CLASSICAL FOURIER ANALYSIS: INTERPOLATION OF L^p SPACES

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Abstract. In this written seminar work I will basically follow the section *Interpolation* in the book *Classical Fourier Analysis, third Edition* by Loukas Grafakos. I will review three basic but important theorems on interpolation of operators on L^p spaces, namely the *Marcinkiewicz Interpolation Theorem*, the *Riesz-Thorin Interpolation Theorem* and finally an extension of the Riesz-Thorin Interpolation Theorem to analytic families of operators (the so-called *Stein's theorem on interpolation of analytic families of operators*). We are mainly concerned with the notion of linear operators as well as slight generalizations of them.

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1. Introduction and Basic Definitions. Suppose $(p_0, q_0), (p_1, q_1) \in [1, \infty] \times [1, \infty]$ are two pairs of indices and assume that the estimates

$$\|T(f)\|_{L^{q_0}} \leq M_0 \|f\|_{L^{p_0}} \quad \text{and} \quad \|T(f)\|_{L^{q_1}} \leq M_1 \|f\|_{L^{p_1}}$$

hold, where T is an appropriately choosen operator. Does this imply that

$$\|T(f)\|_{L^q} \leq M \|f\|_{L^p} \quad \text{for other pairs } (p, q) \in [1, \infty]?$$

Those and similar questions will be answered by a tool called *interpolation*, in our case interpolation of L^p spaces. Using interpolation it is possible to reduce difficult estimates to endpoint estimates and so interpolation can (but not always does) simplify matters. Among the numerous applications of interpolation is by far the shortest proof of *Young's inequality for convolutions* [Gra14, pp. 22–23]. There is not *the* interpolation theorem, merely a family of theorems which can be roughly divided into two main categories: *real* and *complex* interpolation methods. Real methods use so called *cut-off* functions to divide the functions in the domain of the operator T into a bounded and unbounded part and then establish bounds on each of those parts whereas complex interpolation theorems are based upon standard results in complex analysis and are more restrictive on the operator T in question but yield more natural bounds (even continuous estimates) and will therefore be considered in this task. First we need a rigorous idea of what an appropriately choosen operator means in the context of Lebesgue spaces.

Definition 1.1. *Let (X, μ) and (Y, ν) be measure spaces. Further let T be an operator defined on a linear space of complex-valued measurable functions on X and taking values in the set of all complex-valued, finite almost everywhere, measurable functions on Y. Then T is called* linear *if for all functions f and g in the domain of T and all $z \in \mathbb{C}$ holds*

$$T(f + g) = T(f) + T(g) \qquad T(zf) = zT(f) \tag{1}$$

and quasi-linear *if*

$$|T(f + g)| \leqslant K\left(|T(f)| + |T(g)|\right) \qquad |T(zf)| = |z|\,|T(f)| \tag{2}$$

holds for some real constant $K > 0$. If $K = 1$, T is called sublinear.

2. The Complex Method. This theorem will unfortunately only be applicable to linear operators but will yield a more natural bound of the operator on the intermediate space. The proof will make strong use of complex variables technique. A major tool will be an application of the maximum modulus principle, known as *Hadamard's three lines lemma*.

2.1. Hadamard's Three Lines Lemma. The proof of the Riesz-Thorin interpolation theorem heavily relies on Hadamard's three lines lemma which is itself based on a restatement of the maximum modulus theorem (see [Rud87, p. 212]) found in [Rud87, p. 253]. To do so, we have first to establish some common terminology. A complex-valued function f is said to be *holomorphic* in $\Omega \subseteq \mathbb{C}$ open, if $f'(z)$ exists for any $z \in \Omega$. By a region

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we shall mean a nonempty connected open subset of the complex plane. The restatement reads as follows.

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a bounded region and f be a continuous function on $\overline{\Omega}$ which is holomorphic in Ω. Then*

$$\left|f(z)\right| \leq \sup\left\{\left|f(z)\right| : z \in \partial\Omega\right\}$$

for every $z \in \Omega$. If equality holds at one point $z \in \Omega$, then f is constant.

Lemma 2.1. (Hadamard's three lines lemma) *Let F be a holomorphic function in the strip $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$, continuous and bounded on $\overline{S}$, such that $\left|F(z)\right| \leq B_0$ when $\operatorname{Re} z = 0$ and $\left|F(z)\right| \leq B_1$ when $\operatorname{Re} z = 1$, for some $0 < B_0, B_1 < \infty$. Then $\left|F(z)\right| \leq B_0^{1-\theta} B_1^{\theta}$ when $\operatorname{Re} z = \theta$, for any $0 < \theta < 1$.*

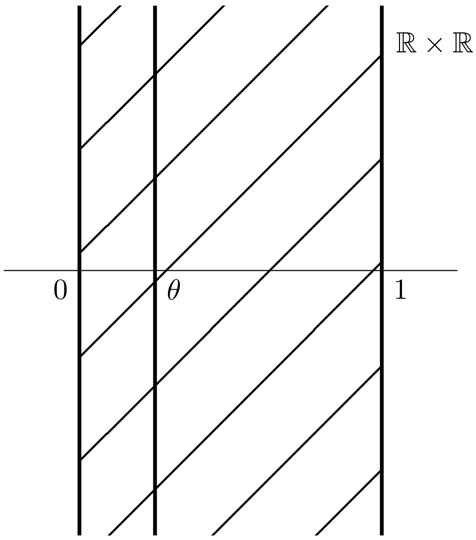

Figure 1. Sketch of the setting of Hadamard's three lines lemma.

Proof. For $z \in \overline{S}$ define

$$G(z) := \frac{F(z)}{B_0^{1-z} B_1^{z}} \qquad G_n(z) := G(z) e^{(z^2-1)/n}, \ n \in \mathbb{N}_{>0}$$

$G(z)$ and $G_n(z)$ are holomorphic in S by

$$G'(z) = \frac{F'(z) - F(z)\log\left(B_1/B_0\right)}{B_0^{1-z} B_1^{z}} \qquad G_n'(z) = G'(z) e^{(z^2-1)/n} + \frac{2}{n} z G_n(z)$$

and $e^z \neq 0$ for every $z \in \mathbb{C}$. Further, we have

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$$\left|B_0^{1-z}B_1^z\right| = \left(B_0^{1-z}B_0^{1-\overline{z}}B_1^zB_1^{\overline{z}}\right)^{1/2} = B_0^{1-\operatorname{Re} z}B_1^{\operatorname{Re} z}$$

Consider $0 \leq \operatorname{Re} z \leq 1$ and $B_0 \geq 1$. Then $B_0^{1-\operatorname{Re} z} \geq 1$ and $B_0^{1-\operatorname{Re} z} \geq B_0$ in the case $B_0 < 1$. Similarily, $B_1^{\operatorname{Re} z} \geq 1$ if $B_1 \geq 1$ and $B_1^{\operatorname{Re} z} \leq B_1$ if $B_1 < 1$. Hence

$$\left|B_0^{1-z}B_1^z\right| \geq \min\{1, B_0\}\min\{1, B_1\} > 0 \tag{3}$$

for all $z \in \overline{S}$. Since F is bounded on $\overline{S}$, we have $\left|F(z)\right| \leq L$ for some $L > 0$ and all $z \in \overline{S}$. Thus by (3)

$$\left|G(z)\right| = \frac{\left|F(z)\right|}{\left|B_0^{1-z}B_1^z\right|} \leq \frac{L}{\min\{1, B_0\}\min\{1, B_1\}} =: M$$

for every $z \in \overline{S}$. Fix $n \in \mathbb{N}_{>0}$ and write $z := x + iy \in \overline{S}$. Then

$$\left|G_n(z)\right| \leq M\left(e^{\left(x^2+2ixy-y^2-1\right)/n}e^{\left(x^2-2ixy-y^2-1\right)/n}\right)^{1/2} = Me^{-y^2/n}e^{\left(x^2-1\right)/n} \leq Me^{-y^2/n}$$

for $0 \leq x \leq 1$. Thus

$$\lim_{y\to\pm\infty}\sup\{\left|G_n(z)\right| : 0 \leq x \leq 1\} = 0$$

by the pinching-principle. Hence there exist $C_0(n), C_1(n) \in \mathbb{R}$, such that

$$\sup\{\left|G_n(z)\right| : 0 \leq x \leq 1\} \leq 1$$

when $y > C_0(n)$ or $y < C_1(n)$. Letting

$$C(n) := \max\left\{\left|C_0(n)\right| + 1, \left|C_1(n)\right| + 1\right\}$$

we conclude $\left|G_n(z)\right| \leq 1$ for all $0 \leq x \leq 1$ when $|y| \geq C(n)$. Now consider the rectangle $R := (0,1) \times \big(-C(n), C(n)\big)$. We have $\left|G_n(z)\right| \leq 1$ on the lines $[0,1] \times \{\pm C(n)\}$. By

$$\left|G_n(iy)\right| = \frac{\left|F(iy)\right|}{\left|B_0^{1-iy}B_1^{iy}\right|}e^{-\left(y^2+1\right)/n} \leq 1 \qquad \left|G_n(1+iy)\right| = \frac{\left|F(1+iy)\right|}{\left|B_0^{-iy}B_1^{1+iy}\right|}e^{-y^2/n} \leq 1$$

we have $\left|G_n(z)\right| \leq 1$ on the lines $\{0\}\times[-C(n), C(n)]$, $\{1\}\times[-C(n), C(n)]$. Thus $\left|G_n(z)\right| \leq 1$ on ∂R. Since $\left|G_n(z)\right|$ is continuous on $\overline{R}$, holomorphic in R and R is a bounded region, the maximum modulus theorem implies

$$\left|G_n(z)\right| \leq \sup\left\{\left|G_n(z)\right| : z \in \partial R\right\} \leq 1$$

for every $z \in R$. Therefore $\left|G_n(z)\right| \leq 1$ on $\overline{R}$ and so $\left|G_n(z)\right| \leq 1$ on $\overline{S}$. Since inequalities are preserved by limits and the modulus is a continuous function, we have that $\left|G(z)\right| = \lim_{n\to\infty}\left|G_n(z)\right| \leq 1$ for $z \in \overline{S}$. We conclude by

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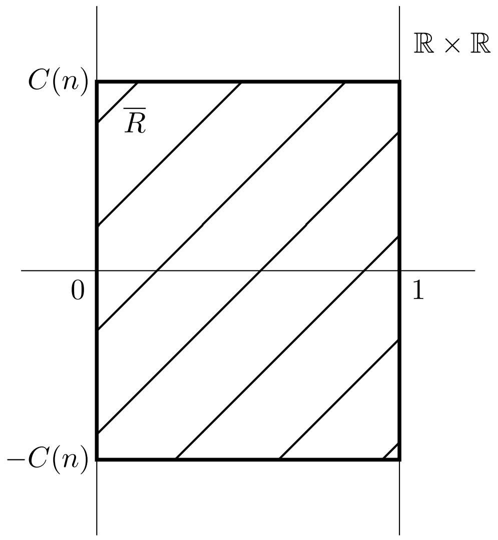

FIGURE 2. Sketch of the rectangle $\overline{R}$.

$$\left|F(\theta + it)\right| = \left|G(\theta + it)\right| |B_0^{1-\theta-it} B_1^{\theta+it}| \leq B_0^{1-\theta} B_1^{\theta}$$

whenever $0 < \theta < 1$, $t \in \mathbb{R}$. $\square$

2.2. The Riesz-Thorin Interpolation Theorem. For two measure spaces (X, μ), (Y, ν) let Σ_X and Σ_Y denote the set of all finitely simple functions on X, Y respectively.

