Applied Analysis

for Engineering Sciences

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Contents

Preface

Throughout the history of civilization, mathematics has served as one of the major tools to analyze real world applications. In turn, through these applications it has been developed and expanded considerably. Moreover, mathematics helps establishing and consolidating the belief in eternal and exact truth, and hence the trust on sciences.

Since the invention of Calculus by Newton and Leibniz in the seventeenth century, mathematics has been overwhelmingly successful in almost every branch of sciences. It is instrumental for scientists and engineers to think, to work and to communicate.

We recall that Calculus is built fundamentally upon the definition of the real numbers. This definition naturally leads to the notion of limit. Two special and most useful limits are the derivative and the integral of a function. Most physical theories are described in terms of differential equations. Modern physics essentially started from Newton's theory of motions, and Newton's second law is a paradigm. The electro-magnetic theory is essentially the studies on the Maxwell equations. The theory of general relativity explores the Einstein equation, and the quantum mechanics uses the Schrödinger equation or the Wigner equation. We discuss the Lagrangian or Hamiltonian systems in mechanics, the biharmonic equation in elasticity, and the Navier-Stokes equations in fluid, etc.

The invention of electronic computers changed fundamentally the way for scientific research. Though we may not obtain analytical solution to a complex system in general, computer allows us to find the solution to a set of continuous differential equations in a discrete manner. Under certain circumstances, we even do not need to go to the continuous form. For instance, a fully discrete binomial algorithm may be used to compute the price of an option. We point out that instead of becoming a substitute for the continuous analysis, scientific computing reaches its best efficiency in real world applications only when we have a good understanding of the physics, the continuous modeling and analysis, the numerical algorithm, and the computer code.

This book is an outcome of an advanced course, conducted in English, for graduate students and senior undergraduate students at Department of Mechanics of Peking University. This course has been offered roughly every other year since 1998. We set forth the following objectives.

- To show some modern (1900-1990?) mathematical methods that are widely used in engineering sciences, nonlinear mechanics and other physical sciences.
- To help initiating research activities, namely, to boost ideas, to formulate the problem, and to explore the mathematics.
- To help bridging the gap between the mathematical tools and the physical understandings taught in other undergraduate courses.

A major ingredient of this course is nonlinearity. As is well known, superposition is the feature that distinguishes linear and nonlinear systems.

In linear algebra, we have

$$Ax_1 = y_1, \quad Ax_2 = y_2 \Rightarrow \quad A(x_1 + x_2) = y_1 + y_2.$$

The differential operator and integral operator are also linear.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\alpha f(x) + \beta g(x)) &= \alpha\frac{\mathrm{d}f}{\mathrm{d}x} + \beta\frac{\mathrm{d}g}{\mathrm{d}x}, \\ \int (\alpha f(x) + \beta g(x))\,\mathrm{d}x &= \alpha\int f(x)\mathrm{d}x + \beta\int g(x)\mathrm{d}x. \end{aligned}$$

Similarly, the Fourier transform and the Laplace transform are linear operators.

Superposition also applies to linear differential equations. For example, if both $x_1(t)$ and $x_2(t)$ are solutions to the equation $ax'' + bx' + cx = 0$, so is $\alpha x_1(t) + \beta x_2(t)$ for any constant α and β. As a matter of fact, the general solution is $x(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}$, where λ_1 and λ_2 are the roots to the quadratic equation $a\lambda^2 + b\lambda + c = 0$.

For instance, we consider the following RCL circuit in Fig. 1. A resistor obeys Ohm's law $V_R = RI$, while an inductor and a capacitor satisfy $V_L = L\frac{dI}{\mathrm{d}t}$ and $I = C\frac{\mathrm{d}V_C}{\mathrm{d}t}$, respectively. Kirchhoff's law gives rise to an integral-differential equation.

$$V = RI + L\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{1}{C}\int_0^t I(s)\mathrm{d}s + V_C(0).$$

We differentiate it once to obtain

$$L\frac{\mathrm{d}^2I}{\mathrm{d}t^2} + R\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{1}{C}I = 0.$$

If V varies along with time, the righthand side does not vanish. This circuit may generate electro-magnetic waves of a certain frequency.

As Einstein pointed out, the Laws of Nature cannot be linear. Linear system is usually a special case or a simplified version, therefore incomplete. Lots of important features of the real world can only be explained under the framework of nonlinear systems. Besides, linear problems are relatively simple, and mathematical tools we have learned before are fairly competent to handle them. We head for challenges and excitements through studies of nonlinear problems.

For an example of nonlinear system, we consider an oversimplified mechanical system consisted of the sun with mass M and the earth with mass m in Fig. 2. Let their positions be y and x, respectively. Newton's second law and the universal gravitation theory lead to the following coupled system,

$$\left\{\begin{aligned} m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= \frac{GmM}{|x-y|^2}, \\ M\frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= -\frac{GmM}{|x-y|^2}. \end{aligned}\right.$$

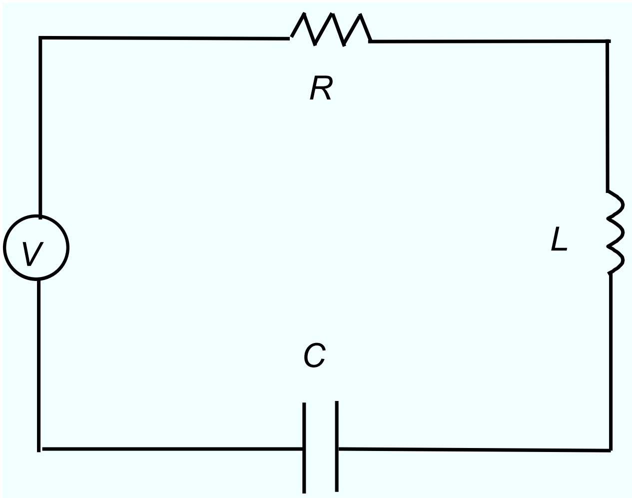

Figure 1: RCL circuit.

For another example, the Navier-Stokes equations for an incompressible fluid are as follows. Let $\mathbf{v}$ be the velocity, p the pressure, and Re the Reynolds number,

$$\begin{cases} \nabla \cdot \mathbf{v} & = & 0, \\ \mathbf{v}_t + (\mathbf{v}\nabla) \cdot \mathbf{v} + \nabla p & = & \dfrac{1}{\mathrm{Re}} \nabla^2 \mathbf{v}. \end{cases}$$

In this course, we shall mainly discuss the following topics.

In Chapter 1, we expose the qualitative theory for ODE systems (4 weeks of teaching). We start with some basic notions. Then we present a basic fixed point theory from functional analysis. This allows us to establish existence results for an ODE system. A further application is also illustrated, namely, iteration methods for solving a linear algebraic system. To understand qual-

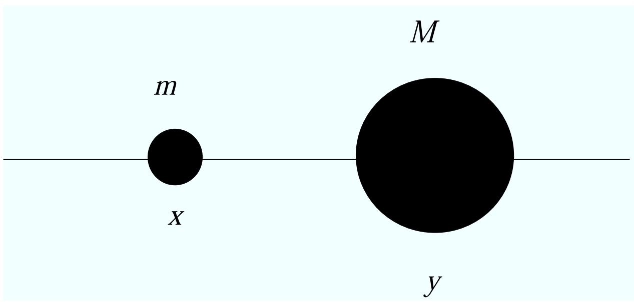

Figure 2: A one-dimensional two-body system of the sun and the earth.

itatively an ODE system, we analyze its critical points. For a second order ODE, the so-called plane analysis may provide substantial understanding. For a general system, there are not as many powerful tools. Stability analysis via the Lyapunov function is an exception. When there is a controlling parameter in a system, bifurcation may occur. We conclude this chapter by an exhibition of chaos in the Lorenz system and the logistic map.

For partial differential equations, we first study reaction-diffusion systems (3 weeks of teaching). We set up BVP (boundary-value problem) and IBVP (initial-boundary-value problem), and then show a simple example of instability at equilibrium. For a linearized problem, its dispersion relation gives a primary linear stability result. For nonlinear systems, an invariant domain approach sometimes works. This is a geometrical way to get *a priori* estimate. For a special example of nonlinear system, we illustrate a perturbation method for its steady states. Next, traveling wave analysis reduces a PDE system to an ODE system, and usually provides explanation to some wave behaviours of the PDE system. Only for very exceptional cases, a nonlinear PDE may be transformed to a linear one, e.g., Burgers' equation by the Cole-Hopf transform. We further illustrate a combination of theoretical and numerical investigations in an example of reaction-diffusion equation, namely, the evolutionary Duffing equation.

In Chapter 3, we discuss elliptic equations (2 weeks of teaching). the main topic is to introduce some basic ideas in the modern theories of partial differential equations. We start with generalized functions and weak derivatives, and introduce briefly the Sobolev spaces, and state the embedding theorem. Weak formulations and minimization procedure are used to establish existence results.

Chapter 4 is devoted to hyperbolic conservation laws (5 weeks of teaching). The most distinct feature of this type of PDE's lies in the inevitable appearance of discontinuities, regardless of smooth initial data. We show shock formation in inviscid Burgers' equation, by a characteristics approach. Then taking the Euler equations for polytropic gas as an example, we discuss the elementary waves, which include shock waves via vanishing viscosity approach, and rarefaction waves via self-similarity solution approach. For a general Riemann problem of gas dynamics, the unique composition of these elementary waves gives the solution, which is a weak one by construction. We further discuss solitons in the KdV equation, for which a brilliant theory of inverse scattering transform is sketched.

As this book is only an introduction of qualitative theories for ODE and PDE systems, further readings are suggested.

1. Smoller J, Shock Waves and Reaction-diffusion Equations. Springer, 1999.

2. Grindrod P, Patterns and Waves. Claredon, 1991.

3. Whitham G B, Linear and Nonlinear Waves. John Wiley & Sons, 1974.

4. Wang L, Wang M Q, Qualitative Analysis for Nonlinear Ordinary Differential Equations (in Chinese). Harbin Institute of Technology Press, 1987.

5. Huang Y N, Lecture Notes on Nonlinear Dynamics (in Chinese). Peking University Press, 2010.

6. Ding T R, Li C Z, A Course on Ordinary Differential Equations (in Chinese). Higher Education Press, 2004.

7. Ye Q X, Li Z Y, Introduction to Reaction-Diffusion Equations (in Chinese). Science Press, 1994.

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Chapter 1

Qualitative Theory for ODE Systems

1.1 Basic notions

For many applications, we describe a system using only one independent variable. A dependent quantity is expressed as a function of this independent variable. An ordinary differential equation (ODE) is an equation that contains an unknown function, called a state variable, together with its derivatives with respect to the single independent variable. For historical reasons, the independent variable is typically denoted as t, representing time. Depending on the applications, actually t may mean some other quantities, such as temperature, height, etc. The order of the highest derivative in an ODE is its order.

An ODE system, also called a dynamical system, is a set of ODE's. Typically each equation in this system is of first order. The order of the ODE system is the number of first order equations in the system.

A high-order ODE can always be recast to an ODE system of the same order. For instance,

$$x'' + xx' + x(1-x) + f(t) = 0 \tag{1.1}$$

can be rewritten as

$$\begin{cases} x' &= y, \\ y' &= -[xy + x(1-x) + f(t)]. \end{cases} \tag{1.2}$$

Therefore, a general ODE system reads

$$x' = f(t,x), \quad \text{with} \quad x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n. \tag{1.3}$$

The ODE system is autonomous if the righthand side depends only on the state variable, that is, $f(x,t) = f(x)$. A non-autonomous system can be trivially reshaped to an autonomous one. In fact, if we take $y = (t, x_1, \cdots, x_n)^T$, then we obtain

$$y' = \begin{pmatrix} 1 \\ f(y) \end{pmatrix}. \tag{1.4}$$

Through this procedure, the order of the system rises by one.

In this course, we are not concerned with a particular solution to an ODE system. Instead, we take a global view of all the solutions to a system, and for the aforementioned reason, an autonomous ODE system. These solutions form a family of (vector) functions. This family is the object for the qualitative theory.

If one such function $x(t)$ satisfies $x(t_0) = x_0$ for a certain time t_0, then we call it an orbit, or a trajectory passing through the point (t_0, x_0). These names reflect that we take a geometrical view. Sometimes we use the notation $x = \phi(t; t_0, x_0)$ to identify the orbit. In contrast, in the previous ODE course, one regards (t_0, x_0) as an initial point, and usually considers the solution $x = x(t)$ only for $t \geq t_0$. The geometrical name for this part of the solution is the positive semi-orbit, denoted by $x = \phi^+(t; t_0, x_0)$. Meanwhile, the solution for $t \in (-\infty, t_0]$ is called as the negative semi-orbit, and denoted as $x = \phi^-(t; t_0, x_0)$. Under such a geometrical view, we regard the function $x(t)$ equivalent to a curve in the space $\mathbb{R}^n$. This space is called a phase space, or a phase plane if $n = 2$.

We remark that sometimes one also specifies boundary data, namely n algebraic equations for n quantities selected from $x_1(a), \cdots, x_n(a), x_1(b), \cdots, x_n(b)$, when one looks for a solution in $t \in [a, b]$.

For an autonomous system, a translation in time is invariant. More precisely, if $x = \phi(t; 0, x_0)$ solves the system with initial data $x(0) = x_0$, then $x = \phi(t + t_0; t_0, x_0)$ solves the system with initial data $x(t_0) = x_0$. Therefore, it suffices to study the problem with initial data at one selected time, which is usually chosen as $t_0 = 0$.

We recall that the existence, uniqueness and continuous dependency hold for quite general cases, e.g., when the source term (righthand side) is continuous. Existence will be discussed later by means of a fixed point theory.

Qualitative theory is concerned with the global structure of trajectories in the phase space, instead of a particular solution for certain given initial data.

At each given point x, the source term introduces a vector $f(x)$ in the phase space. The direction of this vector determines the direction of the trajectory, and the absolute value determines how fast a solution takes to go through this point. We may imagine that there is a particle moving along the trajectory according to the vector field (velocity field). See Fig. 1.1.

