Yannis Bähni
yannis.baehni@uzh.ch

CLASSICAL FOURIER ANALYSIS: INTERPOLATION ON L^p SPACES

Abstract. In this written seminar work I will basically follow [Gra14, pp. 33–48]. I will review three basic but important theorems on interpolation of operators on L^p spaces, namely the *Marcinkiewicz Interpolation Theorem*, the *Riesz-Thorin Interpolation Theorem* and finally an important extension of the Riesz-Thorin Interpolation Theorem to analytic families of operators (the so-called *Stein's theorem on interpolation of analytic families of operators*). We are mainly concerned with the notion of linear operators as well as slight generalizations of them.

Contents

1. Linear Operators. First we need to have a precise and suitable idea of *linear operators* in the generalized setting of measure spaces.

Definition 1.1. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces. Further let T be an operator defined on a linear space of complex-valued measurable functions on X and taking values in the set of all complex-valued, finite almost everywhere, measurable functions on Y. Then T is called* linear *if for all functions f and g in the domain of T and all $z \in \mathbb{C}$ holds*

$$T(f+g) = T(f) + T(g) \qquad T(zf) = zT(f) \tag{1}$$

and quasi-linear *if*

$$|T(f+g)| \leqslant K\left(|T(f)| + |T(g)|\right) \qquad |T(zf)| = |z||T(f)| \tag{2}$$

holds for some real constant $K > 0$. If $K = 1$, T is called sublinear.

Remark 1.1. *For simplicity I will omnit in most of what follows that a complex-valued function assumes values in the measured space $(\mathbb{C}, \mathfrak{B}^2)$ where for $p \in \mathbb{N}_{>0}$ we define $\mathfrak{B}^p := \mathfrak{B}(\mathbb{R}^p) = \sigma(\mathfrak{O}^p) := \sigma(\{U \subseteq \mathbb{R}^p : U \text{ open}\})$, the σ-*algebra of the Borel sets of $\mathbb{R}^p$ *(the notation $\sigma(\mathfrak{C})$ for $\mathfrak{C} \subseteq \mathcal{P}(X)$ of any set X denotes the σ-*algebra generated by $\mathfrak{C}$*). For more details see [Els11, pp. 16–19].*

2. The Real Method. The name originates from the real variables technique used for prooving the theorem.

Theorem 2.1. (The Marcinkiewicz Interpolation Theorem) *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, $(Y, \mathcal{B}, \nu)$ another measure space and $0 < p_0 < p_1 \leqslant +\infty$. Further let T be a sublinear operator defined on*

$$L^{p_0}(X, \mathcal{A}, \mu) + L^{p_1}(X, \mathcal{A}, \mu) := \{f_0 + f_1 : f_0 \in L^{p_0}(X, \mathcal{A}, \mu), f_1 \in L^{p_1}(X, \mathcal{A}, \mu)\}$$

and taking values in the space of measurable functions on Y. Assume that there exist $A_0, A_1 < +\infty$ such that

$$\forall f \in L^{p_0}(X, \mathcal{A}, \mu), \|T(f)\|_{L^{p_0,\infty}(Y,\mathcal{B},\nu)} \leqslant A_0 \|f\|_{L^{p_0}(X,\mathcal{A},\mu)} \tag{3}$$

$$\forall f \in L^{p_1}(X, \mathcal{A}, \mu), \|T(f)\|_{L^{p_1,\infty}(Y,\mathcal{B},\nu)} \leqslant A_1 \|f\|_{L^{p_1}(X,\mathcal{A},\mu)} \tag{4}$$

Then for all $p_0 < p < p_1$ and for all $f \in L^p(X, \mathcal{A}, \mu)$ we have the estimate

$$\|T(f)\|_{L^p(Y,\mathcal{B},\nu)} \leqslant A \|f\|_{L^p(X,\mathcal{A},\mu)} \tag{5}$$

where

$$A := 2 \left(\frac{p}{p - p_0} + \frac{p}{p_1 - p} \right)^{1/p} A_0^{\frac{\frac{1}{p} - \frac{1}{p_1}}{\frac{1}{p_0} - \frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0} - \frac{1}{p}}{\frac{1}{p_0} - \frac{1}{p_1}}} \tag{6}$$

Proof. The proof is subdivided into two main parts, which are further subdivided. In detail, we have the following partitioning:

(i.) $\underline{p_1 < +\infty}$.
  - **a.** Split f using cut-off functions.
  - **b.** Estimate the distribution function $d_{T(f)}$.
  - **c.** Estimate $\|T(f)\|^p_{L^p(Y,\mathcal{B},\nu)}$.

(ii.) $\underline{p_1 = +\infty}$.
  - **a.** Show that $\mu(\{|T(f_1)| > \alpha/2\}) = 0$.
  - **b.** Estimate the distribution function $d_{T(f_0)}$.
  - **c.** Estimate $\|T(f)\|^p_{L^p(Y,\mathcal{B},\nu)}$.

(i.) **a.** Let us first consider the case $\underline{p_1 < +\infty}$. Fix $f \in L^p(X, \mathcal{A}, \mu)$, $\alpha > 0$ and $\delta > 0$ (δ will be determined later). We split f using so-called *cut-off* functions, by stipulating $f \equiv f_0(\cdot; \alpha, \delta) + f_1(\cdot; \alpha, \delta)$, where $f_0(\cdot; \alpha, \delta)$ is the *unbounded part of* f and $f_1(\cdot; \alpha, \delta)$ is the *bounded part of* f, defined by

$$\begin{aligned} f_0(x; \alpha, \delta) &:= \begin{cases} f(x), & |f(x)| > \delta\alpha, \\ 0, & |f(x)| \leqslant \delta\alpha. \end{cases} \\ f_1(x; \alpha, \delta) &:= \begin{cases} f(x), & |f(x)| \leqslant \delta\alpha, \\ 0, & |f(x)| > \delta\alpha. \end{cases} \end{aligned} \tag{7}$$

for $x \in X$. To facilitate reading I will omit the dependency of $f_0(\cdot; \alpha, \delta)$ and $f_1(\cdot; \alpha, \delta)$ upon the parameters α and δ and simply use f_0, f_1 respectively. Since $p_0 < p$ we have

Yannis Bähni
yannis.baehni@uzh.ch

$$
\begin{aligned}
\|f_0\|^{p_0}_{L^{p_0}(X,\mathcal{A},\mu)} &= \int_X |f_0|^{p_0} d\mu \\
&= \int_X |f|^{p_0} \cdot \chi_{\{|f|>\delta\alpha\}} d\mu \\
&\overset{(\dagger)}{=} \int_{\{|f|>\delta\alpha\}} |f|^{p_0} d\mu \\
&= \int_{\{|f|>\delta\alpha\}} |f|^p |f|^{p_0-p} d\mu \\
&= \int_{\{|f|>\delta\alpha\}} \frac{|f|^p}{|f|^{p-p_0}} d\mu \\
&\leqslant \frac{1}{(\delta\alpha)^{p-p_0}} \int_{\{|f|>\delta\alpha\}} |f|^p d\mu \\
&= (\delta\alpha)^{p_0-p} \int_X |f|^p \cdot \chi_{\{|f|>\delta\alpha\}} d\mu \\
&\leqslant (\delta\alpha)^{p_0-p} \int_X |f|^p d\mu \\
&= (\delta\alpha)^{p_0-p} \|f\|^p_{L^p(X,\mathcal{A},\mu)}
\end{aligned} \tag{8}
$$

Since $f \in L^p(X,\mathcal{A},\mu)$ and thus $\|f\|_{L^p(X,\mathcal{A},\mu)} < +\infty$, we have by estimate (8) $f_0 \in L^{p_0}(X,\mathcal{A},\mu)$. Analogously we get $\|f_1\|^{p_1}_{L^{p_1}(X,\mathcal{A},\mu)} \leqslant (\delta\alpha)^{p_1-p}\|f\|^p_{L^p(X,\mathcal{A},\mu)}$ and so $f_1 \in L^{p_1}(X,\mathcal{A},\mu)$. Therefore $f \equiv f_0 + f_1 \in L^{p_0}(X,\mathcal{A},\mu) + L^{p_1}(X,\mathcal{A},\mu)$.

