CLASSICAL FOURIER ANALYSIS: INTERPOLATION ON L^p SPACES

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Abstract. In this written seminar work I will basically follow the section *Interpolation* in the book *Classical Fourier Analysis, third Edition* by Loukas Grafakos. I will review three basic but important theorems on interpolation of operators on L^p spaces, namely the *Marcinkiewicz Interpolation Theorem*, the *Riesz-Thorin Interpolation Theorem* and finally an extension of the Riesz-Thorin Interpolation Theorem to analytic families of operators (the so-called *Stein's theorem on interpolation of analytic families of operators*). We are mainly concerned with the notion of linear operators as well as slight generalizations of them.

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1. Introduction and Basic Definitions.

1.1. Linear Operators. First we need to have a precise and suitable idea of *linear operators* in the generalized setting of measure spaces.

DEFINITION 1.1. *Let* (X, μ) *and* (Y, ν) *be measure spaces. Further let* T *be an operator defined on a linear space of complex-valued measurable functions on* X *and taking*

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values in the set of all complex-valued, finite almost everywhere, measurable functions on Y. Then T is called linear *if for all functions f and g in the domain of T and all $z \in \mathbb{C}$ holds*

$$T(f+g) = T(f) + T(g) \qquad T(zf) = zT(f) \tag{1}$$

and quasi-linear *if*

$$|T(f+g)| \leqslant K(|T(f)| + |T(g)|) \qquad |T(zf)| = |z||T(f)| \tag{2}$$

holds for some real constant $K > 0$. If $K = 1$, T is called sublinear.

2. The Real Method.

2.1. The Marcinkiewicz Interpolation Theorem. The name originates from the real variables technique used for prooving the theorem.

Theorem 2.1. (The Marcinkiewicz Interpolation Theorem) *Let (X, μ) be a σ-finite measure space, (Y, ν) another measure space and $0 < p_0 < p_1 \leqslant \infty$. Further let T be a sublinear operator defined on*

$$L^{p_0} + L^{p_1} := \{f_0 + f_1 : f_0 \in L^{p_0}(X, \mu), f_1 \in L^{p_1}(X, \mu)\}$$

and taking values in the space of measurable functions on Y. Assume that there exist $A_0, A_1 < \infty$ such that

$$\forall f \in L^{p_0}(X,\mu) \ \|T(f)\|_{L^{p_0,\infty}} \leqslant A_0 \|f\|_{L^{p_0}} \tag{3}$$
$$\forall f \in L^{p_1}(X,\mu) \ \|T(f)\|_{L^{p_1,\infty}} \leqslant A_1 \|f\|_{L^{p_1}} \tag{4}$$

Then for all $p_0 < p < p_1$ and for all $f \in L^p(X, \mu)$ we have the estimate

$$\|T(f)\|_{L^p} \leqslant A \|f\|_{L^p} \tag{5}$$

where

$$A := 2\left(\frac{p}{p-p_0} + \frac{p}{p_1 - p}\right)^{1/p} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}} \tag{6}$$

Proof. The proof is subdivided into two main parts, which are further subdivided. In detail, we have the following partitioning:

(i.) $p_1 < \infty$.
  - **a.** Split f using cut-off functions.
  - **b.** Estimate the distribution function $d_{T(f)}$.
  - **c.** Estimate $\|T(f)\|_{L^p}^p$.

(ii.) $p_1 = \infty$.
  - **a.** Show that $\mu(\{|T(f_1)| > \alpha/2\}) = 0$.

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b. Estimate the distribution function $d_{T(f_0)}$.

c. Estimate $\|T(f)\|_{L^p}^p$.

(i.) **a.** Let us first consider the case $\underline{p_1 < \infty}$. Fix $f \in L^p(X, \mu)$, $\alpha > 0$ and $\delta > 0$ (δ will be determined later). We split f using so-called *cut-off* functions, by stipulating $f \equiv f_0(\cdot; \alpha, \delta) + f_1(\cdot; \alpha, \delta)$, where $f_0(\cdot; \alpha, \delta)$ is the *unbounded part of* f and $f_1(\cdot; \alpha, \delta)$ is the *bounded part of* f, defined by

$$\begin{aligned} f_0(x; \alpha, \delta) &:= \begin{cases} f(x), & |f(x)| > \delta\alpha, \\ 0, & |f(x)| \leqslant \delta\alpha. \end{cases} \\ f_1(x; \alpha, \delta) &:= \begin{cases} f(x), & |f(x)| \leqslant \delta\alpha, \\ 0, & |f(x)| > \delta\alpha. \end{cases} \end{aligned} \tag{7}$$

for $x \in X$. To facilitate reading I will omit the dependency of $f_0(\cdot; \alpha, \delta)$ and $f_1(\cdot; \alpha, \delta)$ upon the parameters α and δ in what follows and simply write f_0, f_1 respectively. Since $p_0 < p$ we have

$$\begin{aligned} \|f_0\|_{L^{p_0}}^{p_0} &= \int_X |f_0|^{p_0} d\mu = \int_X |f|^{p_0} \cdot \chi_{\{|f| > \delta\alpha\}} d\mu \overset{(\dagger)}{=} \int_{\{|f| > \delta\alpha\}} |f|^{p_0} d\mu \\ &= \int_{\{|f| > \delta\alpha\}} |f|^p |f|^{p_0 - p} d\mu = \int_{\{|f| > \delta\alpha\}} \frac{|f|^p}{|f|^{p - p_0}} d\mu \\ &\leqslant \frac{1}{(\delta\alpha)^{p - p_0}} \int_{\{|f| > \delta\alpha\}} |f|^p d\mu = (\delta\alpha)^{p_0 - p} \int_X |f|^p \cdot \chi_{\{|f| > \delta\alpha\}} d\mu \\ &\leqslant (\delta\alpha)^{p_0 - p} \int_X |f|^p d\mu = (\delta\alpha)^{p_0 - p} \|f\|_{L^p}^p < \infty \end{aligned} \tag{8}$$

Thus $f_0 \in L^{p_0}(X, \mu)$. Analogously it can be checked, that $f_1 \in L^{p_1}(X, \mu)$ by the estimate $\|f_1\|_{L^{p_1}}^{p_1} \leqslant (\delta\alpha)^{p_1 - p} \|f\|_{L^p}^p$. Therefore $f \equiv f_0 + f_1 \in L^{p_0} + L^{p_1}$.

Proof of the equality ($\dagger$). Assume μ is defined on the σ-algebra $\mathcal{A}$. We have to proove that $\{|f| > \delta\alpha\} \in \mathcal{A}$[1]. Since f is complex-valued, we may write $f \equiv \mathrm{Re} f + i \mathrm{Im} f$ and thus $|f|^2 \equiv \mathrm{Re}^2 f + \mathrm{Im}^2 f$. Since f is measurable by hypothesis this implies that $\mathrm{Re} f$ and $\mathrm{Im} f$ are measurable[2]. Further for measurable real-valued functions $f, g : (X, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathfrak{B}})$[3] the functions $f + g$ and $f \cdot g$ are measurable[4] and thus $|f|^2$ is measurable. Hence $\{\mathrm{Re}^2 f + \mathrm{Im}^2 f > \lambda\} \in \mathcal{A}$[5] for

[1] For $Y \in \mathcal{A}$ the μ-integral of $f : X \to \mathbb{C}$ over Y is defined to be $\int_Y f d\mu := \int_X f \cdot \chi_Y d\mu$. For more details see [Els11, pp. 135–136].

[2] For a proof see [Els11, p. 106]

[3] $\mathfrak{B} := \sigma(\mathbb{R})$ and $\overline{\mathfrak{B}} = \{B \cup E : B \in \mathfrak{B}, E \subseteq \{\pm\infty\}\}$.