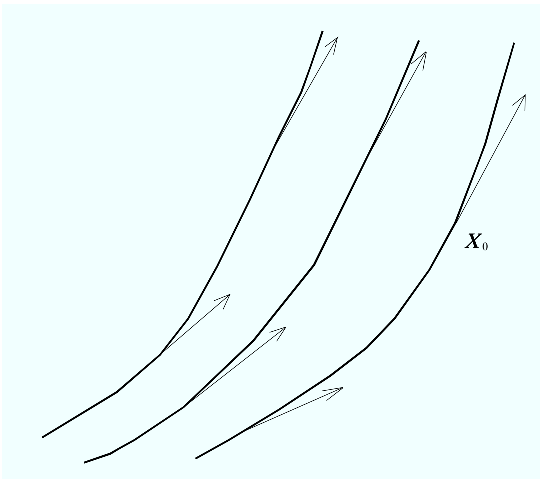


Figure 1.1: Trajectories, vector field and phase space (plane).

Noticing that at a point x where $f(x)$ vanishes, the previous statement become meaningless. This leads to the notion of a critical point (equilibrium point, singular point, stationary point, etc.), which turns out to be crucial in later discussions.

A point x is a critical point where $f(x) = 0$. It is a regular point if f is finite and non-zero. Two trajectories may intersect only at a critical point in the phase space. This can be proved by the uniqueness of solution to the following ODE system in the neighborhood of a regular point x^*. Assuming that $f_1(x) \neq 0$, we have

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{f_2(x)}{f_1(x)}, \cdots, \frac{\mathrm{d}x_n}{\mathrm{d}x_1} = \frac{f_n(x)}{f_1(x)}, \quad (x_2(x_1^*), \cdots, x_n(x_1^*) = (x_2^*, \cdots, x_n^*)). \tag{1.5}$$

Furthermore, a trajectory usually starts from/ends at a critical point or the infinity, or forms a closed orbit. Under certain circumstances, a chaotic orbit may appear.

In the subsequent sections, we shall discuss the local existence for ODE systems, using a fixed point theory. Then we shall perform detailed analysis in the vicinity of a critical point.

1.2 Local existence

The local existence for an ODE system may be proven through a Picard iteration procedure. We present here a systematic approach, using a fixed point theory from functional analysis.

1.2.1 Normed spaces and fixed point theorem

One of the fixed point theorems in Calculus is as follows. If a mapping $f : \mathbb{R} \to \mathbb{R}$ is a contraction, then there exists one and only one fixed point.

Here are some concepts involved. By a contraction, we mean that $\exists \alpha \in [0, 1)$, $\forall x, y \in \mathbb{R}$, it holds that $|f(x) - f(y)| \leq \alpha |x - y|$. A contractive function, is necessarily continuous. A point $x \in \mathbb{R}$ is called a fixed point if $f(x) = x$, i.e., f maps this point to itself. The proof for this type of theorems uses an iteration procedure.

Our interest goes beyond this simple situation. We intend to develop tools to analyze differential equations. For this purpose, we work in function spaces, and define contractive operators.

Space, similar to set, is a primitive concept. It consists of elements/points. In a space, different points are not related to each other in general, except that they are in the same space. A vector space is a space X with points (called vectors now) $x, y, \cdots$, in which the vector addition and the multiplication by scalars are defined. These two operations provide the algebraic structure for a space.

More precisely, vector addition is a binary operation that puts any two vectors x, y to another vector $z = x + y \in X$, called the sum of x and y. It is Abelian and associative, i.e., for any vectors $x, y, z \in X$, one has

$$x + y = y + x, \qquad \text{(Abelian)} \tag{1.6}$$

$$(x + y) + z = x + (y + z). \qquad \text{(associative)} \tag{1.7}$$

Moreover, there exists a zero vector $0 \in X$, which satisfies $0 + x = x, \forall x \in X$. For each $x \in X$, there is a unique $-x \in X$, and it holds that $x + (-x) = 0$.

Multiplication of a vector $x \in X$ by a scalar $\lambda \in \mathbb{R}$ defines a product $\lambda x \in X$. It satisfies the following rules,

$$\alpha(\beta x) = (\alpha\beta)x, \quad \forall \alpha, \beta \in \mathbb{R}, \forall x \in X; \tag{1.8}$$

$$1 \cdot x = x, \quad \forall x \in X; \tag{1.9}$$

$$\alpha(x + y) = \alpha x + \alpha y, \quad \forall \alpha \in \mathbb{R}, x, y \in X; \tag{1.10}$$

$$(\alpha + \beta)x = \alpha x + \beta x, \quad \forall \alpha, \beta \in \mathbb{R}, x \in X. \tag{1.11}$$

Same as in linear algebra, we call a given finite set $M = \{x_1, \cdots, x_n\}$ linearly independent if $\sum\limits_{i=1}^{n} \lambda_i x_i = 0$ implies that $\lambda_i = 0, i = 1, \cdots, n$. We call an infinite set M to be linearly independent if any finite subset of M is linearly independent.

A vector space X is r-dimensional, if any $(r + 1)$ vectors in this space are linearly dependent, while there exist r vectors which are linearly independent. An infinite dimensional space refers to a vector space that does not have a finite dimension, i.e., there exist n vectors which are linearly independent for any $n \in \mathbb{N}$.

We illustrate this with an example of $\mathbb{R}^2$. We know that the vectors $(1, 0)$ and $(0, 1)$ are linearly independent. On the other hand, any three vectors must be linearly dependent. In fact, if the first two vectors are linearly independent, then the third one can be expressed as a linear combination of the first two. See Fig. 1.2.

Now we introduce a geometric structure to a vector space. A norm is a function on a vector space X. That is, for each point $x \in X$, we denote its norm as $\|x\| \in \mathbb{R}$. This norm function satisfies the following four axioms.

1. $\|x\| \geq 0, \ \forall x \in X$;
2. $\|x\| = 0 \Leftrightarrow x = 0$;
3. $\|\alpha x\| = |\alpha| \|x\|, \ \forall \alpha \in \mathrm{R}, \forall x \in X$;
4. $\|x + y\| \leq \|x\| + \|y\|, \ \forall x, y \in X$. (triangular inequality)

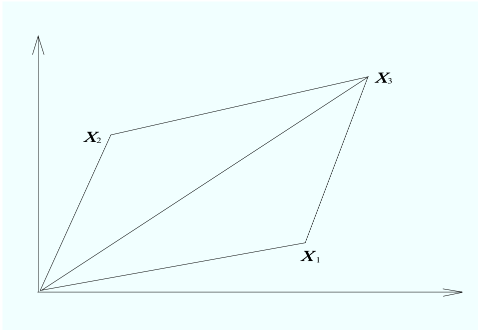


Figure 1.2: Linear dependency of three vectors in $\mathbb{R}^2$.

The distance between two vectors x and y is defined as $\|x - y\|$. We notice that when distance is defined, the definition of limit follows.

A normed space is a vector space with a norm. We remark that a vector space may take different norms, and the resulting normed spaces are different. In another word, only when both the algebraic structure and the geometric structure are the same, may we identify two normed spaces.

The Euclidean space $\mathbb{R}^n$ with $\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$ is a simple example. One may easily verify that this defines a normed space. Historically, people sometimes write $\mathbb{E}^n$ to emphasize that the norm is the Euclidean one. In the mean time, $\|x\|_{\max} = \max_{1\le i\le n} |x_i|$ also satisfies the axioms and defines another norm on $\mathbb{R}^n$. We remark that in case of finite dimensional spaces, it may be proved that different norms are equivalent. Moreover, to certain extent, an n-dimensional normed space is essentially $\mathbb{R}^n$. However, for infinite dimensional spaces, different norms make two normed spaces with the same underlying vector space different.

Now we are ready to describe function spaces. In particular, we consider the space $C[a, b]$, which contains all functions that are continuous over a closed interval $[a, b]$. Each function is a point in this space. The algebraic operations are defined as usual by

$$(f + g)(t) = f(t) + g(t), \quad (\alpha f)(t) = \alpha f(t). \tag{1.12}$$

This defines an infinite dimensional vector space. For instance, we consider a specific example of $C[0, 1]$. For any given integer n, $f_n(t) = t^n$ defines a point in $C[0, 1]$. Moreover, $f_1, \cdots, f_n$ are linearly independent.

For the geometric structure, we define

$$\|f\| = \max_{t\in[a,b]} |f(t)|. \tag{1.13}$$

Because $f \in C[a, b]$, this maximum is attained, and the norm is well defined. All four axioms are readily verified. This maximum norm is analogous to $\|x\|_{\max}$in $\mathbb{R}^n$.

Meanwhile, we may define an L^2 norm

$$\|f\|_2 = \left[\int_a^b |f(t)|^2 \mathrm{d}t\right]^{1/2}. \tag{1.14}$$

It also satisfies all axioms, and resembles the Euclidean norm for $\mathbb{R}^n$. However, we claim that under these two norms, the function spaces are different, from the perspective of completeness.

A point $x \in X$ is the limit of a sequence $\{x_n\} \subset X$, if $\lim\limits_{n\to\infty} \|x_n - x\| = 0$. In another word, we call $\lim\limits_{n\to\infty} x_n = x$ if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$, such that $\|x_n - x\| < \varepsilon$ provided $n > N$.

In Calculus, we have learned that a Cauchy sequence or fundamental sequence is a sequence $\{x_n\} \subset X$, for which $\forall \varepsilon > 0, \exists N \in \mathbb{N}$, such that $\|x_n - x_m\| < \varepsilon$ provided $n, m > N$. Moreover, it is proved that a Cauchy sequence is exactly a convergent sequence. That is, a Cauchy sequence converges; and a convergent sequence must be a Cauchy sequence.

In a normed space X, a convergent sequence must be a Cauchy sequence, by virtue of the triangular inequality. More precisely, if $\lim\limits_{n\to\infty} x_n = x$, then $\forall \varepsilon > 0, \exists N \in \mathbb{N}$, such that $\|x_n - x\| < \varepsilon$ provided $n > N$. Therefore, we have

$$\|x_n - x_m\| \le \|x_n - x\| + \|x_m - x\| < 2\varepsilon, \text{ for } n, m > N. \tag{1.15}$$

However, the other direction of deduction does not hold in general. If it holds, we call X a complete normed space, or a Banach space.

In particular, $\mathbb{R}^n$ is a Banach space. For $C[a, b]$, it turns out that different norms lead to different answers.

If we take the norm $\|f\| = \max\limits_{t\in[a,b]} |f(t)|$, we claim that the space $C[a, b]$ is complete.

Assume that there is a Cauchy sequence $\{f_n\} \subset C[a, b]$. Then $\forall \varepsilon > 0$, there exists $N \in \mathbb{N}$, such that $\|f_m - f_n\| < \varepsilon$, provided $m, n > N$.

This implies that at each fixed $t \in [a, b]$, $|f_m(t) - f_n(t)| < \varepsilon$. Therefore, a real number sequence $\{f_n(t)\} \subset \mathbb{R}$ is Cauchy. Since $\mathbb{R}$ is complete, this sequence is convergent, and we denote the limit as

$$f_0(t) \equiv \lim_{n\to\infty} f_n(t). \tag{1.16}$$

When $t \in [a, b]$ varies, a function $f_0(t)$ is obtained.

If $f_0(t) \in C[a, b]$, we take the limit $n \to \infty$ in $|f_m(t) - f_n(t)| < \varepsilon$, and obtain

$$\|f_m(t) - f_0(t)\| \le \varepsilon, \text{ for } m > N. \tag{1.17}$$

This proves that under the maximum norm in $C[a,b]$, it holds $\lim_{n\to\infty} f_n = f_0$.

Finally, we verify that $f_0(t)$ lies in $C[a,b]$.

In fact, f_{N+1} is continuous in the closed interval $[a,b]$, so it is uniformly continuous. That is, for the above $\varepsilon > 0$, $N \in \mathbb{N}$, and any $t_1, t_2 \in [a,b]$, there exists $\delta > 0$, such that

$$|f_{N+1}(t_1) - f_{N+1}(t_2)| < \varepsilon, \text{ provided } |t_1 - t_2| < \delta. \tag{1.18}$$

This leads to

$$\begin{aligned} & |f_0(t_1) - f_0(t_2)| \\ \leq \; & |f_0(t_1) - f_{N+1}(t_1)| + |f_{N+1}(t_1) - f_{N+1}(t_2)| + |f_{N+1}(t_2) - f_0(t_2)| \\ < \; & 3\varepsilon. \end{aligned}$$

We conclude that $C[a,b]$ with the aforementioned norm is a Banach space.

In contrast, if we take the L^2 norm, $C[a,b]$ is no longer complete. To see this, we construct a sequence of continuous functions,

$$f_n(t) = \begin{cases} 1, & \text{if } t - (a+b)/2 < -1/n. \\ -n\left[t - (a+b)/2\right], & \text{if } |t - (a+b)/2| \leq 1/n, \\ -1, & \text{if } t - (a+b)/2 > 1/n. \end{cases} \tag{1.19}$$

This is a Cauchy sequence, because

$$\|f_n - f_m\|_2^2 = \int_a^b (f_n(t) - f_m(t))^2 \mathrm{d}t = \frac{2(m-n)^2}{3mn\max\{m,n\}} \leq \frac{2}{3N}, \text{ if } m,n > N. \tag{1.20}$$

However, the pointwise limit defines a function

$$f_0(t) = \begin{cases} 1, & \text{if } t < (a+b)/2, \\ 0, & \text{if } t = (a+b)/2, \\ -1, & \text{if } t > (a+b)/2. \end{cases} \tag{1.21}$$

It is discontinuous. This shows that $C[a,b]$ is not complete under the L^2 norm. See Fig. 1.3.

