

CLASSICAL FOURIER ANALYSIS: INTERPOLATION OF L^p SPACES

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DEFINITION. Let (X, μ) and (Y, ν) be measure spaces. Further let T be an operator defined on a linear space of complex-valued measurable functions on X and taking values in the set of all complex-valued, finite almost everywhere, measurable functions on Y . Then T is called linear if for all functions f and g in the domain of T and all $z \in \mathbb{C}$ holds

$$T(f + g) = T(f) + T(g) \quad T(zf) = zT(f) \quad (1)$$

and quasi-linear if

$$|T(f + g)| \leq K(|T(f)| + |T(g)|) \quad |T(zf)| = |z| |T(f)| \quad (2)$$

holds for some real constant $K > 0$. If $K = 1$, T is called sublinear.

A complex-valued function f is said to be *holomorphic* in $\Omega \subseteq \mathbb{C}$ open, if $f'(z)$ exists for any $z \in \Omega$. By a region we shall mean a nonempty connected open subset of the complex plane. The proof of Hadamard's three lines lemma heavily relies on a restatement of the maximum modulus theorem (see [Rud87, p. 212]) found in [Rud87, p. 253].

THEOREM. Let $\Omega \subseteq \mathbb{C}$ be a bounded region and f be a continuous function on $\overline{\Omega}$ which is holomorphic in Ω . Then

$$|f(z)| \leq \sup \{|f(z)| : z \in \partial\Omega\}$$

for every $z \in \Omega$. If equality holds at one point $z \in \Omega$, then f is constant.

LEMMA. (Hadamard's three lines lemma) Let F be a holomorphic function in the strip $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$, continuous and bounded on \overline{S} , such that $|F(z)| \leq B_0$ when $\operatorname{Re} z = 0$ and $|F(z)| \leq B_1$ when $\operatorname{Re} z = 1$, for some $0 < B_0, B_1 < \infty$. Then $|F(z)| \leq B_0^{1-\theta} B_1^\theta$ when $\operatorname{Re} z = \theta$, for any $0 \leq \theta \leq 1$.

Proof. For $z \in \overline{S}$ define

$$G(z) := \frac{F(z)}{B_0^{1-z} B_1^z} \quad G_n(z) := G(z) e^{(z^2-1)/n}, \quad n \in \mathbb{N}_{>0}$$

$G(z)$ and $G_n(z)$ are holomorphic in S by

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$$G'(z) = \frac{F'(z) - F(z) \log(B_1/B_0)}{B_0^{1-z} B_1^z} \quad G'_n(z) = G'(z) e^{(z^2-1)/n} + \frac{2}{n} z G_n(z)$$

and $e^z \neq 0$ for every $z \in \mathbb{C}$. Further, we have

$$|B_0^{1-z} B_1^z| = (B_0^{1-z} B_0^{1-\bar{z}} B_1^z B_1^{\bar{z}})^{1/2} = B_0^{1-\operatorname{Re} z} B_1^{\operatorname{Re} z}$$

Consider $0 \leq \operatorname{Re} z \leq 1$ and $B_0 \geq 1$. Then $B_0^{1-\operatorname{Re} z} \geq 1$ and $B_0^{1-\operatorname{Re} z} \geq B_0$ in the case $B_0 < 1$. Similarly, $B_1^{\operatorname{Re} z} \geq 1$ if $B_1 \geq 1$ and $B_1^{\operatorname{Re} z} \leq B_1$ if $B_1 < 1$. Hence

$$|B_0^{1-z} B_1^z| \geq \min\{1, B_0\} \min\{1, B_1\} > 0 \quad (3)$$

for all $z \in \bar{S}$. Since F is bounded on \bar{S} , we have $|F(z)| \leq L$ for some $L > 0$ and all $z \in \bar{S}$. Thus by (3)

$$|G(z)| = \frac{|F(z)|}{|B_0^{1-z} B_1^z|} \leq \frac{L}{\min\{1, B_0\} \min\{1, B_1\}} =: M$$

