Knot Theory

V. O. Manturov

To my mother, Elena Ivanovna Manturova

Gelegentlich ergreifen wir die Feder Und schreiben Zeichen auf ein weißes Blatt, Die sagen dies und das, es kennt sie jeder, Es ist ein Spiel, das seine Regeln hat.

Doch wenn ein Wilder oder Mondmann käme Und solches Blatt, solch furchig Runenfeld Neugierig forschend vor die Augen nähme, Ihm starrte draus ein fremdes Bild der Welt, Ein fremder Zauberbildersaal entgegen. Er sähe A und B als Mensch und Tier, Als Augen, Zungen, Glieder sich bewegen, Bedächtig dort, gehetzt von Trieben hier, Er läse wie im Schnee den Krähentritt, Er liefe, ruhte, litte, flöge mit Und sähe aller Schöpfung Möglichkeiten Durch die erstarrten schwarzen Zeichen spuken, Durch die gestabten Ornamente gleiten, Säh Liebe glühen, sähe Schmerzen zucken. Er würde staunen, lachen, weinen, zittern, Da hinter dieser Schrift gestabten Gittern Die ganze Welt in ihrem blinden Drang Verkleinert ihm erschiene, in die Zeichen Verzwergt, verzaubert, die in steifem Gang Gefangen gehn und so einander gleichen, Daß Lebensdrang und Tod, Wollust und Leiden Zu Brüdern werden, kaum zu unterscheiden...

Und endlich würde dieser Wilde schreien
Vor unerträglicher Angst, und Feuer schüren
Und unter Stirnaufschlag und Litaneien
Das weiße Runenblatt den Flammen weihen.
Dann würde er vielleicht einschlummernd spüren,
Wie diese Un-Welt, dieser Zaubertand,
Dies Unerträgliche zurück ins Niegewesen
Gesogen würde und ins Nirgendland,
Und würde seufzen, lächeln und genesen.

Herman Hesse, "Buchstaben" (Das Glasperlenspiel).

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Preface

Knot theory now plays a large role in modern mathematics, and the most significant results in this theory have been obtained in the last two decades. For scientific research in this field, Jones, Witten, Drinfeld, and Kontsevich received the highest mathematical award, the Fields medals. Even after these outstanding achievements, new results were obtained and even new theories arose as ramifications of knot theory. Here we mention Khovanov's categorification of the Jones polynomial, virtual knot theory proposed by Kauffman and the theory of Legendrian knots.

The aim of the present monograph is to describe the main concepts of modern knot theory together with full proofs that would be both accessible to beginners and useful for professionals. Thus, in the first chapter of the second part of the book (concerning braids) we start from the very beginning and in the same chapter construct the Jones two-variable polynomial and the faithful representation of the braid groups. A large part of the present title is devoted to rapidly developing areas of modern knot theory, such as virtual knot theory and Legendrian knot theory.

In the present book, we give both the "old" theory of knots, such as the fundamental group, Alexander's polynomials, the results of Dehn, Seifert, Burau, and Artin, and the newest investigations in this field due to Conway, Matveev, Jones, Kauffman, Vassiliev, Kontsevich, Bar–Natan and Birman. We also include the most significant results from braid theory, such as the full proof of Markov's theorem, Alexander's and Vogel's algorithms, Dehornoy algorithm for braid recognition, etc. We also describe various representations of braid groups, e.g., the famous Burau representation and the newest (1999–2000) faithful Krammer–Bigelow representation. Furthermore, we give a description of braid groups in different spaces and simple newest recognition algorithms for these groups. We also describe the construction of the Jones two–variable polynomial.

In addition, we pay attention to the theory of coding of knots by d-diagrams, described in the author's papers [Ma1, Ma2, Ma3]. Also, we give an introduction to virtual knot theory, proposed recently by Louis H. Kauffman [KaVi]. A great part of the book is devoted to the author's results in the theory of virtual knots.

Proofs of theorems involve some constructions from other theories, which have their own interest, i.e., quandle, product integral, Hecke algebras, connection theory and the Knizhnik–Zamolodchikov equation, Hopf algebras and quantum groups, Yang–Baxter equations, LD–systems, etc.

The contents of the book are not covered by existing monographs on knot theory; the present book has been taken a much of the author's Russian lecture notes

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book [Ma0] on the subject. The latter describes the lecture course that has been delivered by the author since 1999 for undergraduate students, graduate students, and professors of the Moscow State University.

The present monograph contains many new subjects (classical and modern), which are not represented in [Ma0].

While describing the skein polynomials we have added the Przytycky–Traczyk approach and Conway algebra. We have also added the complete knot invariant, the distributive grouppoid, also known as a quandle, and its generalisation. We have rewritten the virtual knot and link theory chapter. We have added some recent author's achievements on knots, braids, and virtual braids. We also describe the Khovanov categorification of the Jones polynomial, the Jones two–variable polynomial via Hecke algebras, the Krammer–Bigelow representation, etc.

The book is divided into thematic parts. The first part describes the state of "pre-Vassiliev" knot theory. It contains the simplest invariants and tricks with knots and braids, the fundamental group, the knot quandle, known skein polynomials, Kauffman's two–variable polynomial, some pretty properties of the Jones polynomial together with the famous Kauffman–Murasugi theorem and a knot table.

The second part discusses braid theory, including Alexander's and Vogel's algorithms, Dehornoy's algorithm, Markov's theorem, Yang–Baxter equations, Burau representation and the faithful Krammer–Bigelow representation. In addition, braids in different spaces are described, and simple word recognition algorithms for these groups are presented. We would like to point out that the first chapter of the second part (Chapter 8) is central to this part. This gives a representation of the braid theory in total: from various definitions of the braid group to the milestones in modern knot and braid theory, such as the Jones polynomial constructed via Hecke algebras and the faithfulness of the Krammer–Bigelow representation.

The third part is devoted to the Vassiliev knot invariants. We give all definitions, prove that Vassiliev invariants are stronger than all polynomial invariants, study structures of the chord diagram and Feynman diagram algebras, and finally present the full proof of the invariance for Kontsevich's integral. Here we also present a sketchy introduction to Bar-Natan's theory on Lie algebra representations and knots. We also give estimates of the dimension growth for the chord diagram algebra.

In the fourth part we describe a new way for encoding knots by d-diagrams proposed by the author. This way allows us to encode topological objects (such as knot, links, and chord diagrams) by words in a finite alphabet. Some applications of d-diagrams (the author's proof of the Kauffman–Murasugi theorem, chord diagram realisability recognition, etc.) are also described.

The fifth part contains virtual knot theory together with "virtualisations" of knot invariants. We describe Kauffman's results (basic definitions, foundation of the theory, Jones and Kauffman polynomials, quandles, finite—type invariants), and the work of Vershinin (virtual braids and their representation). We also included our own results concerning new invariants of virtual knots: those coming from the "virtual quandle", matrix formulae and invariant polynomials in one and several variables, generalisation of the Jones polynomials via curves in 2–surfaces, "long virtual link" invariants, and virtual braids.

The final part gives a sketchy introduction to two theories: knots in 3-manifolds (e.g., knots in \mathbf{RP}^3 with Drobotukhina's generalisation of the Jones polynomial), the introduction to Kirby's calculus and Witten's theory, and Legendrian knots and links after Fuchs and Tabachnikov. We recommend the newest book on 3-manifolds by Matveev [Matv4].

At the end of the book, a list of unsolved problems in knot and link theory and the knot table are given.

The description of the mathematical material is sufficiently closed; the monograph is quite accessible for undergraduate students of younger courses, thus it can be used as a course book on knots. The book can also be useful for professionals because it contains the newest and the most significant scientific developments in knot theory. Some technical details of proofs, which are not used in the sequel, are either omitted or printed in small type.

Besides the special course at the Moscow State University, I have also held a seminar "Knots and Representation Theory" since 2000, where many aspects of modern knot theory were discussed. Until 2002 this seminar was held together with Professor Valery Vladimirovich Trofimov (1952–2003). I am deeply indebted to him for his collaboration over many years and for fruitful advice.

It is a great pleasure to express my gratitude to all those who helped me at different stages of the present book. I am grateful to my father, Professor Oleg V. Manturov, for attention to my mathematical work during all my life. I wish to thank Professor Victor A. Vassiliev for constant attention to my scientific papers on knots, fruitful ideas and comments.

I am glad to express my gratitude to Profs. Louis H. Kauffman, Roger A. Fenn, Heiner Zieschang, and Alexey V. Chernavsky, Sergei V. Matveeev, Nikita Yu. Netsvetaev, Joan Birman, Patrick Dehornoy, Kent E. Orr, Michiel Hazewinkel, Vladimir V. Vershinin, Drs. Vladimir P. Lexin, Sergei K. Lando, Sergei V. Duzhin, Alexei V. Shchepetilov for many useful comments concerning my book and papers.

I am also grateful to the participants of the seminar "Knots and Representation Theory," especially to Evgeny V. Teplyakov.

It is a pleasure for me to thank my friend and colleague Dr. Rutwig Campoamor–Stursberg for constant consultations and correspondence.

The book was written using knots.tex fonts containing special symbols from knot theory, such as the analysis of the area of the special symbols from knot theory, such as the area of the special symbols from knot theory, such as the area of the special symbols from knot theory, such as the area of the special symbols from knot theory, such as the area of the special symbols from knot theory, such as the special symbols from the special symbols from the special symbols from knot theory, such as the special symbols from the specia

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Part I

Knots, links, and invariant polynomials

Chapter 1

Introduction

As a mathematical theory, knot theory appeared at the end of the 18th century. It should be emphasised that for more than two years knot theory was studied by A.T. Vandermonde, C.-F. Gauss (who found the famous electromagnetic link coefficient formula [Gau]), F. Klein, and M. Dehn [Dehn]. Systematic study of knot theory begins at the end of the 19th century, when mathematicians and physicists started to tabulate knots. A very interesting (but incorrect!) idea belonged to W. Thompson (later known as Lord Kelvin). He thought that knots should correspond to chemical elements. However, the most significant results in knot theory took place in the second part of the 20th century. These achievements are closely connected with the names of J.H. Conway, V.F.R. Jones, V.A. Vassiliev, M.L. Kontsevich, V.G. Turaev, M.N. Goussarov, J.H. Birman, L.H. Kauffman, D. Bar–Natan, and many others.

Knot theory originates from a beautiful and quite simple (or so it can seem) topological problem. In order to solve this problem, one should involve a quite complicated and sophisticated mathematical approach, coming from topology, discriminant theory, Lie theory, product integral theory, quantum algebras, and so on. Knot theory is rapidly developing; it stimulates constructions of new branches of mathematics. One should mention that in the recent years Jones, Witten, Drinfeld (1990) and Kontsevich (1998) were awarded their Fields medals for work in knot theory.

And even after these works, new directions of knot theory appeared. Here we would like to mention the beautiful construction of Khovanov [Kh1] who proposed a categorification of the Jones polynomial — a new knot invariant based on brand new ideas. We shall also touch on the theory of Legendrian knots, lying at the junction of knot theory and contact geometry and topology. The most significant contributions to this (very young) science were made in the last few years (Fuchs, Tabachnikov, Chekanov, Eliashberg).

Another beautiful ramification is virtual knot theory proposed by Kauffman in 1996. The main results in this area are still to be obtained.

Besides this, knot theory is instrumental in constructing other theories; a very important example is the Kirby calculus, i.e., the theory of encoding 3–manifolds by means of links with a special structure — framed links.

1.1 Basic definitions

By a *knot* is meant a smooth embedding of the circle S^1 in \mathbb{R}^3 (or in the sphere S^3)¹ as well as the image of this embedding.

While deforming the ambient space \mathbb{R}^3 , our knot (image) will be deformed as well and hence be embedded. Two knots are called *isotopic*, if one of them can be transformed to the other by a diffeomorphism of the ambient space onto itself; here we require that this homeomorphism should be homotopic to the identical one in the class of diffeomorphisms.

The main question of knot theory is the following: which two knots are (*isotopic*) and which are not. This problem is called *the knot recognition problem*. Having an isotopy equivalence relation, one can speak about *knot isotopy classes*. When seen in this context, we shall say "knot" when referring to the knot isotopy class. One can also talk about *knot invariants*, i.e., functions on knot isotopy classes or functions on knots invariant under isotopy.

A partial case of the knot recognition problem is the *trivial knot recognition* problem. Here by *trivial knot* is meant the simplest knot that can be represented as the boundary of a 2-disc embedded in \mathbb{R}^3 .

Both questions are very difficult. Though they are solved their solution requires many techniques (see [He]) and cannot in fact be implemented for practical purposes. The main stages of the complete solutions of these (and some other) problems can be read in [Mat].

In the present book we shall give partial answers to these questions. As usual, in order to prove that two knots are isotopic, one should present a step-by-step isotopy transforming one knot to the other. Later we shall present the list of Reidemeister moves, which are indeed step-by-step isotopy moves. To show that two knots are not isotopic, one usually finds an invariant having different values on these two knots.

Usually, knots are encoded as follows. Fix a knot, i.e., a map $f: S^1 \to \mathbf{R}^3$. Consider a plane $h \subset \mathbf{R}^3$ and the projection of the knot on it. Without loss of generality, one can assume that h = Oxy. In the general position case, this projection is a quadrivalent graph embedded in the plane. Usually, we shall call a part (the image of an interval) of a knot a branch of it. Each vertex V of this graph (also called a crossing) is endowed with the following structure. Let a, b be two branches of a knot, whose projections intersect in the point V. Since a and b do not intersect in \mathbf{R}^3 , the two preimages of V have different z-coordinates. So, we can say which branch (a or b) comes over, or forms an overcrossing; the other one forms an undercrossing (see Fig. 1.1).

The quadrivalent projection graph without an over– and undercrossing structure is called the *shadow* of the knot. The *complexity* of a knot is the minimal number of crossings for knots of given isotopy type.

The following exercise is left for the reader.

Exercise 1.1. Show that any two knots having the same combinatorial structure of planar diagrams (i.e. isomorphic embeddings with the same crossing structure) are isotopic.

 $^{^{1}}$ These two theories are equivalent because of codimension reasons. In the sequel, we shall deal with knots in \mathbb{R}^{3} , unless otherwise specified.

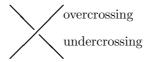


Figure 1.1. Local structure of a crossing

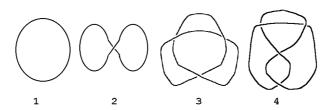


Figure 1.2. The simplest knots

Let us now give some examples.

Example 1.1. The knot having a diagram without crossings (see Fig. 1.2.1) is called the unknot or the trivial knot. Figure **1.2.2.** represents another planar diagram of the unknot. The knot shown in Fig. **1.2.3.** is called the trefoil, and that in Fig. **1.2.4.** is called the figure eight knot. Both knots are not trivial; they are not isotopic to each other.

Definition 1.1. A knot diagram L is called ascending (starting from a point A on it different from any vertex) if while walking along L from A (in some direction) each crossing is first passed under and then over.

Exercise 1.2. Show that each ascending diagram represents an unknot.

For each knot, one can construct its *mirror image*, i.e. the knot obtained from the initial one by reflecting it in some plane. Typical diagrams of a knot and its mirror image can be obtained by switching all crossing types (overcrossing replaces undercrossing and vice versa). A knot is called *amphicheiral* if it is isotopic to its mirror image.

Thus, e.g., the trefoil knot is not amphicheiral. We shall prove this fact later. This allows us to speak about two trefoil knots, the *right* one and the *left one*, see Fig. 1.3.

Exercise 1.3. Show that figure eight is an amphicheiral knot.

One can also speak about *oriented* knots, i.e., smooth mappings (images) of an oriented circle in \mathbb{R}^3 . By an *isotopy* of oriented knots is meant an isotopy of knots preserving orientation.

Considering several circles instead of one circle, one comes to the notion of a link. A link is a smooth embedding (image) of several disjoint circles in \mathbb{R}^3 . Each

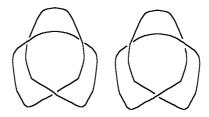


Figure 1.3. a. Left trefoil b. Right trefoil

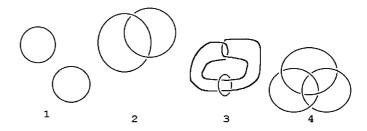


Figure 1.4. The simplest links

knot, representing the image of one of these circle is called a *link component*. One can naturally define *link isotopy* (by using an orientation-preserving diffeomorphism of the ambient space), *link planar diagrams* and *link invariants*. By an *oriented link* is meant a smooth mapping (image) of the disjoint union of several oriented circles.

There is another approach to link isotopy, when each link component is allowed to have self–intersections during the isotopy, but intersections of different components are forbidden. This theory is described in a beautiful paper [Mi] of John Willard Milnor, who introduced his famous μ –invariants for classification of links up to this "isotopy."

The trivial n-component link or n-unlink is a link represented by diagram consisting of n circles without crossings.

Example 1.2. Figure 1.4.1. represents the trivial two-component link. Figures 1.4.2., 1.4.3., and 1.4.4. show us links, representing the Hopf link, the Whitehead link and the Borromean rings, respectively. The latter link is named in honour of the famous Italian family Borromeo, whose coat of arms was decorated by these rings. All these three links are not trivial (this will be shown later, when we are able to calculate values of some invariants). Borromean links demonstrate an interesting effect: while deleting each of three link components, one obtains the trivial link, whence the total link is not trivial.

Exercise 1.4. Show that the Whitehead link has component symmetry: there is an isotopy to itself permuting the link components.

Let us now talk about knot (and link) invariants. The first well–known link invariant (after the Gauss electromagnetic linking coefficient) is the *fundamental* group of the complement to the knot (link). This invariant is purely topological; it

distinguishes different knots quite well (in particular, it recognises the unknot as well as the trivial link with arbitrary many components). However, this "solution" of the knot recognition problem is not complete because we only reduce this problem to the group recognition problem, which is, generally, indecidable.

In 1923, the famous American mathematician James Alexander [Ale, Ale2] derived a polynomial invariant of knots and links from the fundamental group. This invariant is, certainly, weaker than the fundamental group itself, but the invariant polynomial is much easier to recognise: one can easily compare two polynomials (unlike groups given by their presentation).

In 1932, the German topologist K. Reidemeister published his book *Knotenthe-orie* (English translation: [Rei]), in which he presented a list of local moves (known as Reidemeister moves) and proved that any two planar diagrams generate isotopic knots (links) if and only if there exist a finite chain of Reidemeister moves from one of them to the other. In addition, he tabulated knot isotopy classes up to complexity seven, inclusively.

Since that time, to prove the invariance for some function on knots, one usually checks its invariance under Reidemeister moves.

Among books describing the state of knot theory at that time, we would like to point out those by Ashley [Ash], Crowell and Fox [CF] and that by Burde and Zieschang [BZ].

The next stage of development of knot theory was the discovery of the Conway polynomial [Con]. This discovery is based on so-called *skein relations*. These relations are purely combinatorial and based on the notion of the planar diagram. The Alexander polynomial [Ale] can also be interpreted in terms of skein relations. Moreover, Alexander knew about this. However, Conway was the first to show that skein relations can be used axiomatically for defining a knot invariant.

This discovery stimulated further beautiful work presenting polynomials based on skein relations. By using these polynomials, some old problems were solved, e.g., Tait's problem [Tait].

Among the other skein polynomials, we would like to emphasise the HOMFLY polynomial and the Kauffman polynomial; HOMFLY is the abbreviation of the first letters of the authors: Hoste, Ocneanu, Millett, Freyd, Lickorish and Yetter (see [HOMFLY]). This polynomial was also discovered by Przytycki and Traczyk [PT].

The most powerful of the skein polynomials is the Jones polynomial of two variables; each of those named above can be obtained from it by a variable change. About Jones polynomials of one and two variables one can see [Jon1, Jon2].

The planar diagram approach for coding links is not the only possibility. Besides knot theory one should point out another theory — the theory of braids. Braids were proposed by the German mathematician Emil Artin, who gave initial definitions and proved basic theorems on the subject [Art1]; English translation [Art2]. There are four classical definitions of the braid group. Braids are closely connected with polynomials without multiple roots, discriminant theory, representation theory, etc. By an n-strand braid we mean a set of n ascending simple non-intersecting piecewise-linear curves (strands), connecting points A_1, \ldots, A_n on a line with points B_1, \ldots, B_n on a parallel line. Analogously to the case of knots, one can describe braids by their planar diagrams; the equivalence of braids is defined as an isotopy of strands preserving all strands vertically. The product of two braids a and b is

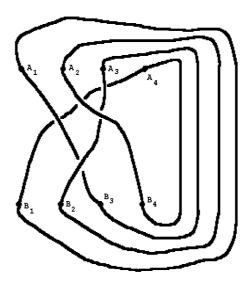


Figure 1.5. Closure of a braid

obtained by juxtaposing one braid under the other and rescaling the height coordinate.

It is easy to see that by *closing* the braid in the most natural way (i.e., by connecting A_i with B_i , i = 1, ..., n, see Fig. 1.5) we obtain a link diagram.

In the present book we describe the three important theorems on braids and links. The Alexander theorem states that each link isotopy class can be obtained as a closure of braids. The Artin theorem [Art1] gives a presentation of the braid group by generators and relations. The Markov theorem [Mar'] gives a list of sufficient relations transforming one braid to the other in the case when their closures represent isotopic knots. We also present algorithms for braid recognition: a simple algorithm described in [GM] and that of Dehornoy.

The great advantage of braid theory (unlike knot theory) is that braids form a group. This simplifies some problems (i.e., reduces the word recognition problem to the trivial word recognition problem).

We would like to recommend the following monographs on those parts of knot theory described above: Louis Kauffman's two books [Ka1, Ka2], and those of Adams [Ada], Kawauchi [Kaw] and Atiyah [Atiyah].

Suppose we have a knot and we want to change its isotopy class, by changing the smooth map of the circle in \mathbb{R}^3 . By definition, it is impossible to do this without intersection. Thus, the most important moment of this map is the intersection moment. If there exists only a finite number of transversal intersections, one can speak about a singular knot. The space of all singular knots (that contains the space of all knots as a subspace) is called the discriminant space.

By studying the properties of the discriminant sets, V.A. Vassiliev proposed the notion of finite type invariants, later known as *Vassiliev's invariants*. Initially, Vassiliev's knot invariants required a complicated and non-trivial mathematical approach. However, a purely combinatorial interpretation of them was found. In the

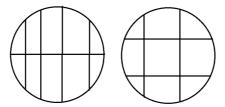


Figure 1.6. Left trefoil; right trefoil.

present book, we shall give a proof of the fact that the Vassiliev knot invariants are stronger than all the invariants named above.

The initial proof of the existence of the Vassiliev knot invariants is given in [Vas]; the structure of these invariants was obtained by M.L. Kontsevich by means of his remarkable integral construction now known as the Kontsevich integral. The work of Kontsevich is published in [Kon]. We would also recommend the profound and detailed description of the Kontsevich integral in [BN]. There one can also find good points of view for the connection between knots, Vassiliev's invariants, and the representation of Lie algebras.

The calculation of the Kontsevich integral had been very difficult before the work by Le and Murakami [LM] appeared. In this work, they present explicit techniques for the Kontsevich integral calculations. This techniques is, however, very difficult.

One should also mention the outstanding work by Khovanov [Kh1] from 1997 who gave a generalisation of the Jones polynomial in terms of homologies of some formal algebraic complex. A new knot invariant appeared after so much had been discovered.

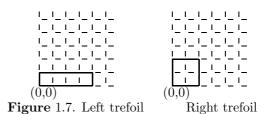
Here we would like to mention another way of representing knots and links. This is based on the notion of *d-diagram* (see [Ma3]). A *d*-diagram can be seen as a circle with two families of non-intersecting chords where no point can be the end of two different chords. This theory has its origin in atoms and Hamiltonian systems. However, the *d*-diagram theory allows us to represent all links by using words in a finite alphabet, see [Ma3]. Here we have advantage in comparison with, say, encoding knots by braids: in the latter case one requires an infinite number of letters. This encoding can be easily generalised for cases of braids, singular knots and certain other purposes.

By using d-diagrams, one can represent all links as loops on checked paper; these loops should lie inside the first quadrant and come from the origin of coordinates.

The simplest d-diagrams corresponding to the left and right trefoil knots are shown in Fig. 1.6.

Their bracket structures look like

 $\big(\big(\big(\big(\big[\big] \big) \big) \big) \big)$ for the left one and $\big(\big(\big[\big[\big] \big) \big) \big] \big]$ for the right one.



Example 1.3. The left trefoil knot can be represented as the rectangle 1×4 . The square 2×2 represents the right trefoil knot (see Fig 1.7).

Actually, the theory of atoms first developed for the classification of Hamiltonian systems can be useful in many areas of geometry and topology, e.g., for coding 3-manifolds.

In the last few years, one of the most important branches of knot theory is the theory of virtual knots. A virtual knot is a combinatorial notion proposed by Louis H. Kauffman in 1996 (see [KaVi]). This notion comes from a generalisation of classical knot diagram with generalised Reidemeister moves. The theory can be interpreted as a "projection" of knot theory from knots in some 3-manifolds. This theory is developed very rapidly. There are many common approaches coming from classical knot theory, and there are "purely virtual invariants" (see [Ma2]).

Chapter 2

Reidemeister moves. Knot arithmetics

In the present chapter, we shall discuss knots, their planar diagrams and the semigroup structure on knots; the latter is isomorphic to that on natural numbers with respect to multiplication. This allows us to investigate *knot arithmetics* and to establish some properties of multiplication and decomposition of knots.

As we have shown in the previous chapter, all links can be encoded by their regular planar diagrams. Obviously, while deforming a link, its planar projection might pass through some singular states. These singular states give the motivation for writing down the list of simplest moves for planar diagrams.

2.1 Polygonal links and Reidemeister moves

Because each knot is a smooth embedding of S^1 in \mathbb{R}^3 , it can be arbitrarily closely approximated by an embedding of a closed broken line in \mathbb{R}^3 . Here we mean a good approximation such that after a very small smoothing (in the neighbourhood of all vertices) we obtain a knot from the same isotopy class. However, generally this might not be the case.

Definition 2.1. An embedding of a disjoint union of n closed broken lines in \mathbf{R}^3 is called a *polygonal* n-component link. A polygonal knot is a polygonal 1-component link.

Definition 2.2. A link is called *tame* if it is isotopic to a polygonal link and *wild* otherwise.

Remark 2.1. The difference between tame and wild knots is of a great importance. To date, the serious systematic study of wild knots has not been started. We shall deal only with tame knots, unless otherwise specified.

All C^1 -smooth knots are tame, for a proof see [CF]. In the sequel, all knots are taken to be smooth, hence, tame.

Definition 2.3. Two polygonal links are *isotopic* if one of them can be transformed to the other by means of an iterated sequence of *elementary isotopies* and

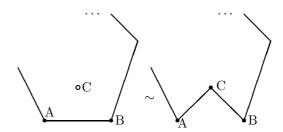


Figure 2.1. Elementary isotopy

reverse transformation. Here the elementary isotopy is a replacement of an edge AB with two edges AC and BC provided that the triangle ABC has no intersection points with other edges of the link, see Fig. 2.1.

It can be proved that the isotopy of smooth links corresponds to that of polygonal links; the proof is technically complicated. It can be found in [CF]. When necessary, we shall use either the smooth or polygonal approach for representing links.

Like smooth links, polygonal links admit *planar diagrams* with overcrossings and undercrossings; having such a diagram one can restore the link up to isotopy.

Exercise 2.1. Show that all polygonal links with less than six edges are trivial.

Exercise 2.2. Draw a polygonal trefoil knot with six edges.

Definition 2.4. By a *planar isotopy* of a smooth link planar diagram we mean a diffeomorphism of the plane onto itself not changing the combinatorial structure of the diagram.

Remark 2.2. The polygonal knot planar diagram is defined analogously to the smooth diagram. However bivalent vertices give us redundant information. Thus, dividing an edge into two edges by an additional vertex we obtain a diagram that is planar–isotopic to the initial one.

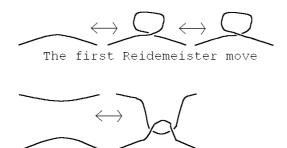
Obviously, planar isotopy is an isotopy, i.e., it does not change the link isotopy type in \mathbb{R}^3 .

Theorem 2.1 (Reidemeister [Rei]). Two diagrams D_1 and D_2 of smooth links generate isotopic links if and only if D_1 can be transformed into D_2 by using a finite sequence of planar isotopy and the three Reidemeister moves $\Omega_1, \Omega_2, \Omega_3$, shown in Fig. 2.2.

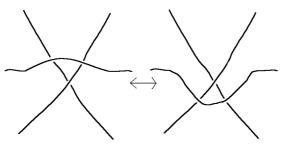
One can prove this fact by using the notion of codimension; it involves some complicated technicalities. Here we give another proof for the case of polygonal links.

Proof. One implication is evident: one should just check that moves $\Omega_1, \Omega_2, \Omega_3$ do not change the link isotopy class.

Now, let us prove the reverse statement of the theorem. Let D_1, D_2 be two planar diagrams of the two isotopic polygonal links K_1 and K_2 . By definition,



The second Reidemeister move



The third Reidemeister move

Figure 2.2. Reidemeister moves $\Omega_1, \Omega_2, \Omega_3$

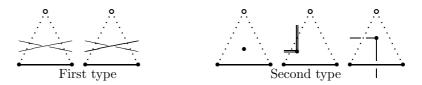
the isotopy between K_1 and K_2 consists of a finite number of elementary isotopies looking like $[AB] \to [AC] \bigcup [CB]$. Here the "link before" is reconstructed to the "link after." Without loss of generality, let us assume that for each step for the triangle ABC, the edges [DA] and [BE], coming from the ends of [AB], do not intersect the interior of ABC. Otherwise, we can obtain it by using Ω_1 .

Let L_0 be the projection of the "link before" on the plane ABC. Let us split the intersection components of ABC and L_0 into two sets: upper and lower according to the location of edges of the link K_0 with respect to the plane ABC.

Let us tile ABC into small triangles of the four types in such a way that edges of small triangles do not contain vertices of L_0 . Each first-type triangle contains only one crossing of L_0 ; here edges of L_0 intersect two sides of the triangle. The second-type triangle contains the only vertex of L_0 and parts of outgoing edges. The third-type triangle contains a part of one edge of L_0 and no vertices. Finally, the fourth-type triangle contains neither vertices nor edges. All triangle types except the fourth (empty) one are illustrated in Fig. 2.3.

Such a triangulation of ABC can be constructed as follows. First, we cut all vertices and all crossings by triangles of the first and the second type, respectively. Then we tile the remaining part of ABC and obtain triangles of the last two types.

The plan of the proof is now the following. Instead of performing the elementary isotopy to ABC, we perform step-by-step elementary isotopies for small triangles, composing ABC. It is clear that these elementary moves (isotopies) can be repre-



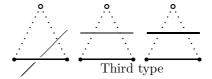


Figure 2.3. Types of small triangles

sented as combinations of Reidemeister moves and planar isotopies.

More precisely, the first-type triangle generates a combination of the second and third Reidemeister moves, the second and the third-type triangles generate Ω_2 or planar isotopy. The fourth-type triangle generates planar isotopy.

Exercise 2.3. Show that two variants of the third Reidemeister moves, shown in Fig. 2.2, are not independent. More precisely, each of them can be obtained from the other and the second Reidemeister move.

Difficult exercise 2.1. Show that all three Reidemeister moves $\Omega_1, \Omega_2, \Omega_3$ are independent.

The above reasonings show that, in a general position, links have projections with many intersection points, which are simple and transverse. However, "the opposite case" is also interesting.

Difficult exercise 2.2 (H. Brunn). Prove that for each link $L \subset L'$, there exists a link L' and a plane P such that the projection of L' on P has the only intersection point, which is transverse for all branches.

Definition 2.5. A knot is called *invertible* if it is isotopic to the knot obtained from the initial one by the orientation change.

Remark 2.3. Do not confuse knot invertibility and the existence of the inverse knot (in the sense of concatenation, see Def. 2.2).

Exercise 2.4. By using Reidemeister moves, show that each of the two trefoil knots is invertible.

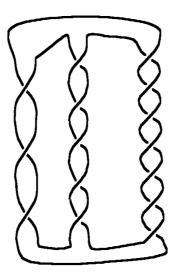


Figure 2.4. Non-invertible knot

The existence of non-invertible knots had been an open problem for a long time. This problem was solved positively in 1964 by Trotter [Tro]. The simplest non-invertible knot is 8_{17} , see the knot table at the end of the book.

An example of a non-invertible knot is shown in Fig. 2.4.

Definition 2.6. A link diagram is called *alternating* if while moving along each component, one passes overcrossings and undercrossings alternately.

The simplest non–alternating knot (i.e., a knot not having alternating diagrams) is shown in Fig. 2.5.

2.2 Knot arithmetics and Seifert surfaces

Let us now discuss the algebraic structure on the set of knot isotopy classes. Let K_1 and K_2 be two oriented knots.

Definition 2.7. By a composition, concatenation, or connected sum of knots K_1 and K_2 is meant the oriented knot obtained by attaching the knot K_2 to the knot K_1 with respect to the orientation of both knots, see Fig. 2.6. Notation: $K_1 \# K_2$.

Exercise 2.5. Show that the concatenation operation is well defined, i.e., the isotopy class of $K_1 \# K_2$ does not depend on the two places of attachment.

Exercise 2.6. Show that the concatenation operation is commutative: for any K_1, K_2 the knots $K_1 \# K_2$ and $K_2 \# K_1$ are isotopic.

The exercise above is intuitively clear. However, the desired isotopy can be performed for the case of knots with fixed ends (or $long\ knots$).