Proof of the equality (†). We have to proove that $\{|f| > \delta\alpha\} \in \mathcal{A}$[1]. Since f is complex-valued, we may write $f \equiv \mathrm{Re}f + i\mathrm{Im}f$ and thus $|f|^2 \equiv \mathrm{Re}^2 f + \mathrm{Im}^2 f$. Since f is measurable by hypothesis this implies that $\mathrm{Re}f$ and $\mathrm{Im}f$ are measurable[2]. Further for measurable real-valued functions $f, g : (X,\mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathfrak{B}})$[3] the functions $f+g$ and $f \cdot g$ are measurable[4] and thus $|f|^2$ is measurable. Hence $\{\mathrm{Re}^2 f + \mathrm{Im}^2 f > \lambda\} \in \mathcal{A}$[5] for any $\lambda \in \mathbb{R}$. So especially for $\lambda := (\delta\alpha)^2$ we have $\{|f| > \delta\alpha\} \in \mathcal{A}$[6]. In a similar manner it can also be prooven that $\{|f| \leqslant \delta\alpha\} \in \mathcal{A}$. Let us proove a useful

Lemma 2.1. *Let* $A \in \wp(X)$ *and* $\chi_A : (X,\mathcal{A}) \to (\mathbb{C}, \mathfrak{B}^2)$ *be the* characteristic function of the set A. *Then* χ_A *is measurable if and only if* A *is measurable.*

Proof. Assume χ_A is measurable. Then $\mathrm{Re}\chi_A$ and $\mathrm{Im}\chi_A$ are measurable. Especially for $0 < \lambda < 1$ we have that $\{\mathrm{Re}\chi_A > \lambda\} = A \in \mathcal{A}$. Conversly, assume A is measurable. For $\lambda \in \mathbb{R}_{<0}$ we have $\{\mathrm{Re}\chi_A > \lambda\} = X \in \mathcal{A}$, $\lambda \in [0,1[$, $\{\mathrm{Re}\chi_A > \lambda\} = A \in \mathcal{A}$ and

[1] For $Y \in \mathcal{A}$ the μ-integral of $f : X \to \mathbb{C}$ over Y is defined to be $\int_Y f d\mu := \int_X f \cdot \chi_Y d\mu$. For more details see [Els11, pp. 135–136].

[2] For a proof see [Els11, p. 106]

[3] $\mathfrak{B} := \sigma(\mathbb{R})$ and $\overline{\mathfrak{B}} = \{B \cup E : B \in \mathfrak{B}, E \subseteq \{\pm\infty\}\}$.

[4] For a proof see [Els11, p. 107].

[5] For a proof see [Els11, pp. 105–106]

[6] This follows from the fact that $x < y$ if and only if $x^n < y^n$ for $n \in \mathbb{N}_{>0}$ and some real numbers $x, y > 0$

Yannis Bähni
yannis.baehni@uzh.ch

$\{\mathrm{Re}\chi_A > \lambda\} = \emptyset \in \mathcal{A}$ for $\lambda \in \mathbb{R}_{\geqslant 1}$. Since $\mathrm{Im}\chi_A \equiv 0$ we have $\{\mathrm{Im}\chi_A > \lambda\} = X \in \mathcal{A}$ if $\lambda \in \mathbb{R}_{<0}$ and $\{\mathrm{Im}\chi_A > \lambda\} = \emptyset \in \mathcal{A}$ if $\lambda \in \mathbb{R}_{\geqslant 0}$. $\square$

By Lemma 2.1 and the fact that $f \cdot g$ is measurable for two measurable functions $f, g : (X, \mathcal{A}) \to (\mathbb{C}, \mathfrak{B}^2)$[7], f_0 and f_1 are measurable since $f_0 \equiv f \cdot \chi_{\{|f|>\delta\alpha\}}$ and $f_1 \equiv f \cdot \chi_{\{|f|\leqslant\delta\alpha\}}$.

One subtility is left to clear: the μ-integrability of either $|f_1|^{p_0}$ or $|f_1|^{p_1}$ requires that $|f_0|^{p_0}$ and $|f_1|^{p_1}$ are measurable functions. By the fact that any continuous map $g : (X, d_X) \to (Y, d_Y)$ between metric spaces is Borel-measurable (see [Els11, p. 86]) and that the composition of measurable functions is again measurable (see [Els11, p. 87]), the measurability of either f_0 or f_1 follows by $|f_0|^{p_0} \equiv \cdot^{p_0} \circ |f \cdot \chi_{\{|f|>\delta\alpha\}}|$ and $|f_1|^{p_1} \equiv \cdot^{p_1} \circ |f \cdot \chi_{\{|f|\leqslant\delta\alpha\}}|$ by stipulating $\cdot^p : (\mathbb{R}_{\geqslant 0}, |\cdot|) \to (\mathbb{C}, |\cdot|)$, $x^p := \exp(p\log(x))$ for $p > 0$ and $x \in \mathbb{R}_{>0}$ and $x^p := 0$ if $x = 0$.

b. Since T is a sublinear operator we have $|T(f)| = |T(f_0 + f_1)| \leqslant |T(f_0)| + |T(f_1)|$. Thus for any $y \in Y$ with $|T(f)(y)| > \alpha$ we therefore have either $|T(f_0)(y)| > \frac{\alpha}{2}$ or $|T(f_1)(y)| > \frac{\alpha}{2}$ [8]. Hence

$$\{|T(f)| > \alpha\} \subseteq \{|T(f_0)| > \alpha/2\} \cup \{|T(f_1)| > \alpha/2\}$$

and so by the monotonicity [9] and subadditivity [10] property of the measure $\mu : \mathcal{A} \to \overline{\mathbb{R}}$ we have

$$\begin{aligned} d_{T(f)}(\alpha) &= \mu(\{|T(f)| > \alpha\}) \\ &\leqslant \mu(\{|T(f_0)| > \alpha/2\} \cup \{|T(f_1)| > \alpha/2\}) \\ &\leqslant \mu(\{|T(f_0)| > \alpha/2\}) + \mu(\{|T(f_1)| > \alpha/2\}) \\ &= d_{T(f_0)}(\alpha/2) + d_{T(f_1)}(\alpha/2) \end{aligned} \tag{9}$$

Now by the hypothesis $\|T(f)\|_{L^{p_0,\infty}(Y,\mathcal{B},\nu)} \leqslant A_0\|f\|_{L^{p_0}(X,\mathcal{A},\mu)}$ we have for $d_{T(f_0)}(\alpha/2)$ the estimate

$$\begin{aligned} d_{T(f_0)}(\alpha/2) &= \left(\frac{\alpha/2}{\alpha/2}\right)^{p_0} d_{T(f_0)}(\alpha/2) \\ &\leqslant \left(\frac{1}{\alpha/2}\right)^{p_0} \left[\sup\left\{\gamma d_{T(f_0)}(\gamma)^{1/p_0} : \gamma \in \mathbb{R}_{>0}\right\}\right]^{p_0} \\ &= \left(\frac{1}{\alpha/2}\right)^{p_0} \|T(f_0)\|^{p_0}_{L^{p_0,\infty}(Y,\mathcal{B},\nu)} \\ &\leqslant \left(\frac{A_0}{\alpha/2}\right)^{p_0} \|f_0\|^{p_0}_{L^{p_0}(X,\mathcal{A},\mu)} \end{aligned} \tag{10}$$

[7] Els11, p. 107.

[8] Without loss of generality assume $|T(f_0)(y)| \leqslant |T(f_1)(y)|$. Then we have $\alpha < |T(f)(y)| \leqslant |T(f_0)(y)| + |T(f_1)(y)| \leqslant 2|T(f_1)(y)|$ (this is possible since $\mathbb{R}$ is an ordered field).

[9] $A, B \in \mathcal{A}$ with $A \subseteq B$ implies $\mu(A) \leqslant \mu(B)$. This follows immediately from the observation that $B = (B\setminus A) \cup A$ and $(B \setminus A) \cap A = \emptyset$ implies by the σ-additivity of $\mathcal{A}$ directly the inequality $\mu(A) \leqslant \mu(B \setminus A) + \mu(A) = \mu(B)$ since $B \setminus A$ is measurable and $\mu(B \setminus A) \geqslant 0$.