for every $z \in \bar{S}$. Fix $n \in \mathbb{N}_{>0}$ and write $z := x + iy \in \bar{S}$. Then

$$|G_n(z)| \leq M \left(e^{(x^2+2ixy-y^2-1)/n} e^{(x^2-2ixy-y^2-1)/n} \right)^{1/2} = M e^{-y^2/n} e^{(x^2-1)/n} \leq M e^{-y^2/n}$$

for $0 \leq x \leq 1$. Thus

$$\lim_{y \rightarrow \pm\infty} \sup\{|G_n(z)| : 0 \leq x \leq 1\} = 0$$

by the pinching-principle. Hence there exist $C_0, C_1 \in \mathbb{R}$, such that

$$\sup\{|G_n(z)| : 0 \leq x \leq 1\} \leq 1$$

when $y > C_0$ or $y < C_1$. Letting

$$C(n) := \max\{|C_0| + 1, |C_1| + 1\}$$

we conclude $|G_n(z)| \leq 1$ for all $0 \leq x \leq 1$ when $|y| \geq C(n)$. Now consider the rectangle $R := (0, 1) \times (-C(n), C(n))$. We have $|G_n(z)| \leq 1$ on the lines $[0, 1] \times \{\pm C(n)\}$. By

$$|G_n(iy)| = \frac{|F(iy)|}{|B_0^{1-iy} B_1^{iy}|} e^{-(y^2+1)/n} \leq 1 \quad |G_n(1+iy)| = \frac{|F(1+iy)|}{|B_0^{-iy} B_1^{1+iy}|} e^{-y^2/n} \leq 1$$

we have $|G_n(z)| \leq 1$ on the lines $\{0\} \times [-C(n), C(n)]$, $\{1\} \times [-C(n), C(n)]$. Thus $|G_n(z)| \leq 1$ on ∂R . Since $|G_n(z)|$ is continuous on \bar{R} , holomorphic in R and R is a bounded region, the maximum modulus theorem implies

$$|G_n(z)| \leq \sup\{|G_n(z)| : z \in \partial R\} \leq 1$$

for every $z \in R$. Therefore $|G_n(z)| \leq 1$ on \overline{R} and so $|G_n(z)| \leq 1$ on \overline{S} . Since inequalities are preserved by limits and the modulus is a continuous function, we have that $|G(z)| = \lim_{n \rightarrow \infty} |G_n(z)| \leq 1$ for $z \in \overline{S}$. We conclude by

$$|F(\theta + it)| = |G(\theta + it)| \left| B_0^{1-\theta-it} B_1^{\theta+it} \right| \leq B_0^{1-\theta} B_1^\theta$$

whenever $0 \leq \theta \leq 1, t \in \mathbb{R}$. □

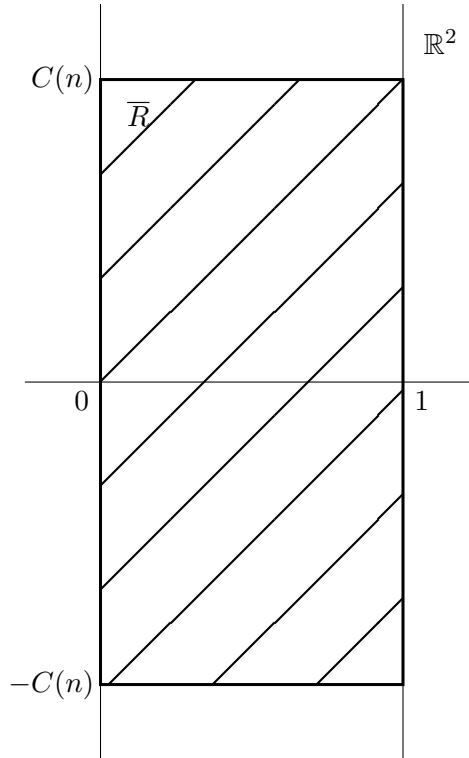


FIGURE 1. Sketch of the rectangle \overline{R} .