THEOREM 2.1. (Riesz-Thorin interpolation theorem) *Suppose that (X, μ), (Y, ν) are measure spaces and $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. If $q_0 = q_1 = \infty$, suppose also that ν is semifinite. Let T be a linear operator defined on Σ_X and taking values in the set of measurable functions on Y, such that for some $0 < M_0, M_1 < \infty$ the estimates*

$$\left\|T(f)\right\|_{L^{q_0}} \leq M_0 \left\|f\right\|_{L^{p_0}} \quad \text{and} \quad \left\|T(f)\right\|_{L^{q_1}} \leq M_1 \left\|f\right\|_{L^{p_1}} \tag{4}$$

hold for all $f \in \Sigma_X$. Then for all $0 \leq \theta \leq 1$ we have

$$\left\|T(f)\right\|_{L^q} \leq M_0^{1-\theta} M_1^{\theta} \left\|f\right\|_{L^p} \tag{5}$$

for all $f \in \Sigma_X$, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Proof. The idea is to bound the quantity (see [Fol99, p. 189])

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$$M_q\left(T(f)\right) = \sup\left\{\left|\int_Y T(f)g\,\mathrm{d}\nu\right| : g \in \Sigma_Y, \|g\|_{L^{q'}} = 1\right\}$$

appropriately. If either $\theta = 0$ or $\theta = 1$, the estimate (5) follows directly from the hypotheses (4) on T. Thus we may assume $\underline{0 < \theta < 1}$. Furthermore, if $f \in \Sigma_X$, $\|f\|_{L^p} = 0$, then $f = 0$ μ-a.e. and either one of the hypotheses on T in (4) implies $T(f) = 0$ μ-a.e. and thus the estimate (5) holds trivially. Therefore we can assume $\underline{\|f\|_{L^p} \neq 0}$. Fix $f \in \Sigma_X$, $g \in \Sigma_Y$ with representation

$$f = \sum_{j=1}^{n} a_j e^{i\alpha_j}\chi_{A_j} \qquad g = \sum_{k=1}^{m} b_k e^{i\beta_k}\chi_{B_k}$$

where $a_j, b_k \neq 0$, $\alpha_j, \beta_k \in \mathbb{R}$ for any $j = 1, \dots, n$, $k = 1, \dots, m$, the sets A_j and B_k are each pairwise disjoint with $\mu\left(A_j\right), \nu\left(B_k\right) < \infty$ and so, that $\|g\|_{L^{q'}} \neq 0$ (recall $q' := q/\left(q-1\right)$). Define

$$P(z) := \frac{p}{p_0}(1-z) + \frac{p}{p_1}z \qquad Q(z) := \frac{q'}{q'_0}(1-z) + \frac{q'}{q'_1}z$$

for $z \in \mathbb{C}$ (since either $p = \infty$ implies $p_0 = p_1 = \infty$ or $q = 1$ implies $q_0 = q_1 = 1$, the functions P, Q are well-defined). Further let

$$f_z := \sum_{j=1}^{n} a_j^{P(z)} e^{i\alpha_j}\chi_{A_j} \qquad g_z := \sum_{k=1}^{m} b_k^{Q(z)} e^{i\beta_k}\chi_{B_k} \tag{6}$$

and

$$F(z) := \int_Y T(f_z)g_z\,\mathrm{d}\nu \tag{7}$$

By (6), (7) and the linearity of the operator T we have

$$F(z) = \sum_{j=1}^{n}\sum_{k=1}^{m} a_j^{P(z)} b_k^{Q(z)} e^{i\alpha_j} e^{i\beta_k} \int_Y T(\chi_{A_j})\chi_{B_k}\,\mathrm{d}\nu$$

Applying Hölder's inequality yields

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$$\begin{aligned}\left|\int_Y T(\chi_{A_j})\chi_{B_k}\,\mathrm{d}\nu\right| &\leq \int_Y \left|T(\chi_{A_j})\chi_{B_k}\right|\mathrm{d}\nu \\ &= \left\|T(\chi_{A_j})\chi_{B_k}\right\|_{L^1} \\ &\leq \left\|T(\chi_{A_j})\right\|_{L^{q_0}} \left\|\chi_{B_k}\right\|_{L^{q_0'}} \\ &\leq M_0 \left\|\chi_{A_j}\right\|_{L^{p_0}} \left\|\chi_{B_k}\right\|_{L^{q_0'}} \\ &\leq M_0 \mu\left(A_j\right)^{1/p_0} \nu\left(B_k\right)^{1/q_0'}\end{aligned} \tag{8}$$

for each $j = 1, \ldots, n$, $k = 1, \ldots, m$ (even in the cases where either $p_0 = \infty$ or $q_0' = \infty$, or both, by observing that $\|\chi_A\|_{L^\infty} \leq 1$ for any measurable set A). Thus the function F is well-defined on $\mathbb{C}$. Let $t \in \mathbb{R}$. For $p, p_0 \neq \infty$

$$\begin{aligned}\|f_{it}\|_{L^{p_0}} &= \left(\sum_{j=1}^n \int_{A_j} |f_{it}|^{p_0}\,\mathrm{d}\mu + \int_{X\setminus\bigcup_{j=1}^n A_j} |f_{it}|^{p_0}\,\mathrm{d}\mu\right)^{1/p_0} \\ &= \left(\sum_{j=1}^n \left|a_j^{P(it)}e^{i\alpha_j}\right|^{p_0} \int_X \chi_{A_j}\,\mathrm{d}\mu\right)^{1/p_0} \\ &= \left(\sum_{j=1}^n a_j^{p_0 \operatorname{Re} P(it)} \mu\left(A_j\right)\right)^{1/p_0} \\ &= \left(\sum_{j=1}^n a_j^p \mu\left(A_j\right)\right)^{p/(p_0 p)} \\ &= \|f\|_{L^p}^{p/p_0}\end{aligned}$$

holds. Let $p_0 = \infty$, $p \neq \infty$. Then $\|f_{it}\|_{L^\infty} = 1$ since $\left|a_j^{P(it)}\right| = a_j^{p/p_0} = 1$ and that there exists some index j, such that $\mu\left(A_j\right) \neq 0$. If $p = \infty$, then $p_0 = p_1 = \infty$ and thus $P(it) = 1$. By the same considerations we have $\|g_{it}\|_{L^{q_0'}} = \|g\|_{L^{q'}}^{q'/q_0'}$. Hence

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$$\begin{aligned}
\left|F(it)\right| &\leq \int_Y \left|T(f_{it})g_{it}\right| \mathrm{d}\nu \\
&= \left\|T(f_{it})g_{it}\right\|_{L^1} \\
&\leq \left\|T(f_{it})\right\|_{L^{q_0}} \left\|g_{it}\right\|_{L^{q_0'}} \\
&\leq M_0 \left\|f_{it}\right\|_{L^{p_0}} \left\|g_{it}\right\|_{L^{q_0'}} \\
&= M_0 \left\|f\right\|_{L^p}^{p/p_0} \left\|g\right\|_{L^{q'}}^{q'/q_0'}
\end{aligned}$$

by Hölder's inequality. In an analogous manner we derive

$$\left\|f_{1+it}\right\|_{L^{p_1}} = \left\|f\right\|_{L^p}^{p/p_1} \qquad \left\|g_{1+it}\right\|_{L^{q_1'}} = \left\|g\right\|_{L^{q'}}^{q'/q_1'}$$

and thus

$$\left|F(1+it)\right| \leq M_1 \left\|f\right\|_{L^p}^{p/p_1} \left\|g\right\|_{L^{q'}}^{q'/q_1'}$$

Further by estimate (8)

$$\begin{aligned}
\left|F(z)\right| &\leq \sum_{j=1}^n \sum_{k=1}^m \left|a_j^{P(z)}\right| \left|b_k^{Q(z)}\right| \left|\int_Y T(\chi_{A_j})\chi_{B_k} \,\mathrm{d}\nu\right| \\
&\leq M_0 \sum_{j=1}^n \sum_{k=1}^m a_j^{\operatorname{Re} P(z)} b_k^{\operatorname{Re} Q(z)} \mu\left(A_j\right)^{1/p_0} \nu\left(B_k\right)^{1/q_0'} \\
&\leq M_0 \sum_{j=1}^n \sum_{k=1}^m \max\left\{1, a_j^{p/p_0 + p/p_1}\right\} \max\left\{1, b_k^{q'/q_0' + q'/q_1'}\right\} \mu\left(A_j\right)^{1/p_0} \nu\left(B_k\right)^{1/q_0'}
\end{aligned}$$

Hence F is bounded on $\overline{S}$ by some constant depending on f and g only. By

$$\begin{aligned}
F'(z) = \sum_{j=1}^n \sum_{k=1}^m a_j^{P(z)} \log\left(a_j\right) \left(\frac{p}{p_1} - \frac{p}{p_0}\right) b_k^{Q(z)} e^{i\alpha_j} e^{i\beta_k} \int_Y T(\chi_{A_j})\chi_{B_k} \,\mathrm{d}\nu \\
+ \sum_{j=1}^n \sum_{k=1}^m a_j^{P(z)} b_k^{Q(z)} \log\left(b_k\right) \left(\frac{q'}{q_1'} - \frac{q'}{q_0'}\right) e^{i\alpha_j} e^{i\beta_k} \int_Y T(\chi_{A_j})\chi_{B_k} \,\mathrm{d}\nu
\end{aligned}$$

it is immediate, that F is an entire function and thus holomorphic in S and continuous on $\overline{S}$. Therefore, Hadamard's three lines lemma yields

$$\left|F(z)\right| \leq \left(M_0 \left\|f\right\|_{L^p}^{p/p_0} \left\|g\right\|_{L^{q'}}^{q'/q_0'}\right)^{1-\theta} \left(M_1 \left\|f\right\|_{L^p}^{p/p_1} \left\|g\right\|_{L^{q'}}^{q'/q_1'}\right)^{\theta} = M_0^{1-\theta} M_1^{\theta} \left\|f\right\|_{L^p} \left\|g\right\|_{L^{q'}}$$

for $\operatorname{Re} z = \theta$, $0 < \theta < 1$. We have

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$$\{T(f) \neq 0\} = \bigcup_{n=1}^{\infty} \{|T(f)| > 1/n\}$$

and by Chebychev's inequality either

$$\nu\Big(\big\{|T(f)| > 1/n\big\}\Big) \leq n^{q_0} \left\|T(f)\right\|_{L^{q_0}}^{q_0} \leq n^{q_0} M_0^{q_0} \left\|f\right\|_{L^{p_0}}^{q_0}$$

or

$$\nu\Big(\big\{|T(f)| > 1/n\big\}\Big) \leq n^{q_1} \left\|T(f)\right\|_{L^{q_1}}^{q_1} \leq n^{q_1} M_1^{q_1} \left\|f\right\|_{L^{p_1}}^{q_1}$$

whenever $q_0 \neq \infty$ or $q_1 \neq \infty$. Therefore, the set $\big\{T(f) \neq 0\big\}$ is σ-finite unless $q_0 = q_1 = \infty$. Further we have $P(\theta) = Q(\theta) = 1$. Thus by

$$\begin{aligned} M_q\left(T(f)\right) &= \sup\left\{\left|\int_Y T(f) g \,\mathrm{d}\nu\right| : g \in \Sigma_Y, \|g\|_{L^{q'}} = 1\right\} \\ &= \sup\left\{|F(\theta)| : g \in \Sigma_Y, \|g\|_{L^{q'}} = 1\right\} \\ &\leq M_0^{1-\theta} M_1^{\theta} \left\|f\right\|_{L^p} \end{aligned}$$

we conclude

$$\left\|T(f)\right\|_{L^q} = M_q\left(T(f)\right) \leq M_0^{1-\theta} M_1^{\theta} \left\|f\right\|_{L^p}$$

for any $f \in \Sigma_X$. □

Remark 2.1. *A more general version of the Riesz-Thorin interpolation theorem can be found in* [Fol99, pp. 200–202]. *There, a linear map $T : L^{p_0} + L^{p_1} \to L^{q_0} + L^{q_1}$ is considered. This follows using a density argument from the current version of the theorem and will not be prooven here.*

Remark 2.2. *Using the previous remark, a standard application of the Riesz-Thorin interpolation theorem is to prove Young's inequality for convolutions* [Gra14, pp. 22–23]. *Let G be a locally compact group and λ be a left invariant Haar measure on G, furthermore we assume that G is a countable union of compact subsets, hence the pair (G, λ) forms a σ-finite measure space.*