Besides the completeness for a normed space, a fixed point theory works for a mapping, which is now called an operator. In fact, a function is a mapping from $\mathbb{R}$ (or its subspace) to $\mathbb{R}$. Similarly, an operator is a mapping from a normed space X to another normed space Y,

$$\begin{aligned} T : X &\to Y \\ x &\mapsto y = Tx. \end{aligned} \tag{1.22}$$

A simple example is a linear transform, which is represented by a matrix.

$$\begin{aligned} T : \mathbb{R}^m &\to \mathbb{R}^n \\ x &\mapsto y = Ax, \end{aligned} \tag{1.23}$$

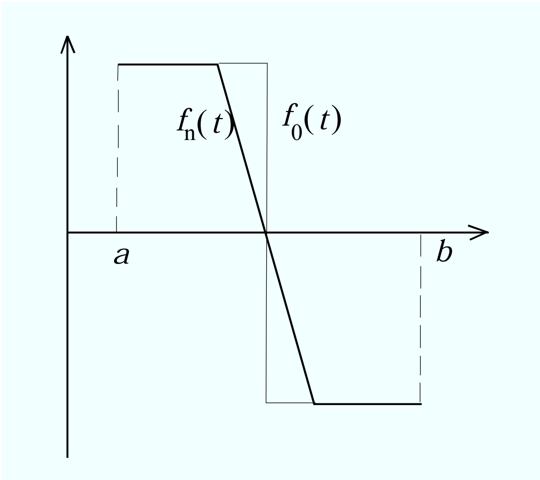


Figure 1.3: Limit of a function sequence in $C[a, b]$ with the L^2 norm.

where A is an $n \times m$ matrix.

In particular, if $Y = \mathbb{R}$, then $f : X \to \mathbb{R}$ is a functional. For special cases when X is a function space, then T is actually a function of functions. An example is the definite integral operator,

$$\begin{aligned} f : C[a, b] &\to \mathbb{R} \\ x(t) &\mapsto \int_a^b x(t)\mathrm{d}t. \end{aligned} \tag{1.24}$$

Consider an operator from a normed space X to itself. It is contractive if $\exists \alpha \in (0, 1)$, such that $\|Tx - Ty\| \le \alpha \|x - y\|$ for $\forall x, y \in X$. A fixed point of T is $x \in X$ such that $Tx = x$.

The Banach fixed point theorem is as follows.

Theorem 1.2.1. *If $T : X \to X$ is a contraction on a Banach space $X \neq \Phi$, then there exists exactly one fixed point $x^* \in X$.*

The proof is done by iteration.

Step 1. We take an arbitrary point $x_0 \in X$, and let $x_n = Tx_{n-1}$ for $n \in \mathbb{N}$.

Step 2. We show that $\{x_n\}$ is a Cauchy sequence.

Because T is a contraction, there is $\alpha \in (0, 1)$, and

$$\|x_{n+1} - x_n\| \le \alpha \|x_n - x_{n-1}\| \le \cdots \le \alpha^n \|Tx_0 - x_0\|. \tag{1.25}$$

This yields

$$\|x_{n+p} - x_n\| \le \left(\alpha^{n+p-1} + \cdots + \alpha^n\right) \|Tx_0 - x_0\| \le \frac{\alpha^n}{1-\alpha} \|Tx_0 - x_0\|. \tag{1.26}$$

So, $\{x_n\}$ is a Cauchy sequence.

Step 3. We find the fixed point. The space X is complete, therefore $\{x_n\}$ converges. Denote $\lim_{n\to\infty} x_n = x^*$.

Step 4. We verify that x^* is a fixed point.

$$\|Tx^* - x^*\| \leq \|Tx^* - Tx_n\| + \|x_{n+1} - x^*\| \leq \alpha \|x^* - x_n\| + \|x_{n+1} - x^*\| \to 0. \tag{1.27}$$

This shows that $Tx^* = x^*$.

Step 5. Finally we show the uniqueness. Assume that there is another fixed point x'. Then it holds that $\|Tx^* - Tx'\| = \|x^* - x'\|$. This contradicts with the fact that T is a contraction.

From this proof, we already have an error estimate

$$\|x_n - x^*\| \leq \frac{\alpha^n}{1-\alpha} \|Tx_0 - x_0\|. \tag{1.28}$$

The theorem actually holds for a closed subspace of a Banach space. Let X be a Banach space equipped with a norm $\|\cdot\|$, and $Y \subset X$ is a closed subspace. By a subspace, we mean that Y is a subset of X. Y inherits from X the norm hence the geometrical structure. By a closed subspace we mean that the limit of any convergent sequence $\{y_n\}_{n=1}^{\infty} \subset Y$(converge in X) converges to a point in Y. Furthermore, a Cauchy sequence $\{y_i\} \subset Y$ is naturally a Cauchy sequence in X. Because X is complete, this sequence converges to certain $y_0 \in X$. Because Y is closed, therefore we have $y_0 \in Y$. Because the norms are the same, $\lim_{n\to\infty} y_n = y_0$ in X means exactly $\lim_{n\to\infty} y_n = y_0$ in Y. This proves that a closed subspace of a Banach space is complete.

We further note that if Y is closed for vector addition and multiplication by scalars, i.e., $\forall y_1, y_2 \in Y, \lambda \in \mathbb{R}$, it holds $y_1 + y_2 \in Y, \lambda y_1 \in Y$, then Y is a vector space, and a Banach space. In extending the Banach fixed point theorem, however, the sub-vectorspace is not necessary.

Checking the five steps in the proof of the Banach fixed point theorem, we may obtain the unique fixed point in Y through the iteration procedure.

1.2.2 Applications to ODE system and linear algebraic system

Now we apply this theorem to an ordinary differential equation

$$x'(t) = f(t, x), \quad x(t_0) = x_0. \tag{1.29}$$

Here we assume that $f(t, x)$ is bounded and Lipschitz with respect to x, namely, there are $c, k > 0$, such that

$$|f(t, x)| \leq c, \quad |f(t, x) - f(t, y)| \leq k |x - y|. \tag{1.30}$$

Taking a time interval $J = [t_0 - \beta, t_0 + \beta]$, we consider the Banach space $C(J)$. We further restrict to a subspace, in the geometrical sense,

$$\tilde{C}(J) = \{ x(t) \in C(J) | \, |x(t) - x_0| \leq c\beta \} . \tag{1.31}$$

It is straightforward to show that $\tilde{C}(J)$ is a closed subspace. Now we construct an operator $T : \tilde{C}(J) \to \tilde{C}(J)$ by

$$Tx(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau)) \mathrm{d}\tau. \tag{1.32}$$

First, we verify $Tx \in \tilde{C}(J)$. In fact, we have

$$|Tx(t_1) - Tx(t_2)| = \left| \int_{t_1}^{t_2} f(\tau, x(\tau)) \mathrm{d}\tau \right| \leq c|t_1 - t_2|. \tag{1.33}$$

In particular, we have

$$|Tx(t) - x_0| = \left| \int_{t_0}^{t} f(\tau, x(\tau)) \mathrm{d}\tau \right| \leq c\, |t - t_0| \leq c\beta. \tag{1.34}$$

Secondly, we check if it is a contraction. Let $x, y \in \tilde{C}(J)$, we have

$$|Tx(t) - Ty(t)| = \left| \int_{t_0}^{t} [f(\tau, x(\tau)) - f(\tau, y(\tau))] \mathrm{d}\tau \right| \leq k\beta \, \|x - y\| . \tag{1.35}$$

If we take $\beta < 1/k$, then T is a contraction.

Now by the Banach fixed point theorem, there is a unique fixed point $x(t) \in \tilde{C}(J)$ for T, namely,

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau)) \mathrm{d}\tau. \tag{1.36}$$

This gives a solution to the original ODE. We notice that the meaning of a solution needs generalization at a time t^* if $f(t, x(t))$ is discontinuous with respect to t at t^*.

The fixed point theorem also provides a way to solve the ODE by iteration. In fact, taking any $x_0(t) \in \tilde{C}(J)$, we may iterate with

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(\tau, x_n(\tau)) \mathrm{d}\tau. \tag{1.37}$$

The limit gives the fixed point, and therefore the solution. This is called the Picard iteration.

This result may be directly extended to the case of an ODE system.

The fixed point theorem has more general applications. For instance, we may adopt it to solve linear algebraic equations. Consider the space $\mathbb{R}^n$, with a norm

$$\|x\| = \max_{1 \leq j \leq n} |x_j| . \tag{1.38}$$

A linear mapping T takes the form of $Tx = Ax + b$, where $A = (a_{ij})$, and $b = (b_j)$.

$$\|Tx - Ty\| = \|A(x-y)\| = \max_{1\le j\le n}\left|\sum_{k=1}^{n} a_{jk}(x_k - y_k)\right| \le \|x-y\| \max_{1\le j\le n}\sum_{k=1}^{n}|a_{jk}|. \tag{1.39}$$

Therefore, if a row sum criterion $\max_{1\le j\le n}\sum_{k=1}^{n}|a_{jk}| < 1$ is satisfied, then T is a contraction, and a unique fixed point may be obtained through iteration. The fixed point solves a linear equation $(I - A)x = b$.

Consider a linear algebraic system $Cx = d$. We reformulate it with $C = E - F$. Thus we have $Ex = Fx + d$, or $x = E^{-1}(Fx + d)$. So, we shall take $A = E^{-1}F$, and $b = E^{-1}d$.

The key issue then becomes the decomposition of C. To form a good iteration, there are three aspects one needs to consider. First, E should be easy to invert. Secondly, $E^{-1}F$ should be a contraction. Thirdly, a small contraction constant α is preferred.

Let us describe the Jacobian iteration by taking $E = \mathrm{diag}(c_{ii})$, which is the diagonal part of C. To use this method, we require that the diagonal entries dominate corresponding rows, namely,

$$\sum_{k=1,k\ne j}^{n}|c_{jk}| < |c_{jj}|, \quad \forall j = 1,\cdots,n. \tag{1.40}$$

This immediately leads to $\max_{1\le j\le n}\sum_{k=1}^{n}|a_{jk}| < 1$, and the operator T is a contraction. The iteration method actually can be written explicitly, since E is easily inverted,

$$x_j^{(m+1)} = \frac{1}{c_{jj}}\left(d_j - \sum_{k=1,k\ne j}^{n} c_{jk}x_k^{(m)}\right). \tag{1.41}$$

As an example, we find the formulation of the Jacobian iteration for the following matrix,

$$C = \begin{bmatrix} 10 & 5 & 4 \\ 7 & 10 & 2 \\ 1 & 3 & 10 \end{bmatrix}. \tag{1.42}$$

It is a diagonally dominated matrix. We take for the Jacobian iteration,

$$E = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & -5 & -4 \\ -7 & 0 & -2 \\ -1 & -3 & 0 \end{bmatrix}. \tag{1.43}$$

Then the contractive matrix is

$$A = E^{-1}F = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{2}{5} \\ -\frac{7}{10} & 0 & -\frac{1}{5} \\ -\frac{1}{10} & -\frac{3}{10} & 0 \end{bmatrix}. \tag{1.44}$$

We remark that there are other ways to decompose the matrix C and form an iteration. For instance, we may take the upper triangular sub-matrix as E, and the lower triangular matrix as F, respectively,

$$E = \begin{bmatrix} 10 & 5 & 4 \\ 0 & 10 & 2 \\ 0 & 0 & 10 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 & 0 \\ -7 & 0 & 0 \\ -1 & -3 & 0 \end{bmatrix}. \tag{1.45}$$

1.3 Critical point

As a trajectory goes locally tangent to the vector field around a regular point, the critical points play an important role in the whole picture of the solutions to an autonomous ODE system. Understanding of critical points helps us capturing the key features of the trajectories.

Around a critical point x_0, we perform Taylor expansion as follows.

$$f(x) = \nabla f(x_0) \cdot (x - x_0) + o(|x - x_0|). \tag{1.46}$$

The Jacobian matrix $A = \nabla f(x_0)$, which is assumed to be invertible, determines the topological structure of trajectories near the critical point. From linear algebra, we know that A is similar to its Jordan normal form. That is, there is an invertible matrix P, such that $PAP^{-1} = B$, where B consists of diagonal entries, as well as Jordan blocks. Now let $y = P(x - x_0)$, the governing equation for y is

$$y' = By + o(|y|). \tag{1.47}$$

While the invertible matrix P may correspond to a combination of rotation, reflection and dilation, the trajectories of the y-system are topologically the same as the x-system. Moreover, it may be proved that except for certain degenerate cases, the topological structure of the trajectories is the same as the linearized system

$$f(x) = \nabla f(x_0) \cdot (x - x_0), \text{ or } y' = By. \tag{1.48}$$

For the sake of clarity, we confine ourselves to the case of a second order ODE system, that is, A and B are 2 by 2 matrices. For the linearized problem (1.48), there are following possibilities. See Fig. 1.4.

- If $B = \begin{pmatrix} \lambda & \\ & \mu \end{pmatrix}$ with $\lambda > \mu > 0$, then the critical point is a source (unstable node), where all trajectories leave x_0. The case with $\mu > \lambda > 0$ is also a source, with a interchanged role of y_1 and y_2.

  The general solution is $y = \begin{pmatrix} c_1 e^{\lambda t} \\ c_2 e^{\mu t} \end{pmatrix}$, where c_1, c_2 are arbitrary constants. For $c_1 = 0$, the trajectory is a straight line $y_1 = 0$. For $c_1 \neq 0$, a solution may be expressed in the phase plane as $y_2 = c y_1^{\mu/\lambda}$. In a degenerate or special case $\mu = \lambda$, the trajectories are straight lines. Otherwise, they are curves. A representative case is $\mu = 2\lambda$, for which we have parabolas. Finally, we need to check the direction in a trajectory. Noticing that the absolute values of y_1, y_2 increase as time evolves, each trajectory points outward from the origin.

  After we obtain a trajectory in terms of y, we may draw it in the x plane by rotation and distortion, according to the inverse transform of $y = P(x - x_0)$.

- If $B = \begin{pmatrix} \lambda & \\ & \mu \end{pmatrix}$ with $\lambda < \mu < 0$, then the critical point is a sink (stable node), where all trajectories go towards x_0.

  The solution takes the same form as the previous case. Moreover, we may consider the time-reversed system with $\tilde{t} = -t$. The critical point then becomes a source. Hence a trajectory remains the same, but with opposite direction.

  We remark that the case with $\mu < \lambda < 0$ is similar.

- If $B = \begin{pmatrix} \lambda & \\ & \mu \end{pmatrix}$ with $\lambda > 0 > \mu$, then the critical point is a saddle, where all trajectories go towards and then leave away from x_0. The solution takes the previous exponential form. We remark that the case with $\mu > 0 > \lambda$ is similar.

- If $B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ with $a > 0$, then the critical point is an unstable focus, where all trajectories wind away from x_0. The solution is $y = \begin{pmatrix} c_1 e^{at} \cos bt + c_2 e^{at} \sin bt \\ c_2 e^{at} \cos bt - c_1 e^{at} \sin bt \end{pmatrix}$ for certain coefficients c_1 and c_2.