For two measure spaces $(X, \mu), (Y, \nu)$ let Σ_X and Σ_Y denote the set of all finitely simple functions on X, Y respectively.

THEOREM. (Riesz-Thorin Interpolation Theorem) *Suppose that $(X, \mu), (Y, \nu)$ are measure spaces and $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. If $q_0 = q_1 = \infty$, suppose also that ν is semifinite. Let T be a linear operator defined on Σ_X and taking values in the set of measurable functions on Y , such that*

$$\|T(f)\|_{L^{q_0}} \leq M_0 \|f\|_{L^{p_0}} \quad \|T(f)\|_{L^{q_1}} \leq M_1 \|f\|_{L^{p_1}} \quad (4)$$

for all $f \in \Sigma_X$ and $0 < M_0, M_1 < \infty$. Then for all $0 < \theta < 1$ we have

$$\|T(f)\|_{L^q} \leq M_0^{1-\theta} M_1^\theta \|f\|_{L^p} \quad (5)$$

for all $f \in \Sigma_X$, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

Proof. If $f \in \Sigma_X$, $\|f\|_{L^p} = 0$, then $f = 0$ μ -a.e. and either one of the hypotheses on T in (4) implies $T(f) = 0$ μ -a.e. and thus the estimate (5) holds trivially. Therefore we can assume $\|f\|_{L^p} \neq 0$. Fix

$$f := \sum_{j=1}^n a_j e^{i\alpha_j} \chi_{A_j} \in \Sigma_X \quad g := \sum_{k=1}^m b_k e^{i\beta_k} \chi_{B_k} \in \Sigma_Y$$

where $a_j, b_k > 0$ and $\alpha_j, \beta_k \in \mathbb{R}$ for every $j = 1, \dots, n, k = 1, \dots, m$ such that $\|g\|_{L^{q'}} \neq 0$ (recall $q' := q/(q-1)$). Define

$$P(z) := \frac{p}{p_0}(1-z) + \frac{p}{p_1}z \quad Q(z) := \frac{q'}{q'_0}(1-z) + \frac{q'}{q'_1}z$$

for $z \in \overline{S}$ (if $p, q' = \infty$ then also $p_0, p_1, q'_0, q'_1 = \infty$ and hence P, Q are well-defined). Further let

$$f_z := \sum_{j=1}^n a_j^{P(z)} e^{i\alpha_j} \chi_{A_j} \quad g_z := \sum_{k=1}^m b_k^{Q(z)} e^{i\beta_k} \chi_{B_k} \quad (6)$$

and

$$F(z) := \int_Y T(f_z) g_z d\nu \quad (7)$$

By the linearity of the operator T we have

$$F(z) = \sum_{j=1}^n \sum_{k=1}^m a_j^{P(z)} b_k^{Q(z)} e^{i\alpha_j} e^{i\beta_k} \int_Y T(\chi_{A_j}) \chi_{B_k} d\nu$$

and by Hölder's inequality

$$\begin{aligned}
\left| \int_Y T(\chi_{A_j}) \chi_{B_k} d\nu \right| &\leq \int_Y |T(\chi_{A_j}) \chi_{B_k}| d\nu \\
&= \|T(\chi_{A_j}) \chi_{B_k}\|_{L^1} \\
&\leq \|T(\chi_{A_j})\|_{L^{q_0}} \|\chi_{B_k}\|_{L^{q'_0}} \\
&\leq M_0 \|\chi_{A_j}\|_{L^{p_0}} \|\chi_{B_k}\|_{L^{q'_0}} \\
&\leq M_0 \mu(A_j)^{1/p_0} \nu(B_k)^{1/q'_0}
\end{aligned} \tag{8}$$