Theorem. (Young's inequality) *Let $1 \leq p, q, r \leq \infty$ satisfy*

$$\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r} \tag{9}$$

Then for all $f \in L^p(G)$, $g \in L^r(G)$ satisfying $\|g\|_{L^r} = \|\tilde{g}\|$ we have $f * g$ exists λ-a.e. and satisfies

$$\|f * g\|_{L^1} \leq \|g\|_{L^r} \|f\|_{L^p}$$

Proof. Fix $g \in L^r(G)$ and let $T(f) := f * g$ be defined on $L^1(G) + L^{r'}(G)$. Obviously, T is a linear operator by the linearity of the integral. By Minkowski's integral inequality (see exercise **1.1.6** [Gra14, p. 13]) we get

$$\begin{aligned}
\|T(f)\|_{L^r} &= \left(\int_G \left|\int_G f(y)g(y^{-1}x)d\lambda(y)\right|^r d\lambda(x)\right)^{1/r} \\
&\leq \int_G \left(\int_G |f(y)|^r|g(y^{-1}x)|^r d\lambda(x)\right)^{1/r} d\lambda(y) \\
&= \int_G |f(y)| \left(\int_G |g(y^{-1}x)|^r d\lambda(y^{-1}x)\right)^{1/r} d\lambda(y) \\
&= \int_G |f(y)| \left(\int_G |g(z)|^r d\lambda(z)\right)^{1/r} d\lambda(y) \\
&\leq \|f\|_{L^1}\|g\|_{L^r}
\end{aligned}$$

for $f \in L^1(g, \mu)$ and $1 \leqslant p < \infty$. The case $r = \infty$ follows from

$$|(f * g)(x)| = \left|\int_G f(y)g(y^{-1}x)d\lambda(y)\right| \leq \int_G |f(y)||g(y^{-1}x)|d\lambda(y) \leq \|g\|_{L^\infty}\|f\|_{L^1}$$

By stipulating $h(y) := g(y^{-1}x)$ we have

$$\begin{aligned}
\big|(f * g)(x)\big| &= \left|\int_G f(y)g(y^{-1}x)d\lambda(y)\right| \leq \int_G |f(y)g(y^{-1}x)|d\lambda(y) \\
&= \|fh\|_{L^1} \leq \|f\|_{L^{r'}}\|h\|_{L^r} = \|f\|_{L^{r'}}\|\tilde{g}\|_{L^r} = \|g\|_{L^r}\|f\|_{L^{r'}}
\end{aligned}$$

for $r < \infty$ and $f \in L^{r'}(G)$, since

$$\|h\|_{L^r}^r = \int_G |g(y^{-1}x)|^r d\lambda(y) = \int_G |\tilde{g}(x^{-1}y)|d\lambda(y) = \|\tilde{g}\|_{L^r}^r$$

The Riesz-Thorin interpolation theorem now yields for any $0 < \theta < 1$

$$\|f * g\|_{L^q} = \big\|T(f)\big\|_{L^q} \leqslant \|g\|_{L^r}^{1-\theta}\|g\|_{L^r}^{\theta}\|f\|_{L^p} = \|g\|_{L^r}\|f\|_{L^p} \tag{10}$$

where

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{r'} \qquad \frac{1}{q} = \frac{1-\theta}{r} + \frac{\theta}{\infty}$$

$\square$

REMARK 2.3. *The proof would be much shorter if we just used Minkowski's inequality* [Gra14, pp. 21–22] *instead of Minkowski's integral inequality. However, the proof given here is an alternative version of the one given already for Minkowski's inequality.*

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3. Interpolation of Analytic Families of Operators. The generalization of the classical Riesz-Thorin interpolation theorem to analytic families of operators is due to *E. M. Stein* and *Guido Weiss*[1]. Crucial for its proof is again an applicatio of advanced topics in complex analysis.

3.1. Extension of Hadamard's Three Lines Lemma. This lemma is inspired by a lemma originally proposed by I.I.Hirschman. I will stick for the most part to the proof given in [Gra14, pp. 43–45], but for some parts I will use the paper by Stein and Weiss.

3.1.1. Auxiliary Lemmata. To shorten the proof of the extension of Hadamard's three lines lemma, I will summarize the most important facts used during the proof.

Lemma 3.1. *Let $D := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc and*

$$h(z) := \frac{1}{\pi i} \log\left(i\frac{1+z}{1-z}\right)$$

for $z \in \overline{D} \setminus \{\pm 1\}$ where we are taking that continuous branch of $\log z$ in the complex plane slit along the negative imaginary axis, $\mathbb{C} \setminus \big(\{0\} \times [0, \infty)\big)$. Then h is a holomorphic function in D which maps $\overline{D} \setminus \{\pm 1\}$ bijectively onto the closure $\overline{S}$ of the strip $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$.

Proof. Define $f(z) := i\frac{1+z}{1-z}$. If we write $z := x + iy \in \overline{D} \setminus \{\pm 1\}$, we have

$$f(z) = \frac{-2y}{(1-x)^2 + y^2} + i\frac{1 - x^2 - y^2}{(1-x)^2 + y^2} \tag{11}$$

Hence $\operatorname{Im} f(z) \geq 0$ on $\overline{D} \setminus \{\pm 1\}$. Stipulating $x := 1 - y$ for y satisfying $y^2 < y$, we get

$$\lim_{y^2 < y, y \to 0^+} \operatorname{Im} f(z) = \lim_{y^2 < y, y \to 0^+} \left(\frac{1}{y} - 1\right) = \infty$$

using the same definition of x we get

$$\lim_{y^2 < y, y \to 0^+} \operatorname{Re} f(z) = -\lim_{y^2 < y, y \to 0^+} \frac{1}{y} = -\infty$$

and by stipulating $x := 1 + y$

$$\lim_{y^2 < -y, y \to 0^-} \operatorname{Re} f(z) = -\lim_{y^2 < -y, y \to 0^-} \frac{1}{y} = \infty$$

Since $2i \neq 0$, f is a linear fractional transformation (see [Rud87, p. 279]) with

$$f^{-1}(z) = \frac{z - i}{z + i}$$

[1] https://projecteuclid.org/euclid.tmj/1178244785, last accessed October 29, 2016.

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Therefore f maps $\overline{D} \setminus \{\pm 1\}$ onto the punctured closed upper half plane $\{z \in \mathbb{C} : \operatorname{Im} z \geq 0\} \setminus \{0\}$. The preceeding logarithm maps this upper half plane onto the strip $\{z \in \mathbb{C} : 0 \leq \operatorname{Im} z \leq \pi\}$. Thus $h(z)$ maps $\overline{D} \setminus \{\pm 1\}$ onto the strip $\overline{S}$. By

$$h'(z) = \frac{2}{\pi i} \frac{1}{1-z} \tag{12}$$

we see that h is a holomorphic function in D. Furthermore, we have

$$h^{-1}(z) = \frac{e^{\pi i z} - i}{e^{\pi i z} + i}$$

□

Lemma. *Let X be a topological space. A function $f : X \to [-\infty, \infty)$ is upper semicontinuous if and only if for all $\alpha \in \mathbb{R}$ the set $f^{-1}\big([-\infty, \alpha)\big)$ is open.*

Proof. Suppose $f : X \to [-\infty, \infty)$ is upper semicontinuous and fix $\alpha \in \mathbb{R}$. We have that

$$f^{-1}\big([-\infty, \alpha)\big) = \bigcup_{x \in \{f < \alpha\}} U_x$$

where U_x is a neighbourhood of x such that $f < \alpha$ for any element in U_x. Conversly, for $x_0 \in X$ and $\alpha > f(x_0)$ we have that $f^{-1}\big([-\infty, \alpha)\big)$ is open and $x_0 \in f^{-1}\big([-\infty, \alpha)\big)$. □

Lemma. *An upper semiconutinuous function $f : X \to [-\infty, \infty)$ on a compact topological space attains its supremum. In particular it is bounded from above.*

Proof. $f(X)$ is bounded from above since otherwise

$$X = \bigcup_{n \in \mathbb{N}} f^{-1}\big([-\infty, n)\big)$$

would not have any finite subcover. Therefore $\sup_{x \in X} f(x)$ exists. Further we have $f(x_0) = \sup_{x \in X} f(x)$ for some $x_0 \in X$ since otherwise

$$X = \bigcup_{n \in \mathbb{N}} f^{-1}\big([-\infty, \sup_{x \in X} f(x) - 1/n)\big)$$

would not have any finite subcover. □

Lemma 3.2. *Let $\Omega \subseteq \mathbb{C}$ and $f : \Omega \to \mathbb{C}$ continuous. Then $\log|f|$ is upper semicontinuous on Ω.*

Proof. Let us consider the topological space $\big(\Omega, |\cdot|\big)$. Let $z_0 \in \Omega$ so such that $f(z_0) \neq 0$. Then $\log|f|$ is continuous as a composition of continuous functions. If $M > f(z_0)$, then $M - \log\big|f(z_0)\big| > 0$ and thus there exists some $\delta > 0$ such that $z \in B_\delta(z_0)$ implies $\Big|\log\big|f(z)\big| - \log\big|f(z_0)\big|\Big| < M - \log\big|f(z_0)\big|$ or equivalently $\log\big|f(z)\big| < M$. Now let $z_0 \in \Omega$ so such that $f(z_0) = 0$. By convention $\log\big|f(z_0)\big| = -\infty$. Furthermore, $M > \log\big|f(z_0)\big|$ for any $M \in \mathbb{R}$. The condition $M > \log\big|f(z)\big|$ is equivalent to $\big|f(z)\big| < e^M$. But $f(z_0) = 0$ and so

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$$\left|f(z)\right| = \left|f(z) - f(z_0)\right| < e^M$$

Since f is continuous at z_0 and $e^M > 0$ we find $\delta > 0$ such that $z \in B_\delta(z_0)$ implies $\left|f(z)\right| < e^M$. $\square$

LEMMA 3.3. *The mapping* $\Phi : \mathbb{R} \to (-\pi, 0)$ *defined by* $\Phi(t) := -i\log\left(h^{-1}(it)\right)$ *is a* C^1-*Diffeomorphism with* $|D\Phi(t)| = \pi\operatorname{sech}(\pi t)$. *In an analogous manner we have that* $\Psi : \mathbb{R} \to (0, \pi)$, $\Psi(t) := -i\log\left(h^{-1}(1+it)\right)$ *is a* C^1-*Diffeomorphism with* $|D\Psi(t)| = \pi\operatorname{sech}(\pi t)$.

Proof. It is easier to consider $\Phi^{-1}(\varphi) = -ih(e^{i\varphi})$ and $\Psi^{-1}(\varphi) = -i\left(h(e^{i\varphi}) - 1\right)$ (this already shows that Φ is a bijective mapping). Since $\left|e^{i\varphi}\right| = 1$ it is immediate by the representation (11) and $y < 0$ that $\operatorname{Im}\Phi(\varphi) = 0$. Furthermore, $\lim_{\varphi\to-\pi}\Phi(\varphi) = \infty$ and $\lim_{\varphi\to 0}\Phi(\varphi) = -\infty$. By (12) Φ is clearly continuously differentiable. Using

$$h^{-1}(it) = \frac{e^{-\pi t} - i}{e^{-\pi t} + i}$$

we get

$$\left|D\Phi(t)\right| = \pi\left|\frac{e^{-\pi t}}{e^{\pi t} - i} - \frac{e^{-\pi t}}{e^{-\pi t} + i}\right| = \pi\left|\frac{2e^{-\pi t}}{e^{-2\pi t} + 1}\right|\pi\left|\frac{2}{e^{-\pi t} + e^{\pi t}}\right| = \pi\operatorname{sech}(\pi t)$$

$\square$

LEMMA 3.4. *Let* $1/(2e-1) \leq \rho < 1$ *and* $\zeta = \rho e^{i\theta}$. *Then*

$$\left|\log\left|\frac{1+\zeta}{1-\zeta}\right|\right| \leq 1 + \log\frac{1}{\left|\cos(\theta/2)\right|} + \log\frac{1}{\left|\sin(\theta/2)\right|}$$

