物理人用微分几何结论速查手册

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1 定义

定义 1. A topological space M is a set of points, endowed with a topology $\mathcal{T}$. This is a collection of open subsets $\mathcal{O}_\alpha \subset M$ which obey:

(1) Both the set M and the empty set $\varnothing$ are open subsets: $M \in \mathcal{T}$ and $\varnothing \in \mathcal{T}$.

(2) The intersection of a finite number of open sets is also an open set. So if $\mathcal{O}_1 \in \mathcal{T}$ and $\mathcal{O}_2 \in \mathcal{T}$ then $\mathcal{O}_1 \cap O_2 \in \mathcal{T}$.

(3) The union of any number (possibly infinite) of open sets is also an open set. So if $\mathcal{O}_\gamma \in \mathcal{T}$ then $\bigcup_\gamma \mathcal{O}_\gamma \in \mathcal{T}$.

定义 2. One further definition (it won't be our last). A homeomorphism between topological spaces $(M, \mathcal{T})$ and $(\tilde{M}, \tilde{\mathcal{T}})$ is a map $f : M \to \tilde{M}$ which is

(1) Injective (or one-to-one): for $p \neq q, f(p) \neq f(q)$.

(2) Surjective (or onto): $f(M) = \tilde{M}$, which means that for each $\tilde{p} \in \tilde{M}$ there exists a $p \in M$ such that $f(p) = \tilde{p}$.

(3) Bicontinuous. This means that both the function and its inverse are continuous. To define a notion of continuity, we need to use the topology. We say that f is continuous if, for all $\tilde{O} \in \tilde{\mathcal{T}}$, $f^{-1}(\tilde{O}) \in \mathcal{T}$.

定义 3. An n-dimensional differentiable manifold is a Hausdorff topological space M such that

(1) M is locally homeomorphic to $\mathbf{R}^n$. This means that for each $p \in M$, there is an open set $\mathcal{O}$ such that $p \in \mathcal{O}$ and a homeomorphism $\phi : \mathcal{O} \to U$ with U an open subset of $\mathbf{R}^n$.

(2) Take two open subsets $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ that overlap, so that $\mathcal{O}_\alpha \cap \mathcal{O}_\beta = \varnothing$. We require that the corresponding maps $\phi_\alpha : \mathcal{O}_\alpha \to U_\alpha$ and $\phi_\beta : \mathcal{O}_\beta \to U_\beta$ are compatible, meaning that the map $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(\mathcal{O}_\alpha \cap \mathcal{O}_\beta) \to \phi_\beta(\mathcal{O}_\alpha \cap \mathcal{O}_\beta)$ is smooth (also known as infinitely differentiable or C^∞), as is its inverse.

定义 4. A tangent vector X_p is an object that differentiates functions at a point $p \in M$. Specifically, $X_p : C^\infty(M) \to \mathbf{R}$ satisfying

(1) Linearity: $X_p(f+g) = X_p(f) + X_p(g)$ for all $f, g \in C^\infty(M)$.

(2) $X_p(f) = 0$ when f is the constant function.

(3) Leibnizarity: $X_p(fg) = f(p)X_p(g) + X_p(f)g(p)$ for all $f, g \in C^\infty(M)$.

定义 5.

- pull-back: If we have a function on $f : N \to \mathbf{R}$, then we can construct a new function that we denote $(\varphi^* f) : M \to \mathbf{R}$,

$$(\varphi^* f)(p) = f(\varphi(p)) \tag{1}$$

- push-forward: If we are given a function $f : N \to \mathbf{R}$, then the vector field $(\varphi_* Y)$ on N acts as

$$(\varphi_* Y)(f) = Y(\varphi^* f) \tag{2}$$

定义 6. A particularly interesting class are totally anti-symmetric $(0, p)$ tensors fields. These are called p-forms. The set of all p-forms over a manifold M is denoted $\Lambda^p(M)$.

A p-form has $\binom{n}{k}$ different components. Forms in $\Lambda^n(M)$ are called top forms.

