电动力学期末

李梓瑞

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1 Recall

Maxwell equation

$$\begin{cases} \nabla\cdot\boldsymbol{D}=\rho_f \\ \nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t} \\ \nabla\cdot\boldsymbol{B}=0 \\ \nabla\times\boldsymbol{H}=\boldsymbol{j}_f+\frac{\partial\boldsymbol{D}}{\partial t} \end{cases},\quad \begin{cases} \boldsymbol{D}=\epsilon_0\boldsymbol{E}+\boldsymbol{P}=\epsilon\boldsymbol{E} \\ \boldsymbol{H}=\frac{\boldsymbol{B}}{\mu_0}-\boldsymbol{M}=\mu^{-1}\boldsymbol{B} \\ \boldsymbol{j}=\sigma\boldsymbol{E} \\ \nabla\cdot\boldsymbol{j}+\frac{\partial\rho}{\partial t}=0 \end{cases},\quad \boldsymbol{F}=q(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}) \tag{1}$$

电场磁场

$$\phi(\boldsymbol{x})=\frac{1}{4\pi\epsilon_0}\int_V\frac{\rho(\boldsymbol{x'})}{r}d\tau',\boldsymbol{E}(\boldsymbol{x})=\frac{1}{4\pi\epsilon_0}\int_V\frac{\rho(\boldsymbol{x'})\boldsymbol{r}}{r^3}d\tau' \tag{2}$$

$$\boldsymbol{A}(\boldsymbol{x})=\frac{\mu_0}{4\pi}\int_V\frac{\boldsymbol{j}(\boldsymbol{x'})}{r}d\tau',\boldsymbol{B}(\boldsymbol{x})=\frac{\mu_0}{4\pi}\oint_L\frac{\boldsymbol{j}(\boldsymbol{x'})\times\boldsymbol{r}}{r^3}d\tau' \tag{3}$$

介质

$$\rho_P=-\nabla\cdot\boldsymbol{P},\quad \boldsymbol{j}_P=\frac{\partial\boldsymbol{P}}{\partial t},\quad \sigma_P=-\boldsymbol{n}\cdot(\boldsymbol{P}_2-\boldsymbol{P}_1) \tag{4}$$

$$\rho_m=0,\quad \boldsymbol{j}_m=\nabla\times\boldsymbol{M},\quad \boldsymbol{\alpha}_m=\boldsymbol{n}\times(\boldsymbol{M}_2-\boldsymbol{M}_1) \tag{5}$$

$$\rho=\rho_f+\rho_P,\quad \boldsymbol{j}=\boldsymbol{j}_f+\boldsymbol{j}_P+\boldsymbol{j}_m \tag{6}$$

$$\epsilon=\epsilon_0\epsilon_r,\ \epsilon_r=1+\chi;\quad \mu=\mu_0\mu_r,\ \mu_r=1+\chi_m \tag{7}$$

边值关系

$$\begin{cases} \boldsymbol{n} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = 0 \\ \boldsymbol{n} \cdot (\boldsymbol{D}_2 - \boldsymbol{D}_1) = \sigma_f \\ \boldsymbol{n} \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0 \\ \boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{\alpha}_f \end{cases} \tag{8}$$

能量

$$\frac{dU}{dt} = -\oint_S \boldsymbol{S} \cdot d\boldsymbol{s} - \frac{d}{dt}\int_V w d\tau, \quad \frac{dU}{dt}|_\infty = -\frac{d}{dt}\int_\infty w d\tau \tag{9}$$

$$\frac{dU}{dt} = \int_V \boldsymbol{E} \cdot \boldsymbol{j} d\tau, \quad \boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B}, \quad U_e = \int_\infty w d\tau \tag{10}$$

$$\frac{\partial w}{\partial t} = \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} \tag{11}$$

辐射功率

$$P = \frac{1}{2}\Re(\boldsymbol{j} \cdot \boldsymbol{E}^*) \tag{12}$$

静电场

$$\begin{cases} \nabla^2 \phi = -\rho/\epsilon \\ \phi_2|_S = \phi_1|_1, \ \epsilon_2 \frac{\partial \phi_2}{\partial n} - \epsilon_1 \frac{\partial \phi_1}{\partial n} = -\sigma \end{cases} \tag{13}$$

$$\phi = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{R} \sum_{n=0}^{\infty} \lambda^n P_n(\cos\theta) \tag{14}$$

$$\phi = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{R} - \boldsymbol{p} \cdot \frac{1}{R} + \frac{1}{6}\vec{\boldsymbol{D}} : \nabla\nabla\frac{1}{R} + \dots\right] \tag{15}$$

$$W = \int_V \rho \phi_e d\tau \tag{16}$$

静磁场

$$\begin{cases}\nabla^2 \boldsymbol{A} = -\mu \boldsymbol{j} \\ \nabla \cdot \boldsymbol{A} = 0\end{cases}, \quad \begin{cases}\nabla^2 \phi_m = -\rho_m/\mu_0 \\ \rho_m = -\mu_0 \nabla \cdot \boldsymbol{M}\end{cases} \tag{17}$$

$$\int_V \boldsymbol{j} \cdot \boldsymbol{E}_e d\tau = \int_V \frac{j^2}{\sigma} d\tau + \oint_S \boldsymbol{S} \cdot d\boldsymbol{s} \tag{18}$$

$$W = \frac{1}{2} \int_V \boldsymbol{j} \cdot \boldsymbol{A}_e d\tau \tag{19}$$

电偶极子 $\phi^{(1)} = \frac{1}{4\pi\epsilon_0} \frac{\boldsymbol{p} \cdot \boldsymbol{R}}{R^3}$

$$W = -\boldsymbol{p} \cdot \boldsymbol{E}, \quad \boldsymbol{F} = \boldsymbol{p} \cdot \nabla \boldsymbol{E}, \quad \boldsymbol{N} = \boldsymbol{p} \times \boldsymbol{E} \tag{20}$$

