

# CLASSICAL FOURIER ANALYSIS: INTERPOLATION OF $L^p$ SPACES

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**Abstract.** In this written seminar work I will basically follow the section *Interpolation* in the book *Classical Fourier Analysis, third Edition* by Loukas Grafakos. I will review three basic but important theorems on interpolation of operators on  $L^p$  spaces, namely the *Marcinkiewicz Interpolation Theorem*, the *Riesz-Thorin Interpolation Theorem* and finally an extension of the Riesz-Thorin Interpolation Theorem to analytic families of operators (the so-called *Stein's theorem on interpolation of analytic families of operators*). We are mainly concerned with the notion of linear operators as well as slight generalizations of them.

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**1. Introduction and Basic Definitions.** If  $1 \leq p < q < r \leq \infty$ , then

$$(L^p \cap L^r) \subseteq L^q \subseteq (L^p + L^r)$$

(see [Fol99, p. 185]). Thus if we have a linear operator  $T$  defined on  $L^p + L^r$ , that is bounded simultaneously on  $L^p$  and  $L^r$  it is useful to know under what circumstances  $T$  is also bounded on  $L^q$ . This question will be answered in the two main theorems: *the Marcinkiewicz interpolation theorem* and *the Riesz-Thorin interpolation theorem*. The next section will provide the fundamental definitions used later on.

**1.1. Linear Operators.** First we need to have a precise and suitable idea of *linear operators* in the generalized setting of measure spaces.

**DEFINITION 1.1.** Let  $(X, \mu)$  and  $(Y, \nu)$  be measure spaces. Further let  $T$  be an operator defined on a linear space of complex-valued measurable functions on  $X$  and taking values in the set of all complex-valued, finite almost everywhere, measurable functions on  $Y$ . Then  $T$  is called *linear* if for all functions  $f$  and  $g$  in the domain of  $T$  and all  $z \in \mathbb{C}$  holds

$$T(f + g) = T(f) + T(g) \quad T(zf) = zT(f) \quad (1)$$

and quasi-linear if

$$|T(f + g)| \leq K(|T(f)| + |T(g)|) \quad |T(zf)| = |z| |T(f)| \quad (2)$$

holds for some real constant  $K > 0$ . If  $K = 1$ ,  $T$  is called *sublinear*.

**2. The Real Method.** A first important theorem on the subject of interpolation of  $L^p$  spaces will be the so-called *Marcinkiewicz Interpolation Theorem* which uses only real variables techniques for its proof (this stands in contrast to the complex variables techniques used for proving the other interpolation theorems).

**2.1. The Marcinkiewicz Interpolation Theorem.** This theorem applies to sublinear operators (as well as for quasilinear operators by a slight change of the constant), which is in comparison to the linearity assumed by the other interpolation theorems more generally applicable.

**THEOREM 2.1.** (The Marcinkiewicz Interpolation Theorem) Let  $(X, \mu)$  be a  $\sigma$ -finite measure space,  $(Y, \nu)$  another measure space and  $0 < p_0 < p_1 \leq \infty$ . Further let  $T$  be a sublinear operator defined on

$$L^{p_0} + L^{p_1} := \{f_0 + f_1 : f_0 \in L^{p_0}(X, \mu), f_1 \in L^{p_1}(X, \mu)\}$$

and taking values in the space of measurable functions on  $Y$ . Assume that there exist  $A_0, A_1 < \infty$  such that

$$\forall f \in L^{p_0}(X, \mu) \quad \|T(f)\|_{L^{p_0}, \infty} \leq A_0 \|f\|_{L^{p_0}} \quad (3)$$

$$\forall f \in L^{p_1}(X, \mu) \quad \|T(f)\|_{L^{p_1}, \infty} \leq A_1 \|f\|_{L^{p_1}} \quad (4)$$

Then for all  $p_0 < p < p_1$  and for all  $f \in L^p(X, \mu)$  we have the estimate

$$\|T(f)\|_{L^p} \leq A \|f\|_{L^p} \quad (5)$$

where

$$A := 2 \left( \frac{p}{p - p_0} + \frac{p}{p_1 - p} \right)^{1/p} A_0^{\frac{\frac{1}{p} - \frac{1}{p_1}}{\frac{1}{p_0} - \frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0} - \frac{1}{p}}{\frac{1}{p_0} - \frac{1}{p_1}}} \quad (6)$$

*Proof.* Let us first consider the case  $p_1 < \infty$ . Fix  $f \in L^p(X, \mu)$ ,  $\alpha > 0$  and  $\delta > 0$  ( $\delta$  will be determined later). We split  $f$  using so-called *cut-off* functions, by stipulating  $f \equiv f_0(\cdot; \alpha, \delta) + f_1(\cdot; \alpha, \delta)$ , where  $f_0(\cdot; \alpha, \delta)$  is the *unbounded part* of  $f$  and  $f_1(\cdot; \alpha, \delta)$  is the *bounded part* of  $f$ , defined by

$$\begin{aligned} f_0(x; \alpha, \delta) &:= \begin{cases} f(x), & |f(x)| > \delta\alpha, \\ 0, & |f(x)| \leq \delta\alpha. \end{cases} \\ f_1(x; \alpha, \delta) &:= \begin{cases} f(x), & |f(x)| \leq \delta\alpha, \\ 0, & |f(x)| > \delta\alpha. \end{cases} \end{aligned} \quad (7)$$

for  $x \in X$ . To facilitate reading I will omit the dependency of  $f_0(\cdot; \alpha, \delta)$  and  $f_1(\cdot; \alpha, \delta)$  upon the parameters  $\alpha$  and  $\delta$  in what follows and simply write  $f_0, f_1$  respectively.

LEMMA 2.1. *The functions  $f_0$  and  $f_1$  defined above satisfy  $f_0 \in L^{p_0}(X, \mu)$  and  $f_1 \in L^{p_1}(X, \mu)$  respectively.*

*Proof.* Since  $p_0 < p$  we have

$$\begin{aligned} \|f_0\|_{L^{p_0}}^{p_0} &= \int_X |f_0|^{p_0} d\mu = \int_X |f|^{p_0} \cdot \chi_{\{|f| > \delta\alpha\}} d\mu \stackrel{(\dagger)}{=} \int_{\{|f| > \delta\alpha\}} |f|^{p_0} d\mu \\ &= \int_{\{|f| > \delta\alpha\}} |f|^p |f|^{p_0 - p} d\mu = \int_{\{|f| > \delta\alpha\}} \frac{|f|^p}{|f|^{p - p_0}} d\mu \\ &\leq \frac{1}{(\delta\alpha)^{p - p_0}} \int_{\{|f| > \delta\alpha\}} |f|^p d\mu = (\delta\alpha)^{p_0 - p} \int_X |f|^p \cdot \chi_{\{|f| > \delta\alpha\}} d\mu \\ &\leq (\delta\alpha)^{p_0 - p} \int_X |f|^p d\mu = (\delta\alpha)^{p_0 - p} \|f\|_{L^p}^p < \infty \end{aligned} \quad (8)$$

Thus  $f_0 \in L^{p_0}(X, \mu)$ . Analogously it can be checked, that  $f_1 \in L^{p_1}(X, \mu)$  by the estimate  $\|f_1\|_{L^{p_1}}^{p_1} \leq (\delta\alpha)^{p_1 - p} \|f\|_{L^p}^p$ .

*Proof of the equality (†).* Assume  $\mu$  is defined on the  $\sigma$ -algebra  $\mathcal{A}$ . We have to prove that  $\{|f| > \delta\alpha\} \in \mathcal{A}$ <sup>1</sup>. Since  $f$  is complex-valued, we may write  $f \equiv \operatorname{Re} f + i\operatorname{Im} f$  and thus  $|f|^2 \equiv \operatorname{Re}^2 f + \operatorname{Im}^2 f$ . Since  $f$  is measurable by hypothesis this implies that  $\operatorname{Re} f$  and  $\operatorname{Im} f$  are measurable<sup>2</sup>. Further for measurable real-valued functions  $f, g : (X, \mathcal{A}) \rightarrow (\mathbb{R}, \mathfrak{B})$ <sup>3</sup> the functions  $f+g$  and  $f \cdot g$  are measurable<sup>4</sup> and thus  $|f|^2$  is measurable. Hence  $\{\operatorname{Re}^2 f + \operatorname{Im}^2 f > \lambda\} \in \mathcal{A}$ <sup>5</sup> for any  $\lambda \in \mathbb{R}$ . So especially for  $\lambda := (\delta\alpha)^2$  we have  $\{|f| > \delta\alpha\} \in \mathcal{A}$ <sup>6</sup>. In a similar manner it can also be proven that  $\{|f| \leq \delta\alpha\} \in \mathcal{A}$ . Let us next prove a useful lemma.

LEMMA 2.2. *Let  $A \in \mathcal{O}(X)$  and  $\chi_A : (X, \mathcal{A}) \rightarrow (\mathbb{C}, \mathfrak{B}^2)$  be the characteristic function of the set  $A$ . Then  $\chi_A$  is measurable if and only if  $A$  is measurable.*

*Proof.* Assume  $\chi_A$  is measurable. Then  $\operatorname{Re} \chi_A$  and  $\operatorname{Im} \chi_A$  are measurable. Especially for  $0 < \lambda < 1$  we have that  $\{\operatorname{Re} \chi_A > \lambda\} = A \in \mathcal{A}$ . Conversely, assume  $A$  is measurable. For  $\lambda < 0$  we have  $\{\operatorname{Re} \chi_A > \lambda\} = X \in \mathcal{A}$ ,  $\lambda \in [0, 1]$ ,  $\{\operatorname{Re} \chi_A > \lambda\} = A \in \mathcal{A}$  and  $\{\operatorname{Re} \chi_A > \lambda\} = \emptyset \in \mathcal{A}$  for  $\lambda \geq 1$ . Since  $\operatorname{Im} \chi_A \equiv 0$  we have  $\{\operatorname{Im} \chi_A > \lambda\} = X \in \mathcal{A}$  if  $\lambda < 0$  and  $\{\operatorname{Im} \chi_A > \lambda\} = \emptyset \in \mathcal{A}$  if  $\lambda \geq 0$ .  $\square$

By Lemma 2.2 and the fact that  $f \cdot g$  is measurable for two measurable functions  $f, g : (X, \mathcal{A}) \rightarrow (\mathbb{C}, \mathfrak{B}^2)$ <sup>7</sup>,  $f_0$  and  $f_1$  are measurable since  $f_0 \equiv f \cdot \chi_{\{|f| > \delta\alpha\}}$  and  $f_1 \equiv f \cdot \chi_{\{|f| \leq \delta\alpha\}}$ .

