TORIC SYMPLECTIC MANIFOLDS

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(Darboux Theorem)

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(!) Throghout the seminar notes we follow closely the exposition of Yannis gathered of material from Lee and Salomon's books. We take inspiration from Anna's stuff.

(1) set Psi to psi?

1. Prerequisites

Remark 1.1. *Must be even dimensional*

All manifolds are smooth unless explicitly noted otherwise as per ana's conventions

chart centered on x means $\varphi(x) = 0$.

precomposition definition of F^* F^* diffeomorphism is linear..

Proposition 1.1 (Cartan's Magic Formula). *(! ref lee)*

Fix a manifold M, a vector field $X \in \mathfrak{X}(M)$, and an $\omega \in \Omega^k(M)$ for some $k \in \mathbb{N}$. Then

$$\mathcal{L}_X \omega = i_X(d\omega) + d(i_X \omega).$$

Theorem 1.1 (Canonical Form Theorem for Symplectic Vector Space). *Fix (V, ω) a symplectic vector space. Then $dimV = 2n$ and there exists a basis $(a_1, ..., a_n, b_1, ..., b_n)$ of V such that:*

$$\omega = \sum_{i=1}^{n} a^i \wedge b^i$$

where $(a^1, ..., a^n, b^1, ..., b^n)$ denotes the dual basis of $(a_1, ..., a_n, b_1, ..., b_n)$.

Definition 1.1. *Fix a manifold M, $J \subseteq \mathbb{R}$ an interval. A time-dependent vector field on M is a smooth map $X : J \times M \to TM$ such that $\forall (t, x) \in J \times M : X(t, x) \in T_x M$ (the section property?).*

Definition 1.2. *An integral curve of a time-dependent vector field X is a curve $\gamma \in \mathcal{C}^\infty(J_0, M)$ such that $\gamma'(t) = X(t, \gamma(t))$ for all $t \in J_0$, where $J_0 \subseteq J$ is an interval by notation as in defintion 1.1.*

Definition 1.3. *Fix a manifold M, $J \subseteq \mathbb{R}$ an open interval, and $X : J \times M \to TM$ a time-dependent vector field. We call a time dependent flow of X an open subset $\mathcal{D} \subseteq J \times J \times M$ paired with a map $\Psi \in \mathcal{C}^\infty(\mathcal{D}, M)$ such that the following holds for $\mathcal{D}^{(t_0,x)} := \{t \in J : (t, t_0, x) \in \mathcal{D}\}$, $\Psi^{(t_0,x)}(t) := \Psi(t, t_0, x)$, $M_{t_1,t_0} := \{x \in M : (t_1, t_0, x) \in \mathcal{D}\}$, $\Psi_{t_1,t_0}(x) := \Psi(t_1, t_0, x)$:*

(1) For any $t_0 \in J, x \in M$, $\mathcal{D}^{(t_0,x)}$ is an open interval such that $t_0 \in \mathcal{D}^{(t_0,x)}$ and $\Psi^{(t_0,x)}(t)$ is the unique maximal integral curve of X with $\Psi^{(t_0,x)}(t_0) = x$.

(2) $t_1 \in \mathcal{D}^{(t_0,x)} \wedge y = \Psi^{(t_0,x)}(t_1) \implies \mathcal{D}^{(t_1,y)} = \mathcal{D}^{(t_0,x)} \wedge \Psi^{(t_1,y)} = \Psi^{(t_0,x)}$

(3) For any $(t_1, t_0) \in J \times J$ we have that M_{t_1,t_0} is open in M and $\Psi_{t_1,t_0} : M_{t_1,t_0} \to M$ is a diffeomorphism from M_{t_1,t_0} onto M_{t_0,t_1} with inverse Ψ_{t_0,t_1}

(4) $x \in M_{t_1,t_0} \wedge \Psi_{t_1,t_0}(x) \in M_{t_2,t_1} \implies$

$$x \in M_{t_2,t_0} \wedge \Psi_{t_2,t_1} \circ \Psi_{t_1,t_0}(x) = \Psi_{t_2,t_0}(x)$$

Definition 1.4. *time dependent differential k-form*

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Theorem 1.2 (Fundamental Theorem of Time-Dependent Flows). *For any time-dependent vector field X, there exists a time-dependend flow of X. (! ref lee)*