Definition 2.8. Analogously, one defines the *disconnected sum*: one just takes two diagrams and situates them inside two non–intersecting domains on the plane.

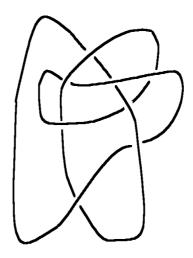


Figure 2.5. Non-alternating knot

Notation: $K_1 \sqcup K_2$.

Theorem 2.2. Let K_1 be a non-trivial knot. Then for each knot K_2 , the knot $K_1 \# K_2$ is not trivial either.

We shall give here three proofs of the theorem. One of them is a bit "speculative" but very clear. Another one is based on a beautiful idea that belongs to J.H. Conway. The last proof is a corollary from a stronger statement.

Let us begin now with the "speculative" one.

Proof. Suppose K_1 is not an unknot and $K_1 \# K_2$ is. Consider the sequence of knots

$$K_1 \# K_2, (K_1 \# K_2) \# (K_1 \# K_2), \ldots,$$

where the knot K_1 lies inside the ball with radius 1, the knot K_2 lies inside the ball of radius $\frac{1}{2}$, the knot K_3 lies inside the ball of radius $\frac{1}{4}$, and so on.

Thus, one can place the infinite series on a finite interval, see Fig. 2.7.

Thus we obtain a knot that will be, possibly, wild. Denote this knot by a. Since the knot $K_1 \# K_2$ is trivial, the knot a is trivial as well. On the other hand, we have the following decomposition: $a = K_1 \# (K_2 \# K_1) \# (K_2 \# K_1) \dots$ Since the concatenation is commutative, the knot $K_2 \# K_1$ is trivial. Thus, the trivial knot a is isotopic to K_1 . This contradiction completes the proof.

There is a beautiful and simple proof of this fact proposed by J. H. Conway.

Proof. Let us look at Fig. 2.8.

Definition 2.9. By a standardly embedded (in \mathbb{R}^3) handle body S_g for a natural number g we mean the small tubular neighbourhood of the graph lying in $\mathbb{R}^2 \subset \mathbb{R}^3$, see Fig. 2.9.

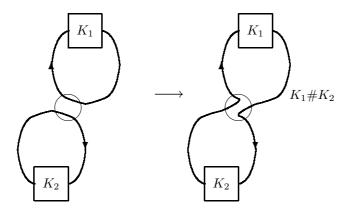


Figure 2.6. Connected sum of knots

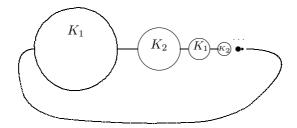


Figure 2.7. A wild knot

By a *standardly embedded* handle surface we mean the boundary of a standardly embedded handle body.

Here we see that the connected sum $K_1 \# K_2$ has a tubular neighbourhood T isomorphic to the natural tubular neighbourhood of K_2 . This neighborhood is such that the intersection of each meridional disc D with $K_1 \# K_2$ is not trivial (homologically).

But for the unknot U the only possible neighbourhood T' with the property described above is a standardly embedded torus.

The knot K_1 is not trivial and hence "thick" knots T' and T are not isotopic. Thus, $K_1 \# K_2$ is not a trivial knot.

Definition 2.10. A knot K is said to be *prime* if for any knots L, M, such that K = L # M, one of the knots L, M is trivial. All other knots are said to be *complicated*.

Definition 2.11. If for some knots K, L, M the statement K = L # M holds then one says that the knots L, M divide the knot K.

Thus, we have proved that all elements of the knot semigroup except the unknot have no inverse elements. Let us study another property of this semigroup.

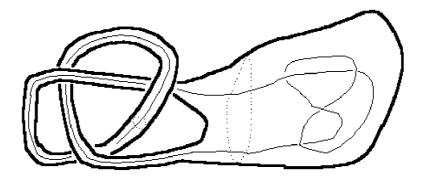


Figure 2.8. The product of non-trivial knots is not trivial

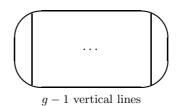


Figure 2.9. A graph on the plane

Exercise 2.7. Show that each knot (link) isotopy class can be represented as a curve (several curves) on some handled body standardly embedded in \mathbb{R}^3 .

Now let us give the definition of the Seifert surface first introduced by Seifert in [Sei1], see also [Pr, CF].

Definition 2.12. Let L be an oriented link. A *Seifert surface* for a link L is a closed compact orientable two–dimensional surface in \mathbb{R}^3 , whose boundary is the link L and the orientation of the link L is induced by the orientation of the surface.

Theorem 2.3. For each link in \mathbb{R}^3 , there exists a Seifert surface of it.

 ${\it Proof.}$ Consider a planar diagram D of the link L. Let us smooth link crossings as shown below.

After such a smoothing, we obtain a set of closed non-intersecting simple curves on the plane.

Definition 2.13. These curves are called *Seifert circles*.

Let us attach discs to these circles. Though the interiors of these circles on the plane might contain one another, discs in 3–space can be attached without intersections.

In the neighbourhood of each crossing, two discs meet each other. Let us choose two closed intervals on the boundary of these discs and connect them by a twisted band, see Fig. 2.11. The boundaries of this band are two branches of the link incident to the chosen crossings. The two positions (upper and lower) in Fig. 2.11

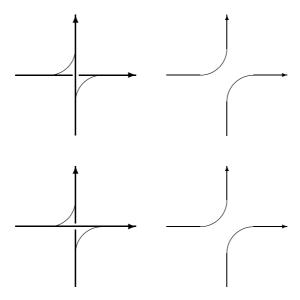


Figure 2.10. Smoothing the diagram crossings

show different ways of twisting (in one case the vertical line lies over the horizontal line, in the other case, vice versa).

Thus we obtain some surface that might not be connected. Connecting different components of these surfaces by thin tubes, we easily obtain a connected surface with the same boundary.

It remains to prove now that the obtained surface is orientable. Actually, consider the plane P of the link projection. Choose a positive frame of reference on P. This generates an orientation for any discs attached to a Seifert circle. Herewith, for two Seifert surfaces adjacent to the same vertex, these orientations are opposite (in the sense of the Seifert surface) since there is a twisted band between them.

It remains to show that for each sequence $C_1, \ldots C_n = C_1$ of Seifert circles, where any two adjacent circles C_i, C_{i+1} have a common vertex, the number n is odd. In other words, while passing from a circle to itself one should pass even number of twistings. The latter follows from the fact that for a polygon with an odd number of sides one cannot choose the orientation of sides in such a way that any two adjacent sides have opposite orientations.

Theorem 2.4. The parity of the number of Seifert surfaces for the diagram of a k-component link with n crossings coincides with the parity of n - k.

Proof. Let L be a diagram of a k-component link with n vertices. Consider a Seifert surface S(L). Let us construct a cell decomposition of it. First let us choose the one-dimensional frame as follows: vertices of the diagram correspond to crossings of L (two vertices near each crossing), and edges correspond to edges of the diagram (two); one edge is associated with each crossing that connects the two vertices, see

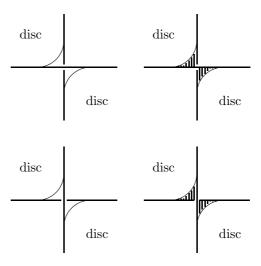


Figure 2.11. Attaching a band to discs

Fig. 2.12. The number of cells of such a decomposition equals the number of Seifert surfaces. Now, let us attach discs to the boundary components (components of the initial link). Thus we obtain a closed oriented manifold. Its Euler characteristic should be even. It equals 2n - 3n + S + k, where n is the number of crossings of the diagram L, and S is the number of Seifert surfaces of this diagram. Taking into account that the number 2n - 3n + S + k is odd, we obtain the claim of the theorem.

The Seifert surface for the knot K is a compact 2–surface whose boundary is K. By attaching a disc to this surface we get a sphere with some handles.

Definition 2.14. A knot K is said to have *genus* g if g is the minimal number of handles for Seifert surfaces corresponding to K.

The notion of knot genus was also introduced by Seifert in [Sei1].

Remark 2.4. Actually, the calculation problem for the knot genus is very complicated but it was solved by Haken. For this purpose, he developed the theory of normal surfaces — surfaces in 3-manifolds lying normally with respect to a cell decomposition, see [Ha]. The trivial knot (knot of genus 0) recognition problem is a partial case of this problem.

Lemma 2.1. The function g is additive, i.e., for any knots K_1, K_2 the equality $g(K_1) + g(K_2) = g(K_1 \# K_2)$ holds.

Proof. First, let us show that $g(K_1 \# K_2) \leq g(K_1) + g(K_2)$.

Consider the Seifert surfaces F_1 and F_2 of minimal genii for the knots K_1 and K_2 . Without loss of generality, we can assume that these surfaces do not intersect each other. Let us connect the two by a band with respect to their orientation.

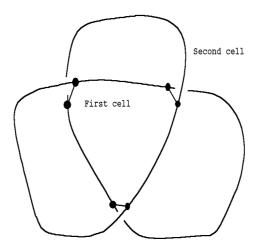


Figure 2.12. Cell decomposition of a Seifert surface

Thus we obtain a Seifert surface for the knot $K_1 \# K_2$ of genus $g(K_1) + g(K_2)$. Thus

$$g(K_1 \# K_2) \le g(K_1) + g(K_2).$$

Now, let us show that $g(K_1 \# K_2) \ge g(K_1) + g(K_2)$. Consider a minimal genus Seifert surface F for the knot $K_1 \# K_2$. There exists a (topological) sphere S^2 , separating knots K_1 and K_2 in the connected sum $K_1 \# K_2$ at some points A, B.

The sphere S^2 intersects the surface F along several closed simple curves and a curve ended at points A, B. Each circle divides the sphere S^2 into two parts; one of them does not contain points of the curve AB. Without loss of generality, assume that the intersection $F \cap S^2$ consists of several closed simple curves a_1, \ldots, a_k and one curve connecting A and B. The neighbourhood of each a_i looks like a cylinder. Let us delete a small cylindrical part that contains the circle (our curve) from the cylinder and glue the remaining parts of the cylinder by small discs. If the obtained surface is not connected, let us take the part of it containing $K_1 \# K_2$. After performing such operations to each circle, we obtain a closed surface F' containing the knot $K_1 \# K_2$ and intersecting S^2 only along AB. The operations described before can only increase the number of handles. Thus, $g(F') \leq g(F)$. Because F has minimal genus, we conclude that $g(F') = g(F) = g(K_1 \# K_2)$.

The sphere S^2 divides the surface F' into surfaces that can be treated as Seifert surfaces for K_1 and K_2 . Thus,

$$g(K_1) + g(K_2) \ge g(F') = g(K_1)$$

Taking into account that we have proved the " \leq "-inequality, we conclude that the genus is additive. \Box

As a corollary, one can conclude that any non-trivial knot has no inverse, since

the unknot has genus zero and the others have genus greater than zero. This is the third proof of the non-invertibility of non-trivial knots.

Exercise 2.8. Show that the trefoil has genus one.

Thus, each knot can be decomposed in no more than a finite number of prime knots. To clarify the situation about knot arithmetics, it remains to prove one more lemma, the lemma on unique decomposition.

Lemma 2.2. Let L and M be knots, and let K be a prime knot dividing L # M. Then either K divides L or K divides M.

Proof. Consider a knot L#M together with some plane p, intersecting it in two points and separating L from M. Since L#M is divisible by K, there exists a (topological) 2–sphere S^2 that intersects the knot L#M in two points and contains the knot K (more precisely, the "long knot" obtained from K by stretching the ends afar) inside.

If this sphere does not intersect p, the problem would be solved. Otherwise, the sphere S^2 intersects the plane at some non–intersecting simple curves (circles). If each of these circles is unlinked with L#M, they can be removed by a simple deformation. In the remaining case, they can also be removed by some deformation of the sphere because of the primitivity of K (because the knot is prime, at least one part of the sphere contains the trivial part of the knot).

Thus, if the connected sum L#M is divisible by K, then either L or M is divisible by K.

Remark 2.5. This basic statement of knot arithmetics belongs to Schubert, see [Schu].

Thus we have:

- 1. Knot isotopy classes form a commutative semigroup related to the concatenation operation; the unit element of the semigroup is the unknot.
- 2. Each non-trivial knot has no inverse element;
- 3. Prime decomposition is unique up to permutation;
- 4. The number of different prime knots is denumerable.

Exercise 2.9. Prove the last statement.

Hence the number of smooth knot isotopy classes is denumerable, and we get the following:

Theorem 2.5. The semigroup of knot isotopy classes with respect to concatenation is isomorphic to the semigroup of natural numbers with respect to multiplication. Here prime knots correspond to prime numbers.

However, the isomorphism described above is not canonical, hence there is no canonical linear order on the set of all knots (prime knots), i.e., it is impossible to say that "the (prime knot) right trefoil corresponds to the prime number three or

to the prime number 97." To do this, one should be able to recongnise knots, which that is quite a difficult problem.

There exists only one semigroup defined by the properties described above (up to isomorphism).

In [Ma3], we propose a purely algebraic description for this semigroup, i.e., we give an explicit isomorphism between this geometric semigroup and some algebraically constructed "bracket semigroup."

Chapter 3

Links in 2-surfaces in R³. Simplest link invariants

Knots (links) embedded in \mathbb{R}^3 can be considered as curves (families of curves) in 2–surfaces, where the latter surfaces are standardly embedded in \mathbb{R}^3 . In this chapter we shall prove that all knots and links can be obtained in this manner.

In the present chapter, we shall describe some series of knots and links, e.g., torus links that can be represented as links lying on the torus standardly embedded in \mathbb{R}^3 .

We shall also present some invariants of knots and links and demonstrate that the trefoil knot is not trivial.

3.1 Knots in 2-surfaces. The classification of torus knots

Consider a handle surface S_g standardly embedded in \mathbf{R}^3 and a curve (knot) K in it. We are going to discuss the following question: which knot isotopy classes can appear for a fixed g?

First, let us note that for g = 0 there exists only one knot embeddable in S^2 , namely, the unknot.

Exercise 3.1. Show, that for each link isotopy class $L \subset \mathbf{R}^3$ there exists a representative link L' (of the class L) lying in some handle surface standardly embedded in \mathbf{R}^3 .

The case g = 1 (torus, torus knots) gives us some interesting information.

Consider the torus as a Cartesian product $S^1 \times S^1$ with coordinates $\phi, \psi \in [0, 2\pi]$, where 2π is identified with 0. In Fig. 3.1, the torus is illustrated as a square with opposite sides identified.

Let us embed this torus standardly in \mathbb{R}^3 ; more precisely:

$$(\phi, \psi) \longrightarrow ((R + r\cos\psi)\cos\phi, (R + r\cos\psi)\sin\phi, r\sin\psi)$$

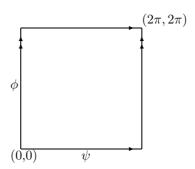


Figure 3.1. Torus

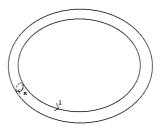


Figure 3.2. Longitude and meridian of the torus

Here R is the outer radius of the torus, r is the small radius (r < R), ϕ is the longitude, and ψ is the meridian, the direction of the longitude and the meridian see in Fig. 3.2.

For the classification of torus knots we shall need the classification of isotopy classes of non-intersecting curves in T^2 : obviously, two curves isotopic in T^2 are isotopic in \mathbb{R}^3 .

Without loss of generality, we can assume that the considered closed curve passes through the point $(0,0)=(2\pi,2\pi)$. It can intersect the edges of the square several times. Without loss of generality, assume all these intersections to be transverse. Let us calculate separately the algebraic number of intersections with horizontal edges and that of intersections with vertical edges. Here passing through the right edge or through the upper edge is said to be positive; that through the left or the lower is negative.

Thus, for each curve of such type we obtain a pair of integer numbers.

Exercise 3.2. Show that if both numbers are equal to zero then the knot is trivial.

Remark 3.1. In the sequel, we shall consider only those knots for which at least one of these numbers is not equal to zero.

The following fact is left for the reader as an exercise.

Exercise 3.3. For a non-self-intersecting curve these numbers are coprime.

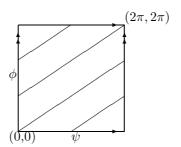


Figure 3.3. Trefoil on the torus

So, each torus knot passes p times the longitude of the torus, and q times its meridian, where GCD(p,q)=1. It is easy to see that for any coprime p and q such a curve exists: one can just take the geodesic line $\{q\phi-p\psi=0 \pmod{2\pi}\}$. Let us denote this torus knot by T(p,q).

Exercise 3.4. Show that curves with the same coprime parameters p, q are isotopic on the torus.

So, in order to classify torus knots, one should consider pairs of coprime numbers p, q and see which of them can be isotopic in the ambient space \mathbb{R}^3 .

The simplest case is when either p or q equals one.

Exercise 3.5. Show that for p = 1, q is arbitrary or q = 1, p is arbitrary we get the unknot.

The next simplest example of a pair of coprime numbers is p=3, q=2 (or p=2, q=3). In each of these cases we obtain the trefoil knot.

Let us prove the following important result.

Theorem 3.1. For any coprime integers p and q, the torus knots (p,q) and (q,p) are isotopic.

Proof. Let us take S^3 as the ambient space for knots (instead of \mathbb{R}^3). As we know, it does not affect isotopy.

It is well known that S^3 can be represented as a union of two full tori attached to each other according to the following boundary diffeomorphism. This diffeomorphism maps the longitude of the first torus to the meridian of the second one, and vice versa. More precisely, $S^3 = \{z, w \in \mathbb{C} \mid |z|^2 + |w|^2 = 1\}$. The two tori (each of them is one half of the sphere) are given by the inequalities $|z|^2 \geq |w|^2$ and $|z|^2 \leq |w|^2$; their common boundary has the equation $|z| = |w| = \sqrt{\frac{1}{2}}$. It is easy to see that the circles defined as

$$|w| = \sqrt{\frac{1}{2}},$$
z — fixed with absolute value $\sqrt{\frac{1}{2}}$

and

$$|z|=\sqrt{\frac{1}{2}},w$$
 — fixed with absolute value $\sqrt{\frac{1}{2}}$

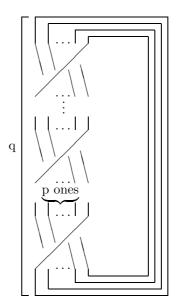


Figure 3.4. Torus p, q-knot diagram

are the longitude and the meridian of the boundary torus.

Thus, the (p,q) torus knot in one full torus is just the (q,p) torus knot in the other one. Thus, mapping one full torus to the other one, we obtain an isotopy of (p,q) and (q,p) torus knots.

Exercise 3.6. Express this homotopy of full tori as a continuous process in S^3 .

Torus knots of type (p,q) can be represented by the following series of planar diagrams, see Fig. 3.4.

Remark 3.2. Figure 3.4 demonstrates a way of coding a knot (link) as a (p-strand) braid closure. We shall speak about braids later in the book.

Analogously to the case of torus knots, one can define *torus links* which are links embedded into the torus standardly embedded in \mathbb{R}^3 .

We know the construction of torus knots. So, in order to draw a torus link one should take a torus knot $K \subset T$ (one can assume that it is represented by a straight-linear curve defined by the equation $q\phi - p\psi = 0 \pmod{2\pi}$) and add to the torus T some closed non-intersecting simple curves; each curve should be non-intersecting and should not intersect K. Thus, these curves should be embedded in $T \setminus K$, i.e., in the open cylinder.

Each closed curve on the cylinder is either contractible or passes the longitude of the cylinder once, see Fig. 3.5.

So, each curve in $T \setminus K$ is either contractible inside $T \setminus K$, or "parallel" to K inside T, i.e. isotopic to the curve given by the equation $q\phi - p\psi = \varepsilon \pmod{2\pi}$ inside $T \setminus K$.

Thus, the following theorem holds.

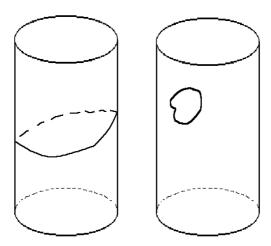


Figure 3.5. Possible simple curves on the cylinder

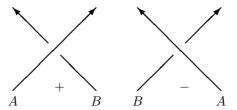


Figure 3.6. Two types of crossings for an oriented link

Theorem 3.2. Each torus knot is isotopic to the disconnected sum of a trivial link and a link that is represented by a set of parallel torus knots of the same type (p,q).

3.2 The linking coefficient

From now on, we shall construct some invariants of links. As we know from the first chapter, a link invariant is a function defined on links that is invariant under isotopies. One can consider separately invariants of knots and links (oriented or non-oriented).

We shall represent links by using their planar diagrams. According to the Reidemeister theorem, in order to prove the invariance of some function on links, it is sufficient to check this invariance under the three Reidemeister moves.

First, let us consider the simplest integer–valued invariant of two–component links.

Let L be a link consisting of two oriented components A and B and let L' be the planar diagram of L. Consider those crossings of the diagram L' where the component A goes over the component B. There are two possible types of such crossings with respect to the orientation, see Fig. 3.6.

For each positive crossing we assign the number (+1), for each negative crossing we assign the number (-1). Let us summarise these numbers along all crossings where the component A goes over the component B. Thus we obtain some integer number.

Exercise 3.7. Show that this number is invariant under Reidemeister moves.

Thus, we have an oriented link invariant.

Definition 3.1. The obtained invariant is called *linking coefficient*.

Remark 3.3. This invariant was first invented by Gauss [Gau]. He calculated it by means of his famous formula. This formula is named the Gauss electromagnetic formula in honour of his. The linking coefficient can be generalised for the case of p- and q-dimensional manifolds in \mathbf{R}^{p+q+1} .

The formula for the parametrised curves $\gamma_1(t)$ and $\gamma_2(t)$ with radius–vectors $r_1(t), r_2(t)$ is given by the following formula

$$lk(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_{\gamma_1} \int_{\gamma_2} \frac{(r_1 - r_2, dr_1, dr_2)}{|r_1 - r_2|}^3.$$

The proof of this fact is left for the reader as an exercise.

Hint 3.1. Prove that this function is the degree of a map from the torus generated by two link components to a sphere and, thus, it is invariant; then prove that it is proportional to the linking coefficient and find the coefficient of proportionality.

The linking coefficient allows us to distinguish some two–component links.

Example 3.1. Let us consider the trivial two-component link and enumerate its components in an arbitrary way. Obviously, their linking coefficient is zero. For the Hopf link, the linking coefficient equals ± 1 depending on the orientation of the components. Hence, the Hopf link is not trivial.

Exercise 3.8. Show that the linking coefficient of two "parallel" torus knots of type (1,n) equals n.

Example 3.2. For any two components of the Borromean rings, the linking coefficient equals zero; each component of this link is a trivial knot. However, the Borromean rings are not isotopic to the trivial three-component link. This will be shown later in the text.

For one-component link diagrams (knot diagrams), one can define the *self-linking coefficient*. To do this, one should take an oriented knot diagram K and draw a parallel copy K' of it on the plane. After this, one takes the linking coefficient of K and K'. It is easy to check that this is invariant under Ω_2 and Ω_2 , but not Ω_1 : adding a loop changes the value by ± 1 .

There exists another approach to the link coefficients, namely that involving Seifert surfaces.

Definition 3.2. Let F be a Seifert surface of an oriented knot J. Assume that an oriented link K intersects F transversely in finitely many points. With each intersection point, we associate a number $\varepsilon_i = \pm 1$ according to the following rule.

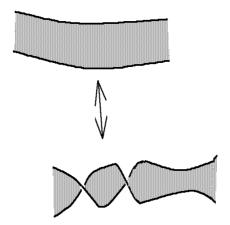


Figure 3.7. Simple twisting

Let us define the orientation of F, assuming the reference point $\{e_1, e_2\}$ positive, where e_1 is the speed vector of J and e_2 is the interior normal vector (it is perpendicular to e_1 and directed inside F). Let e_3 be the speed vector of the curve K at a point $a \in K \cap F$, and $\{e'_1, e'_2\}$ be a positive frame of reference on the surface F at a. Then, $\varepsilon_i = +1$ if the orientation $\{e'_1, e'_2, e_3\}$ coincides with the orientation of the ambient space \mathbf{R}^3 ; otherwise, let us set $\varepsilon_i = -1$.

The sum of all signs ε_i is called the linking coefficient of J and K.

Exercise 3.9. Prove that the linking coefficient (in the latter sense) is well defined (does not depend on the choice of the Seifert surface) and coincides with the initial definition.

Hint 3.2. Use the planar projections and investigate the behaviour of two knot projections in neighbourhoods of crossings.

3.3 The Arf invariant

We have constructed a simple invariant of two-component links, the link coefficient. For unoriented links this construction allows us to define an invariant with coefficients from \mathbb{Z}_2 .

It turns out that there exists a knot invariant valued in \mathbb{Z}_2 that is closely connected with the link coefficient, namely, the so-called Arf invariant. There are many ways to define this invariant; here we follow [Kau2] and [Ada].

The Arf invariant comes from Seifert surfaces. Namely, let K be a knot and l be a band that is a part of the Seifert surface of the knot K, see the upper part of Fig. 3.7.

Let us transform the knot by twisting this band by two full turns, see Fig. 3.8. We obtain a knot K'.

Definition 3.3. Let us say that K and K' are Arf equivalent.

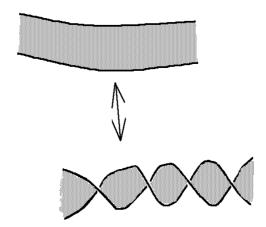


Figure 3.8. Double twisting

Definition 3.4. The *Arf invariant* is the complete invariant of the Arf equivalent classes.

Now, let us denote the Arf invariant by a and decree $A(\bigcirc) = 0$ and $A(\bigcirc) = A(\bigcirc) = 1$ where \bigcirc denotes the unknot.

It turns out that the Arf invariant defined above has only two values, one and zero!

Theorem 3.3. Each knot is either Arf equivalent to the unknot or to the trefoil (both trefoils are Arf equivalent).

The main idea for proving this theorem (for a rigorous proof see, e.g., in [Ada]) is the following. First, let us mention that each Seifert surface can be thought of as a disc with several bands attached to its boundary. Each band can be twisted many times, and bands can be knotted. This observation is left to the reader as an exercise.

The number of half-turn twists for each band can be taken to be zero or one according to the Arf equivalence. Besides, the Arf equivalence allows us to change the disposition of bands in 3–space, for instance, to erase knottedness. Namely, a simple observation shows that passing one band through the other is also an Arf equivalence.

Thus we obtain a Seifert surface of a very simple type: it looks like a disc together with some bands attached to its boundary according to some "chord diagram" law; some of these bands are half-turn twisted; the others are not twisted. After this, one can perform some reducing operations, which will transform our knot into the connected sum of some trefoils. Finally, the only thing to do is the following exercise.

Exercise 3.10. Show that the connected sum of the trefoil with another trefoil is Arf equivalent to the unknot. It does not matter which trefoils you take (right or left).

In the sequel, all calculations concerning the Arf invariant will be performed modulo \mathbf{Z}_2 .

It turns out that the Arf invariant is very easy to calculate.

Let \bigotimes and \bigotimes be two knot diagrams that differ in a small neighbourhood. Then, one of the diagrams \bigotimes is a knot diagram, and the other one is a link diagram. Suppose that \bigotimes is a link diagram, consisting of the two link components l_1 and l_2 . Then the following theorem holds.

Theorem 3.4. Under conditions described above,

$$a(\bigotimes) - a(\bigotimes) \equiv lk(l_1, l_2) \pmod{\mathbf{Z}_2}.$$

This relation in fact allows us to calculate the value of the Arf invariant for different knots. Later, we shall show how to transform any knot diagram to an unknot diagram only by switching some crossing types. This will work in many other situations while calculating some polynomials.

Furthermore, it looks similar to the so–called *skein relations* that will be defined later. The skein relations lead to much stronger knot invariants.

3.4 The colouring invariant

Let us consider now a simple invariant for unoriented links starting from the link diagram. This invariant was well known a long ago. It turns out that it is connected with much stronger invariants of links.

Consider a non-oriented link.

Definition 3.5. By an *arc* of a planar link diagram we mean a connected component of the planar diagram.

Thus, each arc always goes "over"; it starts and stops at undercrossings. For link diagrams in general position, each vertex is incident to three arcs. Some of them may (globally) coincide.

Now, let us associate colours with arcs of a given link. We shall use the three colours: red, blue, white.

Definition 3.6. A colouring of a link diagram is said to be *proper* if for each crossing of the diagram, the three arcs incident to it have either all three different colours or one and the same colour.

Theorem 3.5. The number of proper colourings is an invariant of link isotopy types.

Proof. Let us prove the invariance in the following way. Consider a Reidemeister move and two diagrams L, L' obtained one from the other by using this move. Then we present a one–to–one correspondence of proper colourings for L and L'.

In the case of a Ω_1 -move the situation is clear: the two edges of L' corresponding to one "broken" edge of L should have one colour because we have the situation where two of three edges meeting at a crossing are the same (and hence, there is no possibility to use three colours).

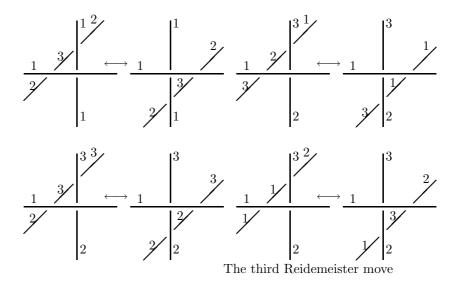
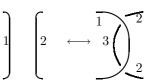


Figure 3.9. Invariance of proper colouring number

Thus, the desired one—to—one correspondence is evident. Analogously, there are "one colour" cases of the second and third Reidemeister moves. The invariance under second Reidemeister move is shown here:



In Fig. 3.9, we give the corresponding colourings for edges taking part in the third Reidemeister moves. For the corresponding colouring all edges outside the area should be coloured identically. Here we show some examples of such colourings and their one—to—one correspondences. The other possibilities can be obtained from these by some permutation of colours (here colours are marked by numbers 1, 2, 3).

Thus, each proper colouring of the initial diagram uniquely corresponds to a colouring of the diagram obtained from the initial one by applying a Reidemeister move. Thus, the number of proper colourings is invariant under Reidemeister moves.

Let us now call the number of proper colourings the *colouring invariant*.

Notation: for a link L we have CI(L).

Obviously, for the unknot, we have $CI(\bigcirc)=3$; for the k-component unlink, we have $CI=3^k$.

Exercise 3.11. Calculate the colouring invariant for the unknot and that for the (right) trefoil knot and show that the trefoil knot is not trivial.

Exercise 3.12. Calculate the colouring invariant for the Whitehead link and show that it is not trivial.

Exercise 3.13. Calculate the values of the colouring invariant for the Borromean rings. Show that they are not trivial.

However, if we observe this invariant on the figure eight knot, we see that each colouring is monochrome. Thus, our invariant does not distinguish between the figure eight knot and the unknot. This encourages us to seek stronger knot and link invariants.

Chapter 4

Fundamental group. The knot group

4.1 Digression. Examples of unknotting

Let us now discuss a sympathetic example (or problem) concerning unknot diagrams and ways of unknotting them.

First, consider an arbitrary diagram of the trivial knot. Let us try to unknot it by using Reidemeister moves. In the "good" case this can be done only by decreasing the number of vertices (we mean the "decreasing" version of the first two Reidemeister moves and the third move).

However, this is not always so. More precisely, there exists an unknot diagram with a non-empty set of vertices such that we can apply neither decreasing versions of the first or second Reidemeister move nor the third Reidemeister move to this diagram. Thus, in order to transform this diagram to the unknot, one should first increase the number of vertices of it. We are going to construct such diagrams.

Example 4.1. Consider the knot diagram shown in Fig. 4.1. Obviously, this is a diagram of the unknot because it can be obtained from the stretched unknotted circle by using a sequence of Ω_2 's.

Example 4.2. Now let us consider the knot diagram shown in Fig. 4.2. This can be obtained from that shown in Fig. 4.1 by throwing the arc AB over infinity (we think of the plane as compactified by the point at infinity). Obviously, this throwing does not change the knot isotopy class. Hence, we see the unknot in Fig. 4.2.

Example 4.3. Let us consider now the knot diagram shown in Fig. 4.3. This can be obtained from that shown in Fig. 4.2 by applying the second Reidemeister move to CD and throwing it over infinity. Hence, the knot diagram shown in Fig. 4.3 also represents the unknot.

Theorem 4.1. Each Reidemeister move that can be applied to the unknot diagram shown in Fig. 4.3 increases the number of vertices of the diagram.

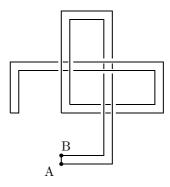


Figure 4.1. First unknot diagram

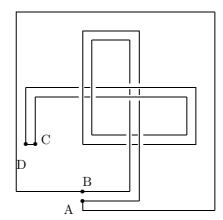


Figure 4.2. Second unknot diagram

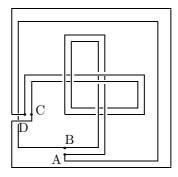


Figure 4.3. Third unknot diagram

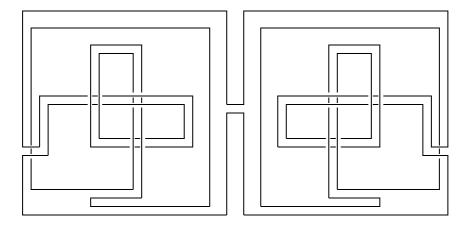


Figure 4.4. Fourth unknot diagram

Proof. Consider the knot diagram shown in Fig. 4.3 as a four-valent graph with overcrossings and undercrossings. It is sufficient to check that in order to perform the first or the second decreasing Reidemeister move, the shadow must contain loops or bigons. Additionally, in order to perform the third Reidemeister move, the shadow must contain a triangle such that no edge of it has overcrossings. It is easy to see that the knot diagram shown in Fig. 4.3 has neither such loops, nor bigons, nor "good" triangles. Thus, each of the Reidemeister moves applicable to this diagram will increase the number of vertices. This completes the proof.