[10] $(A_\iota)_{\iota\in I} \in \mathcal{A}^I$ where $|I| < \aleph_0$ implies $\mu\left(\bigcup_{\iota\in I} A_\iota\right) \leqslant \sum_{\iota\in I} \mu(A_\iota)$. A proof can be found in [Els11, p. 31].

Yannis Bähni
yannis.baehni@uzh.ch

Analogously we get by the second hypothesis $\|T(f)\|_{L^{p_1,\infty}(Y,\mathcal{B},\nu)} \leqslant A_1\|f\|_{L^{p_1}(X,\mathcal{A},\mu)}$ an estimate for $d_{T(f_1)}(\alpha/2)$ of the form $d_{T(f_1)}(\alpha/2) \leqslant \left(\frac{A_1}{\alpha/2}\right)^{p_1} \|f_1\|^{p_1}_{L^{p_1}(X,\mathcal{A},\mu)}$. Combining estimates (9), (10) and (**b.**) and using the definitions of f_0, f_1 we arrive at

$$d_{T(f)}(\alpha) \leqslant \left(\frac{A_0}{\alpha/2}\right)^{p_0} \int_{\{|f|>\delta\alpha\}} |f|^{p_0} d\mu + \left(\frac{A_1}{\alpha/2}\right)^{p_1} \int_{\{|f|\leqslant\delta\alpha\}} |f|^{p_1} d\mu \tag{11}$$

c. By

$$\int_0^{\frac{1}{\delta}|f|} \alpha^{p-p_0-1} d\lambda = \begin{cases} \frac{1}{p-p_0}\frac{1}{\delta^{p-p_0}}|f|^{p-p_0}, & p \geqslant p_0+1 \\ \lim\limits_{\omega\to 0^+} \int_\omega^{\frac{1}{\delta}|f|} \alpha^{p-p_0-1} d\lambda & \\ = \lim\limits_{\omega\to 0^+} \left[\frac{1}{p-p_0}\alpha^{p-p_0}\Big|_\omega^{\frac{1}{\delta}|f|}\right] & \\ = \frac{1}{p-p_0}\left[\frac{1}{\delta^{p-p_0}}|f|^{p-p_0} - \lim\limits_{\omega\to 0^+}\omega^{p-p_0}\right] & \\ = \frac{1}{p-p_0}\frac{1}{\delta^{p-p_0}}|f|^{p-p_0}, & p_0 < p < p_0+1 \end{cases} \tag{12}$$

and

$$\begin{aligned} \int_{\frac{1}{\delta}|f|}^{+\infty} \alpha^{p-p_1-1} d\lambda &= \lim_{\omega\to+\infty}\left[\frac{1}{p-p_1}\alpha^{p-p_1}\right]_{\frac{1}{\delta}|f|}^{\omega} \\ &= \frac{1}{p-p_1}\left[\lim_{\omega\to+\infty}\omega^{p-p_1} - \frac{1}{\delta^{p-p_1}}|f|^{p-p_1}\right] \\ &= \frac{1}{p_1-p}\frac{1}{\delta^{p-p_1}}|f|^{p-p_1} \end{aligned} \tag{13}$$

and the representation $\|f\|^p_{L^p(X,\mathcal{A},\mu)} = p\int_0^{+\infty} \alpha^{p-1} d_f(\alpha) d\lambda$ for $0 < p < +\infty$ we get

Yannis Bähni
yannis.baehni@uzh.ch

$$
\begin{aligned}
&\|T(f)\|^p_{L^p(Y,\mathcal{B},\nu)} \\
=&p\int_0^{+\infty}\alpha^{p-1}d_{T(f)}d\lambda \\
\leqslant& p(2A_0)^{p_0}\int_0^{+\infty}\alpha^{p-p_0-1}\int_{\{|f|>\delta\alpha\}}|f|^{p_0}d\mu d\lambda + p(2A_1)^{p_1}\int_0^{+\infty}\alpha^{p-p_1-1}\int_{\{|f|\leqslant\delta\alpha\}}|f|^{p_1}d\mu d\lambda \\
\overset{(\ddagger)}{=}&p(2A_0)^{p_0}\int_{\{|f|>0\}}|f|^{p_0}\int_0^{\frac{1}{\delta}|f|}\alpha^{p-p_0-1}d\lambda d\mu + p(2A_0)^{p_0}\int_{\{|f|=0\}}|f|^{p_0}\int_0^{\frac{1}{\delta}|f|}\alpha^{p-p_0-1}d\lambda d\mu \\
&+p(2A_1)^{p_1}\int_X|f|^{p_1}\int_{\frac{1}{\delta}|f|}^{+\infty}\alpha^{p-p_1-1}d\lambda d\mu \\
=&p(2A_0)^{p_0}\int_X|f|^{p_0}\int_0^{\frac{1}{\delta}|f|}\alpha^{p-p_0-1}d\lambda d\mu + p(2A_1)^{p_1}\int_X|f|^{p_1}\int_{\frac{1}{\delta}|f|}^{+\infty}\alpha^{p-p_1-1}d\lambda d\mu \\
=&\frac{p(2A_0)^{p_0}}{p-p_0}\frac{1}{\delta^{p-p_0}}\int_X|f|^{p_0}|f|^{p-p_0}d\mu + \frac{p(2A_1)^{p_1}}{p_1-p}\frac{1}{\delta^{p-p_1}}\int_X|f|^{p_1}|f|^{p-p_1}d\mu \\
&= p\left(\frac{(2A_0)^{p_0}}{p-p_0}\frac{1}{\delta^{p-p_0}} + \frac{(2A_1)^{p_1}}{p_1-p}\delta^{p_1-p}\right)\|f\|^p_{L^p(X,\mathcal{A},\mu)}
\end{aligned}
\tag{14}
$$

We pick $\delta > 0$ such that $(2A_0)^{p_0}\frac{1}{\delta^{p-p_0}} = (2A_1)^{p_1}\delta^{p_1-p}$. Solving for δ yields

$$
\delta = \frac{1}{2}\frac{A_0^{\frac{p_0}{p_1-p_0}}}{A_1^{\frac{p_1}{p_1-p_0}}}
\tag{15}
$$

Substituting this in estimate (14) leads to

$$
\begin{aligned}
\|T(f)\|^p_{L^p(Y,\mathcal{B},\nu)} &\leqslant p\left(\frac{(2A_0)^{p_0}}{p-p_0}\frac{2^{p-p_0}A_1^{\frac{p_1(p-p_0)}{p_1-p_0}}}{A_0^{\frac{p_0(p-p_0)}{p_1-p_0}}} + \frac{(2A_1)^{p_1}}{p_1-p}\frac{A_0^{\frac{p_0(p_1-p)}{p_1-p_0}}}{2^{p_1-p}A_1^{\frac{p_1(p_1-p)}{p_1-p_0}}}\right)\|f\|^p_{L^p(X,\mathcal{A},\mu)} \\
&= 2^p p\left(\frac{A_0^{\frac{p_0(p_1-p)}{p_1-p_0}}A_1^{\frac{p_1(p-p_0)}{p_1-p_0}}}{p-p_0} + \frac{A_0^{\frac{p_0(p_1-p)}{p_1-p_0}}A_1^{\frac{p_1(p-p_0)}{p_1-p_0}}}{p_1-p}\right)\|f\|^p_{L^p(X,\mathcal{A},\mu)}
\end{aligned}
\tag{16}
$$

And taking the p-th power further

Yannis Bähni
yannis.baehni@uzh.ch

$$
\begin{aligned}
\|T(f)\|_{L^p(Y,\mathcal{B},\nu)} &\leqslant 2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{p_0(p_1-p)}{p(p_1-p_0)}} A_1^{\frac{p_1(p-p_0)}{p(p_1-p_0)}} \|f\|_{L^p(X,\mathcal{A},\mu)} \\
&= 2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{p_0(p_1-p)}{p(p_1-p_0)}\frac{p_1}{p_1}} A_1^{\frac{p_1(p-p_0)}{p(p_1-p_0)}\frac{p_0}{p_0}} \|f\|_{L^p(X,\mathcal{A},\mu)} \\
&= 2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{\frac{p_1-p}{pp_1}}{\frac{p_1-p_0}{p_0p_1}}} A_1^{\frac{\frac{p-p_0}{p_0p}}{\frac{p_1-p_0}{p_0p_1}}} \|f\|_{L^p(X,\mathcal{A},\mu)} \\
&= 2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}} \|f\|_{L^p(X,\mathcal{A},\mu)}
\end{aligned}
\tag{17}
$$

Proof of the equality (‡). The set $\{|f|=0\}$ is clearly measurable since by the measurability of f the sets $\{|f|\geqslant 0\}$ and $\{|f|\leqslant 0\}$ are measurable and thus by the property of a σ-algebra we have $\{|f|=0\}=\{|f|\geqslant 0\}\cap\{|f|\leqslant 0\}\in\mathcal{A}$ (or by the property of a σ-algebra we have $\{|f|=0\}^c=\{|f|>0\}\in\mathcal{A}$).