Proposition 1.2 (Fisherman's Formula). *(!ref lee) Fix a manifold M. If $X : J \times M \to TM$ is a time-dependent vector field with time-dependent flow $\Psi : \mathcal{D} \to M$ then for any $\omega \in \Omega^k(M)$, $(t_1, t_0, x) \in \mathcal{D}$ we have that*

$$\left.\frac{d}{dt}\right|_{t=t_1} \Psi^*_{t,t_0}\omega = \Psi^*_{t_1,t_0}\left(\mathcal{L}_{X_{t_1}}\omega\right)$$

with $X_{t_1} := X(t_1, \cdot) \in \mathfrak{X}(M)$.

Proposition 1.3 (Fisherman's Formula Adapted). *Fix a manifold M. If $X : J \times M \to TM$ is a time-dependent vector field with time-dependent flow $\Psi : \mathcal{D} \to M$, and further, $\omega : J \times M \to \Lambda^k T^*M$ is a time-dependent differential k-form, then for any $\omega \in \Omega^k(M)$, $(t_1, t_0, x) \in \mathcal{D}$ we have that*

$$\left.\frac{d}{dt}\right|_{t=t_1} \Psi^*_{t,t_0}\omega_t = \Psi^*_{t_1,t_0}\left(\mathcal{L}_{X_{t_1}}\omega_{t_1} + \left.\frac{d}{dt}\right|_{t=t_1}\omega_t\right)$$

with $X_{t_1} := X(t_1, \cdot) \in \mathfrak{X}(M)$.

Proof. For any sufficiently small (!) $\varepsilon > 0$ let

$$F : (t_1 - \varepsilon, t_1 + \varepsilon) \times (t_1 - \varepsilon, t_1 + \varepsilon) \to \Lambda^k T^*M$$

be defined by

$$F(u, v) := \Psi^*_{u,t_0}\omega_v.$$

(!huh?). We compute

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=t_1} \Psi^*_{t,t_0}\omega_t &= \left.\frac{d}{dt}\right|_{t=t_1} F(t,t) && def \\
&= \frac{\partial F}{\partial u}(t_1,t_1) + \frac{\partial F}{\partial v}(t_1,t_1) && cahi \\
&= \left.\frac{d}{du}\right|_{u=t_1} \Psi^*_{u,t_0}\omega_{t_1} + \left.\frac{d}{dv}\right|_{v=t_1} \Psi^*_{t_1,t_0}\omega_v \\
&= \Psi^*_{t_1,t_0}(\mathcal{L}_{X_{t_1}}\omega_{t_1}) + \Psi^*_{t_1,t_0}\left(\left.\frac{d}{dv}\right|_{v=t_1}\omega_v\right) && fisherandcommutessincePsiislinearandindepend \\
&= \Psi^*_{t_1,t_0}\left((\mathcal{L}_{X_{t_1}}\omega_{t_1}) + \left.\frac{d}{dv}\right|_{v=t_1}\omega_v\right) && linearity?isomorp
\end{aligned}$$

to show the required statement. ∎

Lemma 1.1. *Let M be a manifold, $x \in M$ with basis (e_i) for T_xM. Then there exists a chart $(U, x^1, ..., x^n)$ centered on x such that for any $i = 1, ..., n$:*

$$\left.\frac{\partial}{\partial x^i}\right|_x = e_i$$

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Proposition 1.4. *(! ref lee) Every smooth manifold admits a Riemannian metric.*

Definition 1.5 (Tubular Neighbourhood). *(! ref lee) Let (M, g) be a Riemannian manifold, $S \subseteq M$ an embedded submanifold. Denote by $\pi : NS \to S$ the normal bundle of S in M. Restrict the exponential map of M as $exp_S : \mathcal{E} \cap NS \to M$ with $\mathcal{E} \subseteq TM$. A neighbourhood U of S in M is called a tubular neighbourhood of S if there exists a positive continuous function $\delta : S \to \mathbb{R}$ such that U is the diffeomorphic image under exp_S of a subset $V \subseteq \mathcal{E} \cap NS$ of the form*

$$V = \{(x, v) \in NS : |v|_g < \delta(x)\}.$$

We call U a uniform tubular neighbourhood of S if δ is constant.