However, one can consider not only planar knot (link) diagrams, but also *spherical* diagrams. Namely, one can think of the sphere as the plane compactified by the point at infinity. Without loss of generality, one can assume that the shadow of the link does not contain this point. In this case, there appears one more "elementary isotopy," when some edge of the shadow passes through the infinity. This operation is called the *infinity change*.

It is intuitively clear that the infinity change is indeed an isotopy. Actually, it can be represented as a sequence of Reidemeister moves.

Exercise 4.1. Prove this fact directly.

Thus, the knot shown in Fig. 4.3 can be unknotted only by using the second decreasing Reidemeister move, after the preliminary infinity change (throwing an arc over infinity) in the very beginning.

Actually, the knot shown in Fig. 4.4 cannot be unknotted only by non-decreasing Reidemeister moves even if we admit the infinity change (the infinite cell is no longer a bigon).

Now, if we consider the knot diagram from Fig 4.3 as a spherical diagram, we shall see the bigon, containing the infinite point. Thus, after the infinity change, one gets this bigon inside a compact domain. Then one can easily untangle the diagram by using only decreasing moves.

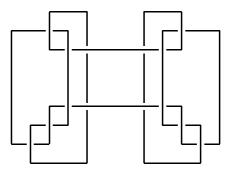


Figure 4.5. An unknot diagram that cannot be decreased in one turn

In Fig. 4.4, we illustrate an example of knot diagram having no loops, no bigons and no "good" triangles with one edge forming two overcrossings. These properties remain true even if considering this as a spherical diagram. Thus, in order to untangle the knot shown in Fig. 4.4, one should first perform some increasing moves.

Another example of such an unknot diagram that cannot be decreased in one turn is shown in Fig. 4.5. This example was invented by the student I.M. Nikonov who attended the author's lecture course.

4.2 Fundamental group. Basic definitions and examples

Let us now start the main part of the present chapter. We shall describe the notion of fundamental group for arbitrary topological spaces and show how to calculate it for link complements. We are going to introduce some presentations of this group.

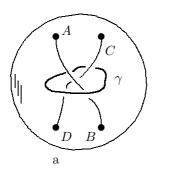
The topological theory of the fundamental group can be read in, e.g., [F, FFG, Va, CF]. The theory of three-manifolds can be read in [Matv4].

Consider a topological space X and a point $x_0 \in X$. Fix a point a on the circle S^1 . Consider the set of continuous mappings $f: S^1 \to X$ such that $f(a) = x_0$. The set of homotopy classes of such mappings admits a group structure. Indeed, the multiplication of two such mappings can be represented by concatenating their paths. The reverse element is obtained by passing over the initial path in the reverse order. Obviously, these operations are well defined up to homotopy.

Definition 4.1. The obtained group is called the *fundamental group* of the space X; it is denoted by $\pi_1(X, x_0)$.

Exercise 4.2. Show that for the case of connected X the group $\pi_1(X, x_0)$ does not depend on the choice of x_0 , i.e., $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.

Remark 4.1. There is no canonical way to define the isomorphism for fundamental groups with different initial points.



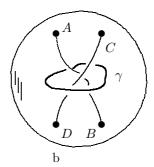


Figure 4.6. Changing the crossing

The fundamental group is a topological space homotopy invariant. Now let K be a link in \mathbb{R}^3 .

Let $M_K = \mathbf{R}^3 \setminus K$ be the complement to K. It is obvious that while performing a smooth isotopy of K in \mathbf{R}^3 the complement always stays isotopic to itself. Hence, the fundamental group of the complement is an invariant of link isotopy classes.

In [GL] it is shown that the complement to the knot (more precisely, to its small tubular neighbourhood in \mathbb{R}^3) is a complete invariant of the knot up to amphicheirality.

However the analogous statement for links is not true. Before constructing a counterexample, let us prove the following lemma.

Lemma 4.1. Let $D^3 \subset \mathbf{R}^3$ be a ball and $T \subset D^3$ be the full torus γ , see Fig. 4.6. There exists a homeomorphism of $\mathbf{R}^3 \setminus T$ onto itself, mapping the curves AB and CD (as they are shown in Fig. 4.6.a) to AB and CD (Fig. 4.6.b) that is constant inside the ball D^3 .

Proof. In order to have a more intuitive outlook, let us imagine that the interior diameter of T is very big (so that the "interior" boundary of it represents a high cylinder) in comparison with the exterior one. Thus, we have a deep hole surrounded by the boundary of the full torus, see fig. 4.7.

Let us consider the ball D^3 and cut a circle from the plane, as shown in Fig. 4.7.a. Let us rotate the part of this cut (that is a circle with two marked points) in the direction indicated by arrows. This operation is possible since the full torus T is deleted from D^3 . Performing the 180–degree turn, we obtain the configuration shown in Fig. 4.7.b. Then, let us make one more turnover. Thus we obtain the embedding represented in Fig. 4.7.c. Thus, each point of the cut returns to the initial position. So, both copies of the cut can be glued together in such a way that the total space remains the same. In this way, we obtain a homeomorphism of the manifold $D^3 \ T$ onto itself. This homeomorphism can be extended to a homeomorphism of $\mathbf{R}^3 \ T$ onto itself, identical inside D^3 . The latter homeomorphism realises the crossing change.

Exercise 4.3. Show that links L_1 and L_2 (Fig. 4.8.a and 4.8.b) are not isotopic, but their complements are homeomorphic.

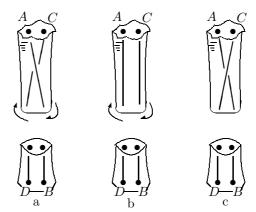


Figure 4.7. Rotating upper part of the cylinder

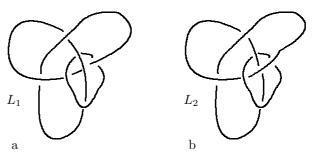


Figure 4.8. Non-isotopic links with isomorphic complements

This gives us an example of non–isotopic links with isomorphic fundamental groups.

Difficult exercise 4.1. Find two non-isotopic (and not mirror) knots with isotopic fundamental groups.

Definition 4.2. The link (knot) complement fundamental group is also called the *link (knot) group*.

Exercise 4.4. Show that the fundamental group of the circle is isomorphic to the fundamental group of the complement to the unknot. Show that they are both isomorphic to **Z**.

Exercise 4.5. Show that the fundamental group of the complement to the trivial n-component link is isomorphic to the free group in n generators.

The link complement fundamental group is a very strong invariant. For instance, it recognises trivial links among links with the same number of components. This results follows from Dehn's theorem.

Theorem 4.2 (Dehn). An m-component link L is trivial if and only if $\pi_1(\mathbf{R}^3 \setminus L)$ is isomorphic to the free group in m generators.

Thus, Dehn's theorem reduces the trivial link recognition problem to the free group recognition problem (for some class of groups). In the general case, the free group recognition problem is undecidable. For more details see [Bir] and [BZ].

Dehn's theorem follows from the following lemma.

Lemma 4.2. Let M be a 3-manifold with boundary and let γ be a closed curve on its boundary ∂M . Then if there exists an immersed 2-disc $D \to M$, such that $\partial D = \gamma$ then there exists an **embedded** disc $D' \subset M$ with the same boundary $\partial D' = \gamma$.

This lemma was first proved by Dehn [Dehn2], but this proof contained lacunas. The rigorous proof was first found by Papakyriakopoulos [Pap]. This proof used the beautiful techniques of towers of 2–folded coverings.

Now the Dehn theorem (for the case of knots) is proved as follows. Having a knot $K \subset \mathbf{R}^3$ with its tubular neighbourhood N(K), let us consider $\pi_1(\mathbf{R}^3 \backslash N(K), A)$ where $A \in T(K) = \partial N(K)$. Obviously, each closed loop can be isotoped to a loop lying on T(K) that can be represented via the longitude and meridian of T(K), which are non-intersecting closed curves. Obviously, the meridian μ , i.e., the simple curve on T(K) that lies in a small neighbourhood of some point on K and has linking coefficient with K equal to one, cannot be contracted to zero. Let λ be the longitude, i.e. a simple curve in T "parallel to K" and having linking coefficient zero with K. The curves π and λ generate the fundamental group of the torus T.

Suppose the group of the knot K is isomorphic to \mathbf{Z} . Obviously, this group contains $\{\mu\} = \mathbf{Z}$. Besides, no power of λ can be equal to a non-trivial exponent of μ (because of linking coefficients). Thus, the curve λ is isotopic to zero in $\pi_1(\mathbf{R}^3 \backslash N(K))$. Thence, there is a singular disc bounded by λ . By Dehn's lemma, there is a disc embedded in $\mathbf{R}^3 \backslash N(K)$ bounded by λ . Contracting N(K) to K, we obtain a disc bounded by K. Thus, K is the unknot. This completes the proof of Dehn's theorem.

The statement of Dehn's theorem shows that the fundamental group is rather a strong invariant. However, it does not allow us to distinguish mirror knots and some other knots. The first example of distinguishing two different mirror knots, namely, the two trefoils, was made by Dehn in [Dehn]. There he considered the group together with the element representing an oriented meridian. This was sufficient to distinguish these two trefoils. Later, we will return to this structure (while speaking of the peripheral structure of the knot complement).

According to modern terminology, we can say that complements to non-trivial links are so–called *sufficiently large* manifolds,

Definition 4.3. A manifold M is called *sufficiently large* if one can embed a handle body (not the sphere) in M in such a way that the image map for the fundamental group has no kernel.

These manifolds are classified by S.V. Matveev; however, the algorithm is quite formal and cannot be performed practically (e.g., by means of a computer program).

In [Mat], he constructed the full invariant of knots, the knot quandle (distributive grouppoid). We shall consider this invariant later.

Exercise 4.6. Show that $\pi_1(A_1 \vee A_2)$ (of the union of spaces A_1 and A_2 with one identified point) is isomorphic to the free product of $\pi_1(A_1)$ and $\pi_1(A_2)$ in the case when both A_1 and A_2 are pathwise connected.

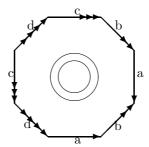


Figure 4.9. Glueing a handle body

Let us demonstrate two ways of calculating a representation for the fundamental group of a knot complement. The first of them is more common; it can be used in many other situations.

Let X be a topological space that admits a decomposition $X=X_1\cup X_2$, where each of the sets $X_1, X_2, X_0=X_1\cap X_2$ is open, pathwise–connected and non-empty. Choose a point $A\in X_0$. Suppose fundamental groups $\pi_1(X_1,A)$ and $\pi_1(X_2,A)$ have presentations $\langle a_1,\ldots | f_1=e,\ldots \rangle$ and $\langle b_1,\ldots | g_1=e,\ldots \rangle$, respectively. Suppose that the generators $c_1,c_2\ldots$ of $\pi_1(X_0,A)$ (which are elements of both groups $\pi_1(X_1,A)$ and $\pi_1(X_2,A)$) are represented as $c_i=c_i(a_1,\ldots)$ and as $c_i=c_i(b_1,\ldots)$ in the terms of $\pi_1(X_1,A)$ and $\pi_1(X_2,A)$, respectively.

Then the following theorem holds.

Theorem 4.3. (The van Kampen theorem) The group $\pi_1(X, A)$ admits a presentation

$$\langle a_i, b_i, | f_i = e, g_i = e, c_i(a) = c_i(b) \rangle.$$

In the case of CW–complices, the proof of the theorem is evident. For more details in the general case, see, e.g. [CF].

Corollary 4.1. If both X_1 and X_2 described above are simply connected then X is simply connected as well.

As an example, let us show how to calculate fundamental groups of 2-manifolds.

Theorem 4.4. The fundamental group of the connected oriented 2-surface of genus g (g > 0) without boundary has a presentation

$$\langle a_1, b_1, \dots, a_g, b_g | a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = e \rangle.$$

Proof. Consider this handle surface as a 4g-gon with some pairs of edges glued together, see Fig. 4.9.

Let us divide this manifold into two parts: one of them is located inside the big circle in Fig. 4.9, the other one is outside the small circle (it contains all edges and all glueings are performed for this part).

The first part is simply connected; the second one is contractible to the union of 2g circles and hence its fundamental group is isomorphic to the free group with generators¹ $a_1, b_1, \ldots, a_g, b_g$.

¹Here each letter means an oriented edge of the octagon; this edge is closed since all vertices are contracted to one point.

The intersection of the two areas described above is isotopic to the circle; hence its fundamental group is isomorphic to \mathbf{Z} . Thus, the only relation we have to add deals with the generator of this \mathbf{Z} is going to be e.

Applying the van Kampen theorem we get that

$$\pi_1(S_g) = \langle a_1, b_1, \dots, a_g, b_g | a_1b_1a_1^{-1}b_1^{-1} \dots a_gb_ga_g^{-1}b_g^{-1} = e \rangle.$$

For each link L, there exists a handle surface S_g standardly embedded in \mathbf{R}^3 and a link $L' \subset S_g \subset \mathbf{R}^3$ isotopic to L in \mathbf{R}^3 . Thus, one can calculate the fundamental group of the complement to L by using the van Kampen theorem: we divide the complement to L' into two parts lying on different sides of S_g .

4.3 Calculating knot groups

Let us demonstrate this techniques for the case of torus knots. Let K be a (p,q)-torus knot, embedded to the standard torus in \mathbb{R}^3 .

Now, consider $\mathbb{R}^3 \backslash K$ as

(Interior full torus without K) \cup (Exterior full torus without K).

Since deleting some set on the boundary does not change the fundamental group (of the full torus), we conclude that both fundamental groups are isomorphic to \mathbf{Z} (one of them has the generator a, the other one has the generator b; they correspond to the longitude and meridian of the torus). The intersection of these parts is homeomorphic to the cylinder; the fundamental group of this cylinder is isomorphic to \mathbf{Z} . The only generator of this group can be expressed as a^p on one hand, and as b^q on the other hand. This implies the following theorem.

Theorem 4.5. The fundamental group of the complement to the (p,q)-torus knot has a presentation with two generators a and b and one relation $a^p = b^q$.

It is obvious that the fundamental group does not distinguish a knot (not necessarily torus) and its mirror image.

As they should be, fundamental groups of isotopic knots are isomorphic. Thus, T(p,q) has the same group as T(q,p), and the group of the knot T(1,n) is isomorphic to **Z**. Torus knots of types (p,q) and (p,-q) are mirror images of each other, thus their groups coincide.

In all the other cases the fundamental group distinguishes torus knots.

Now, we present another way of calculating the fundamental group for arbitrary links. Consider a link L given by some planar diagram \bar{L} . Consider a point x "hanging" over this plane. Let us classify isotopy classes of loops outgoing from this points. It is easy to see (the proof is left to the reader) that one can choose generators in the following way. All generators are classes of loops outgoing from x and hooking the arcs of \bar{L} . Let us decree that the loop corresponding to an oriented edge is a loop turning according to the right–hand screw rule, see Fig. 4.10.

Now, let us find the system of relations for this group.

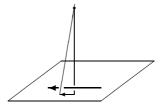


Figure 4.10. Loops corresponding to edges

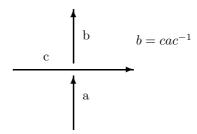


Figure 4.11. Relation for a crossing

It is easy to see the geometrical connection between loops hooking adjacent edges (i.e., edges separated by an overcrossing edge). Actually, we have $b = cac^{-1}$, where c separates a and b, see Fig. 4.11.

Let us show that all relations in the fundamental group of the complement arise from these relations.

Actually, let us consider the projection of a loop on the plane of \bar{L} and some isotopy of this loop. While transforming the loop, its written form in terms of generators changes only when the projection passes through crossings of the link. Such an isotopy is shown in Fig. 4.12. During the isotopy process, the arc connecting P and Q passes under the crossing.

Obviously, the loop shown on the left hand (Fig. 4.12) generates the element $cb^{-1}c^{-1}$, that on the right hand is just a.

Thus, our presentation of the fundamental group of the link complement is constructed as follows: arcs correspond to the generators and the generating relations come from crossings: we take $cac^{-1} = b^{-1}$ for adjacent edges a and b, separated by c, when the edge b lies on the left hand from c with respect to the orientation of c.

Definition 4.4. This presentation of the fundamental group for the knot complement is called *the Wirtinger presentation*.

Exercise 4.7. Find a purely algebraic proof that the Wirtinger presentation for two diagrams of isotopic links generates the same group.

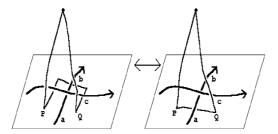


Figure 4.12. Isotopy generating relation

Exercise 4.8. Find a Wirtinger presentation for the trefoil knot and prove that the two groups presented as $\langle a, b | aba = bab \rangle$ and $\langle c, d | c^3 = d^2 \rangle$ are isomorphic.

Remark 4.2. The group with presentation $\langle a, b | aba = bab \rangle$ will appear again in knot theory. It is isomorphic to the three–strand braid group.

It turns out that not only mirror (or equivalent) knots may have isomorphic groups.

Exercise 4.9. Show that for the two trefoils $T_1 = \emptyset$ and $T_2 = \emptyset$, the fundamental groups of complements for $T_1 \# T_1$ and $T_1 \# T_2$ are isomorphic.

Exercise 4.10. Calculate a Wirtinger presentation for the figure eight knot (for the simplest planar diagrams).

Exercise 4.11. Calculate a Wirtinger presentation for the Borromean rings.

Now let us prove the following theorem.

Theorem 4.6. For each knot K, the number CI(K) + 3 is equal to the number of homomorphisms of $\pi_1(\mathbf{R}^3 \setminus K)$ to the symmetric group S_3 .

Proof. Consider a knot K and an arbitrary planar diagram of it. In order to construct a homeomorphism from $\pi_1(\mathbf{R}^3 \setminus K)$ to S_3 , we should first find images of all elements corresponding to arcs of K.

Suppose that there exists at least one such element mapped to an even permutation. Consider the arc s, corresponding to this element. Then any arc s' having a common crossing A with s and separated from s by some overcrossing arc at A, should be mapped to some even permutation. Since K is a knot, we can pass from s to any other arc by means of "passing through undercrossings." Thus, all elements corresponding to arcs of K are mapped to even permutations. Because the group A_3 is commutative, we conclude that all elements corresponding to arcs are mapped to the same element of A_3 (even symmetric group). There are precisely three such mappings.

If at least one element—arc is mapped to an odd permutation then so are all arcs. There are three odd permutations: (12), (23), (31). If we conjugate one of them by means of another one, we get precisely the third one. This operation is well coordinated with the proper colouring rule.

So, all homomorphisms of the group $\pi_1(\mathbf{R}^3\backslash K)$ to S_3 , except three "even" ones, are in one–to–one correspondence with proper colourings of the selected planar diagram of K.

Thus we obtain the following statement.

Corollary 4.2. The colouring invariant does not distinguish mirror knots.

This statement does not obviously follow from the definition of the colouring invariant.

Chapter 5

The knot quandle and the Conway algebra

5.1 Introduction

The aim of the present chapter is to describe the universal knot invariant discovered independently by S.V. Matveev [Matv3] and D. Joyce [Joy]. In Matveev's work and in other works by Russian authors, this invariant is usually called the *distributive grouppoid*; in Western literature it is usually called *quandle*.¹ This invariant is a complete one;² however, it is barely recognisable. In the present chapter, we shall construct some series of "weaker" invariants coming from the knot quandle; the series of invariants to be constructed are easier to calculate and to compare. We shall tell about so-called Conway algebras, describing them according to [PT]. Both these directions, the knot quandle and the Conway algebras, allow us to construct various knot invariants.

First, let us return to the simplest knot invariant, i.e., to the colouring invariant. Why is it possible to construct an invariant function by so simple means?

Even the fact that this invariant is connected with maps from the knot group to the symmetric group S_3 does not tell us very much: an analogous construction with a greater number of colours does not work.

Let us now try to use a greater palette of colours. Let Γ be an arbitrary finite set (here the finiteness will be used in order to be able to *count the number* of colourings); all elements of Γ are to be called *colours*.

Suppose the set Γ is equipped with a binary operation $\alpha: \Gamma \times \Gamma \to \Gamma$; this operation will be denoted like this: $a \circ b \equiv \alpha(a,b)$.

Definition 5.1. By a proper colouring of a diagram D of an oriented link K we mean a way of associating some colour with each arc of D in such a way that for each overcrossing arc (that has colour b), undercrossing arc lying on the left hand (colour a) and undercrossing lying on the right hand (colour c), the relation $a \circ b = c$ holds, see Fig. 5.1.

¹There are some other names for this and similar objects, e.g., crystal and rack.

 $^{^{2}}$ In a slightly weaker sense.

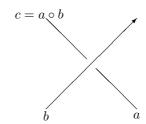


Figure 5.1. The rule of colourings

Which should be the conditions for \circ , such that the number of proper colourings is invariant under Reidemeister moves?

It is easy to show that the invariance of such a colouring function under Ω_1 implies the idempotence relation $a \circ a = a$ for all the elements $a \in \Gamma$ that can be associated to arcs and play the role of colour. However, in order to simplify the situation we shall not restrict ourselves only to this case, and require that $\forall a \in \Gamma : a \circ a = a$.

Analogously, the invariance under Ω_2 requires the left invertibility of the operation \circ : for any a and b from Γ , the equation $x \circ a = b$ should have only the solution $x \in \Gamma$ (in the case of the three–colour palette, the inverse operation for \circ and the operation \circ itself coincide).

Finally, the invariance under Ω_3 implies right self distributivity of the operation \circ , which means that $\forall a, b, c \in \Gamma$ the equation $(a \circ b) \circ c = (a \circ c) \circ (b \circ c)$ holds. In the sequel, each set with an operation \circ satisfying the three properties described above, is called a quandle.

Each quandle generates a rule for proper colouring of link diagrams described above.

Thus, we conclude the following:

Proposition 5.1. The number of proper colourings by elements of any quandle is a link invariant.

In any quandle, the reverse operation for \circ is denoted by /. More precisely, the element b/a is defined to be the unique solution of the equation $x \circ a = b$.

Exercise 5.1. Show that each quandle Γ (with operation \circ) is a quandle with respect to the operation /.

Furthermore, prove the following identities for Γ : $(a \circ b)/c = (a/c) \circ (b/c)$, $(a/b) \circ c = (a \circ c)/(b \circ c)$.

There is a common way for constructing quandles by using their presentations by generators and relations.

Let A be an alphabet consisting of *letters*. A word in the alphabet A is an arbitrary finite sequence of elements of A and symbols $(,), \circ, /$. Now, let us define inductively the set D(A) of admissible words according to the following rules:

1. For each $a \in A$, the word consisting of only the letter a is admissible.

- 2. If two words W_1, W_2 are admissible then the words $(W_1) \circ (W_2)$ and $(W_1)/(W_2)$ are admissible as well.
- 3. There are no other admissible words except for those obtained inductively by rules 1 and 2.

Sometimes we shall omit brackets when the situation is clear from the context. Thus, e.g. for letters a_1, a_2 we write the word $a_1 \circ a_2$ instead of $(a_1) \circ (a_2)$.

Let R be a set of *relations*, i.e., identities of type $r_{\alpha} = s_{\alpha}$, where $r_{\alpha}, s_{\alpha} \in D(A)$ and α runs over some set X of indices. Let us introduce the equivalence relation for D(A), supposing $W_1 \equiv W_2$ if and only if there exists a finite chain of transformations starting from W_1 and finishing at W_2 according to the rules 1-5 described below:

- 1. $x \circ x \iff x$;
- 2. $(x \circ y)/y \iff x;$
- 3. $(x/y) \circ y \iff x$;
- 4. $(x \circ y) \circ z \iff (x \circ z) \circ (y \circ z)$;
- 5. $r_i \iff s_i$.

The set of equivalence classes is denoted by $\Gamma\langle A|R\rangle$. It is easy to check that it is a quandle with respect to the operation \circ .

Let us give one more example of a finite quandle (denoted by G_4). Denote by G_4 the set of four different elements a_1, a_2, a_3, a_4 .

Remark 5.1. This example comes from the universal construction of quandles from groups. Later, this construction will be discussed in detail.

Let us define the operation \circ according to the rule:

0	a_1	a_2	a_3	a_4
a_1	a_1	a_3	a_4	a_2
a_2	a_4	a_2	a_1	a_3
a_3	a_2	a_4	a_3	a_1
a_4	a_3	a_1	a_2	a_4

Here the result $a_i \circ a_j$ occupies the position number j in the i-th line.

Exercise 5.2. Check the quantile axioms for the operation described above.

It is easy to see that the figure eight knot admits non-trivial colourings by elements of G_4 . The same can be said about the trefoil knot.

Remark 5.2. Note that the quandle G_4 is not especially connected with any knot; it is only used for construction of knot invariants.

From proposition 5.1 we see that quandles are useful for constructing knot invariants. It turns out that one can associate with each knot (link) a quandle, that is the universal (almost³ complete) knot invariant. For the sake of simplicity, we shall describe this invariant for the case of a knot. The *universal knot quandle* can be described in two ways: geometrically and algebraically.

³Later, we shall comment on the incompleteness of the quandle in the proper sense.

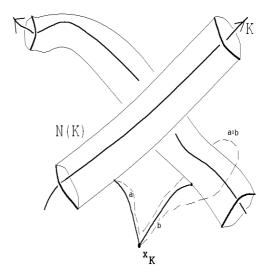


Figure 5.2. Intuitive description of the quandle operation.

5.2 Geometric and algebraic definitions of the knot quandle

5.2.1 Geometric description of the quandle

Let K be an oriented knot in \mathbb{R}^3 , and let N(K) be its small tubular neighbourhood. Let $E(K) = \overline{(\mathbb{R}^3 \backslash N(K))}$ be the complement to this neighbourhood. Fix a base point x_K on E(K). Denote by Γ_K the set of homotopy classes of paths in the space E(K) with fixed initial point at x_K and endpoint on $\partial N(K)$ (these conditions must be preserved during the homotopy). Note that the orientations of \mathbb{R}^3 and K define the orientation of the tubular neighbourhood of the knot (right screw rule). Let m_b be the oriented meridian hooking an arc b. Define $a \circ b = [bm_b b^{-1} a]$, where for $x \in \Gamma_k$ the letter \bar{x} means a representative path, and square brackets denote the class that contains the path [x], see Fig. 5.2.

The quandle axioms can also be checked straightforwardly. Also, one can easily check that the grouppoids corresponding to different points x_k are isomorphic. This statement is left for the reader as an exercise.

There is a natural map from the knot quandle $\Gamma(K)$ to the group $\pi_1(\mathbf{R}^3 \setminus E(K))$. Let us fix a point x outside the tubular neighbourhood. Now, with each element γ of the quandle (path from x to $\partial E(K)$) we associate the loop $\gamma m \gamma^{-1}$, where m is the meridian at the point x.

This interpretation shows that the fundamental group can be constructed by the quandle: all meridians can play the role of generators for the fundamental groups, and all relations of type $a \circ b = c$ have to be replaced with $bab^{-1} = c$.

Besides, the fundamental group has the obvious action on the quandle: for each loop g and element of the quandle γ , the path $g\gamma$ is again an element of the quandle.

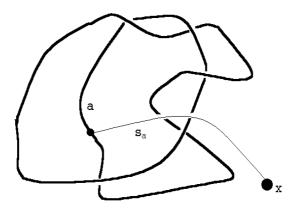


Figure 5.3. Defining the path s_a

5.2.2 Algebraic description of the quandle

Let D be a diagram of an oriented knot K. Denote the set of arcs of D by A_D . Let P be a crossing incident to two undercrossing arcs a and c and an overcrossing arc b. Let us write down the relation: $a \circ b = c$, where a is the arc lying on the left hand with respect to b and c is the arc lying on the right hand with respect to b. Denote the set of all relations for all crossings by R_D . Now, consider the quandle $\Gamma\langle A_D|R_D\rangle$, defined by generators A_D and relations R_D .

Theorem 5.1. Quandles Γ_K and $\Gamma\langle A_D|R_D\rangle$ are isomorphic.

Before proving the theorem, let us first understand its possible interpretations. On one hand, the theorem shows how to describe generators and relations for the geometrical quandle Γ_K . On the other hand, it demonstrates the independence $\Gamma\langle A_D|R_D\rangle$ of the choice of concrete knot diagram. Now, Proposition 5.1 (concerning colouring number) evidently follows from this theorem as a corollary because any proper colouring of a knot diagram by elements of Γ is a presentation of $\Gamma\langle A_D|R_D\rangle$ to Γ .

Proof of Theorem 5.1. With each arc a of the projection D, we associate the path s_a in E(K) in such a way that

- 1. The path s_K connects the base point with a point of the part of the torus ∂N_K corresponding to the arc a;
- 2. At all points where the projection of s_a intersects that of D, the path s_a goes over the knot, see Fig. 5.3.

Obviously, these conditions are sufficient for the definition of the homotopy class of s_a .

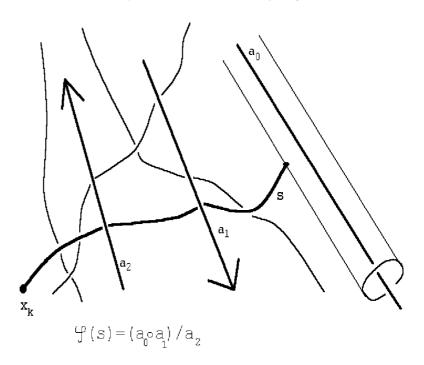


Figure 5.4. Constructing the map $\psi: \Gamma_k \to \Gamma(A_D, R_D)$.

Consequently, to each generator of $\Gamma = \langle A_D | R_D \rangle$, there corresponds an element of the quandle Γ_K . Thus we have defined the homomorphism $\phi : \Gamma \langle A_D | R_D \rangle \to \Gamma_K$. In order to define the inverse homomorphism $\psi : \Gamma_K \to \Gamma \langle A_D | R_D \rangle$, let us fix $s \in \Gamma_K$. Then, the path representing s is constructed in such a way that the projection of the path intersects D transversely and contains no diagram crossing.

Denote by $a_n, a_{n-1}, \ldots, a_1$ those arcs of D going over the path s. Denote by a_0 the arc corresponding to the end of s. Now, for each $s \in \Gamma_K$, let us assign the element $((\ldots(a_0\varepsilon_1a_1)\varepsilon_2\ldots a_{n-1})\varepsilon_na_n$ of the quandle $\Gamma\langle A_D|R_D\rangle$, where ε_i means / if s goes under a_i from the left to the right, or \circ , otherwise, see Fig. 5.4.

It is easy to check that this map is well defined (i.e., it does not depend on the choice of representative s for the element of Γ_k) and that maps ϕ and ψ are inverse to each other. This completes the proof.

The quandle corresponding to knot, is a complete knot invariant. However, it is difficult to recognise quandles by their presentation. This problem is extremely difficult. But it is possible to simplify this invariant making it weaker but more recognizable.

5.3 Completeness of the quandle

Roughly speaking, the quandle is a complete knot invariant because it contains the information about the fundamental group and "a bit more." To prove this fact about the completeness of the quandle, we shall use one very strong result by Waldhausen [Wal] concerning three–dimensional topological surgery.

By Matveev, two (non-isotopic) knots are equivalent if one can be obtained from the other by changing both the orientation of the ambient space and that of the knot. In this sense, the two trefoils are equivalent and have isomorphic quandles.

Here by complete we mean that the quandle distinguishes knots up to equivalence defined above.

The key points of the proof are the following:

- 1. For the unknot the situation is very clear: the fundamental group recognises it by Dehn's theorem.
- 2. If a knot is not trivial then its complement is sufficiently large (by Dehn's theorem); it also has some evident properties to be defined.
- 3. For the class of manifolds satisfying this condition the fundamental group "plus a bit more" is a complete invariant up to equivalence defined above.
- 4. The knot quandle allows us to restore the fundamental group structure for the complement and "a bit more."

For the sake of simplicity, we shall work only with knots. The same results are true for the case of links.

It is easy to see that the fundamental group of the knot complement can be restored from the quandle (this will be shown a bit later).

Let us first introduce some definitions.

Definition 5.2. A surface F in a manifold M is compressible in either of the following cases:

- 1. There is a non-contractible simple closed curve in the interior of F and a disc D in M (whose interior lies in the interior of M) such that $D \cap D = \partial D = k$.
- 2. There is a ball E in M such that $E \cap F = \partial E$.

Otherwise the surface is called *incompressible*.

Definition 5.3. A 3-manifold M is called *irreducible* if any sphere $S^2 \subset M$ is compressible.

A 3–manifold M with boundary is called boundary–irreducible if its boundary ∂M is incompressible.

Let K be an oriented knot. Consider the fundamental group π of $\mathbf{R}^3 \backslash N(K)$ where N is a tubular neighbourhood of K. Obviously, $\partial N = T$ is a torus that has an oriented meridian m (a curve that has linking coefficient 1 with K).

If K is not trivial then the fundamental group $\pi(T)$ is embedded in π . This result follows from Dehn's theorem.

Definition 5.4. For a non-trivial knot K, the embedded system $m \in \pi(T) \subset \pi$ is called a *peripherical system* of K.

The Waldhausen theorem⁴ says the following:

⁴We use the formulation taken from [Matv3]

Theorem 5.2. [Wal] Let M, N be irreducible and boundary–irreducible 3–manifolds. Let M be sufficiently large and let $\psi : \pi_1(N) \to \pi_1(M)$ be an isomorphism preserving the peripherical structure. Then there exists a homeomorphism $f : N \to M$, inducing ψ .

