SYMPLECTIC TORIC MANIFOLDS

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(Darboux Theorem)

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Throughout these seminar notes we follow closely the exposition of Yannis Bähni[1] [Bae19] gathered through material from Lee's [Lee13] [Lee18] and Salamon's books [McD17]. We take inspiration from Ana Cannas da Silva [Sil08] and try to follow her conventions, note thus that all manifolds are smooth unless explicitly stated otherwise.

1. Prerequisites

Remark 1.1. *A chart (U, φ) is called centered on x when $\varphi(x) = 0$.*

Proposition 1.1 (Cartan's Magic Formula). *[Bae19] Fix a manifold M, a vector field $X \in \mathfrak{X}(M)$, and an $\omega \in \Omega^k(M)$ for some $k \in \mathbb{N}$. Then*

$$\mathcal{L}_X \omega = i_X(d\omega) + d(i_X \omega).$$

Theorem 1.1 (Canonical Form Theorem for Symplectic Vector Space). *[Bae19] Fix (V, ω) a symplectic vector space. Then $dimV = 2n$ and there exists a basis $(a_1, ..., a_n, b_1, ..., b_n)$ of V such that:*

$$\omega = \sum_{i=1}^{n} a^i \wedge b^i$$

where $(a^1, ..., a^n, b^1, ..., b^n)$ denotes the dual basis of $(a_1, ..., a_n, b_1, ..., b_n)$.

Definition 1.1. *[Lee13] Fix a manifold M, $J \subseteq \mathbb{R}$ an interval. A time-dependent vector field on M is a smooth map $X : J \times M \to TM$ such that $\forall (t, x) \in J \times M : X(t, x) \in T_x M$.*

Definition 1.2. *[Lee13] An integral curve of a time-dependent vector field X is a curve $\gamma \in \mathcal{C}^\infty(J_0, M)$ such that $\gamma'(t) = X(t, \gamma(t))$ for all $t \in J_0$, where $J_0 \subseteq J$ is an interval by notation as in defintion 1.1.*

Definition 1.3. *[Lee13] Fix a manifold M, $J \subseteq \mathbb{R}$ an open interval, and $X : J \times M \to TM$ a time-dependent vector field. We call a time dependent flow of X an open subset $\mathcal{D} \subseteq J \times J \times M$ paired with a map $\psi \in \mathcal{C}^\infty(\mathcal{D}, M)$ such that the following holds for $\mathcal{D}^{(t_0,x)} := \{t \in J : (t, t_0, x) \in \mathcal{D}\}$, $\psi^{(t_0,x)}(t) := \psi(t, t_0, x)$, $M_{t_1,t_0} := \{x \in M : (t_1, t_0, x) \in \mathcal{D}\}$, $\psi_{t_1,t_0}(x) := \psi(t_1, t_0, x)$:*

1. *For any $t_0 \in J, x \in M$, $\mathcal{D}^{(t_0,x)}$ is an open interval such that $t_0 \in \mathcal{D}^{(t_0,x)}$ and $\psi^{(t_0,x)}(t)$ is the unique maximal integral curve of X with $\psi^{(t_0,x)}(t_0) = x$.*
2. *$t_1 \in \mathcal{D}^{(t_0,x)} \wedge y = \psi^{(t_0,x)}(t_1) \implies \mathcal{D}^{(t_1,y)} = \mathcal{D}^{(t_0,x)} \wedge \psi^{(t_1,y)} = \psi^{(t_0,x)}$*
3. *For any $(t_1, t_0) \in J \times J$ we have that M_{t_1,t_0} is open in M and $\psi_{t_1,t_0} : M_{t_1,t_0} \to M$ is a diffeomorphism from M_{t_1,t_0} onto M_{t_0,t_1} with inverse ψ_{t_0,t_1}*
4. *$x \in M_{t_1,t_0} \wedge \psi_{t_1,t_0}(x) \in M_{t_2,t_1} \implies$*
$$x \in M_{t_2,t_0} \wedge \psi_{t_2,t_1} \circ \psi_{t_1,t_0}(x) = \psi_{t_2,t_0}(x)$$

