Many-body localization, thermalization, and entanglement (I)

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ergodic system: different degrees of freedom exchange energy, information and particles.(thermalization)

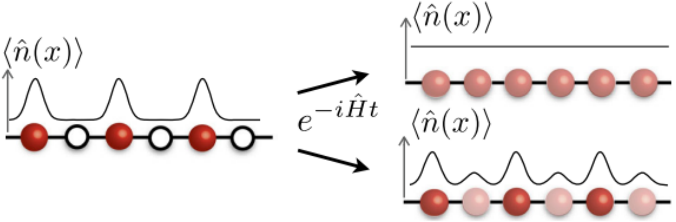


图: quantum quench

Quantum information encoded in the initial state.
Under unitary evolution $e^{-i\hat{H}t}$ $(\hbar = 1)$ sufficiently long times,

$$|\psi(t)\rangle = e^{-i\hat{H}t} |\psi(0)\rangle \, ,$$

where the initial state is a nonequilibrium state.
The state of an ergodic system will have local observables which appear thermal.

Anderson localization(1958): noninteracting, disordered $\rightarrow$ insulator.
The wave functions are localized in the strong-disorder limit, no propagating Bloch states

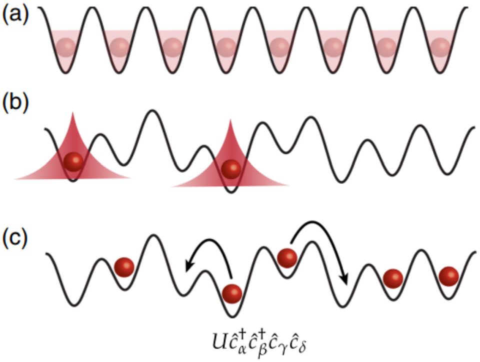


图: (a)Bloch waves;(b)noninteraction;(c)interaction-hopping

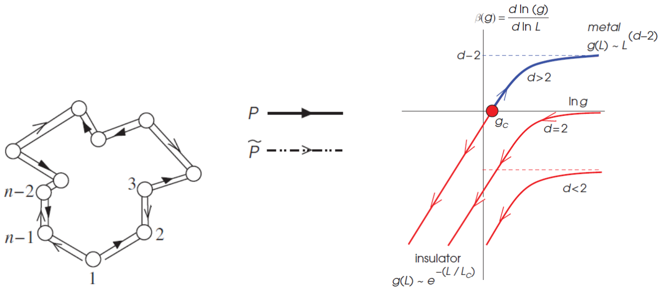


图: Anderson localization

Thouless energy $E_{\text{th}} = \hbar \frac{D}{L^2}$, where $D = v_f^2 \tau / d$ and τ is the time scale induced by Kubo formula.

strong disorder: random potential $W \gg$ tunneling t.
tunneling processes between two sites ($\sim n$ sites apart) occur in the n^{th} perturbation theory, $t_n \sim (t/W)^n$;
gap δ_n will decay only with distance n, $\delta_n \sim W/n^d$, where d is the number of spatial dimensions.
$\Rightarrow$ the absence of diffusion and transportation.
More:scaling theory, RG, metal-insulator transition, intricate effects of symmetries...
Open questions:if it is a true phase of matter?

Perfect interacting insulator at nonzero temperature is said to be many-body localized (MBL).

$\sim$ a dynamical phase of matter. (Lee-yang zeros for Loschmidt amplitude)

cases: quantum dot, higher-dimensional systems...

MBL: robust $\rightarrow$ different from Anderson localization of its ground state.

MBL: with a finite density of excitations above the ground state (or states with a finite energy density)

zero-temperature localization: only the localization of a finite number of excitations in the whole system, corresponding to a vanishing energy density as the system size is taken to infinity.

MBL provides the only known robust mechanism to avoid thermalization in a closed system.
Other examples(not thermalize):
noninteracting systems, Yang-Baxter integrable quantum models in one spatial dimension...