We are going to prove that the knot quandle is a complete knot invariant.

Now let K_1, K_2 be two knots. Suppose that ϕ is an isomorphism $\Gamma(K_1) \to \Gamma(K_2)$ of the quandles. Denote the complements to tubular neighbourhoods of K_1, K_2 by E_{K_1}, E_{K_2} , respectively.

Note that if K_1, K_2 are not trivial then the manifolds E_{K_1}, E_{K_2} are boundary–irreducible, sufficiently large and irreducible (by Dehn's lemma).

Now, let us suppose that one of the two knots (say, K_1) is trivial. Then $\pi(K_1)$ is isomorphic to **Z**. Since the knot group can be restored from the quandle, we have $\pi(K_2)$ is also **Z**. Thus, K_2 is trivial.

Now consider the case when K_1, K_2 are non-trivial.

In this case, we know that the knot group can be restored from the quandle; besides, the meridian can also be obtained from the knot quandle: it can be chosen to be the image of any element of the quandle under the natural morphism.

Now, let us prove that the normaliser of the meridian (as an element of the quandle representing the path from x to x_k) in the fundamental group consists precisely of the fundamental group of the tubular neighbourhood of the knot $\pi_1(T_2) = \mathbf{Z}^2$. Indeed, each element of the $\pi_1(T_2)$ is a path looking like ana^{-1} where a represents the meridian in the quandle (path from the initial point to the point on T^2), and n is a loop on the torus T^2 . So, we have: $ana^{-1} \cdot a = an$ which is homotopic to a in the quandle.

Now, suppose that for some g we have: ga is homotopic to a. Then, there exists a path n on the torus drawn by the endpoint while performing this homotopy. Denote this path by x. So, we have: $gaxa^{-1} = e$, so g is $an^{-1}a^{-1}$ that belongs to the fundamental group of T_2 .

The next step of the proof is the following. The quandle knows the peripheral structure. Let K_1, K_2 be two non-trivial knots with the same peripheral structure. Consider an isomorphism of the knot groups. By the Waldhausen theorem, it generates a homeomorphism between E_{K_1} and E_{K_2} and maps the meridian of the first one to a meridian of the second one. Thus, we have the same information how to attach full tori to E_{K_1} and E_{K_2} in order to obtain \mathbb{R}^3 . The longitudes of these tori are just K_1 and K_2 . So, they are obviously isotopic. To perform all this, we must fix the orientation of E_{K_1} and E_{K_2} . Then we shall be able to choose the orientation of the meridian. If we choose the opposite orientation of them both, we shall obtain an equivalent knot.

However, if the orientation of the meridian is fixed, the knot can be uniquely restored.

5.4 Special realisations of the quandle: colouring invariant, fundamental group, Alexander polynomial

One can easily define the *free quandle* with generators a_1, \ldots, a_k . Namely, one takes into account idempotence, the existence of a unique solution $x \circ b = c$ and self-distributivity. No more conditions will be given for this quandle.

Let us give two more examples.

Example 5.1. Consider the free group with an infinite number of generators. Define the quandle operation on it according to the rules: $a \circ b = bab^{-1}, a/b = b^{-1}ab$. All axioms can be checked straightforwardly. In this case, we have obtained a natural morphism of the free quandle to the free group. As in the case of quandles, free groups can be transformed to arbitrary groups; we only have to describe the quandle relations in terms of groups.

It can be easily checked that the image of the knot quandle is the fundamental group of the knot complement. (To do this, we should just compare the presentation of the quandle and the fundamental group corresponding to a knot diagram.)

Example 5.2. One can consider maps of the quandle to the free module over Laurent polynomial ring (with respect to a variable t) as well. To do this, one should decree

$$a \circ b = ta + (1 - t)b, \quad a/b = \frac{1}{t}a + \left(1 - \frac{1}{t}\right)b.$$

In this case, we obtain the quotient ring that has a quadratic matrix of linear relations for the generators a_1, a_2, \ldots . This matrix is called the Alexander matrix of the knot diagram. The Alexander polynomial of the knot can be defined as follows: we set one variable a_i to be equal to 0 and then we solve the system of n equations for n-1 variables. Finally, we obtain some relation for the ring elements: f(t) = 0, where f is a Laurent polynomial. The function f (defined up to multiplication by $\pm t^k$), is called the Alexander polynomial. It can be calculated by taking each minor of the Alexander matrix of (n-1)-th order.

5.5 The Conway algebra and polynomial invariants

Now we shall describe the construction that allows us to look in the same way at different polynomial invariants of knots: those by Jones, Conway, and HOMFLY. In particular, we shall prove the invariance and uniqueness of the HOMFLY polynomial. Note that HOMFLY is not a surname, but an abbreviation of the first letters of six surnames, [HOMFLY]: Hoste, Ocneanu, Millett, Freyd, Lickorish, and Yetter. This polynomial was later rediscovered by Przytycki and Traczyk [PT]. We shall use the approach proposed in the article [PT].

The main idea is the following. Unlike the previous approach, where we associated a special algebraic object to **each link**, here we construct some algebraic

object and assign some element of this algebraic object to each link. This is going to be the link invariant.

Caveat. We are now going to introduce the two operations, \circ and /. They have another sense and other properties than those described before.

Let A be the algebra with two binary operations \circ and / such that the following properties hold: for all $a,b \in A$ we have $(a \circ b)/b = a, (a/b) \circ b = a$. For each link (diagram) L, let us construct the element W(L) of A as follows. Denote by a_n the element of A corresponding to the n-component trivial link.

Let us also require the following algebraic equation for any *Conway triple* (i.e. three diagrams coinciding outside a small circle and looking like (X), (X), (X) inside this circle; such diagrams are called a *Conway triple*):

$$W(\langle \sum \rangle) = W(\langle \sum \rangle) \circ W(\langle \sum \rangle). \tag{1}$$

The uniqueness of the inverse element means that we must require the existence of the reverse function /, such that $W(\sum)/W(\sum) = W(\sum)$. So, W is going to be a map from the set of all links to the algebraic object to be constructed. Later we shall see that some partial cases of the equation (1) coincide with some skein relations. Let us now take into account the following circumstance: because each link can be transformed to the trivial (ascending) one (Exercise 1.2) by switching some crossing types, the value of the function W on any m-component link with n crossings can be described only by using the value of W for the trivial m-component link and the value of W for some links with fewer crossings.

Indeed, they can also be represented by a_i and values of W on links with less than n-1 crossings. Consequently, for each link L with an arbitrary quantity of crossings, the value of W(L) can be expressed somehow (possibly, not uniquely) in $a_i, i = 1, 2, \ldots$, by using \circ and /.

At the present moment, we do not know whether the function W is well defined and if so, whether it is a link invariant. Let us try to look at the algebra A and find the restrictions for the uniqueness of the definition.

Consider the trivial n-component link diagrams with only one crossing at one (twisted) circle. The value of W on this trivial link should be equal to a_n . Depending on the crossing type, the relation (1) leads to

$$a_n = a_n \circ a_{n+1} \tag{2}$$

and

$$a_n = a_n/a_{n+1}. (3)$$

These two relations should hold for arbitrary $n \geq 1$.

There is one more argument that one may call "the switching order." Consider the diagram of L and choose two (say, positive) crossings p, q of it, see Fig. 5.5.

Denote by $L_{\alpha\beta}$ for $\alpha, \beta \in \{+, -, 0\}$ the link diagram coinciding with L outside small neighbourhoods of p, q and having type α at p and type β at q.

Let us consider the relation (1) at p and then at q.

We get: $W(L_{++}) = W(L_{-+}) \circ W(L_{0+}) = (W(L_{--}) \circ W(L_{-0})) \circ (W(L_{0-}) \circ W(L_{00})).$

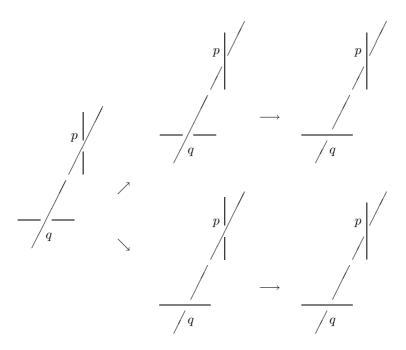


Figure 5.5. Two ways of resolving two crossings

Now, let us consider the same relation for q and later, for p (the other order). We have: $W(L_{++}) = W(L_{+-}) \circ W(L_{+0}) = (W(L_{--}) \circ W(L_{0-})) \circ (W(L_{-0}) \circ W(L_{00}))$. Comparing the obtained equalities, we get:

$$(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d), \tag{4}$$

where $a = W(L_{--}), b = W(L_{0-}), c = W(L_{-0}), d = W(L_{00}).$

We shall require the equation (4) for arbitrary a, b, c, d, which are going to be the elements of the algebra to be constructed.

In the case when both p and q are negative, we get the analogous equation

$$(a/b)/(c/d) = (a/c)/(b/d).$$
 (5)

Analogously, if one crossing is positive, and the other is not, we get the equation

$$(a/b) \circ (c/d) = (a \circ c)/(b \circ d). \tag{6}$$

Thus, we have found some necessary conditions for W(L) to be well defined. Let us show that these conditions are sufficient.

Definition 5.5. An algebra A with two operations \circ and / (reverse to each other) and a fixed sequence a_n of elements is called a *Conway algebra* if the conditions (2)–(6) hold.

Theorem 5.3. For each Conway algebra, there exists a unique function W(L) on link diagrams that has value a_n on the n-component unlink diagrams and satisfies (1). This function is an invariant of oriented links.

Proof. First, let us show that this invariant is well defined on diagrams with numbered components. We shall use induction on the number of crossings.

Let C_k be the class of links having diagrams with no more than k crossings.

The main induction hypothesis. There exists a well-defined function W(L) on C_k which is invariant under those Reidemeister moves, which do not let the diagram leave the set C_k and satisfies the relation (1) for all Conway triples with all elements from C_k .

The induction base (k = 0) is trivial since in this case the class C_k consists only of unlinks and no Reidemeister moves can be performed. In order to perform the induction step, let us first choose a canonical way of associating W(L) with a link $L \in C_{k+1}$ by ordering components and choosing a base point on each of them.

After this, we shall prove the independence from W(L) on the choice of base points, its invariance under Reidemeister moves, and, finally, independence of the order of components.

The construction of W. Let us enumerate all components of L. Fix a point b_1 on the first component, b_2 on the second one, and so forth; base points should not coincide with crossings. Let us now describe how to construct the element $W_b(L)$ (here the index b means the ordered set b_1, b_2, \ldots of base points). Let us walk along the link components according to the orientation. First, take the point b_1 and pass the first component till b_1 , then take b_2 and pass the second component, and so forth. A crossing p is called good if it is first passed under, and then over. All other points are said to be bad.

Now we are going to use the (second) induction on the number of bad points. The induction step is obvious: if all points are good then the link is trivial (the diagram is ascending). In this case, we set by definition $W_b(L) = a_n$, where n is the number of link components.

Suppose we have defined W_b for all links with k+1 crossings and no more than m bad points. Let L be a link diagram with k+1 crossings m+1 bad points.

Let us fix the first bad point (in accordance with the chosen circuit). Without loss of generality, suppose that this crossing is positive. Let us apply the relation (1) to this crossing. Thus we obtain two links: L_- that has one bad point less than the diagram L (consequently, $W_b(L_-)$ is well defined by the induction hypothesis) and L_0 that belongs to C_k (and $W(L_0)$ (consequently, $W_b(L_0)$) is also well defined). By definition, let us put for L

$$W_b(L) = W_b(L_-) \circ W(L_0).$$

Thus, we have defined W_b for C_{k+1} .

Now, let us prove that the function W_b satisfies the relation

$$W_b(\langle \sum \rangle) = W_b(\langle \sum \rangle) \circ W(\langle \sum \rangle)$$

for each crossing q. We have already used this relation **to define** the module. However, we have not yet proved that it does not yield to a contradiction. Thus, the relation (1) deserves proving. Suppose the point q is bad. If it is the first bad point according to our circuit, the desired equality holds by construction. Now, let us apply the induction method (the third induction) on the number N of this bad point. Suppose that if N < m then the Conway relation holds. Let p be the first

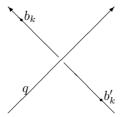


Figure 5.6. Changing the starting point

bad (say, positive) crossing. By using the definition of W_b , the induction hypothesis and the main induction hypothesis, (4), and again, the induction hypothesis and the main induction hypothesis, we obtain:

$$W_b(L_{++}) = W_b(L_{-+}) \circ W(L_{0+}) = (W(L_{--}) \circ W(L_{-0})) \circ (W(L_{0-}) \circ W(L_{00}))$$

$$= (W_b(L_{--}) \circ W(L_{0-})) \circ (W(L_{-0}) \circ W(L_{00})) = W_b(L_{+-}) \circ W(L_{+0}).$$

Here we use the following notation: the first index of $L_{\varepsilon_1\varepsilon_2}$ is related to the point p and the second is related to the point q. Thus, the obtained formula is just what we wanted.

If q is a good point (say, positive) for L_+ , then it is bad for L_- . As we have already proved, the identity $W_b(L_-) = W_b(L_+)/W(L_0)$ holds for this point, and the desired equality $W_b(L_+) = W_b(L_-) \circ W(L_0)$ is just its corollary.

Now, let us prove that the function W_b does not depend on the choice of base points (the order of components remains fixed). It is sufficient to consider only the case when one base point (say, b_k) passes through one crossing (denote it by q) to a position b'_k , see Fig. 5.6.

Suppose the crossing q is positive. If q is good for both base points $b=(b_1,\ldots,b_n)$ and $b'=(b_1,\ldots,b_{k-1},b'_k,b_{k+1},\ldots,b_n)$ then the equality $W_b(L)=W_{b'}(L)$ holds just by definition. If q is bad in both cases then we have $W_b(L_+)=W_b(L_-)\circ W(L_0), W_{b'}(L_+)=W_{b'}(L_-)\circ W(L_0)$. Since q is good for L_- for both choices of base points, we see that $W_b(L_-)=W_{b'}(L_-)$ and hence $W_b(L_+)=W_{b'}(L_+)$. It remains only to consider the case when q is good for b and bad for b' (or vice versa). This happens only in the case when both parts of the link passing through q lie on the same component.

Now we can assume that L has no more bad points (either with respect to b or with respect to b'); in this case all they can be transformed to good ones by using the Conway relation (that preserves the equality $W_b(L) = W_{b'}(L)$).

So, the link L has no bad points with respect to b, hence, $W_b(L) = a_n$. The link L has the only bad point q with respect to b' (by definition, $W_{b'}(L) = W_{b'}(L_-) \circ W(L_0)$). Now, it remains to note that the links L_- and L_0 have no bad points as well. Thus, they are trivial. Besides, $W(L_-) = a_n, W(L_0) = a_{n+1}$. Taking into account $a_n = a_n \circ a_{n+1}$, we get $W_b(L) = W_{b'}(L)$.

Let us prove now that the function $W_b(L)$ is invariant under Reidemeister moves that preserve the link in the class C_{k+1} .

The main idea is the following: suppose we perform some Reidemeister move inside the area U. Thus, we have two pictures inside U: the picture before and the picture after. Outside U, both diagrams have the same crossings and the same shadow. If we can arrange all other crossing types in order to obtain an ascending diagram (with respect to enumerated components) with each of the two fixed crossings inside U, then the invariance is trivial: we just express our diagrams in the same way via diagrams of unlinks. These unlink diagrams differ only inside U. In the other case, we need to perform the relation (1) and then consider the case described above.

Let us be more detailed. First, let us consider the move Ω_1 . By moving the base point, the point added or deleted by Ω_1 can be thought to be good. In this case, the existence of the loop does not affect the result of the calculation of W_b .

For the same reason, the value of W_b does not change under Ω_2 if both points appearing (disappearing) while performing the move are good. If both points are bad and cannot be made good by moving base points (this may happen when two different components are involved in Ω_2), one should apply the Conway relation at both points and note that $L_{-0} \cong L_0$ and $W(L_{-0}) = W(L_{0-})$, see Fig. 5.7.

Because $W_b(L_{-+}) = W_b(L_{--}) \circ W(L_{-0}) = (W_b(L_{+-})/W(L_{0-})) \circ W(L_{-0}) = W_b(L_{+-})$, then the procedure of making all bad points good does not affect the behaviour of W_b under Ω_2 .

Finally, let us consider the case of Ω_3 . Let us assume that all base points are outside the area of Ω_3 . Denote the crossings by x (upper and lower arcs), y (upper and middle) and z (lower and middle), see Fig. 5.8

Denote the link obtained after performing Ω_3 by L'. Let us note that x cannot be the only bad point or the only good point from the set $\{x,y,z\}$. If, for instance, x is good, and y,z are bad then the branch xz is passed before xy with respect to the orientation, the branch xy is before yz, and yz is before xz, which leads to a contradiction.

Thus, if x is good then so is one of y, z. Suppose, y. In this case moving the branch xy over the crossing z does not change the invariant W_b by construction. If the crossing x is bad then one of y, z (say, z) is bad as well. Let us write the Conway relation for the crossings z, x. We get:

$$W_b(L_{++}) = W_b(L_{-+}) \circ W(L_{0+}) = (W_b(L_{--}) \circ W(L_{-0}))) \circ W(L_{0+}),$$

$$W_b(L'_{++}) = W_b(L'_{-+}) \circ W(L'_{0+}) = (W_b(L'_{--}) \circ W(L'_{-0})) \circ W(L'_{0+}).$$

The first index for L.. is related to z and the second one is related to x. Since for the link L_{--} the points x, z are good, then $W_b(L_{--}) = W_b(L'_{--})$ (this case has already been considered). Diagrams L_{-0} and L'_{-0} coincide, and L'_{0+} can be obtained from L_{0+} by applying the move Ω_2 twice, which does not change the value of W_b . Consequently, $W_b(L_{++}) = W_b(L'_{++})$.

Thus, we have constructed a function W that is an invariant of links with marked components.

Let us show that it is a link invariant (no marked point information is needed).

In order to do this, let us introduce the crossing switch operation Ω_0 . Denote the versions of Ω_1, Ω_2 decreasing the number of crossings by Ω_1^-, Ω_2^- .

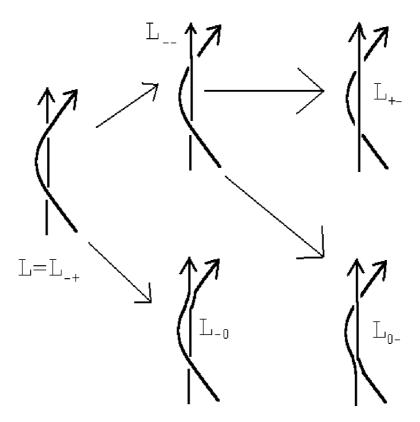


Figure 5.7. Different ways of resolving two crossings

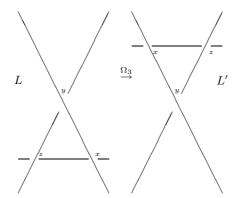


Figure 5.8. The third Reidemeister move

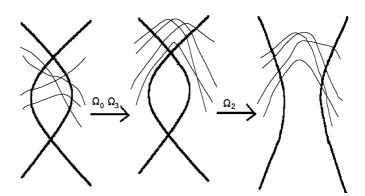


Figure 5.9. Removing a bigon

We shall need the following lemma.

Lemma 5.1. Each link diagram can be transformed to the unlink diagram without crossing by using $\Omega_0, \Omega_1^-, \Omega_2^-, \Omega_3$.

We shall prove this lemma later.

Note that the equation $W_b(L) = W_{b'}(L)$ is invariant under Ω_0 . As we proved earlier, it is also invariant under $\Omega_{1,2,3}$.

Since for the standard diagram of the unlink we have $W_b(L) = W_{b'}(L)$ for any orders of components with base points, then this holds in the general case.

Below, we give a prove of the auxiliary lemma we have used.

Proof of Lemma 5.1. Let L be a link diagram. A branch l of this diagram is called a loop if it starts and ends at the same crossing. A loop is called simple if it has no self-intersection and if in the domain bounded by it there are no other loops. We say that two arcs $l_1, l_2 \supset L$ bound a bigon if they have no selfintersections, have common initial and final points and no other intersections. A bigon is called simple if it does not contain smaller bigons and loops inside. In Fig. 5.9 we show that if L has a simple bigon then the number of its intersection points can be decreased by making this bigon empty (using Ω_0, Ω_3) and deleting it by means of Ω_2^- .

In the case of a simple loop, we can do the same and finally delete it by Ω_1^- .

The only thing to note is that if a link diagram has at least one crossing then it contains either a simple bigon or a simple loop. This is left to the reader as a simple exercise.

Let us show now that Jones, Conway and HOMFLY polynomials are just special cases of the invariant W for some special algebras A.

We have shown that for each Conway algebra A there exists an A-valued invariant of oriented links $W(\cdot) \in A$. Among all such algebras there exists the universal algebra A_U . It is generated by $a_n, n \geq 1$ and has no other relations except for (2)-(6).

The universal link invariant corresponding to the universal algebra is the strongest one among all those obtained in this way. However, it has a significant disadvantage: it is difficult to recognise two different presentations of an element in the algebra A.

We are going to show how to construct a family of Conway algebras. The invariants to be constructed are more convenient than the universal one: they are easier to recognise.

Let A be an arbitrary commutative ring with the unit element, $a_1 \in A$ and α, β be some invertible elements of A. Let us define \circ , / as follows:

$$x \circ y = \alpha x + \beta y \tag{7}$$

and

$$x/y = \alpha^{-1}x - \alpha^{-1}\beta y,\tag{8}$$

where

$$a_n = (\beta^{-1}(1-\alpha))^{n-1}a_1, n \ge 1.$$
(9)

Then the following proposition holds.

Proposition 5.2. For any choice of invertible elements α, β and element a_1 , the ring A endowed with operations \circ , / defined above and with elements a_n , see (9), is a Conway algebra.

The proof follows straightforwardly from the axioms.

5.6 Realisations of the Conway algebra. The Conway–Alexander, Jones, HOMFLY and Kauffman polynomials

Let us give here some examples of simple invariants that originate from the Conway algebra.

Example 5.3. Let A be the ring of polynomials of variable x with integer coefficients. Let $\alpha = 1, \beta = x, a_1 = 1$. Then W(L) coincides with the Conway polynomial (also called the Conway potential function).

The Conway polynomial was the first among the polynomials satisfying the Conway relation. It was proposed in the pioneering work by Conway [Con]. All other polynomials and modifications appeared much later.

In Fig. 5.10, we present the celebrated Kinoshita–Terasaka knot. This knot is not trivial. However, it can be easily checked that this knot has Conway polynomial equal to one. Thus, the Conway (consequently, Alexander) polynomial does not always distinguish the unknot.

This knot has non-trivial Jones polynomial. It is not yet known whether the Jones polynomial always distinguishes the unknot. We will touch on this problem later.

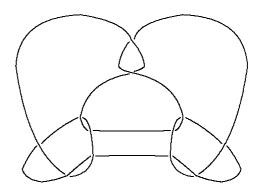


Figure 5.10. The Kinoshita–Terasaka knot

Example 5.4. Let A be the ring of Laurent polynomials in \sqrt{q} , where $\alpha = q^2$, $\beta = q(\sqrt{q} - \frac{1}{\sqrt{q}})$, $\alpha_1 = 1$. In this case, W(L) coincides with the Jones polynomial of q.

Example 5.5. Let A be the integer coefficient Laurent polynomial ring of the variables l, m. Let $\alpha = -\frac{m}{l}, \beta = \frac{1}{l}, a_1 = 1$. Then the obtained invariant $\mathcal{P}(l, m)$ coincides with the HOMFLY polynomial.

Exercise 5.3. Write down the skein relations for these polynomials.

5.7 More on Alexander's polynomial. Matrix representation

There is another way to define the Alexander polynomial more exactly.

We shall not give exact proofs. On one hand, they follow from "quandle properties" of the function $a \circ b$ defined as ta + (1 - t)b. On the other hand, the reader can check them by a direct calculation.

Given an oriented link diagram L with n vertices, let us construct the Alexander matrix M(L) as follows. We shall return to such matrices in the future when studying virtual knot invariants.

Let us enumerate all crossings by natural numbers $1, \ldots, n$. In the general position, there exists precisely one arc outgoing from each crossing (if there are no separated cyclic arcs). It is easy to see that each knot isotopy class has such a diagram. So, we can enumerate outgoing arcs by integers from 1 to n, correspondingly. Now, we construct an incidence matrix, where a crossing corresponds to a row, and an arc corresponds to a column.

Suppose that no crossing is incident twice to one and the same arc (no loops). Then, each crossing (number i) is incident precisely to three arcs: passing through this crossing (number i), incoming (number j) and outgoing (number k).

In this case, the *i*-th row of the Alexander matrix consists of the three elements at places i, j, k. If the *i*-th crossing is positive, then $m_{ii} = 1, m_{ik} = -t, m_{ij} = t - 1$. Otherwise we set $m_{ii} = t, m_{ik} = -1, m_{ij} = 1 - t$.

Obviously, this matrix has determinant zero, because the sum of elements in each row equals zero.

Define the algebraic complement to m_{ij} by Δ_{ij} .

Then the following theorem holds.

Theorem 5.4. All Δ_{ij} coincide.

Denote Δ_{ij} by $\Delta(L)$.

Theorem 5.5. The function Δ defined on links satisfies the following skein relation:

$$\Delta(\langle \sum \rangle) - \Delta(\langle \sum \rangle) = (t^{1/2} - t^{-1/2}) \Delta(\langle \sum \rangle).$$

It is easy to check that for the unknot \bigcirc , we have $\Delta(\bigcirc) = 1$.

Thus, we can conclude that the polynomial Δ coincides with the Conway polynomial up to the variable change $x=t^{1/2}-t^{-1/2}$. So, it is a well–defined link invariant.

Remark 5.3. This way to derive Alexander-like polynomials from groups can be found in, e.g., [CF, Cr].

The Conway polynomial [Con] is obtained from the Alexander polynomial just by a variable change: $x = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})$. Thus, the polynomial (denoted by C) satisfies the skein relation $C(\bigcirc) - C(\bigcirc) = x \cdot C(\bigcirc)$. Conway first proved that this relation (together with $C(\bigcirc) = 1$) can be axiomatic for defining a knot invariant. The approach by Przytycki and Traczyk described in this chapter was a generalisation of the Conway approach.

Chapter 6

Kauffman's approach to Jones polynomial

In the present chapter, we shall describe another approach for constructing invariant polynomials from link diagrams. It was proposed by Kauffman. It is quite expressive and allows us to construct polynomials (known as Kauffman polynomials in one and two variables).

The first polynomial to be constructed coincides with the Jones polynomial up to a suitable variable change. So, one can also speak about the Jones–Kauffman polynomial or the Jones polynomial in Kauffman's form.

The second polynomial includes some more sophisticated techniques in comparison with the first one. So, the second Kauffman polynomial is stronger than the first one. It is "in the general position" with the Jones two–variable polynomial that will be discussed later in the book.

6.1 State models in physics and Kauffman's bracket

First, let us seek an invariant polynomial for unoriented links. Let \bar{L} be an unoriented link diagram having n crossings. Each crossing of \bar{L} can be "smoothed" in two ways.

These ways $L \to L_A$ and $L \to L_B$ are shown in Fig. 6.1.

Now, let us try to construct some function (later, it will be called the Kauffman bracket) in the three variables a, b, c satisfying the following axiomatic relations.

$$\langle L \rangle = a \langle L_A \rangle + b \langle L_B \rangle \tag{1}$$

$$\langle L \sqcup \bigcirc \rangle = c \langle L \rangle \tag{2}$$

$$\langle \bigcirc \rangle = 1.$$
 (3)

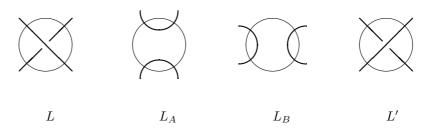


Figure 6.1. Two ways of smoothing the diagram

Here we consider arbitrary diagrams $L = \bigotimes_{i}, L' = \bigotimes_{i}, L_A = \bigotimes_{i}$ and $L_B = \bigotimes_{i}$ which coincide outside a small circle; inside this circle, the diagrams differ as shown in Fig. 6.1.

Herewith, \bigcirc denotes the unknot, and \sqcup denotes the disconnected sum.

Let us try to find the conditions for a, b, c in order to obtain a polynomial invariant under Reidemeister moves.

First, let us test the invariance of the function to be constructed under the second Reidemeister move. By using (1) and (2), we obtain the following properties of the hypothetical function:

$$\begin{split} \langle \bigodot \rangle \rangle &= a \langle \bigodot \rangle \rangle + b \langle \bigodot \rangle \rangle = (a^2 + b^2) \langle \bigodot \rangle \rangle + ab \langle \bigodot \rangle \rangle + ab \langle \bigodot \rangle \rangle \\ &= (a^2 + b^2 + abc) \langle \bigodot \rangle \rangle + ab \langle \bigcirc \bigcirc \rangle \rangle \end{split}$$

Thus,

$$\langle \langle \overleftarrow{\sum} \rangle \rangle = (a^2 + b^2 + abc) \langle \langle \overleftarrow{\sum} \rangle \rangle + ab \langle \langle \overleftarrow{\sum} \rangle \rangle.$$

This equality should hold for all such triples looking like (X), (X) inside some small circle and coinciding outside it.

Thus, the Ω_2 -invariance of the polynomial to be constructed should imply the following relations: ab = 1 and $a^2 + b^2 + abc = 0$.

Let us decree $b = a^{-1}$ and $c = -a^2 - a^{-2}$.

Thus, if the Ω_2 -invariant link polynomial from $\mathbf{Z}[a, a^{-1}]$ exists, then it satisfies the properties described above and it is unique.

It turns out that here the Ω_2 -invariance implies Ω_3 -invariance.

Let us discuss this in more details.

Consider the two diagrams \bowtie and \bowtie where one can be obtained from the other by using Ω_3 . Smoothing them at one vertex we have:

$$\langle \nearrow \rangle = a \langle \nearrow \rangle + a^{-1} \langle \nearrow \rangle$$

and

$$\langle \bigotimes \rangle = a \langle \bigotimes \rangle + a^{-1} \langle \bigotimes \rangle.$$

Let us compare the second parts of these equalities. We have: $\Xi = \Xi$. Furthermore, after applying Ω_2 twice, we obtain:

$$\Rightarrow$$
 = \Rightarrow = \Rightarrow .

Thus we have shown the invariance of the bracket under Ω_3 . Now, one should mention that Ω_3 does not change the writhe number.

This implies the invariance of the "hypothetical" bracket polynomial $\langle L \rangle$ under Ω_3 .

However, while studying the invariance of the hypothetical polynomial under the first Reidemeister move, we meet the following unpleasant circumstance: addition (removal) of a curl multiplies the polynomial by $-a^3$ or by $-a^{-3}$. In fact, by applying (1) to the vertex incident to a curl, we obtain a sum of brackets of two diagrams. One of them gives us $p(-a^2-a^{-2})\langle L\rangle$, the other one gives $p^{-1}\langle L\rangle$, where p is equal to $a^{\pm 1}$. The sign \pm depends on the type of curl twisting.

Taking the sum of these two values, we get $(-a^{\pm 3})\langle L\rangle$.

Thus we have proved some properties that the Kauffman bracket **should** satisfy, i.e., we have deduced these properties from the axioms. However, we have not yet shown the main thing, i.e., the existence of such a polynomial. Let us prove that it exists.

Theorem 6.1. There exists a unique function on unoriented link diagrams valued in $\mathbb{Z}[a, a^{-1}]$ satisfying relations (1)–(3) and invariant under Ω_2, Ω_3 .

Proof. Consider an unoriented diagram \bar{L} of a link L that has n crossings. Let us enumerate all crossings of \bar{L} by integers from 1 to n.

As before, we can smooth each crossing of the diagram in one of two ways, $A:(\bigotimes)\to(\bigotimes)$ or $B:(\bigotimes)\to(\bigotimes)$.

Definition 6.1. By a *state* of a crossing we mean one of the two possible ways of smoothing for it. By a *state* of the diagram L we mean the n states of crossings, one smoothing for each vertex.

Thus, the diagram \bar{L} has 2^n possible states. Choose a state s of the diagram \bar{L} . Obviously, these smoothings turn \bar{L} into a set of non-intersecting curves on the plane.

Let $\alpha(s)$ and $\beta(s)$ be the numbers of crossings in states A and B, respectively. Let γ be the number of circles of the diagram \bar{L} in the state s.

If we "smooth" all crossings of the diagram \bar{L} by means of (1), and then apply the relations (2) and (3) for calculating the bracket polynomials for the obtained diagrams, we get

$$\langle \bar{L} \rangle = \sum_{s} a^{\alpha(s) - \beta(s)} (-a^2 - a^{-2})^{\gamma(s) - 1},$$
 (4)

where the sum is taken over all states s of the diagram \bar{L} .

Thus we have shown the *uniqueness* of the polynomial satisfying (1)–(3).

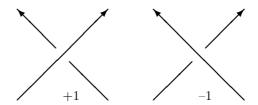


Figure 6.2. Local writh numbers

It remains to show now that the formula (4) shows not only the uniqueness of $\langle \bar{L} \rangle$, but its *existence* too.

Actually, we can just define the polynomial $\langle \bar{L} \rangle$ as in (4).

Being defined by (4), the bracket polynomial evidently satisfies the conditions (2) and (3).

The invariance of the constructed polynomial under Ω_2, Ω_3 can be deduced straightforwardly from the definition. The theorem is proved.

