Ehrenfeucht–Fraïssé Games (EF Games)

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1 Introduction and Motivation

The Ehrenfeucht–Fraïssé game (EF game) provides a fundamental method for comparing mathematical structures from a logical point of view. The central idea is to analyse when two structures agree on all first order sentences of a given logical complexity. Instead of relying solely on syntactic manipulation of formulas, EF games approach this question through strategic interaction. The game is played between two agents who attempt to reveal or conceal structural differences, and the outcome of their interaction reflects the expressive power of first order logic at a given quantifier depth. This perspective turns a semantic question into a combinatorial one and has become one of the most important tools in classical model theory. Hodges introduces this viewpoint as a way to refine Fraïssé's original method of finite partial isomorphisms and explains how EF games give a dynamic understanding of elementary equivalence [1, pp. 95–100].

The motivation comes from the difficulty of proving that two structures satisfy the same first order properties. Direct syntactic comparison of formulas is often unmanageable. EF games replace this by asking whether one player can always respond to the moves of the other in a way that keeps the two structures indistinguishable at the level of finite patterns. If a winning strategy exists for the Duplicator in all finite rounds, then any first order sentence of finite quantifier depth is unable to separate the structures. This correspondence between strategies and logical indistinguishability lies at the centre of the EF method and is stated explicitly in the classical equivalence theorems for these games [2, Chap. 3].

Another source of motivation arises from the study of how much first order logic can express in finite and infinite structures. Many natural properties such as connectivity in graphs or parity in strings cannot be defined in first order logic. EF games offer a precise way to demonstrate these limitations. By constructing pairs of structures that remain indistinguishable through several rounds of the game, one shows that no sentence of the corresponding quantifier depth can express the property in question. This technique is especially important in

finite model theory, where compactness and other classical tools fail, and where a combinatorial characterisation of definability is needed [2, Chap. 3].

A further conceptual motivation arises from the game-theoretic interpretation of logic. Väänänen highlights that central logical notions such as truth, satisfiability, and equivalence can be analysed through games, each capturing a different semantic aspect of first-order logic [3, pp. 1–2]. Within this framework, Ehrenfeucht–Fraïssé games form the component concerned with comparing structures: they reveal how logical indistinguishability corresponds to the existence of suitable response strategies. In this sense, logical notions are translated into strategic interaction, where moves probe structural complexity and counter-moves aim to maintain similarity [3, pp. 14–20]. This perspective becomes even more illuminating in contexts involving infinite games or transfinite rounds, which Väänänen develops as part of the broader theory of semantic games [3, pp. 24–28, Chs. 7 and 9].

Taken together, these motivations show why EF games occupy a central place in the study of model theoretic equivalence. They provide a transparent method for analysing logical similarity, they supply a powerful tool for inexpressibility results, and they integrate naturally into the broader game-theoretic understanding of logic. Because of their ability to capture fine structural differences and to translate logical questions into strategic ones, EF games form a natural starting point for the deeper study of definability and structure in model theory.

2 Basic Definitions

Definition 2.1 (First-order language and structure)**.** A *first-order language L* consists of relation symbols, function symbols, and constant symbols. An *L-structure* $\mathcal{A}$ is a non-empty set A together with interpretations of these symbols: each n-ary relation symbol is assigned a subset of A^n, each function symbol an operation on A, and each constant symbol an element of A [1, pp. 2–5]. This is the standard semantic framework for first-order logic, and Ehrenfeucht–Fraïssé (EF) games are played on pairs of such structures [1, p. 97].

Definition 2.2 (Partial isomorphism)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be structures in the same language. A partial map $f : A \rightharpoonup B$ with finite domain is a *partial isomorphism* if it preserves all atomic formulas: whenever a relation $R(a_1, \ldots, a_n)$ holds in $\mathcal{A}$, the corresponding relation

$$R(f(a_1), \ldots, f(a_n))$$

holds in $\mathcal{B}$, and similarly for equality. Partial isomorphisms formalise the idea that the chosen elements of the two structures "look the same" up to the information tested in the game [1, pp. 95–98]; see also [4, p. 14].