- any multiparticle interaction process can be reduced to two-particle collisions (Sutherland 2004)
- not robust! (D'Alessio 2016)

theory: quantum information (shifted from traditional setups);
experiment: isolated synthetic quantum systems, e.g. quantum quench

ETH: eigenstate thermalization hypothesis
classical: one can track a trajectory in the phase space;
quantum: states in Hilbert space.
an isolated quantum many-body system with $\hat{H}$(local lattice, but can be extend to continuum models).
expand the initial nonequilibrium state $|\psi(0)\rangle = \sum_\alpha A_\alpha |\alpha\rangle$, where $|\alpha\rangle$ are eigenstates.
After evolution, A_α acquires a phase factor:

$$|\psi(t)\rangle = e^{-i\hat{H}t} |\psi(0)\rangle = \sum_\alpha A_\alpha e^{-iE_\alpha t} |\alpha\rangle , \tag{1}$$

and $p_\alpha = |A_\alpha|^2$ does not change.
$\rightarrow$ how to modify the notion of ergodicity?

(physical initial state)observables $\to$ values in microcanonical ensembles at sufficiently long time.
$\hat{O}$ is a linear combination of few-body operators

$$\langle \hat{O} \rangle_\infty = \lim_{T\to\infty} \frac{1}{T} \int_0^T \langle \psi(t) | \hat{O} | \psi(t) \rangle \, \mathrm{d}t = \sum_\alpha p_\alpha \langle \alpha | \hat{O} | \alpha \rangle \,. \tag{2}$$

which is encoded in p_α.
notation: an individual eigenstate of a generic many-body system is inaccessible, as its preparation requires time which is exponentially long in a system's size.

ETH: in ergodic systems individual many-body eigenstates have thermal observables, identical to microcanonical ensemble values at energy $E = E_\alpha$, $\langle\alpha|\,\hat{O}\,|\alpha\rangle \approx \mathcal{O}_{\mathsf{mc}}(E)$.
what's the statment means:
even if the entire system is prepared in an eigenstate, its subsystems experience the remainder as an effective heat bath and explore possible configurations, restricted only by global conservation laws.
(numerical:D'Alessio et al., 2016)

ETH implies thermalization at infinitely long times.
For physical initial states, certain energy $\rightarrow \langle \hat{O} \rangle_\infty \approx \mathcal{O}_{\text{mc}}$.
How to describe the approach to the equilibrium values and bound the temporal fluctuations?
Introduced an ansatz(Srednicki, 1999) for both diagonal and off-diagonal matrix elements of local operators, which is sufficient to ensure thermalization (proved).

$$\langle \alpha | \hat{O} | \beta \rangle = \mathcal{O}_{\text{mc}}(\bar{E})\delta_{\alpha\beta} + e^{-S_{\text{th}}^{\bar{E}}/2} R_{\alpha\beta} f(\omega, \bar{E}), \tag{3}$$

where $\bar{E} = (E_\alpha + E_\beta)/2$ and $\omega = E_\alpha - E_\beta$. $S_{\text{th}}^{\bar{E}}$ is the thermodynamic entropy, $R_{\alpha\beta}$ is a normal-distributed random number and $f(\omega, \bar{E})$ as well as $\mathcal{O}_{\text{mc}}(\bar{E})$ are smooth functions of ω and $\bar{E}$.

For an eigenstate $|\alpha\rangle$ obeying ETH, all observables within a sufficiently small subsystem A will have thermal expectation values. $\rightarrow \rho_A = \mathrm{tr}_B |\alpha\rangle \langle\alpha|$ and the von Neumann entropy is equal to the thermodynamic entropy.

$$S_{\text{ent}}(A) = -\mathrm{Tr}\rho_A \log \rho_A = S_{\text{th}}(A). \tag{4}$$

For highly excited eigenstates, $S_{\text{ent}} \propto$ Volume.
Open questions: if ETH is a necessary condition for thermalization; whether ansatz(3) is also a necessary condition.