6.2 Kauffman's form of Jones polynomial and skein relations

Thus, we have well defined the bracket polynomial (Kauffman's bracket). This polynomial is defined on unoriented link diagrams and is Ω_2 and Ω_3 —invariant. It turns out that it can be transformed into an invariant polynomial of oriented links, i.e., a function on oriented link diagrams invariant under all Reidemeister moves.

Consider an oriented diagram of a link L. Let us define w(L) as follows. With each crossing of L we associate +1 or -1 as shown in Fig. 6.2. This number is called the *local writhe number*. Taking the sum of these numbers at all vertices, we get the writhe number w(L).

It is easy to see that this move is invariant under Ω_2 , Ω_3 , but not invariant under Ω_1 : under this move the writhe number is changed by ± 1 . This circumstance allows us to normalise the bracket. Thus, we can define the **polynomial invariant** of links like this:

$$X(L) = (-a)^{-3w(L)} \langle |L| \rangle, \tag{5}$$

where L is an oriented link diagram, and |L| is the non-oriented diagram obtained from L by "forgetting" the orientation.

Definition 6.2. Let us call the invariant polynomial X according to (5) the Kauffman polynomial.

It turns out that the Kauffman polynomial satisfies a certain skein relation. Actually, let $L_+ = \bigotimes_{i=1}^n L_- = \bigotimes_{i=1}^n \text{ and } L_0 = \bigotimes_{i=1}^n C_i$ be a Conway triple. Without loss of generality, we can assume that $w(L_+) = 1, w(L_-) = -1, w(L_0) = 0$. Consider the non-oriented diagrams $K_+ = |\bigotimes_{i=1}^n | = \bigotimes_{i=1}^n |K_-| = |\bigotimes_{i=1}^n |K_-| =$

$$X(\langle \sum \rangle) = (-a)^{-3} (a\langle K_a \rangle + a^{-1} \langle K_b \rangle) = -a^{-2} \langle K_A \rangle - a^{-4} \langle K_B \rangle$$
 (6)

and, analogously,

$$X(\langle \sum \rangle) = -a^2 \langle K_A \rangle - a^4 \langle K_B \rangle \tag{7},$$

$$X(\langle \sum \langle \cdot \rangle) = \langle K_A \rangle. \tag{8}$$

In order to eliminate K_B from (7) and (8) let us multiply (7) by a^4 and (8) by $(-a)^{-4}$ and take their sum. Thus, we get

$$a^{4}X((\sum)) - a^{-4}X((\sum)) = (a^{-2} - a^{2})X((\sum)). \tag{9}$$

This is the desired skein relation for the Kauffman polynomial. Now, it is evident that for each link, the value of the Kauffman polynomial contains only even degrees of a.

After the change of variables $q = a^{-4}$, we obtain another invariant polynomial called *the Jones polynomial*, originally invented by Jones [Jon1]. Obviously, it satisfies the following skein relation:

$$q^{-1}V((\sum)) - qV((\sum)) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})V((\sum)). \tag{10}$$

Exercise 6.1. Prove that the Jones polynomial never equals zero.

By definition, the value of the Jones polynomial on the n-component unlink is equal to $(-q^{\frac{1}{2}}-q^{-\frac{1}{2}})^{n-1}$

The Jones polynomial allows us to distinguish some mirror knots. For example, we can prove that the right trefoil knot is not isotopic to the left trefoil knot. Namely, from (10) we have:

$$V(\bigcirc) = q^2 \cdot V(\bigcirc) + q(q^{\frac{1}{2}} - q^{-\frac{1}{2}})V(\bigcirc) =$$

(here the Conway relation is applied to the upper left crossing)

$$q^2 + (q^{\frac{3}{2}} - q^{\frac{1}{2}})V(()).$$

Taking into account

$$V(\bigcirc) = q^2(-q^{-\frac{1}{2}} - q^{\frac{1}{2}}) + q(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) = q^{-\frac{5}{2}},$$

we see that

$$V(\bigodot) = q^2 + q(q^{\frac{1}{2}} - q^{-\frac{1}{2}})(-q^{-\frac{5}{2}} - q^{\frac{1}{2}}) = -q^4 + q^3 + q.$$

Analogously,
$$V(\bigcirc) = -q^{-4} + q^{-3} + q$$
.
Thus, $(\bigcirc) \neq (\bigcirc)$.

Later, we shall present the perfect Jones proof of the invariance for a stronger two-variable polynomial, called *the Jones polynomial in two variables*.

6.3 Kauffman's two-variable polynomial

The Kauffman approach also allows us to construct a strong two-variable polynomial. The stronger one does not, however, satisfy, any skein relation.

To construct it, one should consider a more complicated relation than that for the usual Kauffman polynomial. Namely, we first take four diagrams $L = \bigotimes$, $L' = \bigotimes$, $L_A = \bigotimes$, $L_B = \bigotimes$ of unoriented links that differ from each other only inside a small circle

Then we construct the polynomial K(z,a) satisfying the following axioms:

$$D(L) - D(L') = z(D(L_A) - D(L_B))$$
(11);

$$D(\bigcirc) = \left(1 + \frac{a - a^{-1}}{z}\right) \tag{12};$$

$$D(X\#P) = aD(X), D(X\#Q) = a^{-1}D(X), \tag{13}$$

where

$$P = \bigcirc$$
 and $Q = \bigcirc$

are the two loops.

The polynomial D (like Kauffman's bracket) is invariant under Ω_2, Ω_3 .

As in the case of the one–variable Kauffman polynomial, we normalise this function by using w(L), namely for an oriented diagram L of a link, we set $Y(L) = a^{-w(|L|)}D(|L|)$, where |L| is the unoriented diagram obtained from L by forgetting the orientation.

The obtained polynomial is called the two-variable Kauffman polynomial.

After this, one can show the existence, uniqueness and invariance under Reidemeister moves of the polynomial defined axiomatically as above.

For uniqueness, the proof is pretty simple. First we define its values on unlinks.

Later, in order to define the value of the polynomial somehow, we use induction on the number of classical crossings. The induction basis is trivial. The induction step is made by using the relation (11): we can switch all crossing types and express the desired value by the value on the unlink and links with smaller number of crossings.

The existence proof can be found in Kauffman's original book [Ka2].

Just recently, by using the Kauffman approach of statistical sums, B. Bollobás, L. Pebody, and D. Weinreich [BPW] found a beautiful explicit formula for calculating the HOMFLY polynomial. This formula is even easier than that for the Kauffman polynomial in two variables.

Chapter 7

Properties of Jones polynomials. Khovanov's complex

7.1 Simplest properties

In this chapter we shall describe some properties of Jones polynomials and ways how this polynomial can be applied for solving some problems in knot theory.

After this, we shall formulate three celebrated conjectures concerning link diagrams which are related to properties of the Jones polynomial. Finally, we shall present a very sophisticated generalisation of the Jones polynomial, the Khovanov complex.

First, let us prove some properties of the Jones polynomial and deduce some corollaries from them.

Theorem 7.1. 1. The value of the Jones polynomial V(L) is invariant under orientation change for the link diagram.

2. The values of the Jones polynomial on mirror knots differ according to the variable change $q \to q^{-1}$.

Proof. The first statement is evident; it is left for the reader as a simple exercise (use induction on the number of crossings). The proof of the second statement also involves induction on the number of crossings. The induction basis is evident. Let us prove the induction step. To do it, let $L_+ = \langle X \rangle, L_- = \langle X \rangle, L_0 = \langle X \rangle$ be a Conway triple and L'_+, L'_-, L'_0 be the three diagrams obtained from the first ones by switching all crossings. Obviously, $\{L'_-, L'_+, L'_0\}$ is also a Conway triple.

By the induction hypothesis, we can assume that $V(L_0)$ is obtained from $V(L'_0)$ by the change of variables $q \to q^{-1}$. Let us now apply the relation (6.10). We get:

$$V(L_{+}) = q^{2}V(L_{-}) + q(q^{\frac{1}{2}} - q^{-\frac{1}{2}})V(L_{0}),$$

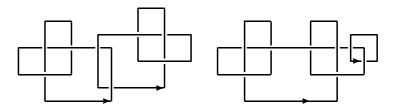


Figure 7.1. Two non-isotopic links with the same Jones polynomial

$$V(L'_{\perp}) = q^{-2}V(L'_{\perp}) - q^{-1}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})V(L'_{0}) = q^{-2}V(L'_{\perp}) + q^{-1}(q^{-\frac{1}{2}} - q^{\frac{1}{2}})V(L'_{0}).$$

Now we can see that if the statement of the theorem holds for L_0 and L_- , then it holds for L_+ as well. This statement holds for L_0 by the induction conjecture. Thus, if it holds for some diagram, then it holds for any diagram with the same shadow.

Since the claim of the theorem is true for each unlink (hence, for each diagram of the unlink with arbitrary shadow), we can conclude that it is true for each diagram with any given shadow. Taking into account that unlinks can have arbitrary shadows, we obtain the desired result.

Theorem 7.2. For arbitrary oriented links K_1 and K_2 the following holds:

$$V(K_1 \# K_2) = V(K_1) \cdot V(K_2).$$

Proof. We shall prove this fact for the Kauffman polynomial.

It suffices to note that the Kauffman bracket is multiplicative with respect to the connected sum operation and w(L) is additive.

We just note that the Kauffman bracket is multiplicative with respect to the connected sum operation (it follows from the definition of the bracket (6.4)) and w(L) is additive with respect to the connected sum.

Remark 7.1. This property is proved for any arbitrary connected sum of two links.

Analogously, one can prove the following.

Theorem 7.3. For arbitrary oriented knots K_1 and K_2 the following equality holds

$$V(K_1 \sqcup K_2) = -(q^{-\frac{1}{2}} + q^{\frac{1}{2}})V(K_1) \cdot V(K_2).$$

Example 7.1. Consider the two links shown in Fig. 7.1. According to Remark 7.1, the Jones polynomials for these links coincide. However, these links are not isotopic since their components are not so.

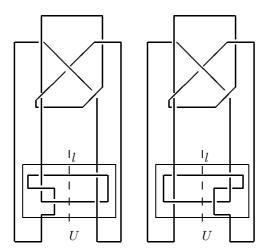


Figure 7.2. Left diagram L; right diagram L'

This example shows that the Jones polynomial does not always distinguish between different links. The reason is that the connected sum is not well defined. It turns out that the Jones polynomial is not a complete knot invariant. To show this, let us do the following.

Let L be a link diagram on the plane P. Suppose there exists a domain $U \subset P$ that is symmetric with respect to a line l in such a way that ∂U is a rectangle whose sides are parallel to coordinate axes. Suppose U intersects the edges of L only transversely at four points: two of theme lie on the upper side of the rectangle and the other two are on the lower sides. Suppose that these points are symmetric with respect to l. Let L' be the diagram that coincides with L outside U, and with reflection of L at l inside U, see Fig.7.2.

Lemma 7.1. In this case V(L) = V(L').

Proof. Reflecting the parts of the diagram at l, we do not change the signs $\varepsilon = \pm 1$ of the crossings. Thus w(L) = w(L'). We only have to show that the Kauffman bracket does not change either. With each state of L, we can naturally associate a state of L'. So, it remains to see that after smoothing all crossings of L and L' (according to corresponding states), the number of circles is the same. The latter is evident.

Example 7.2. It can be shown that the knots shown in Fig. 7.2 are not isotopic. Thus, the Jones polynomial is not a complete knot invariant.

Remark 7.2. Now it is an open problem whether the Jones polynomial distinguishes the unknot, i.e., is it true that V(K) = 1 implies the triviality of K, see the list of unsolved problems in Appendix B. This problem is equivalent to the faithfulness problem of the Burau representation for 4-strand braids (this was proved by Bigelow [Big2]). This problem will be formulated later.

The question of whether the Jones polynomial detects the n-component unlink in the class of n-component links was solved negatively; a series of examples are constructed by Elahou, Kauffman, and Thistletwaite in [EKT].

As we see, it is impossible to define whether a knot is prime, having only the information about the Jones polynomial of this knot.

Hence the Jones polynomial satisfies *skein relations* with only integer or half–integer powers of q; the Jones polynomial is indeed a polynomial in $q^{\pm \frac{1}{2}}$. Furthermore, the following theorem holds.

Theorem 7.4. (a) If an oriented link L has an odd number of components (e.g. it is a knot) then V(L) contains only integer degrees of q.

(b) If the number of components of L is even, then V(L) contains only summands like $q^{\frac{(2k-1)}{2}}, k \in \mathbf{Z}$.

Proof. First, let us note (this follows directly from the definition) that the Jones polynomial evaluated at the m-component unlink is equal to

$$(-q^{-\frac{1}{2}}-q^{\frac{1}{2}})^{m-1}$$
.

Consequently, the claim of the Theorem is true for unlinks.

Then, we can calculate the values of the Jones polynomial for arbitrary links by using the skein relation (6.10) for the Jones polynomial, knowing its values for unlinks.

We shall use induction on the number of crossings. The induction basis is evident. Suppose the induction hypothesis is true for less than n crossings. Let L be a diagram with n crossings. We can transform it to a diagram of the unlink by using the skein relation (6.10). Hence the statement is trivial for the unlink, but we have to check whether it remains true while switching crossing types.

Thus, given three diagrams L_+, L_-, L_0 (the first two having n vertices, the last one having n-1 vertices), the claim of the theorem is true for L_0 and for one of L_+, L_- . Without loss of generality, assume that this is L_- .

To complete the proof, we only have to see that:

- 1) The number of link components of L_+ and L_- coincide and have the same parity:
 - 2) The number of link components of L_{+} and L_{0} have different parity.

Exercise 7.1. Prove analogously that the value of the Conway polynomial on odd-component links (e.g. on knots) contains only members of even degree and that on even-component links contains only monomials of odd degree.

7.2 Tait's first conjecture and Kauffman–Murasugi's theorem

About 100 years ago, the famous English physicist and knot tabulator P.G. Tait formulated three very interesting conjectures. They had been unsolved for many years. Two of them were solved positively; the third one was solved negatively.

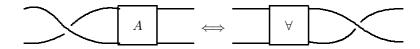


Figure 7.3. The flype move

 ${\bf Remark~7.3.~} \textit{All these conjectures are formulated for links with connected shadow.}$

Definition 7.1. The *length* of a (Laurent) polynomial P is the difference between its leading degree and lower degree.

Notation: Span(P).

Tait's first conjecture (1898) states the following. If a link L with connected shadow has an alternating n-crossing diagram \bar{L} without "splitting" points (i.e. points that split the diagram into two parts), then there is no diagram of L with less than n crossings.

This problem was solved independently by Murasugi [Mur1], Kauffman, and Thistletwaite in 1987.

Theorem 7.5 (The Kauffman–Murasugi theorem). The length of the Jones polynomial for a link with connected shadow is less than or equal to n. The equality holds only for alternating diagrams without splitting points and connected sums of them.

In Chapter 15, we shall give a proof of the Kauffman–Murasugi theorem based on the notion of the atom, and give some generalisations of it.

The first Tait conjecture follows immediately from the Kauffman–Murasugi theorem: if we assume the contrary (i.e., diagrams L, L' represent the same link, L is alternating with n crossings and without splitting points and L' has strictly less than n crossings), we obtain a contradiction when comparing the lengths of polynomials: spanV(L) = n > spanV(L').

7.3 Menasco–Thistletwaite theorem and the classification of alternating links

The Murasugi theorem was a great step in the classification of alternating links. The classification problem for alternating (prime) links is reduced to the case of links with the same number of vertices.

The final step was made by William Menasco and Morwen Thistletwaite, [MT], when they proved the second Tait conjecture. This conjecture (known as the Tait flyping conjecture) was stated a very long ago and solved only in 1993.

In the present section, we consider diagrams up to infinity change.

In fact, Menasco and Thistletwaite proved the following theorem.

Theorem 7.6. Any two diagrams of the same alternating knot can be obtained from each other by using a sequence of flypes, i.e., moves shown in Fig. 7.3.

Obviously, the Menasco-Thistletwaite theorem together with the Kauffman-Murasugi theorem gives a solution for the alternating link classification problem. The algorithm is the following. First, one can consider prime alternating diagrams not having splitting points. Now, consider two alternating link diagrams L, L' without splitting points. Let us see whether they have the same number of crossings. If not, they are not isotopic. If yes, suppose the number of crossings equals n. Then, try to apply all possible flype moves to L in order to obtain L'. We shall start within finite type because the number of link diagrams with n crossings is finite.

For more details, we refer the reader to the original work [MT], where he can find beautiful rigorous proofs based on some inductions and three–dimensional imagination.

7.4 The third Tait conjecture

Any knot whose minimal diagram (with respect to the number of crossings) is odd is not amphicheiral.

The conjecture runs as follows.

If a knot has a minimal **alternating** diagram (without break points) then its Jones polynomial cannot be symmetric with respect to $q^{\frac{1}{2}} \to q^{\frac{1}{2}}$. Thus, it is not amphicheiral. So, the counterexample should not be an alternating knot.

This conjecture was disproved by Thistletwaite in 1998, see [HTW].

7.5 A knot table

Below we give a table of "prime knots diagrams" with less then or equal to seven crossings (up to mirror symmetry). It turns out that the Jones polynomial distinguishes all these diagrams, so, we can construct the simplest table of prime knots. See Fig. 7.4.

7.6 Khovanov's categorification of the Jones polynomial

We are now going to present a very interesting generalisation of the Jones polynomial of one variable, due to Khovanov, see [Kh1, Kh2]. Our description is close to that of the article [BN4] by Bar–Natan. In this article, Bar–Natan gives a clear explanation of Khovanov's theory, calculates various example and shows that the homologies of the Khovanov complex constitute a strictly stronger invariant than the Jones polynomial itself.

Khovanov proposed the following idea: to generalise the notion of Kauffman's bracket using some formal complices and their cohomologies.

First, we give a slight modification of the Jones polynomial and Kauffman bracket due to Khovanov. The (unnormalised) Jones polynomial is the graded Euler characteristic of the Khovanov complex [Kh1]. The version of the Jones polynomial used here differs slightly from that proposed below. They become the same after a suitable variable change.

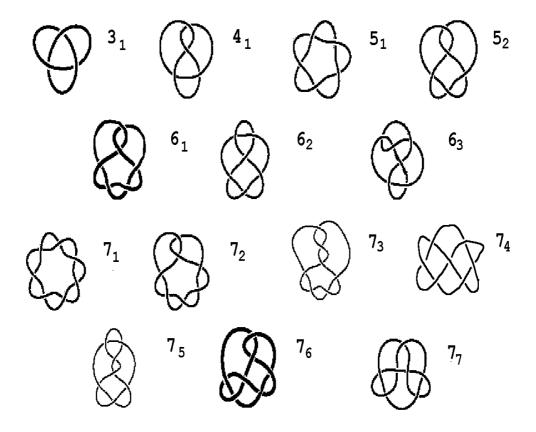


Figure 7.4. Prime knots with less than eight crossings

The axioms for the Kauffman bracket will be the following:

- 1. The Kauffman bracket of the empty set (zero-component link) equals 1.
- 2. $\langle L \sqcup \bigcirc \rangle = (q + q^{-1})\langle L \rangle$.
- 3. For any three diagrams $L=\langle \sum \rangle, L_A=\langle \sum \rangle, L_B=\langle \sum \rangle$ of unoriented links, we have

$$\langle L \rangle = \langle L_A \rangle - q \langle L_B \rangle.$$

Denote the state A of a vertex to be the 0-smoothing, and the state B to be the 1-smoothing. If the vertices are numbered then each way of smoothing for all crossings of the diagram is thought to be a vertex of the n-dimensional cube $\{0,1\}^{\{\mathcal{X}\}}$, where \mathcal{X} is the set of vertices of the diagram.

Let the diagram L have n_+ positive crossings and n_- negative crossings; denote the sum $n_+ + n_-$ by n (that is the total number of crossings).

Denote the unnormalised Jones polynomial by

$$\hat{J}(L) = (-1)^{n_-} q^{n_+ + 2n_-} \langle L \rangle.$$

Let the Jones polynomial (denoted now by J, according to [BN4]) be defined as follows:

$$J(L) = \frac{\hat{J}(L)}{q + q^{-1}}.$$

Thus,

$$J(L) = (-1)^{n_-} q^{n_+ - 2n_-} \sum_s (-q)^{\beta(s)} (q + q^{-1})^{\gamma(s) - 1}.$$

This normalised polynomial J differs from the Jones–Kauffman polynomial by a simple variable change: $a = \sqrt{(-q^{-1})}$. Namely,

$$\begin{split} &(-a)^{-3(n_+-n_-)} \sum_s a^{\alpha(s)-\beta(s)} (-a^2-a^{-2})^{\gamma(s)-1} \\ &= (-1)^n a^{-3(n_+-n_-)} \sum_s a^{-2\beta(s)+n} \cdot (q+q^{-1})^{\gamma(s)-1} \\ &= (-1)^n a^{4n_--2n_+} \sum_s (-q)^{\beta(s)} (q+q^{-1})^{\gamma(s)-1} \\ &= (-1)^{n_-} (-q)^{n_+-2n_-} \sum_s (-q)^{\beta(s)} (q+q^{-1})^{\gamma(s)-1}. \end{split}$$

Khovanov's categorification idea is to replace polynomials by graded vector spaces with some "graded dimension." This makes the Jones polynomial a homological object. On the other hand, the graded dimension allows us to consider the invariant to be constructed as a polynomial in two variables.

We shall construct a "Khovanov bracket" (unnormalised and not complex that plays the same role for the Khovanov complex as the Kauffman bracket for the Jones polynomial). This will be denoted by double square brackets.

Let us start with the basic definitions and introduce the notation (which will differ from that introduced above!)

Definition 7.2. Let $W = \bigoplus_m W_m$ be a graded vector space with homogeneous components W_m . Then the graded dimension of W is defined to be $\operatorname{qdim} W = \sum_m q^m dim W_m$.

Definition 7.3. Let "the degree shift" $\{l\}$ be the operation on the vector space $W = \bigoplus_m W_m$ defined as follows: $[W\{l\}]_m = W_{m-l}$. Thus, $qdimW\{l\} = q^l q dimW$.

Definition 7.4. Let "the height shift" be the operation on chain complices defined as follows. For a chain complex $\bar{C} = \cdots \to \bar{C}^r \xrightarrow{d^r} \bar{C}^{r+1} \ldots$ of (possibly graded) vector spaces (r is called here the "height" of \bar{C}^r), if $C = \bar{C}[s]$ then $C^r = \bar{C}^{r-s}$ (with all differentials shifted accordingly).

Let L, n and n_{\pm} be defined as before. Let \mathcal{X} be the set of all crossings of L. Let V be the graded vector space generated by two basis elements v_{\pm} of degrees ± 1 , respectively. Thus, qdim $V = q + q^{-1}$. With every vertex α of the cube $\{0, 1\}^{\{\mathcal{X}\}}$

we associate the graded vector space $V_{\alpha}(L) = V^{\otimes k}\{r\}$, where k (formerly γ) is the number of circles in the smoothing of L corresponding to α and r is the height $|\alpha| = \sum_i \alpha_i$ of α (so that $\operatorname{qdim} V_{\alpha}(L)$ is the polynomial that appears at the vertex α in the cube). Now, let the r-th chain group $[[L]]^r$ be the direct sum of all vector spaces at height r, that is $\bigoplus_{\alpha} r = |\alpha| V_{\alpha}(L)$.

Let us forget for a moment that [[L]] is not endowed with a differential, and hence, is not a complex. Set $\mathcal{C} := [[L]][-n_-]\{n_+ - 2n_-\}$.

Definition 7.5. The graded Euler characteristic of a graded complex C is the alternating sum of graded dimensions of its homology groups. Notation $\chi_q(L)$.

Remark 7.4. It is easy to show that for a complex C the graded dimension $\chi_q(C)$ equals the alternating sum of the graded dimensions of its chain groups. This is quite analogous to the case of the usual Euler characteristics.

Thus, we can calculate the graded Euler characteristic of C (taking into account only its graded chains); the differential will be introduced later.

Theorem 7.7. The graded Euler characteristic of C(L) is the unnormalised Jones polynomial \hat{J} of L.

Proof. This theorem is almost trivial. One should just take the alternating sum of graded dimensions of chain groups and mention that $\operatorname{qdim}(V^{\oplus n}) = n\operatorname{qdim}(V)$. The remaining part follows straightforwardly.

Now, let us prove that the Khovanov complex is indeed a complex. So, let us introduce the differentials for it. First, we set all $[[L]]^r$ to be the direct sums of the vector spaces appearing in the vertices of the cube with precisely r coordinates equal to 1.

The edges of the cube $\{0,1\}^{\mathcal{X}}$ can be labelled by sequences in $\{0,1,*\}$ of length n having precisely one *. This means that the edge connects two vertices, obtained from this sequence by replacing * with one or zero.

Definition 7.6. The height $|\xi|$ of the edge ξ is defined to be the height of its tail (the end having lower height).

Thus, if the maps for the edges are called d_{ξ} , then we get $d^r = \sum_{\{|\xi|=r\}} (-1)^{\xi} d_{\xi}$.

Now, we have to explain the sign $(-1)^{\xi}$ and to define the edge maps d_{ξ} . Indeed, in order to get a "good" differential operator d, such that $d \circ d = 0$, it suffices to show that all square faces of the cube anticommute.

This can be done in the following way. First, we make all faces commutative, and then we multiply each d_{ξ} by $(-1)^{\xi} = (-1)^{\sum_{i < j} \xi_i}$, where j is the position of * in ξ .

Exercise 7.2. Show that such coefficients really make any commutative cube skew-commutative.

Thus, we should find maps that can make our cube commutative.

Each edge represents some switch of the state for our diagram at some vertex. So, this means either dividing one cycle into two cycles, or joining two cycles together. In these cases, we shall use the comultiplication Δ and multiplication m maps defined as follows.

The map m:

$$\left\{ \begin{aligned} v_{+} \otimes v_{-} &\mapsto v_{-}, v_{+} \otimes v_{+} &\mapsto v_{+}, \\ v_{-} \otimes v_{+} &\mapsto v_{-}, v_{-} \otimes v_{-} &\mapsto 0 \end{aligned} \right.$$

The map Δ :

$$\begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- \mapsto v_- \otimes v_-. \end{cases}$$

Because of the degree shifts, our maps m and Δ are chosen to have degree (-1). Now, the only thing to check is that the faces of our cube for d_{ξ} (without ± 1 coefficients) commute. This follows from a routine verification.

The most interesting fact here is the invariance of *all homologies* of the Khovanov complex under all Reidemeister moves. Let us speak about this in more detail.

For a link diagram L, denote by Kh(L) the expression

$$\sum_{r} t^{r} \operatorname{qdim} \mathcal{H}^{r}(L).$$

Remark 7.5. When we wish to emphasize the field \mathbf{F} , we write $Kh_{\mathbf{F}}(L)$.

Theorem 7.8 (the main theorem). The graded dimensions of the homology groups $\mathcal{H}^r(L)$ are links invariants, hence Kh(L) is a link invariant polynomial (of the variables t, q) that gives the unnormalised Jones' polynomial being evaluated at t = -1.

Proof. We shall restrict ourselves only to three versions of the Reidemeister moves (one of Ω_1 , one of Ω_2 , and one of Ω_3). The other cases can be reduced to those we are going to consider.

In the case of the Kauffman bracket and the Jones polynomial, the invariance can be proved by reducing the Kauffman bracket of the "complicated case" of the move by using the rule $(\langle L \rangle = \langle L_A \rangle - q \langle L_B \rangle)$. Here we will do almost the same, but since we deal with complices and homologies rather than with polynomials, we must interpret it in another language. Namely, we are going to use the following "cancellation principle."

Let \mathcal{C} be a chain complex and let $\mathcal{C}' \subset \mathcal{C}$ be a subchain complex of \mathcal{C} . Then the following two statements hold.

Lemma 7.2 (Cancellation principle). 1. If C' is acyclic then H(C) = H(C/C').

2. If \mathcal{C}/\mathcal{C}' is acyclic (has no homology) then $H(\mathcal{C}) = H(\mathcal{C}')$.

Both statements follow straightforwardly from the following exact sequence:

$$\cdots \to H^r(\mathcal{C}') \to H^r(\mathcal{C}) \to H^r(\mathcal{C}/\mathcal{C}') \to \ldots$$

associated with the short exact sequence

$$0 \to \mathcal{C}' \to \mathcal{C} \to \mathcal{C}/\mathcal{C}' \to 0.$$

Now, let us prove the invariance of $Kh(\cdot)$ under the three Reidemeister moves.

Invariance under Ω_1 .

Consider the three diagrams (Q), (Q), and (Q).

While computing $\mathcal{H}(P)$, we encounter the complex

$$\mathcal{C} = [[\bigcirc{\bigcirc}]] = \left([[\bigcirc{\bigcirc}]] \stackrel{m}{\to} [[\bigcirc{\bigcirc}]]\{1\}\right).$$

This means that the total n-dimensional cube for \bigcirc is divided into two (n-1)-dimensional cubes, corresponding to two smoothed diagrams (one of them is shifted); the differentials between these two cubes are all represented via m by definition.

As we can easily see, all chains in \bigcirc where the small circle \circ is v_+ , "kill" all cycles in \bigcirc according to our differential, because v_+ plays the role of the unit element in V with respect to the multiplication m. Thus, the only homologies we can have lie in $[[\bigcirc]]$ when the small circle is marked by v_- . It is easy to see, that after the necessary normalisation, these homologies precisely coincide with those of $[[\bigcirc]]$.

The case of the other curl (2) can be considered analogously.

In the case of Ω_2 , we shall consider the only case. In this case, the $[(\mathcal{J})]$ will be represented in the terms of brackets of (\mathcal{J}) , (\mathcal{J}) , (\mathcal{J}) , (\mathcal{J}) and differentials between them:

$$\mathcal{C} = \begin{array}{ccc} [[(\bigcirc)]]\{1\} & \rightarrow & [[(\bigcirc)]]\{2\} \\ \uparrow & & m \uparrow \\ [[(\bigcirc)]] & \stackrel{\Delta}{\rightarrow} & [[(\bigcirc)]]\{1\} \end{array}$$

Thus, we have four cubes of codimension two and we know what the differentials in these small cubes look like: we may catch the cohomology elements in terms of these differentials. So, we only have to check whether they really represent cohomologies in the big cube.

The lower–left part of the diagram contains the diagram () (more precisely, all states corresponding to this local state).

Observation 1. It is easy to see that the members of this state cannot be cohomologies of the complex: their differentials have non-trivial projection to (\circ) {1}.

Observation 2. All members corresponding to $[[\mathcal{K}]{1}]$ are not boundaries of members corresponding to $[[\mathcal{K}]]$: the differential of each member from $[[\mathcal{K}]]$ also has an impact on $[[\mathcal{K}]]$.

Observation 3. The complex $[[(\circ)]]\{1\}_{v_+} \stackrel{m}{\to} [[(\circ)]]\{2\}$ is acyclic.

Observation 4. Each boundary element x in $[[(\underbrace{\bigvee})]]\{2\}$ coming from an element $z \in [[(\underbrace{\bigvee})]]\{1\}$ has a unique compensating element in $y \in [[(\underbrace{\bigvee})]]\{1\} \cup [[(\underbrace{\bigvee})]]\{2\}$ such that $\partial y = \partial z = x$. This follows from observation 3. Thus, there exists a y in this complex such that $\partial y = \partial x$.

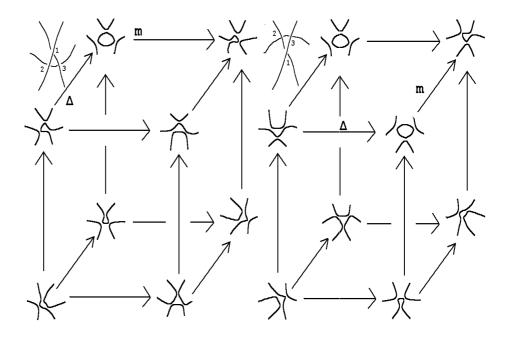


Figure 7.5. Behaviour of Khovanov's complex under Ω_3

Taking into account observations 2 and 4 we conclude that all cohomologies containing elements from \bigcirc are in one-to-one correspondence with homologies of the complex $C[[\bigcirc]]$.

It is easy to check that the complex \mathcal{C} has no other homologies (this follows from observations 1 and 3; the proof is left for the reader).

This results in the invariants of homologies up to height and degree shifts. Taking into account the normalisation constants, we obtain the invariance of the Khovanov complex under the second Reidemeister move Ω_2 .

The invariance proof for the other cases of Ω_2 is quite analogous to the case considered above. The direct calculation via Ω_2 does not work, thus we have to use the cancellation method described above.

In the case of the third Reidemeister move Ω_3 the situation is more difficult than the similar one for the case of the Kauffman polynomial.

In this case we have the following local pictures, see Fig. 7.5.

Let us recall the invariance proof for the Jones one–variable polynomial under Ω_3 . First, we smooth one crossing and then we see that this invariance follows from the invariance under Ω_2 . We are going to do something similar: we consider our three–dimensional cubes and take their top layers that differ by a move Ω_2 (bottom layers of these cubes coincide).

If we consider the situation that occurs while performing the move Ω_2 , we have the following complex.

The initial complex $\mathcal C$ looks like

This complex contains the subcomplex C' that looks as follows

$$\mathcal{C}' = \begin{array}{ccc} [[(\bigodot)]]_{v_+} \{1\} & \longrightarrow & [[(\bigodot)]] \{2\} \\ \uparrow & & m \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

The acyclicity of the complex C' is obvious.

After factorising the complex C by C', we obtain the complex

Now, if we consider the special case of the top layer shown in Fig. 7.5, we see that the complex C' contains a subcomplex

$$\mathcal{C}''' = \Delta \uparrow \qquad \begin{array}{c} \beta & \longrightarrow & 0 \\ \tau = d_{*0} \Delta^{-1} & \uparrow \\ \alpha & \stackrel{d_{*0}}{\longrightarrow} & \tau \beta, \end{array}$$

which is acyclic because Δ is an isomorphic map.

