TORIC SYMPLECTIC MANIFOLDS

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(Darboux Theorem)

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(!) Throghout the seminar notes we follow closely the exposition of Yannis gathered of material from Lee and Salomon's books. We take inspiration from Anna's stuff.

1. Prerequisites

Remark 1.1. *Must be even dimensional*
All manifolds are smooth unless noted otherwise as per ana's conventions
chart centered on x means $\varphi(x) = 0$.
precomposition definition of F^* F^* diffeomorphism is linear..

Proposition 1.1 (Cartan's Magic Formula). *(! ref lee)*
Fix a manifold M, a vector field $X \in \mathfrak{X}(M)$, and an $\omega \in \Omega^k(M)$ for some $k \in \mathbb{N}$. Then

$$\mathcal{L}_X \omega = i_X(d\omega) + d(i_X \omega).$$

Theorem 1.1. *Canonical Form theorem*

Definition 1.1. *time dependent vector field*

Definition 1.2. *time dependent flow of X*

Definition 1.3. *time dependent differential k-form*

Proposition 1.2 (Fisherman's Formula). *(!ref lee) Fix a manifold M. If $X : J \times M \to TM$ is a time-dependent vector field with time-dependent flow $\Psi : \mathcal{D} \to M$ then for any $\omega \in \Omega^k(M)$, $(t_1, t_0, x) \in \mathcal{D}$ we have that*

$$\left.\frac{d}{dt}\right|_{t=t_1} \Psi^*_{t,t_0}\omega = \Psi^*_{t_1,t_0}\left(\mathcal{L}_{X_{t_1}}\omega\right)$$

with $X_{t_1} := X(t_1, \cdot) \in \mathfrak{X}(M)$.

Proposition 1.3 (Fisherman's Formula Adapted). *Fix a manifold M. If $X : J \times M \to TM$ is a time-dependent vector field with time-dependent flow $\Psi : \mathcal{D} \to M$, and further, $\omega : J \times M \to \Lambda^k T^* M$ is a time-dependent differential k-form, then for any $\omega \in \Omega^k(M)$, $(t_1, t_0, x) \in \mathcal{D}$ we have that*

$$\left.\frac{d}{dt}\right|_{t=t_1} \Psi^*_{t,t_0}\omega_t = \Psi^*_{t_1,t_0}\left(\mathcal{L}_{X_{t_1}}\omega_{t_1} + \left.\frac{d}{dt}\right|_{t=t_1}\omega_t\right)$$

with $X_{t_1} := X(t_1, \cdot) \in \mathfrak{X}(M)$.

Proof. todo chain rule amazing □

Lemma 1.1. *Let M be a manifold, $x \in M$ with basis (e_i) for $T_x M$. Then there exists a chart $(U, x^1, ..., x^n)$ centered on x such that for any $i = 1, ..., n$:*

$$\left.\frac{\partial}{\partial x^i}\right|_x = e_i$$

Definition 1.4 (Tubular Neighbourhood). *(! ref lee)*

Theorem 1.2 (Existence of Tubular N). *ahhh*

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PROPOSITION 1.4 (Homotopy Formula). *(!ref canas). Fix U, a tubular neighborhood of a submanifold S embedded in M. If $\omega \in \Omega^k(U)$ is closed and $i^*\omega = 0$ for some $i : S \hookrightarrow U$, then there exists an $\eta \in \Omega^{k-1}(U)$ with $\omega = d\eta$ and $\forall x \in S : \eta_x = 0$.*

Proof. By definition of tubular neighbourhood, we have a continous function $\delta : S \to \mathbb{R}$ with

$$U = exp_S(\{(x, v) \in NS : |v|_g < \delta(x)\})$$

..... todo

□

PROPOSITION 1.5 (Existence of Vector Field). *Fix (M, ω) a symplectic manifold and $\eta \in \Omega^1(M)$. Then there exists a unique vector field $X \in \mathfrak{X}(M)$ such that $i_X\omega = \eta$.*

2. MOSER TRICK

REMARK 2.1. *Moser was at ETH etc. (!) This trick is very useful..*

THEOREM 2.1 (Moser Trick). *(! ref salamon) Fix M a compact manifold. Suppose for some open interval $0 \in J \subseteq \mathbb{R}$ we have a smooth family (!) of symplectic forms $(\omega_t)_{t\in J} \in Omega^2(M)$ such that there exists another smooth family $(\eta_t)_{t\ inK} \in \Omega^1(M)$ with*

$$\frac{d}{dt}\omega_t = d\eta_t.$$

Then there exists a family of diffeomorphisms $(\Psi_t)_{t\in J} \in Diff(M)$ with

$$\Psi_t^*\omega_t = \omega_0.$$

Proof. bla bla bla (!)

note: $\Psi_t := \Psi_{t,0}$ with t0 set as 0 (can translate etc.)