One subtlety is left to clear: the  $\mu$ -integrability of either  $|f_1|^{p_0}$  or  $|f_1|^{p_1}$  requires that  $|f_0|^{p_0}$  and  $|f_1|^{p_1}$  are measurable functions. By the fact that any continuous map  $g : (X, d_X) \rightarrow (Y, d_Y)$  between metric spaces is Borel-measurable (see [Els11, p. 86]) and that the composition of measurable functions is again measurable (see [Els11, p. 87]), the measurability of either  $f_0$  or  $f_1$  follows by  $|f_0|^{p_0} \equiv \cdot^{p_0} \circ |f \cdot \chi_{\{|f| > \delta\alpha\}}|$  and  $|f_1|^{p_1} \equiv \cdot^{p_1} \circ |f \cdot \chi_{\{|f| \leq \delta\alpha\}}|$  by stipulating  $\cdot^p : (\mathbb{R}_{\geq 0}, |\cdot|) \rightarrow (\mathbb{C}, |\cdot|)$ ,  $x^p := \exp(p \log(x))$  for  $p > 0$  and  $x \in \mathbb{R}_{>0}$  and  $x^p := 0$  if  $x = 0$ .  $\square$

By lemma (2.1) we therefore have  $f \equiv f_0 + f_1 \in L^{p_0} + L^{p_1}$ .

LEMMA 2.3. *For fixed  $\alpha > 0$ , the distribution function  $d_{T(f)}(\alpha)$  obeys an upper bound of the form*

$$d_{T(f)}(\alpha) \leq \left( \frac{A_0}{\alpha/2} \right)^{p_0} \|f_0\|_{L^{p_0}}^{p_0} + \left( \frac{A_1}{\alpha/2} \right)^{p_1} \|f_1\|_{L^{p_1}}^{p_1}$$

<sup>1</sup> For  $Y \in \mathcal{A}$  the  $\mu$ -integral of  $f : X \rightarrow \mathbb{C}$  over  $Y$  is defined to be  $\int_Y f d\mu := \int_X f \cdot \chi_Y d\mu$ . For more details see [Els11, pp. 135–136].

<sup>2</sup>For a proof see [Els11, p. 106]

<sup>3</sup> $\mathfrak{B} := \sigma(\mathbb{R})$  and  $\mathfrak{B}^2 = \{B \cup E : B \in \mathfrak{B}, E \subseteq \{\pm\infty\}\}$ .

<sup>4</sup>For a proof see [Els11, p. 107].

<sup>5</sup>For a proof see [Els11, pp. 105–106]

<sup>6</sup>This follows from the fact that  $x < y$  if and only if  $x^n < y^n$  for  $n \in \mathbb{N}_{>0}$  and some real numbers  $x, y > 0$  (see [Zor04, p. 119]).

<sup>7</sup>Els11, p. 107.

*Proof.* Since  $T$  is a sublinear operator we have  $|T(f)| = |T(f_0 + f_1)| \leq |T(f_0)| + |T(f_1)|$ . Thus for any  $y \in Y$  with  $|T(f)(y)| > \alpha$  we therefore have either  $|T(f_0)(y)| > \alpha/2$  or  $|T(f_1)(y)| > \alpha/2$ <sup>8</sup>. Hence

$$\{|T(f)| > \alpha\} \subseteq \{|T(f_0)| > \alpha/2\} \cup \{|T(f_1)| > \alpha/2\}$$

and so by the monotonicity and subadditivity property of the measure  $\mu$  we have

$$\begin{aligned} d_{T(f)}(\alpha) &= \mu(\{|T(f)| > \alpha\}) \\ &\leq \mu(\{|T(f_0)| > \alpha/2\} \cup \{|T(f_1)| > \alpha/2\}) \\ &\leq \mu(\{|T(f_0)| > \alpha/2\}) + \mu(\{|T(f_1)| > \alpha/2\}) \\ &= d_{T(f_0)}(\alpha/2) + d_{T(f_1)}(\alpha/2) \end{aligned} \tag{9}$$

Now by hypothesis (3) we can estimate  $d_{T(f_0)}(\alpha/2)$  as follows

$$\begin{aligned} d_{T(f_0)}(\alpha/2) &= \left(\frac{\alpha/2}{\alpha/2}\right)^{p_0} d_{T(f_0)}(\alpha/2) \\ &\leq \left(\frac{1}{\alpha/2}\right)^{p_0} \left[\sup \left\{ \gamma d_{T(f_0)}(\gamma)^{1/p_0} : \gamma > 0 \right\}\right]^{p_0} \\ &= \left(\frac{1}{\alpha/2}\right)^{p_0} \|T(f_0)\|_{L^{p_0, \infty}}^{p_0} \\ &\leq \left(\frac{A_0}{\alpha/2}\right)^{p_0} \|f_0\|_{L^{p_0}}^{p_0} \end{aligned} \tag{10}$$

Analogously, we get  $d_{T(f_1)}(\alpha/2) \leq \left(\frac{A_1}{\alpha/2}\right)^{p_1} \|f_1\|_{L^{p_1}}^{p_1}$  by hypothesis (4).  $\square$

By

$$\int_0^{\frac{1}{\delta}|f|} \alpha^{p-p_0-1} d\lambda = \begin{cases} \frac{1}{p-p_0} \frac{1}{\delta^{p-p_0}} |f|^{p-p_0}, & p \geq p_0 + 1 \\ = \lim_{\omega \rightarrow 0^+} \int_{\omega}^{\frac{1}{\delta}|f|} \alpha^{p-p_0-1} d\lambda \\ = \lim_{\omega \rightarrow 0^+} \left[ \frac{1}{p-p_0} \alpha^{p-p_0} \right]_{\omega}^{\frac{1}{\delta}|f|} \\ = \frac{1}{p-p_0} \left[ \frac{1}{\delta^{p-p_0}} |f|^{p-p_0} - \lim_{\omega \rightarrow 0^+} \omega^{p-p_0} \right] \\ = \frac{1}{p-p_0} \frac{1}{\delta^{p-p_0}} |f|^{p-p_0}, & p_0 < p < p_0 + 1 \end{cases} \tag{11}$$

and

<sup>8</sup>Without loss of generality assume  $|T(f_0)(y)| \leq |T(f_1)(y)|$ . Then we have  $\alpha < |T(f)(y)| \leq |T(f_0)(y)| + |T(f_1)(y)| \leq 2|T(f_1)(y)|$  (this is possible since  $\mathbb{R}$  is an ordered field).

$$\begin{aligned}
\int_{\frac{1}{\delta}|f|}^{\infty} \alpha^{p-p_1-1} d\lambda &= \lim_{\omega \rightarrow \infty} \left[ \frac{1}{p-p_1} \alpha^{p-p_1} \right]_{\frac{1}{\delta}|f|}^{\omega} \\
&= \frac{1}{p-p_1} \left[ \lim_{\omega \rightarrow \infty} \omega^{p-p_1} - \frac{1}{\delta^{p-p_1}} |f|^{p-p_1} \right] \\
&= \frac{1}{p_1-p} \frac{1}{\delta^{p-p_1}} |f|^{p-p_1}
\end{aligned} \tag{12}$$

and the representation  $\|f\|_{L^p}^p = p \int_0^\infty \alpha^{p-1} d_f(\alpha) d\lambda$  for  $0 < p < \infty$  we get

$$\begin{aligned}
\|T(f)\|_{L^p}^p &= p \int_0^\infty \alpha^{p-1} d_{T(f)} d\lambda \\
&\leq p (2A_0)^{p_0} \int_0^\infty \alpha^{p-p_0-1} \int_{\{|f|>\delta\alpha\}} |f|^{p_0} d\mu d\lambda \\
&\quad + p (2A_1)^{p_1} \int_0^\infty \alpha^{p-p_1-1} \int_{\{|f|\leq\delta\alpha\}} |f|^{p_1} d\mu d\lambda \\
&= p (2A_0)^{p_0} \int_{\{|f|>0\}} |f|^{p_0} \int_0^{\frac{1}{\delta}|f|} \alpha^{p-p_0-1} d\lambda d\mu \\
&\quad + p (2A_0)^{p_0} \int_{\{|f|=0\}} |f|^{p_0} \int_0^{\frac{1}{\delta}|f|} \alpha^{p-p_0-1} d\lambda d\mu \\
&\quad + p (2A_1)^{p_1} \int_X |f|^{p_1} \int_{\frac{1}{\delta}|f|}^\infty \alpha^{p-p_1-1} d\lambda d\mu \\
&= p (2A_0)^{p_0} \int_X |f|^{p_0} \int_0^{\frac{1}{\delta}|f|} \alpha^{p-p_0-1} d\lambda d\mu \\
&\quad + p (2A_1)^{p_1} \int_X |f|^{p_1} \int_{\frac{1}{\delta}|f|}^\infty \alpha^{p-p_1-1} d\lambda d\mu \\
&= \frac{p (2A_0)^{p_0}}{p-p_0} \frac{1}{\delta^{p-p_0}} \int_X |f|^{p_0} |f|^{p-p_0} d\mu \\
&\quad + \frac{p (2A_1)^{p_1}}{p_1-p} \frac{1}{\delta^{p-p_1}} \int_X |f|^{p_1} |f|^{p-p_1} d\mu \\
&= p \left( \frac{(2A_0)^{p_0}}{p-p_0} \frac{1}{\delta^{p-p_0}} + \frac{(2A_1)^{p_1}}{p_1-p} \delta^{p_1-p} \right) \|f\|_{L^p}^p
\end{aligned} \tag{13}$$

We pick  $\delta > 0$  such that  $(2A_0)^{p_0} \delta^{p_0-p} = (2A_1)^{p_1} \delta^{p_1-p}$ . Solving for  $\delta$  yields

$$\delta = \frac{1}{2} \left( \frac{A_0}{A_1} \right)^{p_1/(p_1-p_0)} \tag{14}$$

Substituting this in estimate (13) leads to

$$\begin{aligned}
\|T(f)\|_{L^p}^p &\leq p \left( \frac{(2A_0)^{p_0}}{p-p_0} \frac{2^{p-p_0} A_1^{\frac{p_1(p-p_0)}{p_1-p_0}}}{A_0^{\frac{p_0(p-p_0)}{p_1-p_0}}} + \frac{(2A_1)^{p_1}}{p_1-p} \frac{A_0^{\frac{p_0(p_1-p)}{p_1-p_0}}}{2^{p_1-p} A_1^{\frac{p_1(p_1-p)}{p_1-p_0}}} \right) \|f\|_{L^p}^p \\
&= 2^p p \left( \frac{A_0^{\frac{p_0(p_1-p)}{p_1-p_0}} A_1^{\frac{p_1(p-p_0)}{p_1-p_0}}}{p-p_0} + \frac{A_0^{\frac{p_0(p_1-p)}{p_1-p_0}} A_1^{\frac{p_1(p-p_0)}{p_1-p_0}}}{p_1-p} \right) \|f\|_{L^p}^p
\end{aligned} \tag{15}$$