(ii.) **a.** Now consider the case $p_1=+\infty$. We again use the cut-off functions defined in 7 to write $f\equiv f_0+f_1\in L^{p_0}(X,\mathcal{A},\mu)+L^{p_1}(X,\mathcal{A},\mu)$. Since $\{|f_1|>\delta\alpha\}=\emptyset$, we have

$$
\begin{aligned}
\|T(f_1)\|_{L^\infty(Y,\mathcal{B},\nu)} &\leqslant A_1\|f_1\|_{L^\infty(X,\mathcal{A},\mu)} \\
&= A_1\inf\{B\in\mathbb{R}_{>0}:\mu(\{|f_1|>B\})=0\} \\
&\leqslant A_1\delta\alpha \\
&= \frac{\alpha}{2}
\end{aligned}
\tag{18}
$$

Provided we stipulate $\delta:=\frac{1}{2A_1}$. Therefore the set $\{|T(f_1)|>\alpha/2\}$ has measure zero (this is immediate since $\|T(f_1)\|_{L^\infty(Y,\mathcal{B},\nu)}=\inf\{B\in\mathbb{R}_{>0}:\mu(\{|T(f_1)|>B\})=0\}\leqslant\alpha/2$ and any subset of a set with measure zero has itself measure zero [11]). Thus similar to part **b.** of **(i.)** we get $d_{T(f)}(\alpha)\leqslant d_{T(f_0)}(\alpha/2)$.

b. By the hypothesis $\|T(f)\|_{L^{p_0,\infty}(Y,\mathcal{B},\nu)}\leqslant A_0\|f\|_{L^{p_0}(X,\mathcal{A},\mu)}$ for all $f\in L^{p_0}(X,\mathcal{A},\mu)$ we have again the estimate $d_{T(f_0)}(\alpha/2)\leqslant\left(\frac{A_0}{\alpha/2}\right)^{p_0}\int_{\{2A_1|f|>\alpha\}}|f|^{p_0}d\mu$.

c. Thus by **a.** and **b.**

[11] Let $A,B\in\mathcal{A}$ with $A\subseteq B$ and $\mu(B)=0$. Then $0\leqslant\mu(A)\leqslant\mu(B)=0$.

Yannis Bähni
yannis.baehni@uzh.ch

$$
\begin{aligned}
\|T(f)\|_{L^p(Y,\mathcal{B},\nu)}^p &= p\int_0^{+\infty} \alpha^{p-1} d_{T(f)} d\lambda \\
&\leqslant p(2A_0)^{p_0} \int_0^{+\infty} \alpha^{p-p_0-1} \int_{\{2A_1|f|>\alpha\}} |f|^{p_0} d\mu d\lambda \\
&= p(2A_0)^{p_0} \int_X |f|^{p_0} \int_0^{2A_1|f|} \alpha^{p-p_0-1} d\lambda d\mu \\
&= \frac{2^p p A_0^{p_0} A_1^{p-p_0}}{p-p_0} \int_X |f|^p d\mu \\
&= \frac{2^p p A_0^{p_0} A_1^{p-p_0}}{p-p_0} \|f\|_{L^p(X,\mathcal{A},\mu)}^p
\end{aligned} \tag{19}
$$

That the constant found in (19) really agrees with the one stated in the theorem, can be seen by passing the constant (6) to the limit $p_1 \to +\infty$. We get

$$
\begin{aligned}
\lim_{p_1\to+\infty} A = &\lim_{p_1\to+\infty}\left[2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}}\right] \\
=&2\exp\left[\frac{1}{p}\log\left(\frac{p}{p-p_0}+\lim_{p_1\to+\infty}\frac{1}{p_1}\frac{p}{1-p\lim_{p_1\to+\infty}\frac{1}{p_1}}\right)\right] \\
&\cdot \lim_{p_1\to+\infty} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} \cdot \lim_{p_1\to+\infty} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}} \\
=&2\left(\frac{p}{p-p_0}\right)^{1/p} \exp\left[\frac{\frac{1}{p}-\lim_{p_1\to+\infty}\frac{1}{p_1}}{\frac{1}{p_0}-\lim_{p_1\to+\infty}\frac{1}{p_1}}\log(A_0)\right]\exp\left[\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\lim_{p_1\to+\infty}\frac{1}{p_1}}\log(A_1)\right] \\
=&2\left(\frac{p}{p-p_0}\right)^{1/p} A_0^{\frac{p_0}{p}} A_1^{1-\frac{p_0}{p}}
\end{aligned} \tag{20}
$$

Taking the p-th power in the estimate (19) finally yields the desired result.

□

3. The Complex Method. This theorem will unfortunately only be applicable to linear operators but will yield a more natural bound of the operator on the intermediate space. The proof will make strong use of complex variables technique. A major tool will be an application of the maximum modulus principle, known as *Hadamard's three lines lemma.*

3.1. Hadamard's Three Lines Lemma.

LEMMA 3.1. *Hadamard's three lines lemma) Let F be an analytic function on the strip $S := \{z \in \mathbb{C} : 0 < \mathrm{Re}z < 1\}$, continuous and bounded on $\overline{S}$, such that $|F(z)| \leqslant B_0$ when $\mathrm{Re}z = 0$ and $|F(z)| \leqslant B_1$ when $\mathrm{Re}z = 1$, for some $0 < B_0, B_1 < \infty$. Then $|F(z)| \leqslant B_0^{1-\theta} B_1^{\theta}$ when $\mathrm{Re}z = \theta$, for any $0 \leqslant \theta \leqslant 1$.*

Proof. For $z \in \overline{S}$ define

$$G(z) := \frac{F(z)}{B_0^{1-z} B_1^z} \qquad \forall n \in \mathbb{N}_{>0} : G_n(z) := G(z)e^{(z^2-1)/n} \tag{21}$$

Obviously, $G(z)$ and $G_n(z)$ are analytic functions on S for $n \in \mathbb{N}_{>0}$[12]. Further, we have

$$|B_0^{1-z} B_1^z|^2 = |B_0^{1-z}|^2 |B_1^z|^2 \overset{(\dagger)}{=} B_0^{1-z} B_0^{1-\overline{z}} B_1^z B_1^{\overline{z}} = \left(B_0^{1-\mathrm{Re}z}\right)^2 \left(B_1^{\mathrm{Re}z}\right)^2 \tag{22}$$

