CLASSICAL FOURIER ANALYSIS: INTERPOLATION OF L^p SPACES

YANNIS BÄHNI

Abstract. In this written seminar work I will basically follow the section *Interpolation* in the book *Classical Fourier Analysis, third Edition* by Loukas Grafakos. I will review three basic but important theorems on interpolation of operators on L^p spaces, namely the *Marcinkiewicz Interpolation Theorem*, the *Riesz-Thorin Interpolation Theorem* and finally an extension of the Riesz-Thorin Interpolation Theorem to analytic families of operators (the so-called *Stein's theorem on interpolation of analytic families of operators*). We are mainly concerned with the notion of linear operators as well as slight generalizations of them.

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(Yannis Bähni) University of Zurich, Rämistrasse 71, 8006 Zurich
E-mail address: yannis.baehni@uzh.ch.

Yannis Bähni
yannis.baehni@uzh.ch

1. Introduction and Basic Definitions. What follows is a short summary of the important terms used in this paper.

1.1. Linear Operators. First we need to have a precise and suitable idea of *linear operators* in the generalized setting of measure spaces.

Definition 1.1. *Let (X, μ) and (Y, ν) be measure spaces. Further let T be an operator defined on a linear space of complex-valued measurable functions on X and taking values in the set of all complex-valued, finite almost everywhere, measurable functions on Y. Then T is called* linear *if for all functions f and g in the domain of T and all $z \in \mathbb{C}$ holds*

$$T(f+g) = T(f) + T(g) \qquad T(zf) = zT(f) \tag{1}$$

and quasi-linear *if*

$$|T(f+g)| \leqslant K(|T(f)| + |T(g)|) \qquad |T(zf)| = |z||T(f)| \tag{2}$$

holds for some real constant $K > 0$. If $K = 1$, T is called sublinear.

2. The Real Method. A first important theorem on the subject of interpolation of L^p spaces will be the so-called *Marcinkiewicz Interpolation Theorem* which uses only real variables techniques for its proof (this stands in contrast to the complex variables techniques used for prooving the other interpolation theorems).

2.1. The Marcinkiewicz Interpolation Theorem. This theorem applies to sublinear operators (aswell as for quasilinear operators by a slight change of the constant), which is in comparison to the linearity assumed by the other interpolation theorems more generally applicable.

Theorem 2.1. (The Marcinkiewicz Interpolation Theorem) *Let (X, μ) be a σ-finite measure space, (Y, ν) another measure space and $0 < p_0 < p_1 \leqslant \infty$. Further let T be a sublinear operator defined on*

$$L^{p_0} + L^{p_1} := \{f_0 + f_1 : f_0 \in L^{p_0}(X, \mu), f_1 \in L^{p_1}(X, \mu)\}$$

and taking values in the space of measurable functions on Y. Assume that there exist $A_0, A_1 < \infty$ such that

$$\forall f \in L^{p_0}(X, \mu) \ \|T(f)\|_{L^{p_0,\infty}} \leqslant A_0 \|f\|_{L^{p_0}} \tag{3}$$
$$\forall f \in L^{p_1}(X, \mu) \ \|T(f)\|_{L^{p_1,\infty}} \leqslant A_1 \|f\|_{L^{p_1}} \tag{4}$$

Then for all $p_0 < p < p_1$ and for all $f \in L^p(X, \mu)$ we have the estimate

$$\|T(f)\|_{L^p} \leqslant A \, \|f\|_{L^p} \tag{5}$$

Yannis Bähni
yannis.baehni@uzh.ch

where

$$A := 2\left(\frac{p}{p-p_0} + \frac{p}{p_1 - p}\right)^{1/p} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}} \tag{6}$$

Proof. Let us first consider the case $\underline{p_1 < \infty}$. Fix $f \in L^p(X,\mu)$, $\alpha > 0$ and $\delta > 0$ (δ will be determined later). We split f using so-called *cut-off* functions, by stipulating $f \equiv f_0(\cdot;\alpha,\delta) + f_1(\cdot;\alpha,\delta)$, where $f_0(\cdot;\alpha,\delta)$ is the *unbounded part of* f and $f_1(\cdot;\alpha,\delta)$ is the *bounded part of* f, defined by

$$\begin{aligned} f_0(x;\alpha,\delta) &:= \begin{cases} f(x), & |f(x)| > \delta\alpha, \\ 0, & |f(x)| \leqslant \delta\alpha. \end{cases} \\ f_1(x;\alpha,\delta) &:= \begin{cases} f(x), & |f(x)| \leqslant \delta\alpha, \\ 0, & |f(x)| > \delta\alpha. \end{cases} \end{aligned} \tag{7}$$

for $x \in X$. To facilitate reading I will omit the dependency of $f_0(\cdot;\alpha,\delta)$ and $f_1(\cdot;\alpha,\delta)$ upon the parameters α and δ in what follows and simply write f_0, f_1 respectively.

LEMMA 2.1. *The functions f_0 and f_1 defined above satisfy $f_0 \in L^{p_0}(X,\mu)$ and $f_1 \in L^{p_1}(X,\mu)$ respectively.*

Proof. Since $p_0 < p$ we have

$$\begin{aligned} \|f_0\|_{L^{p_0}}^{p_0} &= \int_X |f_0|^{p_0} d\mu = \int_X |f|^{p_0} \cdot \chi_{\{|f|>\delta\alpha\}} d\mu \overset{(\dagger)}{=} \int_{\{|f|>\delta\alpha\}} |f|^{p_0} d\mu \\ &= \int_{\{|f|>\delta\alpha\}} |f|^p |f|^{p_0-p} d\mu = \int_{\{|f|>\delta\alpha\}} \frac{|f|^p}{|f|^{p-p_0}} d\mu \\ &\leqslant \frac{1}{(\delta\alpha)^{p-p_0}} \int_{\{|f|>\delta\alpha\}} |f|^p d\mu = (\delta\alpha)^{p_0-p} \int_X |f|^p \cdot \chi_{\{|f|>\delta\alpha\}} d\mu \\ &\leqslant (\delta\alpha)^{p_0-p} \int_X |f|^p d\mu = (\delta\alpha)^{p_0-p} \|f\|_{L^p}^p < \infty \end{aligned} \tag{8}$$

Thus $f_0 \in L^{p_0}(X,\mu)$. Analogously it can be checked, that $f_1 \in L^{p_1}(X,\mu)$ by the estimate $\|f_1\|_{L^{p_1}}^{p_1} \leqslant (\delta\alpha)^{p_1-p}\|f\|_{L^p}^p$.

Proof of the equality ($\dagger$). Assume μ is defined on the σ-algebra $\mathcal{A}$. We have to proove that $\{|f| > \delta\alpha\} \in \mathcal{A}$[1]. Since f is complex-valued, we may write $f \equiv \mathrm{Re} f + i\mathrm{Im} f$ and thus $|f|^2 \equiv \mathrm{Re}^2 f + \mathrm{Im}^2 f$. Since f is measurable by hypothesis this implies that $\mathrm{Re} f$ and $\mathrm{Im} f$

[1] For $Y \in \mathcal{A}$ the μ-integral of $f : X \to \mathbb{C}$ over Y is defined to be $\int_Y f d\mu := \int_X f \cdot \chi_Y d\mu$. For more details see [Els11, pp. 135–136].

 Yannis Bähni
yannis.baehni@uzh.ch

are measurable[2]. Further for measurable real-valued functions $f, g : (X, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathfrak{B}})$[3] the functions $f+g$ and $f \cdot g$ are measurable[4] and thus $|f|^2$ is measurable. Hence $\{\mathrm{Re}^2 f + \mathrm{Im}^2 f > \lambda\} \in \mathcal{A}$[5] for any $\lambda \in \mathbb{R}$. So especially for $\lambda := (\delta\alpha)^2$ we have $\{|f| > \delta\alpha\} \in \mathcal{A}$[6]. In a similar manner it can also be prooven that $\{|f| \leqslant \delta\alpha\} \in \mathcal{A}$. Let us next proove a useful lemma.

Lemma 2.2. *Let $A \in \wp(X)$ and $\chi_A : (X, \mathcal{A}) \to (\mathbb{C}, \mathfrak{B}^2)$ be the* characteristic function of the set A. *Then χ_A is measurable if and only if A is measurable.*

Proof. Assume χ_A is measurable. Then $\mathrm{Re}\chi_A$ and $\mathrm{Im}\chi_A$ are measurable. Especially for $0 < \lambda < 1$ we have that $\{\mathrm{Re}\chi_A > \lambda\} = A \in \mathcal{A}$. Conversly, assume A is measurable. For $\lambda < 0$ we have $\{\mathrm{Re}\chi_A > \lambda\} = X \in \mathcal{A}$, $\lambda \in [0, 1[$, $\{\mathrm{Re}\chi_A > \lambda\} = A \in \mathcal{A}$ and $\{\mathrm{Re}\chi_A > \lambda\} = \emptyset \in \mathcal{A}$ for $\lambda \geqslant 1$. Since $\mathrm{Im}\chi_A \equiv 0$ we have $\{\mathrm{Im}\chi_A > \lambda\} = X \in \mathcal{A}$ if $\lambda < 0$ and $\{\mathrm{Im}\chi_A > \lambda\} = \emptyset \in \mathcal{A}$ if $\lambda \geqslant 0$. □

By Lemma 2.2 and the fact that $f \cdot g$ is measurable for two measurable functions $f, g : (X, \mathcal{A}) \to (\mathbb{C}, \mathfrak{B}^2)$[7], f_0 and f_1 are measurable since $f_0 \equiv f \cdot \chi_{\{|f| > \delta\alpha\}}$ and $f_1 \equiv f \cdot \chi_{\{|f| \leqslant \delta\alpha\}}$.

