# CLASSICAL FOURIER ANALYSIS: INTERPOLATION OF $L^p$ SPACES

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Suppose $(p_0, q_0), (p_1, q_1) \in [1,\infty] \times [1,\infty]$ are two pairs of indices and assume that the estimates

$$\|T(f)\|_{L^{q_0}} \leqslant M_0 \|f\|_{L^{p_0}} \qquad \|T(f)\|_{L^{q_1}} \leqslant M_1 \|f\|_{L^{p_1}}$$

holds where $T$ is an apropriate operator. Does this imply that

$$\|T(f)\|_{L^q} \leqslant M \|f\|_{L^p}$$

for other pairs $(p,q) \in [1,\infty]$? Those and similar questions will be answered by a tool called *interpolation*, in our case interpolation of $L^p$ spaces. Using interpolation it is possible to reduce difficult estimates to endpoint estimates and so interpolation can (but not always does) simplify matters. To give one of numerous applications of interpolation is a quick proof of *Young's inequality for convolutions* [Gra14, pp. 22–23]. There is not *the* interpolation theorem, merely a family of theorems which can be roughly divided into two main categories: *real* and *complex* interpolation methods. Real methods use so called *cut-off* functions to divide the functions in the domain of the operator $T$ into a bounded and unbounded part and then to establish bounds on each of those parts. However, Complex interpolation theorems are based upon standard results in complex analysis and are more restrictive on the operator $T$ in question but yield more natural bounds (even continuous estimates) and will therefore be considered in this task. First we need a rigorous idea of what apropriate operator means in the context of Lebesgue spaces.

DEFINITION 1.1. *Let $(X,\mu)$ and $(Y,\nu)$ be measure spaces. Further let $T$ be an operator defined on a linear space of complex-valued measurable functions on $X$ and taking values in the set of all complex-valued, finite almost everywhere, measurable functions on $Y$. Then $T$ is called* linear *if for all functions $f$ and $g$ in the domain of $T$ and all $z \in \mathbb{C}$*

$$T(f+g) = T(f) + T(g) \qquad T(zf) = zT(f)$$

*holds and* quasi-linear *if*

$$|T(f+g)| \leqslant K(|T(f)| + |T(g)|) \qquad |T(zf)| = |z|\,|T(f)|$$

*holds for some constant $K > 0$. If $K = 1$, $T$ is called* sublinear.

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For two measure spaces $(X, \mu)$, $(Y, \nu)$ let $\Sigma_X$ and $\Sigma_Y$ denote the set of all finitely simple functions on $X$, $Y$ respectively.

THEOREM 1.1. (Riesz-Thorin interpolation theorem) *Suppose that $(X, \mu)$, $(Y, \nu)$ are measure spaces and $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$. If $q_0 = q_1 = \infty$, suppose also that $\nu$ is semifinite. Let $T$ be a linear operator defined on $\Sigma_X$ and taking values in the set of measurable functions on $Y$, such that*

$$\|T(f)\|_{L^{q_0}} \leqslant M_0 \|f\|_{L^{p_0}} \qquad \|T(f)\|_{L^{q_1}} \leqslant M_1 \|f\|_{L^{p_1}} \tag{1}$$

*for all $f \in \Sigma_X$ and $0 < M_0, M_1 < \infty$. Then for all $0 \leqslant \theta \leqslant 1$ we have*

$$\|T(f)\|_{L^q} \leqslant M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p} \tag{2}$$

*for all $f \in \Sigma_X$, where*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \textit{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

The proof of Hadamard's three lines lemma heavily relies on a restatement of the maximum modulus theorem (see [Rud87, p. 212]) found in [Rud87, p. 253]. To do so we have to first establish some common terminology. A complex-valued function $f$ is said to be *holomorphic* in $\Omega \subseteq \mathbb{C}$ open, if $f'(z)$ exists for any $z \in \Omega$. By a region we shall mean a nonempty connected open subset of the complex plane. The restatement reads as follows.