定义 7. wedge product

$$(\omega \wedge \eta)_{\mu_1 \dots \mu_p \nu_1 \dots \nu_q} = \frac{(p+q)!}{p!q!} \omega_{[\mu_1 \dots \mu_p} \eta_{\nu_1 \dots \nu_q]} \tag{3}$$

定义 8. exterior derivative, $d : \Lambda^p(M) \to \Lambda^{p+1}(M)$.

$$d\omega = \frac{1}{p!} \frac{\partial \omega_{\mu_1 \dots \mu_p}}{\partial x^\nu} dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \tag{4}$$

$$(d\omega)_{\mu_1 \dots \mu_{p+1}} = (p+1) \partial_{[\mu_1} \omega_{\mu_2 \dots \mu_{p+1}]} \tag{5}$$

A p-form ω is said to be closed if $d\omega = 0$ everywhere. It is exact if $\omega = d\eta$ everywhere for some η. Because $d^2 = 0$, an exact form is necessary closed.

定义 9. interior product, $\iota_X : \Lambda^p(M) \to \Lambda^{p-1}(M)$.

$$\iota_X \omega(Y_1, \dots, Y_{p-1}) = \omega(X, Y_1, \dots, Y_{p-1}), \quad \iota_X f = 0 \tag{6}$$

定义 10. We denote the set of all closed p-forms on a manifold M as $Z^p(M)$. Equivalently, $Z^p(M)$ is the kernel of the map $d : \Lambda^p(M) \to \Lambda^{p+1}(M)$.

We denote the set of all exact p-forms on a manifold M as $B^p(M)$. Equivalently, $B^p(M)$ is the range of $d : \Lambda^{p-1}(M) \to \Lambda^p(M)$.

The pth de Rham cohomology group is defined to be

$$H^p(M) = Z^p(M)/B^p(M) \tag{7}$$

Two closed forms $\omega, \omega' \in Z^p(M)$ are said to be equivalent if $\omega = \omega' + \eta$ for some $\eta \in B^p(M)$. We say that ω and ω' sit in the same equivalence class $[\omega]$.

The Betti numbers B_p of a manifold M are defined as

$$B_p = \dim H^p(M) \tag{8}$$

The Euler character is defined as the alternating sum of Betti numbers,

$$\chi(M) = \sum_p (-1)^p B_p \tag{9}$$

定义 11. A volume form, or orientation on a manifold of dimension $\dim(M) = n$ is a nowhere-vanishing top form v. Any top form has just a single component and can be locally written as

$$v = v(x) dx^1 \wedge \cdots \wedge dx^n \tag{10}$$

where we require $v(x) \neq 0$. If such a top form exists everywhere on the manifold, then M is said to be orientable.

定义 12. In a chart $\phi : \mathcal{O} \to U$, with coordinates x^μ, we have

$$\int_{\mathcal{O}} f v = \int_U dx_1 \ldots dx_n f(x) v(x) \tag{11}$$

定义 13. A manifold Σ with dimension $k < n$ is a submanifold of M if we can find a map $\phi : \Sigma \to M$ which is one-to-one (which ensures that Σ doesn't intersect itself in M) and $\phi_* : T_p(\Sigma) \to T_\phi(p)(M)$ is one-to-one.

We can then integrate a k-form ω on M over a k-dimensional submanifold Σ.

$$\int_{\phi(\Sigma)} \omega = \int_\Sigma \phi^* \omega \tag{12}$$

定义 14. A metric g is a $(0,2)$ tensor field that is:
(1) Symmetric: $g(X, Y) = g(Y, X)$.
(2) Non-Degenerate: If, for any $p \in M$, $g(X, Y)|_p = 0$ for all $Y \in T_p(M)$ then $X_p = 0$.

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu,\, g_{\mu\nu}(x) = g\left(\partial_\mu, \partial_\nu\right), \quad ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu \tag{13}$$

定义 15. A manifolds in which all diagonal entries of the metric are positive is called a Riemannian manifold. The simplest example is Euclidean space $\mathbf{R}^n$ which, in Cartesian coordinates, is equipped with the metric

$$g = dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n \tag{14}$$

The components of this metric are simply $g_{\mu\nu} = \delta_{\mu\nu}$.