磁偶极子 $\boldsymbol{A}^{(1)}(\boldsymbol{x}) = \frac{\mu_0}{4\pi} \frac{\boldsymbol{m} \times \boldsymbol{R}}{R^3}$, $\phi_m^{(1)} = \frac{\boldsymbol{m} \cdot \boldsymbol{R}}{4\pi R^3}$

$$W = \boldsymbol{m} \cdot \boldsymbol{E}, \quad \boldsymbol{F} = \boldsymbol{m} \cdot \nabla \boldsymbol{B}, \quad \boldsymbol{N} = \boldsymbol{m} \times \boldsymbol{B} \tag{21}$$

2 形式化的电磁理论

取度规 $(1, -1, -1, -1)$, 无电磁场耦合:

$$S = \int L dt = -mc \int ds = -mc \int d\tau \sqrt{\left(\frac{dx^\mu}{d\tau}\right)\left(\frac{dx_\mu}{d\tau}\right)} \tag{22}$$

$$L = -mc^2 \sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}}, \quad ds = c d\tau, \quad ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{23}$$

$$\boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}} = \gamma m \boldsymbol{v}, \quad E = \boldsymbol{p} \cdot \boldsymbol{v} - L = \gamma mc^2, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \boldsymbol{v}/c \tag{24}$$

$$u_\mu = \frac{dx_\mu}{dt}, \quad p^\mu = (E/c, \boldsymbol{p}) \tag{25}$$

有电磁场耦合:

$$S = -mc \int ds - \frac{e}{c} \int A_\mu(x) dx^\mu, \quad A^\mu(x) = (\phi(x), \boldsymbol{A}(x)) \tag{26}$$

$$L = -mc^2 \sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}} + \frac{e}{c} \boldsymbol{v} \cdot \boldsymbol{A} - e\phi, \quad \boldsymbol{P} = \boldsymbol{p} + \frac{e}{c} \boldsymbol{A} \tag{27}$$

$$H = \sqrt{(mc^2)^2 + c^2 \left(\boldsymbol{P} - \frac{e}{c}\right)^2} + e\phi \tag{28}$$

规范对称性

$$A_\mu(x) \to A_\mu(x) + \partial_\mu f(x) \tag{29}$$

运动方程

$$mc\frac{du_\mu}{ds} = \frac{e}{c}F_{\mu\nu}u^\nu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_{[\mu}A_{\nu]} \tag{30}$$

$$F_{0i} = E_1,\ F_{12} = -B_3,\ F_{13} = +B_2,\ F_{23} = -B_1 \tag{31}$$

更进一步

$$\begin{cases} \boldsymbol{E'} = \gamma(\boldsymbol{E} + \beta \times \boldsymbol{B}) - \frac{\gamma^2}{1+\gamma}(\beta \cdot \boldsymbol{E})\beta \\ \boldsymbol{B'} = \gamma(\boldsymbol{B} - \beta \times \boldsymbol{E}) - \frac{\gamma^2}{1+\gamma}(\beta \cdot \boldsymbol{B})\beta \end{cases} \tag{32}$$

$$Bianchi\ identity:\ \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} + \partial_\alpha F_{\mu\nu} = 0 \tag{33}$$

$$S_{em} = -\frac{1}{16\pi c}\int d^4x F_{\mu\nu}F^{\mu\nu}\ (Gauss) \tag{34}$$

有源

$$J^\mu = \rho\frac{dx^\mu}{dt} = (c\rho, \boldsymbol{J}), \quad \partial_\mu J^\mu = 0 \tag{35}$$

$$S = -\frac{1}{c^2}\int d^4x A_\mu J^\mu - \frac{1}{16\pi c}\int d^4x F_{\mu\nu}F^{\mu\nu} \tag{36}$$

运动方程

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}J^\nu \tag{37}$$

引入微分几何

$$F = dA = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu \tag{38}$$

$$S = \int\left(-\frac{1}{2}F \wedge \star F + A \wedge \star J\right) \tag{39}$$

$$d \star J = 0, \quad d \star F = \star J \tag{40}$$

$$Q_e = \int_{\partial\Sigma}\star F, \quad Q_m = \int_{\partial\Sigma}F \tag{41}$$

3 电磁波的传播

3.1 平面电磁波

真空中

$$\begin{cases} \nabla^2 \boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0 \\ \nabla^2 \boldsymbol{B} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{B}}{\partial t^2} = 0 \end{cases} \tag{42}$$

有介质

$$\boldsymbol{D}(\omega) = \epsilon(\omega)\boldsymbol{E}(\omega), \quad \boldsymbol{B}(\omega) = \mu(\omega)\boldsymbol{H}(\omega) \tag{43}$$

单色波

$$\boldsymbol{E}(\boldsymbol{x}, t) = \boldsymbol{E}(\boldsymbol{x})e^{-i\omega t}, \quad \boldsymbol{B}(\boldsymbol{x}, t) = \boldsymbol{B}(\boldsymbol{x})e^{-i\omega t} \tag{44}$$

$$\begin{cases} \nabla \times \boldsymbol{E} = i\omega\mu\boldsymbol{H} \\ \nabla \times \boldsymbol{H} = -i\omega\epsilon\boldsymbol{E} \\ \nabla \cdot \boldsymbol{E} = 0 \\ \nabla \cdot \boldsymbol{H} = 0 \end{cases} \tag{45}$$