Remark 7.6. Here the arrow τ is not a differential. In the sequel, the diagonal arrow like $\beta = \tau \beta$ means that we identify two elements of the cube (arrows do not represent differentials).

After this, we see that

$$(\mathcal{C}/\mathcal{C}')/\mathcal{C}''' = \begin{array}{ccc} \beta & \longrightarrow & 0 \\ \uparrow & \searrow & \uparrow \\ 0 & \longrightarrow & \gamma. \end{array}$$

By the cancellation principle, we can perform this operation (factorising by C' and C''' defined for the top layers of the 3-cube) for the two cubes shown in Fig. 7.5 (only to the top layers of them). The resulting cubes are shown in Fig. 7.6.

Now, these two complices really are isomorphic via the map \mathfrak{Y} which keeps the bottom layers shown in Fig. 7.6. in their place and transposes the top layers by mapping the pair (β_1, γ_1) to the pair β_2, γ_2 .

The fact that $\mathfrak Y$ is really an isomorphism of spaces is obvious. To show that it is really an isomorphism of complices, we need to know that it commutes with the edge maps. In this case, only the vertical edges require a proof. The proof of this fact, namely that $\tau_1 \circ d_{1*01} = d_{2*01}$ and $d_{1*10} = \tau_2 \circ d_{2*10}$, is left to the reader as an exercise.

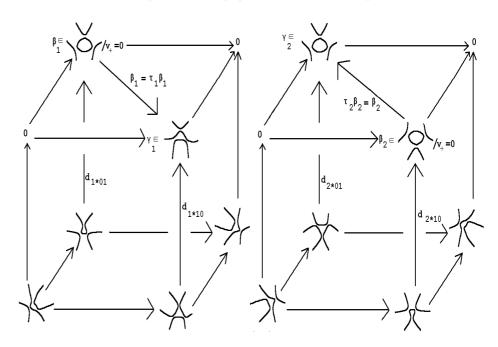


Figure 7.6. Invariance under Ω_3

As we have said before, the Khovanov polynomial (with rational homologies) is strictly stronger than the Jones polynomial. The example of two knots for which the Jones polynomial coincides and Khovanov's homologies do not, is shown in Fig. 7.7

Exercise 7.3. Perform the calculation check for this example.

7.6.1 The two phenomenological conjectures

Obviously, the Khovanov complex (respectively, invariant polynomial) can be considered over an arbitrary field. We are interested in the two cases: \mathbf{Q} and \mathbf{Z}_2 .

Notation: $Kh_{\mathbf{Q}}, Kh_{\mathbf{Z}_2}$

Below we give the two phenomenological conjectures from [BN4]. They belong to Bar–Natan, Khovanov, and Garoufalidis.

Conjecture 7.1. For any prime knot L there exist an even s = s(L) and a polynomial Kh'(L) in $t^{\pm 1}, q^{\pm 1}$ with only non-negative coefficients such that

$$Kh_{\mathbf{Q}}(L) = q^{s-1}(1+q^2+(1+tq^4)Kh'(L))$$

$$Kh_{\mathbf{Z}_2}(L) = q^{s-1}(1+q^2)(1+tq^2)Kh'(L).$$

Conjecture 7.2. For the case of a prime alternating knot L, the number s(L) equals the signature of L, and the polynomial Kh'(L) contains only powers of (tq^2) .

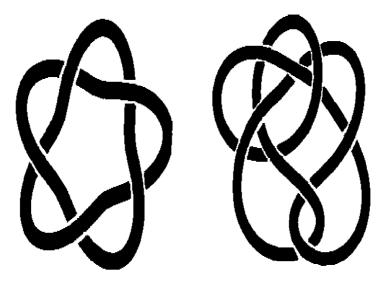


Figure 7.7. Khovanov's Q-homologies are stronger than the Jones polynomial

These two conjectures were checked by Bar–Natan for knots with a reasonably small number of crossings (seven for \mathbf{Q} and eleven for \mathbf{Z}_2).

It is easy to see that for the case of alternating prime knots, these two conjectures imply that the Khovanov polynomial is defined by the Jones polynomial.

A further phenomenological conjecture is presented in Garoufalidis' work [Ga]. All further information concerning these conjectures can be found in Bar–Natan's homepage [BNh].

The conjecture concerning alternating diagrams was solved positively by E.S. Lee, see [Lee].

It is worth mentioning that Khovanov's homologies are functorial. This magnificent result is due to Magnus Jacobsson, see [Jac].

We also recommend to read the paper by O.Ya. Viro [Viro] where a new "simple" approach to Khovanov's homologies is proposed.

Part II Theory of braids

Chapter 8

Braids, links and representations of braid groups

8.1 Four definitions of the braid group

In the previous chapters, we considered only one way of encoding links, namely, link planar diagrams.¹ In the present chapter, we are going to give an introduction to the theory of braids. On one hand, this theory gives us another point of view to knot theory. On the other hand, the theory of braids has some nice intrinsically interesting properties which are worth studying. Namely, the braid groups can be defined in many ways that lead to connections with different theories. Below, we are going to give some definitions of the braid groups and to discuss some properties of them.

8.1.1 Geometrical definition

Consider the lines $\{y=0, z=1\}$ and $\{y=0, z=0\}$ in \mathbf{R}^3 and choose m points on each of these lines having abscissas $1, \dots m$.

Definition 8.1. An m-strand braid is a set of m non-intersecting smooth paths connecting the chosen points on the first line with the points on the second line (in arbitrary order), such that the projection of each of these paths to Oz represents a diffeomorphism.

These smooth paths are called *strands* of the braid.

An example of a braid is shown in Fig. 8.1.

It is natural to consider braids up to isotopy in \mathbb{R}^3 .

Definition 8.2. Two braids B_0 and B_1 are *equal* if they are *isotopic*, i.e., if there exists a continuous family of braids B_t , $\{t \in \{0,1\}\}$ of braids starting at B_0 and finishing at B_1 .

 $^{^{1}}$ Later, we shall also use the d-diagrams mentioned in the Introduction.

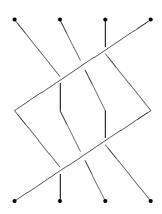


Figure 8.1. A braid

Definition 8.3. The set of all m-strand braids generates a group. The operation in this group is just juxtaposing one braid under the other and rescaling the z-coordinate.

The *unit element* or the *unity* of this group is the braid represented by all vertical parallel strands. The reverse element for a given braid is just its mirror image, see Fig. 8.3.

Exercise 8.1. Check that the group structure on the set of braids is well defined.

Definition 8.4. The Artin m-strand braid group² is the group of braids with the operation defined above.

Notation: Br(m).

One can consider only braids whose strands connect points with equal abscissas.

Definition 8.5. These braids are said to be *pure*. Pure braids form a subgroup of the braid group.

Notation: PB(m).

8.1.2 Topological definition

Definition 8.6. Given a topological space X. The unordered m-configuration space for X is the space (endowed with the natural topology) of all unordered sets of m pairwise different points of X.

Notation: B(X, m).

Analogously, one can define the m-ordered configuration space.

Notation: F(X, m). Now, let $X = \mathbf{R}^2 = \mathbf{C}^1$.

Definition 8.7. The m-strand braid group is defined to be isomorphic to the fundamental group $\pi_1(B(X, m))$.

²In fact, there are other braid groups called Brieskorn braid groups. They are closely connected with Coxeter–Dynkin diagrams and symmetries. For more details see [Bri1, Bri2].

Definition 8.8. The group $\pi_1(F(X,m))$ is called the pure m-strand braid group.

8.1.3 Algebro–geometrical definition

Consider the set of all polynomials of degree m in one complex variable z with leading coefficient equal to one.

Obviously, this set (together with its intrinsic topological structure) is isomorphic to \mathbb{C}^n : its coefficients can be considered as its complex coordinates.

Now, delete the space Σ_m of all polynomials that have multiple roots (at least one). We obtain the set $\mathbb{C}^m \setminus \Sigma_m$.

Definition 8.9. The *m*-strand braid group is the group $\pi_1(\mathbf{C}^m \setminus \Sigma_m)$.

8.1.4 Algebraic definition

Definition 8.10. The m-strand braid group is the group given by the presentation with (m-1) generators $\sigma_1, \ldots, \sigma_{m-1}$ and the following relations

$$\sigma_i \sigma_i = \sigma_i \sigma_i$$

for $|i-j| \geq 2$ and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

for $1 \le i \le m-2$.

These relations are called Artin's relations.

Definition 8.11. Words in the alphabet of σ 's and σ^{-1} 's will be referred to as braid words.

8.1.5 Equivalence of the four definitions

Theorem 8.1. The four definitions of the braid group Br(m) given above are equivalent.

Proof. The easiest part of the proof is to establish the equivalence of the topological and algebro–geometric definitions. Indeed, it is obvious that the two spaces, the space of polynomials of degree m without multiple roots with leading coefficient one and the unordered m-configuration space for \mathbb{C}^1 , are homeomorphic. Thus, their fundamental groups are isomorphic.

Let us now show the equivalence of the geometrical and topological definitions. As we know, the fundamental group does not depend on the choice of the initial point in the connected space. Thus, the initial point A of the unordered m-configuration space can be chosen as the set of integer points (1, 2, ..., m). Consider the space \mathbf{R}^3 as the product $\mathbf{C}^1 \times \mathbf{R}^1$.

With each closed loop, outgoing from A and lying in $B(\mathbf{C}^1, m)$, let us associate a set of lines in \mathbf{R}^3 as follows. Each of these (curvilinear) lines represents the motion of a point on the complex line \mathbf{C}^1 with respect to the time t, where t is the real coordinate, see Fig. 8.2.

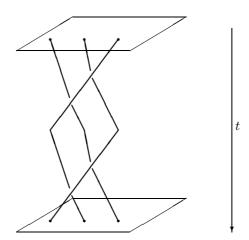


Figure 8.2. A braid in 3–space

Thus, with each geometric braid we have uniquely associated a topological braid. Obviously, with two isotopic (equal) geometric braids we associate the same topological braids.

So, it remains to show the equivalence of geometric and algebraic notions. In order to do this, let us introduce the notion of the *planar braid diagram*, analogous to the planar link diagram.

To see what this is, let us project a braid on the plane Oxz.

In the general case we obtain a diagram that can be described as follows.

Definition 8.12. A braid planar diagram (for the case of m strands) is a graph lying inside the rectangle $[1, m] \times [0, 1]$ endowed with the following structure and having the following properties:

- 1. Points [0, i] and [1, i], i = 1, ..., m are vertices of valency one, the other points of type [0, t] and [1, t] are not graph vertices.
- 2. All other graph vertices (crossings) have valency four; opposite edges at such vertices make angles π .
- 3. Unicursal curves, i.e., lines consisting of edges of the graph, passing from an edge to the opposite one, go from vertices with ordinate one and come to vertices with ordinate zero; they must be descending.
- 4. Each vertex of valency four is endowed with an overcrossing and undercrossing structure.

Analogously to the planar isotopty of link diagrams, one defines the planar isotopy of braid diagrams.

Obviously, all isotopy classes of geometrical braids can be represented by their planar diagrams. Moreover, after a small perturbation, all crossings of the braid can be set to have different ordinates.

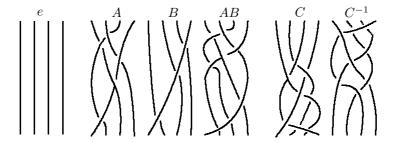


Figure 8.3. Unity. Operations in the braid group

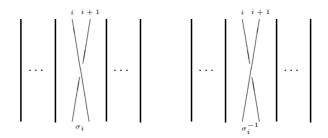


Figure 8.4. Generators of the braid group

It is easy to see that each element of the geometrical braid group can be decomposed into a product of the following generators σ_i 's: the element σ_i for $i=1,\ldots,m-1$ consists of m-2 intervals connecting [k,1] and $[k,0], k \neq i, k \neq i+1$, and two intervals [i,0]-[i+1,1], [i+1,0]-[i,1], where the latter goes over the first one, see Fig. 8.4.

Different braid diagrams can generate the same braid. Thus we obtain some relations in $\sigma_1, \ldots, \sigma_m$.

Let us suppose that we have two equal geometrical braids B_1 and B_2 . Let us represent the process of their isotopy in terms of their planar diagrams. Each interval of this isotopy either does not change the disposition of their vertex ordinates, or in this interval at least two crossings have (in a moment) the same ordinate; in the latter case the diagram becomes irregular.

We are interested in those moments where the algebraic description of our braid changes. We see that there are only three possible cases (all others can be reduced to these). In the first case (see Fig. 8.5.a) just one couple of crossings has the same ordinate. In the second case (see Fig. 8.5.b), two strands are tangent. In the third case (Fig. 8.5.c) we have a triple intersection point.

It is easy to see that the first case gives us the relation $\sigma_i \sigma_j = \sigma_j \sigma_i, |i-j| \ge 2$ (this relation is called far commutativity) or an equivalent relation $\sigma_i^{\pm 1} \sigma_j^{\pm 1} = \sigma_j^{\pm 1} \sigma_i^{\pm 1}, |i-j| \ge 2$, in the second case we get $aa^{-1} = 1$ (or $a^{-1}a = 1$), and in the third case we obtain one of the following three relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},$$

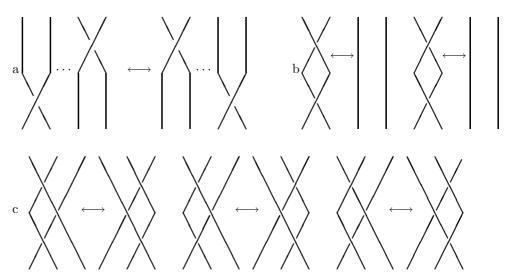


Figure 8.5. Intuitive expression of braid diagram isotopies

$$\sigma_i \sigma_{i+1} \sigma_i^{-1} = \sigma_{i+1}^{-1} \sigma_i \sigma_{i+1}, \quad \sigma_i^{-1} \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1}.$$

Obviously, each of the latter two relations can be reduced from the first one. This simple observation is left to the reader as an exercise. This completes the proof of the theorem. \Box

In the m-strand braid group one can naturally define the subgroup PB(m) of pure braids.

Exercise 8.2. Show that PB(m) is a normal subgroup in Br(m), and the quotient group Br(m)/PB(m) is isomorphic to the permutation group S(m).

8.1.6 The stable braid group

For natural numbers m < n, there exists the natural embedding $Br(m) \subset Br(n)$: a braid from Br(m) can be treated as a braid from Br(n) where the last (n-m) strands are vertical and unlinked (separated) with the others.

Definition 8.13. The *stable braid group* Br is the limit of groups Br(n) as $n \to \infty$ with respect to these embeddings.

The group Br has the representation with generators $\sigma_1, \sigma_2, \ldots$, and the following relations $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i-j| \geq 2$, $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.

8.1.7 Pure braids

With each braid one can associate its permutation: this permutation takes an element k to m if the strand starting with the k-th upper point finishes at the m-th lower point.

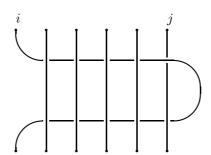


Figure 8.6. Generator b_{ij} of the pure braid group

Definition 8.14. A braid is said to be *pure* (cf. subsection 8.1.2) if its permutation is identical. Obviously, pure braids generate a subgroup $PB_n \subset B_n$.

There are other interpretations of PB_n . For instance, instead of the configuration space of unordered points of \mathbb{R}^2 , one can consider the configuration space of ordered points.

The fundamental group of this space is obviously isomorphic to PB_n .

An interesting problem is to find an explicit finite presentation of the pure braid group on n strands.

Here we shall present some concrete generators (according to [Art2]). A presentation of this group can be found in e.g., [Maka].

Pure n-strand braids correspond to loops in the space of **ordered** point sets on the plane. They generate a finite-index subgroup in the braid group.

There exists an algebraic *Reidemeister–Schreier method* that allows us to construct a presentation of a finite–index subgroup having a presentation of a finitely defined group, see e.g., [CF].

Here we give some generators of the pure braid group [Art1].

The following theorem holds.

Theorem 8.2. The group PB(m) is generated by braids

$$b_{ij}, 1 \le i < j \le n \tag{1}$$

(see Fig. 8.6).

To prove the theorem, we shall use induction on the number n of strands. For n=2 the statement is obvious: each 2-strand pure braid is some power of the braid $b_{12}=\sigma_1^2$.

Suppose that the statement is proved for some n. Consider some pure (n+1)-strand braid d_{n+1} . If we delete the first strand of it, we obtain some pure n-strand braid a_n . Now we can write $d_{n+1} = (d_{n+1}a_n^{-1})a_n$. By the induction hypothesis, the braid a_n can be decomposed in generators (1). The last n strands of $d_{n+1}a_n^{-1}$ are unlinked. Let us straighten them, i.e., make them vertical. In this case, the first strand is braided around them. Now, it is easy to see that this braid can be represented as a product of b_{ij} , b_{ij}^{-1} . This can be done as follows: every time when the first strand goes under some strand, it must be pulled back under all strands

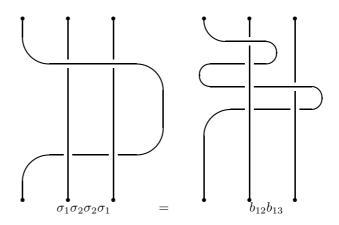


Figure 8.7. Decomposing a pure braid

till the left margin, and after that returned to the previous place under all strands, see Fig. 8.7.

Thus, the product d_{n+1} can be decomposed in b_{ij}, b_{ij}^{-1} .

Pure braid groups and mapping classes

Now, let us give another description of the pure braid group (see, e.g., [PS]).

Denote by H_n the group of isotopy classes of homeomorphisms of the n-punctured disc on itself, constant on the boundary. It turns out that the pure braid group is closely connected with H_n . First, let us consider H_0 .

Theorem 8.3 (Alexander's theorem on homeomorphism). The group H_0 is isomorphic to the unity group, i.e., each homeomorphism of the disc that is constant on the boundary is homotopic to the identity map; moreover, such a homotopy can be found among those constant on the boundary.

Proof. Consider the disc $|z| \leq 1$ in the complex space \mathbb{C}^1 . Let h_0 be the identity map and h_1 be an arbitrary homeomorphism of the disc onto itself that is constant on the boundary. According to a known theorem, h_1 has a fixed point inside the circle. Without loss of generality, let us assume that this point coincides with the centre of the circle. Now, let us construct our homotopy. For $t \in (0,1)$, let us construct the homeomorphism h_t as follows: h_t is identical inside the ring $t \leq |z| \leq 1$. Inside the disc $|z| \leq t$ we decree $h_t(z) = t(h_1(\frac{z}{t}))$. Obviously, h_t is a homotopy that satisfies the condition of the theorem.

Let us consider now the group H_n . Let $g \in H_n$ be a homeomorphism of the n-punctured disc that is identical on the boundary of the punctured disc, i.e., on the boundary of the disc and on the boundaries of the holes. This homeomorphism can be extended to the homeomorphism of the entire disc, since g can be extended

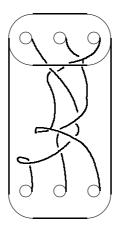


Figure 8.8. Thickened braid

from the boundary of any hole to the hole itself (this can be done, say, by mapping the interior of the hole identically). Denote the obtained homeomorphism by h_1 .

According to the previous theorem, there exists an isotopy h_t , connecting h_1 with $h_0 = id$. Fix the points x_1, \ldots, x_n inside the holes. For each $i = 1, \ldots, n$ the set $(t, h_t(x_0)), 0 \le t \le 1$ is an arc connecting the points on the upper and lower base of the cylinder $I \times D$, where I is the interval of time and D is the disc. In Fig. 8.8 the spurs of fixed points are shown.

Thus we have obtained a pure braid. This braid "knows" a lot about h, but not all. Actually, during the isotopy one can watch the moving of the fixed point of the circle. The circle turns around and its spur represents a cylinder. To describe h up to isotopy, it is sufficient to consider all cylinders and to mark the spurs of fixed points on their boundaries. The pure braid with this additional information is called a *thickened braid*.

Obviously, to each homeomorphism of the disc with n punctures that is constant on the boundary there corresponds some thickened braid.

The reverse statement is true as well. Let us show that with each thickened braid one can associate such a homeomorphism.

Without loss of generality, assume that the thickened braid does not leave the cylinder. The bases of the cylinder are discs with n circular holes.

Let us lower the circle with holes in such a way that the points of the interior boundary move vertically downwards parallel to the axis of the cylinder, and the disc always stays planar. The boundaries of these holes move downwards and twist following the point of the strand. When the circle reaches the lower base, we obtain the required homeomorphism.

Let M^2 be an orientable 2-manifold (with or without boundary) and γ be a closed curve lying inside M^2 . Consider a small neighbourhood U of the curve γ that is homeomorphic to $S^1 \times [0,1]$. Let γ_1 and γ_2 be the boundaries of this neighbourhood.

Definition 8.15. The *Dehn twisting* of a 2-surface M^2 along a curve γ is the homeomorphism of M^2 onto itself, which is constant outside U that is represented

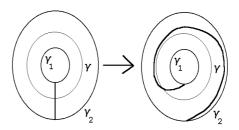


Figure 8.9. Dehn twisting

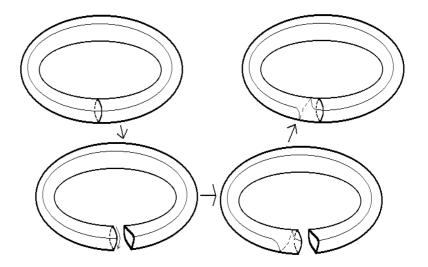


Figure 8.10. Twisting a torus along the meridian

by a full-turn twist of the curve γ_2 with curve γ_1 fixed inside U. It is shown in Fig. 8.9 how the Dehn twisting acts on curves connecting a point from γ_1 with a point from γ_2 . In this figure, we show the image of the straight line connecting two circles

A typical example of Dehn's twisting is the homeomorphism of a torus generated by twisting along the meridian, see Fig. 8.10.

Remark 8.1. The group of thickened braids is an extension of the pure braid group K_n . Moreover, it is easy to check that H_n is the direct sum of K_n and \mathbf{Z}^n (each group \mathbf{Z} corresponds to twistings along boundaries of holes).

Remark 8.2. Actually, the braid group can be considered as the mapping class group of the punctured disc. Namely, we consider all homeomorphisms of P_n onto itself which are constant on the boundary and then factorise these homeomorphismsms by homeomorphisms isotopic to the identity.

This approach is discussed in the book [Bir]. This allows us to construct various representations of braid groups, and to solve some problems.

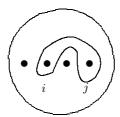


Figure 8.11. The loop around two punctures

To clarify the situation completely, we have to prove the following theorem.

Theorem 8.4. The group H_n is generated by twistings along a finite number of closed curves in the circle.

Proof. As it was proved before, the group H_n is isomorphic to the group of thickened braids. Suppose the homeomorphism $h \in H_n$ corresponds to thickened braid α'_n which is the sum $\alpha_n + \alpha$, where $\alpha_n \in K_n$, and $a \in \mathbf{Z}^n$. The pure braid α_n can be represented in generators b_{ij} that correspond to twisting along curves going once around points i and j, see Fig. 8.11

The thickened braid α is generated by the full-turn Dehn twisting along the curve going around the point a.

Summarising the facts described above, we obtain the claim of the theorem.

8.2 Links as braid closures

With each braid diagram, one can associate a planar knot (or link) diagram as follows.

Definition 8.16. The *closure* of a braid b is the link Cl(b) obtained from b by connecting the lower ends of the braid with the upper ends, see Fig. 8.12.

Obviously, isotopic braids generate isotopic links.

Remark 8.3. Closures of braids are usually taken to be oriented: all strands of the braid are oriented from the top to the bottom.

Some links generate knots, the others generate links. In order to calculate the number of components of the corresponding link, one should take into account the following simple observation. In fact, there exists a simple natural epimorphism from the braid group onto the permutation group $\Sigma : Br(n) \to S_n$, defined by $\sigma_i \to s_i$, where s_i are natural generators of the permutation group.

Consider a braid B. Obviously, for all numbers p belonging to the same orbit of the natural permutation action (of $\Sigma(B)$) on the set $1, \ldots, n$, all upper vertices with abscissas (p,0) belong to the same link component.

Consequently, we obtain the following proposition.

Proposition 8.1. The number of link components of the link of the closure Cl(B) equals the number of orbits of action for $\Sigma(B)$.

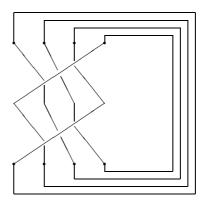


Figure 8.12. A braid closure

Exercise 8.3. Construct braids whose closures represent both trefoils and the figure eight knot.

Obviously, non-isotopic braids might generate isotopic links. We will touch on this question later.

An interesting question is to define the minimal number of strands of a braid whose closure represents the given link isotopy class L. Denote this number by Braid(L).

An interesting theorem on this theme belongs to Birman and Menasco.

Theorem 8.5. For any knots K_1 and K_2 , the following equation holds:

$$Braid(K_1 \# K_2) = Braid(K_1) + Braid(K_2) - 1.$$

In Fig. 8.13 we show that if the knot K_1 can be represented by an n-strand braid, and K_2 can be represented by an m-strand braid, than $K_1 \# K_2$ can be represented by an (n + m - 1)-strand braid. This proves the inequality " \leq ".

A systematic study of links via braid closures (including Markov's and Alexander's theorems, which will be discussed later) was done in the series of works by Birman [B1] and Birman and Menasco [BM2],[BM3],[BM4],[BM5],[BM6].

8.3 Braids and the Jones polynomial

First, let us formulate the celebrated Alexander and Markov theorems that we will use in this section.

Theorem 8.6 (Alexander's theorem, proof will be given later). For each link L, there exists a braid B such that Cl(B) = L.

Theorem 8.7 (Markov's theorem, proof will be given later). The closures of two braids β_1 and β_2 represent isotopic links if and only if β_1 can be transformed

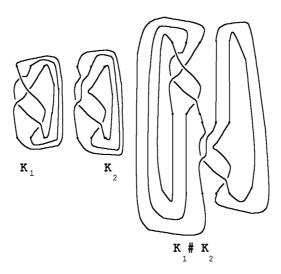


Figure 8.13. Representing the connected sum of braids

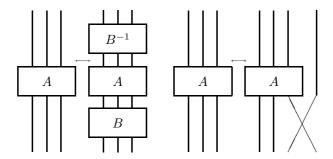


Figure 8.14. Markov's moves

to β_2 by using a sequence of two transformations (Markov's moves), shown in Fig. 8.14 (on the right, both types of the additional crossing are admissible).

The main idea for constructing the Jones two—variable polynomial is the following. One can consider some functions looking like representations of braid groups and investigate their properties. It turns out that some of these functions (so–called Ocneanu's trace) have a good behaviour under Markov's moves. This idea was given to Jones by Joan Birman and led to the beautiful discovery of the Jones polynomial.

First, let us note that if we add the relations $\sigma_i^2 = 1$ to the standard presentation of the braid group B_n , we obtain the permutation group S_n .

Definition 8.17. The *Hecke algebra* H(q, n) is the algebra generated by the following presentation:

$$\langle g_1, \dots, g_{n-1} | g_i^2 = (q-1)g_i + q,$$

 $i = 1, \dots, n-1, g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, i = 1, 2, \dots, n-2,$

$$g_i g_j = g_j g_i, |i - j| \ge 2. \tag{2}$$

Remark 8.4. Jones uses different (reverse) notation for the braid generators; namely, $\sigma_i = \bigotimes_i \text{ and } \sigma_i^{-1} = \bigotimes_i$.

Let us now formulate the theorem on Ocneanu's trace.

Theorem 8.8 (Ocneanu, [HOMFLY]). For each $z \in \mathbb{C}$ there exists a linear trace tr (that can be treated as a function in z,q) on H(q,n) uniquely defined by the following axioms:

1. tr(ab) = tr(ba); 2. tr(1) = 1; 3. $tr(xg_n) = ztr(x)$ for any $x \in H(q, n).$

Below, we present the proof of Jones [Jon1]. This proof differs slightly from the original Ocneanu proof and leads to the construction of the Jones polynomial.

Proof. The main idea of the proof is the following. If we add the new relations $g_i^2 = e$ to the braid group (with generators g_i instead of σ_i), we get exactly the permutation group. This group is finite and quite pleasant to work with.

Namely, all elements of the permutation group with generators p_1, \ldots, p_{n-1} , where p_i permutes the *i*-th and (i+1)-th elements, have a unique representation of the form:

$$\{(p_{i_1}p_{i_1-1}\dots p_{i_1-k_1})(p_{i_2}\dots p_{i_2-k_2})\dots (p_{i_j}\dots p_{i_j-k_j})\}$$

for some

$$1 \le i_1 < i_2 < \dots < i_p \le n - 1.$$

But, in the Hecke algebra we have another quadratic relation instead of $g_i^2 = 1$:

$$g_i^2 = (q-1)g_i + q. (3)$$

It turns out that this one is not worse than that of the symmetric group. Namely, each braid can be reduced to "basic" braids with some coefficients, for which we can easily define the Ocneanu trace.

This set of "basic" braids consists of just the same words as in the permutation group

$$\{(g_{i_1}g_{i_1-1}\dots g_{i_1-k_1})(g_{i_2}\dots g_{i_2-k_2})\dots (g_{i_p}\dots g_{i_p-k_p})\}\tag{4}$$

for

$$1 \le i_1 < i_2 < \dots < i_p \le n - 1.$$

It is sufficient to prove that for any word W of type (4) and for any generator g_i , the words Wg_i and Wg_i^{-1} can be represented as linear combinations of words of type (4). Actually, taking into account the relation $g_i^2 = (q-1)g_i + q$, it is sufficient to consider only the first case.

Now, suppose W is decomposed as a product $W_1 \cdots W_k$, where the g_i 's in each W_j have decreasing order, and the first letters of the W_i 's have increasing order (according to (4)). After this, the additional generator g_i can be "taken through" W_k . Then we take it through W_{k-1}, \ldots until the procedure stops.

For the sake of simplicity suppose the word W_k looks like

$$(g_{i_1}g_{i_1-1}\ldots g_{i_1-k_1}).$$

Then in the case $i = i_1 - k_1 - 1$, the new generator is just added to the word W_k , so there is nothing to prove.

If $i > i_1$, then the word Wg_i is already of type (4) because q_i can be treated as a new word W_{k+1} .

If $i < i_1 - k_1 - 1$, then we commute g_i with the word W_k and do not make any further changes with W_k : we shall work only with $W_1 \dots W_{k-1} g_i$.

In the case when $i = i_1 - k_1 - 1$, the situation is again simple: at the end we obtain g_i^2 which can be transformed into a linear combination of g_i and e. So, the whole word will be $(q-1)W + qW_1 \dots W_{k-1}W'_k$, where under W'_k we mean $g_{i_1} \dots g_{i_1-k_1-1}$ (if $k_1=0$, this word is empty).

In the case when $i_1-k_1-1 \le i \le i_1$, we have the following situation: we commute g_i with the last elements of W_k while possible, and then obtain the following subword in W_k : $g_ig_{i-1}g_i$, which equals $g_{i-1}g_ig_{i-1}$. Now, all letters in W_k before these subwords commute with g_{i-1} . So, we can take g_{i-1} to the left.

Thus, we have obtained a linear combination of words $W_1 \dots W_{k-1} g_{i'} W_k$ and $W_1 \dots W_{k-1} g_i W_k'$ for some i'. In all these cases i' is smaller than the index of the first letter in W_k .

The next step is just as the previous one: we take $g_{i'}$ through W_{k-1} . Then we perform the same with W_{k-2} and so on. The only thing we have to mention here is that the letter on the left side has index always smaller than the initial letter of the last passed word W_i .

Thus we see that the dimension of H(q,n) equals n! (the number of permutations). The proof of the fact that the algebra does not collapse at all is well established. It can be found, e.g. in [Bourb].

The construction above shows that for (n + 1)-strand braids it is sufficient to consider only those generators containing the generator g_n once. Now, we are ready to define Ocneanu's trace explicitly (by using the induction method on the number of strands) by means of the following initial formulae:

$$tr(1) = 1$$

and

$$tr(xg_ny) = z \cdot tr(xy)$$

for all $x, y \in H(q, n)$.

The main problem is to prove the property tr(ab) = tr(ba). By induction (on n), let us suppose that it is true for $x, y \in H(q, n)$.

Now, the only case that does not follow immediately from the definition is $tr(g_nxg_ny)=tr(xg_nyg_n)$. Namely, when we wish to show that for some element $A \in H(n+1)$ for any $B \in H(n+1)$ we have tr(AB)=tr(BA) it is sufficient to check it only for $B=g_n$ and for $B \in H_n$. The latter follows from the definition.

It suffices to prove this for the case when one multiplicator is g_n and the other one lies in H_{n-1} . Obvously, it is sufficient to consider the following three cases:

- 1. $x, y \in H_{n-1}$;
- 2. one of x, y lies in H_{n-1} , the other equals $ag_{n-1}b$;
- 3. $x = ag_{n-1}b, y = cg_{n-1}d$, where $a, b, c, d \in H_{n-1}$.

The first case is trivial because the generator g_n commutes with all elements from H_{n-1} .