Definition 2.3 (k-round Ehrenfeucht–Fraïssé game)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be L-structures. The *k-round Ehrenfeucht–Fraïssé game*, denoted $EF_k(\mathcal{A}, \mathcal{B})$, is played between two players—*Spoiler* and *Duplicator*—on the structures $\mathcal{A}$ and $\mathcal{B}$. In each of the

k rounds, Spoiler selects an element from one of the structures, and Duplicator must respond by choosing an element from the other structure. After k rounds the players have produced tuples

$$(a_1, \ldots, a_k) \in A^k, \qquad (b_1, \ldots, b_k) \in B^k.$$

Duplicator wins the game if the map $a_i \mapsto b_i$ defines a partial isomorphism between the substructures induced by these tuples; otherwise, Spoiler wins. This definition appears in essentially the same form in Hodges [1, pp. 95–98], Libkin [2, Chap. 3], and Väänänen [3, pp. 14–16].

Definition 2.4 (k-equivalence and quantifier rank)**.** Two structures $\mathcal{A}$ and $\mathcal{B}$ are said to be k-*equivalent*, written

$$\mathcal{A} \equiv_k \mathcal{B},$$

if they satisfy exactly the same first-order sentences whose *quantifier rank* is at most k. The quantifier rank of a formula is the maximal nesting depth of quantifiers, a standard measure of logical complexity [1, §3.3]; see also [2, Chap. 3].

Remark 2.5. The fundamental connection is that Duplicator has a winning strategy in $EF_k(\mathcal{A}, \mathcal{B})$ if and only if $\mathcal{A} \equiv_k \mathcal{B}$, showing that the game precisely captures the expressive power of first-order formulas of bounded quantifier depth [1, §3.3]; see also [2, Sec. 3.2].

3 Main Ideas and Examples

The central idea of Ehrenfeucht–Fraïssé games is that the expressive power of first-order logic at a fixed quantifier depth can be understood by analysing how well Duplicator can match Spoiler's moves. If Duplicator can always respond in a way that preserves partial isomorphism, then no formula of the corresponding quantifier complexity can distinguish the structures. Examples illustrate how these strategies reflect structural similarities and differences, and they form an essential part of the EF-game method [2, Chap. 3]; see also [4, pp. 14–16].

Example 3.1 (Duplicator wins on isomorphic paths)**.** Consider two path graphs of length three:

$$\mathcal{A} = \bullet - \bullet - \bullet, \qquad \mathcal{B} = \bullet - \bullet - \bullet.$$

Because the graphs are isomorphic, Duplicator has a straightforward winning strategy in $\mathrm{EF}_k(\mathcal{A}, \mathcal{B})$ for every k. Whenever Spoiler selects an endpoint, Duplicator chooses an endpoint in the other structure; if Spoiler selects the middle vertex, Duplicator selects the middle vertex. In each round the adjacency relations among the chosen tuples are preserved, so the map sending a_i to b_i remains a partial isomorphism. This is a canonical example of a *back-and-forth* strategy: Duplicator mirrors Spoiler's moves to maintain structural similarity [1, §3.2]; see also [2, Chap. 3].

Example 3.2 (Duplicator wins on dense linear orders)**.** A more interesting example involves dense linear orders without endpoints. Consider two copies of $(Q, <)$. Even though Spoiler may attempt to pick points revealing gaps or endpoints, none exist, and Duplicator can always choose a corresponding point in the other structure while preserving all order relations among previously chosen points. Hence Duplicator wins $\mathrm{EF}_k((Q, <), (Q, <))$ for all k.

This illustrates the general back-and-forth argument for countable dense linear orders without endpoints [1, §3.2, Example 3].