Theorem 1.3 (Existence of Tubular N). *(! ref lee) For every embedded submanifold of a Riemannian manifold (M, g), there exists a tubular neighbourhood in M. If the submanifold is compact, there exists a uniform tubular neighbourhood.*

Proposition 1.5 (Homotopy Formula). *(!ref canas). Fix U, a tubular neighburhood of a submanifold S embedded in M. If $\omega \in \Omega^k(U)$ is closed and $i^*\omega = 0$ for some $i : S \hookrightarrow U$, then there exists an $\eta \in \Omega^{k-1}(U)$ with $\omega = d\eta$. It is possible to ensure that $\forall x \in S : \eta_x = 0$.*

Proof. By definition of tubular neighbourhood, we have a positive continous function $\delta : S \to \mathbb{R}$ with

$$U = exp_S(\{(x, v) \in NS : |v|_g < \delta(x)\})$$

Fix a $t \in I = [0, 1]$. Let $\Psi_t : U \to U$ be defined by

$$\Psi_t(exp_S(x, v)) := exp_S(x, tv).$$

Since exp_S is injective and we have an smooth inverse $exp_S(x, tv) \mapsto exp_S(x, v)$, Ψ_t is a diffeomorphism for $t > 0$. The proof is complete if we find a map $H : \Omega^k(U) \to \Omega^{k-1}(U)$ with

$$H \circ d + d \circ H = \Psi_1^* - \Psi_0^* = id - \iota^*$$

since it then follows by the assumptions on ω that

$$\begin{aligned} Hd(\omega) + dH(\omega) &= id(\omega) - \iota^*(\omega) \\ d(H\omega) &= \omega. \end{aligned}$$

<u>Claim:</u> We can define such a map as:

$$\begin{aligned} (H(\omega))_x(v) &:= \int_0^1 (\Psi_t^*(i_{X_t}\omega))_x(v)dt \\ &= \int_0^1 \omega_{\Psi_t(x)} \left(\frac{d}{dt}\Psi_t(x), D(\Psi_t)_x(v) \right) dt \end{aligned}$$

for $x \in U, v \in T_xU$, and $X_t \in \mathfrak{X}(U)$ given for $t > 0$ by

$$X_t := \left(\frac{d}{dt}\Psi_t \right) \circ \Psi_t^{-1}.$$

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Proof of Claim: We compute

$$\begin{aligned}
H(d\omega)+d(H\omega) &= \int_0^1 \Psi_t^*(i_{X_t}(d\omega)) + d(\Psi_t^*(i_{X_t}\omega))dt && defintionandleipnitz \\
&= \int_0^1 \Psi_t^*(i_{X_t}(d\omega) + di_{X_t}\omega)dt && linearityandcommutes \\
&= \int_0^1 \Psi_t^*(\mathcal{L}_{X_t}\omega)dt && cartans \\
&= \int_0^1 \frac{d}{dt}\Psi_t^*\omega dt && fishermans(\Psi_t time-dep.flowof X_t) \\
&= \Psi_1^*\omega - \Psi_0^*\omega
\end{aligned}$$

Since we have $\Psi_t|_S = id_S$ (!) for $t \in I$, with S seen as a subset of NS via the zero section, it follows that X_t vanishes on S and so will η. □

Remark 1.2. *(!) this is calles homotopy formula since bla bla... de rahm*

Proposition 1.6 (Existence of Vector Field). *Fix (M,ω) a symplectic manifold and $\eta \in \Omega^1(M)$. Then there exists a unique vector field $X \in \mathfrak{X}(M)$ such that $i_X\omega = \eta$.*

2. Moser Trick

Remark 2.1. *Moser was at ETH etc. (!) This trick is very useful..*

Theorem 2.1 (Moser Trick). *(! ref salamon) Fix M a compact manifold. Suppose for some open interval $0 \in J \subseteq \mathbb{R}$ we have a smooth family (!) of symplectic forms $(\omega_t)_{t\in J} \in Omega^2(M)$ such that there exists another smooth family $(\eta_t)_{t\ inK} \in \Omega^1(M)$ with*

$$\frac{d}{dt}\omega_t = d\eta_t.$$

Then there exists a family of diffeomorphisms $(\Psi_t)_{t\in J} \in Diff(M)$ with

$$\Psi_t^*\omega_t = \omega_0.$$

Proof. bla bla bla (!)

note: $\Psi_t := \Psi_{t,0}$ with t0 set as 0 (can translate etc.)