- If $B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ with $a < 0$, then the critical point is a stable focus, where all trajectories wind towards x_0. The solution takes the same form as that for the unstable focus.

- If $B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ with $a = 0$, then the critical point is a center, where all trajectories wind around x_0. The solution is $y = \begin{pmatrix} c \sin b(t - t_0) \\ c \cos b(t - t_0) \end{pmatrix}$.

When we say toward/away from, we refer to the direction of the trajectory (particle) when time t increases. In the figures, we plot arrows along the direction of increasing t. It is also clear from

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f_i(x) \tag{1.49}$$

that it takes infinite time to enter an unstable critical point, or leave a stable one.

The above classification is not exclusive. There are degenerate cases. For instance, we consider the following situation,

$$B = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}. \tag{1.50}$$

It may be shown that the critical point is a stable node if $a < 0$, and an unstable node if $a > 0$. The trajectories look qualitatively like those for the sources and sinks.

For a nonlinear system, it is not easy to obtain a global picture. However, it may be shown that around a critical point except for the center, the structure keeps the same as the linearized system. We also mention that the degenerate cases, such as $\lambda = \mu$, are of particular interest when we study the transition from one structure to the other for the geometry of the trajectories. We shall precise this point later.

The local behavior for a nonlinear system may be explored by linearization. Consider the following ODE system,

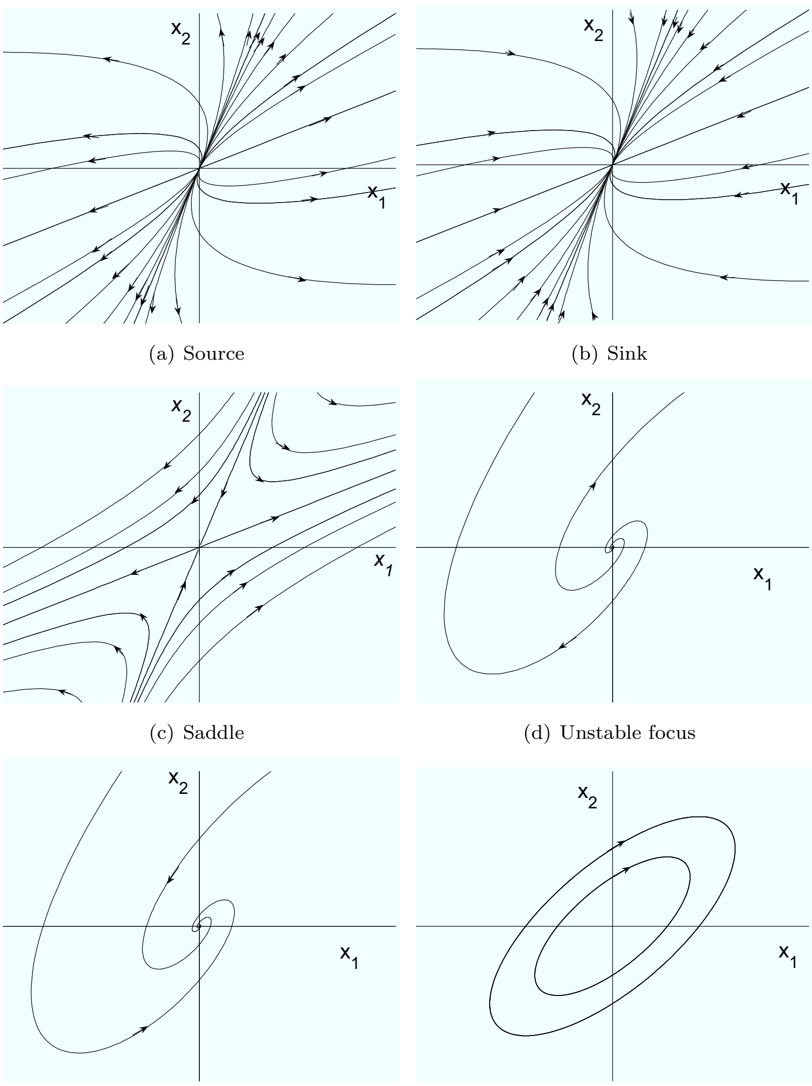
$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}. \tag{1.51}$$

First, we find (x_0, y_0) satisfying

$$\begin{cases} f(x_0, y_0) = 0, \\ g(x_0, y_0) = 0. \end{cases} \tag{1.52}$$

Secondly, we perform Taylor expansion as follows,

$$\begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} = \nabla_{(x,y)} \begin{pmatrix} f \\ g \end{pmatrix} (x_0, y_0) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + o\left(|x - x_0|, |y - y_0|\right). \tag{1.53}$$



(a) Source (b) Sink

(c) Saddle (d) Unstable focus

(e) Stable focus (f) Center

Figure 1.4: Critical points for second order ODE systems.

Thirdly, we let $A = \nabla_{(x,y)} \begin{pmatrix} f \\ g \end{pmatrix} (x_0, y_0) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$ and $\begin{cases} \tilde{x} = x - x_0 \\ \tilde{y} = y - y_0 \end{cases}$. Then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = A \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}. \tag{1.54}$$

Fourthly, we find an invertible matrix P, such that $PAP^{-1} = B$, where B consists of diagonal entries, as well as Jordan blocks. Then we let $\begin{pmatrix} u \\ v \end{pmatrix} = P \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$. This gives a canonical form

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} u \\ v \end{pmatrix} = B \begin{pmatrix} u \\ v \end{pmatrix}. \tag{1.55}$$

With the canonical form, the structure of the trajectories around a critical point can be analyzed according to the previous classification.

1.4 Plane analysis for the Duffing equation

Trajectories of a second order autonomous system lie in the phase plane $\mathbb{R}^2$.

In a linear system, there is either only one critical point, or a line of critical points. Closed orbits exist around a center, with the same period. Complexity may arise in a nonlinear system. First of all, there can be more than one isolated critical points. Secondly, it is possible that a trajectory connects two critical points, or forms a loop from and toward the same critical point. The former is a heteroclinic orbit, and the latter a homoclinic one.

We consider an autonomous system

$$\begin{cases} x' &= f(x,y), \\ y' &= g(x,y). \end{cases} \tag{1.56}$$

Eliminating the time variable by dividing the two equations, we easily obtain the equation that governs the trajectories in the phase plane,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x,y)}{f(x,y)}. \tag{1.57}$$

In many important cases, this equation can be readily integrated. This may provide substantial understanding to the underlying physical system.

As an example, we perform plane analysis to the following Duffing equation,

$$u - u^3 + u'' = 0. \tag{1.58}$$

This equation arises in many applications. For instance, we consider the Lagrangian that describes a particle motion in a double-humped potential field,

$$L = \frac{\dot{u}^2}{2} - \left(\frac{u^2}{2} - \frac{u^4}{4}\right). \tag{1.59}$$

The dynamics is governed by the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{u}}\right) = \frac{\partial L}{\partial u} = -u + u^3. \tag{1.60}$$

Or, equivalently,

$$u'' = -u + u^3. \tag{1.61}$$

This gives the Duffing equation.

Another derivation of this motion is to express the corresponding total energy as a Hamiltonian, with u the displacement and v the velocity (momentum).

$$H(u, v) = \frac{v^2}{2} + \left(\frac{u^2}{2} - \frac{u^4}{4}\right). \tag{1.62}$$

The Hamilton equation reads

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial H}{\partial v} = v, \quad \frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{\partial H}{\partial u} = -u + u^3. \tag{1.63}$$

Substituting the first equation into the second one, we obtain the Duffing equation.

In the following we present two ways to explore the Duffing equation.

First, we recast it into the ODE system (1.63). There are three critical points, i.e., $(u_+, 0) = (1, 0)$,$(u_0, 0) = (0, 0)$ and $(u_-, 0) = (-1, 0)$.

At a critical point $(u^*, 0)$, the Jacobian matrix is

$$J(u^*, v^*) = \begin{bmatrix} 0 & 1 \\ -1 + 3(u^*)^2 & 0 \end{bmatrix}. \tag{1.64}$$

An eigenvalue λ satisfies

$$\lambda^2 + (1 - 3(u^*)^2) = 0. \tag{1.65}$$

The eigenvector is found to be $(1, \lambda)^T$.

At $(0, 0)$, the eigenvalues are $\pm i$. Linear theory suggests that it is a center. However, for a nonlinear system, a trajectory at the vicinity of such a critical point may not close in general due to the nonlinear terms. It hence becomes a focus, either stable or unstable. We refer to this change of critical point type due to nonlinearity as the structural instability. For the current system, however, we observe there is a symmetry $(u, v; t) \to (u, -v; -t)$. More precisely, if $(u(t), v(t))$ is a solution, so is $(u(-t), -v(-t))$. Therefore, if there is a segment in the upper half plane $v \geq 0$, there must be a symmetric one in the lower half plane $v \leq 0$. When this segment has two intersection points with the u-axis, a closed orbit is formed. Hence the critical point $(0, 0)$ is actually a center.

Actually, we further notice another symmetry for this system $(u, v; t) \to (-u, v, -t)$. That is, if $(u(t), v(t))$ is a solution, so is $(-u(-t), v(-t))$. Therefore, the trajectories have symmetry not only with respect to the u-axis, but also to the v-axis.

Moreover, from $u' = v$, we know that along a trajectory u increases in the upper half plane, and decreases in the lower half plane. This helps determining the time direction along a trajectory. In particular, as $u' = 0$ on the v-axis, all trajectories are locally perpendicular to it.

Next, at $(1, 0)$, the eigenvalues are $\pm\sqrt{2}$. The eigenvectors are$(1, \sqrt{2})^T$ and $(1, -\sqrt{2})^T$, respectively. This means that the stable and unstable manifolds emanate from the saddle with an angle of $\pm \arctan \sqrt{2}$. To see this, we perform a linearization around this saddle point as follows.

Let $(\tilde{u}, \tilde{v}) = (u - u_+, v - v^*)$. Taylor expansion for the source term around $(1, 0)$ gives

$$\begin{pmatrix} v \\ -u + u^3 \end{pmatrix} = J(u_+, v^*) \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} + o(\|(\tilde{u}, \tilde{v})\|). \tag{1.66}$$

Therefore the linearized equations read

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = J(u_+, v^*) \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}. \tag{1.67}$$

Now we take a transform matrix

$$P = \begin{bmatrix} 1 & 1 \\ \sqrt{2} & -\sqrt{2} \end{bmatrix}. \tag{1.68}$$

It holds that

$$JP = P \begin{bmatrix} \sqrt{2} & \\ & -\sqrt{2} \end{bmatrix}. \tag{1.69}$$

Therefore, taking a new set of variables

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = P^{-1} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}, \tag{1.70}$$

we obtain a decoupled system

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{bmatrix} \sqrt{2} & \\ & -\sqrt{2} \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \tag{1.71}$$

This system is readily solved and leads to

$$\begin{pmatrix} u \\ v \end{pmatrix} \approx \begin{pmatrix} u_+ \\ v^* \end{pmatrix} + C_1 e^{\sqrt{2}t} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + C_2 e^{-\sqrt{2}t} \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}. \tag{1.72}$$

The particular solution with $C_2 = 0$ corresponds to the unstable manifold of $(1, 0)$, which is along the direction of $(1, \sqrt{2})$ in the phase plane. Similarly, the particular solution with $C_1 = 0$ corresponds to the stable manifold of $(1, 0)$, along the direction of $(1, -\sqrt{2})$.

Because of symmetry, similar arguments apply to $(-1, 0)$.

Furthermore, still due to the symmetry, as the segment of unstable manifold from $(-1, 0)$ intersects the v-axis, there is a segment that is symmetric with respect to the v-axis and goes precisely toward the critical point $(1, 0)$ along its stable manifold. Therefore, we obtain a trajectory that starts from $(-1, 0)$ and ends up at $(1, 0)$. This is a heteroclinic orbit, which means the trajectory connects two different critical points.

The symmetry with respect to the u-axis shows that yet another heteroclinic orbit exists in the lower half plane. These two orbits separate the phase plane into two regions. Inside them, all trajectories are closed orbits. No closed orbit exists outside of them. They form a separatrix.

The trajectories are schematically summarized in Fig. 1.5.

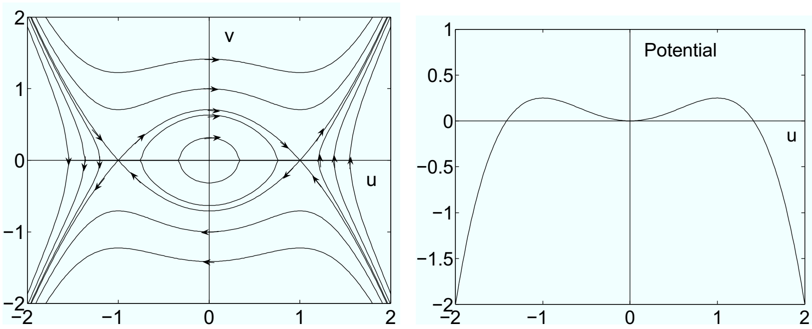
There is an alternative way to analyze the Duffing equation in a more quantitative manner. We multiply the equation by u' and then integrate once to get

$$u^2 - u^4/2 + (u')^2 = C. \tag{1.73}$$

Here $C = 2H$ is an arbitrary constant, representing twice the total energy. We first draw the potential in the right subplot of Fig. 1.5. The maximal potential energy is $1/4$, attained at $u = \pm 1$. A local minimal energy $C = 0$ is reached at $u = 0$. Other two displacements that also reach the same total energy are $u = \pm\sqrt{2}$.

For a total energy larger than $1/4$, the kinetic energy (hence the absolute value of velocity) increases in the intervals $(-1, 0) \bigcup (1, +\infty)$, and decreases in $(-\infty, -1) \bigcup (0, 1)$. Such a whole trajectory, both the positive and negative segments, may reach infinity from both sides.

