QUICKEST REAL-TIME DETECTION OF ARBITRAGE OPPORTUNITIES

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Abstract

For an n-dimensional standard independent Brownian motion with no initial drift, we assume that a non-zero drift is introduced to the process at a random time θ being exponentially distributed. Instead of the change point detection which looks back in time, we are observing the process in real-time and try to produce the best probabilistic estimate of the time. This problem can be discussed in three broad cases in terms of the number of processes, k, that get the drift. 1) $n = 1, k = 1$ is the case of a one-dimensional Brownian motion getting 1 drift at time θ. 2) $k = 1$ is when one out of the n Brownian motions gets the drift. 3) $1 < k < n - 1$ is when a number of drifts are introduced to the n-dimensional Brownian motion at θ. These quickest detection problems can be transformed into optimal stopping problems and solved by free-boundary problems. The first case can be solved by solving the ordinary differential equation from the free-boundary problem. The second case admits an elliptic characterization of problem while the third case admits a hypoelliptic characterization of problem. The ellipticity and hypoellipticity helps in each case to formulate the free-boundary problem that can be solved using non-linear Fredholm integral equations of the optimal stopping boundary function. These problems can be extended to mathematical finance for finding an optimal time to engage in the financial market to maximize wealth using stocks that are modelled as exponential Brownian motions.

Key words: optimal stopping problems, free-boundary problems, quickest detection, mathematical finance

Declaration

No portion of the work referred to in the dissertation has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Intellectual Property Statement

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Chapter 1

Introduction

Traders in the financial market are constantly looking for opportunities to maximize their wealth. Building on the foundation laid by [3], this dissertation focuses on examining the movement of n stock prices modeled as independent geometric Brownian motions. At a random and unobservable point in time, k out of n stocks acquire a permanent non-zero drift. The problem was to detect the random time as accurately as possible. This led to an optimal stopping problem with minimizing the probability of false alarm and expected delay. Throughout the dissertation, we follow the notations used in [3] and denote θ as the *disorder time* and μ as the new drift been introduced. In order to capture the randomness of the disorder time, θ is assumed to follow exponential distribution with $\theta \sim \text{Expo}(\lambda)$ and is considered independent of the initial coordinate processes. The problem then can be formulated into an optimal stopping problem where we try to find an optimal stopping time τ_* that minimizes the probability of false alarm, $\mathbb{P}_\pi[\tau < \theta]$, and expected delay in detection $\mathbb{E}_\pi[\tau - \theta]^+$ where $\pi \in [0, 1]$ is the probability of $\theta = 0$ under $\mathbb{P}_\pi$.

To tackle this problem, this dissertation first investigates the disorder problem for the case of one-dimensional Brownian motion that was studied by Gapeev and Peskir in [4] with a finite-horizon focus and in Chapter VI Section 22 in [8] with both infinite and finite horizon perspectives. Then the dissertation will go beyond the one-dimensional case and examine the multi-dimensional Brownian motion with one drift change that was studied by Ernst and Peskir in [2]. With these foundations, the dissertation digs into the most challenging case of quickest detection of multiple drifts that was studied in [3]. The application of the results in finance was not mentioned in the above literature. To expand from the mathematical results into financial terms, it is simple enough to adapt the classical model for stock price in [14], i.e., the geometric Brownian motion. We consider the motion of n stock prices whose coordinate processes are independent geometric Brownian motions $dS_t^i = r^i S_t^i dt + \sigma^i S_t^i dB_t^i$ for $i \in \{1, \dots, n\}$ solved by

$$S_t^i = \exp\left(\sigma^i B_t^i + (r^i - \frac{(\sigma^i)^2}{2})t\right) \tag{1.0.1}$$

where r^i is the expected return on the stock, $\sigma^i > 0$ is the volatility of the stock, and $S_0^i = 1$. Taking the logarithm of the solution we get

$$\ln(S_t^i) = \sigma^i B_t^i + \left(r^i - \frac{(\sigma^i)^2}{2}\right) t. \tag{1.0.2}$$

Observe that for $\mu_0^i = (r^i - \frac{(\sigma^i)^2}{2})$ the log stock price is a driftted standard Brownian motion. At a random time θ, k out of the n stocks gets drift μ, then we have $X = (X^1, \dots, X^n)$ solving the following stochastic differential equations:

$$dX_t^i = d\ln(S_t^i) = \left(\mu_0^i + \mu \mathbb{1}\left(i \in \beta, t \geq \theta\right)\right) dt + \sigma^i dB_t^i \tag{1.0.3}$$

where $B = (B^1, \dots, B^n)$ is an n-dimensional standard Brownian motion; and β indicates the set of possible combinations of coordinate processes to introduce the drift, which will be specified below in Chapter 3. For simplicity of exposition, we assume that all stocks have the same expected return $r^i \equiv 1/2$ (which could be possible under a risk-neutral market assumption) and $\sigma^i \equiv 1$ so that $\mu_0 = 0$ and

$$dX_t^i = \mu \mathbb{1}\left(i \in \beta, t \geq \theta\right) dt + dB_t^i \tag{1.0.4}$$

which is exactly the same formulation for the problem that was dealt with in [3].

1.1 Optimal stopping problems

In this section, we will work through some of the basic results in [8] on optimal stopping problems. We consider a strong Markov process $X = (X_t)_{t \geq 0}$ which takes value in $(E, \mathcal{E})$, where $\mathcal{E} = \mathcal{B}(\mathbb{R}^d)$ and a *gain function* $G : E \to \mathbb{R}$. X is continuous over all stopping times defined on the filtered probability space $(\Omega,\ \mathcal{F},\ (\mathcal{F}_t)_{t \geq 0},\ \mathbb{P}_x)$, where $\mathbb{P}_x[X_0 = x] = 1$. The optimal stopping problem is defined as follows:

$$V(x) = \sup_{\tau} \mathbb{E}\left[G(X_\tau)\right] \tag{1.1.1}$$

The standard argument from [8] shows that if V is *lower semicontinuous (lsc)* and G is *upper semicontinuous (usc)*, then the first entry time $\tau_D := \inf\{t \geq 0 \mid X_t \in D\}$ to the stopping set $D = \{x \in E \mid V(x) = G(x)\}$ is the optimal. Solving for V is to find the smallest superharmonic function that majorizes the function G on the sate space E. Thus,

$$V(x) = \mathbb{E}_x\left[G(X_{\tau_D})\right]. \tag{1.1.2}$$

Similarly, it follows that for

$$V(x) = \inf_{\tau} \mathbb{E}\left[G(X_\tau)\right], \tag{1.1.3}$$

solving for V is to find the largest superharmonic function that is dominated by the gain function G. This will be essential for solving the problem in this dissertation.

There are several main methods commonly used to find V. In this dissertation, we will be focusing on the *free-boundary problem* method.

There are certain extra conditions that V and the continuation set $C = \{x \in E \mid V(x) > G(x)\}$ (for positive programmes) should satisfy to ensure the superharmonic characterization and thus the optimality of V and τ_D:

$$\mathcal{A}_X V \leq 0 \quad \textit{(superharmonic)} \tag{1.1.4}$$

$$V \geq G \tag{1.1.5}$$

where $\mathcal{A}_X$ is the *infinitesimal generator* of the Markov process X. If X is a diffusion process without jumps, condition (1.1.5) can be extended into

$$\mathcal{A}_X V(x) = 0 \quad \text{for x} \in \text{C} \tag{1.1.6}$$

$$\frac{\partial V(x)}{\partial x} = \frac{\partial G(x)}{\partial x} \quad \text{for x} \in \partial C \quad \textit{(smooth fit)} \tag{1.1.7}$$

[See Chapter I in [8] for details]

Another useful tool that we will be using is the Mayer and Lagrange functional and PIDE (Partial Integro Differential Equations) problems that were discussed in Chapter III of [8]. While [8] discussed MLS (Mayer-Lagrange-Superemum) formulation of optimal stopping, we will only present the general results for Mayer and Lagrange formulation in this dissertation. The Mayer and Lagrange formulation of optimal stopping problems is of the form:

$$V = \sup_{\tau} \mathbb{E}\left[M(X_\tau) + \int_0^\tau L(X_t)dt\right] \tag{1.1.8}$$

for regular boundary ∂C and $M : \partial C \to \mathbb{R}$, $L : C \to \mathbb{R}$ being two continuous functions. For

$$U(x) = E_x \left[M(X_{\tau_D}) \right] \tag{1.1.9}$$

$$V(x) = E_x \left[\int_0^{\tau_D} L(X_t) dt \right] \tag{1.1.10}$$

the function $U(x)$ solves the Dirichlet problem as follows:

$$\mathcal{A}_X U(x) = 0 \quad \text{for x} \in \text{C} \tag{1.1.11}$$

$$U(x) = M(x) \quad \text{for x} \in \partial C \tag{1.1.12}$$

while the function $V(x)$ solves the Dirichlet-Poisson problem as follows:

$$\mathcal{A}_X V(x) = -L(x) \quad \text{for x} \in \text{C} \tag{1.1.13}$$

$$V(x) = M(x) \quad \text{for x} \in \partial C. \tag{1.1.14}$$

The discounted (killed) problems of this kind were discussed in section 6.3 of [8] and takes the form:

$$V = \sup_{\tau} \mathbb{E} \left[e^{-\lambda\tau} \left(M(X_\tau) + \int_0^\tau L(X_t) dt \right) \right] \tag{1.1.15}$$

where $\lambda_t = \int_0^t \lambda(X_s) ds$ is the discounting (killing) process and $\lambda : E \to \mathbb{R}_+$ is the discounting rate function. The problem reduces to a normal MLS problem with the process X replaced by $\tilde{X} = \lambda(X)$. The infinitesimal generator of the discounted process $\tilde{X}$ is given by

$$\mathcal{A}_{\tilde{X}} = \mathcal{A}_X - \lambda I \quad (I \text{ is the identity operator}) \tag{1.1.16}$$

Therefore, the first conditions for the Dirichlet and Dirichlet-Poisson problem above can be formulated into

$$\mathcal{A}_X U(x) - \lambda U(x) = 0 \tag{1.1.17}$$

$$\mathcal{A}_X V(x) - \lambda V(x) = -L(x) \tag{1.1.18}$$

for $x \in C$.

1.2 Extended Itô formula

In this section, we provide a reminder of some topics needed from Itô calculus especially the local time that was discussed in Chapter VI of [12]. We assume that f is a convex function and X is a continuous semimartingale. We then have the following formula

$$f(X_t) = f(X_0) + \int_0^t \frac{df^-}{dx}(X_s)dX_s + \frac{1}{2}A_t^f \tag{1.2.1}$$

where A_t^f is a continuous and increasing process. When $f \in C^2$, $A_t^f = \int_0^t \frac{d^2 f^-}{dx^2}(X_s)d\langle X, X\rangle_s$, and the formula (1.2.1) is the classical Itô formula. For f that are not necessarily in C^2, the expression for A_t^f is not explicitly defined from the above formula.

Theorem 1.2 from Chapter VI of [12] presents a *Tanaka formula* for the function $|x| = x^+ + x^-$ where $x^+ = x \vee 0 = \max(x, 0)$ and $x^- = -(x \wedge 0) = -\min(x, 0)$.

$$|X_t - a| = |X_0 - a| + \int_0^t sgn(X_s - a)dX_s + L_t^a \tag{1.2.2}$$

where $a \in \mathbb{R}$ and $(|x|)'^- = sgn(x)$.

$$(X_t - a)^\pm = (X_0 - a)^\pm \pm \int_0^t \mathbb{1}(X_s > a \quad \text{or} \quad X \leq a)dX_s + \frac{1}{2}L_t^a \tag{1.2.3}$$

where the local time of a semimartingale is defined as

$$L_t^a(X) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}(a - \varepsilon < X_s < a + \varepsilon)\, d\langle X, X\rangle_s. \tag{1.2.4}$$

With these, the *Itô-Tanaka formula* was presented in Theorem 1.5 in Chapter VI of [12] as

$$f(X_t) = f(X_0) + \int_0^t \frac{df^-}{dx}(X_s)dX_s + \frac{1}{2}\int_{\mathbb{R}} L_t^a \frac{d^2 f}{dx^2}(X_s)da \tag{1.2.5}$$

where f is the difference between two convex functions, $f(X_t)$ is a semimartingale, and L_t^a is defined as in (1.2.4).

Chapter 2

Problem formulations

2.1 Disorder problem for one-dimensional Brownian motion

In this section, we will start to tackle the quickest detection problem in the simplest setting, i.e., only one Wiener (standard Brownian motion) process gets introduced a new drift μ at a random, exponentially distributed "disorder time" θ. Therefore, the process is defined to solve the following stochastic differential equation:

$$dX_t = \mu \mathbb{1}(t \geq \theta)dt + dB_t \tag{2.1.1}$$

on a probability-statistical space $(\Omega, \mathcal{F}, \mathbb{P}_\pi, \pi \in [0,1])$, and the probability measure is defined as

$$\mathbb{P}_\pi = \pi \mathbb{P}^0 + (1-\pi) \int_0^\infty \lambda e^{-\lambda s} \mathbb{P}^s ds \tag{2.1.2}$$

where $\mathbb{P}^t[X \in \cdot] = \mathbb{P}_\pi[X \in \cdot \mid \theta = t]$ for $t \geq 0$ is the probability measure under which the drift is introduced to the process at time t.

The Bayesian formulation of the problem follows [16]

$$\begin{aligned} V(\pi) &= \inf_\tau \left(a\mathbb{P}_\pi[\tau \leq \theta] + b\mathbb{E}_\pi[\tau - \theta]^+ \right) \\ &= a \inf_\tau \left(\mathbb{P}_\pi[\tau \leq \theta] + \frac{b}{a}\mathbb{E}_\pi[\tau - \theta]^+ \right) \\ &= \inf_\tau \left(\mathbb{P}_\pi[\tau \leq \theta] + c\mathbb{E}_\pi[\tau - \theta]^+ \right) \end{aligned} \tag{2.1.3}$$

where a, b and $c = \frac{b}{a}$ are weights of the two terms being considered. c can also be understood as the Lagrange multiplier in this problem. We define a *posterior* *probability distribution process* $\Pi_t = \mathbb{P}_\pi\left[\theta \le t \mid \mathcal{F}_t^X\right]$ as the best probabilistic estimate of whether θ has occurred by time t. It follows from a continuous analogue of the arguments used by Shiryaev in [15] that

$$\begin{aligned}\mathbb{P}_\pi[\tau < \theta] = 1 - \mathbb{P}_\pi[\theta \le \tau] &= 1 - \mathbb{E}_\pi[\mathbb{1}(\theta \le \tau)] \\ &= 1 - \mathbb{E}_\pi\left[\mathbb{E}_\pi\left[\mathbb{1}(\theta \le \tau) \mid \mathcal{F}_\tau^X\right]\right] \\ &= 1 - \mathbb{E}_\pi\left[\mathbb{P}_\pi\left[\theta \le \tau \mid \mathcal{F}_\tau^X\right]\right] \\ &= 1 - \mathbb{E}_\pi\left[\Pi_\tau\right] \\ &= \mathbb{E}_\pi\left[1 - \Pi_\tau\right] \end{aligned} \tag{2.1.4}$$

and

$$\begin{aligned}\mathbb{E}_\pi[\tau - \theta]^+ &= \mathbb{E}_\pi\left[\mathbb{E}_\pi\left[(\theta - \tau)^+ \mid \mathcal{F}_\tau^X\right]\right] \\ &= \mathbb{E}_\pi\left[\mathbb{E}_\pi\left[\int_0^\tau \mathbb{1}(\theta \le t)dt \mid \mathcal{F}_\tau^X\right]\right] \\ \text{(Fubibni)} &= \mathbb{E}_\pi\left[\int_0^\tau \mathbb{E}_\pi\left[\mathbb{1}(\theta \le t) \mid \mathcal{F}_t^X\right] dt\right] \\ &= \mathbb{E}_\pi\left[\int_0^\tau \mathbb{P}_\pi\left[\theta \le t \mid \mathcal{F}_t^X\right] dt\right] \\ &= \mathbb{E}_\pi\left[\int_0^\tau \Pi_t dt\right]. \end{aligned} \tag{2.1.5}$$

Therefore, the problem could be reformulated as

$$V(\pi) = \inf_\tau \mathbb{E}_\pi\left[1 - \Pi_\tau + c\int_0^\tau \Pi_t dt\right]. \tag{2.1.6}$$

It is evident that this formulation of the problem is the Mayer-Lagrange formulation, where $M(\pi) = 1 - \pi$ and $L(\pi) = c\pi$, if the posterior probability distribution process $\Pi = (\Pi_t)_{t\ge 0}$ is a (strong) Markov process or diffusion process. The following arguments use Bayes and Itô formula to verify that Π is indeed a strong Markov process under $\mathbb{P}_\pi$.

By the abstract version of Bayes formula that was introduced in Section 7.9 in [6]:

$$\begin{aligned}\Pi_t &= \pi\frac{d(\mathbb{P}^0 \mid \mathcal{F}_t^X)}{d(\mathbb{P}_\pi \mid \mathcal{F}_t^X)} + (1-\pi)\int_0^t \lambda e^{-\lambda s}\frac{d(\mathbb{P}^s \mid \mathcal{F}_t^X)}{d(\mathbb{P}_\pi \mid \mathcal{F}_t^X)}ds \\ &= \pi\frac{d\mathbb{P}^0}{d\mathbb{P}_\pi}(t, X) + (1-\pi)\int_0^t \lambda e^{-\lambda s}\frac{d\mathbb{P}^s}{d\mathbb{P}_\pi}(t, X)ds \end{aligned} \tag{2.1.7}$$

and

$$1 - \Pi_t = \mathbb{P}_\pi[\theta > t \mid \mathcal{F}_t^X] = (1-\pi)\int_t^\infty \lambda e^{-\lambda s} \frac{d(\mathbb{P}^s \mid \mathcal{F}_t^X)}{d(\mathbb{P}_\pi \mid \mathcal{F}_t^X)} ds$$
$$= (1-\pi)e^{-\lambda t}\frac{d\mathbb{P}^\infty}{d\mathbb{P}_\pi}(t, X). \tag{2.1.8}$$

We now investigate the Radon-Nikodym derivative of measure $(\mathbb{P}^0 \mid \mathcal{F}_t^X)$ and $(\mathbb{P}^\infty \mid \mathcal{F}_t^X)$ by defining $L_t := \frac{d\mathbb{P}^0}{d\mathbb{P}^\infty}(t, X)$. By definition of the two probability measures, we can see that under $\mathbb{P}^0$

$$dX_t = \mu dt + dB_t \tag{2.1.9}$$

and under $\mathbb{P}^\infty$

$$dX_t = dB_t. \tag{2.1.10}$$

We require that $\mathrm{Law}(\mu t + B_t \mid \mathbb{P}^0) = \mathrm{Law}(B_t \mid \mathbb{P}^\infty)$. By Girsanov's theorem for $\mu dt + dB_t$ to be a standard Brownian motion under $\mathbb{P}^\infty$, we set $N = \int_0^t 1 dB_s$ and $\langle N, M\rangle_t = \int_0^t -\mu ds$. The the stochastic exponential of M is given by

$$Z_t = \exp(-\mu B_t - \frac{1}{2}\mu^2 t)$$
$$= \exp(-\mu X_t + \frac{1}{2}\mu^2 t) \tag{2.1.11}$$

under $\mathbb{P}^0$ since $\mathbb{P}^\infty[F] = \int_F Z_t d\mathbb{P}^0$ for measurable F and $B_t = X_t - \mu t$ under $\mathbb{P}^0$. It then follows that

$$\frac{1}{Z_t} = \frac{d\mathbb{P}^0}{d\mathbb{P}^\infty}(t, X) = L_t$$
$$L_t = \exp(\mu X_t - \frac{\mu^2}{2}t) \tag{2.1.12}$$

under $\mathbb{P}^\infty$.