[4] For a proof see [Els11, p. 107].

[5] For a proof see [Els11, pp. 105–106]

any $\lambda \in \mathbb{R}$. So especially for $\lambda := (\delta\alpha)^2$ we have $\{|f| > \delta\alpha\} \in \mathcal{A}$[6]. In a similar manner it can also be prooven that $\{|f| \leqslant \delta\alpha\} \in \mathcal{A}$. Let us next proove a useful lemma.

LEMMA 2.1. *Let $A \in \wp(X)$ and $\chi_A : (X, \mathcal{A}) \to (\mathbb{C}, \mathfrak{B}^2)$ be the* characteristic function of the set A. *Then χ_A is measurable if and only if A is measurable.*

Proof. Assume χ_A is measurable. Then $\mathrm{Re}\chi_A$ and $\mathrm{Im}\chi_A$ are measurable. Especially for $0 < \lambda < 1$ we have that $\{\mathrm{Re}\chi_A > \lambda\} = A \in \mathcal{A}$. Conversly, assume A is measurable. For $\lambda < 0$ we have $\{\mathrm{Re}\chi_A > \lambda\} = X \in \mathcal{A}$, $\lambda \in [0, 1[$, $\{\mathrm{Re}\chi_A > \lambda\} = A \in \mathcal{A}$ and $\{\mathrm{Re}\chi_A > \lambda\} = \emptyset \in \mathcal{A}$ for $\lambda \geqslant 1$. Since $\mathrm{Im}\chi_A \equiv 0$ we have $\{\mathrm{Im}\chi_A > \lambda\} = X \in \mathcal{A}$ if $\lambda < 0$ and $\{\mathrm{Im}\chi_A > \lambda\} = \emptyset \in \mathcal{A}$ if $\lambda \geqslant 0$. □

By Lemma 2.1 and the fact that $f \cdot g$ is measurable for two measurable functions $f, g : (X, \mathcal{A}) \to (\mathbb{C}, \mathfrak{B}^2)$[7], f_0 and f_1 are measurable since $f_0 \equiv f \cdot \chi_{\{|f| > \delta\alpha\}}$ and $f_1 \equiv f \cdot \chi_{\{|f| \leqslant \delta\alpha\}}$.

One subtility is left to clear: the μ-integrability of either $|f_1|^{p_0}$ or $|f_1|^{p_1}$ requires that $|f_0|^{p_0}$ and $|f_1|^{p_1}$ are measurable functions. By the fact that any continuous map $g : (X, d_X) \to (Y, d_Y)$ between metric spaces is Borel-measurable (see [Els11, p. 86]) and that the composition of measurable functions is again measurable (see [Els11, p. 87]), the measurability of either f_0 or f_1 follows by $|f_0|^{p_0} \equiv \cdot^{p_0} \circ |f \cdot \chi_{\{|f| > \delta\alpha\}}|$ and $|f_1|^{p_1} \equiv \cdot^{p_1} \circ |f \cdot \chi_{\{|f| \leqslant \delta\alpha\}}|$ by stipulating $\cdot^p : (\mathbb{R}_{\geqslant 0}, |\cdot|) \to (\mathbb{C}, |\cdot|)$, $x^p := \exp(p \log(x))$ for $p > 0$ and $x \in \mathbb{R}_{>0}$ and $x^p := 0$ if $x = 0$.

b. Since T is a sublinear operator we have $|T(f)| = |T(f_0 + f_1)| \leqslant |T(f_0)| + |T(f_1)|$. Thus for any $y \in Y$ with $|T(f)(y)| > \alpha$ we therefore have either $|T(f_0)(y)| > \alpha/2$ or $|T(f_1)(y)| > \alpha/2$ [8] . Hence

$$\{|T(f)| > \alpha\} \subseteq \{|T(f_0)| > \alpha/2\} \cup \{|T(f_1)| > \alpha/2\}$$

and so by the monotonicity and subadditivity property of the measure μ we have

$$\begin{aligned} d_{T(f)}(\alpha) &= \mu(\{|T(f)| > \alpha\}) \\ &\leqslant \mu(\{|T(f_0)| > \alpha/2\} \cup \{|T(f_1)| > \alpha/2\}) \\ &\leqslant \mu(\{|T(f_0)| > \alpha/2\}) + \mu(\{|T(f_1)| > \alpha/2\}) \\ &= d_{T(f_0)}(\alpha/2) + d_{T(f_1)}(\alpha/2) \end{aligned} \tag{9}$$

Now by hypothesis (3) we can estimate $d_{T(f_0)}(\alpha/2)$ as follows

[6] This follows from the fact that $x < y$ if and only if $x^n < y^n$ for $n \in \mathbb{N}_{>0}$ and some real numbers $x, y > 0$ (see [Zor04, p. 119]).

[7] Els11, p. 107.

[8] Without loss of generality assume $|T(f_0)(y)| \leqslant |T(f_1)(y)|$. Then we have $\alpha < |T(f)(y)| \leqslant |T(f_0)(y)| + |T(f_1)(y)| \leqslant 2|T(f_1)(y)|$ (this is possible since $\mathbb{R}$ is an ordered field).

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$$\begin{aligned}
d_{T(f_0)}(\alpha/2) &= \left(\frac{\alpha/2}{\alpha/2}\right)^{p_0} d_{T(f_0)}(\alpha/2) \\
&\leqslant \left(\frac{1}{\alpha/2}\right)^{p_0} \left[\sup\left\{\gamma d_{T(f_0)}(\gamma)^{1/p_0} : \gamma > 0\right\}\right]^{p_0} \\
&= \left(\frac{1}{\alpha/2}\right)^{p_0} \|T(f_0)\|_{L^{p_0,\infty}}^{p_0} \\
&\leqslant \left(\frac{A_0}{\alpha/2}\right)^{p_0} \|f_0\|_{L^{p_0}}^{p_0}
\end{aligned} \tag{10}$$

Analogously, we get $d_{T(f_1)}(\alpha/2) \leqslant \left(\frac{A_1}{\alpha/2}\right)^{p_1} \|f_1\|_{L^{p_1}}^{p_1}$ by hypothesis (4).

c. By

$$\int_0^{\frac{1}{\delta}|f|} \alpha^{p-p_0-1} d\lambda = \begin{cases} \frac{1}{p-p_0}\frac{1}{\delta^{p-p_0}}|f|^{p-p_0}, & p \geqslant p_0+1 \\ \lim\limits_{\omega\to 0^+} \int_\omega^{\frac{1}{\delta}|f|} \alpha^{p-p_0-1} d\lambda & \\ = \lim\limits_{\omega\to 0^+} \left[\frac{1}{p-p_0}\alpha^{p-p_0}\Big|_\omega^{\frac{1}{\delta}|f|}\right] & \\ = \frac{1}{p-p_0}\left[\frac{1}{\delta^{p-p_0}}|f|^{p-p_0} - \lim\limits_{\omega\to 0^+}\omega^{p-p_0}\right] & \\ = \frac{1}{p-p_0}\frac{1}{\delta^{p-p_0}}|f|^{p-p_0}, & p_0 < p < p_0+1 \end{cases} \tag{11}$$

and

$$\begin{aligned}
\int_{\frac{1}{\delta}|f|}^{\infty} \alpha^{p-p_1-1} d\lambda &= \lim_{\omega\to\infty}\left[\frac{1}{p-p_1}\alpha^{p-p_1}\right]_{\frac{1}{\delta}|f|}^{\omega} \\
&= \frac{1}{p-p_1}\left[\lim_{\omega\to\infty}\omega^{p-p_1} - \frac{1}{\delta^{p-p_1}}|f|^{p-p_1}\right] \\
&= \frac{1}{p_1-p}\frac{1}{\delta^{p-p_1}}|f|^{p-p_1}
\end{aligned} \tag{12}$$

and the representation $\|f\|_{L^p}^p = p\int_0^\infty \alpha^{p-1} d_f(\alpha) d\lambda$ for $0 < p < \infty$ we get

