

for $0 \leqslant x \leqslant 1$. Thus

$$\lim_{y \to \pm\infty} \sup\{|G_n(z)| : 0 \leqslant x \leqslant 1\} = 0$$

by the pinching-principle. Hence there exist $C_0, C_1 \in \mathbb{R}$, such that

$$\sup\{|G_n(z)| : 0 \leqslant x \leqslant 1\} \leqslant 1$$

when $y > C_0$ or $y < C_1$ and so there exists some $C(n) > 0$ such that $|G_n(z)| \leqslant 1$ for all $0 \leqslant x \leqslant 1$ when $|y| \geqslant C(n)$. Now consider the rectangle $R := (0,1) \times (-C(n), C(n))$. We have $|G_n(z)| \leqslant 1$ on the lines $[0,1] \times \{\pm C(n)\}$. By

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$$|G_n(iy)| = |F(iy)|/B_0 \leqslant 1 \qquad |G_n(1+iy)| = |F(1+iy)|/B_1 \leqslant 1$$

we have $|G_n(z)| \leqslant 1$ on the line $\{0\} \times [-C(n), C(n)]$ and $\{1\} \times [-C(n), C(n)]$. Thus $|G_n(z)| \leqslant 1$ on ∂R. Since $|G_n(z)|$ is continuous on $\overline{R}$, R is a bounded region and holomorphic in R, the maximum modulus theorem implies

$$|G_n(z)| \leqslant \sup\{|G_n(z)| : z \in \partial R\} \leqslant 1$$

for every $z \in R$. Therefore $|G_n(z)| \leqslant 1$ on $\overline{R}$ and so $|G_n(z)| \leqslant 1$ on $\overline{S}$. Since inequalities are preserved by limits and the modulus is a continuous function, we have that $|G(z)| = \lim_{n\to\infty} |G_n(z)| \leqslant 1$ for $z \in \overline{S}$. We conclude by

$$|F(\theta+it)| = |G(\theta+it)| \left|B_0^{1-\theta-it} B_1^{\theta+it}\right| \leqslant B_0^{1-\theta} B_1^{\theta}$$

whenever $0 \leqslant \theta \leqslant 1$, $t \in \mathbb{R}$. $\square$

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Theorem 1.1. (Riesz-Thorin Interpolation Theorem) *Let (X, μ) be a measure space, (Y, ν) a semifinite measure space and T be a linear operator defined on Σ_X and taking values in the set of measurable functions on Y. Let $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$ and assume that*

$$\|T(f)\|_{L^{q_0}} \leqslant M_0 \|f\|_{L^{p_0}} \qquad \|T(f)\|_{L^{q_1}} \leqslant M_1 \|f\|_{L^{p_1}} \tag{4}$$

for all $f \in \Sigma_X$ and $M_0, M_1 < \infty$. Then for all $0 < \theta < 1$ we have

$$\|T(f)\|_{L^q} \leqslant M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p} \tag{5}$$

for all $f \in \Sigma_X$, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \tag{6}$$

Lemma 1.2. (Hadamard's three lines lemma, extension) *Let F be a holomorphic function in the strip $S := \{z \in \mathbb{C} : 0 < \mathrm{Re} z < 1\}$ and continuous on $\overline{S}$, such that for some $0 < A < \infty$ and $\tau_0 \in (0, \pi)$ we have $\log |F(z)| \leqslant A e^{\tau_0 |\mathrm{Im}\, z|}$ for every $z \in \overline{S}$. Then*

$$|F(z)| \leqslant \exp\left(\frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \left[\frac{\log|F(it+iy)|}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log|F(1+it+iy)|}{\cosh(\pi t) + \cos(\pi x)}\right] d\lambda(t)\right)$$

whenever $z := x + iy \in S$.

Definition 1.2. (Analytic family, admissible growth) *Let (X, μ) be a measure space, (Y, ν) be a semifinite measure spaces and $(T_z)_{z \in \overline{S}}$, where T_z is defined Σ_X and taking values in the space of all measurable functions on Y such that*

$$\int_Y |T_z(\chi_A)\chi_B| \, d\nu \tag{7}$$

whenever $\mu(A), \nu(B) < \infty$. The family $(T_z)_{z \in \overline{S}}$ is said to be analytic *if for all $f \in \Sigma_X$, $g \in \Sigma_Y$ we have that*

$$z \mapsto \int_Y T_z(f) g d\nu \tag{8}$$

is analytic on S and continuous on $\overline{S}$. Further, an analytic family $(T_z)_{z \in \overline{S}}$ is called of admissible growth, *if there is a constant $\tau_0 \in (0, \pi)$, such that for all $f \in \Sigma_X$, $g \in \Sigma_Y$ a constant $C(f, g)$ exists with*

$$\log \left| \int_Y T_z(f) g d\nu \right| \leqslant C(f, g) e^{\tau_0 |\mathrm{Im}\, z|} \tag{9}$$

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for all $z \in \overline{S}$.

THEOREM 1.2. (Stein's Theorem on Interpolation of Analytic Families of Operators) *Let* $(T_z)_{z\in\overline{S}}$ *be an analytic family of admissible growth,* $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$ *and suppose that* M_0, M_1 *are positive functions on the real line such that for some* $\tau_1 \in (0, \pi)$

$$\sup\left\{e^{-\tau_1|y|}\log M_0(y) : y \in \mathbb{R}\right\} < \infty \qquad \sup\left\{e^{-\tau_1|y|}\log M_1(y) : y \in \mathbb{R}\right\} < \infty \tag{10}$$

Fix $0 < \theta < 1$ *and define*

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \tag{11}$$

Further suppose that for all $f \in \Sigma_X$ *and* $y \in \mathbb{R}$ *we have*

$$\|T_{iy}(f)\|_{L^{q_0}} \leqslant M_0(y)\|f\|_{L^{p_0}} \qquad \|T_{1+iy}(f)\|_{L^{q_1}} \leqslant M_1(y)\|f\|_{L^{p_1}} \tag{12}$$

Then for all $f \in \Sigma_X$ *we have*

$$\|T_\theta(f)\|_{L^q} \leqslant M(\theta)\|f\|_{L^p}$$

where for $0 < x < 1$

$$M(x) = \exp\left(\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log M_0(t)}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log M_1(t)}{\cosh(\pi t) + \cos(\pi x)}\right]d\lambda(t)\right)$$

THEOREM 1.3. (The Marcinkiewicz Interpolation Theorem) *Let* (X, μ) *be a* σ*-finite measure space,* (Y, ν) *another measure space and* $0 < p_0 < p_1 \leqslant \infty$. *Further let* T *be a sublinear operator defined on*

$$L^{p_0} + L^{p_1} := \{f_0 + f_1 : f_0 \in L^{p_0}(X, \mu), f_1 \in L^{p_1}(X, \mu)\}$$

and taking values in the space of measurable functions on Y. *Assume that there exist* $A_0, A_1 < \infty$ *such that*

$$\forall f \in L^{p_0}(X, \mu)\ \|T(f)\|_{L^{p_0,\infty}} \leqslant A_0\|f\|_{L^{p_0}} \tag{13}$$

$$\forall f \in L^{p_1}(X, \mu)\ \|T(f)\|_{L^{p_1,\infty}} \leqslant A_1\|f\|_{L^{p_1}} \tag{14}$$

Then for all $p_0 < p < p_1$ *and for all* $f \in L^p(X, \mu)$ *we have the estimate*

$$\|T(f)\|_{L^p} \leqslant A\,\|f\|_{L^p} \tag{15}$$

where

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$$A := 2\left(\frac{p}{p-p_0} + \frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}} \tag{16}$$