Example 3.3 (Spoiler wins in two rounds)**.** Consider two directed graphs $\mathcal{A}$ and $\mathcal{B}$. In $\mathcal{A}$ every vertex has at least one outgoing edge, whereas in $\mathcal{B}$ there is a distinguished vertex b^* with no outgoing edges. The first-order sentence

$$\exists x \, \forall y \, \neg E(x, y)$$

expresses that some vertex has no outgoing edges. Its quantifier rank is 2, and the sentence is true in $\mathcal{B}$ but false in $\mathcal{A}$. Since a formula of quantifier rank 2 separates the structures, Spoiler must have a winning strategy in the 2-round Ehrenfeucht–Fraïssé game $EF_2(\mathcal{A}, \mathcal{B})$.

A simple winning strategy is as follows. In the first round, Spoiler selects the vertex b^* in $\mathcal{B}$. Suppose Duplicator responds with some vertex a_1 in $\mathcal{A}$. In the second round, Spoiler plays in $\mathcal{A}$ and chooses a successor a_2 of a_1; such a successor must exist because every vertex of $\mathcal{A}$ has at least one outgoing edge. Duplicator would then have to respond with a vertex b_2 such that $E(b^*, b_2)$ holds in $\mathcal{B}$, but this is impossible since b^* has no outgoing edges. Thus Duplicator has no legal move in the second round, and Spoiler wins the game in two rounds [5, Example 2.4].

4   Main Results and Simple Proofs

The central theorem about EF games states that they capture first-order equivalence at bounded quantifier rank. Recall that $\mathcal{A} \equiv_k \mathcal{B}$ means that $\mathcal{A}$ and $\mathcal{B}$ satisfy the same first-order sentences of quantifier rank at most k.

Theorem 4.1 (Ehrenfeucht)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be structures in the same language and let $k \in N$. Then Duplicator has a winning strategy in $\mathrm{EF}_k(\mathcal{A}, \mathcal{B})$ if and only if $\mathcal{A} \equiv_k \mathcal{B}$.*

The proof proceeds by induction on the quantifier rank and shows how winning strategies correspond to satisfaction of formulas; full proofs can be found in Hodges [1, §3.3, Thm. 3.3.2 (Fraïssé–Hintikka theorem) and Cor. 3.3.3] and Libkin [2, Thms. 3.9 and 3.18].

As a simple but important application, we record the following lemma.

Lemma 4.2. *If $\mathcal{A}$ and $\mathcal{B}$ are isomorphic structures, then Duplicator has a winning strategy in $\mathrm{EF}_k(\mathcal{A}, \mathcal{B})$ for every $k \in N$.*

Proof. Let $f\colon A \to B$ be an isomorphism from $\mathcal{A}$ to $\mathcal{B}$. Duplicator always plays according to f: if Spoiler chooses $a \in A$, Duplicator responds with $f(a)$, and if Spoiler chooses $b \in B$, Duplicator responds with $f^{-1}(b)$. After k rounds we have tuples $(a_1, \ldots, a_k)$ in A and $(b_1, \ldots, b_k)$ in B with $b_i = f(a_i)$ for all i. Since f is an isomorphism, it preserves and reflects all atomic formulas, so the map $a_i \mapsto b_i$ is a partial isomorphism. Thus Duplicator wins EF_k for every k [1, §3.2]. □

Remark 4.3. Together with the directed graph example above, this lemma provides two concrete instances of the general correspondence between winning strategies and first-order distinguishability.

5 Historical Notes and Contributors

The back-and-forth method that underlies EF games goes back to Roland Fraïssé, who used finite partial isomorphisms to compare countable structures and to analyse elementary equivalence. The explicit game-theoretic formulation is due to Andrzej Ehrenfeucht, whose work introduced what are now called Ehrenfeucht–Fraïssé games and applied them to completeness questions [1, 3].

6 Conclusion

Ehrenfeucht–Fraïssé games give a transparent way to analyse when two structures are indistinguishable by first-order logic. By replacing syntactic manipulation of formulas with a game between Spoiler and Duplicator, they turn questions about quantifier rank and definability into combinatorial problems about strategies. The basic examples and results already show how EF games can both witness elementary equivalence and provide sharp inexpressibility arguments, especially in finite model theory where other tools are limited. This combination of conceptual clarity and technical usefulness explains why EF games have become a standard instrument in modern model theory and its applications.

References

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