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$$
\begin{aligned}
\|T(f)\|_{L^p}^p =& p\int_0^\infty \alpha^{p-1} d_{T(f)} d\lambda \\
\leqslant& p(2A_0)^{p_0}\int_0^\infty \alpha^{p-p_0-1}\int_{\{|f|>\delta\alpha\}} |f|^{p_0}\, d\mu d\lambda \\
&+ p(2A_1)^{p_1}\int_0^\infty \alpha^{p-p_1-1}\int_{\{|f|\leqslant\delta\alpha\}} |f|^{p_1}\, d\mu d\lambda \\
=& p(2A_0)^{p_0}\int_{\{|f|>0\}} |f|^{p_0}\int_0^{\frac{1}{\delta}|f|} \alpha^{p-p_0-1} d\lambda d\mu \\
&+ p(2A_0)^{p_0}\int_{\{|f|=0\}} |f|^{p_0}\int_0^{\frac{1}{\delta}|f|} \alpha^{p-p_0-1} d\lambda d\mu \\
&+ p(2A_1)^{p_1}\int_X |f|^{p_1}\int_{\frac{1}{\delta}|f|}^\infty \alpha^{p-p_1-1} d\lambda d\mu \\
=& p(2A_0)^{p_0}\int_X |f|^{p_0}\int_0^{\frac{1}{\delta}|f|} \alpha^{p-p_0-1} d\lambda d\mu \\
&+ p(2A_1)^{p_1}\int_X |f|^{p_1}\int_{\frac{1}{\delta}|f|}^\infty \alpha^{p-p_1-1} d\lambda d\mu \\
=& \frac{p(2A_0)^{p_0}}{p-p_0}\frac{1}{\delta^{p-p_0}}\int_X |f|^{p_0}|f|^{p-p_0}\, d\mu \\
&+ \frac{p(2A_1)^{p_1}}{p_1-p}\frac{1}{\delta^{p-p_1}}\int_X |f|^{p_1}|f|^{p-p_1}\, d\mu \\
=& p\left(\frac{(2A_0)^{p_0}}{p-p_0}\frac{1}{\delta^{p-p_0}} + \frac{(2A_1)^{p_1}}{p_1-p}\delta^{p_1-p}\right)\|f\|_{L^p}^p
\end{aligned} \tag{13}
$$

We pick $\delta > 0$ such that $(2A_0)^{p_0}\delta^{p_0-p} = (2A_1)^{p_1}\delta^{p_1-p}$. Solving for δ yields

$$
\delta = \frac{1}{2}\left(\frac{A_0}{A_1}\right)^{p_1/(p_1-p_0)} \tag{14}
$$

Substituting this in estimate (13) leads to

$$
\begin{aligned}
\|T(f)\|_{L^p}^p &\leqslant p\left(\frac{(2A_0)^{p_0}}{p-p_0}\frac{2^{p-p_0}A_1^{\frac{p_1(p-p_0)}{p_1-p_0}}}{A_0^{\frac{p_0(p-p_0)}{p_1-p_0}}} + \frac{(2A_1)^{p_1}}{p_1-p}\frac{A_0^{\frac{p_0(p_1-p)}{p_1-p_0}}}{2^{p_1-p}A_1^{\frac{p_1(p_1-p)}{p_1-p_0}}}\right)\|f\|_{L^p}^p \\
&= 2^p p\left(\frac{A_0^{\frac{p_0(p_1-p)}{p_1-p_0}}A_1^{\frac{p_1(p-p_0)}{p_1-p_0}}}{p-p_0} + \frac{A_0^{\frac{p_0(p_1-p)}{p_1-p_0}}A_1^{\frac{p_1(p-p_0)}{p_1-p_0}}}{p_1-p}\right)\|f\|_{L^p}^p
\end{aligned} \tag{15}
$$

And taking the p-th power further

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$$
\begin{aligned}
\|T(f)\|_{L^p} &\leqslant 2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{p_0(p_1-p)}{p(p_1-p_0)}} A_1^{\frac{p_1(p-p_0)}{p(p_1-p_0)}}\|f\|_{L^p} \\
&= 2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{p_0(p_1-p)}{p(p_1-p_0)}\frac{p_1}{p_1}} A_1^{\frac{p_1(p-p_0)}{p(p_1-p_0)}\frac{p_0}{p_0}}\|f\|_{L^p} \\
&= 2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{\frac{p_1-p}{pp_1}}{\frac{p_1-p_0}{p_0p_1}}} A_1^{\frac{\frac{p-p_0}{p_0p}}{\frac{p_1-p_0}{p_0p_1}}}\|f\|_{L^p} \\
&= 2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}}\|f\|_{L^p}
\end{aligned} \tag{16}
$$

(ii.) **a.** Assume $p_1 = \infty$. We again use the cut-off functions defined in (7) to decompose f. Since $\{|f_1| > \delta\alpha\} = \emptyset$, we have

$$
\|T(f_1)\|_{L^\infty} \leqslant A_1\|f_1\|_{L^\infty} = A_1 \inf\{B > 0 : \mu(\{|f_1| > B\}) = 0\} \leqslant A_1\delta\alpha = \alpha/2
$$

Provided we stipulate $\delta := 1/(2A_1)$. Therefore the set $\{|T(f_1)| > \alpha/2\}$ has measure zero (this is immediate since $\|T(f_1)\|_{L^\infty} = \inf\{B > 0 : \mu(\{|T(f_1)| > B\}) = 0\} \leqslant \alpha/2$ and any subset of a set with measure zero has itself measure zero). Thus similar to part **b.** of **(i.)** we get $d_{T(f)}(\alpha) \leqslant d_{T(f_0)}(\alpha/2)$.

b. Hypothesis (3) yields the estimate $d_{T(f_0)}(\alpha/2) \leqslant \left(\frac{A_0}{\alpha/2}\right)^{p_0} \int_{\{2A_1|f|>\alpha\}} |f|^{p_0} d\mu$.

c. Thus by **a.** and **b.**

$$
\begin{aligned}
\|T(f)\|_{L^p}^p &= p\int_0^\infty \alpha^{p-1} d_{T(f)} d\lambda \\
&\leqslant p(2A_0)^{p_0}\int_0^\infty \alpha^{p-p_0-1}\int_{\{2A_1|f|>\alpha\}} |f|^{p_0} d\mu d\lambda \\
&= p(2A_0)^{p_0}\int_X |f|^{p_0}\int_0^{2A_1|f|} \alpha^{p-p_0-1} d\lambda d\mu \\
&= \frac{2^p p A_0^{p_0} A_1^{p-p_0}}{p-p_0}\int_X |f|^p d\mu \\
&= \frac{2^p p A_0^{p_0} A_1^{p-p_0}}{p-p_0}\|f\|_{L^p}^p
\end{aligned} \tag{17}
$$

That the constant $2^p p A_0^{p_0} A_1^{p-p_0}/(p-p_0)$ found in (17) is the p-th power of the one stated in the theorem can be seen by passing the constant (6) to the limit $p_1 \to \infty$:

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$$
\begin{aligned}
\lim_{p_1\to\infty} A = \lim_{p_1\to\infty} &\left[2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}}\right] \\
=&2\exp\left[\frac{1}{p}\log\left(\frac{p}{p-p_0}+\lim_{p_1\to\infty}\frac{1}{p_1}\frac{p}{1-p\lim_{p_1\to\infty}\frac{1}{p_1}}\right)\right] \\
&\cdot\lim_{p_1\to\infty} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}}\cdot\lim_{p_1\to\infty} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}} \\
=&2\left(\frac{p}{p-p_0}\right)^{1/p}\exp\left[\frac{\frac{1}{p}-\lim_{p_1\to+\infty}\frac{1}{p_1}}{\frac{1}{p_0}-\lim_{p_1\to\infty}\frac{1}{p_1}}\log(A_0)\right] \\
&\cdot\exp\left[\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\lim_{p_1\to\infty}\frac{1}{p_1}}\log(A_1)\right] \\
=&2\left(\frac{p}{p-p_0}\right)^{1/p} A_0^{\frac{p_0}{p}} A_1^{1-\frac{p_0}{p}}
\end{aligned}
$$

□

3. The Complex Method. This theorem will unfortunately only be applicable to linear operators but will yield a more natural bound of the operator on the intermediate space. The proof will make strong use of complex variables technique. A major tool will be an application of the maximum modulus principle, known as *Hadamard's three lines lemma*.