进一步得到 Helmholtz 方程 (基本方程)

$$\begin{cases} \nabla^2 \boldsymbol{E} + k^2\boldsymbol{E} = 0 \\ \nabla^2 \boldsymbol{H} + k^2\boldsymbol{H} = 0 \end{cases}, \quad k = \omega\sqrt{\mu\epsilon} \tag{46}$$

在一定频率下,Maxwell's equations 可以化为

$$\begin{cases} \nabla^2 \boldsymbol{E} + k^2\boldsymbol{E} = 0 \\ \nabla \cdot \boldsymbol{E} = 0 \\ \boldsymbol{B} = -\frac{i}{\omega}\nabla \times \boldsymbol{E} \end{cases} \tag{47}$$

或者

$$\begin{cases} \nabla^2 \boldsymbol{B} + k^2 \boldsymbol{B} = 0 \\ \nabla \cdot \boldsymbol{B} = 0 \\ \boldsymbol{E} = -\frac{i}{\omega\mu\epsilon} \nabla \times \boldsymbol{B} \end{cases} \tag{48}$$

解得平面电磁波

$$\boldsymbol{E}(\boldsymbol{x}, t) = \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}, \quad \boldsymbol{B}(\boldsymbol{x}, t) = \boldsymbol{B}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}, \quad \boldsymbol{k} = k\boldsymbol{n} \tag{49}$$

注意单色电磁波有

$$\nabla \to i\boldsymbol{k}, \ \frac{\partial}{\partial t} \to -i\omega, \quad \boldsymbol{k} \cdot \boldsymbol{E}_0 = \boldsymbol{k} \cdot \boldsymbol{B}_0 = 0 \tag{50}$$

$$\boldsymbol{k} \times \boldsymbol{E} = \omega \boldsymbol{B}, \quad \boldsymbol{k} \times \boldsymbol{B} = -\omega\mu\epsilon \boldsymbol{E}, \quad \frac{|\boldsymbol{E}|}{|\boldsymbol{B}|} = \frac{1}{\sqrt{\mu\epsilon}} = v \tag{51}$$

$$w = \epsilon E^2 = \frac{1}{\mu} B^2, \quad \boldsymbol{S} = \sqrt{\frac{\epsilon}{\mu}} E^2 \boldsymbol{n} = vw\boldsymbol{n} \tag{52}$$

3.2 介质表面的折射反射

介质表面通常没有自由电荷和传导电流. 在一定频率的情况下, 这组边界方程 (边值关系) 不是完全独立的.

$$\omega = \omega' = \omega'', \ k_x = k'_x = k''_x, \ k_y = k'_y = k''_y, \tag{53}$$

$$k = \omega\sqrt{\mu_1\epsilon_1}, \ k' = \omega'\sqrt{\mu_1\epsilon_1}, \ k'' = \omega''\sqrt{\mu_2\epsilon_2} \tag{54}$$

$$\theta = \theta', \quad \frac{\sin\theta}{\sin\theta''} = \frac{\sqrt{\mu_2\epsilon_2}}{\sqrt{\mu_1\epsilon_1}} = \frac{n_2}{n_1} = n_{21} \tag{55}$$

菲涅耳公式: 1.E 垂直入射面

$$\frac{E'_{0\perp}}{E_{0\perp}} = \frac{\cos\theta - \sqrt{\epsilon_2/\epsilon_1}\cos\theta''}{\cos\theta + \sqrt{\epsilon_2/\epsilon_1}\cos\theta''} = -\frac{\sin(\theta - \theta'')}{\sin(\theta + \theta'')} \tag{56}$$

$$\frac{E''_{0\perp}}{E_{0\perp}} = \frac{2\cos\theta}{\cos\theta + \sqrt{\epsilon_2/\epsilon_1}\cos\theta''} = \frac{2\cos\theta\sin\theta''}{\sin(\theta + \theta'')} \tag{57}$$

2. E 平行入射面

$$\frac{E'_{0\parallel}}{E_{0\parallel}} = \frac{\sqrt{\epsilon_2/\epsilon_1}\cos\theta - \cos\theta''}{\sqrt{\epsilon_2/\epsilon_1}\cos\theta + \cos\theta''} = -\frac{\tan(\theta-\theta'')}{\tan(\theta+\theta'')} \tag{58}$$

$$\frac{E''_{0\parallel}}{E_{0\parallel}} = \frac{2\cos\theta}{\sqrt{\epsilon_2/\epsilon_1}\cos\theta + \cos\theta''} = \frac{2\cos\theta\sin\theta''}{\sin\theta\cos\theta + \sin\theta''\cos\theta''} \tag{59}$$

偏振:

1. Brewster's angle: $\theta + \theta'' = \pi/2,\ \theta_b = \tan^{-1} n_{21}$

2. 当平面波从光疏介质入射到光密介质时, 半波损失; 当平面波从光密介质入射到光疏介质时, 没有半波损失

3.

$$R_\perp = \frac{\sin^2(\theta-\theta'')}{\sin^2(\theta+\theta'')}, \quad T_\perp = \frac{\sin 2\theta \sin 2\theta''}{\sin^2(\theta+\theta'')} \tag{60}$$

$$R_\parallel = \frac{\tan^2(\theta-\theta'')}{\tan^2(\theta+\theta'')}, \quad T_\parallel = \frac{\sin 2\theta \sin 2\theta''}{\sin^2(\theta+\theta'')\cos^2(\theta-\theta'')} \tag{61}$$