定义 16. A manifold in which one of the diagonal entries of the metric is negative is called Lorentzian. The simplest example of a Lorentzian metric is Minkowski space. This is $\mathbf{R}^n$ equipped with the metric

$$\eta = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + \cdots + dx^{n-1} \otimes dx^{n-1} \tag{15}$$

The components of the Minkowski metric are $_{\mu\nu} = \mathrm{diag}(-1, +1, ..., +1)$. As this example shows, on a Lorentzian manifold we usually take the coordinate index x_μ to run from $0, 1, \ldots, n-1$.

定义 17. At any point p, a vector $X_p \in T_p(M)$ is said to be timelike if $g(X_p, X_p) < 0$, null if $g(X_p, X_p) = 0$, and spacelike if $g(X_p, X_p) > 0$.

定义 18. Given a parametrisation $x_\mu(t)$, this distance is,

$$\tau = \int_a^b dt \sqrt{-g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}} \tag{16}$$

This is called the proper time.

定义 19. The metric also gives us a natural volume form on the manifold M. On a Riemannian manifold, this is defined as

$$v = \sqrt{g}dx^1 \wedge \cdots \wedge dx^n, \quad \sqrt{g} = \sqrt{\det g_{\mu\nu}} \tag{17}$$

On a Lorentzian manifold, the determinant is negative and we instead have

$$v = \sqrt{-g}dx^0 \wedge \cdots \wedge dx^{n-1} \tag{18}$$

定义 20. On an oriented manifold M, we can use the totally anti-symmetric tensor $\epsilon_{\mu_1,\ldots,\mu_n}$ to define a map which takes a p-form $\omega \in \Lambda^p(M)$ to an $(n-p)$-form, denoted $(\star\omega) \in \Lambda^{n-p}(M)$, defined by

$$(\star\omega)_{\mu_1\ldots\mu_{n-p}} = \frac{1}{p!}\sqrt{|g|}\epsilon_{\mu_1\ldots\mu_{n-p}\nu_1\ldots\nu_p}\omega^{\nu_1\ldots\nu_p} \tag{19}$$

This map is called the Hodge dual. It is independent of the choice of coordinates.

定义 21. The Hodge dual allows us to define an inner product on each $\Lambda^p(M)$. If $\omega, \eta \in \Lambda^p(M)$, we define

$$\langle \eta, \omega \rangle = \int_M \eta \wedge \star\omega \tag{20}$$

which makes sense because $\star\omega \in \Lambda^{n-p}(M)$ and so $\eta \wedge \star\omega$ is a top form that can be integrated over the manifold. The inner product is positive-definite.

定义 22. We can combine d and $d^\dagger$ to construct the Laplacian, $\triangle : \Lambda^p(M) \to \Lambda^p(M)$, defined as

$$\triangle = (d + d^\dagger)^2 = dd^\dagger + d^\dagger d \tag{21}$$

定义 23.

$$\triangle\gamma = 0 \tag{22}$$

Such forms are said to be harmonic. An harmonic form is necessarily closed, meaning $d\gamma = 0$, and co-closed, meaning $d^\dagger\gamma = 0$.

The space of harmonic p-forms on a manifold M is denoted $\mathrm{Harm}^p(M)$.

定义 24. A connection is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$. We usually write this as $\nabla(X, Y) = \nabla_X Y$ and the object ∇_X is called the covariant derivative. It satisfies the following properties for all vector fields X, Y and Z,

(1) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$

(2) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$ for all functions f, g.

(3) $\nabla_X(fY) = f\nabla_X Y + (\nabla_X f)Y$ where we define $\nabla_X f = X(f)$.

定义 25. Even though the connection is not a tensor, we can use it to construct two tensors. The first is a rank $(1, 2)$ tensor T known as torsion. It is defined to act on $X, Y \in \mathfrak{X}(M)$ and $\omega \in \Lambda^1(M)$ by

$$T(\omega; X, Y) = \omega(\nabla_X Y - \nabla_Y X - [X, Y]) \tag{23}$$

The other is a rank $(1,3)$ tensor R, known as curvature. It acts on $X, Y, Z \in \mathfrak{X}(M)$ and $\omega \in \Lambda^1(M)$ by

$$R(\omega; X, Y, Z) = \omega(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) \tag{24}$$

The curvature tensor is also called the Riemann tensor.