Consider $0 \leqslant \mathrm{Re}z \leqslant 1$ and $B_0 \geqslant 1$. Then $B_0^{1-\mathrm{Re}z} = \exp\left((1 - \mathrm{Re}z)\log B_0\right) \geqslant 1$ and $B_0^{1-\mathrm{Re}z} \geqslant B_0$ in the case $B_0 < 1$. A similar estimation of $B_1^{\mathrm{Re}z}$ leads to

$$|B_0^{1-z} B_1^z| \geqslant \min\{1, B_0\} \min\{1, B_1\} \tag{23}$$

for all $z \in \overline{S}$. By this, $G(z)$ is bounded on $\overline{S}$ (by the boundedness of F). Let $M > 0$, such that $|G(z)| \leqslant M$ for $z \in \overline{S}$. Fix $n \in \mathbb{N}_{>0}$ and write $z := x + iy \in \overline{S}$. Since

$$\begin{aligned} |G_n(z)|^2 &= |G(z)|^2 |e^{((x+iy)^2-1)/n}|^2 \\ &\leqslant M^2 e^{(x^2+2ixy-y^2-1)/n} e^{(x^2-2ixy-y^2-1)/n} \\ &= M^2 \left(e^{-y^2/n}\right)^2 \left(e^{(x^2-1)/n}\right)^2 \\ &\leqslant M^2 \left(e^{-y^2/n}\right)^2 \\ &= M^2 \left(e^{-|y|^2/n}\right)^2 \end{aligned} \tag{24}$$

we have $\lim_{y \to \pm\infty} \sup\{|G_n(z)| : x \in [0,1]\} = 0$ by the pinching-principle. Hence there exists some $C(n) \in \mathbb{R}_{>0}$, such that $|G_n(z)| \leqslant 1$ for all $|y| \geqslant C(n)$ and all $x \in [0,1]$. Consider the rectangle $R := [0,1] \times [-C(n), C(n)]$. Now $|G_n(z)| \leqslant 1$ on the lines $[0,1] \times \{\pm C(n)\}$ and since $|G(z)| = |F(z)|/B_0 \leqslant 1$, $|G(z)| = |F(z)|/B_1 \leqslant 1$ on the line $\{0\} \times [-C(n), C(n)]$ and

[12] Recall, that a function f is called *analytic on* U, $U \subseteq \mathbb{C}$ open, if f is analytic at every $z_0 \in U$, that is, there exists a power series $\sum_{n\in\mathbb{N}} a_n(z - z_0)^n$ and some $r > 0$, such that the series converges absolutely for $|z - z_0| < r$, and such that for such z, we have $f(z) = \sum_{n\in\mathbb{N}} a_n(z - z_0)^n$ (as defined in [Lan93, pp. 68–69]). If f and g are analytic on $U \subseteq \mathbb{C}$, so are $f + g$, $f \cdot g$. Also f/g is analytic on the open subset of $z \in U$ such that $g(z) \neq 0$. If $g : U \to V$ and $f : V \to C$ are analytic so is $f \circ g$. Further, if $f(z) = \sum_{n\in\mathbb{N}} a_n z^n$ is a power series with radius of convergence r, f is analytic on $B_r(0)$ (for a proof see [Lan93, pp. 69–70]).

Yannis Bähni
yannis.baehni@uzh.ch

$\{1\} \times [-C(n), C(n)]$ respectively by assumption, we have $|G_n(z)| \leqslant 1$ on ∂S. By the maximum modulus principle [13] we have $|G_n(z)| \leqslant 1$ on R and thus $|G_n(z)| \leqslant 1$ on $\overline{S}$. Since inequalities are preserved by limits and the modulus is a continuous function, we have that $|G(z)| = \lim_{n\to\infty} |G_n(z)| \leqslant 1$ on $\overline{S}$. Taking $z := \theta + it$, where $0 \leqslant \theta \leqslant 1$ and $t \in \mathbb{R}$, we conclude $|F(z)| = |G(z)||B_0^{1-z} B_1^z| \leqslant B_0^{1-\theta} B_1^{\theta}$, which completes the proof.

Proof of the equality (†). For any $\alpha \in \mathbb{R}_{>0}$ and $z \in \mathbb{C}$ we have $\alpha^z = \exp(z \log(\alpha))$. Since the exponential function is convergent on the whole complex plane, for fixed $\varepsilon > 0$ we find $C \in \mathbb{N}$ such that $|\sum_{k=0}^N \frac{z^k}{k!} - \exp(z)| < \varepsilon$ whenever $N > C$. But by the properties of the complex conjugate we get $|\sum_{k=0}^N \frac{\overline{z}^k}{k!} - \overline{\exp(z)}| = |\overline{\sum_{k=0}^N \frac{z^k}{k!} - \exp(z)}| = |\sum_{k=0}^N \frac{z^k}{k!} - \exp(z)| < \varepsilon$. Therefore $\overline{\exp(z)} = \sum_{k\in\mathbb{N}} \frac{\overline{z}^k}{k!} = \exp(\overline{z})$ and thus $\overline{\alpha^z} = \alpha^{\overline{z}}$. □

Remark 3.1. *To apply the maximum modulus principle it is mandatory for G_n to be non-constant. That the constant case is obviously true can be seen as follows. Assume $G_n(z) \equiv w \in \mathbb{C}$ for $z \in S$. This immediately implies $F(z) = wB_0^{1-z}B_1^z e^{(1-z^2)/n}$. Hence $F(z) = F(z;n)$. Thus the only possible case left is $w = 0$ and so $F \equiv 0$. But then the lemma holds trivially.*

3.2. The Theorem. Now we are able to proove the Riesz-Thorin Interpolation theorem without an interruption.

Theorem 3.1. (Riesz-Thorin Interpolation Theorem) *Let $(X, \mathcal{A}, \mu)$ be a measure space, $(Y, \mathcal{B}, \nu)$ a σ-finite measure space and T be a linear operator defined on the set of all finitely simple functions on X and taking values in the set of measurable functions on Y. Let $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$ and assume that*

$$\|T(f)\|_{L^{q_0}(Y,\mathcal{B},\nu)} \leqslant M_0 \|f\|_{L^{p_0}(X,\mathcal{A},\mu)} \qquad \|T(f)\|_{L^{q_1}(Y,\mathcal{B},\nu)} \leqslant M_1 \|f\|_{L^{p_1}(X,\mathcal{A},\mu)} \tag{25}$$

holds for all finitely simple functions f on X and $0 < M_0, M_1 < \infty$. Then for all $0 < \theta < 1$ we have

$$\|T(f)\|_{L^q(Y,\mathcal{B},\nu)} \leqslant M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p(X,\mathcal{A},\mu)} \tag{26}$$

for all finitely simple functions f on X, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \tag{27}$$

:

[13] The theorem can be found in[Lan93, pp. 91–92]. I will reproduce it here.

Lemma 3.2. (Maximum Modulus Principle, global version) *Let $U \subseteq \mathbb{C}$ be a connected open set, and let f be an analytic function on U. If $z_0 \in U$ is a maximum point for $|f|$, that is $|f(z_0)| \geqslant |f(z)|$ for all $z \in U$, then f is constant on U.*

For our purpose the following corollary is more appropriate.

Corollary 3.1. *Let $U \subseteq \mathbb{C}$ be a connected open set and f be a continuous function on $\overline{U}$, analytic and non-constant on U. If $z_0 \in \overline{U}$ is a maximum for f, that is $|f(z_0)| \geqslant |f(z)|$ for all $z \in \overline{U}$, then $z_0 \in \partial U$.*

Yannis Bähni
yannis.baehni@uzh.ch

Proof. We will use the fact that the $L^p(Y, \mathcal{B}, \nu)$ norm of a function can be obtained via duality for $1 < p \leqslant \infty$ (for $p = \infty$ the underlying space has to be σ-finite according to [Els11, pp. 288–289]) by

$$\|f\|_{L^p(Y,\mathcal{B},\nu)} = \sup\left\{\left|\int_Y fgd\nu\right| : \|g\|_{L^{p'}(Y,\mathcal{B},\nu)} = 1\right\}$$

with $p' := \frac{p}{p-1}$ for $p \in]1, \infty[$ and $p' := 1$ for $p = \infty$. Since we will also make use of

$$\|f\|_{L^p(Y,\mathcal{B},\nu)} = \sup\left\{\left|\int_Y fgd\nu\right| : \|g\|_{L^{p'}(Y,\mathcal{B},\nu)} \leqslant 1\right\}$$