全反射

$$\theta_0 = \sin^{-1} n_{21}, \quad k''_z = i\alpha\,(\sin\theta > n_{21}),\ \alpha = k\sqrt{\sin^2\theta - n_{21}^2}, k = \omega/v_1 \tag{62}$$

$$\boldsymbol{E''} = \boldsymbol{E''_0} e^{-\alpha z} e^{i(kx\sin\theta - \omega t)}, \quad \langle S''_z \rangle = 0 \tag{63}$$

$$\frac{E'_{0\perp}}{E_{0\perp}} = e^{-i2\phi}, \quad \tan\phi = \frac{\sqrt{\sin^2\theta - n_{21}^2}}{\cos\theta} \tag{64}$$

$$\frac{E'_{0\parallel}}{E_{0\parallel}} = e^{-i2\psi}, \quad \tan\psi = \frac{\sqrt{\sin^2\theta - n_{21}^2}}{n_{21}^2\cos\theta} \tag{65}$$

一个线偏振波入射在介质界面上经过反射成了一个椭园偏振波

3.3 导体

良导体条件 $\sigma/\epsilon\omega \gg 1$ 表示导体中传导电流与位移电流之比、导体中磁场能与电场能之比. 在讨论电磁波在导体中的传播问题时, 可以认为

$\rho = 0, \boldsymbol{j} = \sigma \boldsymbol{E}$.

$$\nabla \times H = -i\omega\epsilon' \boldsymbol{E}, \quad \epsilon' = \epsilon + i\frac{\sigma}{\omega} \tag{66}$$

$$\begin{cases} \nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} = 0 \\ \nabla^2 \boldsymbol{H} + k^2 \boldsymbol{H} = 0 \end{cases}, \quad k^2 = \omega^2 \mu \epsilon', \quad \boldsymbol{k} = \boldsymbol{\beta} + i\boldsymbol{\alpha} \tag{67}$$

$$\alpha_x = \alpha_y = \beta_y = 0,\ \beta_x = \omega\sqrt{\mu_0\epsilon_0}\sin\theta,\ \alpha_z\beta_z = \frac{\omega\mu\sigma}{2}, \tag{68}$$

$$\boldsymbol{\beta}^2 - \boldsymbol{\alpha}^2 = \omega^2\mu\epsilon \Rightarrow \beta_z^2 \approx \alpha_z^2 - \beta_x^2 \tag{69}$$

略去 β_x

$$\alpha_z = \beta_z = \sqrt{\frac{\omega\mu\sigma}{2}}, \quad \beta_z \gg \beta_x = \omega\sqrt{\mu_0\epsilon_0}\sin\theta \tag{70}$$

折射角 $\theta'' = \tan^{-1}\frac{\beta_x}{\beta_z}$

取 $\beta_x = 0$, 精确解

$$\beta = \omega\sqrt{\mu\epsilon}\sqrt{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right]},\ \alpha = \omega\sqrt{\mu\epsilon}\sqrt{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1\right]} \tag{71}$$

在导体中波长变短, 相速度 $v_p = \omega/\beta$. 磁场相位比电场相位滞后 $\pi/4$, 金属内主要是储存磁能.

反射

$$\cos\theta'' = \sqrt{\frac{\epsilon' - \epsilon_0\sin^2\theta}{\epsilon'}} \tag{72}$$

$$\frac{E'_{0\perp}}{E_{0\perp}} = \frac{\sqrt{\epsilon_0}\cos\theta - \sqrt{\epsilon'}\cos\theta''}{\sqrt{\epsilon_0}\cos\theta + \sqrt{\epsilon'}\cos\theta''}, \frac{E''_{0\perp}}{E_{0\perp}} = \frac{2\sqrt{\epsilon_0}\cos\theta}{\sqrt{\epsilon_0}\cos\theta + \sqrt{\epsilon'}\cos\theta''} \tag{73}$$

$$\frac{E'_{0\parallel}}{E_{0\parallel}} = \frac{\sqrt{\epsilon'}\cos\theta - \sqrt{\epsilon_0}\cos\theta''}{\sqrt{\epsilon'}\cos\theta + \sqrt{\epsilon_0}\cos\theta''}, \frac{E''_{0\parallel}}{E_{0\parallel}} = \frac{2\sqrt{\epsilon_0}\cos\theta}{\sqrt{\epsilon'}\cos\theta + \sqrt{\epsilon_0}\cos\theta''} \tag{74}$$

若垂直入射

$$\frac{E'_{0\perp}}{E_{0\perp}} = -\frac{E'_{0\parallel}}{E_{0\parallel}} = \frac{k - k''}{k + k''}, \quad k'' = \frac{1+i}{\delta},\ \delta = \sqrt{\frac{2}{\mu\sigma\omega}} \tag{75}$$

$$R_\perp = R_\parallel = \frac{\left(1 - \sqrt{2\epsilon_0\omega/\sigma}\right)^2 + 1}{\left(1 + \sqrt{2\epsilon_0\omega/\sigma}\right)^2 + 1} \approx 1 - 2\sqrt{\frac{2\epsilon_0\omega}{\sigma}} \tag{76}$$

3.4 波导

横电磁波, 简称 TEM 波

矩形波导

$$\begin{cases} E_x = B_1 \cos k_x x \sin k_y y e^{i(k_z z - \omega t)} \\ E_y = B'_1 \sin k_x x \cos k_y y e^{i(k_z z - \omega t)} \\ E_z = \frac{B_1 k_x + B'_1 k_y}{i k_z} \sin k_x x \sin k_y y e^{i(k_z z - \omega t)} \end{cases}, \quad \nabla \times \boldsymbol{E} = i\omega\mu\boldsymbol{H} \tag{77}$$

$$k_x = m\pi/a, \quad k_y = n\pi/b, \quad k_z = \sqrt{\frac{\omega^2}{v^2} - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \tag{78}$$

电场和磁场不能同时为横波, 通常选一种波模为 $E_z = 0$ 的波称为横电波 (TEW). 另一种波模为 $H_z = 0$ 的波称为横磁波 (TMW) . TEW 和 TMW 又按 (m,n) 值的不同而分为 TE_{mn} 波和 TM_{mn} 波. 一般情况下, 在波导中可以存在这些波的叠加.