One subtility is left to clear: the μ-integrability of either $|f_1|^{p_0}$ or $|f_1|^{p_1}$ requires that $|f_0|^{p_0}$ and $|f_1|^{p_1}$ are measurable functions. By the fact that any continuous map $g : (X, d_X) \to (Y, d_Y)$ between metric spaces is Borel-measurable (see [Els11, p. 86]) and that the composition of measurable functions is again measurable (see [Els11, p. 87]), the measurability of either f_0 or f_1 follows by $|f_0|^{p_0} \equiv \cdot^{p_0} \circ |f \cdot \chi_{\{|f| > \delta\alpha\}}|$ and $|f_1|^{p_1} \equiv \cdot^{p_1} \circ |f \cdot \chi_{\{|f| \leqslant \delta\alpha\}}|$ by stipulating $\cdot^p : (\mathbb{R}_{\geqslant 0}, |\cdot|) \to (\mathbb{C}, |\cdot|)$, $x^p := \exp(p \log(x))$ for $p > 0$ and $x \in \mathbb{R}_{>0}$ and $x^p := 0$ if $x = 0$. □

By lemma (2.1) we therefore have $f \equiv f_0 + f_1 \in L^{p_0} + L^{p_1}$.

Lemma 2.3. *For fixed $\alpha > 0$, the distribution function $d_{T(f)}(\alpha)$ obeys an upper bound of the form*

$$d_{T(f)}(\alpha) \leqslant \left(\frac{A_0}{\alpha/2}\right)^{p_0} \|f_0\|_{L^{p_0}}^{p_0} + \left(\frac{A_1}{\alpha/2}\right)^{p_1} \|f_1\|_{L^{p_1}}^{p_1}$$

Proof. Since T is a sublinear operator we have $|T(f)| = |T(f_0 + f_1)| \leqslant |T(f_0)| + |T(f_1)|$. Thus for any $y \in Y$ with $|T(f)(y)| > \alpha$ we therefore have either $|T(f_0)(y)| > \alpha/2$ or $|T(f_1)(y)| > \alpha/2$ [8] . Hence

$$\{|T(f)| > \alpha\} \subseteq \{|T(f_0)| > \alpha/2\} \cup \{|T(f_1)| > \alpha/2\}$$

[2] For a proof see [Els11, p. 106]

[3] $\overline{\mathfrak{B}} := \sigma(\overline{\mathbb{R}})$ and $\overline{\mathfrak{B}} = \{B \cup E : B \in \mathfrak{B}, E \subseteq \{\pm\infty\}\}$.

[4] For a proof see [Els11, p. 107].

[5] For a proof see [Els11, pp. 105–106]

[6] This follows from the fact that $x < y$ if and only if $x^n < y^n$ for $n \in \mathbb{N}_{>0}$ and some real numbers $x, y > 0$ (see [Zor04, p. 119]).

[7] Els11, p. 107.

[8] Without loss of generality assume $|T(f_0)(y)| \leqslant |T(f_1)(y)|$. Then we have $\alpha < |T(f)(y)| \leqslant |T(f_0)(y)| + |T(f_1)(y)| \leqslant 2|T(f_1)(y)|$ (this is possible since $\mathbb{R}$ is an ordered field).

Yannis Bähni
yannis.baehni@uzh.ch

and so by the monotonicity and subadditivity property of the measure μ we have

$$\begin{aligned} d_{T(f)}(\alpha) &= \mu(\{|T(f)| > \alpha\}) \\ &\leqslant \mu(\{|T(f_0)| > \alpha/2\} \cup \{|T(f_1)| > \alpha/2\}) \\ &\leqslant \mu(\{|T(f_0)| > \alpha/2\}) + \mu(\{|T(f_1)| > \alpha/2\}) \\ &= d_{T(f_0)}(\alpha/2) + d_{T(f_1)}(\alpha/2) \end{aligned} \tag{9}$$

Now by hypothesis (3) we can estimate $d_{T(f_0)}(\alpha/2)$ as follows

$$\begin{aligned} d_{T(f_0)}(\alpha/2) &= \left(\frac{\alpha/2}{\alpha/2}\right)^{p_0} d_{T(f_0)}(\alpha/2) \\ &\leqslant \left(\frac{1}{\alpha/2}\right)^{p_0} \left[\sup\left\{\gamma d_{T(f_0)}(\gamma)^{1/p_0} : \gamma > 0\right\}\right]^{p_0} \\ &= \left(\frac{1}{\alpha/2}\right)^{p_0} \|T(f_0)\|_{L^{p_0,\infty}}^{p_0} \\ &\leqslant \left(\frac{A_0}{\alpha/2}\right)^{p_0} \|f_0\|_{L^{p_0}}^{p_0} \end{aligned} \tag{10}$$

Analogously, we get $d_{T(f_1)}(\alpha/2) \leqslant \left(\frac{A_1}{\alpha/2}\right)^{p_1} \|f_1\|_{L^{p_1}}^{p_1}$ by hypothesis (4). $\square$

By

$$\int_0^{\frac{1}{\delta}|f|} \alpha^{p-p_0-1} d\lambda = \begin{cases} \frac{1}{p-p_0}\frac{1}{\delta^{p-p_0}}|f|^{p-p_0}, & p \geqslant p_0 + 1 \\ = \lim_{\omega\to 0^+} \int_\omega^{\frac{1}{\delta}|f|} \alpha^{p-p_0-1} d\lambda & \\ = \lim_{\omega\to 0^+} \left[\frac{1}{p-p_0}\alpha^{p-p_0}\right]_\omega^{\frac{1}{\delta}|f|} & \\ = \frac{1}{p-p_0}\left[\frac{1}{\delta^{p-p_0}}|f|^{p-p_0} - \lim_{\omega\to 0^+}\omega^{p-p_0}\right] & \\ = \frac{1}{p-p_0}\frac{1}{\delta^{p-p_0}}|f|^{p-p_0}, & p_0 < p < p_0 + 1 \end{cases} \tag{11}$$

and

$$\begin{aligned} \int_{\frac{1}{\delta}|f|}^\infty \alpha^{p-p_1-1} d\lambda &= \lim_{\omega\to\infty} \left[\frac{1}{p-p_1}\alpha^{p-p_1}\right]_{\frac{1}{\delta}|f|}^{\omega} \\ &= \frac{1}{p-p_1}\left[\lim_{\omega\to\infty}\omega^{p-p_1} - \frac{1}{\delta^{p-p_1}}|f|^{p-p_1}\right] \\ &= \frac{1}{p_1-p}\frac{1}{\delta^{p-p_1}}|f|^{p-p_1} \end{aligned} \tag{12}$$

and the representation $\|f\|_{L^p}^p = p\int_0^\infty \alpha^{p-1} d_f(\alpha) d\lambda$ for $0 < p < \infty$ we get

Yannis Bähni
yannis.baehni@uzh.ch

$$\begin{aligned}
\|T(f)\|_{L^p}^p &= p\int_0^\infty \alpha^{p-1} d_{T(f)} d\lambda \\
&\leqslant p\,(2A_0)^{p_0}\int_0^\infty \alpha^{p-p_0-1}\int_{\{|f|>\delta\alpha\}} |f|^{p_0}\, d\mu d\lambda \\
&\quad + p(2A_1)^{p_1}\int_0^\infty \alpha^{p-p_1-1}\int_{\{|f|\leqslant\delta\alpha\}} |f|^{p_1}\, d\mu d\lambda \\
&= p\,(2A_0)^{p_0}\int_{\{|f|>0\}} |f|^{p_0}\int_0^{\frac{1}{\delta}|f|}\alpha^{p-p_0-1} d\lambda d\mu \\
&\quad + p\,(2A_0)^{p_0}\int_{\{|f|=0\}} |f|^{p_0}\int_0^{\frac{1}{\delta}|f|}\alpha^{p-p_0-1} d\lambda d\mu \\
&\quad + p\,(2A_1)^{p_1}\int_X |f|^{p_1}\int_{\frac{1}{\delta}|f|}^\infty \alpha^{p-p_1-1} d\lambda d\mu \\
&= p\,(2A_0)^{p_0}\int_X |f|^{p_0}\int_0^{\frac{1}{\delta}|f|}\alpha^{p-p_0-1} d\lambda d\mu \\
&\quad + p\,(2A_1)^{p_1}\int_X |f|^{p_1}\int_{\frac{1}{\delta}|f|}^\infty \alpha^{p-p_1-1} d\lambda d\mu \\
&= \frac{p\,(2A_0)^{p_0}}{p-p_0}\frac{1}{\delta^{p-p_0}}\int_X |f|^{p_0}\,|f|^{p-p_0}\, d\mu \\
&\quad + \frac{p\,(2A_1)^{p_1}}{p_1-p}\frac{1}{\delta^{p-p_1}}\int_X |f|^{p_1}\,|f|^{p-p_1}\, d\mu \\
&= p\left(\frac{(2A_0)^{p_0}}{p-p_0}\frac{1}{\delta^{p-p_0}} + \frac{(2A_1)^{p_1}}{p_1-p}\delta^{p_1-p}\right)\|f\|_{L^p}^p
\end{aligned} \tag{13}$$