定义 26. Alternatively, we could think of torsion as a map $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \tag{25}$$

Similarly, the curvature R can be viewed as a map from $\mathfrak{X}(M) \times \mathfrak{X}(M)$ to a differential operator acting on $\mathfrak{X}(M)$,

$$R(X, Y, Z) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \tag{26}$$

定义 27. parallel transport:

Take a vector field X and consider some associated integral curve C, with coordinates $x^\mu(\tau)$, such that

$$X^\mu|_C = \frac{dx^\mu(\tau)}{d\tau} \tag{27}$$

We say that a tensor field T is parallely transported along C if

$$\nabla_X T = 0 \tag{28}$$

定义 28. A geodesic is a curve tangent to a vector field X that obeys

$$\nabla_X X = 0 \tag{29}$$

定义 29. normal coordinates:

On a Riemannian manifold, in the neighbourhood of a point $p \in M$, we can always find coordinates such that

$$g_{\mu\nu}(p) = \delta_{\mu\nu}, \quad g_{\mu\nu,\rho}(p) = 0 \tag{30}$$

The same holds for Lorentzian manifolds, now with $g_{\mu\nu}(p) = \eta_{\mu\nu}$.

the Christoffel symbols vanish

定义 30. The Exponential Map:

$$\text{Exp: } T_p(M) \to M \tag{31}$$

Given $X_p \in T_p(M)$, construct the appropriate geodesic and the follow it for some affine distance which we take to be $\tau = 1$. This gives a point $q \in M$.

The Equivalence Principle: normal coordinates are called a local inertial frame.

定义 31. Consider now a one-parameter family of geodesics, with coordinates $x^\mu(\tau; s)$. Here τ is the affine parameter along the geodesics. Meanwhile, s labels the different geodesics.

$$X^\mu = \left.\frac{\partial x^\mu}{\partial \tau}\right|_s, \quad S^\mu = \left.\frac{\partial x^\mu}{\partial s}\right|_\tau \tag{32}$$

The tangent vector S^μ is sometimes called the deviation vector

定义 32. Ricci tensor

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} \tag{33}$$

Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu} \tag{34}$$

定义 33. Einstein Tensors

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{35}$$

定义 34. vielbeins or tetrads

$$\hat{e}_a = e_a{}^\mu \partial_\mu \tag{36}$$

on a Riemannian\Lorentzian manifold,

$$g(\hat{e}_a, \hat{e}_b) = g_{\mu\nu} e_a{}^\mu e_b{}^\nu = \delta_{ab}, \quad g(\hat{e}_a, \hat{e}_b) = g_{\mu\nu} e_a{}^\mu e_b{}^\nu = \eta_{ab} \tag{37}$$

(On an n-dimensional manifold, these objects are usually called "German word for n" -beins.) The vielbeins aren't unique.

定义 35. The dual basis of one-forms $\{\hat{\theta}^a\}$ is defined by $\hat{\theta}^a(\hat{e}_b) = \delta^b_a$. They are related to the coordinate basis by

$$\hat{\theta}^a = e^a{}_\mu dx^\mu \tag{38}$$

$e^a{}_\mu$ is the inverse of $e_a{}^\mu$. In the non-coordinate basis, the metric on a Lorentzian manifold takes the form

$$g_{\mu\nu} = e^a{}_\mu e^b_\nu \eta_{ab} \tag{39}$$

For Riemannian manifolds, we replace η_{ab} with δ_{ab}.

定义 36. Given a non-coordinate basis $\{\hat{e}_a\}$, we can define the components of a connection

$$\nabla_{\hat{e}_c} \hat{e}_b = \Gamma^a_{cb} \hat{e}_a \tag{40}$$

define the matrix-valued connection one-form as

$$\omega^a{}_b = \Gamma^a_{cb} \hat{\theta}^c \tag{41}$$

定义 37. the components of the Riemann tensor in our non-coordinate basis

$$R^a{}_{bcd} = R(\hat{\theta}^a; \hat{e}_c, \hat{e}_d, \hat{e}_b) \tag{42}$$

The anti-symmetry of the last two indices, $R^a{}_{bcd} = -R^a{}_{bdc}$, makes this ripe for turning into a matrix of two-forms,

$$\mathcal{R}^a{}_b = \frac{1}{2} R^a{}_{bcd} \hat{\theta}^c \wedge \hat{\theta}^d \tag{43}$$