Proof. This proof is due to Prof. Schlein. We have on the one hand

$$|1+\zeta| \leq 1 + |\zeta| = 1 + \rho$$

and on the other hand

$$|1-\zeta| \geq |\operatorname{Im}\zeta| = \rho\left|\sin(\theta)\right|$$

Hence

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$$\begin{aligned}\log\frac{|1+\zeta|}{|1-\zeta|} &\leq \log\frac{1+\rho}{\rho\left|\sin(\theta)\right|}\\ &= \log\frac{1+\rho}{2\rho\left|\sin(\theta/2)\right|\left|\cos(\theta/2)\right|}\\ &= \log\frac{1+\rho}{2\rho} + \log\frac{1}{\left|\sin(\theta/2)\right|} + \log\frac{1}{\left|\cos(\theta/2)\right|}\\ &\leq 1 + \log\frac{1}{\left|\sin(\theta/2)\right|} + \log\frac{1}{\left|\cos(\theta/2)\right|}\end{aligned}$$

since

$$\frac{1+\rho}{2\rho} = \frac{1}{2} + \frac{1}{2\rho} \leq e$$

Now by

$$-\log\frac{|1+\zeta|}{|1-\zeta|} = \log\frac{|1-\zeta|}{|1+\zeta|}$$

which corresponds to considering $-\zeta = e^{i\pi}\zeta = e^{i(\pi+\theta)}$ in the first case, yields by invoking the identities

$$\cos\left(\frac{\pi+\theta}{2}\right) = -\sin(\theta/2) \qquad \sin\left(\frac{\pi+\theta}{2}\right) = \cos(\theta/2)$$

the bound

$$-\log\frac{|1+\zeta|}{|1-\zeta|} \leq 1 + \log\frac{1}{\left|\sin(\theta/2)\right|} + \log\frac{1}{\left|\cos(\theta/2)\right|}$$

and we are done. □

Lemma 3.5. *Let $0 \leq \tau_0 < \pi$. Then*

$$\frac{1}{\left|\cos(\theta/2)\right|^{\tau_0/\pi}}\frac{1}{\left|\sin(\theta/2)\right|^{\tau_0/\pi}} \in L^1[-\pi,\pi]$$

Proof. The case $\tau_0 = 0$ is trivial. We use [Els11, pp. 153–154]. By the symmetry of the integrand it is enough to consider

$$\int_0^\pi \frac{1}{\left|\cos(\theta/2)\right|^{\tau_0/\pi}}\frac{1}{\left|\sin(\theta/2)\right|^{\tau_0/\pi}}\,\mathrm{d}\theta$$

Thus we may perform the splitting

$$\int_0^1 \frac{1}{\left|\cos(\theta/2)\right|^{\tau_0/\pi}}\frac{1}{\left|\sin(\theta/2)\right|^{\tau_0/\pi}}\,\mathrm{d}\theta + \int_1^\pi \frac{1}{\left|\cos(\theta/2)\right|^{\tau_0/\pi}}\frac{1}{\left|\sin(\theta/2)\right|^{\tau_0/\pi}}\,\mathrm{d}\theta$$

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Let us consider only the first integral, the second one is similar. Then we have

$$\frac{1}{\left|\cos(\theta/2)\right|^{\tau_0/\pi}} \leq 1$$

for $0 \leq \theta \leq 1$ and

$$\begin{aligned}
\lim_{\theta \searrow 0} \frac{1/\sin(\theta/2)^{\tau_0/\pi}}{1/(\theta/2)^{\tau_0/\pi}} &= \lim_{\theta \to 0^+} \frac{(\theta/2)^{\tau_0/\pi}}{\sin(\theta/2)^{\tau_0/\pi}} \\
&= \lim_{\theta \searrow 0} \frac{(\theta/2)^{\tau_0/\pi}}{\left(\frac{\theta}{2} + O\left(\frac{\theta^3}{8}\right)\right)^{\tau_0/\pi}} \\
&= \lim_{\theta \searrow 0} \frac{(\theta/2)^{\tau_0/\pi}}{(\theta/2)^{\tau_0/\pi}\left(1 + O\left(\frac{\theta^2}{4}\right)\right)^{\tau_0/\pi}} \\
&= \lim_{\theta \searrow 0} \frac{(\theta/2)^{\tau_0/\pi}}{(\theta/2)^{\tau_0/\pi}\left(1 + O\left(\frac{\theta^2}{4}\right)\right)} \\
&= 1
\end{aligned}$$

Thus

$$\frac{1}{\left|\sin(\theta/2)\right|^{\tau_0/\pi}} \sim \frac{1}{\left(\theta/2\right)^{\tau_0/\pi}}$$

when $\theta \searrow 0$ and since $\tau_0 < \pi$, the latter integral converges. Thus we conclude by the comparison theorem. □

3.1.2. The Lemma. Recall, that a function $f : X \to [-\infty, \infty[$ defined on a topological space X is said to be *upper semicontinuous* if for every point $x_0 \in X$ and each $M > f(x_0)$ a neighbourhood U of x_0 exists such that $M > f(x)$ for every $x \in U$ (see [HS91, p. 199]).

Lemma 3.6. (Hadamard's three lines lemma, extension) *Let F be a holomorphic function in the strip $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ and continuous on $\overline{S}$, such that for some $0 < A < \infty$ and $0 \leq \tau_0 < \pi$ we have $\log|F(z)| \leq Ae^{\tau_0|\operatorname{Im} z|}$ for every $z \in \overline{S}$. Then*

$$|F(z)| \leq \exp\left(\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log|F(it+iy)|}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log|F(1+it+iy)|}{\cosh(\pi t) + \cos(\pi x)}\right]\mathrm{d}t\right)$$

whenever $z := x + iy \in S$.

Proof. We will first prove the case $\underline{y = 0}$. Assume F to be not identically zero (the case where F is identically zero is trivial). Let h be as in lemma (3.1) and let $\zeta = \rho e^{i\theta}$,

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$0 \leq \rho < 1$. Since $\zeta \in D$, we have $0 < \operatorname{Re} h(\zeta) < 1$ and thus the hypothesis on F and lemma (3.4) yields

$$\log|F(h(\zeta))| \leq Ae^{\frac{\tau_0}{\pi}\left|\log\left|\frac{1+\zeta}{1-\zeta}\right|\right|} \leq Ae^{\tau_0/\pi}\frac{1}{\left|\cos(\theta/2)\right|^{\tau_0/\pi}}\frac{1}{\left|\sin(\theta/2)\right|^{\tau_0/\pi}} \tag{13}$$

for $1/(2e-1) \leq \rho$. Since $0 < \tau_0 < \pi$, inequality (13) asserts, that $\log|F(h(\zeta))$ is bounded from above by an integrable function of θ, independently of $\rho \geq 1/(2e-1)$. Furthermore we have

$$M := \sup\left\{\log|F(h(\zeta))| : \zeta \in \overline{B}_{1/(2e-1)}\right\} < \infty \tag{14}$$

since a upper semicontinuous function on a compact space attains its supremum (see lemma 3.1.1). Hence

$$\log|F(h(\rho e^{i\theta}))| \leq \max\left\{M, Ae^{\tau_0/\pi}\frac{1}{\left|\cos(\theta/2)\right|^{\tau_0/\pi}}\frac{1}{\left|\sin(\theta/2)\right|^{\tau_0/\pi}}\right\} =: g\,(\theta) \tag{15}$$

for any $0 \leq \rho < 1$ where $g \in L^1[-\pi, \pi]$. Let $0 \leq \rho < R < 1$ and $a_1, \ldots, a_n$ denote the zeros of $F(h(\zeta))$ for $|\zeta| < R$ (since $F \circ h$ is holomorphic for $|\zeta| < 1$ there are indeed only finitely many ones) multiple zeros being repeated. Then for $F(h(\zeta)) \neq 0$ we have by the *Poisson-Jensen formula* (see [Ahl79, p. 208])

$$\log|F(h(\zeta))| = -\sum_{k=1}^{n}\log\left|\frac{R^2 - \overline{a}_k\zeta}{R\,(\zeta - a_k)}\right| + \frac{1}{2\pi}\int_{-\pi}^{\pi}\operatorname{Re}\left[\frac{Re^{it}+\zeta}{Re^{it}-\zeta}\right]\log|F(h(Re^{it}))|\,\mathrm{d}t \tag{16}$$

Therefore by

$$\begin{aligned}
\operatorname{Re}\left[\frac{Re^{it}+\zeta}{Re^{it}-\zeta}\right] &= \operatorname{Re}\left[\frac{R^2 - 2i\operatorname{Im}\left[\zeta Re^{-it}\right] - |\zeta|^2}{R^2 - 2\operatorname{Re}\left[\zeta Re^{-it}\right] + |\zeta|^2}\right] \\
&= \operatorname{Re}\left[\frac{R^2 - 2iR\rho\sin(\theta - t) - \rho^2}{R^2 - 2R\rho\cos(\theta - t) + \rho^2}\right] \\
&= \frac{R^2 - \rho^2}{R^2 - 2R\rho\cos(\theta - t) + \rho^2}
\end{aligned}$$

and since $\left(R^2 - |a_k|^2\right)\left(R^2 - \rho^2\right) \geq 0$ for all $k = 1, \ldots, n$ implies $|R^2 - \overline{a}_k\zeta| \geq |R\,(\zeta - a_k)|$ the estimate

$$\log|F(h(\zeta))| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{R^2 - \rho^2}{R^2 - 2R\rho\cos(\theta - t) + \rho^2}\log|F(h(Re^{it}))|\,\mathrm{d}t \tag{17}$$

is valid for every $|\zeta| < R$. By [Rud87, p. 236] we have

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$$\frac{R-\rho}{R+\rho} \leq \frac{R^2-\rho^2}{R^2-2R\rho\cos(\theta-\varphi)+\rho^2} \leq \frac{R+\rho}{R-\rho} \tag{18}$$

for $0 \leq \rho < R$. Combining (15) and (18) yields

$$\frac{R^2-\rho^2}{R^2-2R\rho\cos(\theta-t)+\rho^2}\log|F(h(\zeta))| \leq \frac{R+\rho}{R-\rho}g(\theta) =: G(\theta) \tag{19}$$

where $G \in L^1[-\pi,\pi]$. For $0 < R < 1$ let

$$f_R(\varphi) := \log|F(h(Re^{i\varphi}))|\frac{R^2-\rho^2}{R^2-2R\rho\cos(\theta-\varphi)+\rho^2}$$

and for $\varphi \neq 0, \pi$

$$f(\varphi) := \log|F(h(e^{i\varphi}))|\frac{1-\rho^2}{1-2\rho\cos(\theta-\varphi)+\rho^2}$$

Since $\log|F(h(\zeta))|$ is upper semicontinuous on $\overline{D}\setminus\{\pm 1\}$ by lemma 3.2 we get

$$\begin{aligned}
\limsup_{R\nearrow 1} f_R(\varphi) &= \limsup_{R\nearrow 1}\left[\log|F(h(Re^{i\varphi}))|\frac{R^2-\rho^2}{R^2-2R\rho\cos(\theta-\varphi)+\rho^2}\right]\\
&= \limsup_{R\nearrow 1}\log|F(h(Re^{i\varphi}))|\lim_{R\nearrow 1}\frac{R^2-\rho^2}{R^2-2R\rho\cos(\theta-\varphi)+\rho^2}\\
&= \log|F(h(e^{i\varphi}))|\frac{1-\rho^2}{1-2\rho\cos(\theta-\varphi)+\rho^2}\\
&= f(\varphi)
\end{aligned}$$

using [Bou95, p. 363] and proposition A.1. The functions $G - f_R$ being non-negative, an application of Fatou's lemma yields

$$\int_{-\pi}^{\pi}\liminf_{R\nearrow 1}\left[G(\varphi)-f_R(\varphi)\right]\mathrm{d}\varphi \leq \liminf_{R\nearrow 1}\int_{-\pi}^{\pi}\left[G(\varphi)f_R(\varphi)\right]\mathrm{d}\varphi$$