I will proove their equivalence. If we define $\varphi_f(g) : L^{p'}(Y, \mathcal{B}, \nu) \to \mathbb{C}$, $\varphi_f(g) := \int_Y fgd\mu$, φ_f is clearly a linear functional (to be precise, a continuous linear functional by [Els11, p. 289]). Hence let $(V, \|\cdot\|)$ and $(W, \|\cdot\|)$ be two normed vector spaces over $\mathbb{C}$ and $L \in \mathrm{Hom}_{\mathbb{C}}(V, W)$ continuous. Then we define $v_n := \left(1 - \frac{1}{n}\right) v$ for $v \in V$ with $\|v\| = 1$ and $n \in \mathbb{N}_{>0}$. We have $\|v_n\| = 1 - \frac{1}{n} \leqslant 1$. Thus $\|L(v_n)\| \leqslant \sup\{\|L(v)\| : \|v\| \leqslant 1\}$ and so $\lim_{n\to\infty} \|L(v_n)\| = \|L(v)\| \leqslant \sup\{\|L(v)\| : \|v\| \leqslant 1\}$. On the other hand we have $\|L(v)\| \leqslant \frac{1}{\|v\|}\|L(v)\| = \left\|L\left(\frac{v}{\|v\|}\right)\right\| \leqslant \sup\{\|L(v)\| : \|v\| = 1\}$ for any $v \in V$ with $\|v\| \leqslant 1$.
Define $\mathfrak{F} := \mathrm{span}_{\mathbb{C}}\{\chi_E : E \in \mathcal{B}, \nu(E) < \infty\}$, the set of all finitely simple functions on Y[14]. Since $\mathfrak{F}$ is dense in $L^p(Y, \mathcal{B}, \nu)$ for every $0 < p < \infty$[15], we may use the corollary found in [Bou95, p. 76]

Corollary 3.2. (Principle of extension of identities) *Let f, g be two continuous mappings of a topological space X into a Hausdorff space Y. If $f(x) = g(x)$ at all points of a dense subset of X, then $f \equiv g$.*

to see, that also

$$\|f\|_{L^p(Y,\mathcal{B},\nu)} = \sup\left\{\left|\int_Y fgd\mu\right| : g \in \mathfrak{F}, \|g\|_{L^{p'}(Y,\mathcal{B},\nu)} \leqslant 1\right\}$$

Assume $\underline{q > 1}$. Fix $f :\equiv \sum_{k=1}^n a_k e^{i\alpha_k}\chi_{X_k}$, where $n \in \mathbb{N}_{>0}, a_k > 0$, $\alpha_k \in [0, 2\pi[$, $X_i \cap X_j = \emptyset$ for $i, j = 1, \ldots, n$ and $\mu(X_k) < \infty$ for every $k = 1, \ldots, n$. Further let $g :\equiv \sum_{k=1}^m b_k e^{i\beta_k}\chi_{Y_k} \in \mathfrak{F}$, where $m \in \mathbb{N}_{>0}, b_k > 0$ and $\beta_k \in [0, 2\pi[$. Define

$$P(z) := \frac{p}{p_0}(1 - z) + \frac{p}{p_1}z \qquad Q(z) := \frac{q'}{q_0'}(1 - z) + \frac{q'}{q_1'}z$$

for $z \in \overline{S}$ (in the case $p = \infty$ we get also $p_0 = p_1 = \infty$ and hence by stipulating $\frac{\infty}{\infty} := 1$ the function P is well-defined). Further let

[14] This is almost trivial. Consider $Y_1, Y_2 \in \mathcal{B}$ with $\nu(Y_1), \nu(Y_2) < \infty$ and $Y_1 \cap Y_2 \neq \emptyset$. Then $f \equiv z_1\chi_{Y_1} + z_2\chi_{Y_2} \in \mathfrak{F}$ for $z_1, z_2 \in \mathbb{C}$. We see, that $f \equiv z_1\chi_{Y_1\setminus Y_2} + z_2\chi_{Y_2\setminus Y_1} + (z_1 + z_2)\chi_{Y_1\cap Y_2} \in \mathfrak{F}$ where the latter function is a finitely simple one since $\nu(Y_1 \cup Y_2) \leqslant \nu(Y_1) + \nu(Y_2) < \infty$ and $Y_1 \setminus Y_2, Y_2 \setminus Y_1, Y_1 \cap Y_2 \subseteq Y_1 \cup Y_2$.

[15] In [Els11, p. 242] a proof can be found, that $\mathfrak{F}$ is dense in $\mathcal{L}^p$ for $0 < p < \infty$. Now the canonical map $\pi : \mathcal{L}^p \to L^p/\mathcal{N}$ is continuous. Hence we may use the following lemma.

Lemma 3.3. *Let X and Y be topological spaces, $f : X \to Y$ and $A \subseteq X$ dense in X. Then $f(A)$ is dense in Y.*

Proof. By [Mun00, p. 104] we have $Y = f(X) = f(\overline{A}) \subseteq \overline{f(A)} \subseteq Y$. □

Yannis Bähni
yannis.baehni@uzh.ch

$$f_z :\equiv \sum_{k=1}^{n} a_k^{P(z)} e^{i\alpha_k} \chi_{X_k} \qquad g_z :\equiv \sum_{k=1}^{m} b_k^{Q(z)} e^{i\beta_k} \chi_{Y_k} \tag{28}$$

and

$$F(z) := \int_Y T(f_z)(y) g_z(y) d\nu(y) \tag{29}$$

By the linearity of the operator T we have

$$F(z) = \sum_{j=1}^{n} \sum_{k=1}^{m} a_j^{P(z)} b_j^{Q(z)} e^{i\alpha_j} e^{i\beta_k} \int_Y T(\chi_{X_j})(y) \chi_{Y_k}(y) d\nu(y) \tag{30}$$

and by using Hölder's inequality [16]

$$\begin{aligned}
\left| \int_Y T(\chi_{X_j})(y) \chi_{Y_k}(y) d\nu(y) \right| &\leqslant \int_Y |T(\chi_{X_j})(y)| \chi_{Y_k}(y) d\nu(y) \\
&= \|T(\chi_{X_j}) \chi_{Y_k}\|_{L^1(Y,\mathcal{B},\nu)} \\
&\leqslant \|T(\chi_{X_j})\|_{L^{q_0}(Y,\mathcal{B},\nu)} \|\chi_{Y_k}\|_{L^{q_0'}(Y,\mathcal{B},\nu)} \\
&\leqslant M_0 \|\chi_{X_j}\|_{L^{p_0}(X,\mathcal{A},\mu)} \|\chi_{Y_k}\|_{L^{q_0'}(Y,\mathcal{B},\nu)} \\
&= M_0 \left(\int_X |\chi_{X_j}(x)|^{p_0} d\mu(x) \right)^{1/p_0} \left(\int_Y |\chi_{Y_k}(y)|^{q_0'} d\nu(y) \right)^{1/q_0'} \\
&= M_0 \left(\int_X \chi_{X_j}(x) d\mu(x) \right)^{1/p_0} \left(\int_Y \chi_{Y_k}(y) d\nu(y) \right)^{1/q_0'} \\
&= M_0 \mu(X_j)^{p_0} \nu(Y_k)^{q_0'} \\
&< \infty
\end{aligned} \tag{31}$$

for $p_0 < \infty$ and each $j = 1, \ldots, n$, $k = 1, \ldots, m$ we get that $F(z)$ is analytic on S. The case $p_0, q_0' = \infty$ is trivial since $\|\chi_{X_j}\|_{L^\infty(X,\mathcal{A},\mu)}, \|\chi_{Y_k}\|_{L^\infty(Y,\mathcal{B},\nu)} \leqslant 1$. Now

[16] A proof can be found in [Els11, p. 223].