And taking the  $p$ -th power further

$$\begin{aligned}
\|T(f)\|_{L^p} &\leq 2 \left( \frac{p}{p-p_0} + \frac{p}{p_1-p} \right)^{1/p} A_0^{\frac{p_0(p_1-p)}{p(p_1-p_0)}} A_1^{\frac{p_1(p-p_0)}{p(p_1-p_0)}} \|f\|_{L^p} \\
&= 2 \left( \frac{p}{p-p_0} + \frac{p}{p_1-p} \right)^{1/p} A_0^{\frac{p_0(p_1-p)}{p(p_1-p_0)} \frac{p_1}{p_1}} A_1^{\frac{p_1(p-p_0)}{p(p_1-p_0)} \frac{p_0}{p_0}} \|f\|_{L^p} \\
&= 2 \left( \frac{p}{p-p_0} + \frac{p}{p_1-p} \right)^{1/p} A_0^{\frac{\frac{p_1-p}{p p_1}}{\frac{p_1-p_0}{p_0 p_1}}} A_1^{\frac{\frac{p-p_0}{p_0 p_1}}{\frac{p_1-p_0}{p_1 p_1}}} \|f\|_{L^p} \\
&= 2 \left( \frac{p}{p-p_0} + \frac{p}{p_1-p} \right)^{1/p} A_0^{\frac{\frac{1}{p} - \frac{1}{p_1}}{\frac{1}{p_0} - \frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0} - \frac{1}{p}}{\frac{1}{p_0} - \frac{1}{p_1}}} \|f\|_{L^p}
\end{aligned} \tag{16}$$

Assume  $p_1 = \infty$ . We again use the cut-off functions defined in (7) to decompose  $f$ . Since  $\{|f_1| > \delta\alpha\} = \emptyset$ , we have

$$\|T(f_1)\|_{L^\infty} \leq A_1 \|f_1\|_{L^\infty} = A_1 \inf \{B > 0 : \mu(\{|f_1| > B\}) = 0\} \leq A_1 \delta\alpha = \alpha/2$$

Provided we stipulate  $\delta := 1/(2A_1)$ . Therefore the set  $\{|T(f_1)| > \alpha/2\}$  has measure zero (this is immediate since  $\|T(f_1)\|_{L^\infty} = \inf \{B > 0 : \mu(\{|T(f_1)| > B\}) = 0\} \leq \alpha/2$  and any subset of a set with measure zero has itself measure zero). Thus similar to part **b.** of (i.) we get  $d_{T(f)}(\alpha) \leq d_{T(f_0)}(\alpha/2)$ .

Hypothesis (3) yields the estimate  $d_{T(f_0)}(\alpha/2) \leq \left( \frac{A_0}{\alpha/2} \right)^{p_0} \int_{\{2A_1|f| > \alpha\}} |f|^{p_0} d\mu$ .

Thus by **a.** and **b.**

$$\begin{aligned}
\|T(f)\|_{L^p}^p &= p \int_0^\infty \alpha^{p-1} d_{T(f)} d\lambda \\
&\leq p(2A_0)^{p_0} \int_0^\infty \alpha^{p-p_0-1} \int_{\{2A_1|f|>\alpha\}} |f|^{p_0} d\mu d\lambda \\
&= p(2A_0)^{p_0} \int_X |f|^{p_0} \int_0^{2A_1|f|} \alpha^{p-p_0-1} d\lambda d\mu \\
&= \frac{2^p p A_0^{p_0} A_1^{p-p_0}}{p-p_0} \int_X |f|^p d\mu \\
&= \frac{2^p p A_0^{p_0} A_1^{p-p_0}}{p-p_0} \|f\|_{L^p}^p
\end{aligned} \tag{17}$$

That the constant  $2^p p A_0^{p_0} A_1^{p-p_0} / (p-p_0)$  found in (17) is the  $p$ -th power of the one stated in the theorem can be seen by passing the constant (6) to the limit  $p_1 \rightarrow \infty$ :

$$\begin{aligned}
\lim_{p_1 \rightarrow \infty} A &= \lim_{p_1 \rightarrow \infty} \left[ 2 \left( \frac{p}{p-p_0} + \frac{p}{p_1-p} \right)^{1/p} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}} \right] \\
&= 2 \exp \left[ \frac{1}{p} \log \left( \frac{p}{p-p_0} + \lim_{p_1 \rightarrow \infty} \frac{1}{p_1} \frac{p}{1-p \lim_{p_1 \rightarrow \infty} \frac{1}{p_1}} \right) \right] \\
&\quad \cdot \lim_{p_1 \rightarrow \infty} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} \cdot \lim_{p_1 \rightarrow \infty} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}} \\
&= 2 \left( \frac{p}{p-p_0} \right)^{1/p} \exp \left[ \frac{\frac{1}{p} - \lim_{p_1 \rightarrow +\infty} \frac{1}{p_1}}{\frac{1}{p_0} - \lim_{p_1 \rightarrow \infty} \frac{1}{p_1}} \log(A_0) \right] \\
&\quad \cdot \exp \left[ \frac{\frac{1}{p_0} - \frac{1}{p}}{\frac{1}{p_0} - \lim_{p_1 \rightarrow \infty} \frac{1}{p_1}} \log(A_1) \right] \\
&= 2 \left( \frac{p}{p-p_0} \right)^{1/p} A_0^{\frac{p_0}{p}} A_1^{1-\frac{p_0}{p}}
\end{aligned}$$

□

**3. The Complex Method.** This theorem will unfortunately only be applicable to linear operators but will yield a more natural bound of the operator on the intermediate space. The proof will make strong use of complex variables technique. A major tool will be an application of the maximum modulus principle, known as *Hadamard's three lines lemma*.



**3.1. Hadamard's Three Lines Lemma.** A complex-valued function  $f$  is said to be *holomorphic* in  $\Omega \subseteq \mathbb{C}$  open, if  $f'(z)$  exists for any  $z \in \Omega$ . By a region we shall mean a nonempty connected open subset of the complex plane. The proof of Hadamard's three lines lemma heavily relies on a restatement of the maximum modulus theorem (see [Rud87, p. 212]) found in [Rud87, p. 253].

**THEOREM 3.1.** *Let  $\Omega \subseteq \mathbb{C}$  be a bounded region and  $f$  be a continuous function on  $\overline{\Omega}$  which is holomorphic in  $\Omega$ . Then*

$$|f(z)| \leq \sup \{|f(z)| : z \in \partial\Omega\}$$

for every  $z \in \Omega$ . If equality holds at one point  $z \in \Omega$ , then  $f$  is constant.

**LEMMA 3.1.** (Hadamard's three lines lemma) *Let  $F$  be a holomorphic function in the strip  $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ , continuous and bounded on  $\overline{S}$ , such that  $|F(z)| \leq B_0$  when  $\operatorname{Re} z = 0$  and  $|F(z)| \leq B_1$  when  $\operatorname{Re} z = 1$ , for some  $0 < B_0, B_1 < \infty$ . Then  $|F(z)| \leq B_0^{1-\theta} B_1^\theta$  when  $\operatorname{Re} z = \theta$ , for any  $0 \leq \theta \leq 1$ .*

*Proof.* For  $z \in \overline{S}$  define

$$G(z) := \frac{F(z)}{B_0^{1-z} B_1^z} \quad G_n(z) := G(z) e^{(z^2-1)/n}, \quad n \in \mathbb{N}_{>0}$$

$G(z)$  and  $G_n(z)$  are holomorphic in  $S$  by

$$G'(z) = \frac{F'(z) - F(z) \log(B_1/B_0)}{B_0^{1-z} B_1^z} \quad G'_n(z) = G'(z) e^{(z^2-1)/n} + \frac{2}{n} z G_n(z)$$

and  $e^z \neq 0$  for every  $z \in \mathbb{C}$ . Further, we have

$$|B_0^{1-z} B_1^z| = (B_0^{1-z} B_0^{1-\bar{z}} B_1^z B_1^{\bar{z}})^{1/2} = B_0^{1-\operatorname{Re} z} B_1^{\operatorname{Re} z}$$

Consider  $0 \leq \operatorname{Re} z \leq 1$  and  $B_0 \geq 1$ . Then  $B_0^{1-\operatorname{Re} z} \geq 1$  and  $B_0^{1-\operatorname{Re} z} \geq B_0$  in the case  $B_0 < 1$ . Similarly,  $B_1^{\operatorname{Re} z} \geq 1$  if  $B_1 \geq 1$  and  $B_1^{\operatorname{Re} z} \leq B_1$  if  $B_1 < 1$ . Hence

$$|B_0^{1-z} B_1^z| \geq \min\{1, B_0\} \min\{1, B_1\} > 0 \quad (18)$$

for all  $z \in \overline{S}$ . Since  $F$  is bounded on  $\overline{S}$ , we have  $|F(z)| \leq L$  for some  $L > 0$  and all  $z \in \overline{S}$ . Thus by (18)

$$|G(z)| = \frac{|F(z)|}{|B_0^{1-z} B_1^z|} \leq \frac{L}{\min\{1, B_0\} \min\{1, B_1\}} =: M$$

for every  $z \in \overline{S}$ . Fix  $n \in \mathbb{N}_{>0}$  and write  $z := x + iy \in \overline{S}$ . Then

$$|G_n(z)| \leq M \left( e^{(x^2+2ixy-y^2-1)/n} e^{(x^2-2ixy-y^2-1)/n} \right)^{1/2} = M e^{-y^2/n} e^{(x^2-1)/n} \leq M e^{-y^2/n}$$

for  $0 \leq x \leq 1$ . Thus

$$\lim_{y \rightarrow \pm\infty} \sup\{|G_n(z)| : 0 \leq x \leq 1\} = 0$$

by the pinching-principle. Hence there exist  $C_0, C_1 \in \mathbb{R}$ , such that

$$\sup\{|G_n(z)| : 0 \leq x \leq 1\} \leq 1$$

when  $y > C_0$  or  $y < C_1$ . Letting

$$C(n) := \max\{|C_0| + 1, |C_1| + 1\}$$

we conclude  $|G_n(z)| \leq 1$  for all  $0 \leq x \leq 1$  when  $|y| \geq C(n)$ . Now consider the rectangle  $R := (0, 1) \times (-C(n), C(n))$ . We have  $|G_n(z)| \leq 1$  on the lines  $[0, 1] \times \{\pm C(n)\}$ . By

$$|G_n(iy)| = \frac{|F(iy)|}{|B_0^{1-iy} B_1^{iy}|} e^{-(y^2+1)/n} \leq 1 \quad |G_n(1+iy)| = \frac{|F(1+iy)|}{|B_0^{-iy} B_1^{1+iy}|} e^{-y^2/n} \leq 1$$

we have  $|G_n(z)| \leq 1$  on the lines  $\{0\} \times [-C(n), C(n)]$ ,  $\{1\} \times [-C(n), C(n)]$ . Thus  $|G_n(z)| \leq 1$  on  $\partial R$ . Since  $|G_n(z)|$  is continuous on  $\bar{R}$ , holomorphic in  $R$  and  $R$  is a bounded region, the maximum modulus theorem implies

$$|G_n(z)| \leq \sup\{|G_n(z)| : z \in \partial R\} \leq 1$$

for every  $z \in R$ . Therefore  $|G_n(z)| \leq 1$  on  $\bar{R}$  and so  $|G_n(z)| \leq 1$  on  $\bar{S}$ . Since inequalities are preserved by limits and the modulus is a continuous function, we have that  $|G(z)| = \lim_{n \rightarrow \infty} |G_n(z)| \leq 1$  for  $z \in \bar{S}$ . We conclude by

$$|F(\theta + it)| = |G(\theta + it)| \left| B_0^{1-\theta-it} B_1^{\theta+it} \right| \leq B_0^{1-\theta} B_1^\theta$$

whenever  $0 \leq \theta \leq 1$ ,  $t \in \mathbb{R}$ . □

**3.2. The Riesz-Thorin Interpolation Theorem.** For two measure spaces  $(X, \mu)$ ,  $(Y, \nu)$  let  $\Sigma_X$  and  $\Sigma_Y$  denote the set of all finitely simple functions on  $X, Y$  respectively.