Since L_t is a function of the stochastic process $dX_t = dB_t$ under $\mathbb{P}^\infty$, we can apply Itô formula to the function $f(t, x) = \exp(\mu x - \frac{1}{2}\mu^2 t)$ to find the SDE that L_t solves.

$$dL_t = df(t, x) = -\frac{1}{2}\mu^2 L_t dt + \mu L_t dX_t + \frac{1}{2}\mu^2 d\langle X, X\rangle_t$$
$$= (-\frac{1}{2}\mu^2 L_t + \frac{1}{2}\mu^2 L_t)dt + \mu L_t dX_t$$
$$= \mu L_t dX_t. \tag{2.1.13}$$

The *likelihood ratio process* $\Phi_t = \frac{\Pi_t}{1-\Pi_t}$ can be defined as (Note that $\Phi_0 = \frac{\pi}{1-\pi}$):

$$\begin{aligned}\Phi_t &= \frac{\pi}{1-\pi}e^{\lambda t}\frac{d\mathbb{P}^0}{d\mathbb{P}^\infty} + e^{\lambda t}\int_0^t \lambda e^{-\lambda s}\frac{d\mathbb{P}^s}{d\mathbb{P}^\infty}ds \\ &= \varphi_0 e^{\lambda t}L_t + e^{\lambda t}\int_0^t \lambda e^{-\lambda s}\frac{d\mathbb{P}^s}{d\mathbb{P}^\infty}ds \\ &= e^{\lambda t}L_t\left(\varphi_0 + \lambda L_t^{-1}\int_0^t e^{-\lambda s}\frac{d\mathbb{P}^s}{d\mathbb{P}^\infty}ds\right) \\ &= e^{\lambda t}L_t\left(\varphi_0 + \lambda\int_0^t e^{-\lambda s}\frac{1}{L_s}ds\right). \qquad (2.1.14)\end{aligned}$$

Substituting in L_t,

$$\begin{aligned}\Phi_t &= e^{\lambda t}\exp\left(\mu X_t - \frac{\mu^2}{2}t\right)\left(\varphi_0 + \lambda\int_0^t e^{-\lambda s}\exp\left(-(\mu X_s - \frac{\mu^2}{2}s)\right)ds\right) \\ &= \exp\left(\lambda t + \mu X_t - \frac{\mu^2}{2}t\right)\left(\varphi_0 + \lambda\int_0^t \exp\left(-(\lambda s + \mu X_s - \frac{\mu^2}{2}s)\right)ds\right) \\ &= e^{Y_t}\left(\varphi_0 + \lambda\int_0^t e^{-Y_s}ds\right) \qquad (2.1.15)\end{aligned}$$

where we set

$$Y_t = \mu X_t + (\lambda - \frac{\mu^2}{2})t. \qquad (2.1.16)$$

With the expression of Φ_t in (2.1.14), we can apply Itô formula again for

$$\Phi_t = A_t L_t C_t \qquad (2.1.17)$$

where $A_t = e^{\lambda t}$ and $C_t = \varphi_0 + \lambda\int_0^t e^{-\lambda s}L_s^{-1}ds$, it follows that

$$\begin{aligned}d\Phi_t &= L_tC_tdA_t + A_tC_tdL_t + A_tL_tdC_t \\ &= \lambda e^{\lambda t}L_tC_tdt + A_tC_t\mu L_tdX_t + A_tL_t\lambda e^{-\lambda t}(L_t)^{-1}dt \\ &= \lambda(1+\Phi_t)dt + \mu\Phi_tdB_t. \qquad (2.1.18)\end{aligned}$$

Similarly for $\Pi_t = \frac{\Phi_t}{1+\Phi_t}$,

$$\begin{aligned}d\Pi_t &= \frac{1}{(1+\Phi_t)^2}d\Phi_t - \frac{1}{2}\frac{2}{(1+\Phi_t)^3}d\langle\Phi,\Phi\rangle_t \\ &= (1-\Pi_t)\lambda dt + \Pi_t(1-\Pi_t)\mu dB_t - \Pi_t^2(1-\Pi_t)\mu^2dt \\ &= \lambda(1-\Pi_t)dt + \Pi_t(1-\Pi_t)\mu d\bar{B}_t \qquad (2.1.19)\end{aligned}$$

where

$$d\bar{B}_t = dB_t - \Pi_t\mu dt \qquad (2.1.20)$$

is a standard Brownian motion under $\mathbb{P}_\pi$, and Φ_t, Π_t are one-dimensional (strong) Markov/ diffusion processes with respect to $\mathcal{F}_t^X$.

2.2 Multi-dimensional Brownian motion with one drift change

Similar to the one-dimensional case discussed in the previous section, we can investigate the multi-dimensional case of the quickest detection problem and reduce the Bayesian formulation into an optimal stopping problem of the posterior probability distribution process which occurs to be a (strong) Markov process. Then the general Markovian approach to solve the optimal stopping problem can be applied and will be discussed in Chapter III.

In this subsection, we consider an n-dimensional Brownian motion ($n \geq 2$) and follow closely the derivations in [2]. For $X_t = (X_t^1, X_t^2, \ldots, X_t^n)$ to be a standard drift-less n-dimensional Brownian motion, there is a drift μ introduced to *one* of the coordinate processes at an exponentially distributed time θ that is independent from the n-dimensional Brownian motion. For simplicity, we assume that $n = 2$. Then $X = (X^1, X^2)$ solves

$$dX_t^1 = \mu \mathbb{1}(\beta = 1, t \geq \theta)dt + dB_t^1 \tag{2.2.1}$$

$$dX_t^2 = \mu \mathbb{1}(\beta = 2, t \geq \theta)dt + dB_t^2 \tag{2.2.2}$$

where $\beta = i$ indicates X^i gets the drift at time θ with a probability $\mathbb{P}_\pi[\beta = i] = p_i$ and $p_1 + p_2 = 1$.

On a filtered statistical probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t^X)_{t\geq 0}, \mathbb{P}_\pi, \pi \in [0,1])$, the probability measure has the following decomposition:

$$\mathbb{P}_\pi = p_1\pi\mathbb{P}_1^0 + p_1(1-\pi)\int_0^\infty \lambda e^{-\lambda t}\mathbb{P}_1^t dt + p_2\pi\mathbb{P}_2^0 + p_2(1-\pi)\int_0^\infty \lambda e^{-\lambda t}\mathbb{P}_2^t dt \tag{2.2.3}$$

$$= p_1\mathbb{P}_1 + p_2\mathbb{P}_2 \tag{2.2.4}$$

where

(a) $\mathbb{P}_\pi[\theta = 0] = \pi$ and $\mathbb{P}_\pi[\theta > t|\theta > 0] = e^{-\lambda t}$ (memory-less),

(b) $\mathbb{P}_i^t[X \in \cdot] = \mathbb{P}_\pi[X \in \cdot \mid \beta = i, \theta = t]$,

(c) $\mathbb{P}_i = \pi\mathbb{P}_i^0 + (1-\pi)\int_0^\infty \lambda e^{-\lambda t}\mathbb{P}_i^t dt$ is the probability measure of X^i gets the drift,

(d) $\mathbb{P}_i^\infty = \mathbb{P}^\infty$ is the probability measure of no drift introduced to the processes.

The Bayesian formulation is of the same format as in the one-dimensional case in previous section where

$$V(\pi) = \inf_{\tau} \left(\mathbb{P}_{\pi}[\tau \leq \theta] + c\mathbb{E}_{\pi}[\tau - \theta]^{+} \right), \tag{2.2.5}$$

and can be reformulated into

$$V(\pi) = \inf_{\tau} \mathbb{E}_{\pi} \left[1 - \Pi_{\tau} + c \int_0^{\tau} \Pi_t dt \right] \tag{2.2.6}$$

where

$$\Pi_t = \mathbb{P}_{\pi}[\theta \leq t \mid \mathcal{F}_t^X] = \Pi_t^1 + \Pi_t^2 \tag{2.2.7}$$

$$\Pi_t^i = \mathbb{P}_{\pi}[\beta = i, \theta \leq t \mid \mathcal{F}_t^X]. \tag{2.2.8}$$

Similar to the one-dimensional case, we define a likelihood ratio process $\Phi_t = (\Phi_t^1, \Phi_t^2)$ as

$$\Phi_t^i = \frac{\Pi_t^i}{1 - \Pi_t^i}. \tag{2.2.9}$$

Using the abstract version of Bayes formula again we find

$$\Pi_t^i = p_i \left(\pi \frac{d\mathbb{P}_{i,t}^0}{d\mathbb{P}_{\pi,t}} + (1 - \pi) \int_0^t \lambda e^{-\lambda s} \frac{d\mathbb{P}_{i,t}^s}{d\mathbb{P}_{\pi,t}} ds \right) \tag{2.2.10}$$

$$1 - \Pi_t^i = p_i (1 - \pi) e^{-\lambda t} \frac{d\mathbb{P}_{i,t}^{\infty}}{d\mathbb{P}_{\pi,t}} \tag{2.2.11}$$

where $\mathbb{P}_{i,t}^s$ and $\mathbb{P}_{\pi,t}$ indicate the restriction $(measure)\big|_{\mathcal{F}_t^X}$.

Using Girsanov's theorem again for $Law(\mu t + B_t^i \mid \mathbb{P}_{i,t}^0) = Law(B_t^i \mid \mathbb{P}_t^{\infty})$, we have the Radon-Nikodym derivative of the equivalent measures as

$$L_t^i = \frac{d\mathbb{P}_{i,t}^0}{d\mathbb{P}_t^{\infty}} = \exp(\mu X_t^i - \frac{\mu^2}{2} t) \tag{2.2.12}$$

and

$$\Phi_t^i = e^{\lambda t} L_t^i (\Phi_0^i + \lambda \int_0^t e^{-\lambda s} (L_t^i)^{-1} ds) \tag{2.2.13}$$

$$d\Phi_t^i = \lambda (1 + \Phi_t^i) dt + \mu \Phi_t^i dB_t^i \tag{2.2.14}$$

$$d\Pi_t^i = \lambda (1 - \Pi_t^i) dt + \mu \Pi_t^i (1 - \Pi_t^i) d\bar{B}_t^i. \tag{2.2.15}$$

This follows from exactly the same derivation in the previous section only with an additional superscript $i = 1, 2$ to indicate the multi-dimensional nature of the coordinate process. The system of stochastic differential equations (2.2.14) has a strong unique solution, the Shiryaev process Φ, that follows a stochastic (Markovian) flow if its initial point φ

$$\begin{aligned}\Phi_t^i &= \exp\left(\lambda t + \mu X_t^i - \mu^2/2t\right)\left(\varphi + \lambda\int_0^t \exp\left(-(\lambda s + \mu X_s^i - \mu^2/2s)\right) ds\right) \\ &= e^{Y_t^i}\left(\varphi + \lambda\int_0^t e^{-Y_s^i} ds\right) \end{aligned} \tag{2.2.16}$$

where

$$Y_t = \mu X_t^i + (\lambda - \frac{\mu^2}{2})t \tag{2.2.17}$$

[also see [10]].

2.3 Multi-dimensional Brownian motion with multiple drift changes

The mathematical formulation and setup for this section closely follows [3] where, similar to Section 2.2, we consider an n-dimensional standard Brownian motion with no initial drift but only we have $1 < k < n$ drifts introduced instead of only 1. The setup here in this case is trickier due to the fact that we need to consider different combination of picking k coordinate processes out of n for the drift μ to be introduced upon. As already discussed earlier in Chapter 1, the process should be modelled as

$$dX_t^i = \mu \mathbb{1}\left(i \in \beta, t \geq \theta\right) dt + dB_t^i. \tag{2.3.1}$$

To capture combination, we define a set $C_k^n = \{(n_1, \ldots, n_k) \mid 1 \leq n_1 < \cdots < n_k \leq n\}$ as the set of ordered tuples of combinations of k coordinate processes. For $\beta \in C_k^n$, taking a tuple $\beta = (n_1, \ldots, n_k)$, the coordinate processes $X^{n_1}, \ldots, X^{n_k}$ get the drift with a probability $\mathbb{P}_\pi[\beta = (n_1, \ldots, n_k)] = p_{n_1,\ldots,n_k}$. For simplicity of notation and following [3], we slightly abuse the notation i. Firstly, we have $i = (n_1, \ldots, n_k)$ for $i \in C_k^n$ indicating a possible tuple of k elements and $1 \leq i \leq N$ where $N = \binom{n}{k}$ is the total number of possible combinations of choosing k out of n. Secondly, i takes value in $1, \ldots, n$ for index purposes and is used in (2.3.1).

With the above setup, we define the probability measure $\mathbb{P}_\pi$ as

$$\mathbb{P}_\pi = \sum_{i \in C_k^n} p_i(\pi \mathbb{P}_i^0 + (1-\pi)\int_0^\infty \lambda e^{-\lambda t}\mathbb{P}_i^t dt) \tag{2.3.2}$$

$$= \sum_{i \in C_k^n} p_i \mathbb{P}_i \tag{2.3.3}$$

for $i = (n_1, \dots, n_k)[1 \le i \le N]$. We also use the best probabilistic estimate of θ has occurred by time t, the posterior probability distribution process, Π_t for

$$\Pi_t = \mathbb{P}_\pi[\theta \le t \mid \mathcal{F}_t^X] \tag{2.3.4}$$

$$= \sum_{i=1}^N \Pi_t^i \tag{2.3.5}$$

where

$$\Pi_t^i = \mathbb{P}_\pi\left[\beta = i, \theta \le t \mid \mathcal{F}_t^X\right] \quad \text{for } 1 \le i \le N \text{ in } C_k^n. \tag{2.3.6}$$

Similar to previous settings, we define a likelihood ratio process

$$\Phi_t^i = \frac{\Pi_t^i}{1 - \Pi_t^i} \tag{2.3.7}$$

in order to connect stochastic analysis into the problem.

By Girsanov's theorem again, the Radon-Nikodym derivative of the two measures $\mathbb{P}_{i,t}^0$ and $\mathbb{P}_t^\infty$ can be derived as a linear combination of k Radon-Nikodym derivatives from Section 2.2 where only 1 drift change was to be detected.

$$L_t^i = \frac{d\mathbb{P}_{i,t}^0}{d\mathbb{P}_t^\infty} = \exp(\mu \sum_{j=1}^k X_t^{n_j} - k\frac{\mu^2}{2}t) \tag{2.3.8}$$

for $i = (n_1, \dots, n_k)$. Additionally, with the abstract version of Bayes formula, we can have an expression of Π_t^i and $1 - \Pi_t^i$ like in (2.2.10) and (2.2.11) for $i = (n_1, \dots, n_k)[1 \le i \le N]$, and thus an expression for Φ_t^i is given as

$$\Phi_t^i = e^{\lambda t} L_t^i \left(\Phi_0^i + \lambda \int_0^t e^{-\lambda s}(L_s^i)^{-1} ds \right) \tag{2.3.9}$$

for $i = (n_1, \dots, n_k)[1 \le i \le N]$. Using the multi-dimensional Itô formula, we have

$$d\Phi_t^i = \lambda(1 + \Phi_t^i)dt + \sum_{j=1}^k \mu \Phi_t^i dB_t^{n_j} \tag{2.3.10}$$

$$d\Pi_t^i = \lambda(1 - \Pi_t^i)dt + \sum_{j=1}^k \Pi_t^i(1 - \Pi_t^i)\mu d\bar{B}_t^{n_j} \tag{2.3.11}$$

for $i = (n_1, \ldots, n_k)[1 \le i \le N]$ where

$$d\bar{B}_t^{n_j} = dB_t^{n_j} - k\Pi_t^i \mu dt. \tag{2.3.12}$$

We see that, like before, Φ follows a stochastic (Markovian) flow of its initial point by substituting L_t^i into (2.2.9) or by solving the stochastic differential equation (2.3.10).

$$\Phi_t^i = e^{Y_t^i}\left(\Phi_0^i + \lambda \int_0^t e^{-Y_s^i} ds\right) \tag{2.3.13}$$

where

$$Y_t^i = \mu \sum_{j=1}^{k} X_t^{n_j} + (\lambda - k\frac{\mu^2}{2})t. \tag{2.3.14}$$

We then have the same results from previous sections that the Bayesian quickest detection problem can be reformulated into an optimal stopping problem of a Markov process.

$$V(\pi) = \inf_\tau \mathbb{E}_\pi \left[1 - \Pi_\tau + c\int_0^\tau \Pi_t dt\right]. \tag{2.3.15}$$

Chapter 3

Method of solutions

3.1 One dimension

The case of one-dimensional drift change has been solved in Section 22 of [8] for both a finite horizon and an infinite horizon case. In this section, we first try to solve it in infinite horizon case and then investigate a change of measure approach that was used in [2]. Recall that the optimal stopping problem from Section 2.1 was reduced to a form of Mayer-Lagrange formulation for $M(\pi) = 1 - \pi$ and $L(\pi) = c\pi$:

$$V(\pi) = \inf_{\tau} \mathbb{E}_{\pi} \left[M(\Pi_{\tau}) + \int_0^{\tau} L(\Pi_t) dt \right]. \tag{3.1.1}$$

The solutions and methods used in [8] focused on proceeding from this formulation.

3.1.1 Mayer-Lagrange formulation

We notice that from the strong Markov process $\Pi = (\Pi_t)_{t \geq 0}$ takes value in $[0, 1]$. As $\pi \uparrow 1$, $1 - \pi \downarrow 0$ and $c \int_0^t \pi ds$ increases; therefore, $\exists A \in (0, 1)$ beyond which the return diminishes for further observation of the process, and $\tau_{A_*} = \inf\{t \geq 0 \mid \Pi_t \geq A\}$ is optimal. Since the Π is strong Markov, standard arguments for the Dirichlet & Dirichlet-Poisson problem imply that a free-boundary problem applies:

$$\mathcal{A}_{\Pi} V = -L = -c\pi \quad \text{for } \pi \in (0, 1), \tag{3.1.2}$$

$$V\big|_{\partial C} = M\big|_{\partial C}, \tag{3.1.3}$$

$$\frac{dV}{d\pi}\big|_{\partial C}(\pi) = \frac{dG}{d\pi}\big|_{\partial C}(\pi) = -1 \quad \textit{(smooth fit condition)}, \tag{3.1.4}$$

$$\tag{3.1.5}$$

$$V < G = M \quad \text{on } C, \tag{3.1.6}$$

$$V = G = M \quad \text{on } D, \tag{3.1.7}$$

where $G(\pi) = M(\pi) + \int_0^t L(\pi)ds$ is the gain function; and $\mathcal{A}_\Pi$ is the infinitesimal generator of the process Π given by

$$\mathcal{A}_\Pi = \lambda(1-\pi)\frac{d}{d\pi} + \frac{1}{2}\mu^2\pi^2(1-\pi)^2\frac{d^2}{d\pi^2}. \tag{3.1.8}$$

Therefore, the condition (3.1.2) is a linear ordinary differential equation

$$\begin{gathered} \lambda(1-\pi)\frac{dV}{d\pi} + \gamma\pi^2(1-\pi)^2\frac{d^2V}{d\pi^2} = -c\pi \\ \frac{\lambda}{\gamma}\frac{1}{\pi^2(1-\pi)}V' + V'' = -\frac{c}{\gamma}\frac{1}{\pi(1-\pi)^2} \end{gathered} \tag{3.1.9}$$

for $\gamma = \mu^2/2$. We define the integrating factor as

$$\exp\left(\int \frac{1}{\pi^2(1-\pi)}\frac{\lambda}{\gamma}d\pi\right) := \exp\left(\frac{\lambda}{\gamma}\alpha(\pi)\right). \tag{3.1.10}$$

Therefore, the ODE (3.1.9) can be solved as a linear first order ordinary differential equation for V' as

$$\begin{gathered} \frac{d}{d\pi}(V' e^{\frac{\lambda}{\gamma}\alpha(\pi)}) = -\frac{c}{\gamma}\frac{1}{\pi(1-\pi)^2}e^{\frac{\lambda}{\gamma}\alpha(\pi)} \\ V'(\pi) = e^{-\frac{\lambda}{\gamma}\alpha(\pi)}\left(C - \int_0^\pi \frac{c}{\gamma}\frac{1}{\rho(1-\rho)^2}e^{\frac{\lambda}{\gamma}\alpha(\rho)}d\rho\right). \end{gathered} \tag{3.1.11}$$

To determine the appropriate constant C, we first take a look at the term $e^{A\alpha(\pi)}$ for a constant A. As $\pi \downarrow 0$, $e^{A\alpha(\pi)} \to +\infty$ or 0 depending on A being negative or positive respectively. Therefore, the $V' \to \pm\infty$ depending on C being positive or negative respectively, and an appropriate choice of C is 0. With $C = 0$ chosen, we define $V'(\pi) = \Psi(\pi)$. Thus

$$V_*(\pi) = \begin{cases} (1-A_*) + \int_{A_*}^\pi \Psi(\rho)d\rho & \text{for } \pi \in [0, A_*) \\ 1-\pi & \text{for } \pi \in [A_*, 1] \end{cases} \tag{3.1.12}$$

and

$$V_*'(A_*) = \Psi(A_*) = -1 \tag{3.1.13}$$

for A_* is a unique solution to (3.1.13).

We can apply a verification approach to show that the value function V_* indeed solve the free-boundary problem and $\tau_{A_*} = \inf\{t \geq 0 \mid \Pi_t = A_*\}$ is the optimal stopping time. From (3.1.12) we see that $V_* = (1-\pi)$ on $(A_*, 1]$ thus $\mathcal{A}_\Pi V_*(\pi) = -\lambda + \lambda\pi$ which is greater than $-c\pi$ on $[\lambda/(\lambda+c), 1]$. Since $A_* \geq \lambda/(\lambda+c)$, then $\mathcal{A}_\Pi V_*(\pi) \geq -c\pi = -L(\pi)$ on $(A_*, 1]$. Together with (3.1.2), we see that $\mathcal{A}_\Pi V_* \geq -c\pi$ on $(0, 1]$. By the change of variable formula in (3.5.5) of [8],

$$\begin{aligned} dV_*(\Pi_t) &= \frac{1}{2}\left(V_*'(\Pi_t+) + V_*'(\Pi_t-)\right) d\Pi_t + \frac{1}{2} V_*''(\Pi_t) \mathbb{1}(\Pi_t \neq A_*) d\langle \Pi, \Pi \rangle_t \\ &= \mathcal{A}_\Pi V_* \mathbb{1}(\Pi_t \neq A_*) dt + V_*'(\Pi_t)\mu\Pi_t(1-\Pi_t) d\bar{B}_t \\ &\geq -L(\Pi_t)dt + V_*'(\Pi_t)\mu\Pi_t(1-\Pi_t)d\bar{B}_t. \end{aligned} \tag{3.1.14}$$

From (3.1.6) and (3.1.7), we see that

$$M(\Pi_t) \geq V_*(\Pi_t) \geq V_*(\pi) - \int_0^t L(\Pi_s)ds + \int_0^t V_*'(\Pi_s)\mu\Pi_s(1-\Pi_s)d\bar{B}_s \tag{3.1.15}$$

where $M(\pi) = 1 - \pi$. By optional sampling theorem,

$$\mathbb{E}_\pi\left[M(\Pi_\tau)\right] \geq \mathbb{E}_\pi\left[V_*(\pi)\right] - \mathbb{E}_\pi\left[\int_0^\tau L(\Pi_s)ds\right] + 0 \tag{3.1.16}$$

and taking the infimum over all stopping times for Π

$$V_*(\pi) \leq \inf_\tau \mathbb{E}_\pi\left[M(\Pi_\tau) + \int_0^\tau L(\Pi_t)dt\right] = V(\pi). \tag{3.1.17}$$

Assume that τ_{A_*} achieves the infimum in (3.1.1) and from (3.1.15), we see that

$$M(\Pi_{\tau_{A_*}}) = V_*(\Pi_{\tau_{A_*}}) = V_*(\pi) - \int_0^{\tau_{A_*}} L(\Pi_t)dt + \int_0^{\tau_{A_*}} V_*'(\Pi_t)\mu\Pi_t(1-\Pi_t)d\bar{B}_t. \tag{3.1.18}$$

By optional sampling theorem,

$$V_*(\pi) = \mathbb{E}_\pi\left[M(\Pi_{\tau_{A_*}}) + \int_0^{\tau_{A_*}} L(\Pi_t)dt\right] \geq V(\pi) \tag{3.1.19}$$

since V_* is the result of a specific choice of stopping policy which is greater than the infimum over all τ. From (3.1.17) and (3.1.19), we see that $V_* = V$ and the stopping time τ_{A_*} is optimal.