By [Bou95, p. 354], we get

$$-\int_{-\pi}^{\pi}\limsup_{R\nearrow 1}\left[f_R(\varphi)-G(\varphi)\right]\mathrm{d}\varphi \leq -\limsup_{R\nearrow 1}\int_{-\pi}^{\pi}\left[f_R(\varphi)-G(\varphi)\right]\mathrm{d}\varphi$$

and thus

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$$\begin{aligned}
&\limsup_{R\nearrow 1}\int_{-\pi}^{\pi} f_R(\varphi)\,\mathrm{d}\varphi-\int_{-\pi}^{\pi} G(\varphi)\,\mathrm{d}\varphi\\
&\qquad=\limsup_{R\nearrow 1}\int_{-\pi}^{\pi} f_R(\varphi)\,\mathrm{d}\varphi+\lim_{R\nearrow 1}\int_{-\pi}^{\pi}\big(-G(\varphi)\big)\,\mathrm{d}\varphi\\
&\qquad=\limsup_{R\nearrow 1}\int_{-\pi}^{\pi}\big[f_R(\varphi)-G(\varphi)\big]\,\mathrm{d}\varphi\leq\int_{-\pi}^{\pi}\limsup_{R\nearrow 1}\big[f_R(\varphi)-G(\varphi)\big]\,\mathrm{d}\varphi\\
&\qquad\leq\int_{-\pi}^{\pi}\limsup_{R\nearrow 1} f_R(\varphi)\,\mathrm{d}\varphi+\int_{-\pi}^{\pi}\lim_{R\nearrow 1}\big(-G(\varphi)\big)\,\mathrm{d}\varphi\\
&\qquad=\int_{-\pi}^{\pi}\limsup_{R\nearrow 1} f_R(\varphi)\,\mathrm{d}\varphi-\int_{-\pi}^{\pi} G(\varphi)\,\mathrm{d}\varphi
\end{aligned}$$

by [Bou95, p. 358]. Hence

$$\limsup_{R\nearrow 1}\int_{-\pi}^{\pi} f_R(\varphi)\,\mathrm{d}\varphi\leq\int_{-\pi}^{\pi}\limsup_{R\nearrow 1} f_R(\varphi)\,\mathrm{d}\varphi$$

and so

$$\log|F(h(\zeta))|\leq\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|F(h(e^{i\varphi}))|\frac{1-\rho^2}{1-2\rho\cos(\theta-\varphi)+\rho^2}\,\mathrm{d}\varphi \tag{20}$$

The lemma now follows from (20) by a change of variables. By stipulating $x := h(\zeta)$ we obtain [2]

$$\begin{aligned}
\zeta=h^{-1}(x)=\frac{e^{\pi ix}-i}{e^{\pi ix}+i}&=\frac{\cos(\pi x)+i\sin(\pi x)-i}{\cos(\pi x)+i\sin(\pi x)+i}\\
&=\frac{\cos(\pi x)+i\sin(\pi x)-i}{\cos(\pi x)+i\sin(\pi x)+i}\frac{\cos(\pi x)-i\sin(\pi x)-i}{\cos(\pi x)-i\sin(\pi x)-i}\\
&=-i\frac{\cos(\pi x)}{1+\sin(\pi x)}=\left(\frac{\cos(\pi x)}{1+\sin(\pi x)}\right)e^{-i\pi/2}
\end{aligned} \tag{21}$$

by

$$\begin{aligned}
&\big(\cos(\pi x)+i\sin(\pi x)-i\big)\big(\cos(\pi x)-i\sin(\pi x)-i\big)\\
&\qquad=\cos^2(\pi x)-i\sin(\pi x)\cos(\pi x)-i\cos(\pi x)+i\sin(\pi x)\cos(\pi x)\\
&\qquad\qquad+\sin^2(\pi x)+\sin(\pi x)-i\cos(\pi x)-\sin(\pi x)-1=-2i\cos(\pi x)
\end{aligned}$$

and

[2] Recall, that for $z\in\mathbb{C}$ the trigonometric functions are defined by $\sin(z) := \frac{e^{iz}-e^{-iz}}{2i}$ and $\cos(z) := \frac{e^{iz}+e^{-iz}}{2}$. Hence the identities $e^{iz}=\cos(z)+i\sin(z)$ and $\cos^2(z)+\sin^2(z)=1$ holds for any $z\in\mathbb{C}$ (see [Ahl79, pp. 42–44]).

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$$
\begin{aligned}
&\big(\cos(\pi x) + i\sin(\pi x) + i\big)\big(\cos(\pi x) - i\sin(\pi x) - i\big) \\
&\qquad = \cos^2(\pi x) - i\sin(\pi x)\cos(\pi x) - i\cos(\pi x) + i\sin(\pi x)\cos(\pi x) \\
&\qquad\qquad + \sin^2(\pi x) + \sin(\pi x) + i\cos(\pi x) + \sin(\pi x) + 1 = 2 + 2\sin(\pi x)
\end{aligned}
$$

From equality (21) we deduce $\rho = \frac{\cos(\pi x)}{1+\sin(\pi x)}$, $\theta = \frac{\pi}{2}$ if $0 < x \leq \frac{1}{2}$ and $\rho = -\frac{\cos(\pi x)}{1+\sin(\pi x)}$, $\theta = \frac{\pi}{2}$ if $\frac{1}{2} \leq x < 1$. Let $0 < x \leq \frac{1}{2}$. Then we have

$$
\begin{aligned}
&\frac{1-\rho^2}{1 - 2\rho\cos(\theta - \varphi) + \rho^2} \\
&\qquad = \frac{1 + 2\sin(\pi x) + \sin^2(\pi x) - \cos^2(\pi x)}{1 + 2\sin(\pi x) + \sin^2(\pi x) + 2\cos(\pi x)\sin(\varphi)(1 + \sin(\pi x)) + \cos^2(\pi x)} \\
&\qquad = \frac{\sin(\pi x) + \sin^2(\pi x)}{1 + \sin(\pi x) + \cos(\pi x)\sin(\varphi)(1 + \sin(\pi x))} = \frac{\sin(\pi x)}{1 + \cos(\pi x)\sin(\varphi)}
\end{aligned}
$$

since $\cos\big(-\pi/2 - \varphi\big) = -\sin(\varphi)$. That the case $\frac{1}{2} \leq x < 1$ yields the same result is due to $\cos(\pi/2 - \varphi) = \sin(\varphi)$. Let Φ and Ψ be defined as in lemma (3.3). We have

$$
\begin{aligned}
e^{i\Phi(t)} = h^{-1}(it) &= \frac{e^{-\pi t} - i}{e^{-\pi t} + i}\,\frac{e^{-\pi t} - i}{e^{-\pi t} - i} = \frac{e^{-2\pi t} - 2ie^{-\pi t} - 1}{e^{-2\pi t} + 1} = \frac{e^{-2\pi t} - 1}{e^{-2\pi t} + 1} - \frac{2ie^{-\pi t}}{e^{-2\pi t} + 1} \\
&= \frac{e^{-2\pi t} - 1}{e^{-2\pi t} + 1} - \frac{2i}{e^{-\pi t} + e^{\pi t}} = \frac{1 - e^{2\pi t}}{1 + e^{2\pi t}} - \frac{2i}{e^{-\pi t} + e^{\pi t}} = -\tanh(\pi t) - i\operatorname{sech}(\pi t)
\end{aligned}
$$

and thus

$$
\begin{aligned}
\sin\big(\Phi(t)\big)\cosh(\pi t) &= \sin\Big(-i\log\big(-\tanh(\pi t) - i\operatorname{sech}(\pi t)\big)\Big)\cosh(\pi t) \\
&= \frac{1}{2i}\left[-\tanh(\pi t) - i\operatorname{sech}(\pi t) + \frac{1}{\tanh(\pi t) + i\operatorname{sech}(\pi t)}\right]\cosh(\pi t) \\
&= \frac{1}{2i}\left[\frac{\cosh(\pi t) - \tanh(\pi t)\sinh(\pi t) - 2i\tanh(\pi t) + \operatorname{sech}(\pi t)}{\tanh(\pi t) + i\operatorname{sech}(\pi t)}\right] \\
&= \frac{1}{2i}\left[\frac{\cosh^2(\pi t) - \sinh^2(\pi t) - 2i\sinh(\pi t) + 1}{\sinh(\pi t) + i}\right] \\
&= \frac{1 - i\sinh(\pi t)}{i\sinh(\pi t) - 1} \\
&= -1
\end{aligned}
$$

Therefore the transformation formula yields

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$$\frac{1}{2\pi}\int_{-\pi}^{0}\frac{\sin(\pi x)}{1+\cos(\pi x)\sin(\varphi)}\log|F(h(e^{i\varphi}))|\,\mathrm{d}\varphi = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin(\pi x)}{\cosh(\pi t)-\cos(\pi x)}\log|F(it)|\,\mathrm{d}t \tag{22}$$

and in a similar manner

$$\frac{1}{2\pi}\int_{0}^{\pi}\frac{\sin(\pi x)}{1+\cos(\pi x)\sin(\varphi)}\log|F(h(e^{i\varphi}))|\,\mathrm{d}\varphi = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin(\pi x)}{\cosh(\pi t)+\cos(\pi x)}\log|F(1+it)|\,\mathrm{d}t \tag{23}$$

holds since

$$\begin{aligned}
\sin\big(\Psi(t)\big)\cosh(\pi t) &= \sin\Big(-i\log\big(-\tanh(\pi t)+i\operatorname{sech}(\pi t)\big)\Big)\cosh(\pi t)\\
&= \frac{1}{2i}\left[-\tanh(\pi t)+i\operatorname{sech}(\pi t)-\frac{1}{-\tanh(\pi t)+i\operatorname{sech}(\pi t)}\right]\cosh(\pi t)\\
&= \frac{1}{2i}\left[\frac{-\cosh(\pi t)+\tanh(\pi t)\sinh(\pi t)-2i\tanh(\pi t)-\operatorname{sech}(\pi t)}{-\tanh(\pi t)+i\operatorname{sech}(\pi t)}\right]\\
&= \frac{1}{2i}\left[\frac{-\cosh^2(\pi t)+\sinh^2(\pi t)-2i\sinh(\pi t)-1}{i-\sinh(\pi t)}\right]\\
&= \frac{1+i\sinh(\pi t)}{1+i\sinh(\pi t)}\\
&= 1
\end{aligned}$$

Thus the case $y=0$ is prooven.

The case $y \neq 0$. follows easily from the previous one. Fix $y \neq 0$ and define $G(z) := F(z+iy)$ for $z \in \overline{S}$. Then G is a holomorphic function in S and continuous on $\overline{S}$ as a composition of continuous and holomorphic functions. Moreover, the hypothesis on F yields

$$\log|G(z)| = \log|F(z+iy)| \leq Ae^{\tau|\operatorname{Im} z+y|} \leq Ae^{\tau|\operatorname{Im} z|}e^{\tau|y|} \tag{24}$$

for all $z \in \overline{S}$. The previous case yields for G with A replaced by $Ae^{\tau|y|}$

$$|G(x)| \leq \exp\left(\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log|G(it)|}{\cosh(\pi t)-\cos(\pi x)}+\frac{\log|G(1+it)|}{\cosh(\pi t)+\cos(\pi x)}\right]\mathrm{d}t\right) \tag{25}$$

Now, observing $G(x) = F(x+iy)$, $G(it) = F(it+iy)$ and $G(1+it) = F(1+it+iy)$ yields the desired result. □

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REMARK 3.1. *Exercise **1.3.8.** [Gra14, p. 48] shows that the name extension is an appropriate choice. For $0 < x < 1$ consider*

$$\begin{aligned}
\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\frac{1}{\cosh(\pi t)+\cos(\pi x)}\,\mathrm{d}t &= \frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\frac{1}{\frac{1}{2}\left(e^{\pi t}+e^{-\pi t}\right)+\cos(\pi x)}\,\mathrm{d}t \\
&= \frac{\sin(\pi x)}{\pi}\int_{0}^{\infty}\frac{1}{s^2+2\cos(\pi x)\,s+1}\,\mathrm{d}s \\
&= \frac{\sin(\pi x)}{\pi}\int_{0}^{\infty}\frac{1}{\big(s+\cos(\pi x)\big)^2+\sin^2(\pi x)}\,\mathrm{d}s \\
&= \frac{1}{\pi\sin(\pi x)}\int_{0}^{\infty}\frac{1}{\left(\frac{s+\cos(\pi x)}{\sin(\pi x)}\right)^2+1}\,\mathrm{d}s \\
&= \frac{1}{\pi}\int_{\cot(\pi x)}^{\infty}\frac{1}{u^2+1}\,\mathrm{d}u \\
&= \frac{1}{\pi}\left[\frac{\pi}{2}-\arctan\big(\cot(\pi x)\big)\right] \\
&= x
\end{aligned}$$

and in the same manner

$$\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\frac{1}{\cosh(\pi t)-\cos(\pi x)}\,\mathrm{d}t = 1-x$$

Assume that F is holomorphic in S, continuous and bounded on $\overline{S}$ with $\big|F(z)\big| \leq B_0$ when $\operatorname{Re} z = 0$ and $\big|F(z)\big| \leq B_1$ when $\operatorname{Re} z = 1$ for some $0 < B_0, B_1 < \infty$. If $\big|F(z)\big| \leq M$ for $0 < M < \infty$, F satisfies the hypothesis of lemma 3.6 with $A := \log(M)$ and $\tau_0 = 0$. Therefore

$$\begin{aligned}
|F(z)| &\leq \exp\left(\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log|F(it+iy)|}{\cosh(\pi t)-\cos(\pi x)}+\frac{\log|F(1+it+iy)|}{\cosh(\pi t)+\cos(\pi x)}\right]\mathrm{d}t\right) \\
&\leq \exp\left(\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log B_0}{\cosh(\pi t)-\cos(\pi x)}+\frac{\log B_1}{\cosh(\pi t)+\cos(\pi x)}\right]\mathrm{d}t\right) \\
&= \exp\big(x\log B_0+(1-x)\log B_1\big) \\
&= B_0^x B_1^{1-x}
\end{aligned}$$

whenever $z := x+iy \in S$. Hence lemma 3.6 reduces to lemma 2.1.