To begin with the end in mind, suppose that

$$\frac{d}{dt}\Psi_t = X_t \circ \Psi_t$$

If we would have the X_t and were to induce the flow Ψ_t with the Fundamental Flow theorem (!) we would also receive that

$$\Psi_0 = \Psi_0 \circ \Psi_0 = id_M$$

To satisfy $\Psi_t^*\omega_t$ being constant as desired we set:

$$0 = \frac{d}{dt}\Psi_t^*\omega_t = \Psi_t^*\left(\mathcal{L}_{X_t}\omega_t + \frac{d}{dt}\omega_t\right) \qquad fisherman's formula$$

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$$
\begin{aligned}
&= \Psi_t^* \left(i_{X_t}(d\omega_t) + d(i_{X_t}\omega_t) + \frac{d}{dt}\omega_t \right) && \textit{cartans} \\
&= \Psi_t^* \left(d(i_{X_t}\omega_t) + \frac{d}{dt}\omega_t \right) && d\omega_t = 0 b/c \omega_t closed \\
&= \Psi_t^* \left(d(i_{X_t}\omega_t) + d\eta_t \right) && \textit{assumption}
\end{aligned}
$$

Since Ψ_t^* is an isomorphism and d is a sheaf morphism we can peel away the layers:

$$
\begin{aligned}
&0 = \Psi_t^* \left(d(i_{X_t}\omega_t) + d\eta_t \right) \\
\Leftrightarrow\ &0 = d(i_{X_t}\omega_t) + d\eta_t = d(i_{X_t}\omega_t + \eta_t) \\
\Leftrightarrow\ &0 = i_{X_t}\omega_t + \eta_t
\end{aligned}
$$

We can solve $i_{X_t}\omega_t = -\eta_t$ for X_t using Proposition 1.6. With the Flow Theorem (!) we can now integrate the X_t resulting in the flows Ψ_t such that $\Psi_t^*\omega_t$ is constant, and since $\Psi_0^* = id$ we have $\Psi_t^*\omega_t = \omega_0$.

(!) note smoothly from t -¿ allows applic of flo thm. ☐

Theorem 2.2 (Moser Isotopy). *(!salomon) Fix as M a 2n-dimensional manifold and as $S \subseteq M$ a compact submanifold. If $\omega_0, \omega_1 \in \Omega^2(M)$ are close and*

(1) $\forall x \in S : \omega_0|x = \omega_1|x$

(2) $\forall x \in S : \omega_0|x, \omega_1|x$ are nondegenerate.

then there exist neighbourhoods U_0, U_1 of S in M and a diffeomorphism $F : U_0 \to U_1$ with

$$
\begin{aligned}
F|_S &= id_S \\
F^*(\omega_1|_{U_1}) &= \omega_0|_{U_0}.
\end{aligned}
$$

Proof. Let U be a uniform tubular neighbourhood of S in M by Theorem 1.3. By construction $\overline{U}$ is compact. By the Homotopy Formula 1.5 there exists $\eta \in \Omega^1(U)$ such that

$$\omega_1 - \omega_0 = d\eta.$$

Further we also have that η vanishes on S. Define for $t \in \mathbb{R}$

$$\omega_t := \omega_0 + t(\omega_1 - \omega_0)$$

By construction ω_t is closed (!). To assure that ω_t is non-degenerate, we shrink U to U_0, a new neighbourhood of S in M. In doing this, note that $\omega_t = \omega_0$ on S per assumption and that we may take the union of open neighbourhoods of the non-degenerate points of S to exceed S as it is closed, and by smoothness retain the non-degenerate property.

We then have that:

$$\frac{d}{dt}\omega_t = \omega_1 - \omega_0 = d\eta.$$

Since $\overline{U_0} \subseteq \overline{U}$ is compact due to being a closed subset of a compact space, we can apply now Moser's Trick 2.1 to get a family of diffeomorphisms $(\Psi_t)_{t \in J}$ with

$$\Psi_t^*\omega_t = \omega_0.$$

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Let now $F := \Psi_1$, $U_1 := F(U_0)$. The final property follows from η vanishing on S (!). $\square$

3. Darboux Theorem

Theorem 3.1 (Darboux's Theorem). *Fix (M, ω) a 2n-dimensional symplectic manifold, $x \in M$. Then there exists a chart $(U, x^1, ..., x^n, y^1, ..., y^n)$ centered on x such that:*

$$\omega|_U = \sum_{i=1}^{n} dx^i \wedge dy^i$$

.