3.1.2 Measure-changed Lagrange formulation

It was pointed out in [5] that changing the measure from $\mathbb{P}_\pi$ to $\mathbb{P}^\infty$ would reduce the complexity of the problem. Since Proposition 2 from [5] extends to processes beyond just Bessel process to other general diffusion, we have the following proposition.

Proposition 1. *The value function in* (3.1.1) *for* $M(\pi) = 1 - \pi$ *and* $L(\pi) = c\pi$ *can be reformulated into*

$$V(\pi) = (1-\pi)\left[1 + c\hat{V}(\pi)\right] \tag{3.1.20}$$

where

$$\hat{V}(\pi) = \hat{V}(\varphi) = \inf_{\tau} \mathbb{E}_{\varphi}^{\infty}\left[\int_0^{\tau} e^{-\lambda t}(\Phi_t - \frac{\lambda}{c})dt\right] \tag{3.1.21}$$

for $\varphi \in [0, \infty)$ *under* $\mathbb{P}_{\varphi}^{\infty}$ *and* $\mathbb{P}_{\varphi}^{\infty}[\Phi_0 = \varphi = \pi/(1-\pi)] = 1$.

Proof. This proof follows the same method used in [5] with certain expanded steps for easier understanding. First we note that using (2.1.19),

$$\Pi_{\tau} = \pi + \int_0^{\tau} \lambda(1-\Pi_t)dt + \int_0^{\tau} \mu\Pi_t(1-\Pi_t)d\bar{B}_t. \tag{3.1.22}$$

By Doob's optional sampling theorem, we apply expectation to both side of (3.1.22)

$$\begin{aligned}\mathbb{E}_{\pi}\left[\Pi_{\tau}\right] &= \pi + \lambda\mathbb{E}_{\pi}\left[\int_0^{\tau} 1dt\right] - \lambda\mathbb{E}_{\pi}\left[\int_0^{\tau} \Pi_t dt\right] + 0 \\ &= \pi + \lambda\mathbb{E}_{\pi}\left[\tau\right] - \lambda\mathbb{E}_{\pi}\left[\int_0^{\tau} \Pi_t dt\right] \end{aligned} \tag{3.1.23}$$

and

$$\mathbb{E}_{\pi}\left[\int_0^{\tau} \Pi_t dt\right] = \frac{1}{\lambda}\left(\pi + \lambda\mathbb{E}_{\pi}\left[\tau\right] - \mathbb{E}_{\pi}\left[\Pi_{\tau}\right]\right). \tag{3.1.24}$$

From (2.1.8) we see that

$$\frac{d\mathbb{P}^{\infty}}{d\mathbb{P}_{\pi}}(t, X) = e^{\lambda t}\frac{1-\Pi_t}{1-\pi} \tag{3.1.25}$$

$$\frac{d\mathbb{P}_{\pi}}{d\mathbb{P}^{\infty}}(t, X) = e^{-\lambda t}\frac{1-\pi}{1-\Pi_t} \tag{3.1.26}$$

and thus

$$\mathbb{E}_{\pi}[\cdot] = \mathbb{E}^{\infty}[\quad\cdot\quad e^{-\lambda t}\frac{1-\pi}{1-\Pi_t}]. \tag{3.1.27}$$

With these, the following proof follows exactly the same as in Proposition 2 of [5] and completes the proof. □

This Lagrange-formulated optimal stopping problem (3.1.21) can also be transformed into a Mayer-formulated problem. Although we do not directly solve the problem using the Mayer formulation, it provides an insight into the solution of the Lagrange formulated problem and will be used in the multidimensional case to establish the properties of the optimal stopping boundary.

1. The infinitesimal generator of the process Φ_t is given by

$$\mathcal{A}_\Phi = \lambda(1+\varphi)\frac{d}{d\varphi} + \frac{\mu^2}{2}\varphi^2\frac{d^2}{d\varphi^2}. \tag{3.1.28}$$

Therefore, we can reformulate the Lagrange-formulated optimal stopping problem in (3.1.3) into a Mayer-formulated by finding a function $M : [0,\infty) \to \mathbb{R}$ that solves the ordinary differential equation

$$\mathcal{A}_\Phi M - \lambda M = L \tag{3.1.29}$$

for $L(\varphi) = \varphi - \lambda/c$ since the original Lagrange-functional is in a discounted form.

To solve the ordinary differential equation, we first consider it without the constant term $-\lambda/c$ and solve the ordinary differential equation

$$\lambda(1+\varphi)\frac{dM(\varphi)}{d\varphi} + \frac{\mu^2}{2}\varphi^2\frac{d^2M(\varphi)}{d\varphi^2} - \lambda M(\varphi) = \varphi. \tag{3.1.30}$$

Multiplying both sides of (3.1.30) by $\frac{2}{\mu^2}$, setting $x = \varphi$ and $y(x) = M(\varphi)$, we get

$$\kappa(1+x)y' + x^2y'' - \kappa y = \nu x \tag{3.1.31}$$

where $\kappa := 2\lambda/\mu^2$ and $\nu := 2/\mu^2$. Standard arguments in [2] shows that the function y can be solved as

$$y(x) = (1+x)z(u) \tag{3.1.32}$$

where $u = x/(1+x)$ and $z(u)$ is the solution to another ordinary differential equation

$$u^2(1-u)z'' + \kappa z' = \nu\frac{u}{1-u} \tag{3.1.33}$$

and

$$z(\frac{x}{1+x}) = \nu\int_0^{x/(1+x)}\left(\frac{1-a}{a}\right)^\kappa\int_0^a\frac{b^{\kappa-1}}{(1-b)^{\kappa+2}}e^{-\kappa/b}dbda \tag{3.1.34}$$

where a, b are dummy variables for the integral.

We can verify this by assuming that $y(x)$ indeed takes the form in (3.1.32) and thus

$$\frac{du}{dx} = \frac{1}{(1+x)^2} \quad \frac{d^2u}{dx^2} = -\frac{2}{(1+x)^3} \tag{3.1.35}$$

$$\begin{aligned} y'(x) &= \frac{d}{dx}\left[(1+x)z(\frac{x}{1+x})\right] \\ &= z(\frac{x}{1+x}) + (1+x)\frac{dz}{du}\frac{du}{dx} \\ &= z(u) + \frac{1}{1+x}\frac{dz}{du} \end{aligned} \tag{3.1.36}$$

$$
\begin{aligned}
y''(x) &= \frac{d}{dx}\left[z(u) + \frac{1}{1+x}\frac{dz}{du}\right] \\
&= \frac{dz}{du}\frac{du}{dx} + \frac{d}{dx}\left(\frac{1}{1+x}\frac{dz}{du}\right) \\
&= \frac{1}{(1+x)^3}\frac{d^2z}{du^2}. \qquad (3.1.37)
\end{aligned}
$$

Substitute (3.1.36) and (3.1.37) back into (3.1.31) we get

$$
\begin{aligned}
\kappa(1+x)\left(z(u) + \frac{1}{1+x}\frac{dz}{du}\right) + x^2\left(\frac{1}{(1+x)^3}\frac{d^2z}{du^2}\right) - \kappa(1-x)z(u) &= \nu x \\
\frac{x^2}{(1+x)^3}\frac{d^2z}{du^2} + \kappa\frac{dz}{du} &= \nu x \\
u^2(1-u)z'' + \kappa z' &= \nu\frac{u}{1-u} \qquad (3.1.38)
\end{aligned}
$$

coincides with the ordinary differential equation (3.1.33).

With this we are left to consider the constant $-\lambda/c$ that was ignored in (3.1.30). This was done in Proposition 2 from [2]. We propose a one-dimensional analogue.

Proposition 2. *The value function* $\hat{V}$ *in* (3.1.21) *can be reformulated into*

$$
\hat{V}(\varphi) = \inf_{\tau} \mathbb{E}_{\varphi}^{\infty}\left[e^{-\lambda\tau}M(\Phi_\tau)\right] - M(\varphi) \qquad (3.1.39)
$$

for $\varphi \in [0,\infty)$ *where* $M(\varphi)$ *is defined as*

$$
M(\varphi) = \frac{2}{\mu^2}(1+\varphi)\int_0^{\varphi/(1+\varphi)}\left(\frac{1-a}{a}\right)^{\kappa}\int_0^a \frac{b^{\kappa-1}}{(1-b)^{\kappa+2}}e^{-\kappa/b}\,db\,da + \frac{1}{c} \qquad (3.1.40)
$$

the infimum of all the stopping times of Φ_t.

Proof. We apply Itô formula to the function $e^{-\lambda t}M(\Phi_t)$

$$
d(e^{-\lambda t}M(\Phi_t)) = e^{-\lambda t}dM(\Phi_t) - \lambda e^{-\lambda t}M(\Phi_t)dt \qquad (3.1.41)
$$

since $e^{-\lambda t}$ is of bounded variation.

$$
\begin{aligned}
dM(\Phi_t) &= \frac{dM(\varphi)}{d\varphi}d\Phi_t + \frac{1}{2}\frac{d^2M(\varphi)}{d\varphi}d\left\langle\Phi,\Phi\right\rangle_t \\
&= \mathcal{A}_\Phi M dt + \mu\Phi_t\frac{dM(\varphi)}{d\varphi}dB_t \qquad (3.1.42) \\
d(e^{-\lambda t}M(\Phi_t)) &= e^{-\lambda t}\mathcal{A}_\Phi M dt + e^{-\lambda t}\mu\Phi_t\frac{dM(\varphi)}{d\varphi}dB_t - \lambda e^{-\lambda t}M(\Phi_t)dt \\
&= e^{-\lambda t}(\mathcal{A}_\Phi M - \lambda M)dt + e^{-\lambda t}\mu\Phi_t\frac{dM(\varphi)}{d\varphi}dB_t. \qquad (3.1.43)
\end{aligned}
$$

Applying Doob's optional sampling theorem on (3.1.43) we see that

$$\begin{aligned}\mathbb{E}_{\varphi}^{\infty}\left[e^{-\lambda\tau}M(\Phi_{\tau})\right] &= M(\varphi)+\mathbb{E}_{\varphi}^{\infty}\left[\int_0^{\tau} e^{-\lambda t}(\mathcal{A}_{\Phi}M-\lambda M)dt\right]+0 \\ &= M(\varphi)+\mathbb{E}_{\varphi}^{\infty}\left[\int_0^{\tau} e^{-\lambda t}L(\Phi_t)dt\right] \qquad (3.1.44)\end{aligned}$$

completes the proof. □

Remark. The Mayer-formulated problem will not be solved directly but used to provide insights in the solution of the Lagrange-formulated problem. The above results will also be used again in the multi-dimensional cases later on in the dissertation with slight alternations (PDE instead of ODE) to deal with higher dimensions to help to establish properties of the optimal stopping boundary.

2. We then go back to the Lagrange formulated optimal stopping problem (3.1.21) and try to solve it directly through a free-boundary problem:

$$\mathcal{A}_{\Phi}\hat{V}-\lambda\hat{V} = -L = -(\varphi-\lambda c) \qquad \text{on} \quad C, \qquad (3.1.45)$$

$$\hat{V} = 0 \quad \textit{(instantaneous stopping condition)} \qquad \text{on} \quad \partial C, \qquad (3.1.46)$$

$$\hat{V}' = 0 \quad \textit{(smooth fit condition)} \qquad \text{on} \quad \partial C. \qquad (3.1.47)$$

In the one-dimensional case, without loss of generality, it is feasible to assume that the optimal stopping boundary is certain $\varphi_* \in (\frac{\lambda}{c}, \infty)$ since the gain function in (3.1.21) is negative and sub-optimal for $\varphi < \frac{\lambda}{c}$ ($[0, \lambda/c) \in C$.)

The ordinary differential equation (3.1.45) could be solved as the sum of a general solution to the homogeneous part of it and a specific solution to the non-homogeneous part of it.

$$\hat{V}(\varphi) = \hat{V}_g(\varphi) + \hat{V}_p(\varphi) \qquad (3.1.48)$$

$$\hat{V}_g(\varphi) = A(1+\varphi)\int_{1/2}^{\varphi/(1+\varphi)} (\frac{1-a}{a})^{\kappa} e^{\kappa/a} da + B(1+\varphi) \qquad (3.1.49)$$

$$\hat{V}_p(\varphi) = -\frac{2}{\mu^2}(1+\varphi)\int_{\varphi_*/(1+\varphi_*)}^{\varphi/(1+\varphi)} \left(\frac{1-a}{a}\right)^{\kappa} \int_0^a \frac{b^{\kappa-1}}{(1-b)^{\kappa+2}} e^{-\kappa/b} db da - \frac{1}{c} \qquad (3.1.50)$$

for $\varphi \in [0, \varphi_*]$. $\hat{V}_p$ can be derived by noticing that (3.1.45) implies $\hat{V}_p = -M$ for M in (3.1.40) with the lower limit of the integral changed due to the instantaneous stopping condition. To see $\hat{V}_g$ is of this form, we take a similar approach as we did for finding the M by noticing that the homogeneous part of (3.1.44) is the same as setting the right-hand-side of (3.1.30) to 0. We apply the same reduction from (3.1.30) to (3.1.33) to see that

$$\kappa(1+x)y' + x^2y'' - \kappa y = 0 \tag{3.1.51}$$

where y, y', and y'' are specified in (3.1.32), (3.1.36), and (3.1.37) respectively. Substituting into (3.1.51), we have

$$u^2(1-u)z'' + \kappa z' = 0. \tag{3.1.52}$$

The ordinary differential equation can be solved in two cases

$$\begin{aligned} z'(u) &= A\left(\frac{1-a}{a}\right)^{\kappa} e^{\kappa/a} \quad \text{for } A \neq 0 \\ z(u) &= A\int_{1/2}^{\varphi/(1+\varphi)} \left(\frac{1-a}{a}\right)^{\kappa} e^{\kappa/a} da, \end{aligned} \tag{3.1.53}$$

$$\begin{aligned} z'(u) &= 0 \quad \text{for } A = 0 \\ z(u) &= B. \end{aligned} \tag{3.1.54}$$

We now try to determine the unspecified constants A and B in (3.1.49). This also follows the derivation in [2] through direct differentiation of (3.1.48) which yields that $\hat{V}'(0+) = \pm\infty$ for $A > 0, A < 0$ respectively. Thus $A = 0$ is a natural candidate,

$$\begin{aligned} B(1+\varphi_*) - \frac{1}{c} &= 0 \\ B &= \frac{1}{c(1+\varphi_*)} \end{aligned} \tag{3.1.55}$$

and the (candidate) solution

$$\hat{V}(\varphi) = -\frac{\varphi_* - \varphi}{c(1+\varphi_*)} - \frac{2}{\mu^2}(1+\varphi)\int_{\varphi_*/(1+\varphi_*)}^{\varphi/(1+\varphi)} \left(\frac{1-a}{a}\right)^{\kappa} \int_0^a \frac{b^{\kappa-1}}{(1-b)^{\kappa+2}} e^{-\kappa/b} db da. \tag{3.1.56}$$

As indicated in [2], using Leibniz rule, we can differentiate (3.1.56) to see that for the smooth fit condition (3.1.47) to hold we need

$$\begin{aligned} \hat{V}'(\varphi)\big|_{\varphi=\varphi_*} &= \frac{1}{c(1+\varphi_*)} - \frac{2}{\mu^2}\left(\int_{\varphi_*/(1+\varphi_*)}^{\varphi/(1+\varphi)} G(a)da + (1+\varphi)G(\frac{\varphi}{1+\varphi})\frac{d}{d\varphi}(\frac{\varphi}{1+\varphi})\right)\Big|_{\varphi=\varphi_*} \\ &= \frac{1}{c(1+\varphi_*)} - \frac{2}{\mu^2}\frac{G(\frac{\varphi_*}{1+\varphi_*})}{1+\varphi_*} = 0 \end{aligned} \tag{3.1.57}$$

where

$$G(a) = (\frac{1-a}{a})^{\kappa} e^{\kappa/a} \int_0^a \frac{b^{\kappa-1}}{(1-b)^{\kappa+2}} e^{-\kappa/b} db. \tag{3.1.58}$$

Therefore, φ_* has to solve

$$\frac{\mu^2}{2c} = \frac{e^{\kappa(1+\varphi_*)/\varphi_*}}{\varphi_*^\kappa} \int_0^{\varphi_*/(1+\varphi_*)} \frac{b^{\kappa-1}}{(1-b)^{\kappa+2}} e^{-\kappa/b} db \tag{3.1.59}$$

for (3.1.57) to evaluate to 0.

Upon verification in [2], it is evident that the value function is indeed the sought function in the optimal stopping problem in (3.1.21) and the stopping time

$$\tau_* = \inf\{t \geq 0 \mid \Phi_t \in [\varphi_*, \infty)\} \tag{3.1.60}$$

is optimal for $\varphi_* \in (\lambda/c, \infty)$ is the unique solution to the equation (3.1.59).

Assume $\hat{V}_*$ is optimal and by Itô formula,

$$de^{-\lambda t}\hat{V}_*(\Phi_t) = e^{-\lambda t}(\mathcal{A}_\Phi \hat{V}_* - \lambda \hat{V}_*)dt + e^{-\lambda t}\hat{V}_*' \mu \Phi_t dB_t. \tag{3.1.61}$$

Taking $(\sigma_m)_{m\geq 1}$ as a localizing sequence for the continuous local martingale term in (3.1.61). Applying expectation and changing t to $\sigma_m \wedge t$,

$$\mathbb{E}_\varphi^\infty \left[e^{-\lambda t\wedge\sigma_m}\hat{V}_*(\Phi_{t\wedge\sigma_m})\right] = \hat{V}_*(\varphi) - \mathbb{E}_\varphi^\infty \left[\int_0^{t\wedge\sigma_m} e^{-\lambda s} L(\Phi_t^1)\mathbb{1}(\Phi_s < \varphi_*)ds\right]. \tag{3.1.62}$$

Taking $m \to \infty$,

$$\hat{V}_*(\varphi) = e^{-\lambda t}\mathbb{E}_\varphi^\infty \left[\hat{V}_*(\Phi_t)\right] + \mathbb{E}_\varphi^\infty \left[\int_0^t e^{-\lambda s} L(\Phi_s^1)\mathbb{1}(\Phi_s < \varphi_*)ds\right]. \tag{3.1.63}$$

Taking $t \to \infty$,

$$\begin{aligned}\hat{V}_*(\varphi) &= \mathbb{E}_\varphi^\infty \left[\int_0^\infty e^{-\lambda s} L(\Phi_s^1)\mathbb{1}(\Phi_s < \varphi_*)ds\right] \\ &= \mathbb{E}_\varphi^\infty \left[\int_0^{\tau_*} e^{-\lambda s} L(\Phi_s^1)ds\right]\end{aligned} \tag{3.1.64}$$

coincides with the value function in (3.1.21). This procedure verifies that the value function indeed solves the problem (3.1.21) and could be extended into verifying the solution in higher dimenions as shown later on in the dissertation.

3.2 Multi-dimension (One-drift)

With the solution for the one-dimensional case we now revisit the multi-dimensional case with one drift change that was set up in Section 2.2. From Proposition 1, we have seen that changing the measure from $\mathbb{P}_\pi$ to $\mathbb{P}^\infty$ would help us to formulate the

Mayer-Lagrange optimal stopping problem for the Markov process Π_t into a Lagrange optimal stopping problem for the Markov process Φ_t which we could use for solutions in the one dimensional case. The same arguments apply here in this section with a slightly extended alternation to deal with the multidimensional nature.

Proposition 3. *The optimal stopping problem in* (2.2.6) *can be reformulated into*

$$V(\pi) = (1-\pi)\left[1 + c\hat{V}(\pi)\right] \tag{3.2.1}$$

where

$$\hat{V}(\pi) = \hat{V}(\varphi) = \inf_{\tau} \mathbb{E}_{\varphi}^{\infty}\left[\int_0^{\tau} e^{-\lambda t}(p_1\Phi_t^1 + p_2\Phi_t^2 - \frac{\lambda}{c})dt\right] \tag{3.2.2}$$

for $\varphi \in [0,\infty)\times[0,\infty)$ *under* $\mathbb{P}_{\varphi}^{\infty}$ *and* $\mathbb{P}_{\varphi}^{\infty}[\Phi_0 = \varphi = \pi/(1-\pi)] = 1$.