THEOREM. *Let $\Omega \subseteq \mathbb{C}$ be a bounded region and $f$ be a continuous function on $\overline{\Omega}$ which is holomorphic in $\Omega$. Then*

$$|f(z)| \leqslant \sup\{|f(z)| : z \in \partial\Omega\}$$

*for every $z \in \Omega$. If equality holds at one point $z \in \Omega$, then $f$ is constant.*

LEMMA 1.1. (Hadamard's three lines lemma) *Let $F$ be a holomorphic function in the strip $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$, continuous and bounded on $\overline{S}$, such that $|F(z)| \leqslant B_0$ when $\operatorname{Re} z = 0$ and $|F(z)| \leqslant B_1$ when $\operatorname{Re} z = 1$, for some $0 < B_0, B_1 < \infty$. Then $|F(z)| \leqslant B_0^{1-\theta} B_1^{\theta}$ when $\operatorname{Re} z = \theta$, for any $0 < \theta < 1$.*

*Proof.* For $z \in \overline{S}$ define

$$G(z) := \frac{F(z)}{B_0^{1-z} B_1^{z}} \qquad G_n(z) := G(z) e^{(z^2-1)/n}, \; n \in \mathbb{N}_{>0}$$

$G(z)$ and $G_n(z)$ are holomorphic in $S$ by

$$G'(z) = \frac{F'(z) - F(z) \log(B_1/B_0)}{B_0^{1-z} B_1^{z}} \qquad G_n'(z) = G'(z) e^{(z^2-1)/n} + \frac{2}{n} z G_n(z)$$

and $e^z \neq 0$ for every $z \in \mathbb{C}$. Further, we have

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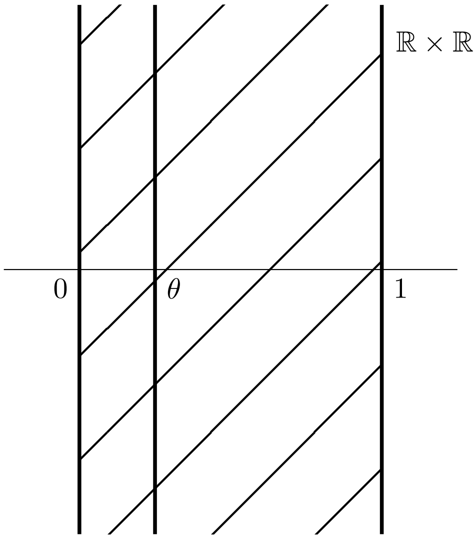


FIGURE 1. Sketch of the setting of Hadamard's three lines lemma.

$$\left|B_0^{1-z}B_1^z\right| = \left(B_0^{1-z}B_0^{1-\overline{z}}B_1^zB_1^{\overline{z}}\right)^{1/2} = B_0^{1-\operatorname{Re}z}B_1^{\operatorname{Re}z}$$

Consider $0 \leqslant \operatorname{Re} z \leqslant 1$ and $B_0 \geqslant 1$. Then $B_0^{1-\operatorname{Re}z} \geqslant 1$ and $B_0^{1-\operatorname{Re}z} \geqslant B_0$ in the case $B_0 < 1$. Similarily, $B_1^{\operatorname{Re}z} \geqslant 1$ if $B_1 \geqslant 1$ and $B_1^{\operatorname{Re}z} \leqslant B_1$ if $B_1 < 1$. Hence

$$\left|B_0^{1-z}B_1^z\right| \geqslant \min\{1, B_0\}\min\{1, B_1\} > 0 \tag{3}$$

for all $z \in \overline{S}$. Since $F$ is bounded on $\overline{S}$, we have $|F(z)| \leqslant L$ for some $L > 0$ and all $z \in \overline{S}$. Thus by (3)

$$|G(z)| = \frac{|F(z)|}{\left|B_0^{1-z}B_1^z\right|} \leqslant \frac{L}{\min\{1, B_0\}\min\{1, B_1\}} =: M$$

for every $z \in \overline{S}$. Fix $n \in \mathbb{N}_{>0}$ and write $z := x + iy \in \overline{S}$. Then