3.1. Hadamard's Three Lines Lemma.

Lemma 3.1. *Hadamard's three lines lemma) Let F be an analytic function on the strip $S := \{z \in \mathbb{C} : 0 < \mathrm{Re} z < 1\}$, continuous and bounded on $\overline{S}$, such that $|F(z)| \leqslant B_0$ when $\mathrm{Re} z = 0$ and $|F(z)| \leqslant B_1$ when $\mathrm{Re} z = 1$, for some $0 < B_0, B_1 < \infty$. Then $|F(z)| \leqslant B_0^{1-\theta} B_1^{\theta}$ when $\mathrm{Re} z = \theta$, for any $0 \leqslant \theta \leqslant 1$.*

Proof. For $z \in \overline{S}$ define

$$G(z) := \frac{F(z)}{B_0^{1-z}B_1^z} \qquad \forall n \in \mathbb{N}_{>0} : G_n(z) := G(z)e^{(z^2-1)/n} \tag{18}$$

Obviously, $G(z)$ and $G_n(z)$ are analytic functions on S for $n \in \mathbb{N}_{>0}$[9]. Further, we have

[9] Recall, that a function f is called *analytic on* U, $U \subseteq \mathbb{C}$ open, if f is analytic at every $z_0 \in U$, that is, there exists a power series $\sum_{n\in\mathbb{N}} a_n(z-z_0)^n$ and some $r > 0$, such that the series converges absolutely for $|z - z_0| < r$, and such that for such z, we have $f(z) = \sum_{n\in\mathbb{N}} a_n(z-z_0)^n$ (as defined in [Lan93, pp. 68–69]). If f and g are analytic on $U \subseteq \mathbb{C}$, so are $f+g$, $f \cdot g$. Also f/g is analytic on the open subset of $z \in U$ such

$$|B_0^{1-z}B_1^z|^2 = |B_0^{1-z}|^2|B_1^z|^2 = B_0^{1-z}B_0^{1-\overline{z}}B_1^zB_1^{\overline{z}} = \left(B_0^{1-\mathrm{Re}z}\right)^2\left(B_1^{\mathrm{Re}z}\right)^2 \tag{19}$$

Consider $0 \leqslant \mathrm{Re}z \leqslant 1$ and $B_0 \geqslant 1$. Then $B_0^{1-\mathrm{Re}z} = \exp((1-\mathrm{Re}z)\log B_0) \geqslant 1$ and $B_0^{1-\mathrm{Re}z} \geqslant B_0$ in the case $B_0 < 1$. A similar estimation of $B_1^{\mathrm{Re}z}$ leads to

$$|B_0^{1-z}B_1^z| \geqslant \min\{1, B_0\}\min\{1, B_1\} \tag{20}$$

for all $z \in \overline{S}$. By this, $G(z)$ is bounded on $\overline{S}$ (by the boundedness of F). Let $M > 0$, such that $|G(z)| \leqslant M$ for $z \in \overline{S}$. Fix $n \in \mathbb{N}_{>0}$ and write $z := x + iy \in \overline{S}$. Since

$$\begin{aligned}
|G_n(z)|^2 &= |G(z)|^2|e^{((x+iy)^2-1)/n}|^2 \\
&\leqslant M^2 e^{(x^2+2ixy-y^2-1)/n}e^{(x^2-2ixy-y^2-1)/n} \\
&= M^2\left(e^{-y^2/n}\right)^2\left(e^{(x^2-1)/n}\right)^2 \\
&\leqslant M^2\left(e^{-y^2/n}\right)^2 \\
&= M^2\left(e^{-|y|^2/n}\right)^2
\end{aligned} \tag{21}$$

we have $\lim_{y\to\pm\infty}\sup\{|G_n(z)| : x \in [0,1]\} = 0$ by the pinching-principle. Hence there exists some $C(n) \in \mathbb{R}_{>0}$, such that $|G_n(z)| \leqslant 1$ for all $|y| \geqslant C(n)$ and all $x \in [0,1]$. Consider the rectangle $R := [0,1] \times [-C(n), C(n)]$. Now $|G_n(z)| \leqslant 1$ on the lines $[0,1] \times \{\pm C(n)\}$ and since $|G(z)| = |F(z)|/B_0 \leqslant 1$, $|G(z)| = |F(z)|/B_1 \leqslant 1$ on the line $\{0\} \times [-C(n), C(n)]$ and $\{1\}\times[-C(n), C(n)]$ respectively by assumption, we have $|G_n(z)| \leqslant 1$ on ∂S. By the maximum modulus principle [10] we have $|G_n(z)| \leqslant 1$ on R and thus $|G_n(z)| \leqslant 1$ on $\overline{S}$. Since inequalities are preserved by limits and the modulus is a continuous function, we have that $|G(z)| = \lim_{n\to\infty}|G_n(z)| \leqslant 1$ on $\overline{S}$. Taking $z := \theta + it$, where $0 \leqslant \theta \leqslant 1$ and $t \in \mathbb{R}$, we conclude $|F(z)| = |G(z)||B_0^{1-z}B_1^z| \leqslant B_0^{1-\theta}B_1^\theta$, which completes the proof.

□

that $g(z) \neq 0$. If $g : U \to V$ and $f : V \to C$ are analytic so is $f \circ g$. Further, if $f(z) = \sum_{n\in\mathbb{N}} a_n z^n$ is a power series with radius of convergence r, f is analytic on $B_r(0)$ (for a proof see [Lan93, pp. 69–70]).

[10] The theorem can be found in[Lan93, pp. 91–92]. I will reproduce it here.

Lemma 3.2. (Maximum Modulus Principle, global version) *Let $U \subseteq \mathbb{C}$ be a connected open set, and let f be an analytic function on U. If $z_0 \in U$ is a maximum point for $|f|$, that is $|f(z_0)| \geqslant |f(z)|$ for all $z \in U$, then f is constant on U.*

For our purpose the following corollary is more appropriate.

Corollary 3.1. *Let $U \subseteq \mathbb{C}$ be a connected open set and f be a continuous function on $\overline{U}$, analytic and non-constant on U. If $z_0 \in \overline{U}$ is a maximum for f, that is $|f(z_0)| \geqslant |f(z)|$ for all $z \in \overline{U}$, then $z_0 \in \partial U$.*

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3.2. The Riesz-Thorin Interpolation Theorem. Now we are able to proove the Riesz-Thorin Interpolation theorem without an interruption.