In the second case (we only consider the case when $y \in H_{n-1}$, so y commutes with g_n ; the case $x \in H_{n-1}$ is completely analogous to the first one) we have:

$$tr(g_n a g_{n-1} b g_n y)$$

$$= tr(a g_n g_{n-1} g_n b y) = tr(a g_{n-1} g_n g_{n-1} b y)$$

$$= z \cdot tr(a g_{n-1}^2 b y)$$

$$= (q-1)z \cdot tr(a g_{n-1} b y) + qz \cdot tr(a b y)$$

and

$$tr(ag_{n-1}bg_nyg_n)$$

$$= tr(ag_{n-1}bg_n^2y) = (q-1)tr(ag_nbg_{n-1}y + qtr(ag_{n-1}by)$$

$$= z(q-1)tr(ag_{n-1}by) + qz \cdot tr(aby).$$

Finally, in the third case we have

$$tr(g_n a g_{n-1} b g_n c g_{n-1} d)$$

$$= tr(a g_n g_{n-1} g_n b c g_{n-1} d) = tr(a g_{n-1} g_n g_{n-1} b c g_{n-1} d)$$

$$= z \cdot tr(a g_{n-1}^2 b c g_{n-1} d) =$$

$$= z(q-1)tr(a g_{n-1} b c g_{n-1} d) + zq \cdot tr(a b c g_{n-1} d) =$$

$$z(q-1)tr(ag_{n-1}bcg_{n-1}d) + z^2q \cdot tr(abcd)$$

and

$$tr(ag_{n-1}bg_{n}cg_{n-1}dg_{n})$$

$$= tr(ag_{n-1}bcg_{n}g_{n-1}g_{n}d) = tr(ag_{n-1}bcg_{n-1}g_{n}g_{n-1}d)$$

$$= z \cdot tr(ag_{n-1}bcg_{n-1}^{2}d) =$$

$$= z(q-1)tr(ag_{n-1}bcg_{n-1}d) + zq \cdot tr(ag_{n-1}bcd) =$$

$$= z(q-1)tr(ag_{n-1}bcg_{n-1}d) + z^{2}q \cdot tr(abcd).$$

Thus, we have completed the induction step and defined correctly the Ocneanu trace.

This completes the proof of the theorem.

It follows directly from the proof that properties 1, 2, and 3 allow us to calculate the trace for any given element of H(q, n). Actually, first we transform this element to a combination of some "basic" braids. Then we use the formula $tr(xg_ny) = ztr(xy)$ and reduce our problem to the case of braids with a smaller number of strands. Then we just apply the induction method on the number of strands.

Exercise 8.4. Let us calculate $tr(g_1g_2g_3g_2)$. We have:

$$tr(g_1g_2g_3g_2)$$
= $z \cdot tr(g_1g_2^2) = z(q-1)tr(g_1g_2) + zq \cdot tr(g_1)$
= $z^3(q-1) + z^2q$.

Let us consider oriented links as braid closures. In order to construct a link invariant, one should check the behaviour of some function defined on braids under the two Markov moves.

The function tr is perfectly invariant under the first Markov move (conjugation). Besides, it behaves quite well under the second move. The only thing to do now is the normalisation.

Let us normalise all g_i 's in such a way that both types of the second Markov move affect the Ocneanu trace in the same way. To do this, let us introduce a variable Θ such that $tr(\Theta g_i) = tr((\Theta g_i)^{-1})$.

Simple calculations give us

$$\Theta^2 = \frac{\left(\frac{z - (q - 1)}{q}\right)}{z} = \frac{z - q + 1}{qz}.$$

Let us make a variable change. Namely, let $\lambda = \Theta^2$. Now we are ready to define the invariant polynomial.

Definition 8.18. The Jones two–variable polynomial $X_L(q,\lambda)$ of an oriented link L is defined by

$$X_L(q,\lambda) = \left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^{n-1} (\sqrt{\lambda})^e tr(\pi(a)),$$

where $\alpha \in B_n$ is any braid whose closure is L, e is the exponent sum of α as a word on the σ_i 's and π the presentation of B_n to $H_n : \pi(\sigma_i) = g_i$.

Notation. To denote the value of a polynomial on a link L, we put L in the lower index of the letter, denoting the polynomial. We do it for the sake of convenience because we are going to consider polynomials in some variables that will be put in brackets.

The invariance of X under conjugation is obvious (because of invariance of trace) and the invariance of X under the second Markov move follows straightforwardly from the properties of γ .

Theorem 8.9. For any Conway triple, the Jones two-variable polynomial satisfies the following skein relation:

$$\frac{1}{\sqrt{\lambda q}} \cdot X - \sqrt{\lambda q} X = \frac{q-1}{\sqrt{q}} X$$

Proof. Indeed, consider three diagrams that differ at one crossing: (X) has σ_i , (X) has σ_i^{-1} and (X) has no crossing at all.

So,
$$X = \sqrt{\lambda} tr(g_i)M, X = \frac{1}{\sqrt{\lambda}} tr(g_i^{-1})M, X = M$$
, where M is common for all three diagrams.

Then, writing down the Hecke algebra relation $g_i = (q-1) + qg_i^{-1}$, we obtain the desired result.

Let us prove now that the HOMFLY polynomial satisfies a certain skein relation. Indeed, let $t = \sqrt{\lambda}\sqrt{q}$, $x = (\sqrt{q} - \frac{1}{\sqrt{q}})$. Denote $X_L(q, \lambda)$ by $P_L(t, x)$. This is the famous HOMFLY polynomial [HOMFLY].

Theorem 8.10. The following skein relation holds:

$$t^{-1}P = xP$$

The proof is quite analogous to that described above.

8.4 Representations of the braid groups

8.4.1 The Burau representation

The most natural way to seek a representation of the braid group is the following. One considers the braid group $B_n (\equiv Br(n))$ and tries to represent it by matrices

 $n \times n$. More precisely, one takes σ_i to some block–diagonal matrix with one 2×2 block lying in two rows (i,i+1) and two columns (i,i+1) and all the other (1×1) unit submatrices lying on the diagonal. Obviously this implies the commutation between images of σ_i, σ_j when $|i-j| \geq 2$. If one takes all σ_i having the same (2×2) –block (in different positions), then we shall only have to check the relation $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$ for 3×3 –matrices. Thus we can easily obtain the representation where the block matrix looks like

$$\begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix}$$
.

This representation is called the *Burau* representation of the braid group. It was proposed by Burau, [Bur].

The faithfulness of this representation has been an open problem for a long time. In [Bir], Joan Birman proved that it was faithful for the case of three strands.

In [Moo91], Moody found the first example of a kernel element for this representation.

To date, the problem is solved positively for $n \leq 3$ and negatively for $n \geq 5$; see, e.g. [Big1]. The case n=4 still remains open. In [Big2], Stephen Bigelow shows that this problem is equivalent to the question whether the Jones polynomial detects the unknot, i.e. Bigelow proved the following theorem.

Theorem 8.11. [Big2] The Jones polynomial in one variable detects the unknot if and only if the Burau representation is faithful for n = 4.

Besides this, Bigelow thinks that faithfulness of the Burau representation in this case seems to be beyond the reach of any known computer algorithm.

Here we shall demonstrate the proof for the case n=3 and give a counterexample for n=5, following Bigelow [Big1, Big2].

First, let us consider the following description of the braid group and the Burau representation. Denote by D_n the unit complex disc D with n punctures x_1, \ldots, x_n on the real line. The set of all automorphisms of D_n considered up to isotopy is precisely the braid group B_n . Let d_0 (the initial point) be -i.

Now, the Burau representation can be treated as follows. For each braid $\beta \in B_n$ one can consider the number of full turns in β in the counterclockwise direction, or, equivalently, the number of generators in any braid—word representing β . Thus, there exists a homomorphism $h: B_n \to \mathbf{Z}$. So, one can construct a cover $\tilde{B_n} \to B_n$ with the action of \mathbf{Z} on it. This group has a generator $\langle q \rangle$. Let us consider the group $H_1(\tilde{B_n})$. According to the arguments above, this group admits a module structure over $\mathbf{Z}[q,q^{-1}]$. Let $\tilde{d_0}$ be a pre-image of d_0 under h.

The Burau representation is just obtained from module homomorphisms. More precisely, let $\bar{\beta}$ be a homomorphism representing the braid $\beta \in B_n$. The induced action of $\bar{\beta}$ on $\pi_1(D_n)$ satisfies $h \circ \beta = h$. Thus, there exists a unique lift $\hat{\beta}$ of β such that the following diagram is commutative:

$$(\tilde{D_n}, \tilde{d_0}) \stackrel{\hat{\beta}}{\to} (\tilde{D_n}, \tilde{d_0})$$

$$\downarrow \qquad \qquad \downarrow$$

$$(D_n, d_0) \stackrel{\beta}{\to} (D_n, d_0).$$

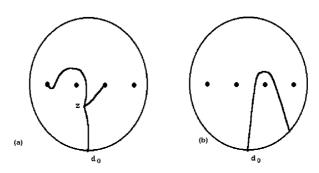


Figure 8.15. (a) A fork;

(b) a noodle.

Furthermore, $\hat{\beta}$ commutes with the action of q. Thus, $\hat{\beta}$ induces a $\mathbf{Z}[q^{\pm 1}]$ -module homomorphism denoted by $\hat{\beta}_*$. Thus, we can define the Burau representation just as

$$Burau(\beta) = \hat{\beta}_*.$$

It is not difficult to check that this definition of the Burau representation coincides with the initial one.

Now let us prove that this representation is faithful in the case of three strands. First, let us introduce the notation of [Big2].

Definition 8.19. A fork is an embedded tree F in D with four vertices d_0, p_i, p_j, z such that:

- 1. the only puncture points of F are p_i, p_j ;
- 2. F meets ∂D_n only at d_0 ;
- 3. all three edges of F have z as a vertex.

Definition 8.20. The edge of F containing d_0 is called the *handle* of F. The union of the other two edges forms one edge; let us call it the *tine edge* of F and denote it by T(F). Let us orient T(F) in such a way that the handle of F lies to the right of T(F).

Definition 8.21. A noodle is an embedded oriented edge N in D_n such that

- 1. N goes from d_0 to another point on ∂D_n ;
- 2. N meets ∂D_n only at endpoints;
- 3. a component $D_n \setminus N$ contains precisely one puncture point.

Let F and N be a fork and a noodle, respectively. Let us define a pairing $\langle N, F \rangle$ in $\mathbf{Z}[q^{\pm 1}]$ as follows. Without loss of generality, let us assume that T(F) intersects N transversely. Let z_1, \ldots, z_k be the intersection points between T(F) and N (with

no order chosen). For each i = 1, ..., k, let γ_i be the arc in D_n which goes from d_0 to z_i along F and then goes back to d_0 along N. Let a_i be the integer such that $h(\gamma_i) = q^{a_i}$. Let ε_i be the sign of the intersection between N and F at z_i . Let

$$\langle N, F \rangle = \sum_{i=1}^{k} \varepsilon_i q^{a_i}. \tag{5}$$

One can easily check that this pairing is independent of the preliminary isotopy (which allows us to assume the transversality of T(F) and N. Besides, this follows from the basic lemma.

The faithfulness proof follows from the two lemmas.

Lemma 8.1 (The basic lemma). Let $\beta: D_n \to D_n$ represent an element of the kernel of the Burau representation. Then $\langle N, F \rangle = \langle N, \beta(F) \rangle$ for any noodle N and fork F.

Lemma 8.2 (The key lemma). In the case n = 3, the equality $\langle N, F \rangle = 0$ holds if and only if T(F) is isotopic to an arc which is disjoint from N.

Let us now deduce the faithfulness for the case of three strands from these two lemmas. Suppose β lies in the kernel of the Burau representation. We shall show that β represents the trivial braid.

Let N be a noodle. Take N to be a horizontal line through D_n such that the puncture points p_1 and p_2 lie above N and p_3 lies below N. Let F be a fork such that T(F) is a straight line from p_1 to p_2 which does not intersect N. Then $\langle N, F \rangle = 0$. By the basic lemma we have $\langle N, \beta(F) \rangle = 0$. By the key lemma, $\beta(T(F))$ is isotopic to an arc which is disjoint from N. By applying isotopy to β , we can assume that $\beta(T(F)) = T(F)$.

Analogously, one can prove that each of the three edges of the triangle connecting p_1, p_2, p_3 is fixed by β . The only possibility for β to be non-trivial is to represent some whole twists of D, but one can easily check that this is not the case. Thus, the Burau representation is faithful for the case of three strands.

For those who are interested in details, we give the proofs of these two lemmas.

Proof of the basic lemma. The main idea is the following. We transform the definition (5) in such a way that it works with loops rather than with forks. Then the invariance for loops follows straightforwardly because our transformation lies in the kernel.

We can assume that the tine edges of both T and $\beta(T)$ intersect N transversely.

Let \tilde{F} be the lift of F to $\tilde{D_n}$ with respect to h, containing $\tilde{d_0}$. Let $\tilde{T}(F)$ be the corresponding lift of T(F). Then $\tilde{T}(F)$ intersects $q^a\tilde{N}$ transversely for any $a \in \mathbf{Z}$. Let $(q^a\tilde{N}, \tilde{T}(F))$ denote the algebraic intersection number of these two arcs. Then the following definition of pairing is equal to (5):

$$\langle N, F \rangle = \sum_{a \in \mathbf{Z}} (q^a \tilde{N}, \tilde{T}(F)) q^a.$$
 (6)

Suppose now that T(F) goes from p_i to p_j . Let $\nu(p_i)$ and $\nu(p_j)$ be small disjoint regular neighbourhoods of p_i and p_j , respectively. Let γ be a subarc of T(F), which starts in $\nu(p_i)$ and ends in $\nu(p_j)$. Let δ_i be a loop in $\nu(p_i)$ with base point $\gamma(0)$, which goes counterclockwise around p_i . Denote by $T_2(F)$ the following loop:

$$T_2(F) = \gamma \delta_j \gamma^{-1} \delta_j^{-1}.$$

Let $\tilde{T}_2(F)$ be the lift of $T_2(F)$ which is equal to $(1-q)\tilde{T}(F)$ outside a small neigbourhood of the punctures. Then the following definition (7) is equivalent to (5) and (6).

$$\langle N, F \rangle = \frac{1}{1 - q} \sum_{a \in \mathbf{Z}} (q^a \tilde{N}, \tilde{T}_2(F)) q^a. \tag{7}$$

Because β lies in the kernel of the Burau representation, the loops $\tilde{T}_2(F)$ and $\tilde{T}_2(\beta(F))$ represent the same element of the homology group $H_1(\tilde{D}_n)$. Thus, they have the same algebraic intersection number with any lift $q^a\tilde{N}$ of N. Thus, the claim of the lemma follows from (7).

Proof of the key lemma. The main idea is to find the fork whose tine edge intersects N at the minimal number of points and to show that if this number is not zero then $\langle N, F \rangle$ cannot be equal to zero.

First, let us recall that by definition (5): $\langle N, F \rangle = \sum_{i=1}^k \varepsilon_i q^{a_i}$.

By applying a homeomorphism to our picture, we can take N to be a horizontal straight line through D_3 with two punctures above it and one puncture point below it (the fork will therefore be twisted). Here we slightly change our convention that the punctures lie on the real line by a small deformation of $D_n \subset D$. Let D_n^+, D_n^- denote the upper and lower components of $D_n \setminus N$, respectively. Let us relabel puncture points in such a way that D_n^+ contains p_1, p_2 and D_n^- contains p_3 . Now, let us consider the intersection of T(F) with D_n^- . It consists of a disjoint collection of arcs having both endpoints on N (possibly, one arc can have p_3 as an endpoint). An arc $T(F) \cap D_n^-$ which has both endpoints on N must enclose p_3 ; otherwise one could remove it together with some intersection points. Thus $T(F) \cap D_n^-$ must consist of a collection of parallel arcs enclosing p_3 , and, possibly, one arc with p_3 as an endpoint.

Similarly, each arc in $T(F) \cap D_n^+$ either encloses one of p_1, p_2 or has an endpoint at p_1 or p_2 . There can be no arc in $T(F) \cap D_n^+$ which encloses both points; otherwise the outermost such arc together with the outermost arc of the lower part will form a closed loop.

Now, we are going to calculate carefully the intersection number and see that all summands evaluated at q = -1 have the same sign.

Indeed, let z_i and z_j be two points of intersection between T(F) and N which are joined by an arc in $T(F) \cap D_n^+$ or $T(F) \cap D_n^-$. This arc, together with a subarc of N, encloses a puncture point. Thus, $a_j = a_i \pm 1$. Besides, the two signs of intersection are opposite: $\varepsilon_j = -\varepsilon_i$. So, $\varepsilon_j(-1)^{a_j} = \varepsilon_i(-1)^{a_i}$. Arguing as above, we prove that all summands for $\langle N, F \rangle$ evaluated at q = -1 have the same sign. Thus, $\langle N, F \rangle$ is not equal to zero.

8.4.2 A counterexample

In fact, the proof of faithfulness does not work in the case of n > 3. Roughly speaking, the problem is that the key lemma uses the intrinsic properties of the three–strand braid groups.

The idea of the example is the following. Each curve α with endpoints at punctures generates an automorphism $\tilde{\tau}_{\alpha}$ of $H_1(D_n)$. This automorphism is generated by Dehn's half-turn twist about α (permutating the endpoints).

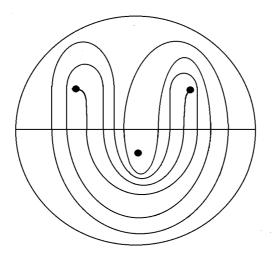


Figure 8.16. A tine edge and a noodle in D_3

A direct calculation shows that if $\langle \alpha, \beta \rangle = 0$ then $\tilde{\tau}(\alpha)$ and $\tilde{\tau}(\beta)$ commute.

To show that the Burau representation is not faithful it suffices to provide an example of oriented embedded arcs $\alpha, \beta \in D_n$ with endpoints at punctures such that $\alpha, \beta = 0$, but the corresponding braids τ_{α} and τ_{β} do not commute.

For $n \geq 6$, the simplest known example is the following. We set $\phi_1 = \sigma_1^2 \sigma_2^{-1} \sigma_5^{-2} \sigma_4$ and $\phi_2 = \sigma_1^{-1} \sigma_2 \sigma_5 \sigma_4^{-1}$. Take γ to be the simplest arc connecting x_3 with x_4 (so that the corresponding braid is σ_3).

After this, a straightforward check shows that for $\alpha = \phi_1(\gamma)$, $\beta = \phi_2(\gamma)$ so that the corresponding braids are $\tau_{\alpha} = \phi_1 \sigma_3 \phi_1^{-1}$ and $\tau_{\beta} = \phi_2 \sigma_3 \phi_2^{-1}$, and we have $\langle \alpha, \beta \rangle = 0$.

It follows from a straightforward check that $\langle \alpha, \beta \rangle = 0$. So, the braid $\tau_{\alpha} \tau_{\beta} \tau_{\alpha}^{-1} \tau_{\beta}^{-1}$ belongs to the kernel of the Burau representation. To prove that this braid is not trivial, one can use an algorithm for braid recognition, say, one of those described later in the book.

This element has length 44 in the standard generators of the group Br(5). In fact, an example of a kernel element for the Burau representation for five strands can be constructed by using similar (but a bit more complicated) techniques. This element has length 120.

8.5 The Krammer–Bigelow representation

While developing the idea that the Burau representation comes from some covering, Bigelow proposed a more sophisticated covering that leads to another representation. Bigelow proved its faithfulness by using the techniques of forks and noodles. We begin with the formal definition according to Krammer's work [Kra1]. After that, we shall describe the main features of Bigelow's work [Big1].

8.5.1 Krammer's explicit formulae

Let n be a natural number and let R be a commutative ring with the unit element. Suppose that $q, t \in R$ are two invertible elements of this ring. Let V be the linear space over R of dimension $\frac{n(n-1)}{2}$ generated by elements $x_{i,j}, 1 \le i < j \le n$. Let us define the action of the braid group Br(n) on the space V according to

Let us define the action of the braid group Br(n) on the space V according to the following rule:

$$\sigma_{k}(x_{i,j}) = \begin{cases}
x_{i,j} & k < i - 1 \text{ or } k > j; \\
x_{i-1,j} + (1-q)x_{i,j} & k = i - 1; \\
tq(q-1)x_{i,i+1} + qx_{i+1,j} & k = i < j - 1; \\
tq^{2}x_{i,j} & k = i = j - 1; \\
x_{i,j} + tq^{k-i}(q-1)^{2}x_{k,k+1} & i < k < j - 1; \\
x_{i,j-1} + tq^{j-i}(q-1)x_{j-1,j} & i < k = j - 1; \\
(1-q)x_{i,j} + qx_{i,j+1} & k = j;
\end{cases} (8)$$

where $\sigma_k, k = 1, \dots, (n-1)$, are generators of the braid group

It can be clearly checked that these formulae give us a representation of the braid group.

Denote the Krammer-Bigelow representation space of the braid group Br(n) by L_n . Because the basis of L_n is a part of the basis for L_{n+1} , we have $L_n \subset L_{n+1}$.

Formula (8) implies that L_n is an invariant space under the action of the representation (8) of Br(n+1) in L_{n+1} .

While passing from matrices of the representation for Br(n) to those for Br(n+1), the upper–left block stabilises. Thus, one may speak of the infinite–dimensional linear representation of the stable braid group.

We shall not give the (algebraic) proof of the faithfulness because it involves a lot of sophisticated constructions. For the details, see the original work by Krammer [Kra2].

8.5.2 Bigelow's construction and main ideas of the proof

Here we shall describe Bigelow's results following [Big1, Tur2]. There he constructs a more sophisticated covering than that corresponding to the Burau representation. This covering gives a faithful representation that coincides with Krammer's.

In fact, the Bigelow representation deals with **the braid group itself as the** mapping class group rather than with any presentation of it. One should point out the work of Lawrence [Law] where he extended the idea of the Burau representation via coverings for the configuration spaces in D_n , and was able to obtain all of the so-called Temperley-Lieb representations. Just the representations of Lawrence were shown to be faithful (by Krammer and by Bigelow).

The Bigelow proof of the faithfulness is based on the ideas used in the proof of faithfulness of the Burau representation for three strands. Namely, we have the following three steps.

- 1. The basic lemma.
- 2. The key lemma.
- 3. Deducing the faithfulness from these lemmas.

Below, we shall modify the scalar product for curves defined above by using a more sophisticated bundle over a four-dimensional space together with action on its 2-homologies. As we have proved above, the basic lemma (new version) works for all n even for the Burau representation. For the remaining two steps we shall use the modified covering and scalar product.

Let D, D_n be as above, the punctured points are $-1 < x_1 \cdots < x_n < 1$ which we shall puncture.

Let C be the space of all unordered pairs of points in D_n . This space is obtained from $D_n \times D_n \setminus diagonal$ by the identification $\{x,y\} = \{y,x\}$ for any distinct points $x,y \in D_n$. It is clear that C is a connected non-compact four-dimensional manifold with boundary. It is endowed with a natural orientation induced by the counter-clockwise orientation of D_n . Let d = -i and let $d' = -ie^{\frac{\varepsilon \pi i}{2}}$ for small positive ε . We take $c_0 = \{d, d'\}$ as the base point of C.

A closed curve $\alpha:[0,1]\to C$ can be written in the form $\{\alpha_1(s),\alpha_2(s)\}$, where $s\in[0,1]$ and α_1,α_2 are arcs in D_n such that $\{\alpha_1(0),\alpha_2(0)\}=\{\alpha_1(1),\alpha_2(1)\}$. In this case, the arcs α_1,α_2 are either both loops or can be composed with each other. Thus, they form a closed oriented 1-manifold α in D_n . Let $a(\alpha)\in \mathbf{Z}$ be the total winding number of this manifold α around all the punctures of D_n .

Consider the map $s:[0,1]\to S^1$ given by $s\mapsto \frac{\alpha_1(s)-\alpha_2(s)}{|\alpha_1(s)-\alpha_2(s)|}$ and the natural projection $S^1\to \mathbf{R}P^1$. Thus, we obtain a loop in $\mathbf{R}P^1$. The corresponding element of $H_1(\mathbf{R}P^1)$ is denoted by $b(\alpha)$. The formula $\alpha\to q^{a(\alpha)}t^{b(\alpha)}$ thus defines a homomorphism ϕ from $H_1(C)$ to the free commutative group generated by q,t. Let $R=\mathbf{Z}[q^{\pm 1},t^{\pm 1}]$ be the group ring of this group.

Let $\tilde{C} \to C$ be the regular covering corresponding to the kernel of ϕ . The generators q,t act on \tilde{C} as commuting covering transformations. The homology group $H_2(\tilde{C}, \mathbf{Z})$ thus becomes an R-module.

Any homeomorphism h of D_n onto itself induces a homeomorphism $C \to C$ by $h(\{x,y\}) = \{h(x), h(y)\}$ (we preserve the notation h).

It is easy to check that $h(c_0) = c_0$ and the action of j on the homologies $H_1(C)$ commutes with ϕ . Thus, the homeomorphism $h: C \to C$ can be lifted uniquely to a map $\tilde{C} \to \tilde{C}$ that fixes the fibre over c_0 pointwise and commutes with the covering transformation. Consider the representation $B_n \to Aut(H_2(\tilde{C}))$ mapping the isotopy class of h to the R-linear automorphism \tilde{h}_* of $H_2(C)$.

Theorem 8.12 ([Big1]). The representation $B_n \to Aut(H_2(\tilde{C}) \text{ is faithful for all } n \geq 1.$

The proof of this fact uses the techniques of noodles and forks.

We kindly ask the reader to be patient while reading all definitions.

For arcs $\alpha, \beta : [0,1] \to D_n$ such that for all $s \in [0,1]$ $\alpha(s) \neq \beta(s)$ denote by $\{\alpha(s), \beta(s)\}$ the arc in C given by $\{\alpha, \beta\}(s) = \{\alpha(s), \beta(s)\}$. We fix a point $\tilde{c}_0 \in \tilde{C}$ lying over $c_0 \in C$.

We should be slightly more precise about the notion of a noodle. Namely, by a noodle we mean an embedded arc $N \subset D_n$ with endpoints d and d'. For any noodle N, the set $\Sigma_N = \{\{x,y\} \in C | x,y \in N, x \neq y\}$ is a surface in C containing c_0 . This surface is homeomorphic to a triangle with one edge removed. We orient N from d to d' and orient Σ_N as follows: at a point $\{x,y\} = \{y,x\} \in \Sigma_N$ such that x is

closer to d than y along N, the orientation of Σ_N is the product of orientations of N at x and at y in this order.

Let $\tilde{\Sigma}_N$ be the lift of Σ_N to \tilde{C} containing the point c_0 . The orientation of Σ_N naturally induces an orientation for $\tilde{\Sigma}_N$. Obviously, $\tilde{\Sigma}_N \cap \partial \tilde{C} = \partial \tilde{\Sigma}_N$

Having a fork $F = T \cup H$ with endpoints $d = d_0, x_i, x_j$ and vertex z, we can push it slightly (fixing x_i and x_j and moving d to d') and obtain a parallel copy F' with tine edges T' and handle H'. Denote $T' \cap H'$ by z'. We can assume that F' intersects F only in common vertices $\{x_i, x_j\}$ and in one more point $H \cap T'$ lying close to z and z'.

We shall use $\tilde{\Sigma}_N$ and $\tilde{\Sigma}_F$ to establish the duality between N and D. Without loss of generality we can assume that N intersects the tine edge T of F transversely at m points z_1, \ldots, z_m . We choose the parallel fork $F' = T' \cup H'$ as above in such a way that T' intersects N transversely in m points z'_1, \ldots, z'_m , where each pair z_i, z'_i is joined by a short arc in N that lies in the narrow strip bounded by $T \cup T'$ and meets no other z_i, z_j . Then, the surfaces Σ_F, Σ_N intersect transversely in m^2 points $\{z_i, z'_j\}$ for $i, j = 1, \ldots, m$. Thus, for any $a, b \in \mathbf{Z}$, the image $q^a t^b \tilde{\Sigma}_N$ under the covering transformation meets $\tilde{\Sigma}_F$ transversely.

Now, we are ready to define the pairing.

Definition 8.22. Consider the algebraic intersection number $q^a t^b \tilde{\Sigma}_N \cdot \tilde{\Sigma}_F \in \mathbf{Z}$ and set

$$\langle N, F \rangle = \sum_{a,b \in \mathbf{Z}} (q^a t^b \tilde{\Sigma}_N \cdot \tilde{\Sigma}_F) q^a t^b.$$
 (9)

Now, we shall highlight the main ideas of proof of the main theorem.

For every pair $i, j \in \{1, 2, ..., m\}$ there exist unique integers $a_{i,j}, b_{i,j} \in \mathbf{Z}$ such that $q^{a_{i,j}} t^{b_{i,j}} \tilde{\Sigma}_N$ intersects $\tilde{\Sigma}_F$ at a point lying over $\{z_i, z_j\} \in C$.

- 1. First, one should check the invariance of pairing under homotopy. This follows from a routine verification.
- 2. It can be observed that the sign $\varepsilon_{i,j}$ of the intersection point $\{i,j\}$ is given by the formula $\varepsilon_{i,j} = -(-1)^{b_{i,i}+b_{j,j}+b_{i,i}}$,.
- 3. The key lemma

Lemma 8.3. If a homeomorphism h of D_n onto itself is an element of the kernel of $B_n \to Aut(H_2(\tilde{C}))$ then for any noodle N and any fork F, we have $\langle N, h(F) \rangle = \langle N, F \rangle$.

The proof follows from a routine verification by rewriting the definition of pairing.

4. The basic lemma:

Lemma 8.4. $\langle N, F \rangle = 0$ if and only if the tine edge T can be isotoped off N.

For the Burau representation we used the fact that we only have three points and thus we obtained that for a minimal state all intersections have the same sign for q = -1. Here we have another argument: two-dimensionality.

So, suppose $\langle N,F\rangle=0$. Assume that the intersection between N and T cannot be removed and consider the minimal intersection. Let us use the lexicographic ordering of monomials $q^at^b>q^ct^d$ if a>c or a=c,b>d. The ordered pair $\{i,j\}$ is maximal if $q^{a_{i,j}}t^{b_{i,j}}\geq q^{a_{k,l}}t^{b_{k,l}}$ for any $k,l\in\{1,\ldots,m\}$. Then the following statement holds.

- 5. If the pair (i, j) is maximal then $b_{i,i} = b_{j,j} = b_{j,i}$. Thus, all entries of the maximal monomial, say, $q^a t^b$ in (9) have the same sign $-(-1)^b$. Thus $\langle N, F \rangle \neq 0$.
- 6. From the key lemma and the basic lemma we see that each element of the kernel of the representation preserves the lines connecting punctures. Thus, the only possibility for this kernel element is to be some power of the half–turn Dehn twist. However, such a twist is a multiplication by $q^{2n}t^2$.

So, we have constructed a faithful representation of the braid group with polynomials in two variables as coefficients. It follows from classical number theorems that there exist a pair of real numbers transforming this representation to a faithful representation with **real coefficients**. Thus, all braid groups are linear.

Chapter 9

Braids and links. Braid construction algorithms

In the present chapter, we shall present two algorithms for constructing a braid B by a given link L such that Cl(B) = L and thus prove Alexander's theorem.

9.1 Alexander's theorem

Throughout the present section, we shall work only with oriented links. For any given unoriented link, we choose an arbitrary orientation of it.

Exercise 9.1. Construct a braid whose closure represents the Borromean rings.

The main statement of this chapter is Alexander's theorem that each link can be obtained as a braid closure. We shall give two proofs of this theorem: the original one by Alexander and the one by Vogel that realises a faster algorithm for constructing a corresponding braid.

Theorem 9.1. (Alexander's theorem [Ale3]) Each link can be represented as the closure of a braid.

Proof. We shall prove this theorem for the case of polygonal links.

Consider a diagram L of an oriented polygonal link and a point O on the plane P of the diagram (this point should not belong to edges and should not coincide with vertices of the diagram). We say that L is braided around O if each edge of L is visible from O as counterclockwise—oriented.

Definition 9.1. For any L and O, let us call edges visible as counterclockwise-oriented *positive*; the other ones will be *negative*.

If there exists a point O such that our link diagram is braided around O, then the statement of Alexander's theorem becomes quite clear: we just cut the diagram along a ray coming from O and "straighten the diagram," see Fig. 9.1.

Thus, in order to prove the general case of the theorem, we shall reconstruct our arbitrary link diagram in order to obtain a diagram braided around some point O.

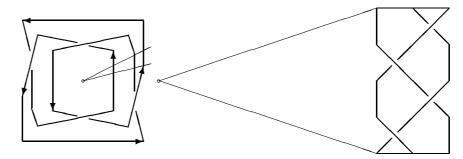


Figure 9.1. Constructing a braid by a braided link

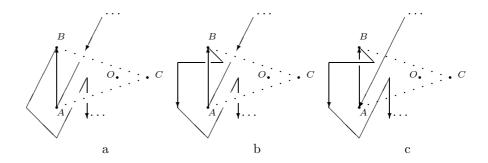


Figure 9.2. Alexander's trick

First, fix a point O. Now, we are going to use the Alexander trick as follows. Consider a negative edge AB of our polygonal link and find some point C on the projection plane P such that the triangle ABC contains O. Then we replace AB by AC and BC. Both edges will evidently be positive, see Fig. 9.2.

We shall use this operation till we get a diagram braided around O.

Let us describe this construction in more detail. In the case when the negative edge AB contains no crossings, the Alexander trick can be easily performed directly, see Fig. 9.2.a. Actually, on can divide the edge AB into two parts (edges) and then push them over O.

The same can be done in the case when AB contains the only crossing that is an overcrossing with respect to the other edge, see Fig. 9.2.b.

Finally, if AB contains the only crossing that is an undercrossing with respect to the other edge, then we can push it under, as shown in Fig. 9.2.c.