3.2. Stein's Theorem on Interpolation of Analytic Families of Operators. Because of the complex nature of the proof of the Riesz-Thorin Interpolation Theorem (2.1), Elias M. Stein realized quickly, that the restritction to consider only one linear operator T

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could easily be omited and instead, an analytic family of operators T_z depending on some complex parameter z could be considered.

DEFINITION 3.1. (Analytic family, admissible growth) *Let (X, μ) be a measure space, (Y, ν) be a semifinite measure spaces and $(T_z)_{z\in\overline{S}}$, where T_z is defined Σ_X and taking values in the space of all measurable functions on Y such that*

$$\int_Y \left|T_z(\chi_A)\chi_B\right| d\nu \tag{26}$$

whenever $\mu(A), \nu(B) < \infty$. The family $(T_z)_{z\in\overline{S}}$ is said to be analytic *if for all $f \in \Sigma_X$, $g \in \Sigma_Y$ we have that*

$$z \mapsto \int_Y T_z(f) g d\nu \tag{27}$$

is analytic on S and continuous on $\overline{S}$. Further, an analytic family $(T_z)_{z\in\overline{S}}$ is called of admissible growth, *if there is a constant $\tau_0 \in (0, \pi)$, such that for all $f \in \Sigma_X$, $g \in \Sigma_Y$ a constant $C(f, g)$ exists with*

$$\log \left| \int_Y T_z(f) g d\nu \right| \leq C(f, g) e^{\tau_0 |\operatorname{Im} z|} \tag{28}$$

for all $z \in \overline{S}$.

Now we are able to write down the theorem.

THEOREM 3.1. (Stein's Theorem on Interpolation of Analytic Families of Operators) *Let $(T_z)_{z\in\overline{S}}$ be an analytic family of admissible growth, $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and suppose that M_0, M_1 are positive functions on the real line such that for some $\tau_1 \in (0, \pi)$*

$$\sup\left\{e^{-\tau_1|y|} \log M_0(y) : y \in \mathbb{R}\right\} < \infty \qquad \sup\left\{e^{-\tau_1|y|} \log M_1(y) : y \in \mathbb{R}\right\} < \infty \tag{29}$$

Fix $0 < \theta < 1$ and define

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \tag{30}$$

Further suppose that for all $f \in \Sigma_X$ and $y \in \mathbb{R}$ we have

$$\|T_{iy}(f)\|_{L^{q_0}} \leq M_0(y) \|f\|_{L^{p_0}} \qquad \|T_{1+iy}(f)\|_{L^{q_1}} \leq M_1(y) \|f\|_{L^{p_1}} \tag{31}$$

Then for all $f \in \Sigma_X$ we have

$$\|T_\theta(f)\|_{L^q} \leq M(\theta) \|f\|_{L^p}$$

where for $0 < x < 1$

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$$M(x) = \exp\left(\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log M_0(t)}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log M_1(t)}{\cosh(\pi t) + \cos(\pi x)}\right] d\lambda(t)\right)$$

Proof. Fix $0 < \theta < 1$ and $f \in \Sigma_X$, $g \in \Sigma_Y$ with $\|f\|_{L^p} = \|g\|_{L^{q'}} = 1$. Define f_z, g_z as in (**??**) and for $z \in \overline{S}$

$$F(z) := \int_Y T_z(f_z) g_z d\nu \tag{32}$$

We have

$$\begin{aligned}
\log\left|F(z)\right| &= \log\left|\sum_{j=1}^{n}\sum_{k=1}^{m} a_j^{P(z)} b_j^{Q(z)} e^{i\alpha_j} e^{i\beta_k}\int_Y T_z(\chi_{A_j})(y)\chi_{B_k}(y)d\nu(y)\right| \\
&\leq \log\left[\sum_{j=1}^{n}\sum_{k=1}^{m}\max\left\{1, a_j^{p/p_0+p/p_1}\right\}\max\left\{1, b_k^{q'/q_0'+q'/q_1'}\right\}\left|\int_{B_k} T_z(\chi_{A_j})d\nu\right|\right] \\
&\leq \log\left[\sum_{j=1}^{n}\sum_{k=1}^{m}(1+a_j)^{p/p_0+p/p_1}(1+b_k)^{q'/q_0'+q'/q_1'} e^{c(A_j,B_k)e^{\tau_0|\operatorname{Im} z|}}\right] \\
&\leq \log\left[\sum_{j=1}^{n}\sum_{k=1}^{m} e^{\log\left((1+a_j)^{p/p_0+p/p_1}(1+b_k)^{q'/q_0'+q'/q_1'}\right)+c(A_j,B_k)e^{\tau_0|\operatorname{Im} z|}}\right] \\
&\leq \log\left[mne^{\sum_{j=1}^{n}\sum_{k=1}^{m}\log\left((1+a_j)^{p/p_0+p/p_1}(1+b_k)^{q'/q_0'+q'/q_1'}\right)+c(A_j,B_k)e^{\tau_0|\operatorname{Im} z|}}\right] \\
&= \log(mn) + \sum_{j=1}^{n}\sum_{k=1}^{m}\log\left((1+a_j)^{p/p_0+p/p_1}(1+b_k)^{q'/q_0'+q'/q_1'}\right) + c(A_j,B_k)e^{\tau_0|\operatorname{Im} z|}
\end{aligned}$$

Since $\tau_0 \in (0, \pi)$ and thus $e^{\tau_0|\operatorname{Im} z|} \geq 1$, F satisfies the hypotheses of the extension of Hadamard's three lines lemma (3.6) with

$$A = \log(mn) + \sum_{j=1}^{n}\sum_{k=1}^{m}\left(\frac{p}{p_0} + \frac{p}{p_1}\right)\log(1+a_j) + \left(\frac{q'}{q_0'} + \frac{q'}{q_1'}\right)\log(1+b_k) + c(A_j, B_k)$$

The same calculations as in the proof of the Riesz-Thorin interpolation theorem (2.1) yields for $y \in \mathbb{R}$

$$\left\|f_{iy}\right\|_{L^{p_0}} = \|f\|_{L^p}^{p/p_0} = 1 = \|g\|_{L^{q'}}^{q'/q_0'} = \left\|g_{iy}\right\|_{L^{q_0'}}$$

and

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$$\left\|f_{1+iy}\right\|_{L^{p_1}} = \|f\|_{L^p}^{p/p_1} = 1 = \|g\|_{L^{q'}}^{q'/q_1'} = \left\|g_{1+iy}\right\|_{L^{q_1'}}$$

Further

$$\left|F(iy)\right| \leq \left\|T_{iy}(f_{iy})\right\|_{L^{q_0}} \left\|g_{iy}\right\|_{L^{q_0'}} \leq M_0(y)\left\|f_{iy}\right\|_{L^{p_0}} \left\|g_{iy}\right\|_{L^{q_0'}} = M_0(y)$$

and

$$\left|F(1+iy)\right| \leq \left\|T_{1+iy}(f_{1+iy})\right\|_{L^{q_1}} \left\|g_{1+iy}\right\|_{L^{q_1'}} \leq M_1(y)\left\|f_{1+iy}\right\|_{L^{p_1}} \left\|g_{1+iy}\right\|_{L^{q_1'}} = M_1(y)$$

by Hölder's inequality and the hypotheses on the analytic family $(T_z)_{z\in\overline{S}}$. Therefore the extension of Hadamard's three lines lemma (3.6) yields

$$\left|F(x)\right| \leq \exp\left(\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log M_0(t)}{\cosh(\pi t)-\cos(\pi x)} + \frac{\log M_1(t)}{\cosh(\pi t)+\cos(\pi x)}\right] d\lambda(t)\right) = M(x)$$

for every $0 < x < 1$. Furthermore observe that

$$F(\theta) = \int_Y T_\theta(f) g d\nu$$

and thus by [Fol99, p. 189] (Σ_Y denotes the set of all finitely simple functions on the semifinite space Y)

$$\begin{aligned} M_q\left(T_\theta(f)\right) &= \sup\left\{\left|\int_Y T_\theta(f) g\right| : g \in \Sigma_Y, \|g\|_{L^{q'}}\right\} \\ &= \sup\left\{\left|F(\theta)\right| : g \in \Sigma_Y, \|g\|_{L^{q'}}\right\} \\ &\leq M(\theta) \end{aligned}$$

Since $M(\theta)$ is an absolutely convergent integral for any $0 < \theta < 1$, $M_q\left(T_\theta(f)\right) < \infty$ and thus $M_q\left(T_\theta(f)\right) = \left\|T_\theta(f)\right\|_{L^q}$ (this is incorporated by the growth conditions on M_0 and M_1). The general statement follows by replacing f with $f/\|f\|_{L^p}$ when $\|f\|_{L^p} \neq 0$. The theorem is trivially true when $\|f\|_{L^p} = 0$. $\square$

4. The Real Method. A first important theorem on the subject of interpolation of L^p spaces will be the so-called *Marcinkiewicz Interpolation Theorem* which uses only real variables techniques for its proof (this stands in contrast to the complex variables techniques used for prooving the other interpolation theorems).

4.1. The Marcinkiewicz Interpolation Theorem. This theorem applies to sublinear operators (aswell as for quasilinear operators by a slight change of the constant), which is in comparison to the linearity assumed by the other interpolation theorems more generally applicable.

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THEOREM 4.1. (The Marcinkiewicz Interpolation Theorem) *Let* (X,μ) *be a* σ*-finite measure space,* (Y,ν) *another measure space and* $0 < p_0 < p_1 \leqslant \infty$*. Further let* T *be a sublinear operator defined on*

$$L^{p_0} + L^{p_1} := \left\{ f_0 + f_1 : f_0 \in L^{p_0}(X,\mu), f_1 \in L^{p_1}(X,\mu) \right\}$$

and taking values in the space of measurable functions on Y*. Assume that there exist* $A_0, A_1 < \infty$ *such that*

$$\forall f \in L^{p_0}(X,\mu) \; \|T(f)\|_{L^{p_0,\infty}} \leqslant A_0 \|f\|_{L^{p_0}} \tag{33}$$

$$\forall f \in L^{p_1}(X,\mu) \; \|T(f)\|_{L^{p_1,\infty}} \leqslant A_1 \|f\|_{L^{p_1}} \tag{34}$$

Then for all $p_0 < p < p_1$ *and for all* $f \in L^p(X,\mu)$ *we have the estimate*

$$\left\|T(f)\right\|_{L^p} \leqslant A \left\|f\right\|_{L^p} \tag{35}$$

where

$$A := 2\left(\frac{p}{p-p_0} + \frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}} \tag{36}$$

Proof. Let us first consider the case $\underline{p_1 < \infty}$. Fix $f \in L^p(X,\mu)$, $\alpha > 0$ and $\delta > 0$ (δ will be determined later). We split f using so-called *cut-off* functions, by stipulating $f \equiv f_0(\cdot;\alpha,\delta) + f_1(\cdot;\alpha,\delta)$, where $f_0(\cdot;\alpha,\delta)$ is the *unbounded part of* f and $f_1(\cdot;\alpha,\delta)$ is the *bounded part of* f, defined by

$$\begin{aligned} f_0(x;\alpha,\delta) &:= \begin{cases} f(x), & |f(x)| > \delta\alpha, \\ 0, & |f(x)| \leqslant \delta\alpha. \end{cases} \\ f_1(x;\alpha,\delta) &:= \begin{cases} f(x), & |f(x)| \leqslant \delta\alpha, \\ 0, & |f(x)| > \delta\alpha. \end{cases} \end{aligned} \tag{37}$$

for $x \in X$. To facilitate reading I will omit the dependency of $f_0(\cdot;\alpha,\delta)$ and $f_1(\cdot;\alpha,\delta)$ upon the parameters α and δ in what follows and simply write f_0, f_1 respectively.