We pick $\delta > 0$ such that $(2A_0)^{p_0}\,\delta^{p_0-p} = (2A_1)^{p_1}\,\delta^{p_1-p}$. Solving for δ yields

$$\delta = \frac{1}{2}\left(\frac{A_0}{A_1}\right)^{p_1/(p_1-p_0)} \tag{14}$$

Substituting this in estimate (13) leads to

$$\begin{aligned}
\|T(f)\|_{L^p}^p &\leqslant p\left(\frac{(2A_0)^{p_0}}{p-p_0}\frac{2^{p-p_0}A_1^{\frac{p_1(p-p_0)}{p_1-p_0}}}{A_0^{\frac{p_0(p-p_0)}{p_1-p_0}}} + \frac{(2A_1)^{p_1}}{p_1-p}\frac{A_0^{\frac{p_0(p_1-p)}{p_1-p_0}}}{2^{p_1-p}A_1^{\frac{p_1(p_1-p)}{p_1-p_0}}}\right)\|f\|_{L^p}^p \\
&= 2^p p\left(\frac{A_0^{\frac{p_0(p_1-p)}{p_1-p_0}}A_1^{\frac{p_1(p-p_0)}{p_1-p_0}}}{p-p_0} + \frac{A_0^{\frac{p_0(p_1-p)}{p_1-p_0}}A_1^{\frac{p_1(p-p_0)}{p_1-p_0}}}{p_1-p}\right)\|f\|_{L^p}^p
\end{aligned} \tag{15}$$

And taking the p-th power further

Yannis Bähni
yannis.baehni@uzh.ch

$$
\begin{aligned}
\|T(f)\|_{L^p} &\leqslant 2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{p_0(p_1-p)}{p(p_1-p_0)}} A_1^{\frac{p_1(p-p_0)}{p(p_1-p_0)}} \|f\|_{L^p} \\
&= 2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{p_0(p_1-p)}{p(p_1-p_0)}\frac{p_1}{p_1}} A_1^{\frac{p_1(p-p_0)}{p(p_1-p_0)}\frac{p_0}{p_0}} \|f\|_{L^p} \\
&= 2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{\frac{p_1-p}{pp_1}}{\frac{p_1-p_0}{p_0p_1}}} A_1^{\frac{\frac{p-p_0}{p_0p}}{\frac{p_1-p_0}{p_0p_1}}} \|f\|_{L^p} \\
&= 2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}} \|f\|_{L^p}
\end{aligned}
\tag{16}
$$

Assume $p_1 = \infty$. We again use the cut-off functions defined in (7) to decompose f. Since $\{|f_1| > \delta\alpha\} = \emptyset$, we have

$$
\|T(f_1)\|_{L^\infty} \leqslant A_1\|f_1\|_{L^\infty} = A_1 \inf\{B > 0 : \mu(\{|f_1| > B\}) = 0\} \leqslant A_1\delta\alpha = \alpha/2
$$

Provided we stipulate $\delta := 1/(2A_1)$. Therefore the set $\{|T(f_1)| > \alpha/2\}$ has measure zero (this is immediate since $\|T(f_1)\|_{L^\infty} = \inf\{B > 0 : \mu(\{|T(f_1)| > B\}) = 0\} \leqslant \alpha/2$ and any subset of a set with measure zero has itself measure zero). Thus similar to part **b.** of **(i.)** we get $d_{T(f)}(\alpha) \leqslant d_{T(f_0)}(\alpha/2)$.

Hypothesis (3) yields the estimate $d_{T(f_0)}(\alpha/2) \leqslant \left(\frac{A_0}{\alpha/2}\right)^{p_0} \int_{\{2A_1|f|>\alpha\}} |f|^{p_0} d\mu$.

Thus by **a.** and **b.**

$$
\begin{aligned}
\|T(f)\|_{L^p}^p &= p\int_0^\infty \alpha^{p-1} d_{T(f)} d\lambda \\
&\leqslant p(2A_0)^{p_0} \int_0^\infty \alpha^{p-p_0-1} \int_{\{2A_1|f|>\alpha\}} |f|^{p_0} d\mu d\lambda \\
&= p(2A_0)^{p_0} \int_X |f|^{p_0} \int_0^{2A_1|f|} \alpha^{p-p_0-1} d\lambda d\mu \\
&= \frac{2^p p A_0^{p_0} A_1^{p-p_0}}{p-p_0} \int_X |f|^p d\mu \\
&= \frac{2^p p A_0^{p_0} A_1^{p-p_0}}{p-p_0} \|f\|_{L^p}^p
\end{aligned}
\tag{17}
$$

That the constant $2^p p A_0^{p_0} A_1^{p-p_0}/(p-p_0)$ found in (17) is the p-th power of the one stated in the theorem can be seen by passing the constant (6) to the limit $p_1 \to \infty$:

Yannis Bähni
yannis.baehni@uzh.ch

$$
\begin{aligned}
\lim_{p_1\to\infty} A = &\lim_{p_1\to\infty}\left[2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}}\right] \\
=&2\exp\left[\frac{1}{p}\log\left(\frac{p}{p-p_0}+\lim_{p_1\to\infty}\frac{1}{p_1}\frac{p}{1-p\lim\limits_{p_1\to\infty}\frac{1}{p_1}}\right)\right] \\
&\cdot\lim_{p_1\to\infty} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}}\cdot\lim_{p_1\to\infty} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}} \\
=&2\left(\frac{p}{p-p_0}\right)^{1/p}\exp\left[\frac{\frac{1}{p}-\lim\limits_{p_1\to+\infty}\frac{1}{p_1}}{\frac{1}{p_0}-\lim\limits_{p_1\to\infty}\frac{1}{p_1}}\log(A_0)\right] \\
&\cdot\exp\left[\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\lim\limits_{p_1\to\infty}\frac{1}{p_1}}\log(A_1)\right] \\
=&2\left(\frac{p}{p-p_0}\right)^{1/p} A_0^{\frac{p_0}{p}} A_1^{1-\frac{p_0}{p}}
\end{aligned}
$$

□

3. The Complex Method. This theorem will unfortunately only be applicable to linear operators but will yield a more natural bound of the operator on the intermediate space. The proof will make strong use of complex variables technique. A major tool will be an application of the maximum modulus principle, known as *Hadamard's three lines lemma.*

3.1. Hadamard's Three Lines Lemma. As the name already says, the lemma yields a natural bound of an analytic function defined on a vertical strip in the complex plane using the bounds of the function on the two parallel lines enclosing the strip.

LEMMA 3.1. *Hadamard's three lines lemma) Let F be a holomorphic function in the strip $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$, continuous and bounded on $\overline{S}$, such that $|F(z)| \leqslant B_0$ when $\operatorname{Re} z = 0$ and $|F(z)| \leqslant B_1$ when $\operatorname{Re} z = 1$, for some $0 < B_0, B_1 < \infty$. Then $|F(z)| \leqslant B_0^{1-\theta} B_1^{\theta}$ when $\operatorname{Re} z = \theta$, for any $0 \leqslant \theta \leqslant 1$.*

Proof. For $z \in \overline{S}$ define

$$G(z) := \frac{F(z)}{B_0^{1-z}B_1^z} \qquad \forall n \in \mathbb{N}_{>0} : G_n(z) := G(z)e^{(z^2-1)/n} \tag{18}$$

Yannis Bähni
yannis.baehni@uzh.ch

Obviously, $G(z)$ and $G_n(z)$ are holomorphic functions on S for $n \in \mathbb{N}_{>0}$[9]. Further, we have

$$\left|B_0^{1-z} B_1^z\right|^2 = \left|B_0^{1-z}\right|^2 \left|B_1^z\right|^2 = B_0^{1-z} B_0^{1-\overline{z}} B_1^z B_1^{\overline{z}} = \left(B_0^{1-\operatorname{Re} z}\right)^2 \left(B_1^{\operatorname{Re} z}\right)^2 \tag{19}$$