波导内不可能传播横电磁波, 一组 (m,n) 的值组成一个模式. 临界状态:$k_z = 0$; 截止频率:

$$\omega_{c,mn} = \frac{\pi}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

. 对于 TE_{10} 波 (又称为主波), 通常对于矩形波导总是取 $a > b$, 于是 $\omega_{c,10}$ 给出矩形波导中的最小截止频率.

相速度可能大于光速

$$u = \frac{dz}{dt} = \frac{\omega}{k_z} \geq v \tag{79}$$

群速度

$$u_g = \frac{d\omega}{dk_z} = \frac{v^2}{u} \leq v \tag{80}$$

谐振腔见书.

4 电磁波的辐射

4.1 矢势、标势

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \quad \boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} \tag{81}$$

规范变换

$$\boldsymbol{A} \to \boldsymbol{A'} = \boldsymbol{A} + \nabla\psi, \quad \phi \to \phi' = \phi - \frac{\partial\psi}{\partial t} \tag{82}$$

库伦规范

$$\nabla \cdot \boldsymbol{A} = 0 \tag{83}$$

洛伦兹规范

$$\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0 \tag{84}$$

达朗贝尔 (d' Alembert) 方程

$$\begin{cases} \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \\ \nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = -\mu_0\boldsymbol{j} \end{cases} \tag{85}$$

其中采用洛仑兹规范. 若采用库伦规范, 得

$$\begin{cases} \nabla^2\phi = -\frac{\rho}{\epsilon_0} \\ \nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} - \frac{1}{c^2}\frac{\partial\nabla\phi}{\partial t} = -\mu_0\boldsymbol{j} \end{cases} \tag{86}$$

以后都采用洛仑兹规范.

4.2 推迟势

达朗贝尔方程的解

$$\phi(\boldsymbol{x},t)=\frac{1}{4\pi\epsilon_0}\int_{V'}\frac{\rho(\boldsymbol{x'},t-\frac{r}{c})}{r}d\tau',\quad \boldsymbol{A}(\boldsymbol{x},t)=\frac{\mu_0}{4\pi}\int_{V'}\frac{\boldsymbol{j}(\boldsymbol{x'},t-\frac{r}{c})}{r}d\tau' \tag{87}$$

电磁作用是以有限速度 $v=c$ 向外传播的. 推迟势满足 Lorentz 规范条件

4.3 电偶极辐射

若

$$\boldsymbol{j}(\boldsymbol{x'},t')=\boldsymbol{j}(\boldsymbol{x'})e^{-i\omega t'},\quad k=\omega/c \tag{88}$$

则

$$\boldsymbol{A}(\boldsymbol{x},t)=\boldsymbol{A}(\boldsymbol{x})e^{-i\omega t},\quad \boldsymbol{A}(\boldsymbol{x})=\frac{\mu_0}{4\pi}\int_{V'}\frac{\boldsymbol{j}(\boldsymbol{x'})e^{ikr}}{r}d\tau' \tag{89}$$

电流分布于小区域而激发的远区场, 展开

$$\boldsymbol{A}(\boldsymbol{x})=\frac{\mu_0 e^{ikR}}{4\pi R}\int_{V'}\boldsymbol{j}(\boldsymbol{x'})\left[1-ik\boldsymbol{n}\cdot\boldsymbol{x'}+\frac{1}{2!}(ik\boldsymbol{n}\cdot\boldsymbol{x})^2+\dots\right]d\tau' \tag{90}$$

偶极辐射对应第一项

$$\boldsymbol{A}_{(1)}(\boldsymbol{x})=\frac{\mu_0 e^{ikR}}{4\pi R}\dot{\boldsymbol{p}},\quad \boldsymbol{p}=\int q\boldsymbol{x'}d\tau',\ \dot{\boldsymbol{p}}=\int\boldsymbol{j}(\boldsymbol{x'})d\tau' \tag{91}$$

$$\boldsymbol{B}=\frac{i\mu_0 k}{4\pi R}e^{ikR}\boldsymbol{n}\times\dot{\boldsymbol{p}}=\frac{(\ddot{\boldsymbol{p}}\times\boldsymbol{n})e^{ikR}}{4\pi\epsilon_0 Rc^3},\quad \boldsymbol{E}=c\boldsymbol{B}\times\boldsymbol{n} \tag{92}$$

球坐标

$$\boldsymbol{B}=\frac{|\ddot{\boldsymbol{p}}|\sin\theta e^{ikR}}{4\pi\epsilon_0 Rc^3}\boldsymbol{e}_\phi,\quad \boldsymbol{E}=\frac{|\ddot{\boldsymbol{p}}|\sin\theta e^{ikR}}{4\pi\epsilon_0 Rc^2}\boldsymbol{e}_\theta \tag{93}$$

能流密度

$$\langle\boldsymbol{S}\rangle=\frac{|\ddot{\boldsymbol{p}}|^2\sin^2\theta}{32\pi^2\epsilon_0 R^2c^3}\boldsymbol{n} \tag{94}$$