Proof. We follow the exact same proof used in [2]. Noticing that from (2.2.4) we could decompose the measure $\mathbb{P}_\pi$ into two measures $\mathbb{P}_1$ and $\mathbb{P}_2$, we then define two new processes

$$\tilde{\Pi}_t^1 = \mathbb{P}_1\left[\theta \le t \mid \mathcal{F}_t^X\right] \quad \& \quad \tilde{\Pi}_t^2 = \mathbb{P}_2\left[\theta \le t \mid \mathcal{F}_t^X\right]. \tag{3.2.3}$$

From (3.1.6) and (3.1.7) in [2], we see that

$$\mathbb{E}_\pi\left[\Pi_\tau^i\right] = p_i\mathbb{E}_i\left[\tilde{\Pi}_\tau^i\right] \quad \& \quad \mathbb{E}_\pi\left[\int_0^\tau \Pi_t^i dt\right] = p_i\mathbb{E}_i\left[\int_0^\tau \tilde{\Pi}_t^i dt\right]. \tag{3.2.4}$$

$$\begin{aligned}
V(\pi) &= \inf_\tau \mathbb{E}_\pi\left[1 - \Pi_\tau + c\int_0^\tau \Pi_t dt\right] \\
&= \inf_\tau \mathbb{E}_\pi\left[1 - (\Pi_\tau^1 + \Pi_\tau^2) + c\int_0^\tau \Pi_t^1 + \Pi_t^2 dt\right] \\
&= 1 - \inf_\tau\left(\sum_{i=1}^2 p_i\mathbb{E}_i\left[\tilde{\Pi}_\tau^i\right] + c\sum_{i=1}^2 p_i\mathbb{E}_i\left[\int_0^\tau \tilde{\Pi}_t^i dt\right]\right) \\
&= \inf_\tau \sum_{i=1}^2 p_i\mathbb{E}_i\left[1 - \tilde{\Pi}_\tau^i + c\int_0^\tau \tilde{\Pi}_t^i dt\right] \\
&= (1-\pi)\left(1 + c\inf_\tau \mathbb{E}^\infty\left[\int_0^\tau e^{-\lambda t}\left(p_1\Phi_t^{1,\pi/(1-\pi)} + p_2\Phi_t^{2,\pi/(1-\pi)} - \frac{\lambda}{c}\right)\right]dt\right)
\end{aligned} \tag{3.2.5}$$

where second equality is from (2.2.7) and the last equality is a direct substitution from (4.12) in [5]. Thus

$$\begin{aligned}
\hat{V}(\pi) &= \inf_\tau \mathbb{E}^\infty\left[\int_0^\tau e^{-\lambda t}\left(p_1\Phi_t^{1,\pi/(1-\pi)} + p_2\Phi_t^{2,\pi/(1-\pi)} - \frac{\lambda}{c}\right)dt\right] \\
&= \inf_\tau \mathbb{E}_\varphi^\infty\left[\int_0^\tau e^{-\lambda t}\left(p_1\Phi_t^1 + p_2\Phi_t^2 - \frac{\lambda}{c}\right)dt\right] = \hat{V}(\varphi)
\end{aligned} \tag{3.2.6}$$

completes the proof. □

With the Lagrange formulated optimal stopping problem (3.2.2), we can now reformulate it into a Mayer formulation like we did in the previous section. This is equivalent to finding a function $M : [0, \infty) \times [0, \infty) \to \mathbb{R}$ which is the solution to the PDE

$$\mathcal{A}_\Phi M - \lambda M = L = p_1\varphi_1 + p_2\varphi_2 - \lambda/c \tag{3.2.7}$$

for $(\varphi_1, \varphi_2) \in [0, \infty) \times [0, \infty)$ where $\mathcal{A}_\Phi$ is the infinitesimal generator for the multidimensional Markov process $\Phi = (\Phi_t^1, \Phi_t^2)$

$$\mathcal{A}_\Phi = \sum_{i=1}^{2} \lambda(1 + \varphi_i)\partial_{\varphi_i} + \frac{1}{2}\sum_{i=1}^{2} \mu^2\varphi_i^2\partial_{\varphi_i\varphi_i} \tag{3.2.8}$$

for $\varphi = (\varphi_1, \varphi_2) \in (0, \infty) \times (0, \infty)$. The problem is equivalent to solving two ordinary differential equations

$$\lambda(1 + \varphi_i)\frac{dM(\varphi_i)}{d\varphi_i} + \frac{\mu^2}{2}\varphi_i^2\frac{d^2M(\varphi_i)}{d\varphi_i^2} - \lambda M(\varphi_i) = p_i\varphi_i \tag{3.2.9}$$

for $i = 1, 2$, which was solved in the same way as from (3.1.30) to (3.1.38) as

$$M(\varphi_i) = \frac{2}{\mu^2}(1 + \varphi_i)\int_0^{\varphi_i/(1+\varphi_i)} \left(\frac{1-a}{a}\right)^{\kappa} e^{\kappa/a}\int_0^a \frac{b^{\kappa-1}}{(1-b)^{\kappa+2}}e^{-\kappa/b}dbda \tag{3.2.10}$$

where $\kappa := 2\lambda/\mu$ and

$$M(\varphi_1, \varphi_2) = p_1M(\varphi_1) + p_2M(\varphi_2) + 1/c. \tag{3.2.11}$$

Proposition 4. *The value function $\hat{V}$ in (3.2.2) could be expressed as*

$$\hat{V}(\varphi) = \inf_\tau \mathbb{E}_\varphi^\infty\left[e^{-\lambda\tau}M(\Phi_\tau^1, \Phi_\tau^2)\right] - M(\varphi) \tag{3.2.12}$$

for $\varphi = (\varphi_1, \varphi_2) \in [0, \infty) \times [0, \infty)$

Proof. The proof follows the same way as in Proposition 2 with an application of multidimensional Itô formula for the function $e^{-\lambda t}M$ and then use Doob's optional sampling theorem to apply an expectation. Thus we omit the proof. □

For an n-dimensional continuous Markov process $\Phi = (\Phi_t)_{t\geq 0}$ with no jump components and solves a system of stochastic differential equations (2.2.14) for $i = 1, \ldots, n$, the infinitesimal generator of the process is given as

$$\mathcal{A}_\Phi = \sum_{i=1}^{d} \mu_i\partial_{x_i} + \sum_{i,j=1}^{d}(\sigma\sigma^T)\partial_{x_ix_j} \tag{3.2.13}$$

where $\mu^i(\varphi) = \lambda(1+\varphi^i)$ and $\sigma^i(\varphi) = \mu\varphi_i$. For $\Phi = (\Phi^1, \ldots, \Phi^n)$ where each Φ^i are independent, the diffusion matrix in (3.2.13) simplifies to $\frac{1}{2}\sum_{i=1}^{d}(\sigma^i)^2\partial_{\varphi_i\varphi_i}$. Extending the (2.12) from [9] from two-dimensional to higher dimensions as we discussed above, we see that the infinitesimal generator of the process Φ is a differential operator where $(\sigma^i)^2 \geq 0$. Plugging in σ^i, we see that $(\sigma^i)^2 \neq 0$, the differential operator is of *elliptic* type.

As we have seen in the system of stochastic differential equations that the process Φ solves in (2.2.14), we see that the all SDEs are non-degenerate and of full rank. The drift coefficient and the diffusion coefficient are Lipschitz and bounded, thus the Stroock-Varadhan theorem applies and the semigroup of the strong Markov process Φ is *strong Feller* [p170 [13]].

3.2.1 Structure of optimal stopping boundary

Our previous derivation from the one-dimensional case offers us a candidate continuation set and stopping set where for φ such that $\hat{V}(\varphi) < 0$ is not optimal and in C while $\hat{V}(\varphi) = 0$ is in D.

Since the Shiryaev process Φ is a stochastic (Markovian) flow (2.2.16) of its initial point and the expectation in the Mayer-formulated problem (3.2.12) becomes a family of continuous functions. Taking the infimum over the family of continuous functions, we conclude that $\hat{V}(\varphi)$ is usc. The gain function in (3.2.12) is continuous and thus lsc. Then the standard arguments we have reviewed in Section 1.1 applies and the first entry time of the process Φ to the stopping set D is optimal.

From (3.1.59) in the one-dimensional case, we see that the optimal stopping point φ_* is a function of two parameters λ/c and λ/μ^2. Thus we propose in two-dimensions,

$$\varphi_i^* = \varphi_*(\lambda/p_i c, \lambda/\mu^2) \tag{3.2.14}$$

for $\varphi_i^* \in [\lambda/c, \infty)$. With these, several properties of the optimal stopping boundary have been derived in Section 6, Proposition 3 from [2]. We repeat the proof of the proposition here with some more details.

Proposition 5.

(1) $\hat{V}$ *is concave and continuous on the state space of* Φ.

(2) *If* $\varphi_1 \le \psi_1$ & $\varphi_2 \le \psi_2$ *then* $\hat{V}(\varphi_1, \varphi_2) \le \hat{V}(\psi_1, \psi_2)$.

(3) *If the state* $(\varphi_1, \varphi_2) \in D$ & $\psi_1 \ge \varphi_1, \psi_2 \ge \varphi_2$ *then the state* $(\psi_1, \psi_2) \in D$.

(4) *The stopping set* D *is a convex set and the trigon* $\{(\varphi_1, \varphi_2) \in [0, \infty) \times [0, \infty) \mid \varphi_1/\varphi_1^* + \varphi_2/\varphi_2^* - 1 \ge 0\} \subseteq D$.

(5) *The triangle* $\{(\varphi_1, \varphi_2) \in [0, \infty) \times [0, \infty) \mid p_1\varphi_1 + p_2\varphi_2 - \lambda/c < 0\} \subseteq C$.

Proof. (1) From the stochastic flow that the process Φ admits, the process is continuous and thus the $L(\Phi_t) = p_1\Phi_t^1 + p_2\Phi_t^2 - \lambda/c$ is a convex combination where $L(\alpha\Phi_t^1 + (1-\alpha)\Phi_t^2) = \alpha L(\Phi_t^1) + (1-\alpha)L(\Phi_t^2)$ for $\alpha \in [0, 1]$. By discounting, applying expectation, and taking the infimum, $\hat{V}(\varphi) \ge \alpha\hat{V}(\varphi) + (1-\alpha)\hat{V}(\varphi)$. Thus $\hat{V}(\varphi)$ is concave and continuous at the boundaries of the state space of Φ, $[0, \infty) \times [0, \infty)$. The concavity implies that $\hat{V}(\varphi)$ is lsc and together with the usc property from earlier, $\hat{V}(\varphi)$ is continuous.

(2)-(3) The stochastic flow of the process Φ is an increasing function, together with the fact that the expectation and infimum operations are continuous, $\hat{V}$ is an increasing function as well. Since $\hat{V}(\varphi) \le 0$, $\hat{V}(\varphi_1, \varphi_2) \le \hat{V}(\psi_1, \psi_2) \le 0$ & $(\varphi_1, \varphi_2) \in D$, then $\hat{V}(\varphi_1, \varphi_2) = 0$. By the squeeze theorem, $\hat{V}(\psi_1, \psi_2) = 0$ & $(\psi_1, \psi_2) \in D$.

(4) The convexity of the set D can be directly seen from the concavity of $\hat{V}$. Since φ_1^* and φ_2^* are the optimal stopping point for two independent one-dimensional problems, points $(0, \varphi_1^*)$ and $(\varphi_2^*, 0)$ are in D. Due to convexity The line segment $\varphi_1/\varphi_1^* + \varphi_2/\varphi_2^* = 1$ is in D, and thus the trigon [see figure 3.1].

(5) For all $(\varphi_1, \varphi_2) \in [0, \infty) \times [0, \infty)$ such that $p_1\varphi_1 + p_2\varphi_2 - \lambda/c < 0$, $L(\Phi) < 0$ in (3.42) and suboptimal. Therefore, the triangle is in C. [see Figure 3.2] □

With the above results and the visualization in Figures 3.1 and 3.2, it is evident that there exists an optimal boundary function $b : [0, \infty) \to [0, \infty)$ between the two line segments in Figure 3.3 such that

$$\partial C = \{(\varphi_1, \varphi_2) \in [0, \infty) \times [0, \infty) \mid \varphi_2 = b(\varphi_1)\} \tag{3.2.15}$$

and

$$C = \{(\varphi_1, \varphi_2) \in [0, \infty) \times [0, \infty) \mid \varphi_2 < b(\varphi_1)\} \tag{3.2.16}$$

$$D = \{(\varphi_1, \varphi_2) \in [0, \infty) \times [0, \infty) \mid \varphi_2 \ge b(\varphi_1)\}. \tag{3.2.17}$$

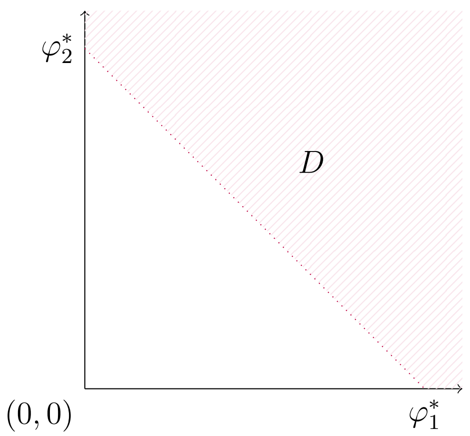

Figure 3.1: Upper bound for the optimal boundary

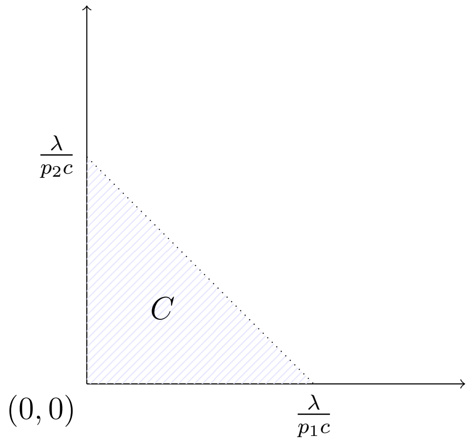

Figure 3.2: Lower bound for the optimal boundary

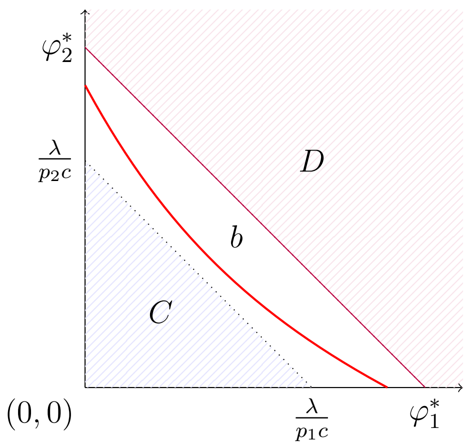

Figure 3.3: Optimal stopping boundary $b(\varphi_1)$

Although evident from the regularity of the diffusion process Φ that the all points in ∂C are regular for the stopping set D, we try to prove it using a discritization of the stopping time $\tau_D = \inf\{t \geq 0 \mid \Phi \in D\}$ that $\{\tau_D = 0\} = \cap_{n=1}^{\infty} \cup_{t \in (0, \frac{1}{n})} \{\Phi_t \in D\}$.

Proposition 6. *The optimal stopping boundary ∂C is probablistically regular for D for the one-drift detection problem* (3.2.2).

Proof. Without loss of generality, we assume that $i = 1, 2$ and let $\varphi = (\varphi_1, \varphi_2)$ be any point in ∂C as initial points given fixed.

$$\begin{aligned}\mathbb{P}_\varphi^\infty \left[\cap_{n=1}^{\infty} \cup_{t \in (0, \frac{1}{n})} \{\Phi_t \in D\}\right] &\geq \lim_{n \to \infty} \sup_{t \in (0, \frac{1}{n})} \mathbb{P}_\varphi^\infty \left[\Phi_t \in D\right] \\ &= \limsup_{t \downarrow 0} \mathbb{P}_\varphi^\infty \left[\Phi_t^1 \geq \varphi_1, \Phi_t^2 \geq \varphi_2\right] \qquad (3.2.18)\end{aligned}$$

from (2.2.13), and we have that

$$\begin{aligned}
\limsup_{t\downarrow 0} \mathbb{P}^{\infty}_{\varphi}\left[\Phi^i_t \geq \varphi_1\right] &= \limsup_{t\downarrow 0} \mathbb{P}^{\infty}_{\varphi}\left[e^{-\lambda t}L^i_t\varphi_i + \lambda e^{-\lambda t}L^i_t\int_0^t e^{-\lambda s}(L^i_s)^{-1}ds \geq \varphi_i\right] \\
&\geq \limsup_{t\downarrow 0} \mathbb{P}^{\infty}\left[L^i_t \geq 1\right] \quad (e^{\lambda t}\to 1 \text{ and second integral} \downarrow 0) \\
&= \limsup_{t\downarrow 0} \mathbb{P}^{\infty}\left[\exp(\mu X^i_t - \mu^2 t/2) \geq 1\right] \quad \text{from (2.2.12)} \\
&= \limsup_{t\downarrow 0} \mathbb{P}^{\infty}\left[X^i_t \geq \mu t/2\right] \\
&= \limsup_{t\downarrow 0} \mathbb{P}\left[B^i_t \geq \mu t/2\right] \qquad (3.2.19)
\end{aligned}$$

for $i = 1, 2$. Then (3.2.18) can be written as

$$\begin{aligned}
\mathbb{P}^{\infty}_{\varphi}\left[\cap_{n=1}^{\infty}\cup_{t\in(0,\frac{1}{n})}\{\Phi_t \in D\}\right] &\geq \limsup_{t\downarrow 0} \mathbb{P}\left[B^1_t \geq \mu t/2, B^2_t \geq \mu t/2\right] \\
&= \limsup_{t\downarrow 0} (\mathbb{P}\left[B^1_t \geq \mu t/2\right])^2 \\
&= \limsup_{t\downarrow 0} (\mathbb{P}\left[B^1_1 \geq \mu\sqrt{t}/2\right])^2 \quad \text{(by Brownian scaling)} \\
&= (\mathbb{P}\left[B^1_1 \geq 0\right])^2 = (\frac{1}{2})^2 > 0. \qquad (3.2.20)
\end{aligned}$$

By Blumental's law for $\tau_D = 0 \in \mathcal{F}^X_{0+}$ and $\mathbb{P}^{\infty}_{\varphi}[\tau_D = 0] > 0$, then $\mathbb{P}^{\infty}_{\varphi}[\tau_D = 0] = 0$ for all points in ∂C, meaning that the process Φ starting at any point in ∂C stops immediately. □

This proof, although not immediately required in this setting, would be required in the scenario with multiple drifts due to the complexity of the process Φ and thus would provide sufficient basis for the proof in the complex case. With the structure of the optimal stopping boundary established, we now ready to solve the free-boundary problem to derive the integral representation of the value function using the optimal stopping boundary.

3.2.2 Free-boundary problem

The infinitesimal generator we have here is an elliptic differential operator as discussed above. Recalling that the optimal stopping problem (3.2.2) is of a discounted Lagrange formulation, a free-boundary problem can be applied here to find $\hat{V}$ and the optimal stopping boundary function b:

$$\mathcal{A}_\Phi \hat{V} - \lambda \hat{V} = -L \quad \text{in } C, \tag{3.2.21}$$

$$\hat{V}\big|_D = 0 \quad \textit{(instantaneous stopping condition)}, \tag{3.2.22}$$

$$\frac{\partial \hat{V}}{\partial \varphi_i}\big|_{\partial C} = 0 \quad \textit{(smooth fit condition)}. \tag{3.2.23}$$

The optimal boundary $\partial C = \{(\varphi_1, \varphi_2) \in [0,\infty) \times [0,\infty) \mid \varphi_2 = b(\varphi_1)\}$. The pair $(\hat{V}, b)$ is the unique solution to the above free-boundary problem among a class of admissible functions, $\mathcal{C}$. For $(U, a) \in \mathcal{C}$, (1) $U \in C^2(C_a) \cap C^1(\bar{C_a})$, continuous and bounded on $[0,\infty)^2$ and (2) $a : [0,\infty) \to [0,\infty)$ is a continuous and decreasing function with $p_1\varphi_1 + a(\varphi_1) - \lambda/c \geq 0$ (in the sense of having the characteristics we identified above.) C_a is defined in the same way as (3.2.15) with function b changed to a general a.

To see that the pair $(\hat{V}, b) \in \mathcal{C}$ and solves (3.2.21) to (3.2.23), we first notice that the Lagrange formulation in (3.2.2) admits a Dirichlet-Poisson problem and (3.2.21) holds [from (1.1.18).] From the first paragraph of Section 3.2.1, we see that the instantaneous stopping condition (3.2.22) holds for the negative programme since the value function equals to the gain function on D and thus $\hat{V} = 0$ in D. Since the optimal stopping boundary ∂C is probabilistically regular for D (Proposition 6) and Φ is a strong Feller process, then the ∂C is Green regular for D (by the Corollary 2 in [1]). Since Φ can be realized as a continuous stochastic (Markovian) flow of its initial point as in (2.2.13) and the gain function in the Mayer formulation (3.2.12) is continuously differentiable, the value function $\hat{V}$ is then continuously differentiable at the optimal stopping boundary ∂C, ensuring that the smooth fit condition (3.2.23) holds. (1) By Theorem 8 from [1], $\hat{V}$ is continuously differentiable on $[0,\infty) \times [0,\infty)$ and thus $\hat{V} \in C^1(\bar{C})$. By ellipticity of the differential operator $\mathcal{A}_\Phi$, the solution $\hat{V} \in C^2(C)$. (2) The properties of the function b in Proposition 4 illustrates that b satisfies condition (2) to be in $\mathcal{C}$. Therefore, the pair $(\hat{V}, b)$ is in the class $\mathcal{C}$. The uniqueness is shown in the integral representation we will be discussing in the following.