for each $j = 1, \dots, n$, $k = 1, \dots, m$ (even in the cases where either $p_0 = \infty$ or $q'_0 = \infty$, or both, by observing that $\|\chi_A\|_{L^\infty} \leq 1$ for any measurable set A). Thus the function F is well-defined on \overline{S} . Let $t \in \mathbb{R}$. For $p, p_0 \neq \infty$

$$\begin{aligned}
\|f_{it}\|_{L^{p_0}} &= \left(\sum_{j=1}^n \int_{A_j} |f_{it}|^{p_0} d\mu + \int_{X \setminus \bigcup_{j=1}^n A_j} |f_{it}|^{p_0} d\mu \right)^{1/p_0} \\
&= \left(\sum_{j=1}^n \left| a_j^{P(it)} e^{i\alpha_j} \right|^{p_0} \int_X \chi_{A_j} d\mu \right)^{1/p_0} \\
&= \left(\sum_{j=1}^n a_j^{p_0 \operatorname{Re} P(it)} \mu(A_j) \right)^{1/p_0} \\
&= \left(\sum_{j=1}^n a_j^p \mu(A_j) \right)^{p/(p_0 p)} \\
&= \|f\|_{L^p}^{p/p_0}
\end{aligned}$$

holds. Let $p_0 = \infty$, $p \neq \infty$. Then $\|f_{it}\|_{L^\infty} = 1$ since $|a_j^{P(it)}| = a_j^{p/p_0} = 1$ and that there exists some index j , such that $\mu(A_j) \neq 0$. If $p = \infty$, observe that $P(z) = 1$ and thus $\|f_{it}\|_{L^\infty} = \|f\|_{L^\infty}$. By the same considerations we have $\|g_{it}\|_{L^{q'_0}} = \|g\|_{L^{q'}}^{q'/q'_0}$. Hence

$$\begin{aligned}
|F(it)| &\leq \int_Y |T(f_{it}) g_{it}| d\nu \\
&= \|T(f_{it}) g_{it}\|_{L^1} \\
&\leq \|T(f_{it})\|_{L^{q_0}} \|g_{it}\|_{L^{q'_0}} \\
&\leq M_0 \|f_{it}\|_{L^{p_0}} \|g_{it}\|_{L^{q'_0}} \\
&= M_0 \|f\|_{L^p}^{p/p_0} \|g\|_{L^{q'}}^{q'/q'_0}
\end{aligned}$$

by Hölder's inequality. In an analogous manner we derive

$$\|f_{1+it}\|_{L^{p_1}} = \|f\|_{L^p}^{p/p_1} \quad \|g_{1+it}\|_{L^{q'_1}} = \|g\|_{L^{q'}}^{q'/q'_1}$$

and thus

$$|F(1+it)| \leq M_1 \|f\|_{L^p}^{p/p_1} \|g\|_{L^{q'}}^{q'/q'_1}$$

Further by estimate (8)

$$\begin{aligned} |F(z)| &\leq \sum_{j=1}^n \sum_{k=1}^m \left| a_j^{P(z)} \right| \left| b_k^{Q(z)} \right| \left| \int_Y T(\chi_{A_j}) \chi_{B_k} d\nu \right| \\ &\leq M_0 \sum_{j=1}^n \sum_{k=1}^m a_j^{\operatorname{Re} P(z)} b_k^{\operatorname{Re} Q(z)} \mu(A_j)^{1/p_0} \nu(B_k)^{1/q'_0} \\ &\leq M_0 \sum_{j=1}^n \sum_{k=1}^m \max \left\{ 1, a_j^{p/p_0+p/p_1} \right\} \max \left\{ 1, b_k^{q'/q'_0+q'/q'_1} \right\} \mu(A_j)^{1/p_0} \nu(B_k)^{1/q'_0} \end{aligned}$$