$$\langle\alpha|\,\hat{O}\,|\beta\rangle = \mathcal{O}_{\mathsf{mc}}(\bar{E})\delta_{\alpha\beta} + e^{-S_{\mathsf{th}}^{\bar{E}}/2}R_{\alpha\beta}f(\omega,\bar{E}),$$

implies the strong sensitivity.
For $\hat{H} \to \hat{H} + \epsilon\hat{O}$, $f(\omega,\bar{E})$ exhibits an algebraic decay for $\omega \lesssim J$, where J is the characteristic energy scale of $\hat{H}$.
Thouless energy(D'Alessio et al., 2016).
the off-diagonal matrix element is exponentially larger than the many-body level spacing, which scales as $\Delta \sim Je^{-S_{\mathsf{th}}^{\bar{E}}}$. $\to \epsilon\hat{O}$ with $\epsilon \ll 1$ has a nonlocal effect in Hilbert space, mixing an exponentially large number of original eigenstates.
chaos

sensitivity $\rightarrow$ level repulsion $s_\alpha = E_{\alpha+1} - E_\alpha$
s_a can be an indicator of quantum chaos for few-body systems (Wigner, 1951), e.g., stadium billiards.
zero dimensional quantum systems ($\sim$ chaotic dynamics) display level repulsion.
The level spacings in such systems obey Wigner-Dyson statistics ($p(s) \sim s^\beta$ as $s \rightarrow 0$ with $\beta = 1, 2, 4$ $\sim$ symmetry).
Wigner-Dyson statistics was also found in thermalizing many-body lattice models.
If the system has an extensive number of integrals of motion (like Bethe-ansatz), the eigenenergies that belong to different sectors behave as independent random variables. Hence, in such systems the distribution of level spacings is Poisson, and $p(0) = 1$.

localization $\leftarrow$ the absence of transport $\begin{cases} \text{dimensionality,} \\ \text{form of disorder.} \end{cases}$

$d = 1, 2$ / random uncorrelated disorder: localized for arbitrarily weak disorder and single-particle.

$d \geq 3$: a metal-insulator transition $\sim$ disorder strength. $\rightarrow$ single-particle mobility edge

*Interactions may open up new transport channels: decay from a high-energy (single-particle).

ζ: localization length; $\lambda \sim$ interactions, given by the ratio of the two-particle transition matrix element to the level spacing δ_ζ of excitations.

Others:

(1) Despite the presence of interactions, these localized states remain close to a noninteracting single-particle excitation with a perturbatively small admixture of a few particle-hole excitations.

(2) connectivity K depends on the temperature T: $K \sim T/\delta_\zeta$.

(3) critical temperature below which the interacting model is localized as $K_c \approx T_c/\delta_\zeta \approx 1/(\lambda \ln|\lambda|)$.

(4) The transition between localized and delocalized many-body eigenstates happens at a finite energy density, which was named a “many-body mobility edge” .

(numerical) disordered lattice models with a finite dimension of the local Hilbert space can remain in the MBL phase even at infinite temperature.
As a specific model, they studied a 1D chain of spinless fermions with an on-site disorder, nearest-neighbor interactions, and hopping between nearest-neighbor and nextnearest-neighbor sites. without longer-range hopping:

$$\hat{H} = t\sum_i(\hat{c}_i^\dagger \hat{c}_{i+1} + \text{H.c.}) + V\sum_i \hat{n}_i\hat{n}_{i+1} + \sum_i \epsilon_i\hat{n}_i \tag{5}$$

where $\epsilon_i \in [-W, W]$.

Jordan-Wigner transformation:

$$\hat{\sigma}_j^+ = \hat{c}_j^\dagger e^{i\hat{\phi}_j},\ \hat{\sigma}_j^- = \hat{c}_j e^{i\hat{\phi}_j},\ \hat{\sigma}_j^z = \hat{c}_j^\dagger \hat{c}_j - \frac{1}{2},\ \hat{\phi}_j = \pi \sum_{1\le m<j} \hat{c}_m^\dagger \hat{c}_m.$$

mapped to random-field XXZ spin chain by using a JW transformation

$$\hat{H}_{XXZ} = \frac{J_\perp}{2}\sum_{i=1}^{L}(\hat{\sigma}_i^x\hat{\sigma}_{i+1}^x + \hat{\sigma}_i^y\hat{\sigma}_{i+1}^y) + \sum_{i=1}^{L}\left(\frac{J_z}{2}\hat{\sigma}_i^z\hat{\sigma}_{i+1}^z + h_i^z\hat{\sigma}_i^z\right) \quad (6)$$

where $h_i^z \in [-W, W]$.