$$|G_n(z)| \leqslant M\left(e^{\left(x^2+2ixy-y^2-1\right)/n}e^{\left(x^2-2ixy-y^2-1\right)/n}\right)^{1/2} = Me^{-y^2/n}e^{\left(x^2-1\right)/n} \leqslant Me^{-y^2/n}$$

for $0 \leqslant x \leqslant 1$. Thus

$$\lim_{y\to\pm\infty}\sup\{|G_n(z)| : 0 \leqslant x \leqslant 1\} = 0$$

by the pinching-principle. Hence there exist $C_0, C_1 \in \mathbb{R}$, such that

$$\sup\{|G_n(z)| : 0 \leqslant x \leqslant 1\} \leqslant 1$$

when $y > C_0$ or $y < C_1$. Letting

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$$C(n) := \max\{|C_0| + 1, |C_1| + 1\}$$

we conclude $|G_n(z)| \leqslant 1$ for all $0 \leqslant x \leqslant 1$ when $|y| \geqslant C(n)$. Now consider the rectangle $R := (0,1) \times (-C(n), C(n))$. We have $|G_n(z)| \leqslant 1$ on the lines $[0,1] \times \{\pm C(n)\}$. By

$$|G_n(iy)| = \frac{|F(iy)|}{\left|B_0^{1-iy} B_1^{iy}\right|} e^{-(y^2+1)/n} \leqslant 1 \qquad |G_n(1+iy)| = \frac{|F(1+iy)|}{\left|B_0^{-iy} B_1^{1+iy}\right|} e^{-y^2/n} \leqslant 1$$

we have $|G_n(z)| \leqslant 1$ on the lines $\{0\} \times [-C(n), C(n)]$, $\{1\} \times [-C(n), C(n)]$. Thus $|G_n(z)| \leqslant 1$ on $\partial R$. Since $|G_n(z)|$ is continuous on $\overline{R}$, holomorphic in $R$ and $R$ is a bounded region, the maximum modulus theorem implies

$$|G_n(z)| \leqslant \sup\{|G_n(z)| : z \in \partial R\} \leqslant 1$$

for every $z \in R$. Therefore $|G_n(z)| \leqslant 1$ on $\overline{R}$ and so $|G_n(z)| \leqslant 1$ on $\overline{S}$. Since inequalities are preserved by limits and the modulus is a continuous function, we have that $|G(z)| = \lim_{n\to\infty} |G_n(z)| \leqslant 1$ for $z \in \overline{S}$. We conclude by

$$|F(\theta + it)| = |G(\theta + it)| \left|B_0^{1-\theta-it} B_1^{\theta+it}\right| \leqslant B_0^{1-\theta} B_1^{\theta}$$

whenever $0 < \theta < 1$, $t \in \mathbb{R}$. □

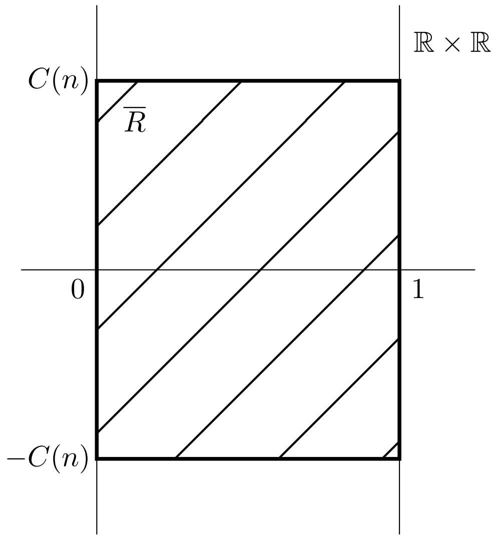


FIGURE 2. Sketch of the rectangle $\overline{R}$.