**THEOREM 3.2.** (Riesz-Thorin Interpolation Theorem) *Suppose that  $(X, \mu)$ ,  $(Y, \nu)$  are measure spaces and  $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ . If  $q_0 = q_1 = \infty$ , suppose also that  $\nu$  is semifinite. Let  $T$  be a linear operator defined on  $\Sigma_X$  and taking values in the set of measurable functions on  $Y$ , such that*

$$\|T(f)\|_{L^{q_0}} \leq M_0 \|f\|_{L^{p_0}} \quad \|T(f)\|_{L^{q_1}} \leq M_1 \|f\|_{L^{p_1}} \quad (19)$$

*for all  $f \in \Sigma_X$  and  $0 < M_0, M_1 < \infty$ . Then for all  $0 < \theta < 1$  we have*

$$\|T(f)\|_{L^q} \leq M_0^{1-\theta} M_1^\theta \|f\|_{L^p} \quad (20)$$

*for all  $f \in \Sigma_X$ , where*

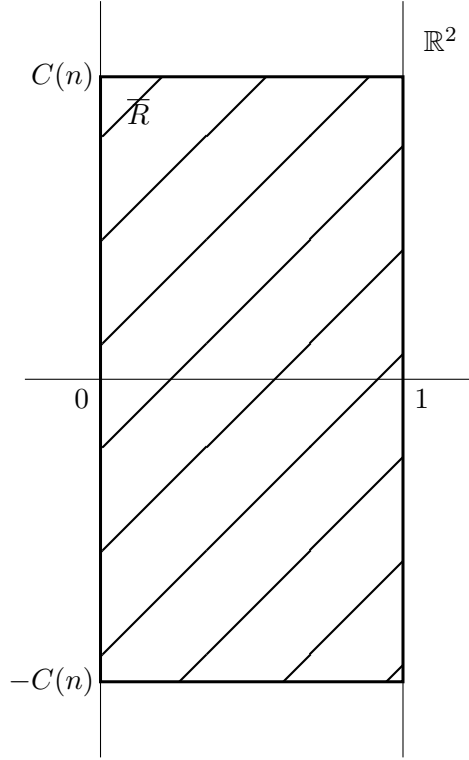


FIGURE 1. Sketch of the rectangle  $\overline{R}$ .

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

*Proof.* If  $f \in \Sigma_X$ ,  $\|f\|_{L^p} = 0$ , then  $f = 0$   $\mu$ -a.e. and either one of the hypotheses on  $T$  in (19) implies  $T(f) = 0$   $\mu$ -a.e. and thus the estimate (20) holds trivially. Therefore we can assume  $\|f\|_{L^p} \neq 0$ . Fix

$$f := \sum_{j=1}^n a_j e^{i\alpha_j} \chi_{A_j} \in \Sigma_X \quad g := \sum_{k=1}^m b_k e^{i\beta_k} \chi_{B_k} \in \Sigma_Y$$

where  $a_j, b_k > 0$  and  $\alpha_j, \beta_k \in \mathbb{R}$  for every  $j = 1, \dots, n, k = 1, \dots, m$  such that  $\|g\|_{L^{q'}} \neq 0$  (recall  $q' := q/(q-1)$ ). Define

$$P(z) := \frac{p}{p_0}(1-z) + \frac{p}{p_1}z \quad Q(z) := \frac{q'}{q'_0}(1-z) + \frac{q'}{q'_1}z$$

for  $z \in \mathbb{C}$  (if  $p, q' = \infty$  then also  $p_0, p_1, q'_0, q'_1 = \infty$  and hence  $P, Q$  are well-defined). Further let

$$f_z := \sum_{j=1}^n a_j^{P(z)} e^{i\alpha_j} \chi_{A_j} \quad g_z := \sum_{k=1}^m b_k^{Q(z)} e^{i\beta_k} \chi_{B_k} \quad (21)$$

and

$$F(z) := \int_Y T(f_z) g_z d\nu \quad (22)$$

By the linearity of the operator  $T$  we have

$$F(z) = \sum_{j=1}^n \sum_{k=1}^m a_j^{P(z)} b_k^{Q(z)} e^{i\alpha_j} e^{i\beta_k} \int_Y T(\chi_{A_j}) \chi_{B_k} d\nu$$

and by Hölder's inequality

$$\begin{aligned} \left| \int_Y T(\chi_{A_j}) \chi_{B_k} d\nu \right| &\leq \int_Y |T(\chi_{A_j}) \chi_{B_k}| d\nu \\ &= \|T(\chi_{A_j}) \chi_{B_k}\|_{L^1} \\ &\leq \|T(\chi_{A_j})\|_{L^{q_0}} \|\chi_{B_k}\|_{L^{q'_0}} \\ &\leq M_0 \|\chi_{A_j}\|_{L^{p_0}} \|\chi_{B_k}\|_{L^{q'_0}} \\ &\leq M_0 \mu(A_j)^{1/p_0} \nu(B_k)^{1/q'_0} \end{aligned} \quad (23)$$

for each  $j = 1, \dots, n$ ,  $k = 1, \dots, m$  (even in the cases where either  $p_0 = \infty$  or  $q'_0 = \infty$ , or both, by observing that  $\|\chi_A\|_{L^\infty} \leq 1$  for any measurable set  $A$ ). Thus the function  $F$  is well-defined on  $\mathbb{C}$ . Let  $t \in \mathbb{R}$ . For  $p, p_0 \neq \infty$

$$\begin{aligned} \|f_{it}\|_{L^{p_0}} &= \left( \sum_{j=1}^n \int_{A_j} |f_{it}|^{p_0} d\mu + \int_{X \setminus \bigcup_{j=1}^n A_j} |f_{it}|^{p_0} d\mu \right)^{1/p_0} \\ &= \left( \sum_{j=1}^n \left| a_j^{P(it)} e^{i\alpha_j} \right|^{p_0} \int_X \chi_{A_j} d\mu \right)^{1/p_0} \\ &= \left( \sum_{j=1}^n a_j^{p_0 \operatorname{Re} P(it)} \mu(A_j) \right)^{1/p_0} \\ &= \left( \sum_{j=1}^n a_j^p \mu(A_j) \right)^{p/(p_0 p)} \\ &= \|f\|_{L^p}^{p/p_0} \end{aligned}$$

holds. Let  $p_0 = \infty$ ,  $p \neq \infty$ . Then  $\|f_{it}\|_{L^\infty} = 1$  since  $|a_j^{P(it)}| = a_j^{p/p_0} = 1$  and that there exists some index  $j$ , such that  $\mu(A_j) \neq 0$ . If  $p = \infty$ , observe that  $P(z) = 1$  and thus  $\|f_{it}\|_{L^\infty} = \|f\|_{L^\infty}$ . By the same considerations we have  $\|g_{it}\|_{L^{q'_0}} = \|g\|_{L^{q'_0}}^{q'/q'_0}$ . Hence

$$\begin{aligned} |F(it)| &\leq \int_Y |T(f_{it})g_{it}| d\nu \\ &= \|T(f_{it})g_{it}\|_{L^1} \\ &\leq \|T(f_{it})\|_{L^{q_0}} \|g_{it}\|_{L^{q'_0}} \\ &\leq M_0 \|f_{it}\|_{L^{p_0}} \|g_{it}\|_{L^{q'_0}} \\ &= M_0 \|f\|_{L^p}^{p/p_0} \|g\|_{L^{q'}}^{q'/q'_0} \end{aligned}$$

by Hölder's inequality. In an analogous manner we derive

$$\|f_{1+it}\|_{L^{p_1}} = \|f\|_{L^p}^{p/p_1} \quad \|g_{1+it}\|_{L^{q'_1}} = \|g\|_{L^{q'_1}}^{q'/q'_1}$$

and thus

$$|F(1+it)| \leq M_1 \|f\|_{L^p}^{p/p_1} \|g\|_{L^{q'_1}}^{q'/q'_1}$$

Further by estimate (23)

$$\begin{aligned} |F(z)| &\leq \sum_{j=1}^n \sum_{k=1}^m |a_j^{P(z)}| |b_k^{Q(z)}| \left| \int_Y T(\chi_{A_j}) \chi_{B_k} d\nu \right| \\ &\leq M_0 \sum_{j=1}^n \sum_{k=1}^m a_j^{\operatorname{Re} P(z)} b_k^{\operatorname{Re} Q(z)} \mu(A_j)^{1/p_0} \nu(B_k)^{1/q'_0} \\ &\leq M_0 \sum_{j=1}^n \sum_{k=1}^m \max \left\{ 1, a_j^{p/p_0 + p/p_1} \right\} \max \left\{ 1, b_k^{q'/q'_0 + q'/q'_1} \right\} \mu(A_j)^{1/p_0} \nu(B_k)^{1/q'_0} \end{aligned}$$

Hence  $F$  is bounded on  $\bar{S}$  by some constant depending on  $f$  and  $g$  only. By

$$\begin{aligned} F'(z) &= \sum_{j=1}^n \sum_{k=1}^m a_j^{P(z)} \log(a_j) \left( \frac{p}{p_1} - \frac{p}{p_0} \right) b_k^{Q(z)} e^{i\alpha_j} e^{i\beta_k} \int_Y T(\chi_{A_j}) \chi_{B_k} d\nu \\ &\quad + \sum_{j=1}^n \sum_{k=1}^m a_j^{P(z)} b_k^{Q(z)} \log(b_k) \left( \frac{q'}{q'_1} - \frac{q'}{q'_0} \right) e^{i\alpha_j} e^{i\beta_k} \int_Y T(\chi_{A_j}) \chi_{B_k} d\nu \end{aligned}$$

it is immediate, that  $F$  is an entire function and thus holomorphic in  $S$  and continuous on  $\bar{S}$ . Therefore, Hadamard's three lines lemma yields

$$|F(z)| \leq \left( M_0 \|f\|_{L^p}^{p/p_0} \|g\|_{L^{q'}}^{q'/q_0'} \right)^{1-\theta} \left( M_1 \|f\|_{L^p}^{p/p_1} \|g\|_{L^{q'}}^{q'/q_1'} \right)^\theta = M_0^{1-\theta} M_1^\theta \|f\|_{L^p} \|g\|_{L^{q'}}$$

for  $\operatorname{Re} z = \theta$ ,  $0 < \theta < 1$ . We have

$$\{T(f) \neq 0\} = \bigcup_{n=1}^{\infty} \{|T(f)| > 1/n\}$$

and by Chebychev's inequality either

$$\nu(\{|T(f)| > 1/n\}) \leq n^{q_0} \|T(f)\|_{L^{q_0}}^{q_0}$$

or

$$\nu(\{|T(f)| > 1/n\}) \leq n^{q_1} \|T(f)\|_{L^{q_1}}^{q_1}$$

whenever  $q_0 \neq \infty$  or  $q_1 \neq \infty$ . Therefore, the set  $\{T(f) \neq 0\}$  is  $\sigma$ -finite unless  $q_0 = q_1 = \infty$ . Further we have  $P(\theta) = Q(\theta) = 1$ . Thus by

$$\begin{aligned} M_q(T(f)) &= \sup \left\{ \left| \int_Y T(f) g d\nu \right| : g \in \Sigma_Y, \|g\|_{L^{q'}} = 1 \right\} \\ &= \sup \{|F(\theta)| : g \in \Sigma_Y, \|g\|_{L^{q'}} = 1\} \\ &\leq M_0^{1-\theta} M_1^\theta \|f\|_{L^p} \end{aligned}$$

we conclude

$$\|T(f)\|_{L^q} = M_q(T(f)) \leq M_0^{1-\theta} M_1^\theta \|f\|_{L^p}$$

for any  $f \in \Sigma_X$ . □

**3.3. Young's inequality.** Using the Riesz-Thorin interpolation theorem, we can give an alternative proof of Young's inequality [Gra14, pp. 22–23].