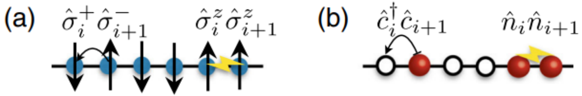
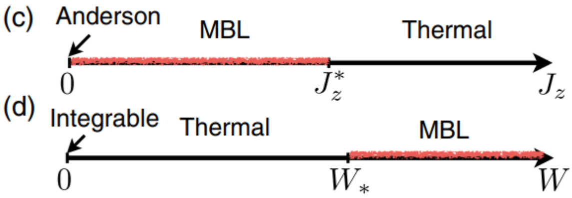


图: (a)XXZ;(b)spinless fermions

$J_z \to 0$, free fermions moving in a disorder potential $\Rightarrow$ Anderson localized.
For a fixed and not too large W, $|J_z| > J_z^*(W)$ lead to delocalization.(a possible recurrent MBL phase for $J_z \gg J_\perp$)
For fixed J_z and $J_\perp$, $W > W_*$ lead to full localization; $W < W_*$, eigenstates in the middle of the band become delocalized leading to a many-body mobility edge.
there is a transition in 1D, which can distinguish MBL from Anderson localization.



Area law: the entanglement entropy of a subsystem A in an MBL egeinstate scales proportional to the volume of the boundary ∂A of A, as both the size of the system and the size of A are taken to infinity $S_{\text{ent}}(A) \propto \text{vol}(\partial A)$.
Area-law entanglement scaling: (1)typical ground states in gapped systems; (2)highly excited states in MBL systems.
thought experiment for the low entanglement (Serbyn, Papic, and Adanin, 2013a)

consider an MBL system with a local Hamiltonian $\hat{H}$ and specify a region A.

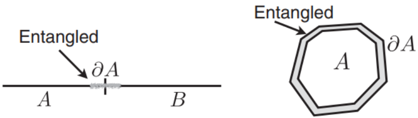
$$\hat{H} = \hat{H}_A + \hat{H}_B + \hat{V}_{AB}$$

turn off the coupling $\hat{V}_{AB} = 0$, and the eigenstates are tensor product states:

$$|I\rangle_{AB} = |\alpha\rangle_A \otimes |\beta\rangle_B \,. \tag{7}$$

$|I\rangle_{AB}$ have zero entanglement entropy for region A now.

turn on the coupling:
(1) local perturbation $\rightarrow$ localization length ξ
$\rightarrow |I'\rangle$ can be obtained from $|I\rangle$ by entangling spins in A and B over a distance $\sim \xi$ away from ∂A.
(2) local perturbation far away from $\partial A \rightarrow$ effects decay exponentially with the distance.
numerical (Bauer and Nayak (2013))(Serbyn, Papic, and Abanin (2013a))



low entanglement $\rightarrow$ product states by a sequence of quasilocal unitary transformations (except for topological order).
map physical degrees of freedom into quasilocal **integrals of motion**.
*proven by (Imbrie, 2016a and Imbrie, 2016b) for 1D and restricted interaction
6 in the limit $J_\perp \rightarrow 0$:

$$\hat{H}_0 = J_z \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z + \sum_i h_i \hat{\sigma}_i^z .$$

$[\hat{H}_0, \hat{\sigma}_i^z] = 0$, and therefore the eigenstates are nonentangled product states:

$$|\{\sigma\}\rangle = |\sigma_1 \sigma_2 \cdots \sigma_N\rangle , \quad \sigma_i = \uparrow, \downarrow . \tag{8}$$

turn on a weak flip-flop (kinetic) term $J_\perp$, remain MBL but no longer diagonal.