Hence F is bounded on \bar{S} by some constant depending on f and g only. By

$$\begin{aligned} F'(z) &= \sum_{j=1}^n \sum_{k=1}^m a_j^{P(z)} \log(a_j) \left(\frac{p}{p_1} - \frac{p}{p_0} \right) b_k^{Q(z)} e^{i\alpha_j} e^{i\beta_k} \int_Y T(\chi_{A_j}) \chi_{B_k} d\nu \\ &\quad + \sum_{j=1}^n \sum_{k=1}^m a_j^{P(z)} b_k^{Q(z)} \log(b_k) \left(\frac{q'}{q'_1} - \frac{q'}{q'_0} \right) e^{i\alpha_j} e^{i\beta_k} \int_Y T(\chi_{A_j}) \chi_{B_k} d\nu \end{aligned}$$

it is immediate, that F is an entire function and thus holomorphic in S and continuous on \bar{S} . Therefore, Hadamard's three lines lemma yields

$$|F(z)| \leq \left(M_0 \|f\|_{L^p}^{p/p_0} \|g\|_{L^{q'}}^{q'/q'_0} \right)^{1-\theta} \left(M_1 \|f\|_{L^p}^{p/p_1} \|g\|_{L^{q'}}^{q'/q'_1} \right)^{\theta} = M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p} \|g\|_{L^{q'}}$$

for $\operatorname{Re} z = \theta$, $0 < \theta < 1$. We have

$$\{T(f) \neq 0\} = \bigcup_{n=1}^{\infty} \{|T(f)| > 1/n\}$$

and by Chebychev's inequality either

$$\nu(\{|T(f)| > 1/n\}) \leq n^{q_0} \|T(f)\|_{L^{q_0}}^{q_0}$$

or

$$\nu(\{|T(f)| > 1/n\}) \leq n^{q_1} \|T(f)\|_{L^{q_1}}^{q_1}$$

whenever $q_0 \neq \infty$ or $q_1 \neq \infty$. Therefore, the set $\{T(f) \neq 0\}$ is σ -finite unless $q_0 = q_1 = \infty$. Further we have $P(\theta) = Q(\theta) = 1$. Thus by

$$\begin{aligned} M_q(T(f)) &= \sup \left\{ \left| \int_Y T(f) g d\nu \right| : g \in \Sigma_Y, \|g\|_{L^{q'}} = 1 \right\} \\ &= \sup \{ |F(\theta)| : g \in \Sigma_Y, \|g\|_{L^{q'}} = 1 \} \\ &\leq M_0^{1-\theta} M_1^\theta \|f\|_{L^p} \end{aligned}$$

we conclude

$$\|T(f)\|_{L^q} = M_q(T(f)) \leq M_0^{1-\theta} M_1^\theta \|f\|_{L^p}$$

for any $f \in \Sigma_X$. □

REMARK. A more general version of the Riesz-Thorin interpolation theorem can be found in [Fol99, pp. 200–202]. There, a linear map

$$T : L^{p_0} + L^{p_1} \rightarrow L^{q_0} + L^{q_1}$$

is considered. This follows using a density argument from the current version of the theorem and will not be proven here.

REMARK. Using the previous remark, a standard application of the Riesz-Thorin interpolation theorem is to prove Young's inequality [Gra14, pp. 22–23].

LEMMA 1.1. (Hadamard's three lines lemma, extension) *Let F be a holomorphic function in the strip $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ and continuous on \overline{S} , such that for some $0 < A < \infty$ and $\tau_0 \in (0, \pi)$ we have $\log |F(z)| \leq A e^{\tau_0 |\operatorname{Im} z|}$ for every $z \in \overline{S}$. Then*

$$|F(z)| \leq \exp \left(\frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \left[\frac{\log |F(it + iy)|}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log |F(1 + it + iy)|}{\cosh(\pi t) + \cos(\pi x)} \right] d\lambda(t) \right)$$

whenever $z := x + iy \in S$.