To begin with the end in mind, suppose that

$$\frac{d}{dt}\Psi_t = X_t \circ \Psi_t$$

If we would have the X_t and were to induce the flow Ψ_t with the Fundamental Flow theorem (!) we would also receive that

$$\Psi_0 = \Psi_0 \circ \Psi_0 = id_M$$

To satisfy $\Psi_t^*\omega_t$ being constant as desired we set:

$$\begin{aligned}
0 = \frac{d}{dt}\Psi_t^*\omega_t = \Psi_t^*\left(\mathcal{L}_{X_t}\omega_t + \frac{d}{dt}\omega_t\right) && fisherman's formula \\
= \Psi_t^*\left(i_{X_t}(d\omega_t) + d(i_{X_t}\omega_t) + \frac{d}{dt}\omega_t\right) && cartans \\
= \Psi_t^*\left(d(i_{X_t}\omega_t) + \frac{d}{dt}\omega_t\right) && d\omega_t = 0 b/c \omega_t closed \\
= \Psi_t^*\left(d(i_{X_t}\omega_t) + d\eta_t\right) && assumption
\end{aligned}$$

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Since Ψ_t^* is an isomorphism and d is a sheaf morphism we can peel away the layers:

$$\begin{aligned} & 0 = \Psi_t^* \left(d(i_{X_t}\omega_t) + d\eta_t \right) \\ \Leftrightarrow\ & 0 = d(i_{X_t}\omega_t) + d\eta_t = d(i_{X_t}\omega_t + \eta_t) \\ \Leftrightarrow\ & 0 = i_{X_t}\omega_t + \eta_t \end{aligned}$$

We can solve $i_{X_t}\omega_t = -\eta_t$ for X_t using Proposition 1.5. With the Flow Theorem (!) we can now integrate the X_t resulting in the flows Ψ_t such that $\Psi_t^*\omega_t$ is constant, and since $\Psi_0^* = id$ we have $\Psi_t^*\omega_t = \omega_0$.

(!) note smoothly from t -¿ allows applic of flo thm. □

THEOREM 2.2 (Moser Isotopy). *(!salomon) Fix as M a 2n-dimensional manifold and as $S \subseteq M$ a compact submanifold. If $\omega_0, \omega_1 \in \Omega^2(M)$ are close and*

(1) $\forall x \in S : \omega_0|x = \omega_1|x$
(2) $\forall x \in S : \omega_0|x, \omega_1|x$ are nondegenerate.

then there exist neighbourhoods U_0, U_1 of S in M and a diffeomorphism $F : U_0 \to U_1$ with

$$\begin{aligned} F|_S &= id_S \\ F^*(\omega_1|_{U_1}) &= \omega_0|_{U_0}. \end{aligned}$$

Proof. Let U be a uniform tubular neighbourhood of S in M by Theorem 1.2. By construction $\overline{U}$ is compact. By the Homotopy Formula 1.4 there exists $\eta \in \Omega^1(U)$ such that

$$\omega_1 - \omega_0 = d\eta.$$

Further we also have that η vanishes on S. Define for $t \in \mathbb{R}$

$$\omega_t := \omega_0 + t(\omega_1 - \omega_0)$$

By construction ω_t is closed (!). To assure that ω_t is non-degenerate, we shrink U to U_0, a new neighbourhood of S in M. In doing this, note that $\omega_t = \omega_0$ on S per assumption and that we may take the union of open neighbourhoods of the non-degenerate points of S to exceed S as it is closed, and by smoothness retain the non-degenerate property.