THEOREM 3.1. (Riesz-Thorin Interpolation Theorem) *Let (X, μ) be a measure space, (Y, ν) a σ-finite measure space and T be a linear operator defined on the set of all finitely simple functions on X and taking values in the set of measurable functions on Y. Let $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$ and assume that*

$$\|T(f)\|_{L^{q_0}} \leqslant M_0 \|f\|_{L^{p_0}} \qquad \|T(f)\|_{L^{q_1}} \leqslant M_1 \|f\|_{L^{p_1}} \tag{22}$$

holds for all finitely simple functions f on X and $0 < M_0, M_1 < \infty$. Then for all $0 < \theta < 1$ we have

$$\|T(f)\|_{L^q} \leqslant M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p} \tag{23}$$

for all finitely simple functions f on X, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \tag{24}$$

Proof. The proof heavily relies on the fact, that the L^p norm of a function can be obtained via duality for $1 \leqslant p \leqslant \infty$ (for $p = \infty$ the underlying space has to be σ-finite according to [Els11, pp. 288–289]) by

$$\|f\|_{L^p} = \sup\left\{\left|\int_Y fg d\nu\right| : \|g\|_{L^{p'}} = 1\right\}$$

with $p' := \frac{p}{p-1}$ for $p \in]1, \infty[$ and $p' := 1$ for $p = \infty$. Let $\mathcal{B}$ denote the domain of ν and define for notational simplification $\mathfrak{F} := \operatorname{span}_{\mathbb{C}}\{\chi_E : E \in \mathcal{B}, \nu(E) < \infty\}$, the set of all finitely simple functions on Y[11]. Since $\mathfrak{F}$ is dense in L^p for every $0 < p < \infty$[12], we may use the corollary found in [Bou95, p. 76]

COROLLARY 3.2. (Principle of extension of identities) *Let f, g be two continuous mappings of a topological space X into a Hausdorff space Y. If $f(x) = g(x)$ at all points of a dense subset of X, then $f \equiv g$.*

to see, that also

[11] This is almost trivial. Consider $Y_1, Y_2 \in \mathcal{B}$ with $\nu(Y_1), \nu(Y_2) < \infty$ and $Y_1 \cap Y_2 \neq \emptyset$. Then $f \equiv z_1\chi_{Y_1} + z_2\chi_{Y_2} \in \mathfrak{F}$ for $z_1, z_2 \in \mathbb{C}$. We see, that $f \equiv z_1\chi_{Y_1\setminus Y_2} + z_2\chi_{Y_2\setminus Y_1} + (z_1+z_2)\chi_{Y_1\cap Y_2} \in \mathfrak{F}$ where the latter function is a finitely simple one since $\nu(Y_1 \cup Y_2) \leqslant \nu(Y_1) + \nu(Y_2) < \infty$ and $Y_1 \setminus Y_2, Y_2 \setminus Y_1, Y_1 \cap Y_2 \subseteq Y_1 \cup Y_2$.

[12] In [Els11, p. 242] a proof can be found, that $\mathfrak{F}$ is dense in $\mathcal{L}^p$ for $0 < p < \infty$. Now the canonical map $\pi : \mathcal{L}^p \to L^p/\mathcal{N}$ is continuous. Hence we may use the following lemma.

LEMMA 3.3. *Let X and Y be topological spaces, $f : X \to Y$ and $A \subseteq X$ dense in X. Then $f(A)$ is dense in Y.*

Proof. By [Mun00, p. 104] we have $Y = f(X) = f(\overline{A}) \subseteq \overline{f(A)} \subseteq Y$. □

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$$\|f\|_{L^p} = \sup\left\{\left|\int_Y fgd\mu\right| : g \in \mathfrak{F}, \|g\|_{L^{p'}} = 1\right\}$$

First we will deal with the case $\underline{q > 1}$. Fix $f :\equiv \sum_{k=1}^n a_k e^{i\alpha_k}\chi_{X_k}$, where $n \in \mathbb{N}_{>0}, a_k > 0$, $\alpha_k \in [0, 2\pi[$, $X_i \cap X_j = \emptyset$ for $i, j = 1, \ldots, n$ and $\mu(X_k) < \infty$ for every $k = 1, \ldots, n$. Further let $g :\equiv \sum_{k=1}^m b_k e^{i\beta_k}\chi_{Y_k} \in \mathfrak{F}$, where $m \in \mathbb{N}_{>0}, b_k > 0$ and $\beta_k \in [0, 2\pi[$. Define

$$P(z) := \frac{p}{p_0}(1 - z) + \frac{p}{p_1}z \qquad Q(z) := \frac{q'}{q_0'}(1 - z) + \frac{q'}{q_1'}z$$

for $z \in \overline{S}$ (in the case $p = \infty$ we get also $p_0 = p_1 = \infty$ and hence by stipulating $\frac{\infty}{\infty} := 1$ the function P is well-defined). Further let

$$f_z :\equiv \sum_{k=1}^n a_k^{P(z)} e^{i\alpha_k}\chi_{X_k} \qquad g_z :\equiv \sum_{k=1}^m b_k^{Q(z)} e^{i\beta_k}\chi_{Y_k} \tag{25}$$

and

$$F(z) := \int_Y T(f_z)(y)g_z(y)d\nu(y) \tag{26}$$

By the linearity of the operator T we have

$$F(z) = \sum_{j=1}^n \sum_{k=1}^m a_j^{P(z)} b_j^{Q(z)} e^{i\alpha_j} e^{i\beta_k} \int_Y T(\chi_{X_j})(y)\chi_{Y_k}(y)d\nu(y) \tag{27}$$

and by using Hölder's inequality [13]

$$\begin{aligned}\left|\int_Y T(\chi_{X_j})(y)\chi_{Y_k}(y)d\nu(y)\right| &\leqslant \int_Y |T(\chi_{X_j})(y)|\chi_{Y_k}(y)d\nu(y) \\ &= \|T(\chi_{X_j})\chi_{Y_k}\|_{L^1} \\ &\leqslant \|T(\chi_{X_j})\|_{L^{q_0}}\|\chi_{Y_k}\|_{L^{q_0'}} \\ &\leqslant M_0\|\chi_{X_j}\|_{L^{p_0}}\|\chi_{Y_k}\|_{L^{q_0'}} \\ &\overset{q_0'\neq\infty}{=} M_0\|\chi_{X_j}\|_{L^{p_0}}\mu(Y_k)^{1/q_0'} \\ &< \infty\end{aligned} \tag{28}$$

for each $j = 1, \ldots, n$, $k = 1, \ldots, m$. In the case $q_0' = \infty$ we simply have $\|\chi_{Y_k}\|_{L^\infty} \leqslant 1$. Thus the function F is well-defined on $\overline{S}$. Now

[13] A proof can be found in [Els11, p. 223].

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$$\begin{aligned}
\|f_{it}\|_{L^{p_0}} &= \left(\sum_{k=1}^{n}\int_X |f_{it}|^{p_0}d\mu + \int_{X\setminus\bigcup_{k=1}^n X_k}|f_{it}|^{p_0}d\mu\right)^{1/p_0} \\
&= \left(\sum_{k=1}^{n}|a_k^{P(it)}e^{i\alpha_k}|^{p_0}\int_X \chi_{X_k}d\mu\right)^{1/p_0} \\
&= \left(\sum_{k=1}^{n} a_k^{p_0\mathrm{Re}P(it)}\mu(X_k)\right)^{1/p_0} \\
&= \left(\sum_{k=1}^{n} a_k^{p}\mu(X_k)\right)^{p/(p_0p)} \\
&= \|f\|_{L^p}^{p/p_0}
\end{aligned} \tag{29}$$