DEFINITION 1.1. (Analytic family, admissible growth) *Let (X, μ) be a measure space, (Y, ν) be a semifinite measure spaces and $(T_z)_{z \in \overline{S}}$, where T_z is defined Σ_X and taking values in the space of all measurable functions on Y such that*

$$\int_Y |T_z(\chi_A) \chi_B| d\nu \tag{9}$$

whenever $\mu(A), \nu(B) < \infty$. The family $(T_z)_{z \in \overline{S}}$ is said to be analytic if for all $f \in \Sigma_X$, $g \in \Sigma_Y$ we have that

$$z \mapsto \int_Y T_z(f) g d\nu \tag{10}$$

is analytic on S and continuous on \overline{S} . Further, an analytic family $(T_z)_{z \in \overline{S}}$ is called of admissible growth, if there is a constant $\tau_0 \in (0, \pi)$, such that for all $f \in \Sigma_X$, $g \in \Sigma_Y$ a constant $C(f, g)$ exists with

$$\log \left| \int_Y T_z(f) g d\nu \right| \leq C(f, g) e^{\tau_0 |\operatorname{Im} z|} \quad (11)$$

for all $z \in \overline{S}$.

THEOREM 1.1. (Stein's Theorem on Interpolation of Analytic Families of Operators) Let $(T_z)_{z \in \overline{S}}$ be an analytic family of admissible growth, $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and suppose that M_0, M_1 are positive functions on the real line such that for some $\tau_1 \in (0, \pi)$

$$\sup \left\{ e^{-\tau_1 |y|} \log M_0(y) : y \in \mathbb{R} \right\} < \infty \quad \sup \left\{ e^{-\tau_1 |y|} \log M_1(y) : y \in \mathbb{R} \right\} < \infty \quad (12)$$

Fix $0 < \theta < 1$ and define

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \quad (13)$$

Further suppose that for all $f \in \Sigma_X$ and $y \in \mathbb{R}$ we have

$$\|T_{iy}(f)\|_{L^{q_0}} \leq M_0(y) \|f\|_{L^{p_0}} \quad \|T_{1+iy}(f)\|_{L^{q_1}} \leq M_1(y) \|f\|_{L^{p_1}} \quad (14)$$

Then for all $f \in \Sigma_X$ we have

$$\|T_\theta(f)\|_{L^q} \leq M(\theta) \|f\|_{L^p}$$

where for $0 < x < 1$

$$M(x) = \exp \left(\frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \left[\frac{\log M_0(t)}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log M_1(t)}{\cosh(\pi t) + \cos(\pi x)} \right] d\lambda(t) \right)$$

THEOREM 1.2. (The Marcinkiewicz Interpolation Theorem) Let (X, μ) be a σ -finite measure space, (Y, ν) another measure space and $0 < p_0 < p_1 \leq \infty$. Further let T be a sublinear operator defined on

$$L^{p_0} + L^{p_1} := \{f_0 + f_1 : f_0 \in L^{p_0}(X, \mu), f_1 \in L^{p_1}(X, \mu)\}$$

and taking values in the space of measurable functions on Y . Assume that there exist $A_0, A_1 < \infty$ such that

$$\forall f \in L^{p_0}(X, \mu) \quad \|T(f)\|_{L^{p_0, \infty}} \leq A_0 \|f\|_{L^{p_0}} \quad (15)$$

$$\forall f \in L^{p_1}(X, \mu) \quad \|T(f)\|_{L^{p_1, \infty}} \leq A_1 \|f\|_{L^{p_1}} \quad (16)$$

Then for all $p_0 < p < p_1$ and for all $f \in L^p(X, \mu)$ we have the estimate

$$\|T(f)\|_{L^p} \leq A \|f\|_{L^p} \quad (17)$$

where

$$A := 2 \left(\frac{p}{p - p_0} + \frac{p}{p_1 - p} \right)^{1/p} A_0^{\frac{\frac{1}{p} - \frac{1}{p_1}}{\frac{1}{p_0} - \frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0} - \frac{1}{p}}{\frac{1}{p_0} - \frac{1}{p_1}}} \quad (18)$$