TORIC SYMPLECTIC MANIFOLDS

SIMON GRÜNING

(Darboux Theorem)

(Simon Grüning) University of Zurich, Rämistrasse 71, 8006 Zurich
E-mail address: simon.gruening@uzh.ch.

Simon Grüning
simon.gruening@uzh.ch

(!) We follow closely the exposition of Yannis gathered of material from Lee and Salomon's books. We take inspiration from Anna's stuff.

1. Prerequisites

Remark 1.1. *Must be even dimensional*

All manifolds are smooth unless noted otherwise as per ana's conventions

chart centered on x means $\phi(x) = 0$.

F diffeomorphism is linear?

precomposition definition of F^*

Proposition 1.1 (Cartan's Magic Formula). *(! ref lee)*

Fix a manifold M, a vector field $X \in \mathfrak{X}(M)$, and an $\omega \in \Omega^k(M)$ for some $k \in \mathbb{N}$. Then

$$\mathcal{L}_X \omega = i_X(d\omega) + d(i_X \omega).$$

Theorem 1.1. *Canonical Form theorem*

Definition 1.1. *time dependent vector field*

Definition 1.2. *time dependent flow of X*

Definition 1.3. *time dependent differential k-form*

Proposition 1.2 (Fisherman's Formula). *(!ref lee) Fix a manifold M. If $X : J \times M \to TM$ is a time-dependent vector field with time-dependent flow $\Psi : \mathcal{D} \to M$ then for any $\omega \in \Omega^k(M)$, $(t_1, t_0, x) \in \mathcal{D}$ we have that*

$$\left.\frac{d}{dt}\right|_{t=t_1} \Psi^*_{t,t_0}\omega = \Psi^*_{t_1,t_0}\left(\mathcal{L}_{X_{t_1}}\omega\right)$$

with $X_{t_1} := X(t_1, \cdot) \in \mathfrak{X}(M)$.

Proposition 1.3 (Fisherman's Formula Adapted). *Fix a manifold M. If $X : J \times M \to TM$ is a time-dependent vector field with time-dependent flow $\Psi : \mathcal{D} \to M$, and further, $\omega : J \times M \to \Lambda^k T^*M$ is a time-dependent differential k-form, then for any $\omega \in \Omega^k(M)$, $(t_1, t_0, x) \in \mathcal{D}$ we have that*

$$\left.\frac{d}{dt}\right|_{t=t_1} \Psi^*_{t,t_0}\omega_t = \Psi^*_{t_1,t_0}\left(\mathcal{L}_{X_{t_1}}\omega_{t_1} + \left.\frac{d}{dt}\right|_{t=t_1}\omega_t\right)$$

with $X_{t_1} := X(t_1, \cdot) \in \mathfrak{X}(M)$.

Proof. todo chain rule amazing $\square$

Lemma 1.1. *Let M be a manifold, $x \in M$ with basis (e_i) for T_xM. Then there exists a chart $(U, x^1, ..., x^n)$ centered on x such that for any $i = 1, ..., n$:*

$$\left.\frac{\partial}{\partial x^i}\right|_x = e_i$$

Definition 1.4 (Tubular Neighbourhood). *(! ref lee)*

Simon Grüning
simon.gruening@uzh.ch

Proposition 1.4 (Homotopy Formula). *(!ref canas). Fix U, a tubular neighburhood of a submanifold S embedded in M. If $\omega \in \Omega^k(U)$ is closed and $i^*\omega = 0$ for some $i : S \hookrightarrow U$, then there exists an $\eta \in \Omega^{k-1}(U)$ with $\omega = d\eta$ and $\forall x \in S : \eta_x = 0$.*

Proof. todo ☐

2. Moser Trick

Theorem 2.1. *Moser Trick*

Theorem 2.2 (Moser Isotopy). *(!salomon) Fix as M a $2n$-dimensional manifold and as $S \subseteq M$ a compact submanifold. If $\omega_0, \omega_1 \in \Omega^2(M)$ are close and*

(1) $\forall x \in S : \omega_0|x = \omega_1|x$
(2) $\forall x \in S : \omega_0|x, \omega_1|x$ are nondegenerate.

then there exist neighbourhoods U_0, U_1 of S in M and a diffeomorphism $F : U_0 \to U_1$ with

$$F|_S = id_S$$
$$F^*(\omega_1|_{U_1}) = \omega_0|_{U_0}.$$

Proof. asf ☐

3. Darboux Theorem

Theorem 3.1 (Darboux's Theorem). *Fix (M, ω) a $2n$-dimensional symplectic manifold, $x \in M$. Then there exists a chart $(U, x^1, ..., x^n, y^1, ..., y^n)$ centered on x such that:*

$$\omega|_U = \sum_{i=1}^{n} dx^i \wedge dy^i$$

.