for $p, p_0 \neq \infty$. Let us consider $p_0 = \infty$, $p \neq \infty$. Then either $\|f_{it}\|_{L^\infty} = 0$ or $\|f_{it}\|_{L^\infty} = 1$. Since $\|\cdot\|_{L^p}$ is a norm for $1 \leqslant p \leqslant \infty$ (see [Els11, p. 231]), we have $f = 0$ μ-a.e. if $\|f_{it}\|_{L^\infty} = 0$. Since f is finitely simple, we may conclude $f \equiv \sum_{k=1}^n a_k e^{i\alpha_k}\chi_{X_k}$, where $\mu(X_k) = 0$ for $k = 1, \dots, n$. But then $\|f_{it}\|_{L^\infty} = \inf\{B > 0 : \mu(\{|f_{it}| > B\}) = 0\} = \inf\{B > 0 : \mu(\{1 > B\}) = 0\} = 0$ since $|a_k^{P(it)}| = \lim_{p_0\to\infty} a_k^{p/p_0} = 1$. In the other case we simply have $\|f_{it}\|_{L^\infty} = 1$ since there exists at least one subset X_k such that $\mu(X_k) \neq 0$. Now consider $p = \infty$. Then $p_0 = p_1 = \infty$. Thus $P(it) = 1$ and so $f_z \equiv f$ and the equation holds trivially. By the same considerations we see that $\|g_{it}\|_{L^{q_0'}} = \|g\|_{L^{q'}}^{q'/q_0'}$ for $q_0 \in [1, \infty]$ (set $\infty' := 1$). Hence

$$\begin{aligned}
|F(it)| &\leqslant \int_Y |T(f_{it})(y)g_{it}(y)|d\nu(y) \\
&= \|T(f_{it})g_{it}\|_{L^1} \\
&\leqslant \|T(f_{it})\|_{L^{q_0}}\|g_{it}\|_{L^{q_0'}} \\
&\leqslant M_0\|f_{it}\|_{L^{p_0}}\|g_{it}\|_{L^{q_0'}} \\
&= M_0\|f\|_{L^p}^{p/p_0}\|g\|_{L^{q'}}^{q'/q_0'} \\
&< \infty
\end{aligned} \tag{30}$$

by Hölder's inequality. By similar calculations we get

$$\|f_{1+it}\|_{L^{p_1}} = \|f\|_{L^p}^{p/p_1} \qquad \|g_{1+it}\|_{L^{q_1'}} = \|g\|_{L^{q'}}^{q'/q_1'} \tag{31}$$

and thus

$$|F(1+it)| \leqslant M_1\|f\|_{L^p}^{p/p_1}\|g\|_{L^{q'}}^{q'/q_1'} \tag{32}$$

Further

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$$
\begin{aligned}
|F(z)| &\leqslant \int_Y |T(f_z)(y)g_z(y)|d\nu(y) \\
&= \|T(f_z)g_z\|_{L^1} \\
&\leqslant \|T(f_z)\|_{L^{q_0}}\|g_z\|_{L^{q_0'}} \\
&\leqslant M_0\|f_z\|_{L^{p_0}}\|g_z\|_{L^{q_0'}} \\
&\overset{p_0,q_0'\neq\infty}{=} M_0\left(\int_X |f_z|^{p_0}d\mu\right)^{1/p_0}\left(\int_Y |g_z|^{q_0'}d\nu\right)^{1/q_0'} \\
&= M_0\left(\sum_{j=1}^n a_j^{p_0\mathrm{Re}P(z)}\mu(X_j)\right)^{1/p_0}\left(\sum_{k=1}^m b_k^{q_0'\mathrm{Re}Q(z)}\nu(Y_k)\right)^{1/q_0'} \\
&= M_0\left(\sum_{j=1}^n a_j^{p(1-\mathrm{Re}z)+(pp_0\mathrm{Re}z)/p_1}\mu(X_j)\right)^{1/p_0}\left(\sum_{k=1}^m b_k^{q'(1-\mathrm{Re}z)+(q'q_0'\mathrm{Re}z)/q_1'}\nu(Y_k)\right)^{1/q_0'} \\
&\leqslant M_0\left(\sum_{j=1}^n a_j^{p+(pp_0)/p_1}\mu(X_j)\right)^{1/p_0}\left(\sum_{k=1}^m b_k^{q'+(q'q_0')/q_1'}\nu(Y_k)\right)^{1/q_0'} \\
&= M_0\|f\|_{L^{p+(pp_0)/p_1}}^{p/p_0+p/p_1}\|g\|_{L^{q'+(q'q_0')/q_1'}}^{q'/q_0'+q'/q_1'} \\
&=: C(f,g)
\end{aligned}
$$

by Hölder's inequality and in the edge cases

$$
\begin{aligned}
p_0 = \infty, q_0' \neq \infty : &\qquad C(f,g) := M_0 \max_{j=1,\dots,n} a_j^{p/p_1}\|g\|_{L^{q'+(q'q_0')/q_1'}}^{q'/q_0'+q'/q_1'} \\
p_0 \neq \infty, q_0' = \infty : &\qquad C(f,g) := M_0\|f\|_{L^{p+(pp_0)/p_1}}^{p/p_0+p/p_1} \max_{k=1,\dots,m} b_k^{q'/q_1'} \\
p_0 = \infty, q_0' = \infty : &\qquad C(f,g) := M_0 \max_{j=1,\dots,n} a_j^{p/p_1} \max_{k=1,\dots,m} b_k^{q'/q_1'}
\end{aligned}
$$

Hence F is bounded on $\overline{S}$. It is obvious, that F is analytic on S and continuous on $\overline{S}$ (as the sum, product,quotient,composition of analytic/continuous functions). Therefore we can apply Hadamard's three lines lemma to get

$$
\begin{aligned}
|F(z)| &\leqslant \left(M_0\|f\|_{L^p}^{p/p_0}\|g\|_{L^{q'}}^{q'/q_0'}\right)^{1-\theta}\left(M_1\|f\|_{L^p}^{p/p_1}\|g\|_{L^{q'}}^{q'/q_1'}\right)^{\theta} \\
&= M_0^{1-\theta}M_1^{\theta}\|f\|_{L^p}\|g\|_{L^{q'}}
\end{aligned}
\tag{33}
$$

for $\mathrm{Re}z = \theta$ where $0 \leqslant \theta \leqslant 1$. Further observe $P(\theta) = Q(\theta) = 1$ for $0 < \theta < 1$ and thus

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$$
\begin{aligned}
\|T(f)\|_{L^q} &= \sup\left\{\left|\int_Y T(f)g d\nu\right| : g \in \mathfrak{F}, \|g\|_{L^{q'}} = 1\right\} \\
&= \sup\left\{|F(\theta)| : g \in \mathfrak{F}, \|g\|_{L^{q'}} = 1\right\} \\
&\leqslant M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p}
\end{aligned}
\tag{34}
$$

Now assume $\underline{q = 1}$. Then $q_0 = q_1 = 1$. Let $g :\equiv \sum_{k=1}^m b_k e^{i\beta_k} \chi_{Y_k}$ be a simple function (this means, that $\nu(Y_k) = \infty$ is possible for some Y_k) with $\|g\|_{L^\infty} = 1$. Define f_z as above and

$$
F(z) := \int_Y T(f_z)(y) g(y) d\nu(y)
\tag{35}
$$

Using the linearity property of T we see again, that $F(z)$ is well-defined on $\overline{S}$. Analogously we get

$$
|F(it)| \leqslant M_0 \|f\|_{L^p}^{p/p_0} \qquad |F(1+it)| \leqslant M_1 \|f\|_{L^p}^{p/p_1}
\tag{36}
$$

Again, F is bounded on $\overline{S}$ by

$$
\begin{aligned}
p_0 \neq \infty, q_0' = \infty : &\qquad C(f,g) := M_0 \|f\|_{L^{p+(pp_0)/p_1}}^{p/p_0 + p/p_1} \max_{k=1,\dots,m} b_k^{q'/q_1'} \\
p_0 = \infty, q_0' = \infty : &\qquad C(f,g) := M_0 \max_{j=1,\dots,n} a_j^{p/p_1} \max_{k=1,\dots,m} b_k^{q'/q_1'}
\end{aligned}
$$