On the other hand, for a total energy smaller than $1/4$, the spatial locations of a trajectory is limited. More precisely, if a particle starts from a position $u_0 \in (-1, 1)$ with zero speed, then it will be confined within the interval between u_0 and $-u_0$. This gives rise to a closed orbit around the center. If the particle starts from a point (with zero velocity) out of this domain, it will go to infinity. Therefore, there are typically three pieces of trajectories in the phase plane for a given total energy between $(0, 1/4)$.



(a) (b)

Figure 1.5: The Duffing equation: (a) trajectories in the phase plane; (b) the potential function.

A closed orbit takes $C \in (0, 1/2)$. The amplitude U_0 of the corresponding solution is related to C by $C = U_0^2 - U_0^4/2$. Moreover, the period L is found to be

$$L(U_0) = 4\int_0^{U_0} \left[U_0^2 - u^2 - U_0^4/2 + u^4/2\right]^{-1/2} \mathrm{d}u = 4\int_0^1 \left[1 - s^2 - U_0^2(1-s^4)/2\right]^{-1/2} \mathrm{d}s. \tag{1.74}$$

This is an elliptic function. It may be show that L increases monotonously from 2π to ∞, as U_0 varies from 0 to 1. In particular, when $U_0 = 1$, there are two trajectories, both with infinite period, which means not periodic. They are the heteroclinic orbits.

Here we give a simple Matlab code for solving the Duffing equation.

First, we write a function meta file named duff.m.

```
function F=duff(t,v)
x=v(1);y=v(2);
F=[y;-x+x^3];
```

Then, we run in the command line of Matlab.

```
tspan=[0,20];
v0=[0,1];
[t,v]=ode45('duff',tspan,v0);
x=v(:,1);y=v(:,2);
z=x.^2-x.^4/2+y.^2;
plot3(x,y,z);
```

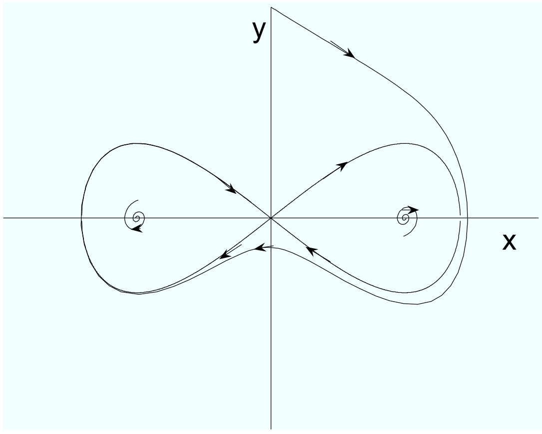


Figure 1.6: Homoclinic orbits and adjacent trajectories.

1.5 Homoclinic orbit and limit cycle

Besides an orbit around a center, there are other types of orbits that form a closed curve in the phase plane, including a homoclinic orbit and a limit cycle.

First, we illustrate a homoclinic orbit by the following example,

$$\left\{ \begin{array}{rcl} x' &=& y + y(1-x^2)\left[y^2 - x^2(1-x^2/2)\right], \\ y' &=& x(1-x^2) - y\left[y^2 - x^2(1-x^2/2)\right]. \end{array} \right. \tag{1.75}$$

It may be shown that $(0,0)$ is a saddle; $(1,0)$ and $(-1,0)$ are unstable foci. In fact, at a critical point$(x^*,0)$, the Jacobian matrix is

$$\left[\begin{array}{cc} 0 & 1 + (1-(x^*)^2)\left[-(x^*)^2(1-(x^*)^2/2)\right] \\ 1-3(x^*)^2 & (x^*)^2(1-(x^*)^2/2) \end{array} \right]. \tag{1.76}$$

Furthermore, it is readily shown that $y^2 = x^2(1-x^2/2)$ solves the equation. As a result, there are two trajectories. Each of them lies at one side of the y axis, and starts from the saddle and comes back. This is a homoclinic orbit.

Limit cycle is an important type of closed orbits in a nonlinear ODE system. Each trajectory near a limit cycle either goes towards, or leaves from it. Limit cycles may be used to model electric oscillators, or heart beat, among many other applications.

In general, it is extremely hard to detect the existence and the number of limit cycle(s) in a system. In certain cases, the existence may be proved with the help of the following theorem. See Fig. 1.7.

Theorem 1.5.1. *(Poincare-Bendixson) Let B be a ring-shaped domain between two simple closed curves L_1 and L_2. If there is no critical point in this domain, and each trajectory that intersects with one of these two curves enters B, then*

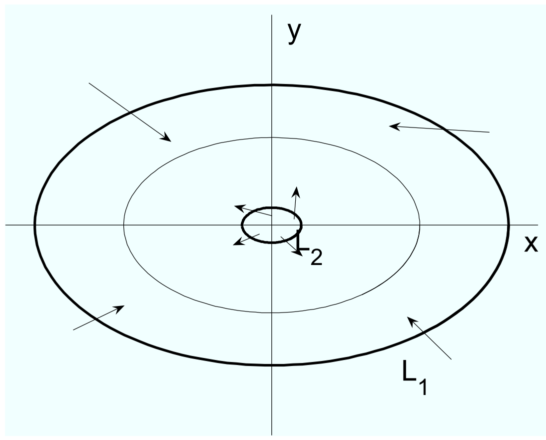


Figure 1.7: Schematic plot for the Poincare-Bendixson theorem.

there exists at least one limit cycle in this domain. The conclusion also holds if the inner boundary curve L_2 shrinks into an unstable node.

We remark how to verify that a trajectory enters the domain. As no explicit expression for a solution is known in general, we check the vector field at the boundary curves. If the vector $f(x, y)$ points toward the domain at a non-zero angle, it is possible to show that the trajectory enters the domain. We remark that a limit cycle exists if every trajectory leaves the ring shaped domain.

As an example, we consider the following dynamical system,

$$\left\{\begin{array}{lcl} x' &=& -y - x(x^2+y^2-1), \\ y' &=& \quad x - y(x^2+y^2-1). \end{array}\right. \tag{1.77}$$

For an outer boundary $L_1 :\ x^2+y^2 = 2$, the outer normal at (x_0, y_0) is (x_0, y_0). The source term vector is $(-y_0 - x_0, x_0 - y_0)$, and gives an inner product

$$(x_0, y_0) \cdot (-y_0 - x_0, x_0 - y_0) = -x_0^2 - y_0^2 = -2 < 0. \tag{1.78}$$

Therefore, the source vector points inward on this boundary curve.

Next, we take an inner boundary $L_1 :\ x^2+y^2 = 1/2$. It is readily shown that the source term vector $(-y_0 + x_0/2, x_0 + y_0/2)$ points outward, namely, again toward the ring-shaped domain between L_1 and L_2. By the Poincare-Bendixson theorem, the existence of a limit cycle is proved.

Indeed, in the polar coordinates, (1.77) reads

$$\left\{\begin{array}{lcl} r' &=& -r(r^2-1), \\ \theta' &=& 1. \end{array}\right. \tag{1.79}$$

Above two curves are $L_1 : r^2 = 2$, and $L_2 : r^2 = 1/2$. It is easy to check that $r'|_{L_1} = -\sqrt{2} < 0$, whereas $r'|_{L_2} = 1/2\sqrt{2} > 0$. This verifies the assumptions of the Poincare-Bendixson theorem, and a limit cycle exists. In fact, the limit cycle is $r = 1$. All trajectories wind towards this limit cycle, so it is stable.

Next, we consider a more realistic example, namely, the van der Pol equation. It was originally proposed by van der Pol, an electrical engineer working at Philips. In fact, the limit cycle discovered here corresponds to an electronic oscillator to generate a signal (wave) with a special frequency.

For a constant parameter ($\varepsilon > 0$), the equation reads

$$x'' + \varepsilon(x^2 - 1)x' + x = 0. \tag{1.80}$$

It may be recast into an ODE system as follows,

$$\begin{cases} x' &= y, \\ y' &= -x + \varepsilon(1 - x^2)y. \end{cases} \tag{1.81}$$

The only critical point $(0, 0)$ is an unstable node. To show the existence of a limit cycle, it suffices to find an outer boundary curve, along which all the trajectories enter the interior domain.

First, we construct an outer boundary in the following way. We select a point $D(x_D, y_D)$ on the left branch of the curve $-x + \varepsilon(1 - x^2)y = 0$. Then we draw the trajectory across this point of the system

$$\begin{cases} x' &= y, \\ y' &= -x + \varepsilon y. \end{cases} \tag{1.82}$$

When we take x_D close enough to -1, this trajectory does not intersect with the aforementioned curve on the right of D. It intersects with the y-axis at a point $E(0, y_E)$.

On the other hand, starting from a point $A(0, -y_A)$ on the lower y-axis with $y_A < -y_E$, we may draw a trajectory of the system

$$\begin{cases} x' &= y, \\ y' &= \varepsilon(1 - x^2)y. \end{cases} \tag{1.83}$$

It intersects with the line $x = -1$ at a point $B(-1, -y_B)$ with $y_B = y_A + 2\varepsilon/3$.

From B, we draw a circle centered at the origin, with a radius $\sqrt{1 + y_B^2}$. The circle intersects with the left branch of the curve $-x + \varepsilon(1 - x^2)y = 0$ at a point $C(x_C, y_C)$.

We remark that if we take D close enough to the line $x = -1$, and take A far away from $(0, -x_E)$, we may ensure that C lies on the left of D. Meanwhile, the system pertains a symmetry. Therefore, we draw reflected curves with respect to the y-axis, and only need to verify the conditions on the left half of the whole boundary. See Fig. 1.8.

Now we verify the conditions in the Poincare-Bendixson theorem. First, at any point on the segment CD, each trajectory is horizontal because $y' = 0$. Moreover, it points to the right as $x' = y > 0$, therefore enters the domain.

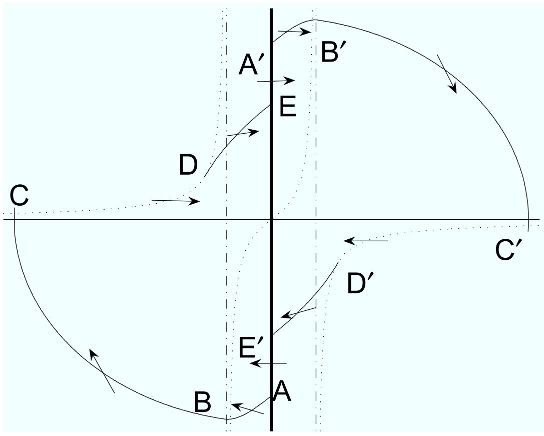


Figure 1.8: Existence of a limit cycle in the van der Pol equation.

Secondly, for the segment DE, again we have $x' = y > 0$. Furthermore, it holds that

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{DE} = \frac{-x+\varepsilon y}{y} > \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{vdP} = \frac{-x+\varepsilon(1-x^2)y}{y}. \tag{1.84}$$

As DE is a monotone curve going from lower-left to upper-right, each trajectory enters the domain.

Thirdly, the segment AB is expressed explicitly by $y = \varepsilon(x - x^3/3) - y_A$. So we have $x' = y < 0$, and

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{AB} = \frac{\varepsilon(1-x^2)y}{y} < \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{vdP} = \frac{-x+\varepsilon(1-x^2)y}{y}. \tag{1.85}$$

So, again each trajectory enters the domain. We note that a special discussion is needed for the point A, which we omit here.

Finally, for the segment BC, we may construct a function, which we shall discuss in more detail in the next section, $V(x,y) = x^2 + y^2$. Along each trajectory, it is found that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 2xx' + 2yy' = 2\varepsilon(1-x^2)y^2. \tag{1.86}$$

As $x^2 > 1$ on BC, a trajectory takes the direction of decreasing V. Thus it enters the domain.

This ends the proof.

Actually, at the corner point B, the trajectory is horizontal. It is not obvious that the trajectory enters the domain. One may rectify this by taking the AB segment slightly longer beyond the line $x = -1$.

We remark that a limit cycle exists only for a nonlinear system. The Poincare-Bendixson theorem only works for second order ODE's. In higher space dimensions, the geometry can be much more involved. It is usually difficult to show the existence of a limit cycle.

We also remark that a closed orbit may take the form of a limit cycle, an orbit around a center, or a homoclinic orbit. In the mean time, an open orbit may be a heteroclinic orbit, or an orbit that lies between two critical points or infinity.

1.6 Stability and Lyapunov function

Besides critical points and closed orbits, stability is one of the key issues in the qualitative theory. Interests on stability originally emerged from celestial mechanics. One naturally asks: is the galaxy or the solar system stable?

The foundation of stability theory was laid essentially by two classical papers, i.e., *Sur les courbes definies par une equation differentielle* by Poincare, and the dissertation by Lyapunov.

We consider an ODE system

$$x' = f(x), \quad x \in \mathbb{R}^n. \tag{1.87}$$

Stability refers to that of a limit set. A limit set is the set of limit points of a trajectory. More specifically, we consider a trajectory $x = \varphi(t; t_0, x_0)$ that issues from a point (t_0, x_0). A point p is called an ω-limit (or α-limit) point of this trajectory, if there exists a sequence $t_n \to +\infty$ (or $t_n \to -\infty$), such that $\lim\limits_{n\to\infty} |\varphi(t_n; t_0, x_0) - p| = 0$. All such limit points form the limit set of this trajectory. Typically for a second order ODE system, the limit set includes critical point(s), limit cycle, heteroclinic orbit, homoclinic orbit, or a closed orbit around a center. In this section, we shall confine ourselves to the stability of a critical point.

To motivate our analysis, we consider two different situations shown in Fig. 1.9. In both cases, the ball is assumed to stay still if initially so. However, the first one is unstable, in the sense that any small perturbation drives it away from the current equilibrium; whereas the second one is stable, in the sense that the ball remains close to the equilibrium under small perturbation.

In a friction free system with a potential function $V(x)$, the corresponding dynamical system is

$$\begin{cases} \dot{x} &= v, \\ \dot{v} &= -V'(x). \end{cases} \tag{1.88}$$

We define the total energy as $E(t) = \dfrac{\dot{x}^2}{2} + V(x)$. It may be shown that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \dot{x}\ddot{x} + V'(x)\dot{x} = \dot{x}(\dot{v} + V'(x)) = 0. \tag{1.89}$$

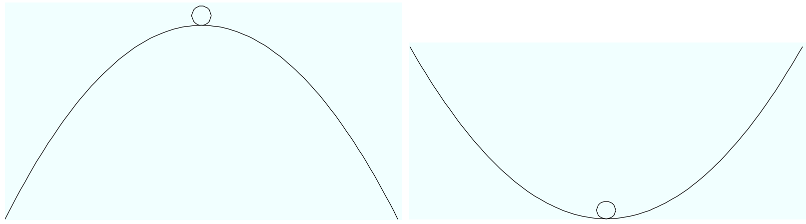


Figure 1.9: Stability of a ball: unstable (left) and stable (right).