角分布

$$\langle f(\theta,\phi)\rangle=\frac{\langle\boldsymbol{S}\rangle\cdot d\boldsymbol{s}}{d\Omega}=\frac{|\ddot{\boldsymbol{p}}|^2\sin^2\theta}{32\pi^2\epsilon_0 c^3} \tag{95}$$

辐射功率

$$\langle P\rangle=\oint_S\langle\boldsymbol{S}\rangle\cdot d\boldsymbol{s}=\frac{|\ddot{\boldsymbol{p}}|^2}{12\pi\epsilon_0 c^3} \tag{96}$$

4.4 磁偶极辐射, 电四极辐射

矢势展开第二项

$$\boldsymbol{A}_{(2)}(\boldsymbol{x}) = -\frac{ik\mu_0}{4\pi R}e^{ikR}\left[-\boldsymbol{n}\times\boldsymbol{m}+\frac{1}{6}\boldsymbol{n}\cdot\vec{\boldsymbol{D}}\right], \tag{97}$$

$$\boldsymbol{m} = \frac{1}{2}\int \boldsymbol{x'}\times\boldsymbol{j}(\boldsymbol{x'})d\tau',\ \vec{\boldsymbol{D}} = 3\int q\boldsymbol{x'}\boldsymbol{x'}d\tau' \tag{98}$$

磁偶极辐射项

$$\boldsymbol{A}^m_{(2)}(\boldsymbol{x}) = \frac{\mu_0 e^{ikR}}{4\pi R}\nabla\times\boldsymbol{m} \tag{99}$$

$$\boldsymbol{B} = \frac{\mu_0 e^{ikR}}{4\pi Rc^2}(\ddot{\boldsymbol{m}}\times\boldsymbol{n})\times\boldsymbol{n}, \quad \boldsymbol{E} = \frac{\mu_0 e^{ikR}}{4\pi Rc}(\ddot{\boldsymbol{m}}\times\boldsymbol{n}) \tag{100}$$

对比

$$\boldsymbol{p}\leftrightarrow\boldsymbol{m}/c, \quad \boldsymbol{E}_p\leftrightarrow c\boldsymbol{B}_m, \quad c\boldsymbol{B}_p\leftrightarrow -\boldsymbol{E}_m \tag{101}$$

其能流密度

$$\langle\boldsymbol{S}\rangle = \frac{\mu_0\omega^4|\ddot{\boldsymbol{m}}|^2\sin^2\theta}{32\pi^2R^2c^3}\boldsymbol{n} \tag{102}$$

总辐射功率

$$\langle P\rangle = \frac{\mu_0\omega^4|\ddot{\boldsymbol{m}}|^2}{12\pi c^3} \tag{103}$$

电四极辐射项

$$\boldsymbol{A}^e_{(2)}(\boldsymbol{x}) = \frac{\ddot{\boldsymbol{D}}e^{ikR}}{24\pi\epsilon_0 Rc^3}, \quad \boldsymbol{D} = \boldsymbol{n}\cdot\vec{\boldsymbol{D}} \tag{104}$$

$$\boldsymbol{B} = \frac{(\dddot{\boldsymbol{D}}\times\boldsymbol{n})e^{ikR}}{24\pi\epsilon_0 Rc^4}, \quad \boldsymbol{E} = \frac{(\dddot{\boldsymbol{D}}\times\boldsymbol{n})\times\boldsymbol{n}e^{ikR}}{24\pi\epsilon_0 Rc^3} \tag{105}$$

其能流密度

$$\langle\boldsymbol{S}\rangle = \frac{1}{4\pi\epsilon_0}\frac{|\dddot{\boldsymbol{D}}\times\boldsymbol{n}|^2}{288\pi R^2c^5} \tag{106}$$

4.5 干涉, 衍射

合成振幅

$$\boldsymbol{E} = 2\boldsymbol{E}_0\cos(\pi\frac{\Delta}{\lambda})\cos 2\pi\left(\frac{t}{T}-\frac{r_1+r_2}{2\lambda}\right), \quad \Delta = r_2 - r_1 \tag{107}$$

当光程差为半光波长的偶数倍时, 合成波振幅最大; 当光程差为半波长的奇数倍时, 合成波振幅为 0.

相干条件:

1. 它们的电场强度和磁场强度都必须分别具有相同的振动方向.
2. 它们的频率必须相同.
3. 两列波的光程差不能太大.
4. 两列波的振幅不能悬殊太大.

标量场的衍射理论

$$\nabla^2\psi + k^2\psi = 0, \quad \nabla^2 G + k^2 G = -4\pi\delta(\boldsymbol{x}-\boldsymbol{x'}),\, G(\boldsymbol{x},\boldsymbol{x'}) = \frac{e^{ikr}}{r} \tag{108}$$

基尔霍夫公式

$$\psi(\boldsymbol{x}) = -\frac{1}{4\pi}\oint_{S'}\frac{e^{ikr}}{r}\boldsymbol{n}\cdot\left[\nabla'\psi(\boldsymbol{x'}) + (ik-\frac{1}{r})\frac{\boldsymbol{r}}{r}\psi(\boldsymbol{x'})\right]ds' \tag{109}$$

即惠更斯原理.