Proof. The canonical form theorem for symplectic tensors 1.1 provides us a basis $(a_1, ..., a_n, b_1, ..., b_n)$ for T_xM such that for its dual basis $(a^1, ..., a^n, b^1, ..., b^n)$ we have

$$\omega_x = \sum_{i=1}^{n} da^i \wedge db^i.$$

By proposition 1.1 we further have a chart $(U, \tilde{\varphi})$ centered on x with associated coordinates $(\tilde{x}^1, ..., \tilde{x}^n, \tilde{y}^1, ..., \tilde{y}^n)$ such that for $i = 1, ..., n$

$$\left.\frac{\partial}{\partial \tilde{x}^i}\right|_x = a_i$$
$$\left.\frac{\partial}{\partial \tilde{y}^i}\right|_x = b_i$$

Combining the previous two results and traversing again into the dual basis we have

$$\omega_x = \sum_{i=1}^{n} d\tilde{x}^i|_x \wedge d\tilde{y}^i|_x.$$

Define:

Simon Grüning
simon.gruening@uzh.ch

$$\omega_0 := \omega|_U$$
$$\omega_1 := \sum_{i=1}^{n} d\tilde{x}^i \wedge d\tilde{y}^i.$$

Then ω_0, ω_1 are symplectic forms on U (!).

Application of the Moser isotopy 2.2 to the compact submanifold $x \subseteq U$ given ω_0, ω_1 provides the existence of neighbourhoods U_0, U_1 of x in U and a diffeomorphism $F : U_0 \to U_1$ with

$$F(x) = x$$
$$F^*\omega_1 = \omega_0.$$

Define now another chart (U_0, φ) with $\varphi := \tilde{\varphi}|_{U_1} \circ F$. By construction (!) the associated coordinates are

$$x^i = \tilde{x}^i \circ F$$
$$y^i = \tilde{y}^i \circ F.$$

If then follows (!) that $\varphi(x) = \tilde{\varphi}(x) = 0$. The remaining property of our chart (U_0, φ) follows by:

$$\begin{aligned}
\omega|_{U_0} &= \omega_0|_{U_0} \\
&= F^*(\omega_1|U_1) \\
&= F^*\left(\sum_{i=1}^{n} d\tilde{x}^i \wedge d\tilde{y}^i\right) \\
&= \sum_{i=1}^{n} F^*(d\tilde{x}^i \wedge d\tilde{y}^i) \\
&= \sum_{i=1}^{n} F^*(d\tilde{x}^i) \wedge F^*(d\tilde{y}^i) \\
&= \sum_{i=1}^{n} dF^*(\tilde{x}^i) \wedge dF^*(\tilde{y}^i) \\
&= \sum_{i=1}^{n} d(\tilde{x}^i \circ F) \wedge d(\tilde{y}^i \circ F) \\
&= \sum_{i=1}^{n} dx^i \wedge dy^i.
\end{aligned}$$

□

Simon Grüning
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4. Discussion / Applications

bla

Appendix A. Basics

(1) symplectic manifold
(2) smooth manifold
(3) einstein summation convention
(4) T_xM et al.
(5) basis of above and dual basis
(6) coordinates associated to chart?
(7) time dependent vector field and flow
(8) time dep differential k-form

Lemma A.1 (refyan E.). *For a smooth function F from manifolds M to N and $\omega, \eta \in \Omega(N)$ we have*

$$F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$$

Lemma A.2 (refyan E.203). *For a smooth function F from manifolds M to N and $\omega \in \Omega(M)$ we have*

$$F^*(d\omega) = d(F^*\omega).$$