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*Proof.* The idea is to bound the quantity (see [Fol99, p. 189])

$$M_q(T(f)) = \sup\left\{\left|\int_Y T(f)g\,\mathrm{d}\nu\right| : g \in \Sigma_Y, \|g\|_{L^{q'}} = 1\right\} < \infty$$

appropriately. If either $\theta = 0$ or $\theta = 1$, the estimate (2) follows directly from the hypotheses (1) on $T$. Thus we may assume $\underline{0 < \theta < 1}$. Furthermore, if $f \in \Sigma_X$, $\|f\|_{L^p} = 0$, then $f = 0$ $\mu$-a.e. and either one of the hypotheses on $T$ in (1) implies $T(f) = 0$ $\mu$-a.e. and thus the estimate (2) holds trivially. Therefore we can assume $\underline{\|f\|_{L^p} \neq 0}$. Fix $f \in \Sigma_X$, $g \in \Sigma_Y$ with representation

$$f = \sum_{j=1}^n a_j e^{i\alpha_j}\chi_{A_j} \qquad g = \sum_{k=1}^m b_k e^{i\beta_k}\chi_{B_k}$$

where $a_j, b_k \neq 0$, $\alpha_j, \beta_k \in \mathbb{R}$ for any $j = 1, \ldots, n$, $k = 1, \ldots, m$, the sets $A_j$ and $B_k$ are each pairwise disjoint with $\mu(A_j), \nu(B_k) < \infty$ and so, that $\|g\|_{L^{q'}} \neq 0$ (recall $q' := q/(q-1)$). Define

$$P(z) := \frac{p}{p_0}(1-z) + \frac{p}{p_1}z \qquad Q(z) := \frac{q'}{q_0'}(1-z) + \frac{q'}{q_1'}z$$

for $z \in \mathbb{C}$ (since either $p = \infty$ implies $p_0 = p_1 = \infty$ or $q = 1$ implies $q_0 = q_1 = 1$, the functions $P$, $Q$ are well-defined). Further let

$$f_z := \sum_{j=1}^n a_j^{P(z)} e^{i\alpha_j}\chi_{A_j} \qquad g_z := \sum_{k=1}^m b_k^{Q(z)} e^{i\beta_k}\chi_{B_k} \tag{4}$$

and

$$F(z) := \int_Y T(f_z)g_z\,\mathrm{d}\nu \tag{5}$$

By the linearity of the operator $T$ we have

$$F(z) = \sum_{j=1}^n\sum_{k=1}^m a_j^{P(z)} b_k^{Q(z)} e^{i\alpha_j} e^{i\beta_k}\int_Y T(\chi_{A_j})\chi_{B_k}\,\mathrm{d}\nu$$

and by Hölder's inequality

$$\begin{aligned}\left|\int_Y T(\chi_{A_j})\chi_{B_k}\,\mathrm{d}\nu\right| &\leqslant \int_Y \left|T(\chi_{A_j})\chi_{B_k}\right|\mathrm{d}\nu \\ &= \left\|T(\chi_{A_j})\chi_{B_k}\right\|_{L^1} \\ &\leqslant \left\|T(\chi_{A_j})\right\|_{L^{q_0}}\left\|\chi_{B_k}\right\|_{L^{q_0'}} \\ &\leqslant M_0\left\|\chi_{A_j}\right\|_{L^{p_0}}\left\|\chi_{B_k}\right\|_{L^{q_0'}} \\ &\leqslant M_0\mu(A_j)^{1/p_0}\nu(B_k)^{1/q_0'}\end{aligned} \tag{6}$$

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for each $j = 1, \dots, n$, $k = 1, \dots, m$ (even in the cases where either $p_0 = \infty$ or $q_0' = \infty$, or both, by observing that $\|\chi_A\|_{L^\infty} \leqslant 1$ for any measurable set $A$). Thus the function $F$ is well-defined on $\mathbb{C}$. Let $t \in \mathbb{R}$. For $p, p_0 \neq \infty$