Theorem 1.2 (Fundamental Theorem of Time-Dependent Flows). *[Lee13] For any time-dependent vector field X, there exists a time-dependend flow of X.*

[1] who was of great help in preparing these notes, thank you

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Proposition 1.2 (Fisherman's Formula). *[Lee13] Fix a manifold M. If $X : J \times M \to TM$ is a time-dependent vector field with time-dependent flow $\psi : \mathcal{D} \to M$ then for any $\omega \in \Omega^k(M)$, $(t_1, t_0, x) \in \mathcal{D}$ we have that*

$$\left.\frac{d}{dt}\right|_{t=t_1} \psi^*_{t,t_0}\omega = \psi^*_{t_1,t_0}\left(\mathcal{L}_{X_{t_1}}\omega\right)$$

with $X_{t_1} := X(t_1, \cdot) \in \mathfrak{X}(M)$.

Proposition 1.3 (Fisherman's Formula Adapted). *[Lee13] Fix a manifold M. If $X : J \times M \to TM$ is a time-dependent vector field with time-dependent flow $\psi : \mathcal{D} \to M$, and further, $\omega : J \times M \to \Lambda^k T^*M$ is a time-dependent differential k-form, then for any $\omega \in \Omega^k(M)$, $(t_1, t_0, x) \in \mathcal{D}$ we have that*

$$\left.\frac{d}{dt}\right|_{t=t_1} \psi^*_{t,t_0}\omega_t = \psi^*_{t_1,t_0}\left(\mathcal{L}_{X_{t_1}}\omega_{t_1} + \left.\frac{d}{dt}\right|_{t=t_1}\omega_t\right)$$

with $X_{t_1} := X(t_1, \cdot) \in \mathfrak{X}(M)$.

Proof. For any sufficiently small $\varepsilon > 0$ (i.e. such that we remain within J) let

$$F : (t_1 - \varepsilon, t_1 + \varepsilon) \times (t_1 - \varepsilon, t_1 + \varepsilon) \to \Lambda^k T^*M$$

be defined by

$$F(u, v) := \psi^*_{u,t_0}\omega_v.$$

We compute

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=t_1} \psi^*_{t,t_0}\omega_t &= \left.\frac{d}{dt}\right|_{t=t_1} F(t,t) \\
&= \frac{\partial F}{\partial u}(t_1, t_1) + \frac{\partial F}{\partial v}(t_1, t_1) \\
&= \left.\frac{d}{du}\right|_{u=t_1} \psi^*_{u,t_0}\omega_{t_1} + \left.\frac{d}{dv}\right|_{v=t_1} \psi^*_{t_1,t_0}\omega_v \\
&= \psi^*_{t_1,t_0}(\mathcal{L}_{X_{t_1}}\omega_{t_1}) + \psi^*_{t_1,t_0}\left(\left.\frac{d}{dv}\right|_{v=t_1}\omega_v\right) && \text{fisherman} \\
&= \psi^*_{t_1,t_0}\left(\mathcal{L}_{X_{t_1}}\omega_{t_1}) + \left.\frac{d}{dv}\right|_{v=t_1}\omega_v\right)
\end{aligned}$$

to show the required statement. □

Lemma 1.1. *[Bae19] Let M be a manifold, $x \in M$ with basis (e_i) for T_xM. Then there exists a chart $(U, x^1, ..., x^n)$ centered on x such that for any $i = 1, ..., n$:*

$$\left.\frac{\partial}{\partial x^i}\right|_x = e_i$$

Proposition 1.4. *[Lee13] Every smooth manifold admits a Riemannian metric.*

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Definition 1.4 (Tubular Neighbourhood). *[Lee18] Let (M, g) be a Riemannian manifold, $S \subseteq M$ an embedded submanifold. Denote by $\pi : NS \to S$ the normal bundle of S in M. Restrict the exponential map of M as $exp_S : \mathcal{E} \cap NS \to M$ with $\mathcal{E} \subseteq TM$. A neighbourhood U of S in M is called a tubular neighbourhood of S if there exists a positive continuous function $\delta : S \to \mathbb{R}$ such that U is the diffeomorphic image under exp_S of a subset $V \subseteq \mathcal{E} \cap NS$ of the form*

$$V = \{(x, v) \in NS : |v|_g < \delta(x)\}.$$

We call U a uniform tubular neighbourhood of S if δ is constant.