$\hat{U}$ is quasilocal if it can be factored into a sequence of n-site($n \geq 2$) unitary operators:

$$\hat{U} = \prod_i \cdots \hat{U}^{(3)}_{i,i+1,i+2} \hat{U}^{(2)}_{i,i+1}.$$

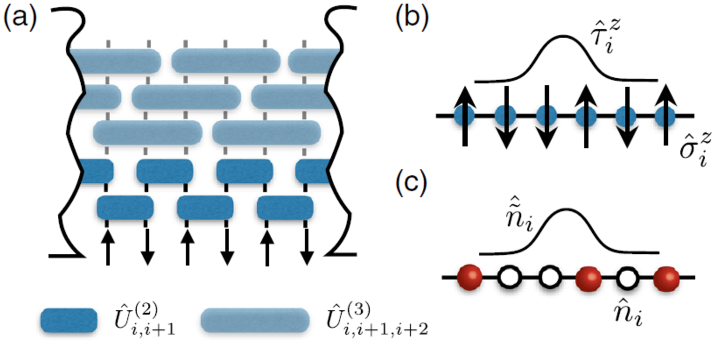
the long-range unitary operators have progressively decreasing rotation angles:

$$\left\| 1 - \hat{U}^{(n)}_{i,i+1,\ldots,i+n} \right\|_F^2 < e^{-n/\xi},$$

where $\|\cdot\|_F$ is the Frobenius operator norm.

For MBL phase, $\hat{U}$ is quasilocal; for thermalizing phase, $\hat{U}$ that diagonalizes $\hat{H}$ is highly nonlocal.
The unitary operator $\hat{U}$ transforms $\hat{\sigma}_i^z$ of $\hat{H}_0$ into $\hat{\tau}_i^z = \hat{U}\hat{\sigma}_i^z\hat{U}^\dagger$ of $\hat{H}$.
quasilocal, the $\hat{\tau}_i^z$ are close to $\hat{\sigma}_i^z$ at least at strong disorder.

$$\hat{\tau}_i^z = Z\hat{\sigma}_i^z + \sum_{n=1}^{\infty} V_i^{(n)}\hat{O}_i^{(n)} \tag{9}$$


(a)
(b)
$\hat{\tau}_i^z$
$\hat{\sigma}_i^z$
(c)
$\hat{\tilde{n}}_i$
$\hat{n}_i$
$\hat{U}^{(2)}_{i,i+1}$
$\hat{U}^{(3)}_{i,i+1,i+2}$

$$\hat{\tau}_i^z = Z\hat{\sigma}_i^z + \sum_{n=1}^{\infty} V_i^{(n)} \hat{O}_i^{(n)},$$

where $\hat{O}_i^{(n)}$ contains up to (2n+1)-body operators(long-range), from $i-n$ to $i+n$ and $\left\| \hat{O}_i^{(n)} \right\|_F = 1$.
$V_i^{(n)} \sim e^{-n/\xi}$, which is the key property distinguishing the MBL phase from the thermal phase.
The length scale ξ can be viewed as the localization length in the MBL phase.

The operators $\hat{\tau}_i^z$ form a complete set of independent quasilocal integrals of motion (**LIOM**s), also called localized bits or l-bits. One can view each $\hat{\tau}_i^z$ as an emergent conserved pseudospin-like degree of freedom, and $\langle \hat{\tau}_i^z \rangle$ cannot decay during evolution. While in thermal systems, overlap $Z \to 0$.
form a complete basis of operators:

$$\hat{\tau}_i^{x,y,z} = \hat{U} \hat{\sigma}_i^{x,y,z} \hat{U}^\dagger,$$

called τ representation.