The energy conserves. As $V(x)$ attains its minimum at the equilibrium, the particle will remain close to the equilibrium if initially so in the second case of Fig. 1.9. In contrast, $V(x)$ is concave in the first case, and the velocity increases indefinitely after an arbitrarily small perturbation to the equilibrium.

Next, we include friction. The governing equations become

$$\begin{cases} \dot{x} &= v, \\ \dot{v} &= -V'(x) - \alpha v. \end{cases} \tag{1.90}$$

Correspondingly, we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\alpha v^2. \tag{1.91}$$

If the potential is convex, the energy decreases, and the particle moves closer and closer to the equilibrium as time elapses.

There are different stabilities defined historically. We list some of these definitions. For simplicity, we consider a critical point $(0,0)$ of a second order ODE system.

- A critical point (0,0) is stable, if $\forall \varepsilon > 0, \exists \delta > 0$, such that $||\varphi(t; t_0, x_0)|| < \varepsilon, \forall t > t_0, \forall ||x_0|| < \delta$.
- A critical point (0,0) is asymptotically stable, if it is stable and $\exists \delta > 0$, such that $\lim\limits_{t \to +\infty} \varphi(t; t_0, x_0) = 0, \forall ||x_0|| < \delta$.
- A critical point (0,0) is exponentially stable, if $\exists \alpha > 0$, for $\forall \varepsilon > 0, \exists \delta > 0$, such that $||\varphi(t, t_0, x_0)|| \leq \varepsilon e^{-\alpha(t-t_0)}, \forall ||x_0|| < \delta$.

Indeed, what we have here is uniformly stable, uniformly asymptotically stable, and uniformly exponentially stable, where uniform means the choice of δ is independent of the time t_0.

Stability of a critical point has already been discussed before in a linear system. With the exact solutions, we know that source, saddle and unstable focus are unstable, whereas sink, stable focus and center are stable. However, a critical point of a nonlinear system, which is a center for its linearized system, is usually a focus. Therefore, center is structurally unstable.

To determine the stability of a critical point for a nonlinear system, there are the Lyapunov first method and the Lyapunov second method.

In the first method, one first examines the linear stability of the linearized problem

$$\dot{y} = \nabla f(x_0) \cdot y. \tag{1.92}$$

Linear stability holds if $\nabla f(x_0)$ is negative-definite. Furthermore, the following theorem ensures the nonlinear stability.

Theorem 1.6.1. *If all eigenvalues of $\nabla f(x_0)$ have negative real part, then the critical point x_0 is asymptotically stable for the nonlinear ODE system.*

For the second Lyapunov method, or sometimes called the direct Lyapunov method, we construct a Lyapunov function. Without loss of generality, we assume that the critical point is $x_0 = 0$.

Theorem 1.6.2. *If there exists a Lyapunov function $V(x)$, which is continuously differentiable with respect to x, positive definite, and non-increasing along each trajectory $x(t)$, i.e.*

$V(0) = 0$; $V(x) > 0$ *for* $x \neq 0$;

$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{x(t)} = \nabla_x V \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = \nabla_x V \cdot f(x) \leq 0$ *in a neighborhood* $\breve{U}(0, \delta)$.

Then the critical point 0 is uniformly stable. If $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{x(t)} < 0$ *in a neighborhood $\breve{U}(0, \delta)$, then it is asymptotically stable. If the inequality holds for $\forall x \in \mathbb{R}^n$, and $V(x)$ is radially unbounded, i.e., $\lim_{|x| \to \infty} V(x) \to +\infty$, then we have global (asymptotical or exponential) stability.*

We also remark that a sufficient condition for global exponential stability is that there exist positive constants c_1, c_2, c_3, such that

$c_1|x|^2 \leq V(x) \leq c_2|x|^2$, and $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{x(t)} \leq -c_3|x|^2$.

This is illustrated in Fig. 1.10.

A Lyapunov function has two properties. First, it is similar to a norm in the phase space, in a geometrical sense. Secondly, it decreases along with time.

The level curves $V(x) = C$ form a parametrization of the phase plane. These curves, with a decreasing $V(x)$, imply a confinement for the trajectories. Typically, the Lyapunov function is designed in a quadratic form, namely, $V(x) = x^T Ax$.

As an example, we consider the motion of a particle with mass $m = 1$ in a conservative force field $\nabla V(x)$. The equation reads, according to Newton's second law,

$$\ddot{x} = -\nabla V(x). \tag{1.93}$$

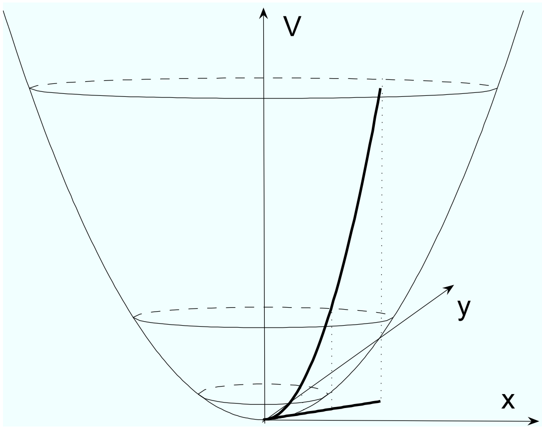


Figure 1.10: Lyapunov function and a trajectory.

Let velocity be $v = \dot{x}$, and rewrite the equation as

$$\begin{cases} \dot{x} &= v, \\ \dot{v} &= -\nabla V(x). \end{cases} \tag{1.94}$$

It reduces to (1.88) if the space is one dimensional. A critical point must be stationary $v = 0$, and locate at an extremal point of V. Without loss of generality, let us assume the extremal point locates at 0.

Consider the total energy $E = v^2/2 + V(x)$. We construct a function as

$$\mathcal{V}(x, v) = mv^2/2 + V(x) - V(0). \tag{1.95}$$

It turns out that $\dot{\mathcal{V}} = 0$, which means the conservation of energy. So, if we have $V(x) > V(0)$ for a certain neighborhood $\breve{U}(0, \delta)$, then $\mathcal{V}$ is a Lyapunov function, and the equilibrium is stable.

This is exactly the Lagrange theorem, which says that an equilibrium position is stable for a conservative force field, if the potential $V(x)$ reaches its local minimum.

1.7 Bifurcation

In applications, a system usually has certain tunable parameters. For instance, there may be a variable resistance or capacitor in an electronic circuit. Consequently, as observed in practice, the system may undergo certain structural change, namely, either its limit set changes or the stability of the limit set changes. Accordingly we observe the change in a physical system when such structural change occurs. As is well known, all that can be observed/measured for a physical system are stable.

To study this in an ODE system, we consider the following parametric system, with $\lambda \in \mathbb{R}$ a parameter,

$$x' = f(x, \lambda). \tag{1.96}$$

At each fixed λ, the analysis in the previous sections applies. More precisely, we first find the critical points. Secondly, we check the type of each critical point using the Jacobian matrix. Thirdly, we draw the representative trajectories schematically.

The first example is the following equation

$$x' = \lambda - x^2. \tag{1.97}$$

There is no critical point for $\lambda < 0$, and two critical points $\pm\sqrt{\lambda}$ for $\lambda > 0$. As we have learned, critical points are essential in determining the structure of trajectories. Therefore, we expect a drastic difference when λ crosses 0. In fact, for $\lambda < 0$, because $\lambda - x^2 < 0$, x keeps decreasing towards $-\infty$, regardless to the initial data.

We further remark that the exact solution with initial data $x(t_0) = x_0$ is

$$x(t) = -\mu \tan\left(\mu(t - t_0) - \arctan(x_0/\mu)\right), \quad \mu = \sqrt{-\lambda}. \tag{1.98}$$

It is defined only for a finite time interval. The state variable x tends to $-\infty$ as t approaches $t_0 + (\pi/2 + \arctan(x_0/\mu))/\mu$. We refer to this as a finite time blow-up. For qualitative analysis, we usually ignore this, as the trajectories do not contain much time information except the direction.

On the other hand, for $\lambda = 0$, the exact solution is

$$x = \frac{x_0}{x_0(t - t_0) + 1}. \tag{1.99}$$

So, if $x_0 > 0$, x tends to the critical point; and if $x_0 < 0$, x tends to $-\infty$. Different from what we have discussed before, the critical point 0 is neither a source nor a sink. In terms of stability, it is stable from the right semi-axis and unstable from the left semi-axis. It is unstable in a general sense. In fact, if we check the Jacobian $\dfrac{\mathrm{d}f}{\mathrm{d}x} = -2x^* = 0$. This is a degenerate case.

If λ further increases to $\lambda > 0$, then x tends to $\sqrt{\lambda}$ if $x_0 > -\sqrt{\lambda}$; and tends to $-\infty$ otherwise. The Jacobian $\dfrac{\mathrm{d}f}{\mathrm{d}x} = -2x^*$ is negative for $x^* = \sqrt{\lambda}$ and positive for $x^* = -\sqrt{\lambda}$. Along with the change in λ, we observe a bifurcation phenomenon, when the controlling parameter λ crosses the bifurcation point $\lambda = 0$.

To illustrate this, we draw a bifurcation diagram in Fig. 1.11. We plot x versus λ for the above first order system. In a multi-dimensional setting, i.e., $x \in \mathbb{R}^n$, we select certain aspect of x for such an illustration, usually an entry

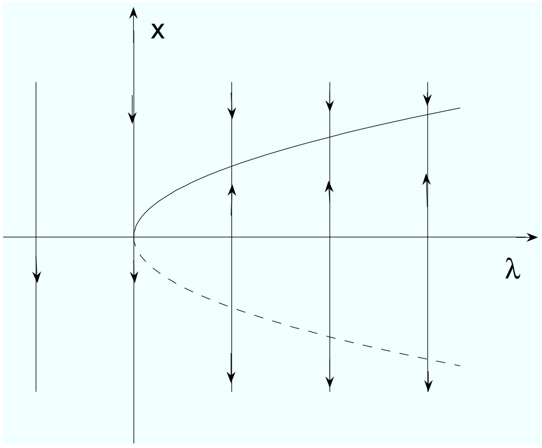


Figure 1.11: Bifurcation diagram: saddle-node bifurcation.

x_i, or the norm $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$. At each representative value for λ, we draw the trajectory with suitable direction of time. For $\lambda > 0$, the stable critical point lies in the solid line (stable branch), whereas the unstable one in the dashed line (unstable branch).

This type of bifurcation is called a tangential bifurcation. It is also called as a saddle-node bifurcation. The meaning becomes clearer if we supplement a decoupled equation to (1.97),

$$\begin{cases} \dot{x} &= \lambda - x^2, \\ \dot{y} &= -y. \end{cases} \tag{1.100}$$

The second variable converges toward the x-axis exponentially, regardless to the dynamics in x. Now for $\lambda > 0$ we get a saddle $(-\sqrt{\lambda}, 0)$ and a node (sink) $(\sqrt{\lambda}, 0)$. See Fig. 1.12.

Now we consider an example of transcritical bifurcation in the equation

$$x' = \lambda x - x^2. \tag{1.101}$$

We notice that there are always two critical points, $x_1 = 0, x_2 = \lambda$. For $\lambda < 0$, x_1 is stable, and x_2 is unstable. For $\lambda > 0$, x_1 becomes unstable, and x_2 is stable. An exchange of stability occurs. Please refer to the bifurcation diagram in Fig. 1.13.

Next, consider the supercritical bifurcation in

$$x' = \lambda x - x^3. \tag{1.102}$$

When $\lambda \le 0$, the only critical point is $x_0 = 0$, which is stable. When $\lambda > 0$, there appear two more critical points $x_1 = \sqrt{\lambda}$ and $x_2 = -\sqrt{\lambda}$. These two critical points are both stable, and $x_0 = 0$ becomes unstable. See Fig. 1.14.

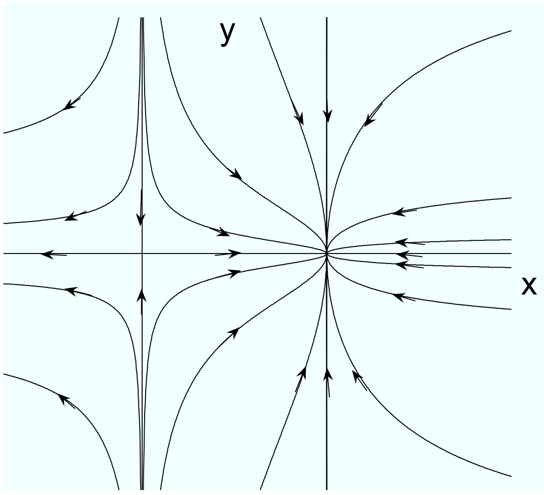


Figure 1.12: Saddle-node bifurcation in a second order system with $\lambda > 0$.

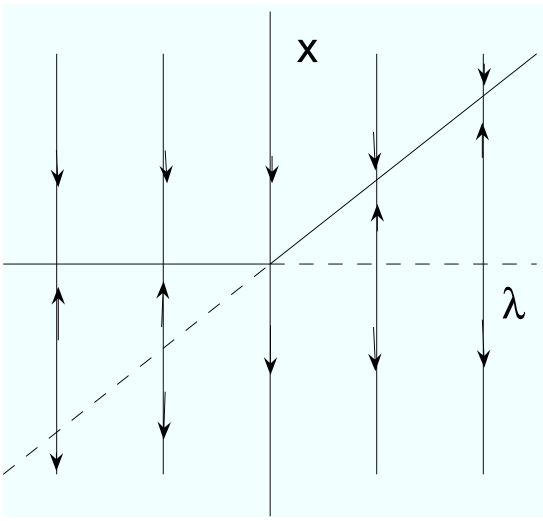


Figure 1.13: Bifurcation diagram: transcritical bifurcation.