Theorem 1.3 (Existence of Tubular N). *[Lee18] For every embedded submanifold of a Riemannian manifold (M, g), there exists a tubular neighbourhood in M. If the submanifold is compact, there exists a uniform tubular neighbourhood.*

Proposition 1.5 (Homotopy Formula). *[Sil08] Fix U, a tubular neighburhood of a submanifold S embedded in M. If $\omega \in \Omega^k(U)$ is closed and $\iota^*\omega = 0$ for some $\iota : S \hookrightarrow U$, then there exists an $\eta \in \Omega^{k-1}(U)$ with $\omega = d\eta$. It is possible to ensure that $\forall x \in S : \eta_x = 0$.*

Proof. By definition of tubular neighbourhood, we have a positive continous function $\delta : S \to \mathbb{R}$ with

$$U = exp_S(\{(x, v) \in NS : |v|_g < \delta(x)\})$$

Fix a $t \in I = [0, 1]$. Let $\psi_t : U \to U$ be defined by

$$\psi_t(exp_S(x, v)) := exp_S(x, tv).$$

Since exp_S is injective and we have an smooth inverse $exp_S(x, tv) \mapsto exp_S(x, v)$, ψ_t is a diffeomorphism for $t > 0$. The proof is complete if we find a map $H : \Omega^k(U) \to \Omega^{k-1}(U)$ with

$$H \circ d + d \circ H = \psi_1^* - \psi_0^* = id - \iota^*$$

since it then follows by the assumptions on ω that

$$\begin{aligned} Hd(\omega) + dH(\omega) &= id(\omega) - \iota^*(\omega) \\ d(H\omega) &= \omega. \end{aligned}$$

<u>Claim:</u> We can define such a map as:

$$\begin{aligned} (H(\omega))_x(v) &:= \int_0^1 (\psi_t^*(i_{X_t}\omega))_x(v)dt \\ &= \int_0^1 \omega_{\psi_t(x)} \left(\frac{d}{dt}\psi_t(x), D(\psi_t)_x(v) \right) dt \end{aligned}$$

for $x \in U, v \in T_xU$, and $X_t \in \mathfrak{X}(U)$ given for $t > 0$ by

$$X_t := \left(\frac{d}{dt}\psi_t \right) \circ \psi_t^{-1}.$$

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Proof of Claim: We compute

$$\begin{aligned}
H(d\omega) + d(H\omega) &= \int_0^1 \psi_t^*(i_{X_t}(d\omega)) + d(\psi_t^*(i_{X_t}\omega))dt \\
&= \int_0^1 \psi_t^*(i_{X_t}(d\omega) + di_{X_t}\omega)dt \\
&= \int_0^1 \psi_t^*(\mathcal{L}_{X_t}\omega)dt && \text{cartan} \\
&= \int_0^1 \frac{d}{dt}\psi_t^*\omega dt && \text{fisherman} \\
&= \psi_1^*\omega - \psi_0^*\omega
\end{aligned}$$

Since we have $\psi_t|_S = id_S$ for $t \in I$, with S seen as a subset of NS via the zero section, it follows that X_t vanishes on S and so will η. □

Proposition 1.6 (Existence of Vector Field). *[Bae19] Fix (M, ω) a symplectic manifold and $\eta \in \Omega^1(M)$. Then there exists a unique vector field $X \in \mathfrak{X}(M)$ such that $i_X\omega = \eta$.*

2. Moser Trick

Remark 2.1. *See* `https://en.wikipedia.org/wiki/J%C3%BCrgen_Moser` *for history.*

Theorem 2.1 (Moser Trick). *[McD17] Fix M a compact manifold. Suppose for some open interval $0 \in J \subseteq \mathbb{R}$ we have a smooth family of symplectic forms $(\omega_t)_{t\in J} \in \Omega^2(M)$ such that there exists another smooth family $(\eta_t)_{t\ inK} \in \Omega^1(M)$ with*

$$\frac{d}{dt}\omega_t = d\eta_t.$$

Then there exists a family of diffeomorphisms $(\psi_t)_{t\in J} \in Diff(M)$ with

$$\psi_t^*\omega_t = \omega_0.$$

Proof. The Moser trick is to see the ψ_t as time-dependent flows induced by some X_t time-dependent vector fields. Note that here we use the convention $\psi_t := \psi_{t,0}$ where t_0 is fixed as 0.