We then have that:

$$\frac{d}{dt}\omega_t = \omega_1 - \omega_0 = d\eta.$$

Since $\overline{U_0} \subseteq \overline{U}$ is compact due to being a closed subset of a compact space, we can apply now Moser's Trick 2.1 to get a family of diffeomorphisms $(\Psi_t)_{t \in J}$ with

$$\Psi_t^*\omega_t = \omega_0.$$

Let now $F := \Psi_1$, $U_1 := F(U_0)$. The final property follows from η vanishing on S (!). □

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3. Darboux Theorem

Theorem 3.1 (Darboux's Theorem). *Fix (M,ω) a 2n-dimensional symplectic manifold, $x \in M$. Then there exists a chart $(U, x^1, ..., x^n, y^1, ..., y^n)$ centered on x such that:*

$$\omega|_U = \sum_{i=1}^{n} dx^i \wedge dy^i$$

.

Proof. The canonical form theorem for symplectic tensors 1.1 provides us a basis $(a_1, ..., a_n, b_1, ..., b_n)$ for T_xM such that for its dual basis $(a^1, ..., a^n, b^1, ..., b^n)$ we have

$$\omega_x = \sum_{i=1}^{n} da^i \wedge db^i.$$

By proposition 1.1 we further have a chart $(U, \tilde{\varphi})$ centered on x with associated coordinates $(\tilde{x}^1, ..., \tilde{x}^n, \tilde{y}^1, ..., \tilde{y}^n)$ such that for $i = 1, ..., n$

$$\left.\frac{\partial}{\partial \tilde{x}^i}\right|_x = a_i$$

$$\left.\frac{\partial}{\partial \tilde{y}^i}\right|_x = b_i$$

Combining the previous two results and traversing again into the dual basis we have

$$\omega_x = \sum_{i=1}^{n} d\tilde{x}^i|_x \wedge d\tilde{y}^i|_x.$$

Define:

$$\omega_0 := \omega|_U$$

$$\omega_1 := \sum_{i=1}^{n} d\tilde{x}^i \wedge d\tilde{y}^i.$$

Then ω_0, ω_1 are symplectic forms on U (!).

Application of the Moser isotopy 2.2 to the compact submanifold $x \subseteq U$ given ω_0, ω_1 provides the existence of neighbourhoods U_0, U_1 of x in U and a diffeomorphism $F : U_0 \to U_1$ with

$$F(x) = x$$

$$F^*\omega_1 = \omega_0.$$

Define now another chart (U_0, φ) with $\varphi := \tilde{\varphi}|_{U_1} \circ F$. By construction (!) the associated coordinates are

$$x^i = \tilde{x}^i \circ F$$

$$y^i = \tilde{y}^i \circ F.$$

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If then follows (!) that $\varphi(x) = \tilde{\varphi}(x) = 0$. The remaining property of our chart (U_0, φ) follows by:

$$\begin{aligned}
\omega|_{U_0} &= \omega_0|_{U_0} \\
&= F^*(\omega_1|U_1) \\
&= F^*\left(\sum_{i=1}^n d\tilde{x}^i \wedge d\tilde{y}^i\right) \\
&= \sum_{i=1}^n F^*(d\tilde{x}^i \wedge d\tilde{y}^i) \\
&= \sum_{i=1}^n F^*(d\tilde{x}^i) \wedge F^*(d\tilde{y}^i) \\
&= \sum_{i=1}^n dF^*(\tilde{x}^i) \wedge dF^*(\tilde{y}^i) \\
&= \sum_{i=1}^n d(\tilde{x}^i \circ F) \wedge d(\tilde{y}^i \circ F) \\
&= \sum_{i=1}^n dx^i \wedge dy^i.
\end{aligned}$$

□

4. Discussion / Applications

bla

Appendix A. Basics

(1) symplectic manifold
(2) symplectic form
(3) smooth manifold
(4) einstein summation convention
(5) T_xM et al.
(6) basis of above and dual basis
(7) coordinates associated to chart?
(8) time dependent vector field and flow
(9) time dep differential k-form
(10) interior multiplication i_X.
(11) exp..
(12) NS
(13) Diff(M)

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(14) sheaf morphism

Lemma A.1 (refyan E.). *For a smooth function F from manifolds M to N and $\omega, \eta \in \Omega(N)$ we have*

$$F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$$

Lemma A.2 (refyan E.203). *For a smooth function F from manifolds M to N and $\omega \in \Omega(M)$ we have*

$$F^*(d\omega) = d(F^*\omega).$$