Consider $0 \leqslant \operatorname{Re} z \leqslant 1$ and $B_0 \geqslant 1$. Then $B_0^{1-\operatorname{Re} z} = \exp\left((1 - \operatorname{Re} z) \log B_0\right) \geqslant 1$ and $B_0^{1-\operatorname{Re} z} \geqslant B_0$ in the case $B_0 < 1$. A similar estimation of $B_1^{\operatorname{Re} z}$ leads to

$$\left|B_0^{1-z} B_1^z\right| \geqslant \min\{1, B_0\} \min\{1, B_1\} \tag{20}$$

for all $z \in \overline{S}$. By this, $G(z)$ is bounded on $\overline{S}$ (by the boundedness of F). Let $M > 0$, such that $|G(z)| \leqslant M$ for $z \in \overline{S}$. Fix $n \in \mathbb{N}_{>0}$ and write $z := x + iy \in \overline{S}$. Since

$$\begin{aligned}
|G_n(z)|^2 &= |G(z)|^2 \left|e^{((x+iy)^2-1)/n}\right|^2 \\
&\leqslant M^2 e^{(x^2+2ixy-y^2-1)/n} e^{(x^2-2ixy-y^2-1)/n} \\
&= M^2 \left(e^{-y^2/n}\right)^2 \left(e^{(x^2-1)/n}\right)^2 \\
&\leqslant M^2 \left(e^{-y^2/n}\right)^2 \\
&= M^2 \left(e^{-|y|^2/n}\right)^2
\end{aligned} \tag{21}$$

we have $\lim_{y \to \pm\infty} \sup\{|G_n(z)| : x \in [0, 1]\} = 0$ by the pinching-principle. Hence there exists some $C(n) > 0$, such that $|G_n(z)| \leqslant 1$ for all $|y| \geqslant C(n)$ and all $x \in [0, 1]$. Consider the rectangle $R := [0, 1] \times [-C(n), C(n)]$. Now $|G_n(z)| \leqslant 1$ on the lines $[0, 1] \times \{\pm C(n)\}$ and since $|G(z)| = |F(z)| / B_0 \leqslant 1$, $|G(z)| = |F(z)| / B_1 \leqslant 1$ on the line $\{0\} \times [-C(n), C(n)]$ and $\{1\} \times [-C(n), C(n)]$ respectively by assumption, we have $|G_n(z)| \leqslant 1$ on ∂S. By the maximum modulus principle [10] we have $|G_n(z)| \leqslant 1$ on R and thus $|G_n(z)| \leqslant 1$ on $\overline{S}$. Since inequalities are preserved by limits and the modulus is a continuous function, we have that $|G(z)| = \lim_{n \to \infty} |G_n(z)| \leqslant 1$ on $\overline{S}$. Taking $z := \theta + it$, where $0 \leqslant \theta \leqslant 1$ and $t \in \mathbb{R}$, we conclude $|F(z)| = |G(z)| \left|B_0^{1-z} B_1^z\right| \leqslant B_0^{1-\theta} B_1^\theta$, which completes the proof. □

3.2. The Riesz-Thorin Interpolation Theorem. Now we are able to proove the Riesz-Thorin Interpolation theorem without an interruption. To simplify notation, let Σ_X, Σ_Y denote the set of all finitely simple functions on X and Y respectively.

Theorem 3.1. (Riesz-Thorin Interpolation Theorem) *Let (X, μ) be a measure space, (Y, ν) a semifinite measure space and T be a linear operator defined on Σ_X and taking*

[9] I adapt here the terminology established in [Rud87, p. 197]. A complex-valued function f is said to be *holomorphic* (or *analytic*) in $\Omega \subseteq \mathbb{C}$ open, if $f'(z)$ exists for any $z \in \Omega$.

[10] Let Ω be a bounded region of the complex plane, f be a complex-valued continuous function on $\overline{\Omega}$ which is holomorphic in Ω. Then $|f(z)| \leqslant \sup\{|f(z)| : z \in \partial\Omega\}$ for every $z \in \Omega$. See [Rud87, p. 253].

Yannis Bähni
yannis.baehni@uzh.ch

values in the set of measurable functions on Y. Let $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$ and assume that

$$\|T(f)\|_{L^{q_0}} \leqslant M_0 \|f\|_{L^{p_0}} \qquad \|T(f)\|_{L^{q_1}} \leqslant M_1 \|f\|_{L^{p_1}} \tag{22}$$

for all $f \in \Sigma_X$ and $M_0, M_1 < \infty$. Then for all $0 < \theta < 1$ we have

$$\|T(f)\|_{L^q} \leqslant M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p} \tag{23}$$

for all $f \in \Sigma_X$, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \tag{24}$$

Proof. Fix

$$f :\equiv \sum_{j=1}^{n} a_j e^{i\alpha_j} \chi_{A_j} \in \Sigma_X \qquad g :\equiv \sum_{k=1}^{m} b_k e^{i\beta_k} \chi_{B_k} \in \Sigma_Y$$

where $a_j, b_k > 0$ and $\alpha_j, \beta_k \in \mathbb{R}$ for every $j = 1, \ldots, n$, $k = 1, \ldots, m$. Define

$$P(z) := \frac{p}{p_0}(1-z) + \frac{p}{p_1} z \qquad Q(z) := \frac{q'}{q'_0}(1-z) + \frac{q'}{q'_1} z$$

for $z \in \overline{S}$ (if $p, q' = \infty$ then also $p_0, p_1, q'_0, q'_1 = \infty$ and hence P, Q are well defined). Further let

$$f_z :\equiv \sum_{j=1}^{n} a_j^{P(z)} e^{i\alpha_j} \chi_{A_j} \qquad g_z :\equiv \sum_{k=1}^{m} b_k^{Q(z)} e^{i\beta_k} \chi_{B_k} \tag{25}$$

and

$$F(z) := \int_Y T(f_z)(y) g_z(y) d\nu(y) \tag{26}$$

By the linearity of the operator T we have

$$F(z) = \sum_{j=1}^{n} \sum_{k=1}^{m} a_j^{P(z)} b_k^{Q(z)} e^{i\alpha_j} e^{i\beta_k} \int_Y T(\chi_{A_j})(y) \chi_{B_k}(y) d\nu(y) \tag{27}$$

and by Hölder's inequality [11]

[11] A proof can be found in [Els11, p. 223].

Yannis Bähni
yannis.baehni@uzh.ch

$$
\begin{aligned}
\left|\int_Y T(\chi_{A_j})(y)\chi_{B_k}(y)d\nu(y)\right| &\leqslant \int_Y \left|T(\chi_{A_j})(y)\chi_{B_k}(y)\right| d\nu(y)\\
&= \left\|T(\chi_{A_j})\chi_{B_k}\right\|_{L^1}\\
&\leqslant \left\|T(\chi_{A_j})\right\|_{L^{q_0}} \left\|\chi_{B_k}\right\|_{L^{q_0'}}\\
&\leqslant M_0 \left\|\chi_{A_j}\right\|_{L^{p_0}} \left\|\chi_{B_k}\right\|_{L^{q_0'}}\\
&\overset{p_0,q_0'\neq\infty}{=} M_0\mu\left(A_j\right)^{1/p_0}\nu\left(B_k\right)^{1/q_0'}\\
&< \infty
\end{aligned}
\tag{28}
$$

for each $j = 1,\dots,n$, $k = 1,\dots,m$. In the case where either $p_0 = \infty$ or $q_0' = \infty$, consider that $\left\|\chi_{A_j}\right\|_{L^\infty}, \left\|\chi_{B_k}\right\|_{L^\infty} \leqslant 1$. Thus the function F is well-defined on $\overline{S}$. Let $t \in \mathbb{R}$. For $p, p_0 \neq \infty$

$$
\begin{aligned}
\left\|f_{it}\right\|_{L^{p_0}} &= \left(\sum_{j=1}^n \int_X \left|f_{it}\right|^{p_0} d\mu + \int_{X\setminus\bigcup_{j=1}^n A_j} \left|f_{it}\right|^{p_0} d\mu\right)^{1/p_0}\\
&= \left(\sum_{j=1}^n \left|a_j^{P(it)}e^{i\alpha_j}\right|^{p_0}\int_X \chi_{A_j} d\mu\right)^{1/p_0}\\
&= \left(\sum_{j=1}^n a_j^{p_0\operatorname{Re}P(it)}\mu\left(A_j\right)\right)^{1/p_0}\\
&= \left(\sum_{j=1}^n a_j^{p}\mu\left(A_j\right)\right)^{p/(p_0 p)}\\
&= \left\|f\right\|_{L^p}^{p/p_0}
\end{aligned}
\tag{29}
$$

holds. Let $p_0 = \infty$, $p \neq \infty$. Then either $\left\|f_{it}\right\|_{L^\infty} = 0$ or $\left\|f_{it}\right\|_{L^\infty} = 1$. In the former case $f \equiv 0$ μ-a.e which implies $\mu\left(A_j\right) = 0$ for any $j = 1,\dots,n$ and thus $\left\|f_{it}\right\|_{L^\infty} = 0$ and in the latter case $\left\|f_{it}\right\|_{L^\infty} = 1$ by the simple observation that $\left|a_j^{P(it)}\right| = a_j^{p/p_0} = 1$ and that there exists some index j, such that $\mu\left(A_j\right) \neq 0$. If $p = \infty$, observe that $P(z) = 1$ and thus $\left\|f_{it}\right\|_{L^\infty} = \left\|f\right\|_{L^\infty}$. By the same considerations we see that $\left\|g_{it}\right\|_{L^{q_0'}} = \left\|g\right\|_{L^{q'}}^{q'/q_0'}$ any legitime q_0, q. Hence