**THEOREM 3.3. (Young's inequality)** *Let  $G$  be a locally compact group, which is a countable union of compact subsets, and let  $\eta$  be a left invariant Haar measure. Let  $1 \leq p, q, r \leq \infty$*

$$\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r} \tag{24}$$

*Then for all  $f \in L^p(G, \eta)$  and all  $g \in L^r(G, \eta)$  satisfying  $\|g\|_{L^r} = \|\tilde{g}\|_{L^r}$  we have  $f * g$  exists  $\eta$ -a.e. and satisfies*

$$\|f * g\|_{L^q} \leq \|g\|_{L^r} \|f\|_{L^p} \tag{25}$$

*Proof.* Fix  $g \in L^r(G, \eta)$  and let  $T(f) := f * g$  be defined on  $L^1(G, \eta) + L^{r'}(G, \eta)$ . Obviously,  $T$  is a linear operator by the linearity of the integral. By Minkowski's integral inequality (see exercise **1.1.6** [Gra14, p. 13]) we get

$$\begin{aligned} \|T(f)\|_{L^r} &= \left( \int_G \left| \int_G f(y) g(y^{-1}x) d\eta(y) \right|^r d\eta(x) \right)^{1/r} \\ &\leq \int_G \left( \int_G |f(y)|^r |g(y^{-1}x)|^r d\eta(x) \right)^{1/r} d\eta(y) \\ &= \int_G |f(y)| \left( \int_G |g(y^{-1}x)|^r d\eta(y^{-1}x) \right)^{1/r} d\eta(y) \\ &= \int_G |f(y)| \left( \int_G |g(z)|^r d\eta(z) \right)^{1/r} d\eta(y) \\ &\leq \|f\|_{L^1} \|g\|_{L^r} \end{aligned} \quad (26)$$

for  $f \in L^1(g, \mu)$  and  $1 \leq p < \infty$  (since  $(G, \eta)$  is  $\sigma$ -finite). The case  $r = \infty$  follows from

$$|(f * g)(x)| = \left| \int_G f(y) g(y^{-1}x) d\eta(y) \right| \leq \int_G |f(y)| |g(y^{-1}x)| d\eta(y) \leq \|g\|_{L^\infty} \|f\|_{L^1} \quad (27)$$

By stipulating  $h(y) := g(y^{-1}x)$  we have

$$\begin{aligned} |(f * g)(x)| &= \left| \int_G f(y) g(y^{-1}x) d\eta(y) \right| \leq \int_G |f(y) g(y^{-1}x)| d\eta(y) \\ &= \|fh\|_{L^1} \leq \|f\|_{L^{r'}} \|h\|_{L^r} = \|f\|_{L^{r'}} \|\tilde{g}\|_{L^r} = \|g\|_{L^r} \|f\|_{L^{r'}} \end{aligned} \quad (28)$$

for  $r < \infty$  and  $f \in L^{r'}(g, \eta)$ , since

$$\|h\|_{L^r}^r = \int_G |g(y^{-1}x)|^r d\eta(y) = \int_G |\tilde{g}(x^{-1}y)|^r d\eta(y) = \|\tilde{g}\|_{L^r}^r$$

The Riesz-Thorin interpolation theorem now yields for any  $0 < \theta < 1$

$$\|f * g\|_{L^q} = \|T(f)\|_{L^q} \leq \|g\|_{L^{r'}}^{1-\theta} \|g\|_{L^r}^\theta \|f\|_{L^p} = \|g\|_{L^r} \|f\|_{L^p} \quad (29)$$

where

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{r'} \quad \frac{1}{q} = \frac{1-\theta}{r} + \frac{\theta}{\infty}$$

and by

$$\frac{1}{p} = 1 - \frac{\theta}{r} \quad \frac{1}{q} = \frac{1}{r} - \frac{\theta}{r}$$

we get

$$\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}$$

□

REMARK 3.1. *The proof would be much shorter if we just used Minkowski's inequality [Gra14, pp. 21–22] instead of Minkowski's integral inequality. However, the proof given here is an alternative version of the one given already for Minkowski's inequality.*

**4. Interpolation of Analytic Families of Operators.** The generalization of the classical Riesz-Thorin interpolation theorem to analytic families of operators is due to *E. M. Stein* and *Guido Weiss*<sup>9</sup>. Crucial for its proof is again an application of advanced topics in complex analysis.

**4.1. Extension of Hadamard's Three Lines Lemma.** This lemma is inspired by a lemma originally proposed by I.I.Hirschman. I will stick for the most part to the proof given in [Gra14, pp. 43–45], but for some parts I will use the paper by Stein and Weiss.

**4.1.1. Auxiliary Lemmata.** To shorten the proof of the extension of Hadamard's three lines lemma, I will summarize the most important facts used during the proof.

LEMMA 4.1. *Let  $D := \{z \in \mathbb{C} : |z| < 1\}$  be the open unit disc and*

$$h(z) := \frac{1}{\pi i} \log \left( i \frac{1+z}{1-z} \right)$$

*for  $z \in \overline{D} \setminus \{\pm 1\}$  where we are taking that continuous branch of  $\log z$  in the complex plane slit along the negative imaginary axis,  $\mathbb{C} \setminus (\{0\} \times [0, \infty))$ . Then  $h$  is a holomorphic function in  $D$  which maps  $\overline{D} \setminus \{\pm 1\}$  bijectively onto the closure  $\overline{S}$  of the strip  $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ .*

*Proof.* Define  $f(z) := i \frac{1+z}{1-z}$ . If we write  $z := x + iy \in \overline{D} \setminus \{\pm 1\}$ , we have

$$f(z) = \frac{-2y}{(1-x)^2 + y^2} + i \frac{1-x^2-y^2}{(1-x)^2 + y^2} \quad (30)$$

Hence  $\operatorname{Im} f(z) \geq 0$  on  $\overline{D} \setminus \{\pm 1\}$ . Stipulating  $x := 1 - y$  for  $y$  satisfying  $y^2 < y$ , we get

$$\lim_{y^2 < y, y \rightarrow 0^+} \operatorname{Im} f(z) = \lim_{y^2 < y, y \rightarrow 0^+} \left( \frac{1}{y} - 1 \right) = \infty$$

using the same definition of  $x$  we get

$$\lim_{y^2 < y, y \rightarrow 0^+} \operatorname{Re} f(z) = - \lim_{y^2 < y, y \rightarrow 0^+} \frac{1}{y} = -\infty$$

and by stipulating  $x := 1 + y$

<sup>9</sup><https://projecteuclid.org/euclid.tmj/1178244785>, last accessed September 13, 2016.



$$\lim_{y^2 < -y, y \rightarrow 0^-} \operatorname{Re} f(z) = - \lim_{y^2 < -y, y \rightarrow 0^-} \frac{1}{y} = \infty$$

Since  $2i \neq 0$ ,  $f$  is a linear fractional transformation (see [Rud87, p. 279]) with

$$f^{-1}(z) = \frac{z - i}{z + i}$$

Therefore  $f$  maps  $\overline{D} \setminus \{\pm 1\}$  onto the punctured closed upper half plane  $\{z \in \mathbb{C} : \operatorname{Im} z \geq 0\} \setminus \{0\}$ . The preceding logarithm maps this upper half plane onto the strip  $\{z \in \mathbb{C} : 0 \leq \operatorname{Im} z \leq \pi\}$ . Thus  $h(z)$  maps  $\overline{D} \setminus \{\pm 1\}$  onto the strip  $\overline{S}$ . By

$$h'(z) = \frac{2}{\pi i} \frac{1}{1 - z} \quad (31)$$

we see that  $h$  is a holomorphic function in  $D$ . Furthermore, we have

$$h^{-1}(z) = \frac{e^{\pi i z} - i}{e^{\pi i z} + i}$$

□

LEMMA 4.2. *The mapping  $\Phi : \mathbb{R} \rightarrow (-\pi, 0)$  defined by  $\Phi(t) := -i \log(h^{-1}(it))$  is a  $C^1$ -Diffeomorphism with  $|D\Phi(t)| = \pi \operatorname{sech}(\pi t)$ . In an analogous manner we have that  $\Psi : \mathbb{R} \rightarrow (0, \pi)$ ,  $\Psi(t) := -i \log(h^{-1}(1 + it))$  is a  $C^1$ -Diffeomorphism with  $|D\Psi(t)| = \pi \operatorname{sech}(\pi t)$ .*

*Proof.* It is easier to consider  $\Phi^{-1}(\varphi) = -ih(e^{i\varphi})$  and  $\Psi^{-1}(\varphi) = -i(h(e^{i\varphi}) - 1)$  (this already shows that  $\Phi$  is a bijective mapping). Since  $|e^{i\varphi}| = 1$  it is immediate by the representation (30) and  $y < 0$  that  $\operatorname{Im} \Phi(\varphi) = 0$ . Furthermore,  $\lim_{\varphi \rightarrow -\pi} \Phi(\varphi) = \infty$  and  $\lim_{\varphi \rightarrow 0} \Phi(\varphi) = -\infty$ . By (31)  $\Phi$  is clearly continuously differentiable. Using

$$h^{-1}(it) = \frac{e^{-\pi t} - i}{e^{-\pi t} + i}$$

we get

$$|D\Phi(t)| = \pi \left| \frac{e^{-\pi t}}{e^{\pi t} - i} - \frac{e^{-\pi t}}{e^{-\pi t} + i} \right| = \pi \left| \frac{2e^{-\pi t}}{e^{-2\pi t} + 1} \right| \pi \left| \frac{2}{e^{-\pi t} + e^{\pi t}} \right| = \pi \operatorname{sech}(\pi t)$$

□

LEMMA 4.3. *Let  $1/(2e - 1) \leq \rho < 1$  and  $\zeta = \rho e^{i\theta}$ . Then*

$$\left| \log \left| \frac{1 + \zeta}{1 - \zeta} \right| \right| \leq 1 + \log \frac{1}{|\cos(\theta/2)|} + \log \frac{1}{|\sin(\theta/2)|}$$