Since $[\hat{\tau}_i^z, \hat{H}] = 0$, $\hat{H}$ cannot include $\hat{\tau}_i^{x,y}$, results the general form in terms of LIOMs:

$$\hat{H}_{\mathsf{MBL}} = \sum_i \tilde{h}_i \hat{\tau}_i^z + \sum_{i>j} J_{ij} \hat{\tau}_i^z \hat{\tau}_j^z + \sum_{i>j>k} J_{ijk} \hat{\tau}_i^z \hat{\tau}_j^z \hat{\tau}_k^z + \cdots , \qquad (10)$$

which often viewed as the universal Hamiltonian of the MBL phase. The coupling decay exponentially with separation between the LIOMs:

$$J_{ij} \propto J_0 e^{-|i-j|/\kappa},\ J_{ijk} \propto J_0 e^{-|i-k|/\kappa}, \ldots \qquad (11)$$

Assuming that $\hat{O}_i^{(n)}$ are superposition of Pauli strings (Trotter-Suzuki decompositions) with coefficients that follow a narrow distribution, $\kappa^{-1} \geq (\xi^{-1} + \ln 2)/2$, which implies κ must remain finite even if ξ diverges at the MBL transition.
couplings $\rightarrow \ln 2$, random signs $\rightarrow \frac{1}{2}$
coefficients? resulting bound?

effective MBL Hamiltonian $\leftrightarrow$ Landau's Fermi-liquid theory
While $\hat{\sigma}_i^z \leftrightarrow \hat{n}_i$ with fermion description, the LIOMs $\hat{\tau}_i^z \leftrightarrow$ quasiparticle occupation numbers $\hat{\tilde{n}}_i$.(Bera et al., 2015)

Difference:

(1) Fermi liquids: low-energy limit; MBL: exact description at all energies.

(2) Fermi liquids: close to the Fermi surface; MBL: LIOMs for all eigenstates.

LIOMs have an overlap with conserved densities (energy, particle number), which explains the absence of transport in the MBL phase.
However the existence of global conserved quantities is not essential for the MBL phase, such as energy in some periodically driven systems.

e.g. noninteracting systems, Yang-Baxter integrable systems.(Sutherland, 2004)

Difference:

(1) the integrals of motion: quasilocal operators $\leftrightarrow$ extensive sums of local operators

(2) the emergent integrability: robust (weak but finite perturbation) $\leftrightarrow$ easily destroyed(arbitrarily weak)

(3) a dynamical phase of matter $\leftrightarrow$ isolated points or lines

Similarity with weakly perturbed integrable classical systems:
(1) classical: Kolmogorov-Arnold-Moser (KAM) theory (Vogtmann, Weinstein, and Arnol'd, 2013)
the incommensurability of frequencies $\rightarrow$ the absence of resonant processes
(2) MBL: disorder $\rightarrow$ the incommensurability of frequencies and energies.
*the MBL phase is the only known example of a KAM-type integrable system that survives in the thermodynamic limit.

In particular: analyze the behavior of an isolated MBL system following a quantum quench.
MBL spin chain
initial state: a product state with low entanglement; for simplicity, a product state in the basis of LIOMs.
(numerical: Znidaric, Prosen, and Prelovsek, 2008; Bardarson, Pollmann, and Moore, 2012)

$$|\psi_0\rangle = \bigotimes_i (A_i |\Uparrow_i\rangle + B_i |\Downarrow_i\rangle). \tag{12}$$

where we introduced double arrows $|\Uparrow_i\rangle, |\Downarrow_i\rangle$, which refer to eigenstates of $\hat{\tau}_i^z = \pm 1$, and $|A_i|^2 + |B_i|^2 = 1$.