We now focus on the non-linear integral equation representation of the value function $\hat{V}$ in terms of the optimal stopping boundary b in this dissertation. As we have discussed earlier, Φ is a Markovian flow of its initial point as defined in (2.2.16) where Y in (2.2.17) is an exponential Brownian motion (strong Markov). Then it is evident that $\Phi = (\Phi^1, \Phi^2)$ is a continuous Markov process with Markov transition kernel

$$\mathbb{P}^{\infty}_{\varphi_1,\varphi_2}[\Phi_t \in A] = P(t, \varphi, A) = \iint_A p(t; \varphi_1, \varphi_2, \psi_1, \psi_2) d\psi_1 d\psi_2 \tag{3.2.24}$$

where $p(t; \varphi_1, \varphi_2, \psi_1, \psi_2)$ is the Markov transition density function. Due to the independence of the underlying driving Brownian motions, the transition density can be decomposed as $p(t; \varphi_1, \varphi_2, \psi_1, \psi_2) = p_1(t; \varphi_1, \psi_1)p_2(t; \varphi_2, \psi_2)$ for (φ_1, φ_2) and (ψ_1, ψ_2) $\in [0, \infty) \times [0, \infty)$. We define the expectation of the Lagrange functional evaluated in the continuation set as

$$\begin{aligned} K_b(t; \varphi_1, \varphi_2) &:= \mathbb{E}^{\infty}_{\varphi_1,\varphi_2}\left[L(\Phi^1_t, \Phi^2_t)\mathbb{1}(\Phi^2_t < b(\Phi^1_t))\right] \\ &= \iint_{[0,\infty)^2} L(\psi_1, \psi_2)\mathbf{1}\{\psi_2 < b(\psi_1)\}p(t; \varphi_1, \varphi_2, \psi_1, \psi_2) d\psi_1 d\psi_2 \\ &= \int_0^{\varphi_0} d\psi_1 \int_0^{b(\psi_1)} L(\psi_1, \psi_2)p(t; \varphi_1, \varphi_2, \psi_1, \psi_2) d\psi_2 \end{aligned} \tag{3.2.25}$$

where φ_0 is the smallest 0 on b on $[0, \infty)$. For $\psi_1 > \varphi_0$, the function $b(\psi_1)$ is non-positive, thus the integral over ψ_2 beyond φ_0 is irrelevant for the expectation because $b(\psi_1)$ is 0. The integration upper limit and lower limit is thus effectively $\psi_1 \in [0, \varphi_0)$ since the contribution of the indicator function is zero. The existence and uniqueness theorem of the optimal boundary b being characterized using the kernel K_b was proved in Theorem 5 [2]. We repeat the proof here with some more details.

Theorem 7. *The optimal stopping boundary b in the problem (3.2.2) is the unique solution to the nonlinear Fredholm integral equation*

$$\int_0^{\infty} e^{-\lambda t} K_b(t; \varphi_1, b(\varphi_1)) dt = 0 \tag{3.2.26}$$

among the class of functions b, continuous, decreasing, and convex, such that $p_1, \varphi_1 + p_2 b(\varphi_1) \geq \lambda/c$ for $\varphi_1 \in [0, \varphi_0)$. The value function $\hat{V}$ can be represented as

$$\hat{V}(\varphi_1, \varphi_2) = \int_0^{\infty} e^{-\lambda t} K_b(t; \varphi_1, \varphi_2) dt \tag{3.2.27}$$

and the optimal stopping time is

$$\tau_b = \inf\{t \geq 0 \mid \Phi^2_t \geq b(\Phi^1_t)\}. \tag{3.2.28}$$

Proof. The proof in [2] uses a sequence of sets C_n, D_n and functions $\hat{V}^n$ to approximate C, D, and $\hat{V}$, where $C_n = \{\varphi \in [0, \infty)^2 \mid \hat{V}(\varphi) < -1/n\}$, $D_n = \{\varphi \in$

$[0,\infty)^2 \mid \hat{V}(\varphi) \geq -1/n\}$. Thus $C_n \uparrow C$, $D_n \downarrow D$, as $n \uparrow \infty$. Apply Proposition 4 from above, we see that D_n is convex and $b_n \uparrow b$, the optimal boundary between C_n and D_n, is a decreasing, convex, continuous function of φ_1 on $[0, \varphi_0^n]$ where $\varphi_0^n \uparrow \varphi_0$ is the smallest zero of b_n.

For $\hat{V}^n$ defined on C_n, and thus $\hat{V}^n \in C^2(\bar{C}_n)$ & $C^2(\bar{D}_n)$ as we have seen above. Therefore, the conditions for Theorem 2.1 from [7] are met since $b_n(\Phi^1)$ is a continuous semimartingale due to convexity of b and a change-of-variable formula can be applied here.

$$\begin{aligned} d\hat{V}^n(\Phi_t) =& \sum_{i=1}^{2} \frac{1}{2}\left(\partial_{\varphi_i}\hat{V}^n(\Phi_t^1, \Phi_t^2+) + \partial_{\varphi_i}\hat{V}^n(\Phi_t^1, \Phi_t^2-)\right) d\Phi_t^i \\ &+ \frac{1}{2}\sum_{i,j=1}^{2} \frac{1}{2}\left(\partial_{\varphi_i\varphi_j}\hat{V}^n(\Phi_t^1, \Phi_t^2+) + \partial_{\varphi_i\varphi_j}\hat{V}^n(\Phi_t^1, \Phi_t^2-)\right) d\langle \Phi^i, \Phi^j\rangle_t \\ &+ \frac{1}{2}\left(\partial_{\varphi_2}\hat{V}^n(\Phi_t^1, \Phi_t^2+) - \partial_{\varphi_2}\hat{V}^n(\Phi_t^1, \Phi_t^2-)\right) I(\Phi_t^2 = b_n(\Phi_t^1)) d\ell_t^{b_n}(\Phi) \end{aligned} \tag{3.2.29}$$

where

$$\ell_t^{b_n}(\Phi) = \text{P-}\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon}\int_0^t \mathbb{1}(-\varepsilon < \Phi_s^2 - b_n(\Phi_s^1) < \varepsilon) d\langle \Phi^2 - b_n(\Phi^1)\Phi^2 - b_n(\Phi^1)\rangle_s \tag{3.2.30}$$

is the local time of Φ on the surface (if in higher dimensions) b. Thus $d\hat{V}$ simplifies to

$$\begin{aligned} d\hat{V}^n(\Phi_t) =& \mathcal{A}_\Phi \hat{V}^n dt + \sum_{i=1}^{2} \mu \Phi_t^i \partial_{\varphi_i}\hat{V}^n(\Phi_t) dB_t^i \\ &+ \frac{1}{2}\left(\partial_{\varphi_2}\hat{V}^n(\Phi_t+) - \partial_{\varphi_2}\hat{V}^n(\Phi_t-)\right) I(\Phi_t^2 = b_n(\Phi_t^1)) d\ell_t^{b_n}(\Phi) \end{aligned} \tag{3.2.31}$$

since $\partial_{\varphi_i}\hat{V}^n(\Phi+) = \partial_{\varphi_i}\hat{V}^n(\Phi-)$ and Φ_t^i is independent of Φ_t^j for $i \neq j$. The indicator function in the last term implies that we are looking at the behaviour of $\hat{V}^n$ only at the boundary b_n where the right-hand limit is 0 (in D_n) and thus $\partial_{\varphi_2}\hat{V}^n(\Phi_t^1, \Phi_t^2+) = 0$.

$$\begin{aligned} de^{-\lambda t}\hat{V}^n(\Phi_t) =& e^{-\lambda t}(\mathcal{A}_\Phi \hat{V}^n - \lambda \hat{V}^n)(\Phi_t) dt + e^{-\lambda t}\sum_{i=1}^{2} \partial_{\varphi_i}\hat{V}^n(\Phi_t)\mu\Phi_t^i dB_t^i \\ &- \frac{1}{2} e^{-\lambda t}\partial_{\varphi_2}\hat{V}^n(\Phi_t-) d\ell_t^{b_n}(\Phi) \\ =& -e^{-\lambda t} L(\Phi_t)\mathbb{1}(\Phi_t \in C_n) dt + e^{-\lambda t}\frac{\lambda}{n}\mathbb{1}(\Phi_t \in D_n) dt \\ &- \frac{1}{2} e^{-\lambda t}\partial_{\varphi_2}\hat{V}(\Phi_t) d\ell_t^{b_n}(\Phi) + dM_t^n \end{aligned} \tag{3.2.32}$$

where $dM_t^n = e^{-\lambda t}\sum_{i=1}^{2} \partial_{\varphi_i}\hat{V}^n(\Phi_t)\mu\Phi_t^i \mathbb{1}(\Phi_t \in C_n) dB_t^i$. The second equality holds since (i) $\mathcal{A}_\Phi \hat{V}^n - \lambda \hat{V}^n = -L$ for $\Phi_t \in C_n$ from the free-boundary problem solved by $\hat{V}^n$, D_n

and $\hat{V}^n = -1/n$ for $\Phi_t \in D_n$, $\mathcal{A}_\Phi \hat{V}^n - \lambda \hat{V}^n = \lambda/n$ (differential operator applied to a constant is zero) and (ii) $\hat{V}$ is differentiable thus $\partial_{\varphi_2}\hat{V}^n = \partial_{\varphi_2}\hat{V}$. We then define $f_t(\varphi_1, \varphi_2) = b_n(\varphi_1) - \varphi_2$. By Tanaka formula

$$df_t^+(\Phi_t^1, \Phi_t^2) = \mathbb{1}(f_t > 0)df_t + \frac{1}{2}\ell_t^{b_n}(\Phi) \tag{3.2.33}$$

since

$$\begin{aligned} df_t &= \partial_{\varphi_1} f_t d\Phi_t^1 + \partial_{\varphi_2} f_t d\Phi_t^2 + \frac{1}{2}\int_{\mathbb{R}} L_t^{\psi_1} \partial_{\varphi_1,\varphi_1} f_t d\psi_1 \\ &= b_n'(\Phi_s^1)d\Phi_s^1 - d\Phi_s^2 + \frac{1}{2}\int_0^\infty L_t^{\psi_1} db_n'(\psi_1) \end{aligned} \tag{3.2.34}$$

by Itô-Tanaka formula, then (3.2.33) can be written as

$$\begin{aligned} (b_n(\Phi_t^1) - \Phi_t^2)^+ =& (b_n(\Phi_0^1) - \Phi_0^2)^+ + \frac{1}{2}\int_0^t \mathbb{1}(b_n(\Phi_s^1) - \Phi_s^2 > 0)\left(b_n'(\Phi_t^1)d\Phi_t^1 - d\Phi_t^2\right) \\ &+ \frac{1}{2}\int_0^t \mathbb{1}(b_n(\Phi_s^1) - \Phi_s^2 > 0)\int_0^\infty d\ell_s^{\psi_1}(\Phi^1)db_n'(\psi_1) + \frac{1}{2}\ell_t^{b_n}(\Phi) \end{aligned} \tag{3.2.35}$$

where $db_n'(\psi_1)$ is a non-negative measure and thus we have

$$\begin{aligned} \frac{1}{2}\ell_t^{b_n}(\Phi) \le& (b_n(\Phi_t^1) - \Phi_t^2)^+ - \int_0^t \mathbb{1}(b_n(\Phi_s^1) - \Phi_s^2 > 0)\lambda(1 + \Phi_s^1)b_n'(\Phi_s^1)ds \\ &+ \int_0^t \mathbb{1}(b_n(\Phi_s^1) - \Phi_s^2 > 0)\lambda(1 + \Phi_s^1)ds \\ &- \int_0^t \mathbb{1}(b_n(\Phi_s^1) - \Phi_s^2 > 0)\mu\Phi_s^1 b_n'(\Phi_s^1)dB_s^1 \\ &+ \int_0^t \mathbb{1}(b_n(\Phi_s^1) - \Phi_s^2 > 0)\mu\Phi_s^2 dB_s^2 \end{aligned} \tag{3.2.36}$$

where the last two terms defines a continuous local martingale which could be reduced into martingale through a localizing sequence of stopping times. We take a localizing sequence $(\tau_m)_{m\ge1}$ and define stopping time $\sigma_m = \inf\{t \ge 0 \mid \Phi_t^1 \le 1/m\}$. Then $\rho_m = \tau_m \wedge \sigma_m$ is also a localizing sequence for the continuous local martingale in (3.2.36).

$$\begin{aligned} \frac{1}{2}\mathbb{E}_\varphi^\infty\left[\ell_{t\wedge\rho_m}^{b_n}(\Phi)\right] \le& \varphi_2^* - b_n'(\frac{1}{m})\int_0^t \lambda(1 + \mathbb{E}_\varphi^\infty\left[\Phi_s^1\right])ds \\ &+ \int_0^t \lambda(1 + \mathbb{E}_\varphi^\infty\left[\Phi_s^2\right])ds \le K_m(t) \end{aligned} \tag{3.2.37}$$

since

$$(b_n(\Phi_t^1) - \Phi_t^2)^+ \le \varphi_2^* \quad \text{and} \quad b_n'(\Phi_t^1) \le b_n'(1/m) \tag{3.2.38}$$

where $K_m(t)$ is a constant not depending on n. Since $\hat{V}$ is continuously differentiable on $[0,\infty)\times[0,\infty)$, $\partial_{\varphi_2}\hat{V}$ is uniformly continuous on $\bar{C}$. From the smooth fit condition we see that $\partial_{\varphi_2}\hat{V}(\varphi_1, b(\varphi_1)) = 0$. Combining the two facts and that $b_n \uparrow b$ is a sequence of convex curves approaching b from below, we see $\forall\varepsilon > 0, \exists n$ such that $0 \le \partial_{\varphi_2}\hat{V}(\varphi_1, b_n(\varphi_1)) \le \varepsilon$ for all $\varphi_1 \in [0, \varphi_0^n]$. We apply expectation to both sides of (3.2.32) and change t to $t\wedge\rho_m$:

$$\begin{aligned}\mathbb{E}_\varphi^\infty\left[e^{t\wedge\rho_m}\hat{V}^n(\Phi_{t\wedge\rho_m})\right] =& \mathbb{E}_\varphi^\infty\left[\hat{V}^n(\Phi_0)\right] - \mathbb{E}_\varphi^\infty\left[\int_0^{t\wedge\rho_m} e^{-\lambda s}L(\Phi_s)\mathbb{1}(\Phi_s\in C_n)ds\right]\\ &+ \lambda/n\mathbb{E}_\varphi^\infty\left[\int_0^{t\wedge\rho_m} e^{-\lambda s}\mathbb{1}(\Phi_s\in D_n)ds\right]\\ &- \mathbb{E}_\varphi^\infty\left[\frac{1}{2}\int_0^{t\wedge\rho_m} e^{-\lambda s}\partial_{\varphi_2}\hat{V}(\Phi_s)d\ell_s^{b_n}(\Phi))\right] \qquad (3.2.39)\end{aligned}$$

. Taking $n\to\infty$ in (3.2.39), by monotone convergence we see that the first term on the right-hand side is $\hat{V}(\varphi)$ and the third term vanishes as $\lambda/n\to 0$. The last term

$$\begin{aligned}0 \le \mathbb{E}_\varphi^\infty\left[\frac{1}{2}\int_0^{t\wedge\rho_m} e^{-\lambda s}\partial_{\varphi_2}\hat{V}(\Phi_s)d\ell_s^{b_n}(\Phi))\right] &\le e^{-\lambda t}2\varepsilon K_m(t)\\ &= 2\varepsilon K_t(m)\to 0 \quad \text{since } \varepsilon\downarrow 0 \text{ as } n\uparrow\infty\end{aligned} \qquad (3.2.40)$$

vanishes as well. Taking another limit for $m\to\infty$ we see that by dominated convergence theorem

$$\hat{V}(\varphi) = e^{-\lambda t}\mathbb{E}_\varphi^\infty\left[\hat{V}(\Phi_t)\right] + \mathbb{E}_\varphi^\infty\left[\int_0^t e^{-\lambda s}L(\Phi_s)\mathbb{1}(\Phi_s\in C)ds\right]. \qquad (3.2.41)$$

As $t\to\infty$,

$$\begin{aligned}\hat{V}(\varphi) &= \mathbb{E}_\varphi^\infty\left[\int_0^\infty e^{-\lambda t}L(\Phi_t)\mathbb{1}(\Phi_t\in C)dt\right]\\ &= \int_0^\infty e^{-\lambda t}\mathbb{E}_\varphi^\infty\left[L(\Phi_t)\mathbb{1}(\Phi_t^2 < b(\Phi_t^1))\right]dt\\ &= \int_0^\infty e^{-\lambda t}K_b(t;\varphi_1, b(\varphi_2))dt. \qquad (3.2.42)\end{aligned}$$

□

3.3 Multi-dimension (Multiple-drifts)

We finally deal with the multiple drift quickest detection problem that was set up in Section 2.3 above. Recall that the difficulty here is addressing the need of picking

k coordinate processes out of n to assign the drift μ. For this, we have $N = \binom{n}{k}$ possible combinations of k coordinate processes which leads to a system of N stochastic differential equations as in (2.3.10). It follows that Proposition 3 can be directly applied here for $1 \leq i \leq N$.

$$V(\pi) = (1-\pi)\left[1 + c\hat{V}(\pi)\right] \tag{3.3.1}$$

where

$$\hat{V}(\pi) = \hat{V}(\varphi) = \inf_{\tau} \mathbb{E}_{\varphi}^{\infty}\left[\int_0^{\tau}\left(\sum_{i=1}^{N} p_i \Phi_t^i - \frac{\lambda}{c}\right) dt\right] \tag{3.3.2}$$

for $\Phi = (\Phi^1, \ldots, \Phi^N)$ and $\varphi = (\varphi_1, \ldots, \varphi_N) = \Phi_0$ $\mathbb{P}_{\varphi}^{\infty}$ almost surely and $\sum_{i=1}^{N} p_i = 1$.

The infinitesimal generator of the Markov process $\Phi = (\Phi^1, \ldots, \Phi^N)$ is

$$\mathcal{A}_{\Phi} = \sum_{i=1}^{N} \lambda(1+\varphi_i)\partial_{\varphi_i} + \frac{1}{2}\sum_{i,j=1}^{N} \mu^2 \varphi_i \varphi_j (I_i, I_j)\partial_{\varphi_i \varphi_j} \tag{3.3.3}$$

where $I_i = (I_{i1}, \ldots, I_{in})$ is a form of indicator function for being one of the k drifted coordinate processes. $I_{ip} = \mathbb{1}_i(p) = 1$ if $p \in n_1, \ldots, n_k$ and zero otherwise.

$$(I_i, I_j) = \sum_{p=1}^{n} I_{ip} I_{jp} = \begin{cases} 1 & \text{if and only if} \quad i = j \\ 0 & \text{otherwise} \end{cases} \tag{3.3.4}$$

since there is only one set of k processes chosen.

For $k = 1$ or $k = n-1$, $N = n$ as in the previous setting. However, for $1 < k < n-1$, there will be a problem since $N > n$. In other words, there will be n driving Brownian motions but N stochastic differential equations. In this case, each of the N likelihood processes Φ^i will not be independent of another and the infinitesimal generator is not an elliptic differential operator as in the previous section.

3.3.1 Hypoellipticity

We now show that the differential operator $\mathcal{A}_{\Phi}$ is of hypoelliptic type and so is the backward time-space operator $-\partial_t + \mathcal{A}_{\Phi}$.