*Proof.* This proof is due to Prof. Schlein. We have on the one hand

$$|1 + \zeta| \leq 1 + |\zeta| = 1 + \rho$$

and on the other hand

$$|1 - \zeta| \geq |\operatorname{Im} \zeta| = \rho |\sin(\theta)|$$

Hence

$$\begin{aligned} \log \frac{|1 + \zeta|}{|1 - \zeta|} &\leq \log \frac{1 + \rho}{\rho |\sin(\theta)|} \\ &= \log \frac{1 + \rho}{2\rho |\sin(\theta/2)| |\cos(\theta/2)|} \\ &= \log \frac{1 + \rho}{2\rho} + \log \frac{1}{|\sin(\theta/2)|} + \log \frac{1}{|\cos(\theta/2)|} \\ &\leq 1 + \log \frac{1}{|\sin(\theta/2)|} + \log \frac{1}{|\cos(\theta/2)|} \end{aligned}$$

since

$$\frac{1 + \rho}{2\rho} = \frac{1}{2} + \frac{1}{2\rho} \leq e$$

Now by

$$-\log \frac{|1 + \zeta|}{|1 - \zeta|} = \log \frac{|1 - \zeta|}{|1 + \zeta|}$$

which corresponds to considering  $-\zeta = e^{i\pi}\zeta = e^{i(\pi+\theta)}$  in the first case, yields by invoking the identities

$$\cos\left(\frac{\pi + \theta}{2}\right) = -\sin(\theta/2) \quad \sin\left(\frac{\pi + \theta}{2}\right) = \cos(\theta/2)$$

the bound

$$-\log \frac{|1 + \zeta|}{|1 - \zeta|} \leq 1 + \log \frac{1}{|\sin(\theta/2)|} + \log \frac{1}{|\cos(\theta/2)|}$$

and we are done.  $\square$

**4.1.2. The Lemma.** Recall, that a real-valued function  $f$ , defined on a topological space  $X$ , is said to be *upper semicontinuous* at a point  $a \in X$ , if for each  $k > f(a)$  there is a neighbourhood  $V$  of  $a$  such that  $k > f(x)$  for each  $x \in V$ .

LEMMA 4.4. (Hadamard's three lines lemma, extension) *Let  $F$  be a holomorphic function in the strip  $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$  and continuous on  $\bar{S}$ , such that for some  $0 < A < \infty$  and  $\tau_0 \in (0, \pi)$  we have  $\log |F(z)| \leq Ae^{\tau_0 |\operatorname{Im} z|}$  for every  $z \in \bar{S}$ . Then*

$$|F(z)| \leq \exp \left( \frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \left[ \frac{\log |F(it + iy)|}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log |F(1 + it + iy)|}{\cosh(\pi t) + \cos(\pi x)} \right] d\lambda(t) \right)$$

whenever  $z := x + iy \in S$ .

*Proof.* We will first prove the case  $y = 0$ . Assume  $F$  to be not identically zero (the case where  $F$  is identically zero is trivial). Let  $h$  be as in lemma (4.1). By composition,  $F \circ h$  is holomorphic in  $D$  and thus by [Rud87, p. 336]  $\log |F \circ h|$  is subharmonic in  $D$ . Let  $\zeta = \rho e^{i\theta}$ ,  $0 \leq \rho < 1$ . Since  $\zeta \in D$ , we have  $0 < \operatorname{Re} h(\zeta) < 1$  and thus the hypothesis on  $F$  and lemma (4.3) yields

$$\log |F(h(\zeta))| \leq A e^{\frac{\tau_0}{\pi} \left| \log \left| \frac{1+\zeta}{1-\zeta} \right| \right|} \leq A e^{\tau_0/\pi} \frac{1}{|\cos(\theta/2)|^{\tau_0/\pi}} \frac{1}{|\sin(\theta/2)|^{\tau_0/\pi}} \quad (32)$$

for  $1/(2e - 1) \leq \rho$ . Since  $0 < \tau_0 < \pi$ , inequality (32) asserts, that  $\log |F(h(\zeta))|$  is bounded from above by an integrable function of  $\theta$ , independently of  $\rho \geq 1/(2e - 1)$ . Set  $R := 1/(2e - 1)$  and consider the function

$$H(\rho e^{i\theta}) := \begin{cases} \log |F(h(Re^{i\theta}))| & \rho = R, \\ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |F(h(Re^{i\varphi}))| \frac{R^2 - \rho^2}{R^2 - 2R\rho \cos(\theta - \varphi) + \rho^2} d\lambda(\varphi) & 0 \leq \rho < R \end{cases}$$

Then  $H$  is continuous for  $|z| \leq R$  and harmonic for  $|z| < R$  (see [Rud87, pp. 234–235]). Since  $\log |F(h(Re^{i\theta}))| = H(Re^{i\theta})$  is continuous on the circle with radius  $R$ , by [Rud87, p. 336] we have

$$\begin{aligned} \log |F(h(\rho e^{i\theta}))| &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |F(h(Re^{i\varphi}))| \frac{R^2 - \rho^2}{R^2 - 2R\rho \cos(\theta - \varphi) + \rho^2} d\lambda(\varphi) \\ &\leq \frac{A e^{\tau_0/\pi}}{2\pi} \int_{-\pi}^{\pi} \frac{1}{|\cos(\varphi/2)|^{\tau_0/\pi}} \frac{1}{|\sin(\varphi/2)|^{\tau_0/\pi}} \frac{R^2 - \rho^2}{R^2 - 2R\rho \cos(\theta - \varphi) + \rho^2} d\lambda(\varphi) \end{aligned}$$

for  $0 \leq \rho < R$ . Using

$$\frac{R - \rho}{R + \rho} \leq \frac{R^2 - \rho^2}{R^2 - 2R\rho \cos(\theta - \varphi) + \rho^2} \leq \frac{R + \rho}{R - \rho}$$

which holds for  $0 \leq \rho < R < 1$  (see [Rud87, p. 236]), we conclude

$$\log |F(h(\rho e^{i\theta}))| \leq g(\theta)$$

for all  $\rho < 1$ , where  $g \in L^1[-\pi, \pi]$ . Thus for  $\rho$  fixed, we have

$$\log |F(h(Re^{i\varphi}))| \frac{R^2 - \rho^2}{R^2 - 2R\rho \cos(\theta - \varphi) + \rho^2} \leq G(\varphi)$$

where  $G \in L^1[-\pi, \pi]$ . For  $R < 0$  let

$$f_R(\varphi) := \log |F(h(Re^{i\varphi}))| \frac{R^2 - \rho^2}{R^2 - 2R\rho \cos(\theta - \varphi) + \rho^2}$$

and for  $\varphi \neq 0, \pi$

$$f(\varphi) := \log |F(h(e^{i\varphi}))| \frac{1 - \rho^2}{1 - 2\rho \cos(\theta - \varphi) + \rho^2}$$

The upper semicontinuity of  $\log |F \circ h|$  (by [Bou95, p. 360]) continuity at a point  $a$  is equivalent to lower and upper semicontinuity of a function at  $a$  and  $\log |F \circ h|$  is continuous on  $\overline{D} \setminus \{\pm 1\}$  implies<sup>10</sup>

$$\begin{aligned} \limsup_{R \uparrow 1} f_R(\varphi) &= \limsup_{R \uparrow 1} \left[ \log |F(h(Re^{i\varphi}))| \frac{R^2 - \rho^2}{R^2 - 2R\rho \cos(\theta - \varphi) + \rho^2} \right] \\ &= \limsup_{R \uparrow 1} \log |F(h(Re^{i\varphi}))| \lim_{R \uparrow 1} \frac{R^2 - \rho^2}{R^2 - 2R\rho \cos(\theta - \varphi) + \rho^2} \\ &= \log |F(h(e^{i\varphi}))| \frac{1 - \rho^2}{1 - 2\rho \cos(\theta - \varphi) + \rho^2} \\ &= f(\varphi) \end{aligned}$$

using an extension of [Bou95, p. 359]. The functions  $G - f_R$  being non-negative, an application of Fatou's lemma yields

$$\int_{-\pi}^{\pi} \liminf_{R \uparrow 1} [G(\varphi) - f_R(\varphi)] d\lambda(\varphi) \leq \liminf_{R \uparrow 1} \int_{-\pi}^{\pi} [G(\varphi) f_R(\varphi)] d\lambda(\varphi)$$

By [Bou95, p. 354], we get

$$- \int_{-\pi}^{\pi} \limsup_{R \uparrow 1} [f_R(\varphi) - G(\varphi)] d\lambda(\varphi) \leq - \limsup_{R \uparrow 1} \int_{-\pi}^{\pi} [f_R(\varphi) - G(\varphi)] d\lambda(\varphi)$$

and thus

$$\begin{aligned} \limsup_{R \uparrow 1} \int_{-\pi}^{\pi} f_R(\varphi) d\lambda(\varphi) - \int_{-\pi}^{\pi} G(\varphi) d\lambda(\varphi) &= \limsup_{R \uparrow 1} \int_{-\pi}^{\pi} f_R(\varphi) d\lambda(\varphi) + \lim_{R \uparrow 1} \int_{-\pi}^{\pi} (-G(\varphi)) d\lambda(\varphi) \\ &= \limsup_{R \uparrow 1} \int_{-\pi}^{\pi} [f_R(\varphi) - G(\varphi)] d\lambda(\varphi) \leq \int_{-\pi}^{\pi} \limsup_{R \uparrow 1} [f_R(\varphi) - G(\varphi)] d\lambda(\varphi) \\ &\leq \int_{-\pi}^{\pi} \limsup_{R \uparrow 1} f_R(\varphi) d\lambda(\varphi) + \int_{-\pi}^{\pi} \lim_{R \uparrow 1} (-G(\varphi)) d\lambda(\varphi) \\ &= \int_{-\pi}^{\pi} \limsup_{R \uparrow 1} f_R(\varphi) d\lambda(\varphi) - \int_{-\pi}^{\pi} G(\varphi) d\lambda(\varphi) \end{aligned}$$

by [Bou95, p. 358]. Hence

<sup>10</sup> By [Bou95, p. 360] if  $f$  is upper semicontinuous at a point, then  $-f$  is lower semicontinuous at the same point. Hence by [Bou95, p. 363] we have  $\limsup_{x \rightarrow a} f(x) = - \liminf_{x \rightarrow a} (-f)(x) = -(-f)(a) = f(a)$ .

$$\limsup_{R \uparrow 1} \int_{-\pi}^{\pi} f_R(\varphi) d\lambda(\varphi) \leq \int_{-\pi}^{\pi} \limsup_{R \uparrow 1} f_R(\varphi) d\lambda(\varphi)$$

and so

$$\log \left| F(h(\rho e^{i\theta})) \right| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left| F(h(e^{i\varphi})) \right| \frac{1 - \rho^2}{1 - 2\rho \cos(\theta - \varphi) + \rho^2} d\varphi \quad (33)$$