But we should avoid $A_i = 1$ or $B_i = 1$, which would take an exponentially long time of the order of the inverse level, $t \sim e^{1/\Delta}$, spacing to prepare.
mean-field $\to A_i, B_i$ acquire phases, which leads to entanglement generation.
two-qubit

$$|\psi_2\rangle = \frac{1}{2}(|\Uparrow_1\rangle + |\Downarrow_1\rangle) \otimes (|\Uparrow_2\rangle + |\Downarrow_2\rangle),$$

and

$$\hat{H}_{\text{MBL}} = J_{12}\hat{\tau}_1^z \hat{\tau}_2^z.$$

evolution

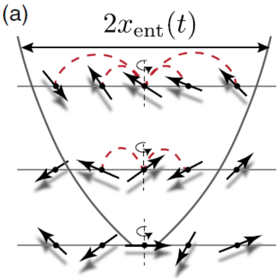
$$
\begin{aligned}
|\psi_2(t)\rangle &= e^{-i\hat{H}_{\mathsf{MBL}}t}\,|\psi_2\rangle \\
&= \frac{1}{2}e^{-iJ_{12}t}(|\Uparrow\Uparrow\rangle + |\Downarrow\Downarrow\rangle) \\
&\quad +\frac{1}{2}e^{iJ_{12}t}(|\Uparrow\Downarrow\rangle + |\Downarrow\Uparrow\rangle).
\end{aligned}
$$

entanglement: $S_{\mathsf{ent}} = -\mathrm{Tr}\rho_1 \ln \rho_1 = \ln 2$ for $J_{12}t = \pi/4$. (Serbyn, Papic, and Abanin, 2013b)
Many spins, phase of the order of one dependent on the state of another spin a distance x away after a time $t(x)$ set by the condition $\tilde{h}_{i,i+x}t(x) \sim 1$.

$$\tilde{h}_{i,i+x} = J_{i,i+x} + J_{i,i+1,i+x}\hat{\tau}^z_{i+1} + \cdots .$$

exponential decay of couplings 11 $\rightarrow \tilde{h}_{i,i+x} \sim J_0 e^{-x/\xi'}$
$\tilde{h}_{i,i+x} t(x) \sim 1$ yields the logartihmic entanglement "light cone"
and logarithmic growth of entanglement

$$x_{\text{ent}}(t) = \xi' \log(J_0 t), \quad S_{\text{ent}}(t) \propto \xi' \log(J_0 t). \tag{13}$$



entanglement spreads over that volume

In a finite-size system, x_{ent} is bounded by the system size; hence the entanglement entropy in a quantum quench saturates to a value that is proportional to the system size $S_{\text{ent}}(\infty) \propto L$.

ξ' controls entanglement growth,

$$\xi'^{-1} \leq \kappa^{-1} + (\ln 2)/2, \text{ and } \xi' \leq 2\xi.$$

$\xi \sim$ typical localization length with interaction,
$\kappa \sim$ coefficients of $\hat{H}_{\mathsf{MBL}}$,
$\xi' \sim$ entanglement growth in dynamical condition.
Establishing whether the three length scales ξ, κ, and ξ_0 are directly related remains an outstanding challenge.

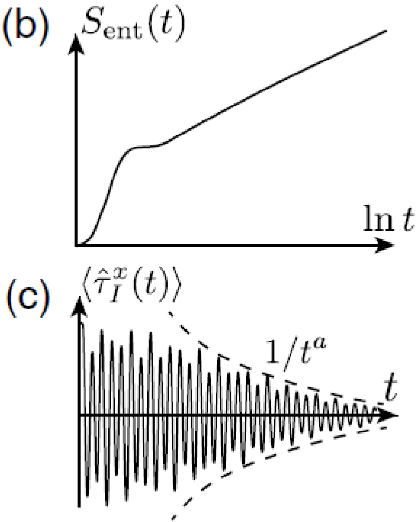
Comments:
(1) the logarithmic growth of entanglement holds generally, because a generic initial state is an extensive superposition of many-body eigenstates, and each τ spin undergoes the dephasing dynamics.
(2) the proportionality coefficient in 13 depends on the diagonal entropy of the initial state, influenced by the disorder strength. (Polkovnikov, 2011; Serbyn, Papić, and Abanin., 2013b)
(3) For initial states of 12 this entropy is determined by the probability distribution of coefficients A_i and B_i.
(4) $S_{\text{ent}}(t) \propto \ln^{\phi} t$ with $\phi > 1$ in the vicinity of a transition between two distinct MBL phases.
(5) in contrast to the ballistic entanglement spreading in ergodic systems (Kim and Huse, 2013) and in Yang-Baxter integrable models (Calabrese and Cardy, 2009). $\rightarrow$ **one of the defining features of MBL**