1. It was reminded in [3] that for a differential operator to be hypoellitpic, it needs to satisfy the *Hörmander's condition.* The infinitesimal generator of the process Φ can

be rewritten as

$$
\begin{aligned}
\mathcal{A}_\Phi &= \sum_{i=1}^{N} \lambda(1+\varphi_i)\partial_{\varphi_i} + \frac{1}{2}\sum_{i,j=1}^{N} \mu\varphi_i\varphi_j(I_i, I_j)\partial_{\varphi_i\varphi_j} \\
&= D_0 + \sum_{i=1}^{N} D_i^2
\end{aligned} \tag{3.3.5}
$$

where

$$
D_0 = \sum_{j=1}^{N} \beta_{0j}\partial_{\varphi_j} \quad \text{and} \quad D_i = \sum_{j=1}^{N} \beta_{ij}\partial_{\varphi_j} \tag{3.3.6}
$$

for

$$
\begin{aligned}
\beta_{0j} &= \lambda(1+\varphi_j) - \frac{1}{2}\sum_{k,l=1}^{N} \mu\varphi_l \mathbb{1}_l(k)\partial_{\varphi_l}(\mu\varphi_j\mathbb{1}_j(k)) \\
&= \lambda(1+\varphi_j) - \frac{1}{2}\sum_{j=1}^{k} \mu^2\varphi_j \\
&= (\lambda - k\mu^2/2)\varphi_j + \lambda = a\varphi_j + \lambda
\end{aligned} \tag{3.3.7}
$$

where $a = \lambda - k\mu^2/2$ and

$$
\beta_{ij} = \begin{cases} \frac{1}{\sqrt{2}}\mu\varphi_j\mathbb{1}_j(i) & (1 \le i \le n) \\ 0 & (n < i \le N) \end{cases} \tag{3.3.8}
$$

for $1 \le i, j \le N$. The indicator $\mathbb{1}_j(i)$ evaluates to 1 if $i = j = m_1, \ldots, m_k \in C_k^n$. With this formulation, we see that D_i is a vector space of N vectors. The *Lie algebra* formulated by D_i is the smallest vector space The Hömander condition is satisfied if the Lie algebra generated by D_i has dimension N, i.e., D_1 to D_N are linearly independent. The Lie algebra generated by D_i is the smallest vector space that contains all linear combinations of the *Lie brackets* defined as

$$
[D_i, D_j] = D_iD_j - D_jD_i \tag{3.3.9}
$$

for $1 \le i, j \le N$. $Lie(D_0, \ldots, D_N) = span\{D_i, [D_i, D_j], [[D_i, D_j], D_k], \cdots \mid 0 \le i, j, k, \cdots \le N\}$. From (3.3.6)-(3.3.8), we see that

$$
D_0 \sim \begin{cases} \sum_{i=1}^{N}(\varphi_i + b)\partial_{\varphi_i} & (a \ne 0) \\ \sum_{i=1}^{N}\partial_{\varphi_i} & (a = 0) \end{cases} \tag{3.3.10}
$$

where $b = \lambda/a$ and

$$D_j = \sum_{i=1}^{N} \beta_{ji}\partial_{\varphi_i} \sim \sum_{i=1}^{N} \varphi_i I_{ij}\partial_{\varphi_i}. \tag{3.3.11}$$

This equivalence extends in determining the Lie algebra as it guarantees a constant multiple of the left-hand side. Therefore, for $j = (n_1, \ldots, n_k) \in C_k^n$ given fixed,

$$[D_0, D_{n_1}] \sim \sum_{i=1}^{N} I_{in_1}\partial_{\varphi_i} \tag{3.3.12}$$

$$[[D_0, D_{n_1}], D_{n_2}] \sim \sum_{i=1}^{N} I_{in_1} I_{in_2}\partial_{\varphi_i} \tag{3.3.13}$$

$$\vdots$$

$$[[[D_0, D_{n_1}], D_{n_2}], \ldots, D_{n_k}] \sim \sum_{i=1}^{N} I_{in_1} I_{in_2} \ldots I_{in_k}\partial_{\varphi_i} \sim \partial_{\varphi_i} \tag{3.3.14}$$

since the series of indicators evaluates to 1 only if $i = j$. ∂_{φ_i} takes value in $\mathbb{R}^N$ thus the Hömander condition is satisfied and $\mathcal{A}_\Phi$ is a hypoelliptic differential operator. By PDE theories, we see that each week solution to the function applied with a hypoelliptic differential operator is a strong solution.

2. Reminded in [3] that for the backward time-space differential operator $-\partial_t + \mathcal{A}_\Phi$ to be hypoelliptic, we need it to satisfy the parabolic Hömander condition:

$$-\partial_t + \mathcal{A}_\Phi = -\partial_{\varphi_0} + \mathcal{A}_\Phi = \bar{D}_0 + \sum_{i=1}^{N} \bar{D}_i^2 \tag{3.3.15}$$

where

$$\bar{D}_0 = \sum_{i=0}^{N} \beta_{0j}\partial_{\varphi_j} \quad \text{and} \quad \bar{D}_i = \sum_{j=0}^{N} \beta_{ij}\partial_{\varphi_j} \tag{3.3.16}$$

for $\beta_{00} = -1$ and $\beta_{i0} = 0$. $\bar{D}_i$ is a vector space of $N+1$ vectors for $0 \leq i \leq N$. Then by enlargement from previous discussion

$$\bar{D}_0 = (-1, D_0) \quad \text{evaluates in} \quad \mathbb{R}^{N+1} \tag{3.3.17}$$

and $\bar{D}_i$ are independent linearly. Therefore, $\dim Lie(\bar{D}_0, \ldots, \bar{D}_N) = N + 1$ and the condition is satisfied for $-\partial_t + \mathcal{A}_\Phi$ to be hypoelliptic.

With the hypoellipticity of the backward time-space operator, we can show that Φ is a strong Feller process. Together with the regularity of the boundary points for the stopping set that will be discussed later on, this will be useful to show that the smooth fit condition in the free boundary problem holds. Taking a bounded measurable function $f : [0,\infty)^N \to \mathbb{R}$ given fixed and defining the transition operator P_t as $P_t f(\varphi) = \mathbb{E}^{\infty}_{\varphi}[f(\Phi_t)]$, by the backward Kolmogorov equation we have $\partial_t P_t f = \mathcal{A}_{\Phi} P_t f$ and $(-\partial_t + \mathcal{A}_{\Phi})P_t f = 0$. The hypoellipticity of $-\partial_t + \mathcal{A}_{\Phi}$ implies that any solution to the homogeneous equation is smooth in t and φ. Therefore, $\varphi \to P_t f \in C^{\infty}$ and is continuous, and we can conclude that the Markov process Φ has strong Feller property and is a strong Feller process.

Similar to previous cases, we can reformulate the Lagrange functional in (3.3.2) into a Mayer-formulated optimal stopping problem by solving the differential equation $\mathcal{A}_{\Phi} M - \lambda M = L$ on $(0,\infty)^N$ for a function $M : [0,\infty)^N \to \mathbb{R}$ and where $L(\varphi_1, \dots, \varphi_N) = \sum_{i=1}^N p_i \varphi_i - \lambda/c$. Since the differential operator $\mathcal{A}_{\Phi}$ is hypoellitpic, the problem is equivalent to solving N ordinary differential equations as follows

$$\lambda(1+\varphi_i)\frac{dM_i}{d\varphi_i} + \frac{\mu^2}{2}\varphi_i^2 \frac{d^2 M_i}{d\varphi_i^2} - \lambda M_i = \varphi_i \tag{3.3.18}$$

for $\sum_{i=1}^N p_i = 1$. The results from previous sections apply and we have

$$M_i(\varphi_i) = \frac{2}{\mu^2}(1+\varphi_i)\int_0^{\varphi_i/(1+\varphi_i)} \left(\frac{1-a}{a}\right)^{\kappa} e^{\kappa/a} \int_0^a \frac{b^{\kappa-1}}{(1-b)^{\kappa+2}} e^{-\kappa/b}\, db\, da \tag{3.3.19}$$

for $\varphi_i \in [0,\infty)$, $1 \le i \le N$ and, $\kappa = 2\lambda/\mu^2$. The Mayer functional then takes the form

$$M(\varphi_1, \dots, \varphi_N) = \sum_{i=1}^N p_i M_i(\varphi_i) + \frac{1}{c} \tag{3.3.20}$$

and

$$\hat{V}(\varphi) = \inf_{\tau} \mathbb{E}^{\infty}_{\varphi}\left[e^{-\lambda\tau} M(\Phi^1_{\tau}, \dots, \Phi^N_{\tau})\right] - M(\varphi). \tag{3.3.21}$$

This can be verified in the same manner using Itô formula and optional sampling theorem as in the previous sections only with an additional summation to indicate that $i \in C^n_k$ as a selection of k coordinates togther with a summation running from 1 to k for each of the selected coordinates.

3.3.2 Structure of the optimal stopping boundary

The structure of the continuation set, stopping set, and optimal stopping boundary can be shown as an extension from our previous section's work on one-drift detection.

The candidate continuation and stopping set are the same as in the previous section with higher dimensions

$$C = \{\varphi \in [0,\infty)^N \mid \hat{V}(\varphi) < 0\} \tag{3.3.22}$$

$$D = \{\varphi \in [0,\infty)^N \mid \hat{V}(\varphi) = 0\}. \tag{3.3.23}$$

From previous results for *lsc* and *usc* of the value function and gain function, the first entry time to the stopping set

$$\tau_D = \inf\{t \geq 0 \mid \Phi_t \in D\} \tag{3.3.24}$$

is optimal. Recall that in the one-dimensional case, we had $\varphi_* = \varphi_*(\lambda/\mu^2, \lambda/c)$, we thus set $\varphi_i^* = \varphi_*(\lambda/k\mu^2, \lambda/p_i c)$.

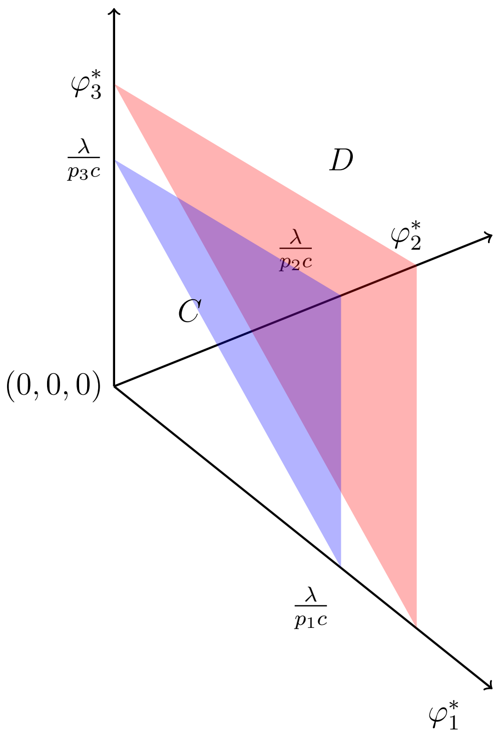

Figure 3.4: Upper and lower boundaries of the optimal stopping Surface

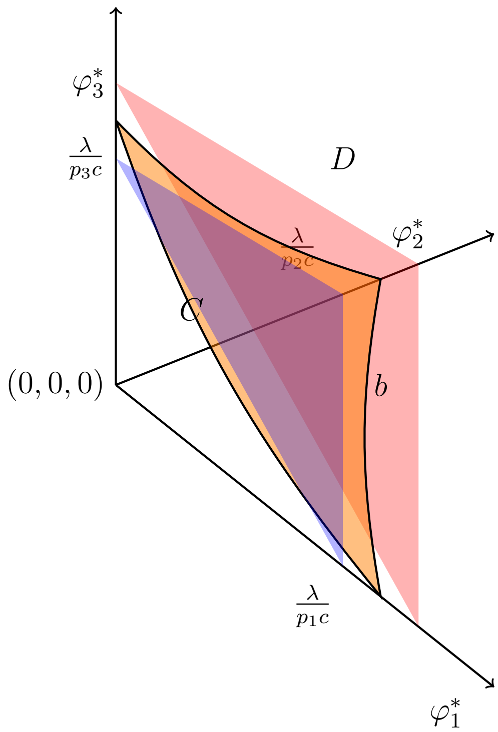

Figure 3.5: Optimal stopping boundary (surface)

Proposition 8.

(1) *$\hat{V}$ is concave and continuous on the state space of Φ.*

(2) *If $\varphi_1 \leq \psi_1, \ldots, \varphi_N \leq \psi_N$ then $\hat{V}(\varphi_1, \ldots, \varphi_N) \leq \hat{V}(\psi_1, \ldots, \psi_N)$.*

(3) *If the state $(\varphi_1, \ldots, \varphi_N) \in D$ & $\psi_1 \geq \varphi_1, \ldots, \psi_N \geq \varphi_N$ then the state $(\psi_1, \ldots, \psi_N) \in D$.*

(4) *The stopping set D is a convex set and the polytope* $\{(\varphi_1, \ldots, \varphi_N) \in [0,\infty)^N \mid \sum_{i=1}^N \varphi_i/\varphi_i^* - 1 \geq 0\} \subseteq D$.

(5) *The simplex* $\{(\varphi_1, \ldots, \varphi_N) \in [0,\infty)^N \mid \sum_{i=1}^N p_i\varphi_i - \lambda/c < 0\} \subseteq C$.

Proof. (1), (2), and (3) follows from the exact same proof as in previous section for Proposition 4 only with the indexes $i = 1, 2$ extended to $i = 1, 2, \ldots, N$.

(4) The convexity of the set D follows from the same proof as in Proposition 4 (4). By pulling p_j in front of the infimum like we did for p_2 in Proposition 4, we see that the rest is solved by the optimal stopping point $(0, \ldots, 0, \varphi_j^*, 0, \ldots, 0)$. Thus $\{(0, \ldots, 0, \varphi_j, 0, \ldots, 0) \mid \varphi \geq \varphi_j^*\} \in D$ due to convexity, the polytope is in D.

(5) For all $(\varphi_1, \ldots, \varphi_N) \in [0,\infty)^N$ such that $\sum_{i=1}^N p_i\varphi_i - \lambda/c < 0$, $L(\Phi) < 0$ and suboptimal. Therefore, the triangle is in C. □

Figure 3.4 shows a three-dimensional version of the polytope and simplex being in D and C. From this we see that there exists an optimal stopping boundary between the upper bound and lower bound such that $b : [0,\infty)^{N-1} \to [0,\infty)$ is a decreasing, convex function as in Figure 3.5. For b to be between the polytope and the sipmlex, we see that

$$\begin{aligned} &\sum_{i=1}^N \frac{\varphi_i}{\varphi_i^*} - 1 \leq 0 && \sum_{i=1}^N p_i\varphi_i - \lambda/c \geq 0 \\ &\varphi_N^* \sum_{i=1}^{N-1} \frac{\varphi_i}{\varphi_i^*} + \varphi_N - \varphi_N^* \leq 0 && \sum_{i=1}^{N-1} \frac{p_i}{p_N}\varphi_i + \varphi_N - \frac{\lambda}{p_N c} \geq 0 \\ &-\sum_{i=1}^{N-1} \frac{\varphi_i}{\varphi_i^*}\varphi_N^* + \varphi_N^* \geq \varphi_N = b, && -\sum_{i=1}^{N-1} \frac{p_i}{p_N}\varphi_i + \frac{\lambda}{p_N c} \leq \varphi_N = b. \end{aligned} \tag{3.3.25}$$

The continuation and stopping set can be defined as

$$C = \{\varphi \in [0,\infty)^N \mid \varphi_N < b(\varphi_1, \ldots, \varphi_{N-1})\} \tag{3.3.26}$$

$$D = \{\varphi \in [0,\infty)^N \mid \varphi_N \geq b(\varphi_1, \ldots, \varphi_{N-1})\} \tag{3.3.27}$$

with ∂C being the set of $(\varphi_1, \ldots, \varphi_N)$ such that $\varphi_N = b(\varphi_1, \ldots, \varphi_{N-1})$. With the definition of the stopping set and continuation set clarified, we can show the following proposition.

Proposition 9. *The boundary* $\partial C = \{\varphi \in [0,\infty)^N \mid \varphi_N = b(\varphi_1, \ldots, \varphi_{N-1})\}$ *is probabilistically regular for the stopping set D in* (3.3.27).

Proof. By a standard discretization argument, the event $\{\tau_D = 0\}$ for the stopping time $\tau_D = \inf\{t \geq 0 \mid \Phi_t \in D\}$ can be rewritten as $\{\tau_D = 0\} = \cap_{n=1}^{\infty} \cup_{t \in (0, \frac{1}{n})} \{\Phi_t \in D\}$. We thus repeat the proof for Proposition 6 above, only noting that the third euqality in (3.2.19) becomes $\limsup_{t \downarrow 0} \mathbb{P}_{\varphi}^{\infty} \left[\exp(\sum_j^k X_t^{n_j^i} - \mu^2 t/2) \geq 1\right]$ for $1 \leq i \leq N$, which is indeed greater than $\limsup_{t \downarrow 0} \mathbb{P}^{\infty}\left[X_t^i \geq \mu t/2\right]$. Then proof in Proposition 5 follows, and we see that $\{\tau_D = 0\} = 1$ by Blumental's law and the boundary ∂C is probabilistically regular for the stopping set D. $\square$

3.3.3 Free-boundary problem

Recall that the optimal stopping problem is of Lagrange formulation as in (3.3.2). Φ is a strong Markov process that solves a system of N stochastic differential equations driven by $B = (B^1, \ldots, B^n)$ $(n \leq N)$. This problem was addressed by the hypoellop-ticity of the infinitesimal generator of the process. Standard arguments for optimal stopping problem of this kind lead to a free-boundary problem which $\hat{V}$ and b solves uniquely:

$$\mathcal{A}_\Phi \hat{V} - \lambda \hat{V} = -L \quad \text{in} \quad C, \tag{3.3.28}$$

$$\hat{V}\Big|_D = 0 \quad \textit{(instantaneous stopping condition)}, \tag{3.3.29}$$

$$\frac{\partial \hat{V}}{\partial \varphi_i}\Big|_{\partial C} = 0 \quad \textit{(smooth fit condition)}. \tag{3.3.30}$$

The optimal stopping boundary $\partial C = \{(\varphi_1, \ldots, \varphi_N) \in [0, \infty)^N \mid \varphi_N = b(\varphi_1, \ldots, \varphi_{N-1})\}$. The pair $(\hat{V}, b)$ is the unique solution to the above free-boundary problem among a class of admissible functions, $\mathcal{C}$. For $(U, a) \in \mathcal{C}$, (1) $U \in C^2(C_a) \cap C^1(\bar{C}_a)$, continuous and bounded on $[0, \infty)^N$ and (2) $a : [0, \infty)^{N-1} \to [0, \infty)$ is a continuous and decreasing function with $\sum_{i=1}^{N-1} p_i \varphi_i + p_N a(\varphi_1, \ldots, \varphi_{N-1}) - \lambda/c \geq 0$ (in the sense of having the characteristics we identified above.) C_a is defined in the same way as (3.3.26) with function b changed to a general a.

Recall that the infinitesimal generator $\mathcal{A}_\Phi$ is a hypoelliptic differential operator and that $\hat{V}$ is a weak solution to (3.3.28) from Corollary 5 in [11], we see that $\hat{V}$ is a strong solution to (3.3.28) due to hypoellipticity in PDE theories. From (3.3.23), the instantaneous stopping condition holds in (3.3.29). Proposition 9 in the above, together with the fact that Φ is a strong Feller process, shows that all points on ∂C is

Green regular for D (by the Corollary 2 in [1]). Since Φ can be realized as a continuous stochastic (Markovian) flow of its initial point as in (2.3.13) and the gain function in the Mayer formulation (3.3.21) is continuously differentiable, the value function $\hat{V}$ is then continuously differentiable at the optimal stopping boundary ∂C. This ensure that the smooth fit condition exists in (3.3.31). Therefore, the free-boundary problem (3.3.28)-(3.3.30) is established. (1) By Theorem 8 in [1], $\hat{V}$ is continuously differentiable on $[0,\infty)^N$ thus in $C^1(\bar{C})$. Since $\mathcal{A}$ is hypoelliptic and the drift and diffusion coefficients of the process Φ are infinitely differentiable from (2.3.10), the solution $\hat{V} \in C^\infty(C)$ by results in [11] thus in $C^2(C)$. (2) b is satisfies (2) and in $\mathcal{C}$ can be seen from the properties of b specified in Proposition 8. Therefore, the pair $(\hat{V}, b)$ is in the class $\mathcal{C}$. The uniqueness is shown in the integral representation we will be discussing in the following.

Since the process Φ is a strong Markov process with a hypoelliptic differential operator, we have $p(t;\varphi_1,\dots,\varphi_N,\psi_1,\dots,\psi_N)$ as the Markov transition density and the Markov transition kernel

$$\mathbb{P}^\infty_\varphi\left[\Phi_t \in A\right] = P(t,\varphi,A) = \int\cdots\int_A p(t;\varphi_1,\dots,\varphi_N,\psi_1,\dots,\psi_N)d\psi_1\dots d\psi_N \quad (3.3.31)$$

for A being a measurable set in the state space of Φ and $\varphi = (\varphi_1,\dots,\varphi_N)$.

The key difference here with the multiple-drifts setting with the one-drift setting is that each Φ^i for $1 \le i \le N$ is not independent and the transition density cannot be seen as a convolution of the transition density for each Φ^i. We define the expectation of the Lagrange functional evaluated in the continuation set as

$$\begin{aligned} K_b(t;\varphi_1,\dots,\varphi_N) &:= \mathbb{E}^\infty_\varphi\left[L(\Phi^1_t,\dots,\Phi^N_t)\mathbb{1}(\Phi^N_t < b(\Phi^1_t,\dots,\Phi^{N-1}_t))\right] \\ &= \int\cdots\int_{\{\Phi^N_t < b(\Phi^1_t,\dots,\Phi^{N-1}_t)\}} L(\psi_1,\dots,\psi_N)p(t;\varphi_1,\dots,\varphi_N,\psi_1,\dots,\psi_N)d\psi_1\dots d\psi_N \end{aligned} \quad (3.3.32)$$

for $L = \sum_{i=1}^N p_i\varphi_i - \lambda/c$.