定义 38. define

$$[\hat{e}_a, \hat{e}_b] = f_{ab}{}^c \hat{e}_c \tag{44}$$

2 结论

结论 1. Lie derivative:

$$\mathcal{L}_X Y = [X, Y], \quad \mathcal{L}_X \mathcal{L}_Y Z - \mathcal{L}_Y \mathcal{L}_X Z = \mathcal{L}_{[X,Y]} Z \tag{45}$$

$$\mathcal{L}_X \omega = (X^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu X^\nu) dx^\mu \tag{46}$$

结论 2.

$$T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu}), \quad T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu}) \tag{47}$$

$$T^\mu_{[\nu|\rho|\sigma]} = \frac{1}{2}(T^\mu_{\nu\rho\sigma} - T^\mu_{\sigma\rho\nu}) \tag{48}$$

结论 3.

$$\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega, \quad \omega \wedge \omega = 0 \tag{49}$$

any p-form ω can be written locally as

$$\omega = \frac{1}{p!} \omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \tag{50}$$

结论 4.

- $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$, where $\omega \in \Lambda^p(M)$.
- $d(\varphi^* \omega) = \varphi^*(d\omega)$ where φ^* is the pull-back associated to the map between manifolds, $\varphi : M \to N$.
- Because the exterior derivative commutes with the pull-back, it also commutes with the Lie derivative. This ensures that we have $d(\mathcal{L}_X \omega) = \mathcal{L}_X (d\omega)$.

$$d(d\omega) = 0 \tag{51}$$

结论 5.

$$\iota_X \iota_Y = -\iota_Y \iota_X, \quad \iota_X(\omega \wedge \eta) = \iota\omega \wedge \eta + (-1)^p \omega \wedge \iota\eta \tag{52}$$

Consider a 1-form ω. Cartan's magic formula:

$$\mathcal{L}_X \omega = (d\iota_X + \iota_X d)\omega \tag{53}$$

结论 6. The Poincaré Lemma: On $M = \mathbf{R}^n$, closed implies exact.

结论 7. The Betti number $B_0 = 1$ for any connected manifold. This can be traced to the existence of constant functions which are clearly closed but, because there are no $p = -1$ forms, are not exact. The higher Betti numbers are non-zero only if the manifold has some interesting topology.

结论 8. Stokes' theorem: Consider a manifold M with boundary ∂M. If the dimension of the manifold is $\dim(M) = n$ then for any $(n-1)$-form ω, we have the following simple result

$$\int_M d\omega = \int_{\partial M} \omega \tag{54}$$

结论 9.

$$\tilde{e}_\mu = \Lambda^\nu{}_\mu e_\nu, \quad \tilde{g}_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu g_{\rho\sigma} \tag{55}$$

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho, \quad \hat{g} = g^{\mu\nu} \partial_\mu \otimes \partial_\nu \tag{56}$$

结论 10.

$$\int_M f v = \int_M d^n x \sqrt{\pm g} f \tag{57}$$

结论 11.

$$\star(\star\omega) = \pm(-1)^{p(n-p)} \omega \tag{58}$$

结论 12. For $\omega \in \Lambda^p(M)$ and $\alpha \in \Lambda^{p-1}(M)$,

$$\langle d\alpha, \omega \rangle = \langle \alpha, d^\dagger \omega \rangle \tag{59}$$

where the adjoint operator $d^\dagger : \Lambda^p(M) \to \Lambda^{p-1}(M)$ is given by

$$d^\dagger = \pm(-1)^{np+n-1} \star d \star, \quad d^2 = d^{\dagger 2} = 0 \tag{60}$$

with, again, the ± sign for Riemannian/Lorentzian manifolds respectively.

结论 13.

$$\triangle(f) = -\frac{1}{\sqrt{|g|}}\partial_\nu\left(\sqrt{|g|}g^{\mu\nu}\partial_\mu f\right) \tag{61}$$

结论 14. Hodge decomposition theorem: any p-form ω on a compact, Riemannian manifold can be uniquely decomposed as

$$\omega = d\alpha + d^\dagger\beta + \gamma \tag{62}$$

where $\alpha \in \Lambda^{p-1}(M)$ and $\beta \in \Lambda^{p+1}(M)$ and $\gamma \in \mathrm{Harm}^p(M)$.