矩形孔的夫琅和费衍射, 入射波是平面波

$$\psi(\boldsymbol{x'}) = -\frac{i\psi_0 e^{ikR}}{4\pi R}(\boldsymbol{k}_1+\boldsymbol{k}_2)\cdot\boldsymbol{n}\cdot 4\frac{\sin(k_{1x}-k_{2x})a}{k_{1x}-k_{2x}}\frac{\sin(k_{1y}-k_{2y})b}{k_{1y}-k_{2y}} \tag{110}$$

$$I(\alpha,\beta,\gamma) = \frac{\psi_0^2}{k^2\pi^2R^2}(1+\cos\gamma)^2\left[\frac{\sin(ak\cos\alpha)\sin(bk\cos\beta)}{\cos\alpha\cos\beta}\right]^2 \tag{111}$$

$b \gg \lambda$, 单缝衍射

$$I(\alpha,\beta) = \frac{\psi_0^2}{k^2\pi^2R^2}(1+\cos\gamma)^2\left[\frac{\sin(ak\cos\alpha)}{\cos\alpha}\right]^2, \quad \beta\to\pi/2 \tag{112}$$

4.6 动量

$$\frac{d\boldsymbol{P}_m}{dt} = -\oint_S \vec{\boldsymbol{T}}\cdot d\boldsymbol{s} - \frac{d}{dt}\int_V \frac{1}{c^2}\boldsymbol{S}d\tau \tag{113}$$

$$\Rightarrow \boldsymbol{P}_e = \int_\infty \frac{1}{c^2}\boldsymbol{S}d\tau = \int_\infty \epsilon_0(\boldsymbol{E}\times\boldsymbol{B})d\tau, \tag{114}$$

$$\boldsymbol{g} = \frac{1}{c^2}\boldsymbol{S} = \epsilon_0(\boldsymbol{E}\times\boldsymbol{B}) \tag{115}$$

其中

$$\vec{\boldsymbol{T}} = -\epsilon_0 \boldsymbol{E}\boldsymbol{E} - \frac{1}{\mu_0}\boldsymbol{B}\boldsymbol{B} + \frac{1}{2}(\epsilon_0 E^2 + \frac{B^2}{\mu_0})\vec{\boldsymbol{I}}, \tag{116}$$

$$\boldsymbol{f} = -\nabla \cdot \vec{\boldsymbol{T}} - \frac{1}{c^2}\frac{\partial \boldsymbol{S}}{\partial t} \tag{117}$$

$\vec{\boldsymbol{T}}$ 为电磁场动量流密度, $-\boldsymbol{n} \cdot \vec{\boldsymbol{T}}$ 称之为 Maxwell 应力张量或张力张量. T_{ij} 的意义是通过垂直于 i 轴的单位面积流过的动量 j 分量.

平面电磁波

$$\boldsymbol{g} = \frac{w}{c}\boldsymbol{n} \tag{118}$$

辐射压力

$$P = \frac{1+R}{6}\langle w \rangle \tag{119}$$

R 为反射系数. 理想导体表面 $R = 1, P = \langle w \rangle/3$.

5 狭义相对论

5.1 基本原理

伽利略变换

$$\begin{cases} \boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{v}t \\ t' = t \end{cases} \tag{120}$$

速度变换 $\boldsymbol{u}' = \boldsymbol{u} - \boldsymbol{v}$

狭义相对论的基本原理

- 相对性原理: 一切物理规律, 无论是力学的, 还是电磁学的, 对于所有惯性系都具有相同的数学形式
- 光速不变原理: 在所有惯性系中, 真空中的光速在任何方向上都恒为 c, 并与光源的运动无关

(限于相互作匀速直线运动的惯性系)

5.2 洛伦兹变换

间隔不变式

$$ds'^2 = ds^2, \quad ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{121}$$

令

$$(x_1, x_2, x_3, x_4) = (x, y, z, ict), \quad x'_\mu = a_{\mu\nu}x_\nu \tag{122}$$

其中

$$a_{\mu\nu}a_{\mu\sigma} = \delta_{\nu\sigma} \tag{123}$$

洛伦兹变换

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma(t - \frac{v}{c^2}x) \end{cases}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} > 1, \quad \beta = v/c \tag{124}$$

矩阵形式

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix}, \quad \Lambda^{-1} = \begin{pmatrix} \gamma & 0 & 0 & -i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \tag{125}$$

5.3 时空性质

间隔

- $s^2 = 0$: 类光间隔
- $s^2 > 0$: 类时间隔
- $s^2 < 0$: 类空间隔

同时的相对性: 若两个事件在某一参考系中为同时异地事件, 那么根据 Lorentz 变换式, 在其他参考系中这两个事件就不是同时的.

Lorentz 收缩:

$$l = \gamma^{-1} l_0 \tag{126}$$

物体沿其长度方向运动时, 其长度缩短. Einstein 延缓:

$$\Delta t = \gamma \Delta \tau \tag{127}$$

运动的时钟所指示的时间间隔比静止的时钟所指示的时间间隔要小.

因果律:$u < c$; 因果事件的四维间隔一定是类时的, 而类空间隔的两事件一定没有因果关系.

速度变换

$$u_x' = \frac{u_x - v}{1 - \frac{vu_x}{c^2}},\ u_y' = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)},\ u_z' = \frac{u_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} \tag{128}$$

一般地

$$\boldsymbol{u'} = \frac{\gamma^{-1}\boldsymbol{u} + (1-\gamma^{-1})\frac{(\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{v}}{v^2} - \boldsymbol{v}}{1 - \frac{\boldsymbol{v}\cdot\boldsymbol{u}}{c^2}} \tag{129}$$

当 $\boldsymbol{v} \perp \boldsymbol{u}$ 时

$$\boldsymbol{u'} = \gamma^{-1}\boldsymbol{u} - \boldsymbol{v} \tag{130}$$

加速度变换

$$a_x' = \frac{a_x}{\gamma^3(1 - \frac{vu_x}{c^2})^3},\ a_y' = \frac{a_y + \frac{vu_y}{c^2 - vu_x}a_x}{\gamma^2(1 - \frac{vu_x}{c^2})^3},\ a_z' = \frac{a_z + \frac{vu_z}{c^2 - vu_x}a_x}{\gamma^2(1 - \frac{vu_x}{c^2})^3} \tag{131}$$