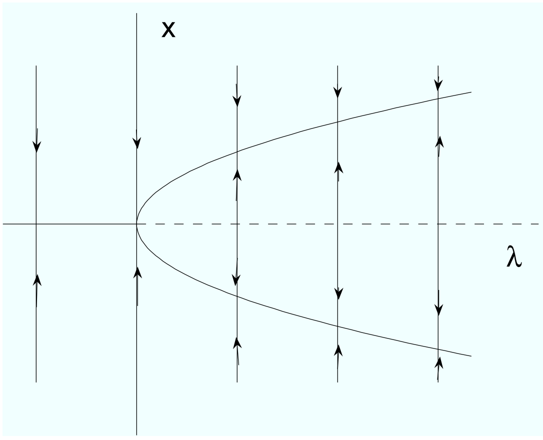


Figure 1.14: Bifurcation diagram: supercritical bifurcation.

The examples we have discussed so far are stationary bifurcations, namely, they concern only with critical points. The structural change may involve closed orbits as well. One most important bifurcation of this type is the Hopf bifurcation. We demonstrate it through a second order ODE system

$$\begin{cases} x' &= -y + x(\lambda - x^2 - y^2), \\ y' &= x + y(\lambda - x^2 - y^2). \end{cases} \tag{1.103}$$

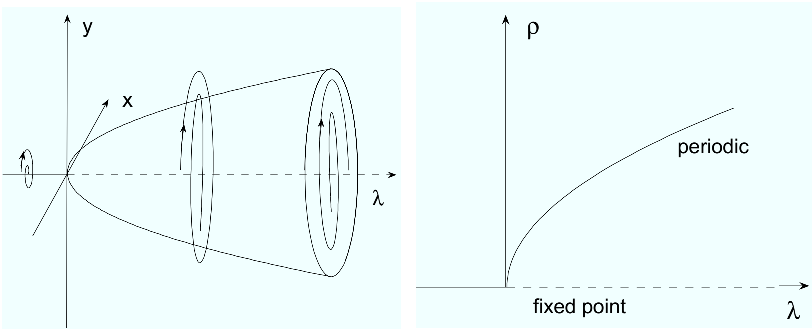
The only critical point is $(x_0, y_0) = (0, 0)$. The Jacobian matrix is $\begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}$, which has eigenvalues $\lambda \pm i$. Therefore, it is a stable focus when $\lambda < 0$, and an unstable focus when $\lambda > 0$. In fact, this system may be better understood in polar coordinates $(x, y) = (\rho\cos\theta, \rho\sin\theta)$. It is readily shown that

$$\begin{cases} \rho' &= \rho(\lambda - \rho^2), \\ \theta' &= 1. \end{cases} \tag{1.104}$$

According to the results of supercritical bifurcation, we know that ρ decreases towards 0 when $\lambda < 0$. Therefore, each trajectory winds counterclockwise towards $\rho = 0$. On the other hand, each trajectory winds counterclockwise towards the periodic orbit $\rho = \sqrt{\lambda}$ for $\lambda > 0$. Thus it is a limit cycle with period 2π.

This bifurcation may be depicted either in a 3-D plot, or a 2-D plot in (ρ, λ)-plane. See Fig. 1.15.

One distinct feature in the Hopf bifurcation lies in the fact that the eigenvalues cross the real axis. Correspondingly a periodic orbit (limit cycle) emerges out of a critical point.



(a) (b)

Figure 1.15: Bifurcation diagram: the Hopf bifurcation.

1.8 Chaos: Lorenz equations and logistic map

So far, qualitative theory provides a satisfactory understanding of ordinary differential equations. In particular, plane analysis provides a clean argument for second order autonomous systems. For systems with more degrees of freedom, local behavior is described by the critical point approach. People once believed that the whole picture is essentially similar to that of a second order system.

But this is not true. Topologically speaking, the structure of $\mathbb{R}^3$ is quite different from that of $\mathbb{R}^2$. In the setting of an ODE system, Lorenz put forward an excellent example to illustrate this difference.

Lorenz tried to disprove the mid-term predictability of weather system. Weather system is largely dominated by convection and heat conduction, and may be described by the Rayleigh-Benard equations,

$$\left\{\begin{aligned} \frac{\partial}{\partial t}\left(\nabla^2\psi\right) &= -\frac{\partial\left(\psi,\nabla^2\psi\right)}{\partial(x,z)} + \nabla^4\psi + \frac{R_a}{\sigma}\frac{\partial\theta}{\partial x}, \\ \frac{\partial}{\partial t}\theta &= -\frac{\partial\left(\psi,\theta\right)}{\partial(x,z)} + \frac{\partial\psi}{\partial x} + \frac{1}{\sigma}\nabla^2\theta. \end{aligned}\right. \tag{1.105}$$

Here ψ is the stream function, θ the temperature, R_a the Rayleigh number, and σ the Prandtl number. The velocities are recovered from

$$u = -\frac{\partial\psi}{\partial z}, \quad w = \frac{\partial\psi}{\partial x}. \tag{1.106}$$

Lorenz made a truncation

$$\left\{\begin{aligned} \psi &= \left(\frac{\pi^2+a^2}{\pi a\sigma}\right)\sqrt{2}X(\tau)\sin ax\sin\pi z, \\ \theta &= \frac{R_c}{\pi R_a}\left[\sqrt{2}Y(\tau)\cos ax\sin\pi z - Z(\tau)\sin 2\pi z\right], \end{aligned}\right. \tag{1.107}$$

with $\tau = \dfrac{R_a}{\sigma}\left(\pi^2+a^2\right)t$, and $R_c = (\pi^2+a^2)^3/a^2$ is the critical Rayleigh number.

Lorenz obtained the following system which is later named after him,

$$\begin{cases} X' &= -\sigma X + \sigma Y, \\ Y' &= -XZ + rX - Y, \\ Z' &= XY - bZ, \end{cases} \tag{1.108}$$

with $r = R_a/R_c$, $b = 4\pi^2/(\pi^2 + a^2)$. This system possesses a symmetry $\{X, Y, Z\} \to \{-X, -Y, Z\}$.

Defining a Lyapunov function

$$V(t) = rX^2 + \sigma Y^2 + \sigma(Z - 2r)^2, \tag{1.109}$$

we have

$$V' = 2\sigma(-rX^2 - Y^2 - bZ^2 + 2brZ). \tag{1.110}$$

We may take

$$c = \begin{cases} b^2r^2/(b - \sigma), & \text{if } b \geq 2\sigma, \ \sigma \leq 1, \\ b^2r^2\sigma/(b - 1), & \text{if } b \geq 2, \ \sigma \geq 1, \\ 4\sigma r^2, & \text{otherwise.} \end{cases} \tag{1.111}$$

Then $V' \leq 0$ for $\forall (X, Y, Z)$ satisfying $V \geq c$. Therefore, a trajectory in the ellipsoid cannot escape. This defines a trapping zone.

Now consider the critical points. They solve a nonlinear algebraic system

$$\begin{cases} 0 &= -\sigma X + \sigma Y, \\ 0 &= -XZ + rX - Y, \\ 0 &= XY - bZ. \end{cases} \tag{1.112}$$

When $r < 1$, the only critical point is $(0, 0, 0)$. The Jacobian matrix is

$$J = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}, \tag{1.113}$$

which has three eigenvalues with non-positive real-part. Therefore, we expect that in this case, all trajectories point towards this node.

In the following, we make analysis under the condition $\sigma - b - 1 > 0$. When $r > 1$, however, one eigenvalue for the Jacobian matrix at $(0, 0, 0)$ becomes positive. It then turns into a saddle. Meanwhile, there appear two new critical points $(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r - 1)$. The eigen-equation for both of them is

$$f(\lambda) = \lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2\sigma b(r - 1) = 0. \tag{1.114}$$

We observe that $f(\pm\infty) = \pm\infty$, and $f'(\lambda) = 3\lambda^2 + 2(b + \sigma + 1)\lambda + b(r + \sigma)$. In particular, when $r = 1$, three eigenvalues are $0, -b, -(\sigma + 1)$. Thus when

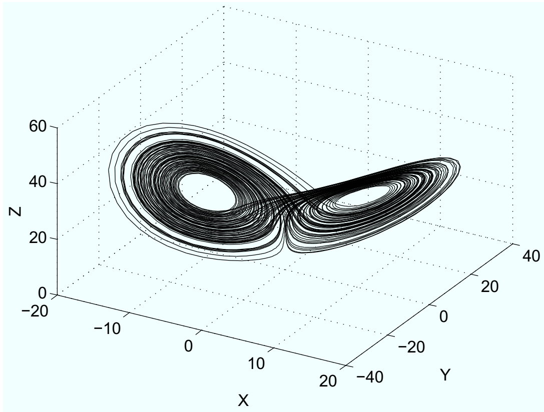


Figure 1.16: A trajectory of the Lorenz equations.

$0 < r - 1 \ll 1$, there are one real eigenvalue with negative sign, and two complex conjugate roots with negative real parts. These two critical points are stable nodes. As r increases, the real eigenvalue becomes smaller and smaller. On the other hand, sum of three roots are fixed as $-(\sigma+1+b)$. Therefore, when r becomes big enough, the real eigenvalue becomes smaller than $-(\sigma+1+b)$. The two complex eigenvalues take positive real parts. The critical points are then unstable, each with a one dimensional stable manifold and two dimensional unstable manifold. In the unstable manifold, trajectories wind away from the critical points.

More detailed analysis shows that at certain r_0, there appear a pair of heteroclinic orbits, connecting $(0,0,0)$ and the unstable critical points. When these heteroclinic orbits break, chaos appears. See Fig. 1.16.

There are many exciting features of chaos. First, there is sensitivity to initial data. All trajectories are bounded within the trapping zone, and there is no aforementioned attractor, such as stable critical point, limit cycle, heteroclinic or homoclinic orbits. Therefore, each trajectory must wind within the ellipsoid forever. This makes a very complex entangled picture. Two trajectories with different yet very close starting points in the phase plane separate from each other after a long run.

Secondly, as pointed out by Lorenz, the deterministic ODE system gives a random-like solution. Because we do not know precisely the initial data, in a long run this imprecision causes different results. In weather forecast, a small perturbation may result in a drastic change after some time, typically in the scale of months.

Another way to study chaos is to go back to the Rayleigh-Benard equation. However, that is too complicated for a substantial understanding. Instead, there are investigations by a better truncation, i.e., with more modes. For instance, by a 14^{th} order ODE system, it was observed that there is a bifurcation process when r increases. More precisely, the system goes to zero equilibrium when $r < 1$; and periodic and quasi-periodic solution appears when r is big enough. When quasi-periodic solution breaks, chaos is observed.

Chaos appears not only in differential equations, but also in mappings. Let us consider the logistic map

$$x_{n+1} = \lambda x_n(1 - x_n). \tag{1.115}$$

Here $\lambda \in [0, 4]$ is a parameter. Assume that $x_0 \in [0, 1]$, then it holds $x_n \in [0, 1], \quad \forall n \in \mathbb{N}$.

When $\lambda = 0$, obviously $x_n = 0, \quad \forall n \in \mathbb{N}$. Meanwhile, we notice that the function $f(x) = \lambda x(1 - x)$ has a derivative

$$|f'(x)| = |\lambda(1 - 2x)| \le \lambda, \quad \forall x \in [0, 1]. \tag{1.116}$$

Therefore, it is a contraction provided $\lambda < 1$. Accordingly, we know $x_n \to 0$, by virtue of the Banach fixed point theorem.

There is an alternative way to see this. When $\lambda \le \lambda_1 = 1$, $x = \xi_1 = 0$ is the only fixed point of $f(x)$. This fixed point is globally stable. To see this, we notice that for $x \ne 0$, it holds that

$$x_{n+1} \le x_n(\lambda(1 - x_n)) < x_n. \tag{1.117}$$

So $\{x_n\}$ forms a decreasing sequence bounded from below by 0. It converges to a certain limit. Due to the continuity of the map, the limit is the fixed point $x = \xi_1 = 0$.

When $\lambda > 1$, $x = 0$ becomes locally unstable, as $f'(0) = \lambda > 1$. There appears another fixed point $\xi_2 = 1 - 1/\lambda$. When $\lambda \in (1, 3)$, from $f'(1 - 1/\lambda) = 2 - \lambda$, we may show local stability for this new fixed point. Actually it is globally stable. To see this, we plot the curve $y = f(x)$ and the line $y = x$ in Fig. 1.17.

When λ exceeds $\lambda_2 = 3$, both fixed points are unstable. However, there appears a 2-periodic state. That is, $f_2(x) = f(f(x))$ has fixed points. They solve

$$x = \lambda(\lambda x(1 - x))(1 - \lambda x(1 - x)). \tag{1.118}$$

This is a fourth order algebraic equation. Fixed points of $f(x)$ remains to be roots, and there are two new ones,

$$\xi_{3,4} = (1 + \lambda \pm \sqrt{(\lambda + 1)(\lambda - 3)})/2\lambda. \tag{1.119}$$

The stability may be studied by analyzing $f_2'(\xi_{3,4})$. By straightforward manipulations, we find that $f_2'(x) = \lambda^2(1-2x)(1-2\lambda x(1-x))$. It turns out that these two roots are stable if $\lambda \le \lambda_3 = 1 + \sqrt{6}$. With an initial value x_0, a typical

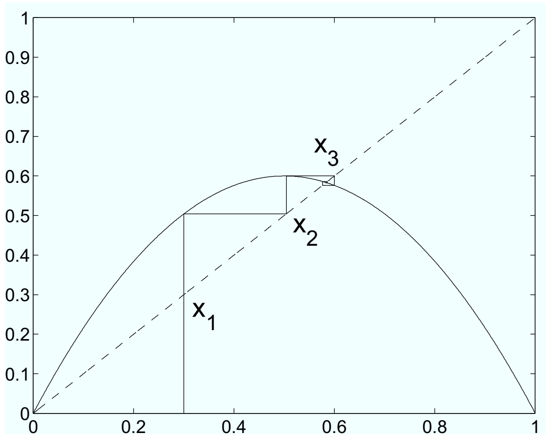


Figure 1.17: Logistic map: $\lambda = 2.4$.