Theorem 10. *The optimal stopping boundary b in the problem* (3.3.2) *is the unique solution to the nonlinear Fredholm integral equation*

$$\int_0^\infty e^{-\lambda t} K_b(t;\varphi_1,\dots,\varphi_{N-1}, b(\varphi_1,\dots,\varphi_{N-1}))dt = 0 \quad (3.3.33)$$

among the class of convex functions b as in $\mathcal{C}$. The value function $\hat{V}$ can be represented *as*

$$\hat{V}(\varphi_1,\ldots,\varphi_N) = \int_0^\infty e^{-\lambda t} K_b(t;\varphi_1,\ldots,\varphi_N)dt \tag{3.3.34}$$

and the optimal stopping time is

$$\tau_b = \inf\{t \geq 0 \mid \Phi_t^N \geq b(\Phi_t^1,\ldots,\Phi_t^{N-1})\}. \tag{3.3.35}$$

Proof. We use a sequence of continuation sets $C_n = \{\varphi \in [0,\infty)^N \mid \hat{V}(\varphi) < -1/n\}$ and stopping sets $D_n = \{\varphi \in [0,\infty)^N \mid \hat{V}(\varphi) \geq -1/n\}$ to approximate the continuation set and stopping set in (3.3.26) - (3.3.27). The optimal boundaries $b_n(\varphi_1,\ldots,\varphi_{N-1})$ between the sets C_n and D_n approaches b as $n \to \infty$. We defined $\hat{V}^n$ as $\hat{V}$ on C_n and $-1/n$ on D_n. $\hat{V}^n \uparrow \hat{V}$ for $\hat{V}^n \in C^2(\bar{C}_n)$ on C_n and $\hat{V}^n \in C^2(\bar{D}_n)$ on D_n. By the change-of-variable formula in [7], we see that

$$\begin{aligned}
d\hat{V}^n(\Phi_t) =& \mathcal{A}_\Phi \hat{V}^n(\Phi_t)dt + \sum_{i=1}^N \sum_{j=1}^k \partial_{\varphi_i}\hat{V}^n(\Phi_t)\mu\Phi_t^i dB_t^{n_j} \\
&- \frac{1}{2}(\partial_{\varphi_N}\hat{V}^n(\Phi_t^1,\ldots,\Phi_t^N+) - \partial_{\varphi_N}\hat{V}^n(\Phi_t^1,\ldots,\Phi_t^N-))\mathbb{1}(\Phi_N = b_n)d\ell_t^{b_n}(\Phi) \\
=& \mathcal{A}_\Phi \hat{V}^n(\Phi_t)dt + \sum_{i=1}^N \sum_{j=1}^k \partial_{\varphi_i}\hat{V}^n(\Phi_t)\mu\Phi_t^i dB_t^{n_j^i} - \frac{1}{2}\partial_{\varphi_N}\hat{V}(\Phi_t^N-)d\ell_t^{b_n}(\Phi)
\end{aligned} \tag{3.3.36}$$

since the right derivative of $\hat{V}^n$ with respect to φ_N at the boundary is 0 and the left derivative equals the total derivative. $\ell_t^{b_n}(\Phi)$ is the local time of the semimartingale Φ on b defined as

$$\begin{aligned}
\ell_t^{b_n}(\Phi) = \text{P-}\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t &\mathbb{1}(-\varepsilon < \Phi_s^N - b_n(\Phi_s^1,\ldots,\Phi_s^{N-1}) < \varepsilon) \\
&d\langle \Phi^N - b_n(\Phi^1,\ldots,\Phi^{N-1}), \Phi^N - b_n(\Phi^1,\ldots,\Phi^{N-1})\rangle_s.
\end{aligned} \tag{3.3.37}$$

$$\begin{aligned}
de^{-\lambda t}\hat{V}^n(\Phi_t) =& e^{-\lambda t}(\mathcal{A}_\Phi\hat{V}^n - \lambda\hat{V}^n)(\Phi_t)dt - \frac{1}{2}e^{-\lambda t}\partial_{\varphi_N}\hat{V}(\Phi_t^N-)d\ell_t^{b_n}(\Phi) \\
&+ e^{-\lambda t}\sum_{i=1}^N\sum_{j=1}^k \partial_{\varphi_i}\hat{V}^n(\Phi_t)\mu\Phi_t^i dB_t^{n_j^i} \\
=& -e^{-\lambda t}L(\Phi_t)\mathbb{1}(\Phi_t \in C_n)dt + \frac{\lambda}{n}e^{-\lambda t}\mathbb{1}(\Phi_t \in D_n)dt \\
&- \frac{1}{2}e^{-\lambda t}\partial_{\varphi_N}\hat{V}(\Phi_t^N-)d\ell_t^{b_n}(\Phi) + M_t
\end{aligned} \tag{3.3.38}$$

where $M_t = e^{-\lambda t}\sum_{i=1}^N \sum_{j=1}^k \partial_{\varphi_i}\hat{V}^n(\Phi_t)\mu\Phi_t^i \mathbb{1}(\Phi_t \in C_n)dB_t^{n_j^i}$ is a continuous martingale. We set function $f_t(\varphi_1,\dots,\varphi_N) = b_n(\varphi_1,\dots,\varphi_{N-1})-\varphi_N$ and apply the Tanaka formula, reminded in Chapter 1, to it

$$df_t^+(\Phi_t) = \mathbb{1}(f>0)df + \frac{1}{2}\ell_t^{b_n}(\Phi). \tag{3.3.39}$$

By Itô formula (since $b_n \in C^2$ by implicit function theorem), we see that

$$\begin{aligned} df &= \sum_{i=1}^N \partial_{\varphi_i} f_t(\Phi_t)d\Phi_t^i + \frac{1}{2}\sum_{i,j=1}^N \partial_{\varphi_i\varphi_j} f_t(\Phi_t)d\langle\Phi^i,\Phi^j\rangle_t \\ &= \sum_{i=1}^{N-1}\partial_{\varphi_i} b_n(\Phi_t^1,\dots,\Phi_t^{N-1})d\Phi_t^i - d\Phi_t^N + \frac{1}{2}\sum_{i,j=1}^N \partial_{\varphi_i\varphi_j} b_n(\Phi_t^1,\dots,\Phi_t^{N-1})d\langle\Phi^i,\Phi^j\rangle_t. \end{aligned} \tag{3.3.40}$$

Then (3.3.39) can be rewritten as

$$\begin{aligned} (b_n-\Phi_t^N)^+ = (b_n-\Phi_0^N)^+ + \int_0^t \mathbb{1}(b_n-\Phi_s^N>0)(\sum_{i=1}^{N-1}\partial_{\varphi_i}b_n d\Phi_s^i - d\Phi_s^N \\ + \frac{1}{2}\sum_{i,j=1}^N \partial_{\varphi_i\varphi_j}b_n d\langle\Phi^i,\Phi^j\rangle_s) + \frac{1}{2}\ell_t^{b_n}(\Phi) \end{aligned} \tag{3.3.41}$$

for $b_n = b_n(\Phi_t^1,\dots,\Phi_t^{N-1})$. Since $d\langle\Phi^i,\Phi^j\rangle_t = \mu^2\Phi_t^i\Phi_t^j(I_i,I_j)\,dt$, the term

$$\int_0^t \mathbb{1}(b_n-\Phi_s^N)\frac{1}{2}\sum_{i,j=1}^N \partial_{\varphi_i\varphi_j}b_n\, d\langle\Phi^i,\Phi^j\rangle_s$$

equals

$$\frac{1}{2}\int_0^t \mathbb{1}(b_n-\Phi_s^N)\sum_{i,j=1}^N \partial_{\varphi_i\varphi_j}b_n\,\mu^2\Phi_s^i\Phi_s^j(I_i,I_j)\,ds \geq 0.$$

Therefore,

$$\begin{aligned} \frac{1}{2}\ell_t^{b_n}(\Phi) \leq &(b_n-\Phi_t^N)^+ - \int_0^t \mathbb{1}(b_n-\Phi_s^N>0)\left(\sum_{i=1}^{N-1}\partial_{\varphi_i}b_n d\Phi_s^i + d\Phi_s^i\right) \\ =&(b_n-\Phi_t^N)^+ - \int_0^t \mathbb{1}(b_n-\Phi_s^N>0)\sum_{i=1}^{N-1}\partial_{\varphi_i}b_n\lambda(1+\Phi_s^i)ds \\ &+ \int_0^t \mathbb{1}(b_n-\Phi_s^N>0)\lambda(1+\Phi_s^N)ds \\ &- \sum_{i=1}^{N-1}\sum_{j=1}^k\int_0^t \mathbb{1}(b_n-\Phi_s^N>0)\partial_{\varphi_i}b_n\mu\Phi_s^i dB_s^{n_j^i} \\ &+ \sum_{j=1}^k\int_0^t \mathbb{1}(b_n-\Phi_s^N>0)\mu\Phi_s^N dB_s^{n_j^N} \end{aligned} \tag{3.3.42}$$

where the last two terms defines a continuous local martingale which could be reduced into martingale through a localizing sequence of stopping times. We take a localizing sequence $(\tau_m)_{m\geq 1}$ and define stopping times $\sigma_m = \inf\{t \geq 0 \mid (\Phi_t^1, \dots, \Phi_t^{N-1}) \in [0, 1/m)^{N-1}\}$. Then $\rho_m = \tau_m \wedge \sigma_m$ is also a localizing sequence for the continuous local martingale in (3.3.42).

$$\frac{1}{2}\mathbb{E}_\varphi^\infty\left[\ell_{t\wedge\rho_m}^{b_n}\right] \leq \varphi_N^* - \sum_{i=1}^{N-1}\partial_{\varphi_i} b_n(1/m, \dots, 1/m)\int_0^t \lambda(1+\mathbb{E}_\varphi^\infty[\Phi_s^i])ds \\ + \int_0^t \lambda(1+\mathbb{E}_\varphi^\infty[\Phi_s^N])ds \leq K_m(t) \tag{3.3.43}$$

since

$$(b_n(\Phi_t^1, \dots, \Phi_t^{N-1}) - \Phi_t^N)^+ \leq \varphi_N^* \quad \text{and} \\ b_n'(\Phi_t^1, \dots, \Phi_t^{N-1}) \leq b_n'(1/m, \dots, 1/m) \tag{3.3.44}$$

where $K_m(t)$ is a constant not depending on n. By the smooth fit condition

$$\partial_{\varphi_N}\hat{V}(\varphi_1, \dots, \varphi_{N-1}, b(\varphi_1, \dots, \varphi_{N-1})) = 0. \tag{3.3.45}$$

Since $b_n \uparrow b$ is a sequence of curves approaching b from downside, $\forall\varepsilon > 0, \exists n$ such that $0 \leq \partial_{\varphi_N}\hat{V}(\varphi_1, \dots, \varphi_{N-1}, b_n(\varphi_1, \dots, \varphi_{N-1})) \leq \varepsilon$. We apply expectation to both sides of (3.3.38) and change t to $t \wedge \rho_m$:

$$\mathbb{E}_\varphi^\infty\left[e^{t\wedge\rho_m}\hat{V}^n(\Phi_{t\wedge\rho_m})\right] = \mathbb{E}_\varphi^\infty\left[\hat{V}^n(\Phi_0)\right] - \mathbb{E}_\varphi^\infty\left[\int_0^{t\wedge\rho_m} e^{-\lambda s}L(\Phi_s)\mathbb{1}(\Phi_s \in C_n)ds\right] \\ + \lambda/n\mathbb{E}_\varphi^\infty\left[\int_0^{t\wedge\rho_m} e^{-\lambda s}\mathbb{1}(\Phi_s \in D_n)ds\right] \\ - \mathbb{E}_\varphi^\infty\left[\frac{1}{2}\int_0^{t\wedge\rho_m} e^{-\lambda s}\partial_{\varphi_N}\hat{V}(\Phi_s)d\ell_s^{b_n}(\Phi)\right]. \tag{3.3.46}$$

Taking $n \to \infty$, by monotone convergence we see that the first term one the right-hand side is $\hat{V}(\varphi)$ and the third term vanishes as $\lambda/n \to 0$. The last term

$$0 \leq \mathbb{E}_\varphi^\infty\left[\frac{1}{2}\int_0^{t\wedge\rho_m} e^{-\lambda s}\partial_{\varphi_N}\hat{V}(\Phi_s)d\ell_s^{b_n}(\Phi))\right] \leq e^{-\lambda t}2\varepsilon K_m(t) \\ = 2\varepsilon K_t(m) \to 0 \quad \text{since } \varepsilon \downarrow 0 \text{ as } n \uparrow \infty \tag{3.3.47}$$

vanishes as well. Taking $m \to \infty$ then $t \wedge \rho_m \to t$ and by dominated convergence theorem

$$\hat{V}(\varphi) = e^{-\lambda t}\mathbb{E}_\varphi^\infty\left[\hat{V}(\Phi_t)\right] + \mathbb{E}_\varphi^\infty\left[\int_0^t e^{-\lambda s}L(\Phi_s)\mathbb{1}(\Phi_s \in C)ds\right]. \tag{3.3.48}$$

As $t \to \infty$

$$\begin{aligned}\hat{V}(\varphi) &= \mathbb{E}_{\varphi}^{\infty}\left[\int_0^{\infty} e^{-\lambda t} L(\Phi_t)\mathbb{1}(\Phi_t \in C)dt\right] \\ &= \int_0^{\infty} e^{-\lambda t}\mathbb{E}_{\varphi}^{\infty}\left[L(\Phi_t)\mathbb{1}(\Phi_t^N < b(\Phi_t^1, \ldots, \Phi_t^{N-1}))\right]dt \\ &= \int_0^{\infty} e^{-\lambda t} K_b(t; \varphi_1, \ldots, \varphi_{N-1} b(\varphi_N))dt. \end{aligned} \tag{3.3.49}$$

$\square$

Chapter 4

Application

In this chapter, we briefly go through some of the applications of the results that was proved in the Chapter 2-3 for one-dimensional Wiener disorder problem, one-drift detection problem in multi-dimensional scenario, and multiple-drift detection problem in multi-dimensional scenario.

1. Recall that in the one-dimensional setting, we have a stock price modeled by geometric Brownian motion as

$$S_t = \exp(\sigma B_t + (r - \frac{\sigma^2}{2}t)) \tag{4.0.1}$$

where r is the expected return on the stock $\sigma > 0$ is the volatility of the stock, and $S_0 = 1$. We model the logarithm stock price with a drift change μ at an exponentially distributed time θ as

$$dX_t = (\mu_0 + \mu\mathbb{1}(t \geq \theta))dt + \sigma dB_t \tag{4.0.2}$$

where $\mu_0 = r - \sigma^2/2$. For simplicity, we assume $r = 0.5$ and $\sigma = 1$ such that $\mu_0 = 0$. The quickest detection of the drift change was solved as an optimal stopping problem in Section 3.1 using the Mayer-Lagrange formulation and a measure-changed Lagrange formulation where the optimal stopping time τ_* is the first time the posterior probability distribution process $(\Pi_t)_{t\geq 0}$ hits the optimal stopping pint A_* or the first time the likelihood process $(\Phi_t)_{t\geq 0}$ hits the optimal stopping point φ_*. As we have discussed in Section 3.1, $A_* \in (\lambda/(\lambda + c), 1)$ is solved by setting (3.1.11) to -1 and $\varphi_* \in (\lambda/c, \infty)$ is the unique solution to the equation (3.1.59). Π and Φ are modelled by

$$d\Pi_t = \lambda(1 - \Pi_t)dt + \mu\Pi_t(1 - \Pi_t)d\bar{B}_t \tag{4.0.3}$$

$$d\Phi_t = \lambda(1 + \Phi_t)dt + \mu\Phi_t dB_t \tag{4.0.4}$$

where $\bar{B}$ is a standard Brownian motion under $\mathbb{P}_\pi$.

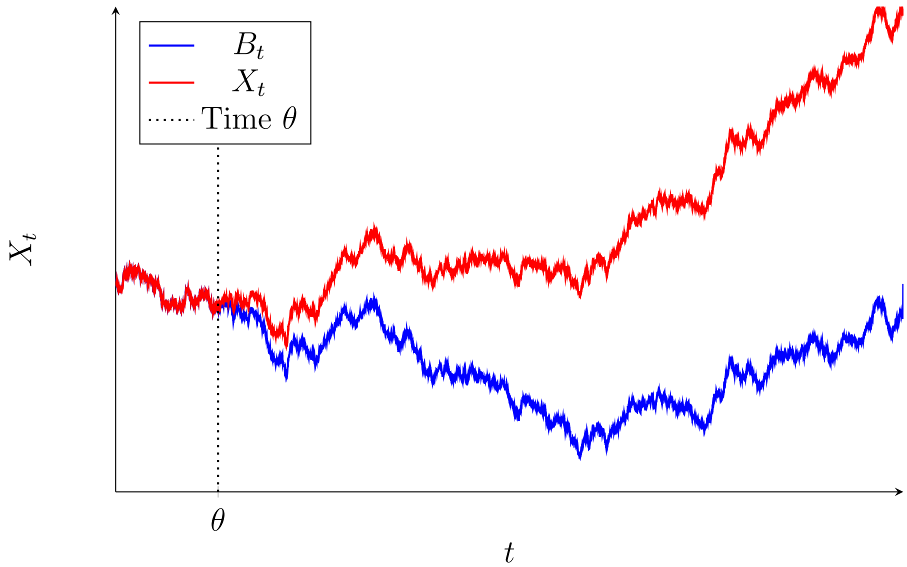

Figure 4.1: Simulated $dX_t = \mu \mathbb{1}(t \geq \theta)dt + dB_t$

Assuming that we are observing the stock price on a 10-day time scale, $\mu = 1$, $\lambda = 1$, and $c = 0.5$. By randomly genrating an exponentially distributed time $\theta = 1.246$ Figure 4.1 is the simulated X_t over the time-horizon where the blue path indicates the original undriftted Brownian motion while the red path indicates the drifted process. Then the unique solution A_* for

$$V'(\pi) = e^{-\frac{\lambda}{\gamma}\alpha(\pi)} \left(\int_0^\pi \frac{c}{\gamma} \frac{1}{\rho(1-\rho)^2} e^{\frac{\lambda}{\gamma}\alpha(\rho)} d\rho \right) = -1 \tag{4.0.5}$$

is given by $A_* = 0.72768 \geq \lambda/(\lambda + c) = 0.66667$. We model the process Π using Euler's method as in Figure 4.2. Therefore, we can see that the optimal stopping time for the process to hit A_* for the first time is $\tau_{A_*} = 1.708$. Using the measure-changed Lagrange method, we solve the problem by finding a unique solution to (3.1.59), $\varphi_* = 2.672$, as the optimal stopping point for the process Φ. We simulate the process Φ using the solution in (2.1.15) + (2.1.16). As indicated in Figure 4.3, the first hitting time of the optimal stopping point is $\tau_* = 1.7.18$. Therefore, an accurate estimate of the stopping time is $\tau = 1.71$. By plain observation of the process X_t (the red path in Figure4.1), it is unlikely that one could assert at time $t = 1.71$ that there has been a positive drift introduced to the process. Upon knowing the optimal stopping time, one could engage in several strategies in the financial market to maximize wealth. The simplest strategy is buy-and-hold the stock at the stopping time and wait for the stock price to drift upwards for a speculation. Another basic method is to use a European call option with terminal time specified after the optimal stopping time to obtain the right to buy the stock with a lower strike price. However, for more risk-averse investors, one could use a covered call or bull spread call, limiting the risk of future decreases in the stock price while limiting the potential profit.

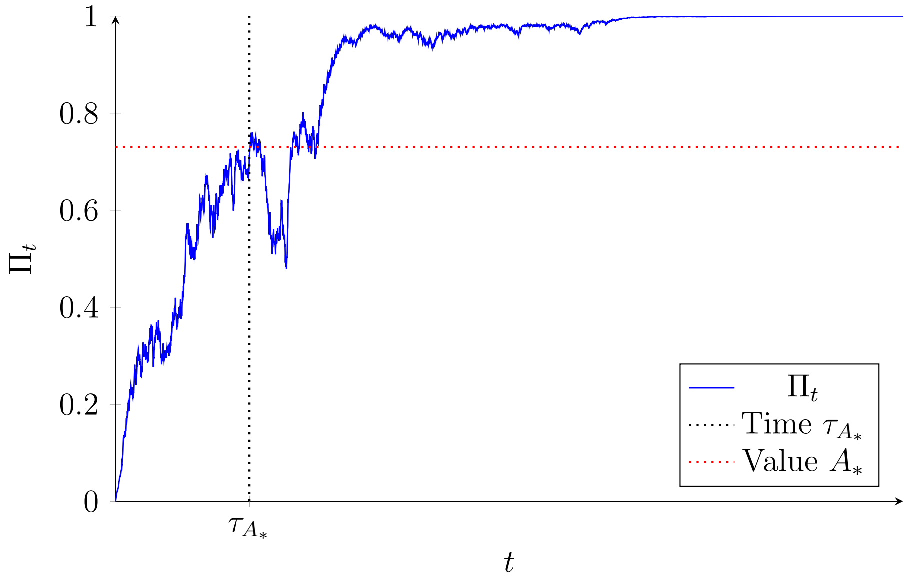

Figure 4.2: Simulated $d\Pi_t = \lambda(1 - \Pi_t)dt + \mu\Pi_t(1 - \Pi_t)d\bar{B}_t$

However, the above simulation is a specific case where obvious delay can be seen from the difference between the actual time $\theta = 1.246$ and the solved stopping time $\tau = 1.71$. This could be traced back to the parameter c which worked as the Lagrange multiplier for the probability of false alarm and expected delay in detection in (2.1.3). The constrained problem for the Lagrangian linear combination was not discussed in the dissertation; however, from an intuitive perspective we could see the role of c in the problem. With a higher c we penalize the delay while increases the risk of having false alarms (early detection). Therefore, finding the c that balances the trade-off is another form of problem that was discussed in part 8 of Section 22 in [8] for the one-dimensional finite horizon case. In our setting we could try to approach this by a simulation method.