LEMMA 4.1. *The functions* f_0 *and* f_1 *defined above satisfy* $f_0 \in L^{p_0}(X,\mu)$ *and* $f_1 \in L^{p_1}(X,\mu)$ *respectively.*

Proof. Since $p_0 < p$ we have

$$
\begin{aligned}
\|f_0\|_{L^{p_0}}^{p_0} &= \int_X |f_0|^{p_0} d\mu = \int_X |f|^{p_0} \cdot \chi_{\{|f|>\delta\alpha\}} d\mu \overset{(\dagger)}{=} \int_{\{|f|>\delta\alpha\}} |f|^{p_0} d\mu \\
&= \int_{\{|f|>\delta\alpha\}} |f|^p |f|^{p_0-p} d\mu = \int_{\{|f|>\delta\alpha\}} \frac{|f|^p}{|f|^{p-p_0}} d\mu \\
&\leqslant \frac{1}{(\delta\alpha)^{p-p_0}} \int_{\{|f|>\delta\alpha\}} |f|^p d\mu = (\delta\alpha)^{p_0-p} \int_X |f|^p \cdot \chi_{\{|f|>\delta\alpha\}} d\mu \\
&\leqslant (\delta\alpha)^{p_0-p} \int_X |f|^p d\mu = (\delta\alpha)^{p_0-p} \|f\|_{L^p}^p < \infty
\end{aligned}
\tag{38}
$$

Thus $f_0 \in L^{p_0}(X, \mu)$. Analogously it can be checked, that $f_1 \in L^{p_1}(X, \mu)$ by the estimate $\|f_1\|_{L^{p_1}}^{p_1} \leqslant (\delta\alpha)^{p_1-p}\|f\|_{L^p}^p$.

Proof of the equality ($\dagger$). Assume μ is defined on the σ-algebra $\mathcal{A}$. We have to proove that $\{|f| > \delta\alpha\} \in \mathcal{A}$[3]. Since f is complex-valued, we may write $f \equiv \mathrm{Re}f + i\mathrm{Im}f$ and thus $|f|^2 \equiv \mathrm{Re}^2 f + \mathrm{Im}^2 f$. Since f is measurable by hypothesis this implies that $\mathrm{Re}f$ and $\mathrm{Im}f$ are measurable[4]. Further for measurable real-valued functions $f, g : (X, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathfrak{B}})$[5] the functions $f+g$ and $f \cdot g$ are measurable[6] and thus $|f|^2$ is measurable. Hence $\{\mathrm{Re}^2 f + \mathrm{Im}^2 f > \lambda\} \in \mathcal{A}$[7] for any $\lambda \in \mathbb{R}$. So especially for $\lambda := (\delta\alpha)^2$ we have $\{|f| > \delta\alpha\} \in \mathcal{A}$[8]. In a similar manner it can also be prooven that $\{|f| \leqslant \delta\alpha\} \in \mathcal{A}$. Let us next proove a useful lemma.

Lemma 4.2. *Let $A \in \wp(X)$ and $\chi_A : (X, \mathcal{A}) \to (\mathbb{C}, \mathfrak{B}^2)$ be the* characteristic function of the set A. *Then χ_A is measurable if and only if A is measurable.*

Proof. Assume χ_A is measurable. Then $\mathrm{Re}\chi_A$ and $\mathrm{Im}\chi_A$ are measurable. Especially for $0 < \lambda < 1$ we have that $\{\mathrm{Re}\chi_A > \lambda\} = A \in \mathcal{A}$. Conversly, assume A is measurable. For $\lambda < 0$ we have $\{\mathrm{Re}\chi_A > \lambda\} = X \in \mathcal{A}$, $\lambda \in [0, 1[$, $\{\mathrm{Re}\chi_A > \lambda\} = A \in \mathcal{A}$ and $\{\mathrm{Re}\chi_A > \lambda\} = \emptyset \in \mathcal{A}$ for $\lambda \geqslant 1$. Since $\mathrm{Im}\chi_A \equiv 0$ we have $\{\mathrm{Im}\chi_A > \lambda\} = X \in \mathcal{A}$ if $\lambda < 0$ and $\{\mathrm{Im}\chi_A > \lambda\} = \emptyset \in \mathcal{A}$ if $\lambda \geqslant 0$. □

By Lemma 4.2 and the fact that $f \cdot g$ is measurable for two measurable functions $f, g : (X, \mathcal{A}) \to (\mathbb{C}, \mathfrak{B}^2)$[9], f_0 and f_1 are measurable since $f_0 \equiv f \cdot \chi_{\{|f|>\delta\alpha\}}$ and $f_1 \equiv f \cdot \chi_{\{|f|\leqslant\delta\alpha\}}$.

[3] For $Y \in \mathcal{A}$ the μ-integral of $f : X \to \mathbb{C}$ over Y is defined to be $\int_Y f d\mu := \int_X f \cdot \chi_Y d\mu$. For more details see [Els11, pp. 135–136].

[4] For a proof see [Els11, p. 106]

[5] $\mathfrak{B} := \sigma(\mathbb{R})$ and $\overline{\mathfrak{B}} = \{B \cup E : B \in \mathfrak{B}, E \subseteq \{\pm\infty\}\}$.

[6] For a proof see [Els11, p. 107].

[7] For a proof see [Els11, pp. 105–106]

[8] This follows from the fact that $x < y$ if and only if $x^n < y^n$ for $n \in \mathbb{N}_{>0}$ and some real numbers $x, y > 0$ (see [Zor04, p. 119]).

[9] Els11, p. 107.

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One subtility is left to clear: the μ-integrability of either $|f_1|^{p_0}$ or $|f_1|^{p_1}$ requires that $|f_0|^{p_0}$ and $|f_1|^{p_1}$ are measurable functions. By the fact that any continuous map $g : (X, d_X) \to (Y, d_Y)$ between metric spaces is Borel-measurable (see [Els11, p. 86]) and that the composition of measurable functions is again measurable (see [Els11, p. 87]), the measurability of either f_0 or f_1 follows by $|f_0|^{p_0} \equiv \cdot^{p_0} \circ |f \cdot \chi_{\{|f|>\delta\alpha\}}|$ and $|f_1|^{p_1} \equiv \cdot^{p_1} \circ |f \cdot \chi_{\{|f|\leqslant\delta\alpha\}}|$ by stipulating $\cdot^p : (\mathbb{R}_{\geqslant 0}, |\cdot|) \to (\mathbb{C}, |\cdot|)$, $x^p := \exp(p \log(x))$ for $p > 0$ and $x \in \mathbb{R}_{>0}$ and $x^p := 0$ if $x = 0$. □

By lemma (4.1) we therefore have $f \equiv f_0 + f_1 \in L^{p_0} + L^{p_1}$.

Lemma 4.3. *For fixed $\alpha > 0$, the distribution function $d_{T(f)}(\alpha)$ obeys an upper bound of the form*

$$d_{T(f)}(\alpha) \leqslant \left(\frac{A_0}{\alpha/2}\right)^{p_0} \|f_0\|_{L^{p_0}}^{p_0} + \left(\frac{A_1}{\alpha/2}\right)^{p_1} \|f_1\|_{L^{p_1}}^{p_1}$$

Proof. Since T is a sublinear operator we have $|T(f)| = |T(f_0 + f_1)| \leqslant |T(f_0)| + |T(f_1)|$. Thus for any $y \in Y$ with $|T(f)(y)| > \alpha$ we therefore have either $|T(f_0)(y)| > \alpha/2$ or $|T(f_1)(y)| > \alpha/2$ [10] . Hence

$$\{|T(f)| > \alpha\} \subseteq \{|T(f_0)| > \alpha/2\} \cup \{|T(f_1)| > \alpha/2\}$$

and so by the monotonicity and subadditivity property of the measure μ we have

$$\begin{aligned} d_{T(f)}(\alpha) &= \mu(\{|T(f)| > \alpha\}) \\ &\leqslant \mu(\{|T(f_0)| > \alpha/2\} \cup \{|T(f_1)| > \alpha/2\}) \\ &\leqslant \mu(\{|T(f_0)| > \alpha/2\}) + \mu(\{|T(f_1)| > \alpha/2\}) \\ &= d_{T(f_0)}(\alpha/2) + d_{T(f_1)}(\alpha/2) \end{aligned} \tag{39}$$

Now by hypothesis (33) we can estimate $d_{T(f_0)}(\alpha/2)$ as follows

$$\begin{aligned} d_{T(f_0)}(\alpha/2) &= \left(\frac{\alpha/2}{\alpha/2}\right)^{p_0} d_{T(f_0)}(\alpha/2) \\ &\leqslant \left(\frac{1}{\alpha/2}\right)^{p_0} \left[\sup\left\{\gamma d_{T(f_0)}(\gamma)^{1/p_0} : \gamma > 0\right\}\right]^{p_0} \\ &= \left(\frac{1}{\alpha/2}\right)^{p_0} \|T(f_0)\|_{L^{p_0,\infty}}^{p_0} \\ &\leqslant \left(\frac{A_0}{\alpha/2}\right)^{p_0} \|f_0\|_{L^{p_0}}^{p_0} \end{aligned} \tag{40}$$

Analogously, we get $d_{T(f_1)}(\alpha/2) \leqslant \left(\frac{A_1}{\alpha/2}\right)^{p_1} \|f_1\|_{L^{p_1}}^{p_1}$ by hypothesis (34). □

[10] Without loss of generality assume $|T(f_0)(y)| \leqslant |T(f_1)(y)|$. Then we have $\alpha < |T(f)(y)| \leqslant |T(f_0)(y)| + |T(f_1)(y)| \leqslant 2|T(f_1)(y)|$ (this is possible since $\mathbb{R}$ is an ordered field).

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By

$$\int_0^{\frac{1}{\delta}|f|} \alpha^{p-p_0-1} d\lambda = \begin{cases} \frac{1}{p-p_0}\frac{1}{\delta^{p-p_0}}|f|^{p-p_0}, & p \geqslant p_0+1 \\ = \lim_{\omega\to 0^+}\int_\omega^{\frac{1}{\delta}|f|}\alpha^{p-p_0-1}d\lambda & \\ = \lim_{\omega\to 0^+}\left[\frac{1}{p-p_0}\alpha^{p-p_0}\right]_\omega^{\frac{1}{\delta}|f|} & \\ = \frac{1}{p-p_0}\left[\frac{1}{\delta^{p-p_0}}|f|^{p-p_0} - \lim_{\omega\to 0^+}\omega^{p-p_0}\right] & \\ = \frac{1}{p-p_0}\frac{1}{\delta^{p-p_0}}|f|^{p-p_0}, & p_0 < p < p_0+1 \end{cases} \tag{41}$$

and

$$\begin{aligned} \int_{\frac{1}{\delta}|f|}^\infty \alpha^{p-p_1-1}d\lambda &= \lim_{\omega\to\infty}\left[\frac{1}{p-p_1}\alpha^{p-p_1}\right]_{\frac{1}{\delta}|f|}^\omega \\ &= \frac{1}{p-p_1}\left[\lim_{\omega\to\infty}\omega^{p-p_1} - \frac{1}{\delta^{p-p_1}}|f|^{p-p_1}\right] \\ &= \frac{1}{p_1-p}\frac{1}{\delta^{p-p_1}}|f|^{p-p_1} \end{aligned} \tag{42}$$

and the representation $\|f\|_{L^p}^p = p\int_0^\infty \alpha^{p-1}d_f(\alpha)d\lambda$ for $0 < p < \infty$ we get