结论 15. Hodge's Theorem: There is an isomorphism

$$\mathrm{Harm}^p(M) \cong H^p(M) \tag{63}$$

where $H^p(M)$ is the de Rham cohomology group. In particular, the Betti numbers can be computed by counting the number of linearly independent harmonic forms,

$$B_p = \dim \mathrm{Harm}^p M \tag{64}$$

结论 16.

$$\nabla_\rho e_\nu = \Gamma^\mu{}_{\rho\nu} e_\mu, \quad (\nabla_\nu Y)^\mu = e_\nu(Y^\mu) + \Gamma^\mu{}_{\nu\rho} Y^\rho \tag{65}$$

In a coordinate basis, in which $e_\mu = \partial_\mu$, the covariant derivative becomes

$$\nabla_\nu Y^\mu \equiv (\nabla_\nu Y)^\mu = \partial_\nu Y^\mu + \Gamma^\mu{}_{\nu\rho} Y^\rho \tag{66}$$

sometimes

$$\nabla_\nu Y \equiv Y^\mu{}_{;\nu}, \quad \partial_\nu Y^\mu \equiv Y^\mu{}_{,\nu} \tag{67}$$

结论 17. The $\Gamma^\mu{}_{\rho\nu}$ defining the connection are not components of a tensor.

$$\tilde{e}_\nu = A^\mu{}_\nu e_\mu, \quad \tilde{\Gamma}^\mu{}_{\rho\nu} = (A^{-1})^\mu{}_\tau A^\sigma{}_\rho A^\lambda{}_\nu \Gamma^\tau{}_{\sigma\lambda} + (A^{-1})^\mu{}_\tau A^\sigma{}_\rho \partial_\sigma A^\tau{}_\nu \tag{68}$$

结论 18. Consider a one-form ω.

$$\nabla_\mu \omega_\rho = \omega_{\rho,\mu} - \Gamma^\nu{}_{\mu\rho}\omega_\nu \tag{69}$$

This kind of argument can be extended to a general tensor field of rank (p, q): for every upper index μ we get a $+\Gamma T$ term, while for every lower index we get a $-\Gamma T$ term.

结论 19. We can evaluate these tensors in a coordinate basis $\{e_\mu\} = \{\partial_\mu\}$, with the dual basis $\{f^\mu\} = \{dx^\mu\}$. The components of the torsion are

$$T^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\mu\nu} - \Gamma^\rho{}_{\nu\mu}, \quad T^\rho{}_{\mu\nu} = -T^\rho{}_{\nu\mu} \tag{70}$$

Connections which are symmetric in the lower indices $T^\rho{}_{\mu\nu} = 0$ are said to be torsion-free.

The components of the curvature tensor are given by

$$R^\sigma{}_{\rho\mu\nu} = \partial_\mu\Gamma^\sigma{}_{\nu\rho} - \partial_\nu\Gamma^\sigma{}_{\mu\rho} + \Gamma^\lambda{}_{\nu\rho}\Gamma^\sigma{}_{\mu\lambda} - \Gamma^\lambda{}_{\mu\rho}\Gamma^\sigma{}_{\nu\lambda} \tag{71}$$

$$R^\sigma{}_{\rho\mu\nu} = -R^\sigma{}_{\rho\nu\mu}, \quad R^\sigma{}_{\rho\mu\nu} = R^\sigma{}_{\rho[\mu\nu]} \tag{72}$$

结论 20. Ricci identity

$$2\nabla_{[\mu}\nabla_{\nu]}Z^\sigma = R^\sigma{}_{\rho\mu\nu}Z^\rho - T^\rho{}_{\mu\nu}\nabla_\rho Z^\sigma \tag{73}$$

where

$$T^\rho{}_{\mu\nu} = 2\Gamma^\rho{}_{[\mu\nu]}, \quad R^\sigma{}_{\rho\mu\nu} = 2\partial_{[\mu}\Gamma^\sigma{}_{\nu]\rho} + 2\Gamma^\sigma{}_{[\mu|\lambda|}\Gamma^\lambda{}_{\nu]\rho} \tag{74}$$

结论 21. The fundamental theorem of Riemannian geometry: There exists a unique, torsion free, connection that is compatible with a metric g, in the sense that

$$\nabla_X g = 0 \tag{75}$$

for all vector fields X.