The growth of entanglement is difficult to measure
The dephasing dynamics in MBL systems leads to equilibration of local observables in a quantum quench setup, which power-law approach to equilibrium values.
Consider $A_i = B_i = 1/\sqrt{2}$, study the single-spin observables (described by operators $\hat{\tau}_I^{x,y}$).
$t = 0$, spin I is pointing in the x direction.
evolution t, $tJ_0 e^{-x_{\text{ent}}(t)/\xi'} \gtrsim 1$, spins within distance $x_{\text{ent}}(t)$ away affect the rotation angle of I.

dephasing $\rightarrow$ the off-diagonal elements of the reduced density matrix or $\langle \hat{\tau}_I^{x,y} \rangle$ decay.
t, the Hilbert space dimension of "environment" $\mathcal{D}(t) \approx 2^{2x_{\mathsf{ent}}(t)}$.

$$\left| \langle \hat{\tau}_I^{x,y}(t) \rangle \right| \sim \frac{1}{\sqrt{\mathcal{D}(t)}} \approx \frac{1}{(J_0 t)^a}, \qquad (14)$$

which describes a power-law decay.
that the exponent of this power law is not universal, and in general is given by $a = \xi' s_0$.
For the initial state with $A_i = B_i = 1/\sqrt{2}$, the diagonal entropy attains its maximal value, $\ln 2$ per spin, leading to $a = \xi' \ln 2$.


(b)
$S_{\rm ent}(t)$
$\ln t$
(c)
$\langle \hat{\tau}_I^x(t) \rangle$
$1/t^a$
t

physical spin operators can be expressed with $\hat{\tau}_i^{x,y,z}$ and their product.
$\hat{\tau}^z$, unchanged; $\hat{\tau}^{x,y}$ decay.
generic local observables approach their long time equilibrium values in a power-law fashion.
But equilibrium values of observables retain the memory of the initial state due to the extensive set of LIOMs.

the standard spin-echo protocol can fully recover the state of a given spin.$\rightarrow$
(1) the intrinsic T_1 relaxation time remains infinite in the MBL phase.
(2) the T_2 time induced by the entanglement dynamics with distant spins increases exponentially with the distance to these spins, reflecting the logarithmic dynamics of entanglement growth.
(3) In practice, this leads only to an incomplete recovery because $\sigma \neq \tau$, but the revival probability is large at strong enough disorder.
Modification: temporal revivals of local observables, double-electron resonance.

Others:
(1) power-law decays due to the same mechanism: mutual information, fluctuations of the out-of-time correlation functions, fluctuations of the Loschmidt echo.
(2) Dissipation is still present in some synthetic quantum systems free from phonons, because being inelastic scattering on lattice lasers and particle loss for atomic and trapped ion experiments. $\rightarrow$

- formulated the Lindblad equation in terms of LIOMs, and reduced it to a classical rate equation. $\Rightarrow$ that relaxation of an initial density modulation $\sim$ a stretched-exponential law
- the spectral function of an MBL system weakly coupled to a heat bath still carries signatures of localization.

the presence of disorder in the MBL phase precludes an explicit analytic construction of LIOMs $\rightarrow$ new numerical tools (highly excited)

(1) area-law entanglement of eigenstates in 1D: MPS and DMRG, MPO($\hat{U}$)

*Extending DMRG techniques to the excited states is highly nontrivial due to the fact that the level spacing becomes very small at a finite energy density.

(2) logarithmic scaling of entanglement: RG, real-space renormalization group(RSRG)...

(3) Transition between different MBL phases: Ising and XYZ chains, random SU(2)$_k$ anyon chains...

(4) Long time dynamics of large open systems: TN

(5) Simplest models, such as (5),(6): ED

*New numerical tools: 1D systems with local Hilbert space larger than 2 (bosons, spinful fermions, higher spins), phase transitions, higher dimensions(**TN**)

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