Hadamard's three lines lemma therefore yields

$$
|F(z)| \leqslant \left(M_0 \|f\|_{L^p}^{p/p_0}\right)^{1-\theta} \left(M_1 \|f\|_{L^p}^{p/p_1}\right)^{\theta} = M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p}
\tag{37}
$$

and observing $P(\theta) = 1$ for $0 < \theta < 1$ yields

$$
\begin{aligned}
\|T(f)\|_{L^1} &= \sup\left\{\left|\int_Y T(f) g d\nu\right| : g \text{ simple}, \|g\|_{L^\infty} = 1\right\} \\
&= \sup\{|F(\theta)| : g \text{ simple}, \|g\|_{L^\infty} = 1\} \\
&\leqslant M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p}
\end{aligned}
\tag{38}
$$

This is justified by the fact, that the simple functions are dense in L^∞ (for a proof see [Coh13, p. 100]). ☐

Remark 3.1. *As you can see in the proof of the case $q = 1$, it is necessary to have $0 < \theta < 1$. Since for example choosing $q_1 = 1$ and $q_0 > 1$ arbitrary leads for $\theta = 1$ to $q = 1$ but then the function g can be choosen so, that the integral in the definition (27) diverges.*

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3.3. Young's inequality. Using the Riesz-Thorin interpolation theorem, we can give an alternative proof of Young's inequality [Gra14, pp. 22–23].

THEOREM 3.2. (Young's inequality) *Let G be a locally compact group, which is a countable union of compact subsets, and let η be a left invariant Haar measure. Let $1 \leqslant p, q, r \leqslant \infty$*

$$\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r} \tag{39}$$

Then for all $f \in L^p(G, \eta)$ and all $g \in L^r(G, \eta)$ satisfying $\|g\|_{L^r} = \|\tilde{g}\|_{L^r}$ we have $f * g$ exists η-a.e. and satisfies

$$\|f * g\|_{L^q} \leqslant \|g\|_{L^r} \|f\|_{L^p} \tag{40}$$

Proof. Fix $g \in L^r(G, \eta)$ and let $T(f) := f * g$ be defined on $L^1(G, \eta) + L^{r'}(G, \eta)$. Obviously, T is a linear operator by the linearity of the integral. By Minkowski's integral inequality (see exercise **1.1.6** [Gra14, p. 13]) we get

$$\begin{aligned} \|T(f)\|_{L^r} &= \left(\int_G \left| \int_G f(y) g(y^{-1}x) d\eta(y) \right|^r d\eta(x) \right)^{1/r} \\ &\leqslant \int_G \left(\int_G |f(y)|^r |g(y^{-1}x)|^r d\eta(x) \right)^{1/r} d\eta(y) \\ &= \int_G |f(y)| \left(\int_G |g(y^{-1}x)|^r d\eta(y^{-1}x) \right)^{1/r} d\eta(y) \\ &= \int_G |f(y)| \left(\int_G |g(z)|^r d\eta(z) \right)^{1/r} d\eta(y) \\ &\leqslant \|f\|_{L^1} \|g\|_{L^r} \end{aligned} \tag{41}$$

for $f \in L^1(g, \mu)$ and $1 \leqslant p < \infty$ (since (G, η) is σ-finite). The case $r = \infty$ follows from

$$|(f * g)(x)| = \left| \int_G f(y) g(y^{-1}x) d\eta(y) \right| \leqslant \int_G |f(y)||g(y^{-1}x)| d\eta(y) \leqslant \|g\|_{L^\infty} \|f\|_{L^1} \tag{42}$$

By stipulating $h(y) := g(y^{-1}x)$ we have

$$\begin{aligned} |(f * g)(x)| &= \left| \int_G f(y) g(y^{-1}x) d\eta(y) \right| \leqslant \int_G |f(y) g(y^{-1}x)| d\eta(y) \\ &= \|fh\|_{L^1} \leqslant \|f\|_{L^{r'}} \|h\|_{L^r} = \|f\|_{L^{r'}} \|\tilde{g}\|_{L^r} = \|g\|_{L^r} \|f\|_{L^{r'}} \end{aligned} \tag{43}$$

for $r < \infty$ and $f \in L^{r'}(g, \eta)$, since

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$$\|h\|_{L^r}^r = \int_G |g(y^{-1}x)|^r d\eta(y) = \int_G |\tilde{g}(x^{-1}y)| d\eta(y) = \|\tilde{g}\|_{L^r}^r$$

The Riesz-Thorin interpolation theorem now yields for any $0 < \theta < 1$

$$\|f * g\|_{L^q} = \|T(f)\|_{L^q} \leqslant \|g\|_{L^r}^{1-\theta} \|g\|_{L^r}^{\theta} \|f\|_{L^p} = \|g\|_{L^r} \|f\|_{L^p} \tag{44}$$

where

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{r'} \qquad \frac{1}{q} = \frac{1-\theta}{r} + \frac{\theta}{\infty}$$

and by

$$\frac{1}{p} = 1 - \frac{\theta}{r} \qquad \frac{1}{q} = \frac{1}{r} - \frac{\theta}{r}$$

we get

$$\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}$$

□

Remark 3.2. *The proof would be much shorter if we just used Minkowski's inequality* [Gra14, pp. 21–22] *instead of Minkowski's integral inequality. However, the proof given here is an alternative version of the one given already for Minkowski's inequality.*

4. Interpolation of Analytic Families of Operators.

4.1. The Poisson Formula. First, we have to extend Hadamard's three lines lemma appropriately (lemma 3.1). To do so, we first need some theorems and definitions of complex analyis.

Theorem 4.1. (Complex Analysis Revisited) *Let $h(e^{i\theta})$ be a continuous function on the unit circle. Then the* Poisson integral

$$\tilde{h}(z) = \int_{-\pi}^{\pi} h\left(e^{i\varphi}\right) P_r(\theta - \varphi) \frac{d\lambda(\varphi)}{2\pi} \qquad z := re^{i\theta} \in \mathbb{D} := \{|z| < 1\}$$

where

$$P_r(\theta) := \sum_{k \in \mathbb{Z}} r^{|k|} e^{ik\theta} \qquad 0 \leqslant r < 1, -\pi \leqslant \theta \leqslant \pi \tag{45}$$

denotes the Poisson kernel function, *is a harmonic function on $\mathbb{D}$ with boundary values $h\left(e^{i\theta}\right)$, that is, $\tilde{h}\left(e^{i\theta}\right)$ tends to $h(\zeta)$ as $z \in \mathbb{D}$ tends to $\zeta \in \partial\mathbb{D}$.*

Proof. A proof can be found in [Gam01, pp. 277–278]. □

Further we introduce the notion of a subharmonic function as found in [Gam01, p. 394].

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Definition 4.1. *Let $D \subseteq \mathbb{C}$ be a domain (open and path-connected), and let $u : D \to [-\infty, \infty[$ be continuous. We say that $u(z)$ is* subharmonic *if for each $z_0 \in D$, there is $\varepsilon > 0$ such that $u(z)$ satisfies the mean value inequality*

$$u(z_0) \leqslant \int_0^{2\pi} u\left(z_0 + re^{i\theta}\right) \frac{d\lambda(\theta)}{2\pi} \qquad 0 < r < \varepsilon \tag{46}$$

And the notion of a conformal mapping ([Gam01, p. 59]).

Definition 4.2. *A smooth complex-valued function $g(z)$ (that is, $g(z)$ has as many derivatives as is necessary for whatever is being asserted to be true) is* conformal at z_0 *if whenever γ_0, γ_1 are two curves terminating at z_0 with non-zero tangents, then the curves $g \circ \gamma_0$, $g \circ \gamma_1$ have non-zero tangents at $g(z_0)$ and the angle from $(g \circ \gamma_0)'(z_0)$ to $(g \circ \gamma_1)'(z_0)$ is the same as the angle from $\gamma_0'(z_0)$ to $\gamma_1'(z_0)$. A* conformal mapping *of one domain D onto another V is a continuously differentiable function that is conformal at each point of D and that maps D one-to-one onto V.*

Now we are able to formulate the proof of the extension of Hadamard's three lines lemma.