Yannis Bähni
yannis.baehni@uzh.ch

$$
\begin{aligned}
|F(it)| &\leqslant \int_Y |T(f_{it})(y)g_{it}(y)|\,d\nu(y) \\
&= \|T(f_{it})g_{it}\|_{L^1} \\
&\leqslant \|T(f_{it})\|_{L^{q_0}} \|g_{it}\|_{L^{q_0'}} \\
&\leqslant M_0 \|f_{it}\|_{L^{p_0}} \|g_{it}\|_{L^{q_0'}} \\
&= M_0 \|f\|_{L^p}^{p/p_0} \|g\|_{L^{q'}}^{q'/q_0'} \\
&< \infty
\end{aligned}
\tag{30}
$$

by Hölder's inequality. In an analogous manner s we can estimate

$$
\|f_{1+it}\|_{L^{p_1}} = \|f\|_{L^p}^{p/p_1} \qquad \|g_{1+it}\|_{L^{q_1'}} = \|g\|_{L^{q'}}^{q'/q_1'}
\tag{31}
$$

and thus

$$
|F(1+it)| \leqslant M_1 \|f\|_{L^p}^{p/p_1} \|g\|_{L^{q'}}^{q'/q_1'}
\tag{32}
$$

Further

$$
\begin{aligned}
|F(z)| &\leqslant \int_Y |T(f_z)(y)g_z(y)|\,d\nu(y) = \|T(f_z)g_z\|_{L^1} \leqslant \|T(f_z)\|_{L^{q_0}} \|g_z\|_{L^{q_0'}} \\
&\leqslant M_0 \|f_z\|_{L^{p_0}} \|g_z\|_{L^{q_0'}} \overset{p_0,q_0'\neq\infty}{=} M_0 \left(\int_X |f_z|^{p_0}\,d\mu\right)^{1/p_0} \left(\int_Y |g_z|^{q_0'}\,d\nu\right)^{1/q_0'} \\
&= M_0 \left(\sum_{j=1}^n a_j^{p_0 \operatorname{Re} P(z)} \mu(A_j)\right)^{1/p_0} \left(\sum_{k=1}^m b_k^{q_0' \operatorname{Re} Q(z)} \nu(B_k)\right)^{1/q_0'} \\
&= M_0 \left(\sum_{j=1}^n a_j^{p(1-\operatorname{Re} z)+(pp_0 \operatorname{Re} z)/p_1} \mu(A_j)\right)^{1/p_0} \left(\sum_{k=1}^m b_k^{q'(1-\operatorname{Re} z)+\left(q'q_0' \operatorname{Re} z\right)/q_1'} \nu(B_k)\right)^{1/q_0'} \\
&\leqslant M_0 \left(\sum_{j=1}^n a_j^{p+(pp_0)/p_1} \mu(A_j)\right)^{1/p_0} \left(\sum_{k=1}^m b_k^{q'+\left(q'q_0'\right)/q_1'} \nu(B_k)\right)^{1/q_0'} \\
&= M_0 \|f\|_{L^{p+(pp_0)/p_1}}^{p/p_0+p/p_1} \|g\|_{L^{q'+\left(q'q_0'\right)/q_1'}}^{q'/q_0'+q'/q_1'} =: C(f,g)
\end{aligned}
$$

by Hölder's inequality and in the edge cases

Yannis Bähni
yannis.baehni@uzh.ch

$$\begin{aligned}
p_0 = \infty, q_0' \neq \infty: &\qquad C(f,g) := M_0 \max_{j=1,\ldots,n} a_j^{p/p_1} \|g\|_{L^{q'+(q'q_0')/q_1'}}^{q'/q_0'+q'/q_1'} \\
p_0 \neq \infty, q_0' = \infty: &\qquad C(f,g) := M_0 \|f\|_{L^{p+(pp_0)/p_1}}^{p/p_0+p/p_1} \max_{k=1,\ldots,m} b_k^{q'/q_1'} \\
p_0 = \infty, q_0' = \infty: &\qquad C(f,g) := M_0 \max_{j=1,\ldots,n} a_j^{p/p_1} \max_{k=1,\ldots,m} b_k^{q'/q_1'}
\end{aligned}$$

Hence F is bounded on $\overline{S}$. By writing

$$F(z) = \sum_{j=1}^{n} \sum_{k=1}^{m} e^{P(z)\log(a_j)} e^{Q(z)\log(b_k)} e^{i\alpha_j} e^{i\beta_k} \int_Y T(\chi_{A_j})(y)\chi_{B_k}(y) d\nu(y)$$

it is immediate that F is continuous on $\overline{S}$ and by

$$F'(z) = \sum_{j=1}^{n} \sum_{k=1}^{m} a_j^{P(z)} \log(a_j) \left(\frac{p}{p_1} - \frac{p}{p_0}\right) b_k^{Q(z)} \log(b_j) \left(\frac{q'}{q_1'} - \frac{q'}{q_0'}\right) e^{i\alpha_j} e^{i\beta_k} \int_Y T(\chi_{A_j})(y)\chi_{B_k}(y) d\nu(y)$$

F is holomorphic in S. Therefore Hadamard's three lines lemma (3.1) yields

$$\begin{aligned}
|F(z)| &\leqslant \left(M_0 \|f\|_{L^p}^{p/p_0} \|g\|_{L^{q'}}^{q'/q_0'}\right)^{1-\theta} \left(M_1 \|f\|_{L^p}^{p/p_1} \|g\|_{L^{q'}}^{q'/q_1'}\right)^{\theta} \\
&= M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p} \|g\|_{L^{q'}}
\end{aligned} \tag{33}$$

for $\operatorname{Re} z = \theta$. By $P(\theta) = Q(\theta) = 1$ and

$$\begin{aligned}
M_q(T(f)) &= \sup\left\{\left|\int_Y T(f) g d\nu\right| : g \in \Sigma_Y, \|g\|_{L^{q'}} = 1\right\} \\
&= \sup\left\{|F(\theta)| : g \in \Sigma_Y, \|g\|_{L^{q'}} = 1\right\} \\
&\leqslant M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p}
\end{aligned} \tag{34}$$

we conclude $\|T(f)\|_{L^q} = M_q(T(f))$ for any $f \in \Sigma_X$ using [Fol99, p. 189] (observe, that $T(f)g \in L^1$ by either one of the hypotheses on the linear operator T). □

Remark 3.1. *It is necessary to have $0 < \theta < 1$, since for example choosing $q_1 = 1$ and $q_0 > 1$ arbitrary leads for $\theta = 1$ to $q = 1$ but then the function g can be choosen so, that the integral in the definition (27) is ∞.*

3.3. Young's inequality. Using the Riesz-Thorin interpolation theorem, we can give an alternative proof of Young's inequality [Gra14, pp. 22–23].

Yannis Bähni
yannis.baehni@uzh.ch

Theorem 3.2. (Young's inequality) *Let G be a locally compact group, which is a countable union of compact subsets, and let η be a left invariant Haar measure. Let* $1 \leqslant p, q, r \leqslant \infty$

$$\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r} \tag{35}$$

Then for all $f \in L^p(G, \eta)$ and all $g \in L^r(G, \eta)$ satisfying $\|g\|_{L^r} = \|\tilde{g}\|_{L^r}$ we have $f * g$ exists η-a.e. and satisfies

$$\|f * g\|_{L^q} \leqslant \|g\|_{L^r} \|f\|_{L^p} \tag{36}$$

Proof. Fix $g \in L^r(G, \eta)$ and let $T(f) := f * g$ be defined on $L^1(G, \eta) + L^{r'}(G, \eta)$. Obviously, T is a linear operator by the linearity of the integral. By Minkowski's integral inequality (see exercise **1.1.6** [Gra14, p. 13]) we get

$$\begin{aligned} \|T(f)\|_{L^r} &= \left(\int_G \left|\int_G f(y) g(y^{-1}x) d\eta(y)\right|^r d\eta(x)\right)^{1/r} \\ &\leqslant \int_G \left(\int_G |f(y)|^r |g(y^{-1}x)|^r d\eta(x)\right)^{1/r} d\eta(y) \\ &= \int_G |f(y)| \left(\int_G |g(y^{-1}x)|^r d\eta(y^{-1}x)\right)^{1/r} d\eta(y) \\ &= \int_G |f(y)| \left(\int_G |g(z)|^r d\eta(z)\right)^{1/r} d\eta(y) \\ &\leqslant \|f\|_{L^1} \|g\|_{L^r} \end{aligned} \tag{37}$$

for $f \in L^1(g, \mu)$ and $1 \leqslant p < \infty$ (since (G, η) is σ-finite). The case $r = \infty$ follows from

$$|(f * g)(x)| = \left|\int_G f(y) g(y^{-1}x) d\eta(y)\right| \leqslant \int_G |f(y)||g(y^{-1}x)| d\eta(y) \leqslant \|g\|_{L^\infty} \|f\|_{L^1} \tag{38}$$