结论 22. Levi-Civita connection.

$$\begin{aligned} g(\nabla_Y X, Z) =& \frac{1}{2}\left[X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y))\right. \\ & \left. - g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y)\right] \end{aligned} \tag{76}$$

We can compute its components in a coordinate basis $\{e_\mu\} = \{\partial_\mu\}$. This is particularly simple because $[\partial_\mu, \partial_\nu] = 0$:

$$g(\nabla_\nu e_\mu, e_\rho) = \Gamma^\lambda{}_{\nu\mu} g_{\lambda\rho} = \frac{1}{2}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right) \tag{77}$$

Christoffel symbols:

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right) \tag{78}$$

结论 23. Lemma: The contraction of the Christoffel symbols can be written as

$$\Gamma^\mu{}_{\mu\nu} = \frac{1}{\sqrt{g}} \partial_\nu \sqrt{g} \tag{79}$$

On Lorentzian manifolds, we should replace $\sqrt{g}$ with $\sqrt{|g|}$.

结论 24. Divergence Theorem: Consider a region of a manifold M with boundary ∂M. Let n^μ be an outward-pointing, unit vector orthogonal to ∂M. Then, for any vector field X^μ on M, we have

$$\int_M d^n x \sqrt{g} \nabla_\mu X^\mu = \int_{\partial M} d^{n-1} x \sqrt{\gamma} n_\mu X^\mu \tag{80}$$

where γ_{ij} is the pull-back of the metric to ∂M, and $\gamma = \det \gamma_{ij}$. On a Lorentzian manifold, a version of this formula holds only if ∂M is purely timelike or purely spacelike, which ensures that $\gamma \neq 0$ at any point.

结论 25. If we now evaluate this on the curve C, we can think of $Y_\mu = Y_\mu(x(\tau))$, which obeys

$$\frac{dY^\mu}{d\tau} + X^\nu \Gamma^\mu{}_{\nu\rho} Y^\rho = 0 \tag{81}$$

Along the curve C, geodesic equation:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{\rho\nu} \frac{dx^\rho}{d\tau} \frac{dx^\nu}{d\tau} = 0 \tag{82}$$

结论 26. For the Levi-Civita connection,

$$\nabla_X g = 0, \quad \frac{d}{d\tau} g(X, Y) = 0 \tag{83}$$

结论 27. we can always pick coordinates s and t on the surface such that $S = \partial/\partial s$ and $X = \partial/\partial t$, ensuring that

$$[S, X] = 0 \tag{84}$$

consider a connection Γ with vanishing torsion

$$\nabla_X \nabla_X S = \nabla_X \nabla_S X = \nabla_S \nabla_X X + R(X, S)X \tag{85}$$

further restrict to an integral curve C associated to the vector field X,

$$\frac{D^2 S^\mu}{D\tau^2} = R^\mu{}_{\nu\rho\sigma} X^\nu X^\rho S^\sigma, \quad \frac{D}{D\tau} = \frac{\partial x^\mu}{\partial \tau} \nabla_\mu \tag{86}$$

this relative acceleration is controlled by the Riemann tensor. Experimentally, such geodesic deviations are called tidal forces.

结论 28. (use the Levi-Civita connection)
If we lower an index on the Riemann tensor, and write $R_{\sigma\rho\mu\nu} = g_{\sigma\lambda} R^\lambda{}_{\rho\mu\nu}$ then the resulting object also obeys the following identities

$$R_{\sigma\rho\mu\nu} = -R_{\sigma\rho\nu\mu}, \ \ R_{\sigma\rho\mu\nu} = -R_{\rho\sigma\mu\nu}, \ \ R_{\sigma\rho\mu\nu} = R_{\mu\nu\sigma\rho}, \ \ R_{\sigma[\rho\mu\nu]} = 0 \tag{87}$$

结论 29. (use the Levi-Civita connection)
The Riemann tensor also obeys the Bianchi identity

$$\nabla_{[\lambda} R_{\sigma\rho]\mu\nu} = 0 \tag{88}$$

Alternatively, we can anti-symmetrise on the final two indices, in which case this can be written as $R^\sigma{}_{\rho[\mu\nu;\lambda]} = 0$.