Lemma 4.1. (Hadamard's three lines lemma, extension) *Let F be an analytic function on the strip $S := \{z \in \mathbb{C} : 0 < \mathrm{Re} z < 1\}$ and continuous on $\overline{S}$, such that for every $z \in \overline{S}$ we have $\log|F(z)| \leqslant Ae^{\tau|\mathrm{Im} z|}$ for some $A < \infty$ and $\tau \in [0, \pi[$. Then*

$$|F(z)| \leqslant \exp\left(\frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \left[\frac{\log|F(it + iy)|}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log|F(1 + it + iy)|}{\cosh(\pi t) + \cos(\pi x)}\right] d\lambda(t)\right)$$

whenever $z := x + iy \in S$.

Proof. Consider the function

$$h(z) := \frac{1}{\pi i} \mathrm{Log}\left(i \frac{1+z}{1-z}\right) \tag{47}$$

on $D := \{z \in \mathbb{C} : |z| < 1\}$. Define $\psi(z) := i(1+z)/(1-z)$. If we write $z := x + iy \in D$, we have

$$\psi(z) = \frac{-2y}{(1-x)^2 + y^2} + i \frac{1 - x^2 - y^2}{(1-x)^2 + y^2} \tag{48}$$

Hence $\mathrm{Im} h(z) > 0$. Stipulating $x := 1 - y$ for y satisfying $y^2 < y$, we get

$$\lim_{y^2 < y, y \to 0^+} \mathrm{Im}\psi(z) = \lim_{y^2 < y, y \to 0^+} \left(\frac{1}{y} - 1\right) = \infty \tag{49}$$

using the same definition of x we get

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$$\lim_{y^2<y, y\to 0^+} \mathrm{Re}\psi(z) = -\lim_{y^2<y, y\to 0^+} \frac{1}{y} = -\infty \tag{50}$$

and by stipulating $x := 1 + y$

$$\lim_{y^2<-y, y\to 0^-} \mathrm{Re}\psi(z) = -\lim_{y^2<-y, y\to 0^-} \frac{1}{y} = \infty \tag{51}$$

Since $2i \neq 0$, ψ is a linear fractional transformation with

$$\psi^{-1}(z) = \frac{z-i}{z+i} \tag{52}$$

Therefore ψ maps the unit circle onto the upper half plane. The principal value of $\log z$ is $\mathrm{Log} z$ defined by

$$\mathrm{Log} z := \log|z| + i\mathrm{Arg} z, \qquad z \neq 0 \tag{53}$$

where $-\pi < \mathrm{Arg} z \leqslant \pi$ is the principal value of the argument of $z \neq 0$ (that is the angle θ in the polar representation $z = r(\cos\theta + i\sin\theta)$). We see that $\pi i h(z)$ maps the upper half plane onto the strip $\mathbb{R}\times]0,\pi[$. Thus $h(z)$ maps the unit circle D onto the strip $]0,1[\times\mathbb{R}$ (the division of h by i is equivalent to multiplying h by i which corresponds to a clockwise rotation of $\pi/2$, and we scale with $1/\pi$). Since h is a sum, product, quotient and composition of analytic functions, by [Rud87, pp. 278–279] and

$$h'(z) = \frac{2}{\pi i}\frac{1}{z-1} \neq 0 \tag{54}$$

on D, h is a conformal map from D onto S.

□

4.2. Stein's Theorem on Interpolation of Analytic Families of Operators.

Definition 4.3. (Analytic family, admissible growth) *Let (X,μ), (Y,ν) be measure spaces and $(T_z)_{z\in\overline{S}}$, where T_z is defined on the space of all finitely simple functions on X and taking values in the space of all measurable functions on Y such that*

$$\int_Y |T_z(\chi_A)\chi_B| d\nu \tag{55}$$

whenever $\mu(A), \nu(B) < \infty$. The family $(T_z)_{z\in\overline{S}}$ is said to be analytic *if for all f, g finitely simple we have that*

$$z \mapsto \int_Y T_z(f) g d\nu \tag{56}$$

is analytic on S and continuous on $\overline{S}$. Further, an analytic family $(T_z)_{z\in\overline{S}}$ is called of admissible growth, *if there is a constant $\tau \in [0,\pi[$, such that for all finitely simple functions f, g a constant $C(f,g)$ exists with*

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$$\log\left|\int_Y T_z(f)gd\nu\right| \leqslant C(f,g)e^{\tau|\mathrm{Im}z|} \tag{57}$$

for all $z \in \overline{S}$.

THEOREM 4.2. (Riesz-Thorin interpolation theorem, extension) *Let* $(T_z)_{z\in\overline{S}}$ *be an analytic family of admissible growth,* $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$ *and suppose that* M_0, M_1 *are positive functions on the real line such that for some* $\tau \in [0, \pi[$

$$\sup\left\{e^{-\tau|y|}\log M_0(y) : y \in \mathbb{R}\right\} < \infty \qquad \sup\left\{e^{-\tau|y|}\log M_1(y) : y \in \mathbb{R}\right\} < \infty \tag{58}$$

Fix $0 < \theta < 1$ *and define*

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \tag{59}$$

Further suppose that for all finitely simple functions f *on* X *and* $y \in \mathbb{R}$ *we have*

$$\|T_{iy}(y)\|_{L^{q_0}} \leqslant M_0(y)\|f\|_{L^{p_0}} \qquad \|T_{1+iy}(y)\|_{L^{q_1}} \leqslant M_1(y)\|f\|_{L^{p_1}} \tag{60}$$

Then for all finitely simple functions f *on* X *we have*

$$\|T_\theta(f)\|_{L^q} \leqslant M(\theta)\|f\|_{L^p}$$

where for $0 < x < 1$

$$M(x) = \exp\left(\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log M_0(t)}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log M_1(t)}{\cosh(\pi t) + \cos(\pi x)}\right]d\lambda(t)\right)$$

Proof. Fix $0 < \theta < 1$ and finitely simple functions f, g on X, Y respectively with $\|f\|_{L^p} = \|g\|_{L^{q'}} = 1$. Define f_z, g_z as in (25) and for $z \in \overline{S}$

$$F(z) := \int_Y T_z(f_z)g_z d\nu \tag{61}$$

Observe, that $|a_j^{P(z)}| \leqslant a_j^{p/p_0+p/p_1}$ and $|b_k^{Q(z)}| \leqslant b_k^{q'/q_0'+q'/q_1'}$ for $z \in \overline{S}$. Hence

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$$
\begin{aligned}
\log|F(z)| &= \log\left|\sum_{j=1}^{n}\sum_{k=1}^{m} a_j^{P(z)} b_j^{Q(z)} e^{i\alpha_j} e^{i\beta_k} \int_Y T_z(\chi_{X_j})(y)\chi_{Y_k}(y)d\nu(y)\right| \\
&\leqslant \log\left(\sum_{j=1}^{n}\sum_{k=1}^{m} |a_j^{P(z)}||b_j^{Q(z)}| \int_Y |T_z(\chi_{X_j})(y)|\chi_{Y_k}(y)d\nu(y)\right) \\
&\leqslant \log\left(\sum_{j=1}^{n}\sum_{k=1}^{m} a_j^{p/p_0+p/p_1} b_k^{q'/q_0'+q'/q_1'} \int_{Y_k} |T_z(\chi_{X_j})|d\nu\right)
\end{aligned}
\tag{62}
$$

$\square$

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