Yannis Bähni
yannis.baehni@uzh.ch

$$\begin{aligned}
\|f_{it}\|_{L^{p_0}(X,\mathcal{A},\mu)} &= \left(\sum_{k=1}^n \int_X |f_{it}|^{p_0} d\mu + \int_{X\setminus\bigcup_{k=1}^n X_k} |f_{it}|^{p_0} d\mu\right)^{1/p_0} \\
&= \left(\sum_{k=1}^n |a_k^{P(it)} e^{i\alpha_k}|^{p_0} \int_X \chi_{X_k} d\mu\right)^{1/p_0} \\
&= \left(\sum_{k=1}^n a_k^{p_0 \mathrm{Re} P(it)} \mu(X_k)\right)^{1/p_0} \\
&= \left(\sum_{k=1}^n a_k^p \mu(X_k)\right)^{p/p_0 p} \\
&= \|f\|_{L^p(X,\mathcal{A},\mu)}^{p/p_0}
\end{aligned} \tag{32}$$

for $p_0 \neq \infty$ and $p < \infty$. Let us consider $p_0 = \infty$. Then either $\|f_{it}\|_{L^\infty(X,\mathcal{A},\mu)} = 0$ or $\|f_{it}\|_{L^\infty(X,\mathcal{A},\mu)} = 1$. Since $\|\cdot\|_{L^p(X,\mathcal{A},\mu)}$ is a norm for $1 \leqslant p \leqslant \infty$ (see [Els11, p. 231]), we have $f = 0 + \mathcal{N}$ if $\|f_{it}\|_{L^\infty(X,\mathcal{A},\mu)} = 0$. Since $f \in \mathfrak{F}$, we may conclude $f \equiv \sum_{k=1}^n a_k e^{i\alpha_k}\chi_{X_k}$, where $\mu(X_k) = 0$ for $k = 1,\dots,n$. But then $\|f_{it}\|_{L^\infty(X,\mathcal{A},\mu)} = \inf\{B > 0 : \mu(\{|f_{it}| > B\}) = 0\} = \inf\{B > 0 : \mu(\{1 > B\}) = 0\} = 0$ since $|a_k^{P(it)}| = \lim_{p_0\to\infty} a_k^{p/p_0} = 1$. In the other case we simply have $\|f_{it}\|_{L^\infty(X,\mathcal{A},\mu)} = 1$ since there exists at least one subset X_k such that $\mu(X_k) \neq 0$. Now consider $p = \infty$. Then $p_0 = p_1 = \infty$. Thus $P(it) = 1$ and so $f_z \equiv f$. By the same considerations we see that $\|g_{it}\|_{L^{q_0'}(Y,\mathcal{B},\nu)} = \|g\|_{L^{q'}(Y,\mathcal{B},\nu)}^{q'/q_0'}$ for $q_0 \in [1,\infty]$ (set $\infty' := 1$). Hence

$$\begin{aligned}
|F(it)| &\leqslant \int_Y |T(f_{it})(y) g_{it}(y)| d\nu(y) \\
&= \|T(f_{it}) g_{it}\|_{L^1(Y,\mathcal{B},\nu)} \\
&\leqslant \|T(f_{it})\|_{L^{q_0}(Y,\mathcal{B},\nu)} \|g_{it}\|_{L^{q_0'}(Y,\mathcal{B},\nu)} \\
&\leqslant M_0 \|f_{it}\|_{L^{p_0}(X,\mathcal{A},\mu)} \|g_{it}\|_{L^{q_0'}(Y,\mathcal{B},\nu)} \\
&= M_0 \|f\|_{L^p(X,\mathcal{A},\mu)}^{p/p_0} \|g\|_{L^{q'}(Y,\mathcal{B},\nu)}^{q'/q_0'}
\end{aligned} \tag{33}$$

by Hölder's inequality. By similar calculations we get

$$\|f_{1+it}\|_{L^{p_1}(X,\mathcal{A},\mu)} = \|f\|_{L^p(X,\mathcal{A},\mu)}^{p/p_1} \qquad \|g_{1+it}\|_{L^{q_1'}(Y,\mathcal{B},\nu)} = \|g\|_{L^{q'}(Y,\mathcal{B},\nu)}^{q'/q_1'} \tag{34}$$

and thus

$$|F(1+it)| \leqslant M_1 \|f\|_{L^p(X,\mathcal{A},\mu)}^{p/p_1} \|g\|_{L^{q'}(Y,\mathcal{B},\nu)}^{q'/q_1'} \tag{35}$$

Since F is analytic on S and continuous on $\overline{S}$ and further

Yannis Bähni
yannis.baehni@uzh.ch

$$
\begin{aligned}
|F(z)| &\leqslant \int_Y |T(f_z)(y)g_z(y)|d\nu(y) \\
&= \|T(f_z)g_z\|_{L^1(Y,\mathcal{B},\nu)} \\
&\leqslant \|T(f_z)\|_{L^{q_0}(Y,\mathcal{B},\nu)}\|g_z\|_{L^{q_0'}(Y,\mathcal{B},\nu)} \\
&\leqslant M_0\|f_z\|_{L^{p_0}(X,\mathcal{A},\mu)}\|g_z\|_{L^{q_0'}(Y,\mathcal{B},\nu)} \\
&= M_0\left(\int_X |f_z|^{p_0}d\mu\right)^{1/p_0}\left(\int_Y |g_z|^{q_0'}d\nu\right)^{1/q_0'} \\
&= M_0\left(\sum_{j=1}^n a_j^{\mathrm{Re}P(z)}\mu(X_j)\right)^{1/p_0}\left(\sum_{k=1}^m b_k^{\mathrm{Re}Q(z)}\nu(Y_k)\right)^{1/q_0'} \\
&\leqslant M_0\left(\sum_{j=1}^n a_j^{p/p_0+p/p_1}\mu(X_j)\right)^{1/p_0}\left(\sum_{k=1}^m b_k^{q'/q_0'+q'/q_1'}\nu(Y_k)\right)^{1/q_0'}
\end{aligned}
\tag{36}
$$

by Hölder's inequality F is bounded on $\overline{S}$ we can apply Hadamard's three lines lemma to get

$$
\begin{aligned}
|F(z)| &\leqslant \left(M_0\|f\|_{L^p(X,\mathcal{A},\mu)}^{p/p_0}\|g\|_{L^{q'}(Y,\mathcal{B},\nu)}^{q'/q_0'}\right)^{1-\theta}\left(M_1\|f\|_{L^p(X,\mathcal{A},\mu)}^{p/p_1}\|g\|_{L^{q'}(Y,\mathcal{B},\nu)}^{q'/q_1'}\right)^{\theta} \\
&= M_0^{1-\theta}M_1^{\theta}\|f\|_{L^p(X,\mathcal{A},\mu)}\|g\|_{L^{q'}(Y,\mathcal{B},\nu)}
\end{aligned}
\tag{37}
$$

for $\mathrm{Re}z = \theta$ where $0 \leqslant \theta \leqslant 1$. Further observe $P(\theta) = Q(\theta) = 1$ and thus

$$
\begin{aligned}
\|T(f)\|_{L^q(Y,\mathcal{B},\nu)} &= \sup\left\{\left|\int_Y T(f)gd\nu\right| : g \in \mathfrak{F}, \|g\|_{L^{q'}(Y,\mathcal{B},\nu)} \leqslant 1\right\} \\
&= \sup\left\{|F(\theta)| : g \in \mathfrak{F}, \|g\|_{L^{q'}(Y,\mathcal{B},\nu)} \leqslant 1\right\} \\
&\leqslant M_0^{1-\theta}M_1^{\theta}\|f\|_{L^p(X,\mathcal{A},\mu)}
\end{aligned}
\tag{38}
$$

Now assume $\underline{q = 1}$. Then $q_0 = q_1 = 1$ and so $Q(z) = 1$ which implies $g_z \equiv g$ for every $z \in \overline{S}$. Assume, that $\|g\|_{L^\infty(Y,\mathcal{B},\nu)} \leqslant 1$. Then the above proof is also valid, if we take the supremum over the simple functions, instead of finitely simple functions, since by [Coh13, p. 100] the simple functions are dense in $L^\infty(Y,\mathcal{B},\nu)$. □

Yannis Bähni
yannis.baehni@uzh.ch

4. Interpolation of Analytic Families of Operators. First, we have to extend Hadamard's three lines lemma appropriately (lemma 3.1). To do so, we first need some theorems and definitions of complex analyis.