By stipulating $h(y) := g(y^{-1}x)$ we have

$$\begin{aligned} |(f * g)(x)| &= \left|\int_G f(y) g(y^{-1}x) d\eta(y)\right| \leqslant \int_G |f(y) g(y^{-1}x)| d\eta(y) \\ &= \|fh\|_{L^1} \leqslant \|f\|_{L^{r'}} \|h\|_{L^r} = \|f\|_{L^{r'}} \|\tilde{g}\|_{L^r} = \|g\|_{L^r} \|f\|_{L^{r'}} \end{aligned} \tag{39}$$

for $r < \infty$ and $f \in L^{r'}(g, \eta)$, since

$$\|h\|_{L^r}^r = \int_G |g(y^{-1}x)|^r d\eta(y) = \int_G |\tilde{g}(x^{-1}y)| d\eta(y) = \|\tilde{g}\|_{L^r}^r$$

The Riesz-Thorin interpolation theorem now yields for any $0 < \theta < 1$

$$\|f * g\|_{L^q} = \|T(f)\|_{L^q} \leqslant \|g\|_{L^r}^{1-\theta}\|g\|_{L^r}^{\theta}\|f\|_{L^p} = \|g\|_{L^r}\|f\|_{L^p} \tag{40}$$

where

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{r'} \qquad \frac{1}{q} = \frac{1-\theta}{r} + \frac{\theta}{\infty}$$

and by

$$\frac{1}{p} = 1 - \frac{\theta}{r} \qquad \frac{1}{q} = \frac{1}{r} - \frac{\theta}{r}$$

we get

$$\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}$$

□

REMARK 3.2. *The proof would be much shorter if we just used Minkowski's inequality* [Gra14, pp. 21–22] *instead of Minkowski's integral inequality. However, the proof given here is an alternative version of the one given already for Minkowski's inequality.*

4. Interpolation of Analytic Families of Operators. This generalization of the classical Riesz-Thorin theorem is due to Elias M. Stein. Crucial for its proof is again a complex-analytic theorem which can be extended on the basis of Hadamard's three lines lemma.

4.1. Extension of Hadamard's Three Lines Lemma. This theorem is analogous to the one originally used by Stein itself and formulated by I. I. Hirschman, Jr.

4.1.1. Auxiliary Lemmata. To shorten the proof of the extension of Hadamard's three lines lemma, I will summarize the most important facts used during the proof.

LEMMA 4.1. *Let* $D := \{z \in \mathbb{C} : |z| < 1\}$ *be the open unit disc and*

$$h(z) := \frac{1}{\pi i}\log\left(i\frac{1+z}{1-z}\right) \tag{41}$$

for $z \in D$ *where we shall interpret* $\log z := \log|z| + i\arg z$ *as the principal value, this means* $-\pi < \arg z \leqslant \pi$*. Then* h *is a holomorphic function which maps* D *bijectively onto the strip* $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$*.*

Proof. Define $f(z) := i\dfrac{1+z}{1-z}$. If we write $z := x + iy \in D$, we have

$$f(z) = \frac{-2y}{(1-x)^2 + y^2} + i\frac{1-x^2-y^2}{(1-x)^2+y^2} \tag{42}$$

Hence $\operatorname{Im} f(z) > 0$ on D. Stipulating $x := 1 - y$ for y satisfying $y^2 < y$, we get

Yannis Bähni
yannis.baehni@uzh.ch

$$\lim_{y^2<y, y\to 0^+} \operatorname{Im} f(z) = \lim_{y^2<y, y\to 0^+} \left(\frac{1}{y} - 1\right) = \infty \tag{43}$$

using the same definition of x we get

$$\lim_{y^2<y, y\to 0^+} \operatorname{Re} f(z) = -\lim_{y^2<y, y\to 0^+} \frac{1}{y} = -\infty \tag{44}$$

and by stipulating $x := 1 + y$

$$\lim_{y^2<-y, y\to 0^-} \operatorname{Re} f(z) = -\lim_{y^2<-y, y\to 0^-} \frac{1}{y} = \infty \tag{45}$$

Since $2i \neq 0$, f is a linear fractional transformation (see [Rud87, p. 279]) with

$$f^{-1}(z) = \frac{z-i}{z+i} \tag{46}$$

Therefore f maps the unit circle D onto the upper half plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. The preceeding logarithm maps the upper half plane onto the strip $\{z \in \mathbb{C} : 0 < \operatorname{Im} z < \pi\}$. Thus $h(z)$ maps the unit circle D onto the strip S. By

$$h'(z) = \frac{2}{\pi i}\frac{1}{1-z} \tag{47}$$

we see that h is a holomorphic function in D. □

Lemma 4.2. *The mapping $\Phi : \mathbb{R} \to (-\pi, 0)$ defined by $\Phi(t) := -i \log\left(h^{-1}(it)\right)$ is a C^1-Diffeomorphism with $|D\Phi(t)| = \pi \operatorname{sech}(\pi t)$. In an analogous manner we have that $\Psi : \mathbb{R} \to (0, \pi)$, $\Psi(t) := -i \log\left(h^{-1}(1 + it)\right)$ is a C^1-Diffeomorphism with $|D\Psi(t)| = \pi \operatorname{sech}(\pi t)$.*

Proof. It is easier to consider $\Phi^{-1}(\varphi) = -ih(e^{i\varphi})$ and $\Psi^{-1}(\varphi) = -i\left(h(e^{i\varphi}) - 1\right)$ (this already shows that Φ is a bijective mapping). Since $\left|e^{i\varphi}\right| = 1$ it is immediate by the representation (42) and $y < 0$ that $\operatorname{Im}\Phi(\varphi) = 0$. Furthermore, $\lim_{\varphi\to-\pi}\Phi(\varphi) = \infty$ and $\lim_{\varphi\to 0}\Phi(\varphi) = -\infty$. By (47) Φ is clearly continuously differentiable. Using

$$h^{-1}(it) = \frac{e^{-\pi t} - i}{e^{-\pi t} + i}$$

we get

$$|D\Phi(t)| = \pi \left|\frac{e^{-\pi t}}{e^{\pi t} - i} - \frac{e^{-\pi t}}{e^{-\pi t} + i}\right| = \pi \left|\frac{2e^{-\pi t}}{e^{-2\pi t} + 1}\right| \pi \left|\frac{2}{e^{-\pi t} + e^{\pi t}}\right| = \pi \operatorname{sech}(\pi t)$$

□

4.1.2. The Lemma. Now we are able to proove the main result in prooving Stein's interpolation theorem.

Yannis Bähni
yannis.baehni@uzh.ch

Lemma 4.3. (Hadamard's three lines lemma, extension) *Let F be a holomorphic function in the strip $S := \{z \in \mathbb{C} : 0 < \mathrm{Re} z < 1\}$ and continuous on $\overline{S}$, such that for some $A < \infty$ and $\tau \in [0, \pi[$ we have $\log |F(z)| \leqslant Ae^{\tau |\operatorname{Im} z|}$ for every $z \in \overline{S}$. Then*

$$|F(z)| \leqslant \exp\left(\frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \left[\frac{\log |F(it + iy)|}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log |F(1 + it + iy)|}{\cosh(\pi t) + \cos(\pi x)}\right] d\lambda(t)\right)$$

whenever $z := x + iy \in S$.

Proof. We will first proove the case $\underline{y = 0}$. Assume F to be not identically zero (the case where F is identically zero is trivial). Consider the function
on D.By composition, $F \circ h$ is holomorphic on D and thus by [Rud87, p. 336] $\log |F \circ h|$ is subharmonic on D. It is easy to verify, that

$$h^{-1}(z) = \frac{e^{\pi i z} - i}{e^{\pi i z} + i} \tag{48}$$

on the unit strip S.

Fix some $0 < R < 1$. Then $\log |F \circ h|$ is continuous for $|z| = R$ and subharmonic in by [Rud87, p. 336]. Define

$$H(re^{i\theta}) := \begin{cases} \log |F(h(Re^{i\theta}))| & r = R, \\ \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} \log |F(h(Re^{it}))| \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} d\lambda(t) & 0 \leqslant r < R \end{cases}$$

Then H is continuous for $|z| \leqslant R$ and harmonic for $|z| < R$ (see [Rud87, pp. 234–235]). Since $\log |F(h(Re^{i\theta}))| = H(Re^{i\theta})$, by [Rud87, p. 336] we have

$$\log |F(h(re^{i\theta})) \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |F(h(Re^{it}))| \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} d\lambda(t)$$

for $0 \leqslant r < R$. Now fix $r < R$, $-\pi < \theta \leqslant \pi$ and let $R := 1 - \frac{1}{n}$ for $n \in \mathbb{N}$ such that $r < R$ holds. Thus

$$\log |F(h(re^{i\theta})) \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} f_n(t) d\lambda(t) \tag{49}$$

Consider $e^{i\theta}$ where $\operatorname{Arg} e^{i\theta} \neq 0, \pi$, we have $\operatorname{Im} \psi(e^{i\theta}) = 0$ and hence $\psi(e^{i\theta}) \in \mathbb{R}$. But then either $\operatorname{Re} h(e^{i\theta}) = 0$, $\psi(e^{i\theta}) > 0$ or $\operatorname{Re} h(e^{i\theta}) = 1$, $\psi(e^{i\theta}) < 0$. Hence the growth property of the hypothesis implies

$$\log |F(h(e^{i\theta}))| \leqslant Ae^{\tau |\operatorname{Im} h(e^{i\theta})|} = Ae^{\tau/\pi |\log |(1+e^{i\theta})(1-e^{i\theta})^{-1}||} = A \left|\frac{1 + e^{i\theta}}{1 - e^{i\theta}}\right|^{\tau/\pi}$$