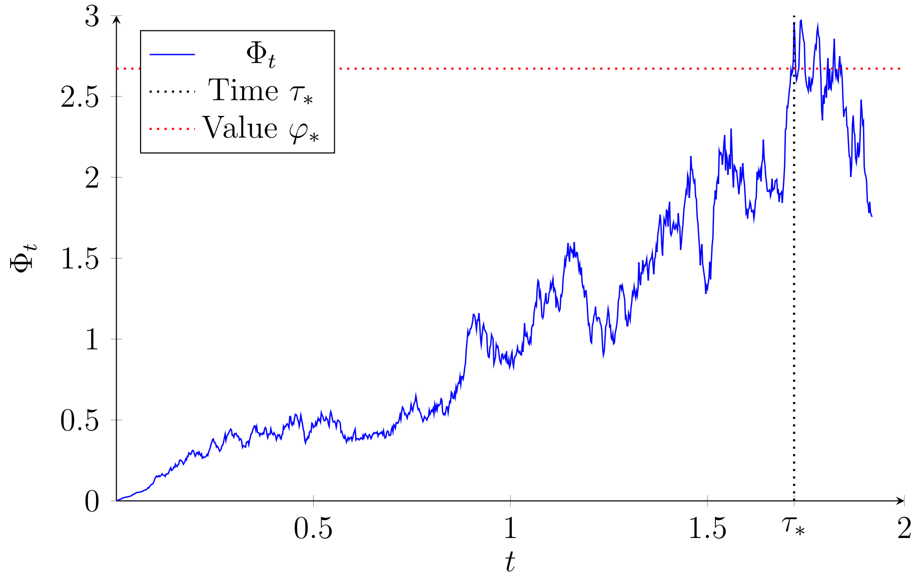

Figure 4.3: Simulated $d\Phi_t = \lambda(1 + \Phi_t)dt + \mu\Phi_t dB_t$

We run $n = 1000$ simulations of θ, X, Π, and Φ for each c ranging from 0.1 to 1.0 and calculate the average θ, average τ_{A_*}, average τ_*, and average difference between the hitting time and θ. As indicated in Table 4, the early detection and delayed detection are minimized at a c between $0.7 - 0.8$ when the performance of the method is the closest towards θ on average.

c	**avg theta**	**avg τ_{A_*}**	**avg τ_***	**avg diff(τ_{A_*})**	**avg diff(τ_*)**
0.1	1.0049	1.4841	2.3435	0.4792	1.3386
0.2	1.0049	1.8672	1.8588	0.8622	0.8539
0.3	1.0049	1.5769	1.5826	0.5720	0.5777
0.4	1.0049	1.3803	1.3824	0.3753	0.3774
0.5	1.0049	1.2287	1.2319	0.2237	0.2270
0.6	1.0049	1.1065	1.1153	0.1015	0.1103
0.7	1.0049	1.0087	1.0127	0.0037	0.0078
0.8	1.0049	0.9267	0.9301	-0.0782	-0.0748
0.9	1.0049	0.8514	0.8630	-0.1535	-0.1419
1.0	1.0049	0.7929	0.7991	-0.2121	-0.2059

Table 4.1: Comparison using different c

2. In the two-dimensional setting with one drift change that we have discussed in Sections 2.2 and 3.2, the solution to the optimal stopping problem was solved by finding the unique solution $b(\varphi_1)$ to the non-linear Fredholm integral equation (3.2.26). The equation involves the the density function $p(t; \varphi_1, \varphi_2, \psi_1, \psi_2)$ in (3.2.25). For this, we could take a look at the method used in [10] for a general Shiryaev process. Since Φ^1

and Φ^2 are independent, $p(t;\varphi_1,\varphi_2,\psi_1,\psi_2) = p_1(t;\varphi_1,\psi_1)p_2(t;\varphi_2,\psi_2)$. From (2.2.14), the drift coefficient and diffusion coefficient for Φ^i is $\mu(\varphi_i) = \lambda(1+\varphi_i)$ and $\sigma(\varphi_i) = \mu\varphi_i$ for $i = 1,2$. Therefore, the scale function of the process is given by

$$\begin{aligned} s(\varphi_i) &= \int_{\varphi_0}^{\varphi_i} \exp(-\int_{\psi_0}^{\psi_i} \frac{2\lambda(1+z_i)}{\mu^2 z_i^2} dz_i) d\psi_i \\ &= \int_{\varphi_0}^{\varphi_i} \exp\left(-\frac{2\lambda}{\mu^2}(-\frac{1}{z_i} + \ln|z_i|)\Big|_{\psi_0}^{\psi_i}\right) d\psi_i \\ &= \int_{\varphi_0}^{\varphi_i} \exp\left(\frac{2\lambda}{\mu^2\psi_i} - \frac{2\lambda}{\mu^2}\ln|\psi_i|\right) \exp\left(\frac{2\lambda}{\mu^2\psi_0} - \frac{2\lambda}{\mu^2}\ln|\psi_0|\right) d\psi_i \\ &= \int_{\varphi_0}^{\varphi_i} \exp\left(\frac{2\lambda}{\mu^2\psi_i} - \frac{2\lambda}{\mu^2}\ln|\psi_i|\right) d\psi_i \\ &= \int_{\varphi_0}^{\varphi_i} e^{\kappa/\psi_i}|\psi_i|^{-\kappa} d\psi_i \\ &= \int_{\varphi_0}^{\varphi_i} e^{\kappa/\psi_i}\psi_i^{-\kappa} d\psi_i \end{aligned} \tag{4.0.6}$$

for $i = 1,2$ and $\kappa := 2\lambda/\mu^2$ where the forth equality holds since ψ_0 is a constant and scale function is unique up to affine transformations. The fifth equality holds since the state space for the process is $[0,\infty)$. With the scale function, we see that the speed measure of the process Φ^i is given by

$$\begin{aligned} m(d\varphi_i) &= \frac{2}{\mu^2\varphi_i^2 s'(\varphi_i)} d\varphi_i \\ &= \frac{2}{\mu^2\varphi_i^2}\frac{1}{e^{\kappa/\varphi_i}\varphi_i^{-\kappa}} d\varphi_i \\ &= \nu/\varphi_i^2 e^{-\kappa/\varphi_i}\varphi_i^{\kappa} d\varphi_i \\ &= \nu\varphi_i^{-2+\kappa}e^{-\kappa/\varphi} d\varphi_i \end{aligned} \tag{4.0.7}$$

for $i = 1,2$ and $\nu := 2/\mu^2$. The invariant density function for the process Φ^i is given by

$$f_i(\varphi_i) = \frac{1}{m([0,\infty))} m(\varphi_i). \tag{4.0.8}$$

$$\begin{aligned} m([0,\infty)) &= \int_0^\infty \nu\varphi_i^{-2+\kappa}e^{-\kappa/\varphi_i} d\varphi_i \\ &= \int_0^\infty \nu\left(\frac{\kappa}{u_i}\right)^{-2+\kappa} e^{-u_i}\left(-\frac{\kappa}{u_i^2}\right) du_i \\ &= \nu\kappa^\kappa \int_0^\infty u_i^{\kappa-1}e^{-u_i} du \\ &= \nu\kappa^\kappa\Gamma(\kappa) \end{aligned} \tag{4.0.9}$$

where $u_i = \kappa/\varphi_i$ and $d\varphi_i = -\kappa/u_i^2 du$ for the change-of-variable in the second equality to hold. Substituting (4.0.9) into (4.0.8) we have the invariant density function of the process Φ^i explicitly stated as

$$f_i(\varphi_i) = \frac{1}{\kappa^\kappa \Gamma(\kappa)} \varphi_i^{-2+\kappa} e^{-\kappa/\varphi_i} \tag{4.0.10}$$

for $i = 1, 2$ where $\kappa := 2\lambda/\mu^2$. With invariant density function defined explicitly, we are left to find the transition density function, $p_i(t, \varphi)$, for the process Φ^i. [10] solved the problem in a specific case using the forward Kolmogorov equation. However, the general solution to find the transition density function of $p_i(t, \varphi)$ was left open for further consideration. With the explicit solution for the transition density function $p(t; \varphi_1, \varphi_2, \psi_1, \psi_2)$, one could use a Picard iteration method to find the numerical solution of the optimal stopping boundary $b = b(\varphi_1)$ and the first time for the process Φ^2 to hit $b(\Phi^1)$ is optimal. Upon knowing the optimal stopping time, one could engage in several strategies in the financial market to maximize wealth. One could engage in a pairs trading strategy by entering into a long position of the undrifted stock and a short position in the drifted stock after the optimal stopping time. The fundamental principle of pairs trading depends on the correlation between the two stocks and the assumption that the stock prices will converge back to a mean value. However, our problem only deals with two stocks modelled by independent exponential Brownian motions; and the drift μ is assumed to be there permanently. Therefore, this highlights the importance of further research for cases where the geometric Brownian motions are driven by correlated n-dimensional Brownian motions. Another potential research that can be derived from this is applying the settings in Merton's controlled diffusion problem to two risky assets in this case to find the optimal allocation.

3. In the multi-dimensional setting with multiple drift changes that we have discussed in Sections 2.3 and 3.3, the solution to the optimal stopping problem was solved by finding the unique solution $b(\varphi_1, \ldots, \varphi_{N-1})$ to the non-linear Fredholm integral equation (3.3.33). One could also apply Picard iteration by setting up an initial guess for the function b and reiterate for the numerical solution of the boundary function b that satisfies the properties proved in Proposition 8. With the optimal stopping boundary specified, the first time for the process Φ^N to hit the boundary $b(\Phi^1, \ldots, \Phi^{N-1})$ is optimal. Similar to the two-dimensional case, one could engage in multi-pair trading instead of monitoring only one pair of stocks in order to profit from the price spread.

Chapter 5

Conclusions

We make a conclusion to the dissertation at this point. Throughout, we have discussed the basics of optimal stopping problems and a specific application of it to the quickest detection of drift(s) in standard Brownian motions for three settings, one-dimensional, multi-dimensional with one drift, and multi-dimensional with multiple drifts. We assumed that at a random time θ, k out of n coordinate processes will be introduced a drift μ. In the one-dimensional case ($n = 1, k = 1$), we went through the standard method used in [8], focusing on the Mayer-Lagrange formulation of optimal stopping problem and the method used in Section 5 of [2], proposing a simplified Lagrange formulation under the measure $\mathbb{P}^\infty$. The multi-dimensional Brownian motion case with one drift can be simplified into a two-dimensional case without loss of generality ($n = 2, k = 1$). The result in Theorem 7 can be extended directly into higher dimensions. The ellipticity of the infinitesimal generator of the likelihood ratio process Φ breaks down when $1 < k < n-1$ in the multiple drift case. We use the hypoellipticity of the generator to proceed and formulated the solution. Both ellipticity in the case $k = 1$ and hypoellitpicity in the case $1 < k < n-1$ were used to set up the free-boundary problem to be solved. The solutions for the three cases can be summarized below

1. **Case 1 (1)** ($n = 1,\ k = 1$) The optimal stopping time is defined as $\tau_{A_*} = \inf\{t \geq 0 \mid \Pi_t \geq A_*\}$ where A_* is the solution to

$$V'(A_*) = e^{-\frac{\lambda}{\gamma}\alpha(A_*)}\left(-\int_0^{A_*} \frac{c}{\gamma}\frac{1}{\rho(1-\rho)^2} e^{\frac{\lambda}{\gamma}\alpha(\rho)} d\rho\right) = -1 \tag{5.0.1}$$

where $\gamma = \mu^2/2$ and the function α is defined in (3.1.10).

2. **Case 1(2)** ($n = 1,\ k = 1$) The optimal stopping time is defined as $\tau_* = \inf\{t \geq 0 \mid e^{\mu X_t + (\lambda - \frac{\mu^2}{2})t}\left(\frac{\pi}{1-\pi} + \lambda \int_0^t e^{-\mu X_s - (\lambda - \frac{\mu^2}{2})s} ds\right) \geq \varphi_*\}$ where φ_* is the solution to
$$\frac{\mu^2}{2c} = \frac{e^{\kappa(1+\varphi_*)/\varphi_*}}{\varphi_*^\kappa} \int_0^{\varphi_*/(1+\varphi_*)} \frac{b^{\kappa-1}}{(1-b)^{\kappa+2}} e^{-\kappa/b} db \tag{5.0.2}$$
where $\kappa = 2\lambda/\mu^2$.

3. **Case 2** ($n = 2,\ k = 1$) The optimal stopping time is defined as
$$\begin{aligned}\tau_* = \inf\Bigg\{ t \geq 0\ | e^{\mu X_t^2 + (\lambda - \frac{\mu^2}{2})t}\left(\frac{\pi}{1-\pi} + \lambda \int_0^t e^{-\mu X_s^2 - (\lambda - \frac{\mu^2}{2})s} ds\right) \geq \\ b\left(e^{\mu X_t^1 + (\lambda - \frac{\mu^2}{2})t}\left(\frac{\pi}{1-\pi} + \lambda \int_0^t e^{-\mu X_s^1 - (\lambda - \frac{\mu^2}{2})s} ds\right)\right)\Bigg\}\end{aligned} \tag{5.0.3}$$
where the function b is a unique solution to the non-linear Fredholm integral equation
$$\int_0^\infty e^{-\lambda t} K_b(t; \varphi_1, b(\varphi_1)) dt = 0 \tag{5.0.4}$$
where K_b is defined in (3.2.25).

4. **Case 3** ($1 < k < n - 1$) The optimal stopping time is defined as
$$\begin{aligned}\tau_* = \inf\Bigg\{ & t \geq 0 \mid e^{\mu \sum_{j=1}^k X_t^{n_j^N} + (\lambda - k\frac{\mu^2}{2})t}\left(\frac{\pi}{1-\pi} + \lambda \int_0^t e^{-\mu \sum_{j=1}^k X_s^{n_j^N} - (\lambda - k\frac{\mu^2}{2})s} ds\right) \\ & \geq b\Bigg(e^{\mu \sum_{j=1}^k X_t^{n_j^1} + (\lambda - k\frac{\mu^2}{2})t}\left(\frac{\pi}{1-\pi} + \lambda \int_0^t e^{-\mu \sum_{j=1}^k X_s^{n_j^1} - (\lambda - k\frac{\mu^2}{2})s} ds\right), \ldots, \\ & e^{\mu \sum_{j=1}^k X_t^{n_j^{N-1}} + (\lambda - k\frac{\mu^2}{2})t}\left(\frac{\pi}{1-\pi} + \lambda \int_0^t e^{-\mu \sum_{j=1}^k X_s^{n_j^{N-1}} - (\lambda - k\frac{\mu^2}{2})s} ds\right)\Bigg)\Bigg\}\end{aligned} \tag{5.0.5}$$
where the function b is a unique solution to the non-linear Fredholm integral equation
$$\int_0^\infty e^{-\lambda t} K_b(t; \varphi_1, \ldots, \varphi_{N-1} b(\varphi_1, \ldots, \varphi_{N-1})) dt = 0 \tag{5.0.6}$$
where K_b is defined in (3.3.32).

Throughout the dissertation, we have only considered the time horizon of the problem to be infinite. In a more realistic case, the problem would involve a finite time horizon for our observation. The problem of finite horizon has been discussed in [8] and [4] for one-dimensional case. The solution and work we have done in this dissertation shows that the finite horizon case for multidimensional case are also solvable with an extension to the differential operator to include time derivative. This extension introduces an additional level of complexity in terms of solving the partial differential equations and in the notation used. As we have discussed in Chapter 4, the case when the n-dimensional Brownian motion that drives the exponential Brownian motions, stock prices, are correlated was not discussed in the dissertation and is open for future analysis and research. Taking a two-dimensional case as an example, we could model 2 stock prices

$$dS^i = \mu_i S_t^i dt + \sigma_i S^i dB_t^i \tag{5.0.7}$$

for $i = 1, 2$ using $B = (B^1, B^2)$ linearly correlated as

$$dB_t^2 = \rho dB_t^1 + \sqrt{1-\rho^2} dW_t \tag{5.0.8}$$

where W_t is another independent standard Brownian motion and ρ is the correlation coefficient between the increments of the two Brownian motions. For $0 < \rho <= 1$, the two stocks are positively correlated and potentially in the same industry. This setting could be used as a more realistic case for pairs trading to be applied to maximize wealth.

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Appendix A

Programming code

1. The code for the simulation in Figure 4.1 is in listing A.1
2. The code for the simulation in Figure 4.2 is in listing A.2
3. The code for the simulation in Figure 4.3 is in listing A.3
4. The code for the simulation in Table 4 is in listing A.4

Listing A.1: Code for Figure 4.1

```
rng = np.random.default_rng(seed=0)
dB = rng.normal(0.0, np.sqrt(dt), size = K)
theta = rng.exponential(lambda_v)

time = np.linspace(0, T, K+1)

B = np.zeros(K+1)
B[0] = 0
for k in range(0, K):
    B[k] = B[k-1] + dB[k]

X = np.zeros(K+1)
dX = np.zeros(K)
for k in range(0,K):
    if time[k] < theta:
        dX[k] = dB[k]
    else:
        dX[k] = mu*dt + dB[k]
    X[k+1] = X[k] + dX[k]
plt.figure(figsize=(10, 6))
plt.plot(time, X, label="X")
plt.plot(time, B, label="B")
plt.axvline(x=theta, color='r', linestyle='--', label=r'$\theta$')
plt.legend()
plt.show()
```

Listing A.2: Code for Figure 4.2

```
PI_t = np.zeros(K+1)
PI_t[0] = pi
for k in range(0, K):
    PI_t[k+1] = PI_t[k] + lambda_v*(1-PI_t[k])*dt + mu*PI_t[k]*(1-PI_t[k
        ])*dB[k]

def alpha(pi):
    return np.log(pi/(1-pi)) - 1/pi
def integrand(rho, gamma, c, lambda_v, epsilon=1e-6):
    return (c / gamma) * (1 / (np.abs(rho + epsilon) * (1 - rho + epsilon
        )**2)) * np.exp(lambda_v / gamma * alpha(rho))
def V_pi(pi, gamma, c, lambda_v, epsilon=1e-6):
    integral_value, _ = QUAD(integrand, epsilon, pi, args=(gamma, c,
        lambda_v))
    return np.exp(-lambda_v / gamma * alpha(pi)) * (-integral_value)
def func_to_solve(pi, gamma, c, lambda_val):
    return V_pi(pi, gamma, c, lambda_val) + 1

gamma = mu**2/2

pi_initial_guess = 0.8
pi_solution = fsolve(func_to_solve, pi_initial_guess, args=(gamma, c,
    lambda_v))
pi_solution[0]

target = pi_solution[0]
index = np.argmax(PI_t >= target)
first_occurrence_time = time[index]
```

Listing A.3: Code for Figure 4.3

```
PHI_t = np.zeros(K+1)
integral_term = np.zeros(K+1)

for k in range(0, K):
    integrand = np.exp(-mu * X[:k] - (lambda_v - mu**2 / 2) * time[:k])
    integral_term[k] = np.sum(integrand) * dt
    PHI_t[k] = np.exp(mu * X[k] + (lambda_v - mu**2 / 2) * time[k]) * (
        phi + lambda_v * integral_term[k])

kappa = 2 * lambda_v / mu**2

def integrand(b, kappa):
    return (b**(kappa-1)) / ((1-b)**(kappa+2)) * np.exp(-kappa/b)

def equation_to_solve(varphi_star):
    integral_value, _ = QUAD(integrand, 0, varphi_star/(1 + varphi_star),
         args=(kappa,))
    left_hand_side = (mu**2) / (2 * c)
    right_hand_side = np.exp(kappa * (1 + varphi_star) / varphi_star) /
        varphi_star**kappa * integral_value
    return left_hand_side - right_hand_side

initial_guess = 0.5
solution = root(equation_to_solve, initial_guess)

varphi_star_solution = solution.x[0]

print(f"The solution for varphi_star is: {varphi_star_solution}")

# The target value
target_value = varphi_star_solution
first_index = np.argmax(PHI_t >= target_value)
first_time = time[first_index]
print("First time PHI_t reaches", target_value, "is at t =", first_time)
```

Listing A.4: Code for Table 4.1

```
def simulation(n):
    first_PI_list = np.zeros(n)
    first_PHI_list = np.zeros(n)
    PI_diff_list = np.zeros(n)
    PHI_diff_list = np.zeros(n)
    theta_list = np.zeros(n)
    for i in range(n):
        dB = rng.normal(0.0, np.sqrt(dt), size=K)
        theta = rng.exponential(lambda_v)
        theta_list[i]= theta
        B = np.zeros(K+1)
        B[0] = 0
        X = np.zeros(K+1)
        dX = np.zeros(K)
        PI_t = np.zeros(K+1)
        PI_t[0] = pi
        PHI_t = np.zeros(K+1)
        integral_term = np.zeros(K+1)
        for k in range(0, K):
            B[k] = B[k-1] + dB[k]
            if time[k] < theta:
                dX[k] = dB[k]
            else:
                dX[k] = mu*dt + dB[k]
            X[k+1] = X[k] + dX[k]

            PI_t[k+1] = PI_t[k] + lambda_v*(1-PI_t[k])*dt + mu*PI_t[k
                ]*(1-PI_t[k])*dB[k]
            integrand = np.exp(-mu * X[:k] - (lambda_v - mu**2 / 2) *
                time[:k])
            integral_term[k] = np.sum(integrand) * dt
            PHI_t[k] = np.exp(mu * X[k] + (lambda_v - mu**2 / 2) * time[k
                ]) * (phi + lambda_v * integral_term[k])
        # For PI
        index_PI = np.argmax(PI_t >= target_PI)
        first_PI = time[index_PI]
        first_PI_list[i] = first_PI
        PIdiff = first_PI - theta
        PI_diff_list[i] = PIdiff
        # For PHI
        index_PHI = np.argmax(PHI_t >= target_PHI)
        first_PHI = time[index_PHI]
        first_PHI_list[i] = first_PHI
        PHIdiff = first_PHI - theta
        PHI_diff_list[i] = PHIdiff
    # Calculate averages
    avg_first_PI = np.mean(first_PI_list)
    avg_first_PHI = np.mean(first_PHI_list)
    avg_PI_diff = np.mean(PI_diff_list)
    avg_PHI_diff = np.mean(PHI_diff_list)
    avg_theta = np.mean(theta_list)
    return avg_first_PI, avg_first_PHI, avg_PI_diff, avg_PHI_diff,
        avg_theta
```