The lemma will now follow from (33) by a change of variables. By stipulating  $x := h(\zeta)$  we obtain <sup>11</sup>

$$\begin{aligned} \zeta = h^{-1}(x) &= \frac{e^{\pi i x} - i}{e^{\pi i x} + i} = \frac{\cos(\pi x) + i \sin(\pi x) - i}{\cos(\pi x) + i \sin(\pi x) + i} \\ &= \frac{\cos(\pi x) + i \sin(\pi x) - i}{\cos(\pi x) + i \sin(\pi x) + i} \frac{\cos(\pi x) - i \sin(\pi x) - i}{\cos(\pi x) - i \sin(\pi x) - i} = -i \frac{\cos(\pi x)}{1 + \sin(\pi x)} \\ &= \left( \frac{\cos(\pi x)}{1 + \sin(\pi x)} \right) e^{-i\pi/2} \end{aligned} \quad (34)$$

by

$$\begin{aligned} &(\cos(\pi x) + i \sin(\pi x) - i)(\cos(\pi x) - i \sin(\pi x) - i) \\ &= \cos^2(\pi x) - i \sin(\pi x) \cos(\pi x) - i \cos(\pi x) + i \sin(\pi x) \cos(\pi x) \\ &\quad + \sin^2(\pi x) + \sin(\pi x) - i \cos(\pi x) - \sin(\pi x) - 1 = -2i \cos(\pi x) \end{aligned}$$

and

$$\begin{aligned} &(\cos(\pi x) + i \sin(\pi x) + i)(\cos(\pi x) - i \sin(\pi x) - i) \\ &= \cos^2(\pi x) - i \sin(\pi x) \cos(\pi x) - i \cos(\pi x) + i \sin(\pi x) \cos(\pi x) \\ &\quad + \sin^2(\pi x) + \sin(\pi x) + i \cos(\pi x) + \sin(\pi x) + 1 = 2 + 2 \sin(\pi x) \end{aligned}$$

From equality (34) we deduce  $\rho = \frac{\cos(\pi x)}{1 + \sin(\pi x)}$ ,  $\theta = \frac{\pi}{2}$  if  $0 < x \leq \frac{1}{2}$  and  $\rho = -\frac{\cos(\pi x)}{1 + \sin(\pi x)}$ ,  $\theta = \frac{\pi}{2}$  if  $\frac{1}{2} \leq x < 1$ . Let  $0 < x \leq \frac{1}{2}$ . Then we have

<sup>11</sup> Recall, that for  $z \in \mathbb{C}$  the trigonometric functions are defined by  $\sin(z) := \frac{e^{iz} - e^{-iz}}{2i}$  and  $\cos(z) := \frac{e^{iz} + e^{-iz}}{2}$ . Hence the identities  $e^{iz} = \cos(z) + i \sin(z)$  and  $\cos^2(z) + \sin^2(z) = 1$  holds for any  $z \in \mathbb{C}$  (see [Ahl79, pp. 42–44]).

$$\begin{aligned}
& \frac{1 - \rho^2}{1 - 2\rho \cos(\theta - \varphi) + \rho^2} \\
&= \frac{1 + 2\sin(\pi x) + \sin^2(\pi x) - \cos^2(\pi x)}{1 + 2\sin(\pi x) + \sin^2(\pi x) + 2\cos(\pi x)\sin(\varphi)(1 + \sin(\pi x)) + \cos^2(\pi x)} \\
&= \frac{\sin(\pi x) + \sin^2(\pi x)}{1 + \sin(\pi x) + \cos(\pi x)\sin(\varphi)(1 + \sin(\pi x))} = \frac{\sin(\pi x)}{1 + \cos(\pi x)\sin(\varphi)}
\end{aligned}$$

since  $\cos(-\pi/2 - \varphi) = -\sin(\varphi)$ . That the case  $\frac{1}{2} \leq x < 1$  yields the same result is due to  $\cos(\pi/2 - \varphi) = \sin(\varphi)$ . Let  $\Phi$  and  $\Psi$  be defined as in lemma (4.2). We have

$$\begin{aligned}
e^{i\Phi(t)} &= h^{-1}(it) = \frac{e^{-\pi t} - i e^{-\pi t} - i}{e^{-\pi t} + i e^{-\pi t} - i} = \frac{e^{-2\pi t} - 2i e^{-\pi t} - 1}{e^{-2\pi t} + 1} = \frac{e^{-2\pi t} - 1}{e^{-2\pi t} + 1} - \frac{2i e^{-\pi t}}{e^{-2\pi t} + 1} \\
&= \frac{e^{-2\pi t} - 1}{e^{-2\pi t} + 1} - \frac{2i}{e^{-\pi t} + e^{\pi t}} = \frac{1 - e^{2\pi t}}{1 + e^{2\pi t}} - \frac{2i}{e^{-\pi t} + e^{\pi t}} = -\tanh(\pi t) - i \operatorname{sech}(\pi t)
\end{aligned}$$

and thus

$$\begin{aligned}
\sin(\Phi(t)) \cosh(\pi t) &= \sin(-i \log(-\tanh(\pi t) - i \operatorname{sech}(\pi t))) \cosh(\pi t) \\
&= \frac{1}{2i} \left[ -\tanh(\pi t) - i \operatorname{sech}(\pi t) + \frac{1}{\tanh(\pi t) + i \operatorname{sech}(\pi t)} \right] \cosh(\pi t) \\
&= \frac{1}{2i} \left[ \frac{\cosh(\pi t) - \tanh(\pi t) \sinh(\pi t) - 2i \tanh(\pi t) + \operatorname{sech}(\pi t)}{\tanh(\pi t) + i \operatorname{sech}(\pi t)} \right] \\
&= \frac{1}{2i} \left[ \frac{\cosh^2(\pi t) - \sinh^2(\pi t) - 2i \sinh(\pi t) + 1}{\sinh(\pi t) + i} \right] \\
&= \frac{1 - i \sinh(\pi t)}{i \sinh(\pi t) - 1} \\
&= -1
\end{aligned}$$

Therefore the transformation formula yields

$$\begin{aligned}
& \frac{1}{2\pi} \int_{-\pi}^0 \frac{\sin(\pi x)}{1 + \cos(\pi x) \sin(\varphi)} \log |F(h(e^{i\varphi}))| d\lambda(\varphi) \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin(\pi x)}{\cosh(\pi t) - \cos(\pi x)} \log |F(it)| d\lambda(t) \quad (35)
\end{aligned}$$

and in a similar manner

$$\begin{aligned} \frac{1}{2\pi} \int_0^\pi \frac{\sin(\pi x)}{1 + \cos(\pi x) \sin(\varphi)} \log |F(h(e^{i\varphi}))| d\lambda(\varphi) \\ = \frac{1}{2} \int_{-\infty}^\infty \frac{\sin(\pi x)}{\cosh(\pi t) + \cos(\pi x)} \log |F(1 + it)| d\lambda(t) \end{aligned} \quad (36)$$

holds since

$$\begin{aligned} \sin(\Psi(t)) \cosh(\pi t) &= \sin(-i \log(-\tanh(\pi t) + i \operatorname{sech}(\pi t))) \cosh(\pi t) \\ &= \frac{1}{2i} \left[ -\tanh(\pi t) + i \operatorname{sech}(\pi t) - \frac{1}{-\tanh(\pi t) + i \operatorname{sech}(\pi t)} \right] \cosh(\pi t) \\ &= \frac{1}{2i} \left[ \frac{-\cosh(\pi t) + \tanh(\pi t) \sinh(\pi t) - 2i \tanh(\pi t) - \operatorname{sech}(\pi t)}{-\tanh(\pi t) + i \operatorname{sech}(\pi t)} \right] \\ &= \frac{1}{2i} \left[ \frac{-\cosh^2(\pi t) + \sinh^2(\pi t) - 2i \sinh(\pi t) - 1}{i - \sinh(\pi t)} \right] \\ &= \frac{1 + i \sinh(\pi t)}{1 + i \sinh(\pi t)} \\ &= 1 \end{aligned}$$

Thus the case  $y = 0$  is proven.

The case  $y \neq 0$  follows easily from the previous one. Fix  $y \neq 0$  and define  $G(z) := F(z + iy)$  for  $z \in \overline{S}$ . Then  $G$  is a holomorphic function in  $S$  and continuous on  $\overline{S}$  as a composition of continuous and holomorphic functions. Moreover, the hypothesis on  $F$  yields

$$\log |G(z)| = \log |F(z + iy)| \leq A e^{\tau |\operatorname{Im} z + y|} \leq A e^{\tau |\operatorname{Im} z|} e^{\tau |y|} \quad (37)$$

for all  $z \in \overline{S}$ . The previous case yields for  $G$  with  $A$  replaced by  $A e^{\tau |y|}$

$$|G(x)| \leq \exp \left( \frac{\sin(\pi x)}{2} \int_{-\infty}^\infty \left[ \frac{\log |G(it)|}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log |G(1 + it)|}{\cosh(\pi t) + \cos(\pi x)} \right] d\lambda(t) \right) \quad (38)$$

Now, observing  $G(x) = F(x + iy)$ ,  $G(it) = F(it + iy)$  and  $G(1 + it) = F(1 + it + iy)$  yields the desired result.  $\square$

**4.2. Stein's Theorem on Interpolation of Analytic Families of Operators.** Because of the complex nature of the proof of the Riesz-Thorin Interpolation Theorem (3.2), Elias M. Stein realized quickly, that the restriction to consider only one linear operator  $T$  could easily be omitted and instead, an analytic family of operators  $T_z$  depending on some complex parameter  $z$  could be considered.

DEFINITION 4.1. (Analytic family, admissible growth) *Let  $(X, \mu)$  be a measure space,  $(Y, \nu)$  be a semifinite measure spaces and  $(T_z)_{z \in \bar{S}}$ , where  $T_z$  is defined  $\Sigma_X$  and taking values in the space of all measurable functions on  $Y$  such that*

$$\int_Y |T_z(\chi_A)\chi_B| d\nu \quad (39)$$

*whenever  $\mu(A), \nu(B) < \infty$ . The family  $(T_z)_{z \in \bar{S}}$  is said to be analytic if for all  $f \in \Sigma_X$ ,  $g \in \Sigma_Y$  we have that*

$$z \mapsto \int_Y T_z(f)g d\nu \quad (40)$$

*is analytic on  $S$  and continuous on  $\bar{S}$ . Further, an analytic family  $(T_z)_{z \in \bar{S}}$  is called of admissible growth, if there is a constant  $\tau_0 \in (0, \pi)$ , such that for all  $f \in \Sigma_X$ ,  $g \in \Sigma_Y$  a constant  $C(f, g)$  exists with*

$$\log \left| \int_Y T_z(f)g d\nu \right| \leq C(f, g) e^{\tau_0 |\operatorname{Im} z|} \quad (41)$$

*for all  $z \in \bar{S}$ .*

Now we are able to write down the theorem.