$$\begin{aligned}
\|f_{it}\|_{L^{p_0}} &= \left( \sum_{j=1}^n \int_{A_j} |f_{it}|^{p_0} \, \mathrm{d}\mu + \int_{X \setminus \bigcup_{j=1}^n A_j} |f_{it}|^{p_0} \, \mathrm{d}\mu \right)^{1/p_0} \\
&= \left( \sum_{j=1}^n \left| a_j^{P(it)} e^{i\alpha_j} \right|^{p_0} \int_X \chi_{A_j} \, \mathrm{d}\mu \right)^{1/p_0} \\
&= \left( \sum_{j=1}^n a_j^{p_0 \operatorname{Re} P(it)} \mu(A_j) \right)^{1/p_0} \\
&= \left( \sum_{j=1}^n a_j^{p} \mu(A_j) \right)^{p/(p_0 p)} \\
&= \|f\|_{L^p}^{p/p_0}
\end{aligned}$$

holds. Let $p_0 = \infty$, $p \neq \infty$. Then $\|f_{it}\|_{L^\infty} = 1$ since $\left| a_j^{P(it)} \right| = a_j^{p/p_0} = 1$ and that there exists some index $j$, such that $\mu(A_j) \neq 0$. If $p = \infty$, then $p_0 = p_1 = \infty$ and thus $P(it) = 1$. By the same considerations we have $\|g_{it}\|_{L^{q_0'}} = \|g\|_{L^{q'}}^{q'/q_0'}$. Hence

$$\begin{aligned}
|F(it)| &\leqslant \int_Y |T(f_{it}) g_{it}| \, \mathrm{d}\nu \\
&= \|T(f_{it}) g_{it}\|_{L^1} \\
&\leqslant \|T(f_{it})\|_{L^{q_0}} \|g_{it}\|_{L^{q_0'}} \\
&\leqslant M_0 \|f_{it}\|_{L^{p_0}} \|g_{it}\|_{L^{q_0'}} \\
&= M_0 \|f\|_{L^p}^{p/p_0} \|g\|_{L^{q'}}^{q'/q_0'}
\end{aligned}$$

by Hölder's inequality. In an analogous manner we derive

$$\|f_{1+it}\|_{L^{p_1}} = \|f\|_{L^p}^{p/p_1} \qquad \|g_{1+it}\|_{L^{q_1'}} = \|g\|_{L^{q'}}^{q'/q_1'}$$

and thus

$$|F(1+it)| \leqslant M_1 \|f\|_{L^p}^{p/p_1} \|g\|_{L^{q'}}^{q'/q_1'}$$

Further by estimate (6)

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$$
\begin{aligned}
|F(z)| &\leqslant \sum_{j=1}^{n}\sum_{k=1}^{m}\left|a_j^{P(z)}\right|\left|b_k^{Q(z)}\right|\left|\int_Y T(\chi_{A_j})\chi_{B_k}\,\mathrm{d}\nu\right| \\
&\leqslant M_0\sum_{j=1}^{n}\sum_{k=1}^{m} a_j^{\operatorname{Re}P(z)} b_k^{\operatorname{Re}Q(z)}\mu(A_j)^{1/p_0}\,\nu(B_k)^{1/q_0'} \\
&\leqslant M_0\sum_{j=1}^{n}\sum_{k=1}^{m}\max\left\{1, a_j^{p/p_0+p/p_1}\right\}\max\left\{1, b_k^{q'/q_0'+q'/q_1'}\right\}\mu(A_j)^{1/p_0}\,\nu(B_k)^{1/q_0'}
\end{aligned}
$$

Hence $F$ is bounded on $\overline{S}$ by some constant depending on $f$ and $g$ only. By

$$
\begin{aligned}
F'(z) = \sum_{j=1}^{n}\sum_{k=1}^{m} a_j^{P(z)}\log(a_j)\left(\frac{p}{p_1}-\frac{p}{p_0}\right) b_k^{Q(z)} e^{i\alpha_j}e^{i\beta_k}\int_Y T(\chi_{A_j})\chi_{B_k}\,\mathrm{d}\nu \\
+\sum_{j=1}^{n}\sum_{k=1}^{m} a_j^{P(z)} b_k^{Q(z)}\log(b_k)\left(\frac{q'}{q_1'}-\frac{q'}{q_0'}\right)e^{i\alpha_j}e^{i\beta_k}\int_Y T(\chi_{A_j})\chi_{B_k}\,\mathrm{d}\nu
\end{aligned}
$$