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$$
\begin{aligned}
\|T(f)\|_{L^p}^p &= p \int_0^\infty \alpha^{p-1} d_{T(f)} d\lambda \\
&\leqslant p\,(2A_0)^{p_0} \int_0^\infty \alpha^{p-p_0-1} \int_{\{|f|>\delta\alpha\}} |f|^{p_0}\, d\mu d\lambda \\
&\quad + p(2A_1)^{p_1} \int_0^\infty \alpha^{p-p_1-1} \int_{\{|f|\leqslant\delta\alpha\}} |f|^{p_1}\, d\mu d\lambda \\
&= p\,(2A_0)^{p_0} \int_{\{|f|>0\}} |f|^{p_0} \int_0^{\frac{1}{\delta}|f|} \alpha^{p-p_0-1} d\lambda d\mu \\
&\quad + p\,(2A_0)^{p_0} \int_{\{|f|=0\}} |f|^{p_0} \int_0^{\frac{1}{\delta}|f|} \alpha^{p-p_0-1} d\lambda d\mu \\
&\quad + p\,(2A_1)^{p_1} \int_X |f|^{p_1} \int_{\frac{1}{\delta}|f|}^\infty \alpha^{p-p_1-1} d\lambda d\mu \\
&= p\,(2A_0)^{p_0} \int_X |f|^{p_0} \int_0^{\frac{1}{\delta}|f|} \alpha^{p-p_0-1} d\lambda d\mu \\
&\quad + p\,(2A_1)^{p_1} \int_X |f|^{p_1} \int_{\frac{1}{\delta}|f|}^\infty \alpha^{p-p_1-1} d\lambda d\mu \\
&= \frac{p\,(2A_0)^{p_0}}{p-p_0} \frac{1}{\delta^{p-p_0}} \int_X |f|^{p_0}\, |f|^{p-p_0}\, d\mu \\
&\quad + \frac{p\,(2A_1)^{p_1}}{p_1-p} \frac{1}{\delta^{p-p_1}} \int_X |f|^{p_1}\, |f|^{p-p_1}\, d\mu \\
&= p \left(\frac{(2A_0)^{p_0}}{p-p_0} \frac{1}{\delta^{p-p_0}} + \frac{(2A_1)^{p_1}}{p_1-p} \delta^{p_1-p} \right) \|f\|_{L^p}^p
\end{aligned}
\tag{43}
$$

We pick $\delta > 0$ such that $(2A_0)^{p_0}\,\delta^{p_0-p} = (2A_1)^{p_1}\,\delta^{p_1-p}$. Solving for δ yields

$$
\delta = \frac{1}{2} \left(\frac{A_0}{A_1} \right)^{p_1/(p_1-p_0)} \tag{44}
$$

Substituting this in estimate (43) leads to

$$
\begin{aligned}
\|T(f)\|_{L^p}^p &\leqslant p \left(\frac{(2A_0)^{p_0}}{p-p_0} \frac{2^{p-p_0} A_1^{\frac{p_1(p-p_0)}{p_1-p_0}}}{A_0^{\frac{p_0(p-p_0)}{p_1-p_0}}} + \frac{(2A_1)^{p_1}}{p_1-p} \frac{A_0^{\frac{p_0(p_1-p)}{p_1-p_0}}}{2^{p_1-p} A_1^{\frac{p_1(p_1-p)}{p_1-p_0}}} \right) \|f\|_{L^p}^p \\
&= 2^p p \left(\frac{A_0^{\frac{p_0(p_1-p)}{p_1-p_0}} A_1^{\frac{p_1(p-p_0)}{p_1-p_0}}}{p-p_0} + \frac{A_0^{\frac{p_0(p_1-p)}{p_1-p_0}} A_1^{\frac{p_1(p-p_0)}{p_1-p_0}}}{p_1-p} \right) \|f\|_{L^p}^p
\end{aligned}
\tag{45}
$$

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And taking the p-th power further

$$\begin{aligned}
\|T(f)\|_{L^p} &\leqslant 2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{p_0(p_1-p)}{p(p_1-p_0)}} A_1^{\frac{p_1(p-p_0)}{p(p_1-p_0)}} \|f\|_{L^p} \\
&= 2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{p_0(p_1-p)}{p(p_1-p_0)}\frac{p_1}{p_1}} A_1^{\frac{p_1(p-p_0)}{p(p_1-p_0)}\frac{p_0}{p_0}} \|f\|_{L^p} \\
&= 2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{\frac{p_1-p}{pp_1}}{\frac{p_1-p_0}{p_0p_1}}} A_1^{\frac{\frac{p-p_0}{p_0p}}{\frac{p_1-p_0}{p_0p_1}}} \|f\|_{L^p} \\
&= 2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}} \|f\|_{L^p}
\end{aligned} \tag{46}$$

Assume $p_1 = \infty$. We again use the cut-off functions defined in (37) to decompose f. Since $\{|f_1| > \delta\alpha\} = \emptyset$, we have

$$\|T(f_1)\|_{L^\infty} \leqslant A_1\|f_1\|_{L^\infty} = A_1 \inf\left\{B > 0 : \mu(\{|f_1| > B\}) = 0\right\} \leqslant A_1\delta\alpha = \alpha/2$$

Provided we stipulate $\delta := 1/(2A_1)$. Therefore the set $\{|T(f_1)| > \alpha/2\}$ has measure zero (this is immediate since $\|T(f_1)\|_{L^\infty} = \inf\left\{B > 0 : \mu(\{|T(f_1)| > B\}) = 0\right\} \leqslant \alpha/2$ and any subset of a set with measure zero has itself measure zero). Thus similar to part **b.** of **(i.)** we get $d_{T(f)}(\alpha) \leqslant d_{T(f_0)}(\alpha/2)$.

Hypothesis (33) yields the estimate $d_{T(f_0)}(\alpha/2) \leqslant \left(\dfrac{A_0}{\alpha/2}\right)^{p_0} \displaystyle\int_{\{2A_1|f|>\alpha\}} |f|^{p_0} d\mu$.

Thus by **a.** and **b.**

$$\begin{aligned}
\|T(f)\|_{L^p}^p &= p\int_0^\infty \alpha^{p-1} d_{T(f)} d\lambda \\
&\leqslant p(2A_0)^{p_0}\int_0^\infty \alpha^{p-p_0-1}\int_{\{2A_1|f|>\alpha\}} |f|^{p_0} d\mu d\lambda \\
&= p(2A_0)^{p_0}\int_X |f|^{p_0}\int_0^{2A_1|f|} \alpha^{p-p_0-1} d\lambda d\mu \\
&= \frac{2^p p A_0^{p_0} A_1^{p-p_0}}{p-p_0}\int_X |f|^p d\mu \\
&= \frac{2^p p A_0^{p_0} A_1^{p-p_0}}{p-p_0}\|f\|_{L^p}^p
\end{aligned} \tag{47}$$

That the constant $2^p p A_0^{p_0} A_1^{p-p_0}/(p-p_0)$ found in (47) is the p-th power of the one stated in the theorem can be seen by passing the constant (36) to the limit $p_1 \to \infty$:

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$$
\begin{aligned}
\lim_{p_1\to\infty} A = \lim_{p_1\to\infty} & \left[2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}}\right] \\
=& 2\exp\left[\frac{1}{p}\log\left(\frac{p}{p-p_0}+\lim_{p_1\to\infty}\frac{1}{p_1}\frac{p}{1-p\lim\limits_{p_1\to\infty}\frac{1}{p_1}}\right)\right] \\
&\cdot \lim_{p_1\to\infty} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} \cdot \lim_{p_1\to\infty} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}} \\
=& 2\left(\frac{p}{p-p_0}\right)^{1/p}\exp\left[\frac{\frac{1}{p}-\lim\limits_{p_1\to+\infty}\frac{1}{p_1}}{\frac{1}{p_0}-\lim\limits_{p_1\to\infty}\frac{1}{p_1}}\log(A_0)\right] \\
&\cdot\exp\left[\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\lim\limits_{p_1\to\infty}\frac{1}{p_1}}\log(A_1)\right] \\
=& 2\left(\frac{p}{p-p_0}\right)^{1/p} A_0^{\frac{p_0}{p}} A_1^{1-\frac{p_0}{p}}
\end{aligned}
$$

□

Appendix A. Limit superior and limit inferior revisited

Definition A.1. *Let (X,d) a metric space, $E \subseteq X$, $f : E \to \mathbb{R}$ and $a \in X$ be a limit point of E. Then we define the* upper limit of f at a *as*

$$\limsup_{x\to a} f(x) := \lim_{\varepsilon\downarrow 0}\left[\sup\left\{f(x) : x \in E \cap \dot{B}_\varepsilon(a)\right\}\right]$$

and the lower limit of f at a *as*

$$\liminf_{x\to a} f(x) := -\limsup_{x\to a}\,(-f)\,(x)$$

Proposition A.1. *Let (X,d) a metric space, $E \subseteq X$, $f,g : E \to \mathbb{R}$, where f is bounded and $a \in X$ be a limit point of E. Then*

$$\limsup_{x\to a}(fg)(x) = \limsup_{x\to a} f(x)\lim_{x\to a} g(x)$$

whenever both sides exist and $\lim_{x\to a} g(x) \geqslant 0$.

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Proof. Write

$$fg = f \lim_{x\to a} g(x) + f\left[g - \lim_{x\to a} g(x)\right]$$

By [Bou95, p. 358] we have

$$\begin{aligned}
\limsup_{x\to a} (fg)(x) &= \limsup_{x\to a} \left(f(x) \lim_{x\to a} g(x) + f(x) \left[g(x) - \lim_{x\to a} g(x)\right]\right) \\
&= \limsup_{x\to a} \left(f(x) \lim_{x\to a} g(x)\right) + \lim_{x\to a} \left(f(x) \left[g(x) - \lim_{x\to a} g(x)\right]\right) \\
&= \limsup_{x\to a} \left(f(x) \lim_{x\to a} g(x)\right)
\end{aligned}$$

since $\lim_{x\to a}\left[g(x) - \lim_{x\to a} g(x)\right] = 0$ and f is bounded. Fix $\varepsilon > 0$. Further by [Bou95, p. 357] and $\lim_{x\to a} g(x) \geqslant 0$

$$\sup\left\{f(x)\lim_{x\to a} g(x) : x \in E \cap \dot{B}_\varepsilon(a)\right\} = \sup\left\{f(x) : x \in E \cap \dot{B}_\varepsilon(a)\right\} \lim_{x\to a} g(x)$$

Hence

$$\begin{aligned}
\limsup_{x\to a}(fg)(x) &= \limsup_{x\to a}\left(f(x)\lim_{x\to a} g(x)\right) \\
&= \lim_{\varepsilon\downarrow 0}\left[\sup\left\{f(x)\lim_{x\to a} g(x) : x \in E \cap \dot{B}_\varepsilon(a)\right\}\right] \\
&= \lim_{\varepsilon\downarrow 0}\left[\sup\left\{f(x) : x \in E \cap \dot{B}_\varepsilon(a)\right\}\right]\lim_{x\to a} g(x) \\
&= \limsup_{x\to a} f(x) \lim_{x\to a} g(x)
\end{aligned} \tag{48}$$

□

Appendix B. Measure Theory

Let (X, μ) be a measure space. Recall, that if for each measurable set E with $\mu(E) = \infty$ there exists a measurable set $F \subseteq E$ and $0 < \mu(F) < \infty$, μ is called *semifinite*.

Lemma B.1. *Every σ-finite measure is semifinite.*

Proof. Let $X = \bigcup_{n\in\mathbb{N}} X_n$ where $\mu(X_n) < \infty$ and E be measurabele with $\mu(E) = \infty$. By letting $Y_n := \bigcup_{k\leqslant n} X_k$, Y_n is an increasing sequence. Then $E \cap Y_n$ is measurable and since $E \cap Y_n \subseteq Y_n$, $\mu(E\cap Y_n) < \infty$ for each $n \in \mathbb{N}$. By the continuity from below (see [Coh13, p. 10] or [Fol99, p. 26]) we have

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$$\infty = \mu(E) = \mu(E \cap X) = \mu\left(E \cap \left(\bigcup_{n\in\mathbb{N}} Y_n\right)\right) = \mu\left(\bigcup_{n\in\mathbb{N}} (E \cap Y_n)\right) = \lim_{n\to\infty} \mu\left(E \cap Y_n\right)$$

Hence for every $C > 0$ there exists $N \in \mathbb{N}$, such that $\infty > \mu(E \cap Y_n) > C$ for $n > N$. $\square$

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