THEOREM 4.1. (Stein's Theorem on Interpolation of Analytic Families of Operators) *Let  $(T_z)_{z \in \bar{S}}$  be an analytic family of admissible growth,  $1 \leq p_0, p_1, q_0, q_1 \leq \infty$  and suppose that  $M_0, M_1$  are positive functions on the real line such that for some  $\tau_1 \in (0, \pi)$*

$$\sup \left\{ e^{-\tau_1 |y|} \log M_0(y) : y \in \mathbb{R} \right\} < \infty \quad \sup \left\{ e^{-\tau_1 |y|} \log M_1(y) : y \in \mathbb{R} \right\} < \infty \quad (42)$$

*Fix  $0 < \theta < 1$  and define*

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \quad (43)$$

*Further suppose that for all  $f \in \Sigma_X$  and  $y \in \mathbb{R}$  we have*

$$\|T_{iy}(f)\|_{L^{q_0}} \leq M_0(y) \|f\|_{L^{p_0}} \quad \|T_{1+iy}(f)\|_{L^{q_1}} \leq M_1(y) \|f\|_{L^{p_1}} \quad (44)$$

*Then for all  $f \in \Sigma_X$  we have*

$$\|T_\theta(f)\|_{L^q} \leq M(\theta) \|f\|_{L^p}$$

*where for  $0 < x < 1$*



$$M(x) = \exp \left( \frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \left[ \frac{\log M_0(t)}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log M_1(t)}{\cosh(\pi t) + \cos(\pi x)} \right] d\lambda(t) \right)$$

*Proof.* Fix  $0 < \theta < 1$  and  $f \in \Sigma_X$ ,  $g \in \Sigma_Y$  with  $\|f\|_{L^p} = \|g\|_{L^{q'}} = 1$ . Define  $f_z, g_z$  as in (21) and for  $z \in \bar{S}$

$$F(z) := \int_Y T_z(f_z) g_z d\nu \quad (45)$$

Observe, that  $|a_j^{P(z)}| \leq a_j^{p/p_0+p/p_1}$  and  $|b_k^{Q(z)}| \leq b_k^{q'/q'_0+q'/q'_1}$  for  $z \in \bar{S}$ . Hence

$$\begin{aligned} \log |F(z)| &= \log \left| \sum_{j=1}^n \sum_{k=1}^m a_j^{P(z)} b_k^{Q(z)} e^{i\alpha_j} e^{i\beta_k} \int_Y T_z(\chi_{A_j})(y) \chi_{B_k}(y) d\nu(y) \right| \\ &\leq \log \left( \sum_{j=1}^n \sum_{k=1}^m a_j^{p/p_0+p/p_1} b_k^{q'/q'_0+q'/q'_1} \left| \int_{B_k} T_z(\chi_{A_j}) d\nu \right| \right) \\ &\leq \log \left( \sum_{j=1}^n \sum_{k=1}^m a_j^{p/p_0+p/p_1} b_k^{q'/q'_0+q'/q'_1} e^{c(A_j, B_k) e^{\tau_0 |\operatorname{Im} z|}} \right) \\ &\leq \log \left( \sum_{j=1}^n \sum_{k=1}^m e^{\left| \log \left( a_j^{p/p_0+p/p_1} b_k^{q'/q'_0+q'/q'_1} \right) \right| + c(A_j, B_k) e^{\tau_0 |\operatorname{Im} z|}} \right) \\ &\leq \log \left( m n e^{\sum_{j=1}^n \sum_{k=1}^m \log \left( a_j^{p/p_0+p/p_1} b_k^{q'/q'_0+q'/q'_1} \right) + c(A_j, B_k) e^{\tau_0 |\operatorname{Im} z|}} \right) \\ &= \log(mn) + \sum_{j=1}^n \sum_{k=1}^m \left| \log \left( a_j^{p/p_0+p/p_1} b_k^{q'/q'_0+q'/q'_1} \right) \right| + c(A_j, B_k) e^{\tau_0 |\operatorname{Im} z|} \end{aligned}$$

Since  $\tau_0 \in (0, \pi)$  and thus  $e^{\tau_0 |\operatorname{Im} z|} > 1$ ,  $F$  satisfies the hypotheses of the extension of Hadamard's three lines lemma (4.4) with

$$A = \log(mn) + \sum_{j=1}^n \sum_{k=1}^m \left( \frac{p}{p_0} + \frac{p}{p_1} \right) |\log(a_j)| + \left( \frac{q'}{q'_0} + \frac{q'}{q'_1} \right) |\log(b_k)| + c(A_j, B_k)$$

The same calculations as in the proof of the Riesz-Thorin interpolation theorem (3.2) yields for  $y \in \mathbb{R}$

$$\|f_{iy}\|_{L^{p_0}} = \|f\|_{L^p}^{p/p_0} = 1 = \|g\|_{L^{q'}}^{q'/q'_0} = \|g_{iy}\|_{L^{q'_0}}$$

and

$$\|f_{1+iy}\|_{L^{p_1}} = \|f\|_{L^p}^{p/p_1} = 1 = \|g\|_{L^{q'}}^{q'/q'_1} = \|g_{1+iy}\|_{L^{q'_1}}$$

Further

$$|F(iy)| \leq \|T_{iy}(f_{iy})\|_{L^{q_0}} \|g_{iy}\|_{L^{q'_0}} \leq M_0(y) \|f_{iy}\|_{L^{p_0}} \|g_{iy}\|_{L^{q'_0}} = M_0(y)$$

and

$$|F(1+iy)| \leq \|T_{1+iy}(f_{1+iy})\|_{L^{q_1}} \|g_{1+iy}\|_{L^{q'_1}} \leq M_1(y) \|f_{1+iy}\|_{L^{p_1}} \|g_{1+iy}\|_{L^{q'_1}} = M_1(y)$$

by Hölder's inequality and the hypotheses on the analytic family  $(T_z)_{z \in \bar{S}}$ . Therefore the extension of Hadamard's three lines lemma (4.4) yields

$$|F(x)| \leq \exp \left( \frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \left[ \frac{\log M_0(t)}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log M_1(t)}{\cosh(\pi t) + \cos(\pi x)} \right] d\lambda(t) \right) = M(x)$$

for every  $0 < x < 1$ . Furthermore observe that

$$F(\theta) = \int_Y T_\theta(f) g d\nu$$

and thus by [Fol99, p. 189] ( $\Sigma_Y$  denotes the set of all finitely simple functions on the semifinite space  $Y$ )

$$\begin{aligned} M_q(T_\theta(f)) &= \sup \left\{ \left| \int_Y T_\theta(f) g \right| : g \in \Sigma_Y, \|g\|_{L^{q'}} \right\} \\ &= \sup \{ |F(\theta)| : g \in \Sigma_Y, \|g\|_{L^{q'}} \} \\ &\leq M(\theta) \end{aligned}$$

Since  $M(\theta)$  is an absolutely convergent integral for any  $0 < \theta < 1$ ,  $M_q(T_\theta(f)) < \infty$  and thus  $M_q(T_\theta(f)) = \|T_\theta(f)\|_{L^q}$  (this is incorporated by the growth conditions on  $M_0$  and  $M_1$ ). The general statement follows by replacing  $f$  with  $f/\|f\|_{L^p}$  when  $\|f\|_{L^p} \neq 0$ . The theorem is trivially true when  $\|f\|_{L^p} = 0$ .  $\square$

## Appendix A. Limit superior and limit inferior revisited

DEFINITION A.1. Let  $(X, d)$  a metric space,  $E \subseteq X$ ,  $f : E \rightarrow \mathbb{R}$  and  $a \in X$  be a limit point of  $E$ . Then we define the upper limit of  $f$  at  $a$  as

$$\limsup_{x \rightarrow a} f(x) := \lim_{\varepsilon \downarrow 0} \left[ \sup \{ f(x) : x \in E \cap \dot{B}_\varepsilon(a) \} \right]$$

and the lower limit of  $f$  at  $a$  as

$$\liminf_{x \rightarrow a} f(x) := - \limsup_{x \rightarrow a} (-f)(x)$$

PROPOSITION A.1. *Let  $(X, d)$  a metric space,  $E \subseteq X$ ,  $f, g : E \rightarrow \mathbb{R}$ , where  $f$  is bounded and  $a \in X$  be a limit point of  $E$ . Then*

$$\limsup_{x \rightarrow a} (fg)(x) = \limsup_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$$

*whenever both sides exist and  $\lim_{x \rightarrow a} g(x) \geq 0$ .*

*Proof.* Write

$$fg = f \lim_{x \rightarrow a} g(x) + f \left[ g - \lim_{x \rightarrow a} g(x) \right]$$

By [Bou95, p. 358] we have

$$\begin{aligned} \limsup_{x \rightarrow a} (fg)(x) &= \limsup_{x \rightarrow a} \left( f(x) \lim_{x \rightarrow a} g(x) + f(x) \left[ g(x) - \lim_{x \rightarrow a} g(x) \right] \right) \\ &= \limsup_{x \rightarrow a} \left( f(x) \lim_{x \rightarrow a} g(x) \right) + \lim_{x \rightarrow a} \left( f(x) \left[ g(x) - \lim_{x \rightarrow a} g(x) \right] \right) \\ &= \limsup_{x \rightarrow a} \left( f(x) \lim_{x \rightarrow a} g(x) \right) \end{aligned}$$

since  $\lim_{x \rightarrow a} [g(x) - \lim_{x \rightarrow a} g(x)] = 0$  and  $f$  is bounded. Fix  $\varepsilon > 0$ . Further by [Bou95, p. 357] and  $\lim_{x \rightarrow a} g(x) \geq 0$

$$\sup \left\{ f(x) \lim_{x \rightarrow a} g(x) : x \in E \cap \dot{B}_\varepsilon(a) \right\} = \sup \left\{ f(x) : x \in E \cap \dot{B}_\varepsilon(a) \right\} \lim_{x \rightarrow a} g(x)$$

Hence

$$\begin{aligned} \limsup_{x \rightarrow a} (fg)(x) &= \limsup_{x \rightarrow a} \left( f(x) \lim_{x \rightarrow a} g(x) \right) \\ &= \lim_{\varepsilon \downarrow 0} \left[ \sup \left\{ f(x) \lim_{x \rightarrow a} g(x) : x \in E \cap \dot{B}_\varepsilon(a) \right\} \right] \\ &= \lim_{\varepsilon \downarrow 0} \left[ \sup \left\{ f(x) : x \in E \cap \dot{B}_\varepsilon(a) \right\} \right] \lim_{x \rightarrow a} g(x) \\ &= \limsup_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x) \end{aligned} \tag{46}$$

□

## Appendix B. Measure Theory

Let  $(X, \mu)$  be a measure space. Recall, that if for each measurable set  $E$  with  $\mu(E) = \infty$  there exists a measurable set  $F \subseteq E$  and  $0 < \mu(F) < \infty$ ,  $\mu$  is called *semifinite*.

LEMMA B.1. *Every  $\sigma$ -finite measure is semifinite.*

*Proof.* Let  $X = \bigcup_{n \in \mathbb{N}} X_n$  where  $\mu(X_n) < \infty$  and  $E$  be measurable with  $\mu(E) = \infty$ . By letting  $Y_n := \bigcup_{k \leq n} X_k$ ,  $Y_n$  is an increasing sequence. Then  $E \cap Y_n$  is measurable and since  $E \cap Y_n \subseteq Y_n$ ,  $\mu(E \cap Y_n) < \infty$  for each  $n \in \mathbb{N}$ . By the continuity from below (see [Coh13, p. 10] or [Fol99, p. 26]) we have

$$\infty = \mu(E) = \mu(E \cap X) = \mu\left(E \cap \left(\bigcup_{n \in \mathbb{N}} Y_n\right)\right) = \mu\left(\bigcup_{n \in \mathbb{N}} (E \cap Y_n)\right) = \lim_{n \rightarrow \infty} \mu(E \cap Y_n)$$

Hence for every  $C > 0$  there exists  $N \in \mathbb{N}$ , such that  $\infty > \mu(E \cap Y_n) > C$  for  $n > N$ .  $\square$

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