it is immediate, that $F$ is an entire function and thus holomorphic in $S$ and continuous on $\overline{S}$. Therefore, Hadamard's three lines lemma yields

$$
|F(z)| \leqslant \left(M_0\|f\|_{L^p}^{p/p_0}\|g\|_{L^{q'}}^{q'/q_0'}\right)^{1-\theta}\left(M_1\|f\|_{L^p}^{p/p_1}\|g\|_{L^{q'}}^{q'/q_1'}\right)^{\theta} = M_0^{1-\theta}M_1^{\theta}\|f\|_{L^p}\|g\|_{L^{q'}}
$$

for $\operatorname{Re} z = \theta$, $0 < \theta < 1$. We have

$$
\{T(f) \neq 0\} = \bigcup_{n=1}^{\infty}\{|T(f)| > 1/n\}
$$

and by Chebychev's inequality either

$$
\nu(\{|T(f)| > 1/n\}) \leqslant n^{q_0}\|T(f)\|_{L^{q_0}}^{q_0}
$$

or

$$
\nu(\{|T(f)| > 1/n\}) \leqslant n^{q_1}\|T(f)\|_{L^{q_1}}^{q_1}
$$

whenever $q_0 \neq \infty$ or $q_1 \neq \infty$. Therefore, the set $\{T(f) \neq 0\}$ is $\sigma$-finite unless $q_0 = q_1 = \infty$. Further we have $P(\theta) = Q(\theta) = 1$. Thus by

$$
\begin{aligned}
M_q(T(f)) &= \sup\left\{\left|\int_Y T(f)g\,\mathrm{d}\nu\right| : g \in \Sigma_Y, \|g\|_{L^{q'}} = 1\right\} \\
&= \sup\left\{|F(\theta)| : g \in \Sigma_Y, \|g\|_{L^{q'}} = 1\right\} \\
&\leqslant M_0^{1-\theta}M_1^{\theta}\|f\|_{L^p}
\end{aligned}
$$

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we conclude

$$\|T(f)\|_{L^q} = M_q(T(f)) \leqslant M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p}$$

for any $f \in \Sigma_X$. □

REMARK. *A more general version of the Riesz-Thorin interpolation theorem can be found in* [Fol99, pp. 200–202]. *There, a linear map*

$$T : L^{p_0} + L^{p_1} \to L^{q_0} + L^{q_1}$$

*is considered. This follows using a density argument from the current version of the theorem and will not be prooven here.*

REMARK. *Using the previous remark, a standard application of the Riesz-Thorin interpolation theorem is to proove Young's inequality* [Gra14, pp. 22–23].

DEFINITION 1.2. (Analytic family, admissible growth) *Let $(X, \mu)$, $(Y, \nu)$ be two $\sigma$-finite measure spaces and for every $z \in \overline{S}$ we have an associated linear operator $T_z$ which is defined on $\Sigma_X$ and taking values in the space of all measurable functions on $Y$ such that*

$$\int_Y |T_z(\chi_A)\chi_B| \,\mathrm{d}\nu < \infty$$

*whenever $\mu(A), \nu(B) < \infty$. The family $(T_z)_{z\in\overline{S}}$ is said to be* analytic *if for all $f \in \Sigma_X$, $g \in \Sigma_Y$ we have that*

$$z \mapsto \int_Y T_z(f) g \,\mathrm{d}\nu$$

*is analytic in $S$ and continuous on $\overline{S}$. Further, an analytic family $(T_z)_{z\in\overline{S}}$ is called* of admissible growth, *if there is a constant $\tau_0 \in (0, \pi)$, such that for all $f \in \Sigma_X$, $g \in \Sigma_Y$ a constant $0 < C(f, g) < \infty$ exists with*

$$\log \left| \int_Y T_z(f) g \,\mathrm{d}\nu \right| \leqslant C(f, g) e^{\tau_0 |\operatorname{Im} z|}$$