一般地

$$\boldsymbol{a'} = \frac{\boldsymbol{a} - \frac{\boldsymbol{v}\times(\boldsymbol{a}\times\boldsymbol{u})}{c^2} - (\gamma - 1)\frac{(\boldsymbol{a}\cdot\boldsymbol{v})\boldsymbol{v}}{v^2}}{\gamma^2(1 - \frac{\boldsymbol{v}\cdot\boldsymbol{u}}{c^2})^3} \tag{132}$$

5.4 相对论力学

四维速度

$$u_\mu = \frac{dx_\mu}{d\tau} = \gamma(v_i, ic), \quad d\tau = \gamma^{-1}dt,\ ds^2 = c^2 d\tau^2 \tag{133}$$

四维动量

$$p_\mu = \gamma(m_0 u_i, i m_0 c) \tag{134}$$

四维力 (闵可夫斯基力)

$$K_\mu = \frac{dp_\mu}{d\tau}, \quad F_i = \gamma^{-1} K_i \tag{135}$$

m_0 为静止质量, 任何大的力都不可能使具有静止质量的质点加速到光速 c.

能量

$$\boldsymbol{K} \cdot \boldsymbol{v} = \frac{d}{d\tau}\left(\gamma m_0 c^2\right) \Rightarrow \boldsymbol{F} \cdot \boldsymbol{v} = \frac{d}{dt}\left(\gamma m_0 c^2\right) \tag{136}$$

$$W = \gamma m_0 c^2 = mc^2, \quad p_4 = \frac{i}{c} W, \; K_\mu = (K_i, K_4) = (\boldsymbol{K}, \frac{i}{c} \boldsymbol{K} \cdot \boldsymbol{v}) \tag{137}$$

Einstein 质能联系定律

$$T = (\gamma - 1) m_0 c^2 = mc^2 - W_0 \tag{138}$$

质量亏损 $\Delta W = (\Delta m) c^2$. 能量、动量和质量的关系式

$$W = \sqrt{(pc)^2 + (m_0 c^2)^2} \tag{139}$$

5.5 相对论电磁学

达朗伯算符 (洛仑兹标量算符)

$$\Box = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \tag{140}$$

四维电流密度矢量, 四维势矢量

$$j_\mu = (\boldsymbol{j}, i c \rho), \quad A_\mu = (\boldsymbol{A}, \frac{i}{c} \phi) \tag{141}$$

势方程

$$\Box A_\mu = j_\mu \tag{142}$$

电磁场张量

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & B_3 & -B_2 & -\frac{i}{c}E_1 \\ -B_3 & 0 & B_1 & -\frac{i}{c}E_2 \\ B_2 & -B_1 & 0 & -\frac{i}{c}E_3 \\ \frac{i}{c}E_1 & \frac{i}{c}E_2 & \frac{i}{c}E_3 & 0 \end{pmatrix}, \quad F' = \Lambda F \Lambda^T \tag{143}$$

Maxwell 方程

$$\begin{cases} \partial_\nu F_{\mu\nu} = \mu_0 j_\mu \\ \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \end{cases} \tag{144}$$

电磁场变换式

$$\begin{cases} E'_x = E_x \\ E'_y = \gamma(E_y - vB_z) \\ E'_z = \gamma(E_z + vB_y) \end{cases}, \quad \begin{cases} B'_x = B_x \\ B'_y = \gamma(B_y + \frac{v}{c^2}E_z) \\ B'_z = \gamma(B_z - \frac{v}{c^2}E_y) \end{cases} \tag{145}$$

或

$$\begin{cases} \boldsymbol{E}'_\parallel = \boldsymbol{E}_\parallel \\ \boldsymbol{E}'_\perp = \gamma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})_\perp \end{cases}, \quad \begin{cases} \boldsymbol{B}'_\parallel = \boldsymbol{B}_\parallel \\ \boldsymbol{B}'_\perp = \gamma(\boldsymbol{B} - \frac{1}{c^2}\boldsymbol{v} \times \boldsymbol{E})_\perp \end{cases} \tag{146}$$

非相对论电磁场变换式 ($v \ll c$)

$$\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}, \quad \boldsymbol{B}' = \boldsymbol{B} - \frac{1}{c^2}\boldsymbol{v} \times \boldsymbol{E} \tag{147}$$

两个不变量

$$\frac{1}{2}F_{\mu\nu}F_{\mu\nu} = B^2 - \frac{1}{c^2}E^2, \quad |F_{\mu\nu}| = -\frac{1}{c^2}(\boldsymbol{E} \cdot \boldsymbol{B})^2 \tag{148}$$

$$\Rightarrow B = \frac{E}{c} \Leftrightarrow B' = \frac{E'}{c}, \quad \cos\langle \boldsymbol{E}, \boldsymbol{B} \rangle = \cos\langle \boldsymbol{E}', \boldsymbol{B}' \rangle \tag{149}$$

单色平面电磁波

$$\begin{cases} k_x' = \gamma(k_x - \frac{v}{c^2}\omega) \\ k_y' = k_y \\ k_z' = k_z \\ \omega' = \gamma(\omega - v k_x) \end{cases}, \quad k_\mu x_\mu = Const \tag{150}$$

其中

$$k_\mu = (\boldsymbol{k}, \frac{i}{c}\omega) \tag{151}$$