Yannis Bähni
yannis.baehni@uzh.ch

Fix some $re^{i\theta}$, $r < R$ and stipulate $x := h(re^{i\theta})$. Then we obtain [12]

$$\begin{aligned}
re^{i\theta} = h^{-1}(x) = \frac{e^{\pi ix} - i}{e^{\pi ix} + i} &= \frac{\cos(\pi x) + i\sin(\pi x) - i}{\cos(\pi x) + i\sin(\pi x) + i} \\
&= \frac{\cos(\pi x) + i\sin(\pi x) - i}{\cos(\pi x) + i\sin(\pi x) + i}\frac{\cos(\pi x) - i\sin(\pi x) - i}{\cos(\pi x) - i\sin(\pi x) - i} = -i\frac{\cos(\pi x)}{1 + \sin(\pi x)} \\
&= \left(\frac{\cos(\pi x)}{1 + \sin(\pi x)}\right) e^{-i\pi/2}
\end{aligned} \tag{50}$$

by

$$\begin{aligned}
&(\cos(\pi x) + i\sin(\pi x) - i)(\cos(\pi x) - i\sin(\pi x) - i) \\
&\quad = \cos^2(\pi x) - i\sin(\pi x)\cos(\pi x) - i\cos(\pi x) + i\sin(\pi x)\cos(\pi x) \\
&\qquad + \sin^2(\pi x) + \sin(\pi x) - i\cos(\pi x) - \sin(\pi x) - 1 = -2i\cos(\pi x)
\end{aligned}$$

and

$$\begin{aligned}
&(\cos(\pi x) + i\sin(\pi x) + i)(\cos(\pi x) - i\sin(\pi x) - i) \\
&\quad = \cos^2(\pi x) - i\sin(\pi x)\cos(\pi x) - i\cos(\pi x) + i\sin(\pi x)\cos(\pi x) \\
&\qquad + \sin^2(\pi x) + \sin(\pi x) + i\cos(\pi x) + \sin(\pi x) + 1 = 2 + 2\sin(\pi x)
\end{aligned}$$

From equality (50) we deduce $r = \frac{\cos(\pi x)}{1+\sin(\pi x)}$, $\theta = \frac{\pi}{2}$ if $0 < x \leqslant \frac{1}{2}$ and $r = -\frac{\cos(\pi x)}{1+\sin(\pi x)}$, $\theta = \frac{\pi}{2}$ if $\frac{1}{2} \leqslant x < 1$. Let $0 < x \leqslant \frac{1}{2}$. Then we have

$$\begin{aligned}
&\frac{1 - r^2}{1 - 2r\cos(\theta - \varphi) + r^2} \\
&\quad = \frac{1 + 2\sin(\pi x) + \sin^2(\pi x) - \cos^2(\pi x)}{1 + 2\sin(\pi x) + \sin^2(\pi x) + 2\cos(\pi x)\sin(\varphi)(1 + \sin(\pi x)) + \cos^2(\pi x)} \\
&\qquad = \frac{\sin(\pi x) + \sin^2(\pi x)}{1 + \sin(\pi x) + \cos(\pi x)\sin(\varphi)(1 + \sin(\pi x))} = \frac{\sin(\pi x)}{1 + \cos(\pi x)\sin(\varphi)}
\end{aligned}$$

since $\cos(-\pi/2 - \varphi) = -\sin(\varphi)$. That the case $\frac{1}{2} \leqslant x < 1$ yields the same result is due to $\cos(\pi/2 - \varphi) = \sin(\varphi)$.
Now we have to reformulate

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin(\pi x)}{1 + \cos(\pi x)\sin(\varphi)} \log|F(h(e^{i\varphi}))| d\lambda(\varphi) \tag{51}$$

[12] Recall, that for $z \in \mathbb{C}$ the trigonometric functions are defined by $\sin(z) := \frac{e^{iz} - e^{-iz}}{2i}$ and $\cos(z) := \frac{e^{iz} + e^{-iz}}{2}$. Hence the identities $e^{iz} = \cos(z) + i\sin(z)$ and $\cos^2(z) + \sin^2(z) = 1$ holds for any $z \in \mathbb{C}$ (see [Ahl79, pp. 42–44]).

Yannis Bähni
yannis.baehni@uzh.ch

Let Φ and Ψ be defined as in lemma (4.2). We have

$$\begin{aligned} e^{i\Phi(t)} = h^{-1}(it) &= \frac{e^{-\pi t} - i}{e^{-\pi t} + i}\frac{e^{-\pi t} - i}{e^{-\pi t} - i} = \frac{e^{-2\pi t} - 2ie^{-\pi t} - 1}{e^{-2\pi t} + 1} = \frac{e^{-2\pi t} - 1}{e^{-2\pi t} + 1} - \frac{2ie^{-\pi t}}{e^{-2\pi t} + 1} \\ &= \frac{e^{-2\pi t} - 1}{e^{-2\pi t} + 1} - \frac{2i}{e^{-\pi t} + e^{\pi t}} = \frac{1 - e^{2\pi t}}{1 + e^{2\pi t}} - \frac{2i}{e^{-\pi t} + e^{\pi t}} = -\tanh(\pi t) - i\,\mathrm{sech}(\pi t) \end{aligned}$$

and thus

$$\begin{aligned} \sin(\Phi(t))\cosh(\pi t) &= \sin(-i\log(-\tanh(\pi t) - i\,\mathrm{sech}(\pi t)))\cosh(\pi t) \\ &= \frac{1}{2i}\left[-\tanh(\pi t) - i\,\mathrm{sech}(\pi t) + \frac{1}{\tanh(\pi t) + i\,\mathrm{sech}(\pi t)}\right]\cosh(\pi t) \\ &= \frac{1}{2i}\left[\frac{\cosh(\pi t) - \tanh(\pi t)\sinh(\pi t) - 2i\tanh(\pi t) + \mathrm{sech}(\pi t)}{\tanh(\pi t) + i\,\mathrm{sech}(\pi t)}\right] \\ &= \frac{1}{2i}\left[\frac{\cosh^2(\pi t) - \sinh^2(\pi t) - 2i\sinh(\pi t) + 1}{\sinh(\pi t) + i}\right] \\ &= \frac{1 - i\sinh(\pi t)}{i\sinh(\pi t) - 1} \\ &= -1 \end{aligned}$$

Therefore the transformation formula yields

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\pi}^{0} \frac{\sin(\pi x)}{1 + \cos(\pi x)\sin(\varphi)} \log|F(h(e^{i\varphi}))| d\lambda(\varphi) \\ &\qquad = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin(\pi x)}{\cosh(\pi t) - \cos(\pi x)} \log|F(it)| d\lambda(t) \end{aligned} \tag{52}$$

and in a similar manner

$$\begin{aligned} &\frac{1}{2\pi}\int_{0}^{\pi} \frac{\sin(\pi x)}{1 + \cos(\pi x)\sin(\varphi)} \log|F(h(e^{i\varphi}))| d\lambda(\varphi) \\ &\qquad = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin(\pi x)}{\cosh(\pi t) + \cos(\pi x)} \log|F(1 + it)| d\lambda(t) \end{aligned} \tag{53}$$

holds since

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yannis.baehni@uzh.ch

$$\begin{aligned}\sin(\Psi(t))\cosh(\pi t) &= \sin(-i\log(-\tanh(\pi t)+i\,\mathrm{sech}(\pi t)))\cosh(\pi t)\\ &= \frac{1}{2i}\left[-\tanh(\pi t)+i\,\mathrm{sech}(\pi t)-\frac{1}{-\tanh(\pi t)+i\,\mathrm{sech}(\pi t)}\right]\cosh(\pi t)\\ &= \frac{1}{2i}\left[\frac{-\cosh(\pi t)+\tanh(\pi t)\sinh(\pi t)-2i\tanh(\pi t)-\mathrm{sech}(\pi t)}{-\tanh(\pi t)+i\,\mathrm{sech}(\pi t)}\right]\\ &= \frac{1}{2i}\left[\frac{-\cosh^2(\pi t)+\sinh^2(\pi t)-2i\sinh(\pi t)-1}{i-\sinh(\pi t)}\right]\\ &= \frac{1+i\sinh(\pi t)}{1+i\sinh(\pi t)}\\ &= 1\end{aligned}$$

Thus the case $y = 0$ is prooven.
The case $\underline{y \neq 0.}$ follows easily from the previous one. Fix $y \neq 0$ and define $G(z) := F(z+iy)$ for $z \in \overline{S}$. Then G is a holomorphic function in S and continuous on $\overline{S}$ as a composition of continuous and holomorphic functions. Moreover, the hypothesis on F yields

$$\log|G(z)| = \log|F(z+iy)| \leqslant Ae^{\tau|\operatorname{Im} z+y|} \leqslant Ae^{\tau|\operatorname{Im} z|}e^{\tau|y|} \tag{54}$$

for all $z \in \overline{S}$. The previous case yields for G with A replaced by $Ae^{\tau|y|}$

$$|G(x)| \leqslant \exp\left(\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log|G(it)|}{\cosh(\pi t)-\cos(\pi x)}+\frac{\log|G(1+it)|}{\cosh(\pi t)+\cos(\pi x)}\right]d\lambda(t)\right) \tag{55}$$

Now, observing $G(x) = F(x+iy)$, $G(it) = F(it+iy)$ and $G(1+it) = F(1+it+iy)$ yields the desired result. □

4.2. Stein's Theorem on Interpolation of Analytic Families of Operators. Because of the complex nature of the proof of the Riesz-Thorin Interpolation Theorem (3.1), Elias M. Stein realized quickly, that the restritction to consider only one linear operator T could easily be omited and instead, an analytic family of operators T_z depending on some complex parameter z could be considered.