Proof. The canonical form theorem for symplectic tensors 1.1 provides us a basis $(a_1, ..., a_n, b_1, ..., b_n)$ for T_xM such that for its dual basis $(a^1, ..., a^n, b^1, ..., b^n)$ we have

$$\omega_x = \sum_{i=1}^{n} da^i \wedge db^i.$$

By proposition 1.1 we further have a chart $(U, \tilde{\varphi})$ centered on x with associated coordinates $(\tilde{x}^1, ..., \tilde{x}^n, \tilde{y}^1, ..., \tilde{y}^n)$ such that for $i = 1, ..., n$

$$\left.\frac{\partial}{\partial \tilde{x}^i}\right|_x = a_i$$
$$\left.\frac{\partial}{\partial \tilde{y}^i}\right|_x = b_i$$

Combining the previous two results and traversing again into the dual basis we have

$$\omega_x = \sum_{i=1}^{n} d\tilde{x}^i|_x \wedge d\tilde{y}^i|_x.$$

Define:

$$\omega_0 := \omega|_U$$
$$\omega_1 := \sum_{i=1}^{n} d\tilde{x}^i \wedge d\tilde{y}^i.$$

Then ω_0, ω_1 are symplectic forms on U (!).

Application of the Moser isotopy 2.2 to the compact submanifold $x \subseteq U$ given ω_0, ω_1 provides the existence of neighbourhoods U_0, U_1 of x in U and a diffeomorphism $F : U_0 \to U_1$ with

$$F(x) = x$$
$$F^*\omega_1 = \omega_0.$$

Define now another chart (U_0, φ) with $\varphi := \tilde{\varphi}|_{U_1} \circ F$. By construction (!) the associated coordinates are

$$x^i = \tilde{x}^i \circ F$$

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$$y^i = \tilde{y}^i \circ F.$$

If then follows (!) that $\varphi(x) = \tilde{\varphi}(x) = 0$. The remaining property of our chart (U_0, φ) follows by:

$$\begin{aligned}
\omega|_{U_0} &= \omega_0|_{U_0} \\
&= F^*(\omega_1|U_1) \\
&= F^*\left(\sum_{i=1}^n d\tilde{x}^i \wedge d\tilde{y}^i\right) \\
&= \sum_{i=1}^n F^*(d\tilde{x}^i \wedge d\tilde{y}^i) \\
&= \sum_{i=1}^n F^*(d\tilde{x}^i) \wedge F^*(d\tilde{y}^i) \\
&= \sum_{i=1}^n dF^*(\tilde{x}^i) \wedge dF^*(\tilde{y}^i) \\
&= \sum_{i=1}^n d(\tilde{x}^i \circ F) \wedge d(\tilde{y}^i \circ F) \\
&= \sum_{i=1}^n dx^i \wedge dy^i.
\end{aligned}$$

□

REMARK 3.1. *This thm allows us to locally proove, invariant, bla bla*

4. Discussion / Applications

bla

Appendix A. Basics

(1) symplectic manifold
(2) symplectic form
(3) riemanian manifold
(4) embedded submanifold
(5) smooth manifold
(6) einstein summation convention
(7) T_xM et al.
(8) basis of above and dual basis
(9) coordinates associated to chart?
(10) time dependent vector field and flow

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(11) time dep differential k-form
(12) interior multiplication i_X.
(13) exp..
(14) NS
(15) Diff(M)
(16) sheaf morphism
(17) $|\cdot|_g$ riemannian metric
(18) pullback
(19) symplectic vector space
(20) ι inclusion
(21) derivative D

Lemma A.1 (refyan E.). *For a smooth function F from manifolds M to N and $\omega, \eta \in \Omega(N)$ we have*

$$F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$$

Lemma A.2 (refyan E.203). *For a smooth function F from manifolds M to N and $\omega \in \Omega(M)$ we have*

$$F^*(d\omega) = d(F^*\omega).$$