结论 30. the first and second Bianchi identities respectively

$$R_{\sigma[\rho\mu\nu]} = 0, \quad \nabla_{[\lambda} R_{\sigma\rho]\mu\nu} = 0 \tag{89}$$

are more general, in the sense that they hold for an arbitrary torsion free connection.

结论 31.

$$R_{\mu\nu} = R_{\nu\mu}, \quad \nabla^\mu G_{\mu\nu} = 0 \tag{90}$$

Bianchi identity for the later.

结论 32. local Lorentz transformation,

$$\tilde{e}_a{}^\mu = e_b{}^\mu (\Lambda^{-1})^b{}_a, \quad \Lambda_a{}^c \Lambda_b{}^d \eta_{cd} = \eta_{ab} \tag{91}$$

$$\tilde{\omega}^a{}_b = \Lambda^a{}_c \omega^c{}_d (\Lambda^{-1})^d{}_b + \Lambda^a{}_c (d\Lambda^{-1})^c{}_b \tag{92}$$

结论 33. the first Cartan structure relations: For a torsion free connection,

$$d\hat{\theta}^a + \omega^a{}_b \wedge \hat{\theta}^b = 0 \tag{93}$$

结论 34. the anti-symmetry condition: For the Levi-Civita connection, the connection one-form is anti-symmetric

$$\omega_{ab} = -\omega_{ba} \tag{94}$$

结论 35. The second Cartan structure relations:

$$\mathcal{R}^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{95}$$

3 物理

3.1 The Electromagnetic Field

结论 36.

$$A = A_\mu(x) dx^\mu, \quad F = dA = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu,\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{96}$$

construct a 4-form to integrate over M

$$S_{\text{top}} = -\frac{1}{2} \int F \wedge F = \int dx^0 dx^1 dx^2 dx^3 \mathbf{E} \cdot \mathbf{B} \tag{97}$$

The existence of a metric allows us to introduce a second two-form, $\star F$, and construct the action

$$S_{\text{Maxwell}} = -\frac{1}{2}\int F\wedge\star F = -\frac{1}{4}\int d^4x\sqrt{-g}F^{\mu\nu}F_{\mu\nu} \tag{98}$$

Maxwell equations:

$$d\star F = 0 \tag{99}$$

couple the gauge field to an electric current:

$$S = \int\left(-\frac{1}{2}F\wedge\star F + A\wedge\star J\right) \tag{100}$$

Maxwell equations:

$$d\star J = 0,\quad d\star F = \star J \tag{101}$$

consider a three-dimensional spatial submanifold Σ. charge:

$$Q_e = \int_{\partial\Sigma}\star F,\quad Q_m = \int_{\partial\Sigma}F \tag{102}$$

结论 37.

$$d\star J = 0 \quad\Leftrightarrow\quad \nabla_\mu J^\mu = 0 \tag{103}$$

$$d\star F = \star J \quad\Leftrightarrow\quad \nabla_\mu F^{\mu\nu} = J^\nu \tag{104}$$

3.2 Yang-Mills Theory

结论 38. Yang-Mills theory is based on a Lie group G which, for this discussion, we will take to be $SU(N)$ or $U(N)$. This is a spacetime “vector” A_μ (gauge potential) which lives in the Lie algebra of G.
under a gauge transformation

$$\tilde{A}_\mu = \Omega A_\mu\Omega^{-1} + \Omega\partial_\mu\Omega^{-1} \tag{105}$$

where $\Omega(x)\in G$. construct a field strength

$$(F_{\mu\nu})^a{}_b = \partial_\mu(A_\nu)^a{}_b - \partial_\nu(A_\mu)^a{}_b + [A_\mu, A_\nu]^a{}_b \tag{106}$$

$$F^a{}_b = dA^a{}_b + A^a{}_c \wedge A^c{}_b \tag{107}$$

another way:

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = F_{\mu\nu} \tag{108}$$

is the Ricci identity for a torsion free connection.