To begin with the end in mind, suppose that

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t$$

If we would have the X_t and were to induce the flow ψ_t with the Fundamental Flow theorem we would also receive that

$$\psi_0 = \psi_0 \circ \psi_0 = id_M$$

To satisfy $\psi_t^*\omega_t$ being constant as desired we set:

$$0 = \frac{d}{dt}\psi_t^*\omega_t = \psi_t^*\left(\mathcal{L}_{X_t}\omega_t + \frac{d}{dt}\omega_t\right) \qquad \text{fisherman}$$

$$
\begin{aligned}
&= \psi_t^* \left(i_{X_t}(d\omega_t) + d(i_{X_t}\omega_t) + \frac{d}{dt}\omega_t \right) && \text{cartan} \\
&= \psi_t^* \left(d(i_{X_t}\omega_t) + \frac{d}{dt}\omega_t \right) && d\omega_t = 0 \text{ since closed} \\
&= \psi_t^* \left(d(i_{X_t}\omega_t) + d\eta_t \right) && \text{assumption}
\end{aligned}
$$

Since ψ_t^* is an isomorphism and d is a sheaf morphism we can peel away the layers:

$$
\begin{aligned}
&0 = \psi_t^* \left(d(i_{X_t}\omega_t) + d\eta_t \right) \\
\Leftrightarrow\ &0 = d(i_{X_t}\omega_t) + d\eta_t = d(i_{X_t}\omega_t + \eta_t) \\
\Leftrightarrow\ &0 = i_{X_t}\omega_t + \eta_t
\end{aligned}
$$

We can solve $i_{X_t}\omega_t = -\eta_t$ for X_t explicitly with $X_t = -\Omega_t^{-1}(\eta_t)$ where Ω_t is the tangent-cotangent bundle isomorphism. With the Fundamental theorem of time-dependent Flow we can now integrate the X_t resulting in the flows ψ_t such that $\psi_t^*\omega_t$ is constant, and since $\psi_0^* = id$ we have $\psi_t^*\omega_t = \omega_0$.

□

Theorem 2.2 (Moser Isotopy). *[McD17] Fix as M a 2n-dimensional manifold and as $S \subseteq M$ a compact submanifold. If $\omega_0, \omega_1 \in \Omega^2(M)$ are close and*

(1) $\forall x \in S : \omega_0|x = \omega_1|x$

(2) $\forall x \in S : \omega_0|x, \omega_1|x$ are nondegenerate.

then there exist neighbourhoods U_0, U_1 of S in M and a diffeomorphism $F : U_0 \to U_1$ with

$$
\begin{aligned}
F|_S &= id_S \\
F^*(\omega_1|_{U_1}) &= \omega_0|_{U_0}.
\end{aligned}
$$

Proof. Let U be a uniform tubular neighbourhood of S in M by the Theorem on the Existence of Tubular Neighbourhoods. By construction $\overline{U}$ is compact, hence the uniform nature. By the Homotopy Formula 1.5 there exists $\eta \in \Omega^1(U)$ such that

$$\omega_1 - \omega_0 = d\eta.$$

Further we can ensure in the application of the Homotopy Formula that η vanishes on S. Define for $t \in \mathbb{R}$

$$\omega_t := \omega_0 + t(\omega_1 - \omega_0)$$

By construction ω_t is closed. To assure that ω_t is non-degenerate, we shrink U to U_0, a new neighbourhood of S in M. In doing this, note that $\omega_t = \omega_0$ on S per assumption and that we may take the union of open neighbourhoods of the non-degenerate points of S to exceed S as it is closed, and by smoothness retain the non-degenerate property.