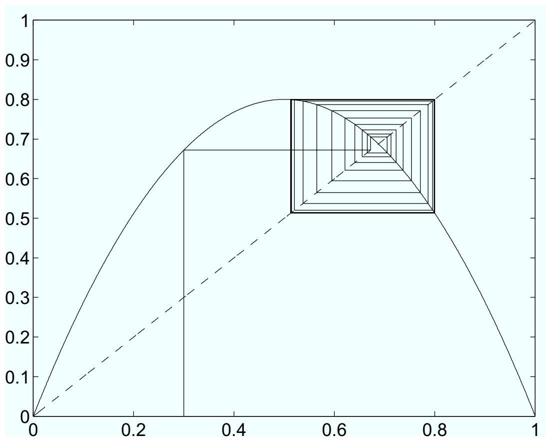
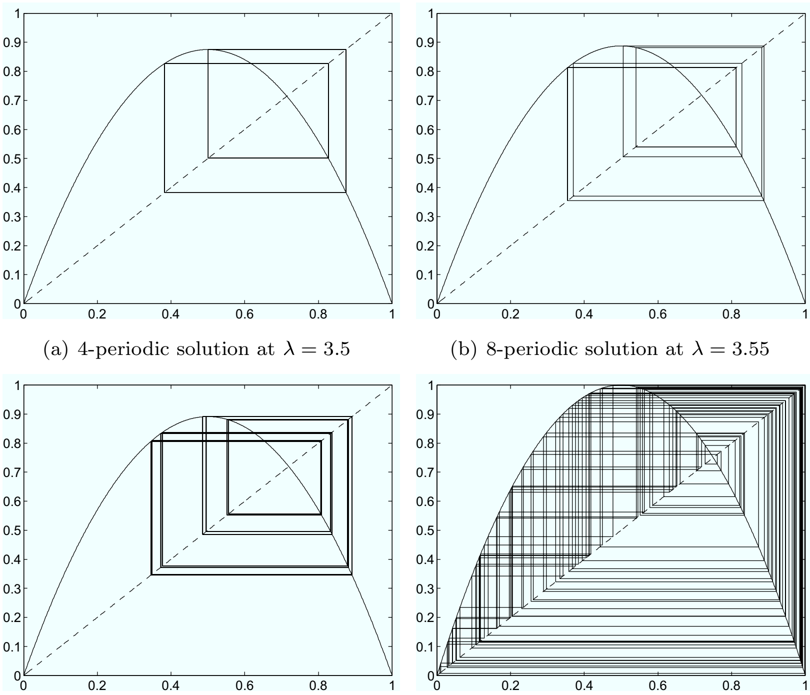


Figure 1.18: Logistic map: 2-periodic solution at $\lambda = 3.2$.



(a) 4-periodic solution at $\lambda = 3.5$

(b) 8-periodic solution at $\lambda = 3.55$

(c) 16-periodic solution at $\lambda = 3.565$

(d) chaotic solution at $\lambda = 4$

Figure 1.19: Logistic map: period doubling and chaos.

picture is shown in Fig. 1.18. We observe that when $n \to \infty$, x_n approaches towards this 2-periodic solution $\{\cdots \to \xi_3 \to \xi_4 \to \xi_3 \to \xi_4 \to \cdots\}$.

When λ further increases, this 2-periodic solution becomes unstable. It turns out that there appears a stable 4-periodic solution. This corresponds to the rest four fixed points $\xi_{5,6,7,8}$ to $f_4(x) = f_2(f_2(x))$, which is a sixteenth-order polynomial. They are stable if $\lambda \leq \lambda_4 = 3.544$. Afterwards, 8-periodic, 16-periodic, 32-periodic, $\cdots$ solutions take place subsequently. See Fig. 1.19.

This is a period doubling bifurcation, and the sequence of λ_n takes the following values

$$1, 3, 3.449490, 3.544090, 3.564407, 3.568759, 3.569692, \cdots \tag{1.120}$$

There is a limit $\lambda_\infty = 3.569945672$. Beyond this threshold, no aforementioned periodic solution is stable, and chaos occurs. There is a uniform constant appears in this bifurcation process

$$\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} = 4.669201661. \tag{1.121}$$

It is called the Feigenbaum number, which turns out to be a universal constant for such bifurcations.

Now we consider the special case $\lambda = 4$ in the chaotic regime.

Let $x = \sin^2(\pi y/2)$ for $y \in [0,1]$, the iteration reads

$$\sin^2(\pi y_{n+1}/2) = 4\sin^2(\pi y_n/2)\cos^2(\pi y_n/2). \tag{1.122}$$

Noticing the range of y_n, we obtain that

$$y_{n+1} = \psi(y_n) = \begin{cases} 2y_n, & y_n \in [0,1/2], \\ 2(1-y_n), & y_n \in [1/2,1]. \end{cases} \tag{1.123}$$

It is plotted in Fig. 1.20.

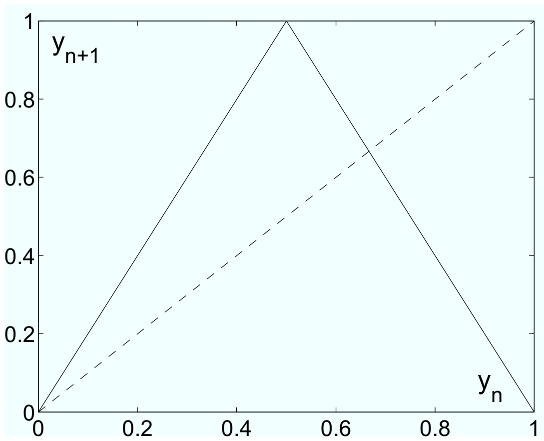


Figure 1.20: Mapping corresponding to the logistic map at $\lambda = 4$.

There are two fixed points, $\eta_1 = 0$ and $\eta_2 = 2/3$, corresponding to $\xi_{1,2}$ respectively. They are also unstable, because the slope is ± 2. In fact, for any $N \in \mathbb{N}$, the N-periodic solutions are unstable.

There is another interpretation of $\psi(y)$. Let $y = 0.p_1p_2\cdots$ in binary digits, namely $p_i = 0, 1$ and $y = \sum_i p_i 2^{-i}$, the iteration actually means $\psi(y) = 0.p_2p_3\cdots$, or $0.\bar{p}_2\bar{p}_3\cdots$. So, this corresponds to a left-shifting.

Sensitivity to initial data is obvious from this interpretation. If $|y - y'| \approx 2^{-k}$, then after k iterations, the difference is amplified to $2^k|y-y'| \approx 1$. Noticing the finite digit representation of a number in computer, we lose the information in initial data completely after certain steps. The second implication is ergodicity, which means that $\{y_n\}$ covers the whole interval of $[0,1]$. In fact, there are three cases for the exact solution. If $y = \sum_{i=1}^{N} p_i 2^{-i}$, then $y_n = 0, \quad \text{for } n > N$. If y is a rational number, yet with denominator not in the form of 2^k, then after certain steps, a periodic solution is selected. Finally, if y is irrational, then $\{y_n\}$ covers $(0,1)$, without repeating. Since there are more irrational numbers than rational ones, we may expect that y_n equally distributed in the interval $[0,1]$. Correspondingly, x_n distributes in the interval $[0,1]$ as well, yet not equally.

Assignments

1. Determine if the following systems are autonomous or not. Rewrite them into a system of first order ODE's.

   (a) $ax'' + bx' + cx = 0$;

   (b) $L\dfrac{\mathrm{d}^2 I}{\mathrm{d}t^2} + R\dfrac{\mathrm{d}I}{\mathrm{d}t} + \dfrac{1}{C}I = V'(t)$;

   (c) $\begin{cases} x'' &= \dfrac{GM}{|x-y|^2}, \\ y'' &= -\dfrac{Gm}{|x-y|^2}. \end{cases}$

2. Find the solution to the following systems.

   (a) $L\dfrac{\mathrm{d}^2 I}{\mathrm{d}t^2} + R\dfrac{\mathrm{d}I}{\mathrm{d}t} + \dfrac{1}{C}I = 0, \quad I(0) = 0, I'(0) = 1$;

   (b) $\begin{cases} x'' &= \dfrac{GM}{|x-y|^2}, \quad x(0) = R, x'(0) = K; \\ y'' &= -\dfrac{Gm}{|x-y|^2}, \quad y(0) = 0, y'(0) = 0. \end{cases}$

3. Show that $\|x\| = \max_{1\le j\le n} |x_j|$ defines a norm on $\mathbb{R}^n$.

4. Is the sequence of functions

$$f_n(t) = \begin{cases} n(t-a), & \text{if } t < a + 1/n, \\ 1, & \text{if } t \ge a + 1/n \end{cases} \tag{1.124}$$

   a Cauchy sequence in $C[a,b]$ with the norm $\|f\| = \max\limits_{t\in[a,b]} |f(t)|$?

5. We call a square matrix A as the representation of an operator T under the basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$, if

$$\begin{aligned} T &: \mathbb{R}^n \to \mathbb{R}^n \\ x &\mapsto y = Ax. \end{aligned} \tag{1.125}$$

   Here $x = (x_1, \cdots, x_n)^T = \sum\limits_{i=1}^{n} x_i \mathbf{e}_i, y = (y_1, \cdots, y_n)^T = \sum\limits_{i=1}^{n} y_i \mathbf{e}_i$. If there is a change of basis $(\mathbf{e}_1, \cdots, \mathbf{e}_n)$ to $(\mathbf{e}'_1, \cdots, \mathbf{e}'_n)$, with $(\mathbf{e}'_1, \cdots, \mathbf{e}'_n) = (\mathbf{e}_1, \cdots, \mathbf{e}_n)\,Q$, where Q is non-singular, please find the matrix that represents the same operator T.

6. Show that the solution to the ODE given by the fixed point theorem is differentiable, and really solves the ODE, provided that $f(t,x)$ is Lipschitz also with respect to t. Discuss a counter example $f(t,x) = \operatorname{sgn}(t)$.

7. In the proof of local existence for an ODE, we have assumed the boundedness and Lipschitz continuity of $f(t,x)$. However, this is required only locally, for $(x,t) \approx (x_0,t_0)$. Please prove the existence of the solution in (t_0-b, t_0+b) by the fixed point theorem under the condition that $\exists a,b,c,k>0$ with $a \geq bc$, such that for $(x,t) \in (x_0-a, x_0+a) \times (t_0-b, t_0+b)$, it holds that

$$|f(t,x)| \leq c, \quad |f(t,x)-f(s,x)| \leq k\,|t-s|\,, |f(t,x)-f(t,y)| \leq k\,|x-y|\,. \tag{1.126}$$

8. Find the operator T and matrix E for the Gauss-Seidel iteration

$$x_j^{(m+1)} = \frac{1}{c_{jj}}\left(d_j - \sum_{k=1}^{j-1} c_{jk}x_k^{(m+1)} - \sum_{k=j+1}^{n} c_{jk}x_k^{(m)}\right). \tag{1.127}$$

Please give a simple sufficient condition for the convergence of the iteration.

9. For a second order system $x' = f(x)$, show that a non-zero vector $f(x) = (f_1(x_1,x_2), f_2(x_1,x_2))^T$ is tangential to the trajectory at the point $x = (x_1,x_2)^T$ in the phase plane.

10. Prove that a trajectory remains unchanged in the phase plane if there undergoes a non-singular time transformation, i.e. $\tau = \tau(t)$ with $\tau'(t) \neq 0$.

11. When we plot the trajectories schematically for the case of a source, we draw all the curves tangential to the y_2-axis, except for the two trajectories along the y_1-axis. Is this correct? Please prove or disprove it.

12. Plot schematically the trajectories for

$$\begin{cases} x' = & 3x - y, \\ y' = & x + y. \end{cases} \tag{1.128}$$

13. Find the critical points of the system

$$\begin{cases} x' & = y(1-x), \\ y' & = (1-y)\sin x. \end{cases} \tag{1.129}$$

Determine their types. Can you also plot schematically the trajectories?

14. Show that $(0,0)$ is a saddle; $(1,0)$ and $(-1,0)$ are unstable foci in equation (1.75).

15. Find a curve around the origin to be the inner boundary for the van der Pol equation, from which each trajectory points towards the domain enclosed by this curve and the outer boundary constructed in the lecture.

16. A critical point (0,0) is called attractive, if and $\exists \delta(t_0) > 0$, such that $\lim_{t \to +\infty} \varphi(t; t_0, x_0) = 0$, for $\forall |x_0| < \delta$. Therefore, it is asymptotically stable if it is stable and attractive. Show that being attractive does not imply stability by analyzing the equation

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &= -x(x-1)y - \dfrac{1}{8}x^3. \end{cases} \tag{1.130}$$

(Remark: See Gao W X, Acta Math Sinica, 32(1):35-41, 1989.)

17. For a linear system $y' = \nabla f(x_0) \cdot y$, if $\nabla f(x_0)$ is negative-definite, find a Lyapunov function.

18. Discuss bifurcation in

$$x' = \lambda + x^2, \tag{1.131}$$

and draw the bifurcation diagram.

19. Discuss the subcritical bifurcation in

$$x' = \lambda x + x^3. \tag{1.132}$$

Draw the bifurcation diagram.

20. For the Fitzhugh-Nagumo equations

$$\begin{cases} x' &= 3(x + y - x^3/3 + \lambda), \\ y' &= -(x - 0.7 + 0.8y)/3, \end{cases} \tag{1.133}$$

compute the critical points, and study their stabilities. Compute when the Hopf bifurcation occurs. Draw a bifurcation diagram in terms of of x versus λ.

21. Consider the following equations,

$$\begin{cases} x' &= y, \\ y' &= x - x^3 + \lambda y + xy/2. \end{cases} \tag{1.134}$$

The critical points are $(0,0), (1,0)$, and $(-1,0)$. Perform numerical computations for $\lambda \in [-0.45, -0.4]$ to investigate when a homoclinic orbit is formed. Plot the representative trajectories for λ around this value. Check the types of the critical points at this value for λ. This is called a homoclinic bifurcation.

22. Please use Matlab to plot some trajectories of the Lorenz equations with $\sigma = 10, r = 28, b = 8/3$.

23. Verify that the Lyapunov function defined by (1.109) satisfies $V' \leq 0$ for $V \geq c$ in the Lorenz equations.