*for all $z \in \overline{S}$.*

THEOREM 1.2. (Stein-Weiss theorem on interpolation of analytic families of operators) *Let $(T_z)_{z\in\overline{S}}$ be an analytic family of admissible growth, $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$ and suppose that $M_0$, $M_1$ are positive functions on the real line such that for some $\tau_1 \in (0, \pi)$*

$$\sup\left\{e^{-\tau_1|y|} \log M_0(y) : y \in \mathbb{R}\right\} < \infty \qquad \sup\left\{e^{-\tau_1|y|} \log M_1(y) : y \in \mathbb{R}\right\} < \infty \quad (7)$$

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*Fix $0 < \theta < 1$ and define*

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \tag{8}$$

*Further suppose that for all $f \in \Sigma_X$ and $y \in \mathbb{R}$ we have*

$$\|T_{iy}(f)\|_{L^{q_0}} \leqslant M_0(y)\,\|f\|_{L^{p_0}} \qquad \|T_{1+iy}(f)\|_{L^{q_1}} \leqslant M_1(y)\,\|f\|_{L^{p_1}} \tag{9}$$

*Then for all $f \in \Sigma_X$ we have*

$$\|T_\theta(f)\|_{L^q} \leqslant M(\theta)\,\|f\|_{L^p}$$

*where for $0 < x < 1$*

$$M(x) = \exp\left(\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log M_0(t)}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log M_1(t)}{\cosh(\pi t) + \cos(\pi x)}\right] \mathrm{d}\lambda(t)\right)$$

Lemma 1.2. (Hadamard's three lines lemma, extension) *Let $F$ be a holomorphic function in the strip $S := \{z \in \mathbb{C} : 0 < \mathrm{Re} z < 1\}$ and continuous on $\overline{S}$, such that for some $0 < A < \infty$ and $\tau_0 \in (0, \pi)$ we have $\log|F(z)| \leqslant Ae^{\tau_0|\operatorname{Im} z|}$ for every $z \in \overline{S}$. Then*

$$\|F(z)\| \leqslant \exp\left(\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log|F(it+iy)|}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log|F(1+it+iy)|}{\cosh(\pi t) + \cos(\pi x)}\right] \mathrm{d}\lambda(t)\right)$$

*whenever $z := x + iy \in S$.*

*Proof.* As mentioned in Terence Tao's blog, Fefferman once noted, that this proof can be obtained from that of the Riesz-Thorin theorem 1.1 simply by adding a single letter of the alphabet. Indeed, this is truly the case, since all hypotheses made in the theorem incorporate the same proof as in the Riesz-Thorin theorem. The only heavy and technical part is the proof of the extension of Hadamard's three lines lemma 1.2. □

Theorem 1.3. (The Marcinkiewicz Interpolation Theorem) *Let $(X, \mu)$ be a $\sigma$-finite measure space, $(Y, \nu)$ another measure space and $0 < p_0 < p_1 \leqslant \infty$. Further let $T$ be a sublinear operator defined on*

$$L^{p_0} + L^{p_1} := \{f_0 + f_1 : f_0 \in L^{p_0}(X, \mu), f_1 \in L^{p_1}(X, \mu)\}$$

*and taking values in the space of measurable functions on $Y$. Assume that there exist $0 < A_0, A_1 < \infty$ such that*

$$\forall f \in L^{p_0}(X, \mu)\ \|T(f)\|_{L^{p_0,\infty}} \leqslant A_0\|f\|_{L^{p_0}} \tag{10}$$

$$\forall f \in L^{p_1}(X, \mu)\ \|T(f)\|_{L^{p_1,\infty}} \leqslant A_1\|f\|_{L^{p_1}} \tag{11}$$

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*Then for all* $p_0 < p < p_1$ *and for all* $f \in L^p(X, \mu)$ *we have the estimate*

$$\|T(f)\|_{L^p} \leqslant A \|f\|_{L^p} \tag{12}$$

*where*

$$A := 2\left(\frac{p}{p-p_0} + \frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}} \tag{13}$$