Definition 4.1. (Analytic family, admissible growth) *Let (X,μ) be a measure space, (Y,ν) be a σ-finite measure spaces and $(T_z)_{z\in\overline{S}}$, where T_z is defined on the space of all finitely simple functions on X and taking values in the space of all measurable functions on Y such that*

$$\int_Y |T_z(\chi_A)\chi_B|d\nu \tag{56}$$

whenever $\mu(A), \nu(B) < \infty$. The family $(T_z)_{z\in\overline{S}}$ is said to be analytic *if for all f, g finitely simple we have that*

Yannis Bähni
yannis.baehni@uzh.ch

$$z \mapsto \int_Y T_z(f) g d\nu \tag{57}$$

is analytic on S and continuous on $\overline{S}$. Further, an analytic family $(T_z)_{z\in\overline{S}}$ is called of admissible growth, *if there is a constant $\tau \in [0,\pi[$, such that for all finitely simple functions f, g a constant $C(f,g)$ exists with*

$$\log\left|\int_Y T_z(f) g d\nu\right| \leqslant C(f,g)e^{\tau|\mathrm{Im} z|} \tag{58}$$

for all $z \in \overline{S}$.

Now we are able to write down the theorem.

Theorem 4.1. (Stein's Theorem on Interpolation of Analytic Families of Operators) *Let $(T_z)_{z\in\overline{S}}$ be an analytic family of admissible growth, $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$ and suppose that M_0, M_1 are positive functions on the real line such that for some $\tau \in [0,\pi)$*

$$\sup\left\{e^{-\tau|y|}\log M_0(y) : y \in \mathbb{R}\right\} < \infty \qquad \sup\left\{e^{-\tau|y|}\log M_1(y) : y \in \mathbb{R}\right\} < \infty \tag{59}$$

Fix $0 < \theta < 1$ and define

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \tag{60}$$

Further suppose that for all finitely simple functions f on X and $y \in \mathbb{R}$ we have

$$\|T_{iy}(y)\|_{L^{q_0}} \leqslant M_0(y)\|f\|_{L^{p_0}} \qquad \|T_{1+iy}(y)\|_{L^{q_1}} \leqslant M_1(y)\|f\|_{L^{p_1}} \tag{61}$$

Then for all finitely simple functions f on X we have

$$\|T_\theta(f)\|_{L^q} \leqslant M(\theta)\|f\|_{L^p}$$

where for $0 < x < 1$

$$M(x) = \exp\left(\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log M_0(t)}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log M_1(t)}{\cosh(\pi t) + \cos(\pi x)}\right] d\lambda(t)\right)$$

Proof. Fix $0 < \theta < 1$ and finitely simple functions f, g on X, Y respectively with $\|f\|_{L^p} = \|g\|_{L^{q'}} = 1$. Define f_z, g_z as in (25) and for $z \in \overline{S}$

$$F(z) := \int_Y T_z(f_z) g_z d\nu \tag{62}$$

Observe, that $\left|a_j^{P(z)}\right| \leqslant a_j^{p/p_0 + p/p_1}$ and $\left|b_k^{Q(z)}\right| \leqslant b_k^{q'/q_0' + q'/q_1'}$ for $z \in \overline{S}$. Hence

Yannis Bähni
yannis.baehni@uzh.ch

$$\begin{aligned}\log|F(z)| &= \log\left|\sum_{j=1}^{n}\sum_{k=1}^{m} a_j^{P(z)} b_j^{Q(z)} e^{i\alpha_j} e^{i\beta_k} \int_Y T_z(\chi_{X_j})(y)\chi_{Y_k}(y)d\nu(y)\right| \\ &\leqslant \log\left(\sum_{j=1}^{n}\sum_{k=1}^{m} a_j^{p/p_0+p/p_1} b_k^{q'/q_0'+q'/q_1'} \left|\int_{Y_k} T_z(\chi_{X_j})d\nu\right|\right)\end{aligned}$$

The same calculations as in the proof of the Riesz-Thorin interpolation theorem (3.1) yields for $y \in \mathbb{R}$

$$\|f_{iy}\|_{L^{p_0}} = \|f\|_{L^p}^{p/p_0} = 1 = \|g\|_{L^{q'}}^{q'/q_0'} = \|g_{iy}\|_{L^{q_0'}}$$

and

$$\|f_{1+iy}\|_{L^{p_1}} = \|f\|_{L^p}^{p/p_1} = 1 = \|g\|_{L^{q'}}^{q'/q_1'} = \|g_{1+iy}\|_{L^{q_1'}}$$

Further

$$|F(iy)| \leqslant \|T_{iy}(f_{iy})\|_{L^{q_0}} \|g_{iy}\|_{L^{q_0'}} \leqslant M_0(y)\|f_{iy}\|_{L^{p_0}} \|g_{iy}\|_{L^{q_0'}} = M_0(y)$$

and

$$|F(1+iy)| \leqslant \|T_{1+iy}(f_{1+iy})\|_{L^{q_1}} \|g_{1+iy}\|_{L^{q_1'}} \leqslant M_1(y)\|f_{1+iy}\|_{L^{p_1}} \|g_{1+iy}\|_{L^{q_1'}} = M_1(y)$$

by Hölder's inequality and the hypotheses on the analytic family $(T_z)_{z\in\overline{S}}$. Therefore the extension of Hadamard's three lines lemma (4.3) yields

$$|F(x)| \leqslant \exp\left(\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log M_0(t)}{\cosh(\pi t)-\cos(\pi x)} + \frac{\log M_1(t)}{\cosh(\pi t)+\cos(\pi x)}\right]d\lambda(t)\right) = M(x)$$

for every $0 < x < 1$. Furthermore observe that

$$F(\theta) = \int_Y T_\theta(f)g\,d\nu$$

and thus by [Fol99, p. 189] (Σ_Y denotes the set of all finitely simple functions on the σ-finite space Y)

$$\begin{aligned}M_q(T_\theta(f)) &= \sup\left\{\left|\int_Y T_\theta(f)g\right| : g \in \Sigma_Y, \|g\|_{L^{q'}}\right\} \\ &= \sup\left\{|F(\theta)| : g \in \Sigma_Y, \|g\|_{L^{q'}}\right\} \\ &\leqslant M(\theta)\end{aligned}$$

Since $M(\theta)$ is an absolutely convergent integral for any $0 < \theta < 1$, $M_q(T_\theta(f)) < \infty$ and thus $M_q(T_\theta(f)) = \|T_\theta(f)\|_{L^q}$. The general statement follows by replacing f with $f/\|f\|_{L^p}$ when $\|f\|_{L^p} \neq 0$. The theorem is trivially true when $\|f\|_{L^p} = 0$. □

Yannis Bähni
yannis.baehni@uzh.ch

Appendix A. Measure Theory

Let (X, μ) be a measure space. Recall, that if for each measurable set Y with $\mu(Y) = \infty$ there exists a measurable set $E \subseteq Y$ and $0 < \mu(E) < \infty$, μ is called *semifinite*.

LEMMA A.1. *Every σ-finite measure is semifinite.*

Proof. Let $X = \bigcup_{n\in\mathbb{N}} X_n$ where $\mu(X_n) < \infty$ and $\mu(Y) = \infty$. By letting $\tilde{X}_N := \bigcup_{n\leqslant N} X_n$, $\tilde{X}_N$ is an increasing sequence. Then $Y \cap \tilde{X}_n$ is measurable for each $n \in \mathbb{N}$ and by [Coh13, p. 10]

$$\infty = \mu(Y) = \mu(Y \cap X) = \mu\left(Y \cap \left(\bigcup_{N\in\mathbb{N}} \tilde{X}_N\right)\right)$$
$$= \mu\left(\bigcup_{N\in\mathbb{N}} \left(Y \cap \tilde{X}_N\right)\right) = \lim_{N\to\infty} \mu\left(Y \cap \tilde{X}_N\right)$$

Since $Y \cap \tilde{X}_N \subseteq \tilde{X}_N$, $\mu(Y \cap \tilde{X}_N) < \infty$ for every $N \in \mathbb{N}$. Hence for every $C > 0$ there exists $M \in \mathbb{N}$, such that

$$\mu(Y \cap \tilde{X}_N) > M$$

for $N > M$. □

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