We then have that:

$$\frac{d}{dt}\omega_t = \omega_1 - \omega_0 = d\eta.$$

Since $\overline{U_0} \subseteq \overline{U}$ is compact due to being a closed subset of a compact space, we can apply now Moser's Trick 2.1 to get a family of diffeomorphisms $(\psi_t)_{t\in J}$ with

$$\psi_t^* \omega_t = \omega_0.$$

Let now $F := \psi_1$, $U_1 := F(U_0)$. The final property follows from η vanishing on S. □

3. Darboux Theorem

Theorem 3.1 (Darboux's Theorem). *[Lee13] Fix (M, ω) a 2n-dimensional symplectic manifold, $x \in M$. Then there exists a chart $(U, x^1, ..., x^n, y^1, ..., y^n)$ centered on x such that:*

$$\omega|_U = \sum_{i=1}^{n} dx^i \wedge dy^i$$

.

Proof. The canonical form theorem for symplectic tensors 1.1 provides us a basis $(a_1, ..., a_n, b_1, ..., b_n)$ for $T_x M$ such that for its dual basis $(a^1, ..., a^n, b^1, ..., b^n)$ we have

$$\omega_x = \sum_{i=1}^{n} da^i \wedge db^i.$$

By proposition 1.1 we further have a chart $(U, \tilde{\varphi})$ centered on x with associated coordinates $(\tilde{x}^1, ..., \tilde{x}^n, \tilde{y}^1, ..., \tilde{y}^n)$ such that for $i = 1, ..., n$

$$\left.\frac{\partial}{\partial \tilde{x}^i}\right|_x = a_i$$
$$\left.\frac{\partial}{\partial \tilde{y}^i}\right|_x = b_i$$

Combining the previous two results and traversing again into the dual basis we have

$$\omega_x = \sum_{i=1}^{n} d\tilde{x}^i|_x \wedge d\tilde{y}^i|_x.$$

Define:

$$\omega_0 := \omega|_U$$
$$\omega_1 := \sum_{i=1}^{n} d\tilde{x}^i \wedge d\tilde{y}^i.$$

Then ω_0, ω_1 are symplectic forms on U.

Application of the Moser isotopy 2.2 to the compact submanifold $\{x\} \subseteq U$ given ω_0, ω_1 provides the existence of neighbourhoods U_0, U_1 of $\{x\}$ in U and a diffeomorphism $F : U_0 \to U_1$ with

$$F(x) = x$$
$$F^* \omega_1 = \omega_0.$$

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Define now another chart (U_0, φ) with $\varphi := \tilde{\varphi}|_{U_1} \circ F$. By construction the associated coordinates are

$$x^i = \tilde{x}^i \circ F$$
$$y^i = \tilde{y}^i \circ F.$$

If then follows that $\varphi(x) = \tilde{\varphi}(x) = 0$, or in other words, the chart remains centered. The remaining property of our chart (U_0, φ) follows by:

$$\begin{aligned}
\omega|_{U_0} &= \omega_0|_{U_0} \\
&= F^*(\omega_1|U_1) \\
&= F^*\left(\sum_{i=1}^n d\tilde{x}^i \wedge d\tilde{y}^i\right) \\
&= \sum_{i=1}^n F^*(d\tilde{x}^i \wedge d\tilde{y}^i) \\
&= \sum_{i=1}^n F^*(d\tilde{x}^i) \wedge F^*(d\tilde{y}^i) \\
&= \sum_{i=1}^n dF^*(\tilde{x}^i) \wedge dF^*(\tilde{y}^i) \\
&= \sum_{i=1}^n d(\tilde{x}^i \circ F) \wedge d(\tilde{y}^i \circ F) \\
&= \sum_{i=1}^n dx^i \wedge dy^i.
\end{aligned}$$

□

Remark 3.1. *This theorem allows us to examine the local nature of symplectic manifolds and discover that they are always similar in nature in this regard.*

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Appendix A. Auxiliary

Lemma A.1. *[Bae19] For a smooth function F from manifolds M to N and $\omega, \eta \in \Omega(N)$ we have*

$$F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$$

Lemma A.2. *[Bae19] For a smooth function F from manifolds M to N and $\omega \in \Omega(M)$ we have*

$$F^*(d\omega) = d(F^*\omega).$$

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