Theorem 4.1. (The Poisson Formula) *Let $h(e^{i\theta})$ be a continuous function on the unit circle. Then the* Poisson integral

$$\tilde{h}\,(z) = \int_{-\pi}^{\pi} h\left(e^{i\varphi}\right) P_r\left(\theta - \varphi\right) \frac{d\lambda(\varphi)}{2\pi} \qquad z := re^{i\theta} \in \mathbb{D} := \{|z| < 1\}$$

where

$$P_r(\theta) := \sum_{k\in\mathbb{Z}} r^{|k|} e^{ik\theta} \qquad 0 \leqslant r < 1, -\pi \leqslant \theta \leqslant \pi \tag{39}$$

denotes the Poisson kernel function*, is a harmonic function on $\mathbb{D}$ with boundary values $h\left(e^{i\theta}\right)$, that is, $\tilde{h}\left(e^{i\theta}\right)$ tends to $h\left(\zeta\right)$ as $z \in \mathbb{D}$ tends to $\zeta \in \partial\mathbb{D}$.*

Proof. A proof can be found in [Gam01, pp. 277–278]. □

Further we introduce the notion of a subharmonic function as found in [Gam01, p. 394].

Definition 4.1. *Let $D \subseteq \mathbb{C}$ be a domain (open and path-connected), and let $u : D \to [-\infty, \infty[$ be continuous. We say that $u(z)$ is* subharmonic *if for each $z_0 \in D$, there is $\varepsilon > 0$ such that $u(z)$ satisfies the mean value inequality*

$$u(z_0) \leqslant \int_0^{2\pi} u\left(z_0 + re^{i\theta}\right) \frac{d\lambda(\theta)}{2\pi} \qquad 0 < r < \varepsilon \tag{40}$$

And the notion of a conformal mapping ([Gam01, p. 59]).

Definition 4.2. *A smooth complex-valued function $g(z)$ (that is, $g(z)$ has as many derivatives as is necessary for whatever is being asserted to be true) is* conformal *at z_0 if whenever γ_0, γ_1 are two curves terminating at z_0 with non-zero tangents, then the curves $g \circ \gamma_0$, $g \circ \gamma_1$ have non-zero tangents at $g(z_0)$ and the angle from $\left(g \circ \gamma_0\right)'(z_0)$ to $\left(g \circ \gamma_1\right)'(z_0)$ is the same as the angle from $\gamma_0'(z_0)$ to $\gamma_1'(z_0)$. A* conformal mapping *of one domain D onto another V is a continuously differentiable function that is conformal at each point of D and that maps D one-to-one onto V.*

Now we are able to formulate the proof of the extension of Hadamard's three lines lemma.

Lemma 4.1. (Hadamard's three lines lemma, extension) *Let F be an analytic function on the strip $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ and continuous on $\overline{S}$, such that for every $z \in \overline{S}$ we have $\log|F(z)| \leqslant Ae^{\tau|\operatorname{Im} z|}$ for some $A < \infty$ and $\tau \in [0, \pi[$. Then*

$$|F(z)| \leqslant \exp\left(\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log|F(it+iy)|}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log|F(1+it+iy)|}{\cosh(\pi t) + \cos(\pi x)}\right] d\lambda(t)\right)$$

whenever $z := x + iy \in S$.

Yannis Bähni
yannis.baehni@uzh.ch

Proof. Consider the function

$$h(z) := \frac{1}{\pi i}\operatorname{Log}\left(\frac{z+1}{iz-i}\right) = \frac{1}{\pi}\left(\operatorname{Arg}\left(\frac{1+z}{1-z}\right) - i\log\left|\frac{1+z}{1-z}\right|\right) \tag{41}$$

which maps $\mathbb{D}$ onto $]0,1[\times\mathbb{R}$. □

DEFINITION 4.3. (Analytic family, admissible growth) *Let (X,μ), (Y,ν) be measure spaces and $(T_z)_{z\in\overline{S}}$, where T_z is defined on the space of all finitely simple functions on X and taking values in the space of all measurable functions on Y such that*

$$\int_Y |T_z(\chi_A)\chi_B| d\nu \tag{42}$$

whenever $\mu(A),\nu(B)<\infty$. The family $(T_z)_{z\in\overline{S}}$ is said to be analytic *if for all f, g finitely simple we have that*

$$z \mapsto \int_Y T_z(f) g d\nu \tag{43}$$

is analytic on S and continuous on $\overline{S}$. Further, an analytic family $(T_z)_{z\in\overline{S}}$ is called of admissible growth, *if there is a constant $\tau \in [0,\pi[$, such that for all finitely simple functions f, g a constant $C(f,g)$ exists with*

$$\log\left|\int_Y T_z(f) g d\nu\right| \leqslant C(f,g)e^{\tau|\operatorname{Im} z|} \tag{44}$$

for all $z \in \overline{S}$.

THEOREM 4.2. (Riesz-Thorin interpolation theorem, extension) *Let $(T_z)_{z\in\overline{S}}$ be an analytic family of admissible growth, $1 \leqslant p_0,p_1,q_0,q_1 \leqslant \infty$ and suppose that M_0, M_1 are positive functions on the real line such that for some $\tau \in [0,\pi[$*

$$\sup\left\{e^{-\tau|y|}\log M_0(y) : y \in \mathbb{R}\right\} < \infty \qquad \sup\left\{e^{-\tau|y|}\log M_1(y) : y \in \mathbb{R}\right\} < \infty \tag{45}$$

Fix $0<\theta<1$ and define

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \tag{46}$$

Further suppose that for all finitely simple functions f on X and $y \in \mathbb{R}$ we have

$$\|T_{iy}(y)\|_{L^{q_0}} \leqslant M_0(y)\|f\|_{L^{p_0}} \qquad \|T_{1+iy}(y)\|_{L^{q_1}} \leqslant M_1(y)\|f\|_{L^{p_1}} \tag{47}$$

Then for all finitely simple functions f on X we have

$$\|T_\theta(f)\|_{L^q} \leqslant M(\theta)\|f\|_{L^p}$$

where for $0<x<1$

Yannis Bähni
yannis.baehni@uzh.ch

$$M(x) = \exp\left(\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log M_0(t)}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log M_1(t)}{\cosh(\pi t) + \cos(\pi x)}\right]d\lambda(t)\right)$$

Proof. Fix $0 < \theta < 1$ and finitely simple functions f, g on X, Y respectively with $\|f\|_{L^p} = \|g\|_{L^{q'}} = 1$. Define f_z, g_z as in (28) and for $z \in \overline{S}$

$$F(z) := \int_Y T_z(f_z)g_z d\nu \tag{48}$$

Observe, that $|a_j^{P(z)}| \leqslant a_j^{p/p_0 + p/p_1}$ and $|b_k^{Q(z)}| \leqslant b_k^{q'/q_0' + q'/q_1'}$ for $z \in \overline{S}$. Hence

$$\begin{aligned}\log|F(z)| &= \log\left|\sum_{j=1}^n\sum_{k=1}^m a_j^{P(z)} b_j^{Q(z)} e^{i\alpha_j} e^{i\beta_k}\int_Y T_z(\chi_{X_j})(y)\chi_{Y_k}(y)d\nu(y)\right| \\ &\leqslant \log\left(\sum_{j=1}^n\sum_{k=1}^m |a_j^{P(z)}||b_j^{Q(z)}|\int_Y |T_z(\chi_{X_j})(y)|\chi_{Y_k}(y)d\nu(y)\right) \\ &\leqslant \log\left(\sum_{j=1}^n\sum_{k=1}^m a_j^{p/p_0+p/p_1} b_k^{q'/q_0'+q'/q_1'}\int_{Y_k}|T_z(\chi_{X_j})|d\nu\right)\end{aligned} \tag{49}$$

□

References

[Bou95] Nicolas Bourbaki. *General Topology - Chapters 1-4*. Elements of Mathematics. Springer-Verlag Berlin Heidelberg, 1995.
[Coh13] Donald L. Cohn. *Measure Theory*. Second edition. Springer, 2013.
[Els11] Jürgen Elstrodt. *Mass- und Integrationstheorie*. 7.,korrigierte und aktualisierte Auflage. Springer Verlag, 2011.
[Gam01] Theodore W. Gamelin. *Complex Analysis*. Springer, 2001.
[Gra14] Loukas Grafakos. *Classical Fourier Analysis*. Third Edition. Springer Science + Business Media New York, 2014.
[Lan93] Serge Lang. *Complex Analysis*. Third Edition. Springer-Verlag, 1993.
[Mun00] James R. Munkres. *Topology*. Second edition. Prentice Hall, 2000.

Index