Exercise 9.2. By using the Alexander trick, construct a braid whose closure represents the connected sum of the two right trefoil knots.

The method of proof for Alexander's theorem described above certainly gives us a concrete algorithm for constructing a braid from a link. However, this algorithm is too slow. Below, we give a simpler algorithm for constructing braids by links.

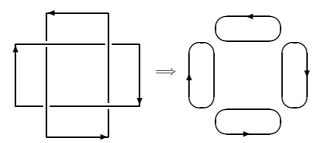


Figure 9.3. Smoothing of crossings and Seifert circles

9.2 Vogel's algorithm

Here we describe the algorithm proposed by Pierre Vogel [Vogel].

We start with a definition.

Definition 9.2. First we say that an oriented link diagram is *braided* if there exists a point on the plane of the diagram for which the link diagram is braided around.

A braided link diagram can be easily represented as a closure of a braid.

Remark 9.1. Obviously, the property of a diagram to be braided does not depend on the crossing structure. We may say that we shall work only with shadows of links. In the sequel, we shall never use this structure.

Given an oriented closed diagram D of a link L, one can correctly define the operation of *crossing smoothing* for it. To do it, we just "smooth" the diagram at each vertex as shown in Fig. 9.3 and consider all Seifert circles of it. Denote this smoothing by σ .

Definition 9.3. Let us say that all Seifert circles of some planar diagram are *nested* if they all induce the same orientation of the plane and bound an enclosed disc system.

Obviously, if all Seifert circles of some planar link diagram are nested, then the corresponding diagram is braided. Moreover, in this case, the number of strands of the braid coincides with the number of Seifert circles.

Let us fix some link diagram D and consider now the shadow of D. This shadow divides the sphere (one–point compactification of the plane) into 2–cells, called *sides*. The *interior* side is that containing infinity.

Definition 9.4. A side S is unordered if it has two edges a, b that belong to different Seifert circles A_1, A_2 and induce the same orientation of S, and ordered otherwise

In the first case we say that the edges A_1, A_2 generate the unordered side.

One can apply the move Ω_2 to unordered sides, as shown in Fig. 9.4. In this case, the set of sides becomes "more ordered." More precisely, the following proposition holds.

Proposition 9.1. If all edges of the side Σ belong to two Seifert circles then this side is ordered.

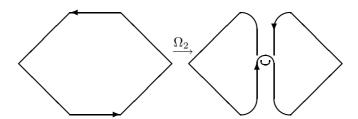


Figure 9.4. Performing the second Reidemeister move

Proof. Actually, consider the edges of this side. It is easy to see that all edges belonging to the same Seifert circle have the same orientation. Consider two adjacent edges belonging to different Seifert circles. They have different orientations. Thus, any two edges of the given diagram belonging to different Seifert circles must have different orientations. Hence, the side is ordered.

Proposition 9.2. If a diagram D of the link L has no unordered sides, then it can be transformed to a braided diagram by using an infinity change.

Proof. Suppose the diagram D has no unordered sides. Consider some side of the planar tiling generated by our link diagram. Any two adjacent edges of this side either have the same orientation (in this case, they belong to the same Seifert circle) or they have different orientations (and belong to different Seifert circles). If we consider the points of adjacent edges belonging to the same Seifert circle as the points of one "long" edge then we obtain some polygon M (or a whole Seifert circle). The edge orientations of the edges of M are alternating. Thus, the number of such edges is even (or equal to one when all edges belong to the same Seifert circle). Since this side is not unordered, all edges of it belong to no more than two different Seifert circles. Thus we conclude that each Seifert circle that defines some edge of the polygon M is adjacent either to one Seifert circle or to two Seifert circles (lying on different sides of M). Otherwise, there would be an unordered side with edges belonging to three different Seifert circles.

The remaining part of the proof is left to the reader as a simple exercise.

The Vogel algorithm works as follows. First we eliminate all crossings by the rule: $(\Sigma) \to (\Sigma)$, $(\Sigma) \to (\Sigma)$. Then, by using Ω_2 we remove unordered sides. Finally, if Seifert circles are not nested, we change the infinity.

Let us describe this algorithm in more detail.

First, let us smooth all crossings of the diagram. Thus we obtain several Seifert circles. Denote the number of these surfaces by s. Some pairs of these circles might generate unordered sides. Let us construct a graph whose vertices are Seifert circles; two vertices should be connected by an edge if there exists a side (ordered or not) that is incident to the two circles. Let us remove from this graph a vertex, corresponding to some "interior" Seifert circle. We obtain some graph Γ_1 . Let us change the notation for the remaining s-1 circles and denote them by $a_1, a_2, \ldots, a_{s-1}$, in

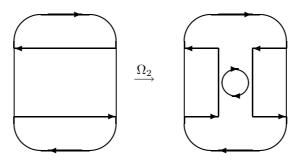


Figure 9.5. Eliminating an unordered side

such a way that a_i and a_{i+1} contain edges that generate an unordered side. This means that our graph Γ_1 is connected. It is easy to see that in the disconnected case we should apply this algorithm to each connected component; it will work even faster.

We shall perform the following operation. Let us take the unordered side, generated by A_1 and A_2 , and perform Ω_2 to it as described above. Instead of circles A_1 and A_2 , we shall get two Seifert circles; one of them lies inside the other. Besides this, they do not generate an unordered side, see Fig. 9.5.

We have got two new circles; one of them lies inside the other. Denote the exterior circle by A_1 and the interior one by A_2 . Because "the former A_2 " generated an unordered side together with A_3 then the new circle A_1 also generates an unordered side together with A_3 (the latter stays the same).

Let us now perform Ω_2 on the circles A_1 and A_3 and change the notation again: the exterior circle will be A_1 and the interior one will be A_3 , and so on. Finally (after s-2 operations Ω_2), we obtain one interior circle A_1 that makes no unordered sides. Now, we shall not touch A_1 , but perform the same procedure with the pairs (A_2, A_3) , (A_2, A_4) , and so on. Then we do the same for $A_3, A_i, i > 3$, and so forth. Thus, we have performed $\frac{(s-1)(s-2)}{2}$ second Reidemeister moves and (possibly) one infinity change and obtained the set of circles $A_1, A_2, \ldots, A_{s-1}$, where each next circle lies inside all previous ones and no two circles generate an unordered side.

Let us show that the remaining circle (that we "removed" in the very beginning) does not make unordered sides either.

Actually, since this circle has some exterior edge it could generate an unordered side only with A_1 , but this is not the case. After this, we should change the infinity (if necessary). Thus, after C_{s-2}^2 operations (for the connected case; in the unconnected case we shall use even less operations) we obtain a braided diagram.

So, we have proved the following.

Theorem 9.2. If the link diagram D has n crossings and s Seifert circles then

- 1. The Vogel algorithm requires no more than C_{s-2}^2 second Reidemeister moves.
- 2. The total number of strands of the obtained braid equals s and the number of crossings does not exceed n + (s 1)(s 2).

Below, we perform the Vogel algorithm for the knot named 5_2 according to the standard classification, given in the end of the book.

To do it, we perform the second Reidemeister move Ω_2 twice and then the infinity change, see Fig. 9.2. Thus, the two moves Ω_2 would be sufficient.

Finally, we get a braided diagram, see the lower part of Fig. 9.2.

Thus we can now construct braids corresponding to given links even faster than by using the Alexander trick.

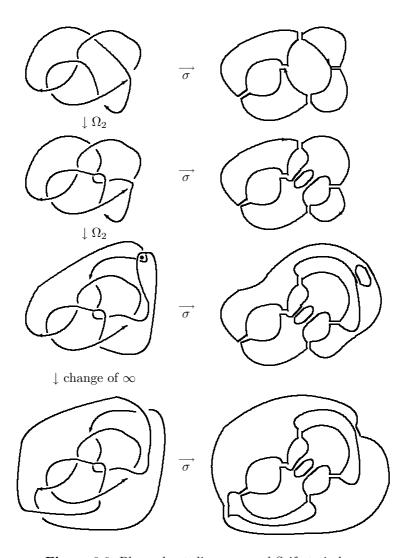


Figure 9.6. Planar knot diagrams and Seifert circles

Chapter 10

Algorithms of braid recognition

Until now, several braid recognition algorithms have been constructed. The first of them was contained in the original work of Artin [Art1] (and, in more detail, [Art2]). However, the approach proposed by Artin was not very clearly explained; both articles are quite difficult to read. There were other works on braid recognition (by Birman [Bir], Garside [Gar], Thurston, et al.) Here we are going to describe a geometrically explicit algorithm (proving the completeness of a slightly modified Artin invariant according to [GM], see also [BZ]) and the algebraic algorithm by Dehornoy.

In [Gar], Garside proposed a method of normal forms; by using this method, he solved not only the word problem for the braid groups, but the conjugation problem as well; the conjugation problem is in fact more complicated. Unfortunately, we do not present here any solution of the conjugation problem. In this chapter, we shall also present a result by M. Berger concerning the minimal braid—word in Br(3) representing the given braid isotopy class.

10.1 The curve algorithm for braids recognition

Below, we shall give a proof of the completeness of one concrete invariant for the braid group elements invented by Artin, see [GM].

10.1.1 Introduction

We are going to describe the construction of the above mentioned invariant for the classical braid group Br(n) for arbitrary n.

The invariant to be constructed has a simple algebraic description as a map (non-homeomorphic) from the braid group Br(n) to the n copies of the free group with n generators.

Several generalisations of this invariant, such as the spherical and cylindrical braid group invariant, are also complete. They will be described later in this chapter.

The key point of such a completeness is that these invariants originate from several curves, and the braid can be uniquely restored from these curves.

10.1.2 Construction of the invariant

Let us begin with the definition of notions that we are going to use, and let us introduce the notation.

Definition 10.1. By an admissible system of n curves we mean a family of n non-intersecting non-self-intersecting curves in the upper half plane $\{y>0\}$ of the plane Oxy such that each curve connects a point having ordinate zero with a point having ordinate one and the abscissas of all curve ends are integers from 1 to n. All points (i,1), where $i=1,\ldots,n$, are called upper points, and all points $(i,0), i=1,\ldots,n$, are called lower points.

Definition 10.2. Two admissible systems of n curves A and A' are equivalent if there exists a homotopy between A and A' in the class of curves with fixed endpoints lying in the upper half plane such that no interior point of any curve can coincide with any upper or lower point during the homotopy.

Analogously, the equivalence is defined for one curve (possibly, self-intersecting) with fixed upper and lower points: during the homotopy in the upper half plane no interior point of the curve can coincide with an upper or lower point.

In the sequel, admissible systems will be considered up to equivalence.

Remark 10.1. Note that curves may intersect during the homotopy.

Remark 10.2. In the sequel, the number of strands of a braid equals n, unless otherwise specified.

Let β be a braid diagram on the plane, connecting the set of lower points $\{(1,0),\ldots,(n,0)\}$ with the set of upper points $\{(1,1),\ldots,(n,1)\}$. Consider the upper crossing C of the diagram β and push the lower branch along the upper braid to the upper point of it as shown in Figure 10.1.

Naturally, this move spoils the braid diagram: the result, shown in Figure 10.1.b is not a braid diagram. The advantage of this "diagram" is that we have a smaller number of crossings.

Now, let us do the same with the next crossing. Namely, let us push the lower branch along the upper branch to the end. If the upper branch is deformed during the first move, we push the lower branch along the deformed branch (see Fig. 10.2).

Reiterating this procedure for all crossings (until the lowest one), we get an admissible system of curves. Denote its equivalence class by $f(\beta)$.

Theorem 10.1. The function f is a braid invariant, i.e., for two diagrams β, β' of the same braid we have $f(\beta) = f(\beta')$.

Proof. Having two braid diagrams, we can write the corresponding braid—words, and denote them by the same letters β, β' . We must prove that the admissible system of curves is invariant under braid isotopies.

The invariance under the commutation relations $\sigma_i \sigma_j = \sigma_j \sigma_i, |i-j| \geq 2$, is obvious: the order of pushing two "far" branches does not change the result. The invariants under $\sigma_i \sigma_i^{-1} = e$ can be readily checked, see Fig. 10.3.

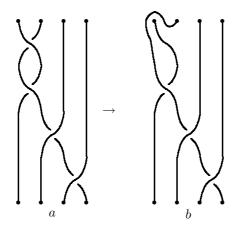


Figure 10.1. Pushing the upper crossing

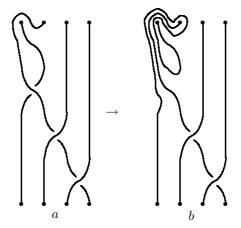


Figure 10.2. Pushing the next crossing

In the leftmost part of Fig. 10.3, the dotted line indicates the arbitrary behaviour for the upper part of the braid diagram. The rightmost part of Fig. 10.3 corresponds to the system of curves without $\sigma_i \sigma_i^{-1}$.

Finally, the invariance under the transformation $\sigma_i \sigma_{i+1} \sigma_i \to \sigma_{i+1} \sigma_i \sigma_{i+1}$ is shown in Fig. 10.4. In the upper part (over the horizontal line) we demonstrate the behaviour of $f(A\sigma_i\sigma_{i+1}\sigma_i)$, and in the lower part we show that of $f(A\sigma_{i+1}\sigma_i\sigma_{i+1})$ for an arbitrary braid A. In the middle–upper part, part of the curve is shown by a dotted line. By removing it, we get the upper–right picture which is just the same that the lower–right picture.

The behaviour of the diagram in the upper part A of the braid diagram is arbitrary. For the sake of simplicity it is pictured by three straight lines.

Thus we have proved that $f(A\sigma_i\sigma_{i+1}\sigma_i) = f(A\sigma_{i+1}\sigma_i\sigma_{i+1})$.

This completes the proof of the theorem.

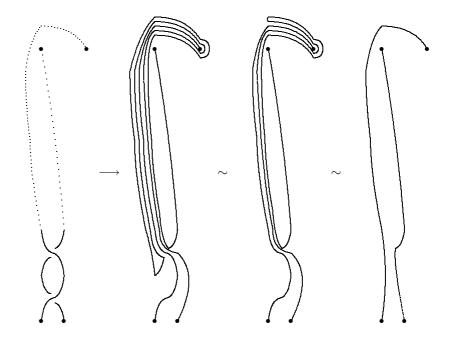


Figure 10.3. Invariance of f under the second Reidemeister move

Now, let us prove the following lemma.

Lemma 10.1. If for two braids a and b we have f(a) = f(b) then for each braid c we obtain f(ac) = f(bc).

Proof. The claim f(ac) = f(bc) follows directly from the construction. Indeed, we just need to attach the braid c to the admissible system of curves corresponding to a (or b) and then to push the crossings of c.

In fact, a much stronger statement holds.

Theorem 10.2 (The main theorem). The function f is a complete invariant.

To prove this statement, we shall use some auxiliary definitions and lemmas. In order to prove the main theorem, we should be able to restore the braid from its admissible system of curves.

In the sequel, we shall deal with braids whose end points are (i,0,0) and (j,1,1) with all strands coming upwards with respect to the third projection coordinates. They obviously correspond to standard braids with upper points (j,0,1). This correspondence is obtained by moving neighbourhoods of upper points along Oy.

Consider a braid b and consider the plane $P = \{y = z\}$ in Oxyz. Let us place b in a small neighbourhood of P in such a way that its strands connect points (i,0,0) and $(j,1,1), i,j=1,\ldots,n$. Both projections of this braid on Oxy and Oxz are braid diagrams. Denote the braid diagram on Oxy by β .

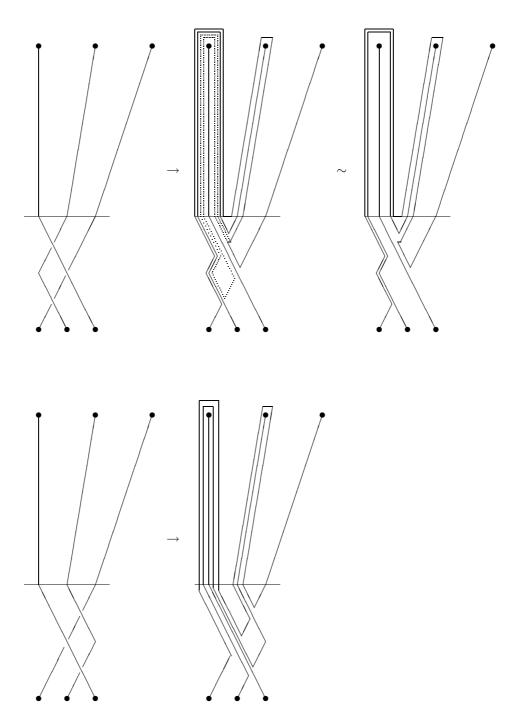


Figure 10.4. Invariance of f under the third Reidemeister move

The next step now is to transform the projection on Oxy without changing the braid isotopy type; we shall just deform the braid in a small neighbourhood of a plane parallel to Oxy.

It turns out that one can change abscissas and ordinates of some intervals of strands of b in such a way that the projection of the transformed braid on Oxy constitutes an admissible system of curves for β .

Indeed, since the braid lies in a small neighbourhood of P, each crossing on Oxz corresponds to a crossing on Oxz. Thus, the procedure of pushing a branch along another branch in the plane parallel to Oxy deletes a crossing on Oxy, preserving that on Oxz.

Thus, we have described the geometric meaning of the invariant f.

Definition 10.3. By an admissible parametrisation (in the sequel, all parametrisations are thought to be smooth) of an admissible system of curves we mean a set of parametrisations for all curves by parameters t_1, \ldots, t_n such that at the upper points all t_i are equal to one, and at the lower points t_i are equal to zero.

Any admissible system A of n curves with admissible parametrisation T generates a braid representative: each curve on the plane becomes a braid strand when we consider its parametrisation as the third coordinate. The corresponding braid has end points (i, 0, 0) and (j, 1, 1), where $i, j = 1, \ldots, n$. Denote it by g(A, T).

Lemma 10.2. The result g(A,T) does not depend on T.

Proof. Indeed, let us consider two admissible parametrisations T_1 and T_2 of the same system A of curves. Let $T_i, i \in [1, 2]$, be a continuous family of admissible parametrisations between T_1 and T_2 , say, defined by the formula $T_i = (i-1)T_1 + (2-i)T_2$. For each $i \in [1, 2]$, the curves from T_i do not intersect each other, and for each $i \in [1, 2]$ the set of curves $g(A, T_i)$ is a braid, thus $g(A, T_i)$ generate the desired braid isotopy.

Thus, the function $g(A) \equiv g(A, T)$ is well defined.

Now we are ready to prove the main theorem.

First, let us prove the following lemma.

Lemma 10.3. Let A, A' be two equivalent admissible systems of n curves. Then g(A) = g(A').

Proof. Let $A_t, t \in [0,1]$, be a homotopy from A to A'. For each $t \in [0,1]$, A_t is a system of curves (possibly, not admissible). For each curve $\{a_{i,t}, i=1,\ldots,n,t\in[0,1]\}$ choose points $X_{i,t}$ and $Y_{i,t}$, such that the interval from the upper point (upper interval) of the curve to $X_{i,t}$ and the interval from the lower point (lower interval) do not contain intersection points. Denote the remaining part of the curve (middle interval) between $X_{i,t}$ and $Y_{i,t}$ by $S_{i,t}$. Now, let us parametrise all curves for all t by parameters $\{s_{i,t} \in [0,1], i=1,\ldots,n\}$ in the following way: for each t, the upper point of each curve has parameter s=1, and the lower point has parameter s=0. Besides, we require that for i < j and for each $x \in S_{i,t}, y \in S_{j,t}$ we have $s_{i,t}(x) < s_{j,t}(y)$. This is possible because we can vary parametrisations of upper and lower intervals on [0,1]; for instance, we parametrise the middle interval of the j-th strand by a parameter on $\{[\frac{j}{n+2},\frac{j+1}{n+2}]\}$.

It is obvious that for t=0 and t=1 these parametrisations are admissible for A and A'. For each $t \in [0,1]$ the parametrisation s generates a braid B_t in R^3 : we just take the parameter $s_{i,t}$ for the strand $a_{i,t}$ as the third coordinate. The strands do not intersect each other because parameters for different middle intervals cannot be equal to each other.

Thus the system of braids B_t induces a braid isotopy between B_0 and B_1 . \square

So, the function g is well defined on equivalence classes of admissible systems of curves.

Now, to complete the proof of the main theorem, we need only to prove the following lemma.

Lemma 10.4. For any braid b, we have g(f(b)) = b.

Proof. Indeed, let us place b in a small neighbourhood of the "inclined plane" P in such a way that the ends of b are (i,0,0) and (j,1,1), $i,j=1,\ldots,n$.

Consider f(b) that lies in Oxy. It is an admissible system of curves for b. So, there exists an admissible parametrisation that restores b from f(b). By Lemma 10.2, each admissible parametrisation of f(b) generates b. So, g(f(b)) = b.

10.1.3 Algebraic description of the invariant

The general situation in the construction of a complete invariant is the following: one constructs a new object that is in one—to—one correspondence with the described object. However, the new object might also be badly recognisable.

Now, we shall describe our invariant algebraically. It turns out that the final result is very easy to recognise. Namely, the problem is reduced to the recognition problem of elements in a free group. So, there exists an injective map from the braid group to the (n copies of) the free group with n generators that is not homomorphic.

Each braid β generates a permutation. This permutation can be uniquely restored from any admissible system of curves corresponding to β . Indeed, for an admissible system A of curves, the corresponding permutation maps i to j, where j is the ordinate of the strand with the upper point (i,1). Denote this permutation by p(A). It is obvious that p(A) is invariant under equivalence of A.

Let n be an integer. Consider the free product G of n groups Z with generators a_1, \ldots, a_n . Denote by E_i the right residue classes in G by $\{a_i\}$, i.e., $g_1, g_2 \in G$ represent the same element of E_i if and only if $g_1 = a^k g_2$ for some k.

Definition 10.4. An *n*-system is a set of elements $e_1 \in E_1, \ldots, e_n \in E_n$.

Definition 10.5. An *ordered* n-system is an n-system together with a permutation from S_n .

Proposition 10.1. There exists an injective map from equivalence classes of admissible systems of curves to ordered n-systems.

Since the permutation for equivalent admissible systems of curves is the same, we can fix the permutation $s \in S_n$ and consider only equivalence classes of admissible systems of curves with permutation s (i.e., with all lower points fixed depending on the upper points in accordance with s). Thus we only have to show that there

exists an injective map from the set of admissible systems of n curves with fixed lower points to n-systems.

To complete the proof of the proposition, it suffices to prove the following.

Lemma 10.5. Equivalence classes of curves with fixed points (i, 1) and (j, 0) are in one-to-one correspondence with E_i .

Proof. Denote $P \setminus \bigcup_{i=1,...,n} (i,1)$ by P_n . Obviously, $\pi_1(P_n) \cong G$. Consider a small circle C centered at (i,1) for some i with the lowest point X on it. Let ρ be a curve with endpoints (i,1) and (j,0). Without loss of generality, assume that ρ intersects C in a finite number of points. Let Q be the first such point that one meets while walking along ρ from (i,1) to (j,0). Thus we obtain a curve ρ' coming from C to (j,0). Now, let us construct an element of $\pi_1(P_n,X)$. First it comes from X to Q along C clockwise. Then it goes along ρ until (j,0). After this, it goes along Ox to the point (i,0). Then it goes vertically upwards till the intersection with C in X. Denote the constructed element by $W(\rho)$.

If we deform ρ outside C, we obtain a continuous deformation of the loop, thus $W(\rho)$ stays the same as the element of the fundamental group. The deformations of ρ inside C might change $W(\rho)$ by multiplying it by a_i on the left side. So, we have constructed a map from equivalence classes of curves with fixed points (i,1) and (i,0) to E_i .

The inverse map can be easily constructed as follows. Let W be an element of $\pi_1(P_n, X)$. Consider a loop L representing W. Now consider the curve L' that first goes from (i,1) to X vertically, then goes along L', after this goes vertically downwards until (i,0) and finally, horizontally till (j,0). Obviously, W(L') = W. It is easy to see that different representatives L of W we obtain the same L'. Besides, for $L_1 = a_i L_2$, the curves L'_1 and L'_2 are isotopic. This completes the proof of the Lemma.

Thus, for a fixed permutation s, admissible systems of curves can be uniquely encoded by n-systems, which completes the proof of the Theorem.

Now, we see that this invariant is a quite simple object: elements of E_i can easily be compared.

Exercise 10.1. Implement a computer program realising this algorithm.

Let us describe the algebraic construction of the invariant f in more details. Let β be a word-braid, written as a product of generators $\beta = \sigma_{i_1}^{\varepsilon_1} \dots \sigma_{i_k}^{\varepsilon_k}$, where each ε_j is either +1 or -1, $i_j = 1, \dots, n-1$ and $\sigma_1, \dots, \sigma_{n-1}$ are the standard generators of the braid group Br(n).

We are going to construct the n-system step-by-step while writing the word β . First, let us write n empty words (in the alphabet a_1, \ldots, a_n). Let the first letter of β be σ_j . Then all words except for the word e_{j+1} should stay the same (i.e., empty), and the word e_{j+1} becomes a_j^{-1} . If the first crossing is negative, i.e., σ_j^{-1} then all words except e_j stay the same and e_j converts to a_{j+1} . While considering each next crossing, we do the following. Let the crossing be $\sigma_j^{\pm 1}$. Let p and q be the numbers of strands coming from the left side and from the right side respectively. If this crossing is positive, i.e., σ_j , then all words except e_q stay the same, and e_q becomes $e_q e_p^{-1} a_p^{-1} e_p$. If it is negative, then all crossings except e_p stay the same,

and e_p becomes $e_p e_q^{-1} a_q e_q$. After processing all the crossings, we get the desired n-system.

Exercise 10.2. For the trivial braid written as $\sigma_1 \sigma_2 \sigma_1 \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1}$ the construction operation works as follows:

$$\begin{split} (e,e,e) &\to (e,a_1^{-1},e) \to (e,a_1^{-1},a_1^{-1}) \to (e,a_1^{-1},b_1^{-1}a_1^{-1}) \to \\ &\to (e,e,b_1^{-1}a_1^{-1}) \to (e,e,b_1^{-1}) \to (e,e,e). \end{split}$$

A priori these words may be non-trivial, they must only represent trivial residue classes, say, (a_1, a_2^2, a_3^{-1}) .

However, it is not the case.

Proposition 10.2. For the trivial braid, the algebraic algorithm described above gives trivial words.

Proof. Indeed, the algebraic number of occurences of a_i in the word e_i equals zero. This can be easily proved by induction on the number of crossings. In the initial position all words are trivial. The induction step is obvious. Thus, the final word e_i equals a_i^p , where p = 0.

From this approach, one can easily obtain the well known invariant (action) as follows. Instead of a set of n words e_1, \ldots, e_n , one can consider the words $e_1a_1e_1^{-1}, \ldots, e_na_ne_n^{-1}$. Since e_i 's are defined up to a multiplication by a_i 's on the left, the obtained elements are well defined in the free groups. Besides, these elements $\mathfrak{E}_i = e_ia_ie_i^{-1}$ are generators of the free group. This can be checked by a step-by-step confirmation. Thus, for each braid b we obtain an set Q(b) of generators for the braid group. So, the braid b defines a transformation of the free group \mathbf{Z}^{*n} . It is easy to see that for two braids, the transformation corresponding to the product equals the composition of transformation. Thus, one can speak about the action of the braid group on the free group. Since f is a complete invariant, this action has empty kernel.

Definition 10.6. This action is called *the Hurwitz action* of the braid group B_n on the free group \mathbf{Z}^{*n} .

10.2 LD-systems and the Dehornoy algorithm

Another algorithm for braid recognition is purely algebraic. It was proposed by French mathematician Patrick Dehornoy [Deh]. The algorithm to be described is rather fast.

The idea to be used is very close to that used in the distributive grouppoid (quandle). We take some set (of colours) and associate colours with arcs of the braid from this set. Then we show how can the braid be reduced to the trivial braid and if it can not be reduced, why it is not trivial (because of some colour reasons). More precisely, for "good" colour systems (having structure similar to that of grouppoids), each braid defines an operator on this colour system, and this operator can not be trivial for the case of a non-trivial braid.

Let us first remind that braids have a group structure. Thus, in order to compare some two braids a and b it is sufficient to check whether the braid ab^{-1} is trivial.

Let us start with the definition.

Given a braid written algebraically as a word W in the alphabet $\sigma_i^{\pm 1}$, i = 1, 2, ..., n.

Definition 10.7. We say that the word W is a 1-positive braid if it has a representation by a word W', where the letter σ_1 occurs only in positive powers (and does not occur in negative powers). Analogously, one defines a 1-negative braid.

If a braid can be written by a braid-word W' without σ_1 and σ_1^{-1} , we say that this braid is 1-neutral.

Remark 10.3. Since any braid can be encoded by different braid words, one cannot say a priori that the classes described above have no intersections. Later, we shall prove that it is not the case.

Theorem 10.3. Each 1-positive (respectively, 1-negative) braid is not trivial.

Remark 10.4. Actually, a much stronger statement holds: each braid represented by a braid-word containing σ_i^e but not σ_i^{-e} for some $e = \pm 1$ is not trivial. We shall not prove this statement, see [Deh].

The first aim of this section is to show that these three sets are in fact non-intersecting. We are going to present an algorithm that transforms each braid word to an equivalent one that is either unity or positive or negative (the set of 1–neutral braids is actually subdivided into more sets according to the next strands starting from the second one).

To prove Theorem 10.3, we shall need some auxiliary definitions and lemmas. Consider a positive braid—word β and all *lower arcs* of it.

Definition 10.8. By a *lower arc* we mean a part of the braid diagram going from one overcrossing to the next one and passing only undercrossings. Lower arcs correspond to arcs of the mirror image of (upper) arcs.

We wish to label the braid diagram by elements from M in the following manner: we are going to associate with each arc some element of M in such a way that:

- 1. All lower arcs outgoing from upper ends of the braid are marked by variables which are allowed to have values in M;
- 2. Each "lower" label is uniquely defined by all "upper" labels over it;
- 3. The operator f expressing lower labels by upper labels is invariant under isotopies of braids.

To set such a labelling, we have to consider the crossings of the diagram. Suppose the crossing is incident to lower arcs a, b, and c. Let us write down the following relation c = a * b as shown in Fig. 10.5

Let us analyse the invariance of f under isotopies. Each elementary isotopy is associated with one of the following formulae:

$$\sigma_i \sigma_i^{-1} = e, i = 1, \dots, n - 1,$$

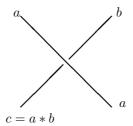


Figure 10.5. A relation

(move Ω_3)

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, i = 1, \dots n-2,$$

(far commutativity)

$$\sigma_i \sigma_j = \sigma_j \sigma_i, |i - j| \ge 2, 1 \le i, j \le n - 1.$$

The relation $\sigma_i \sigma_i^{-1} = \varepsilon$ will be considered later (now we consider only braid words with positive exponents of generators).

The function to be constructed is invariant under far commutativity by construction.

The move Ω_3 give us the self-distributivity relation (in the case of a quandle we needed right self-distributivity):

$$a * (b * c) = (a * b) * (a * c).$$
(1)

Definition 10.9. A set M with a left self-distributive operation * is called an LD-system or LD-set.

Remark 10.5. It is obvious that the operation * is not sufficient to define the operator f for arbitrary braid words: a letter σ_i^{-1} spoils the situation.

In the sequel, we shall add two more operations \vee and \wedge on M as follows.

Obviously, the map f is invariant under the commutation relation (transposing σ_i and σ_j for $|i-j| \geq 2$).

It remains to check the invariance of the map f under Ω_3 , i.e., under transformation $a_i a_{i+1} a_i \to a_{i+1} a_i a_{i+1}$. It is easy to see (Fig. 10.6) that the invariance under Ω_3 means the left distributivity operation.

Thus, having an LD-set M, we can label lower arcs of a positive braid—word by elements of M; the set of elements at lower points is uniquely defined by the set of elements at upper points.

Denote the latter by p_i, i_1, \ldots, i_n ; the elements at lower points will be denoted by q_i . For instance, for the trivial braid, we have $p_i = q_i$.

For an LD–set one can define a partial order relation <. Namely, for each $a,c\in M$ a< c if $\exists b\in M:c=a*b.$

Definition 10.10. A partially ordered LD-set M is called *ordered* if < is acyclic, i.e., there exists no sequence $a_1 < a_2 < \cdots < a_n < a_1$.

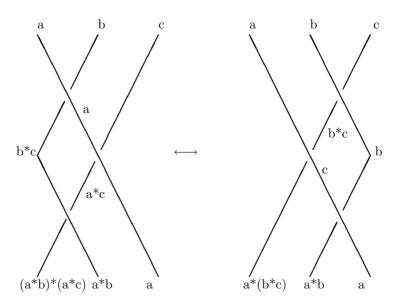


Figure 10.6. Invariance under Ω_3 implies distributivity

Example 10.1. The sets $\mathbf R$ and $\mathbf Q$ admit some left-distributive (but not acyclic) operations:

1. a * b = max(a, b)

2. $a * b = \frac{(a+b)}{2}$

3. a * b = (a + 1).

Let us now give an example of an acyclic LD-system. To do this, we have to introduce more difficult structures, enclosing the LD-structure. Denote the semigroup of non-negative braids (or *positive braid monoid* by $Br(n)^+$.

Remark 10.6. This monoid played the key role in Garside's theory of normal form.

Definition 10.11. Let (M, \wedge) be a set endowed with a binary operation. Let us define the right action of the semigroup $Br(n)^+$ on $M^n = \underbrace{M \times \cdots \times M}_{}$ inductively.

First, for $\vec{a} = (a_1, \dots, a_n)$ we set

$$(\vec{a})\varepsilon = \vec{a}, \quad (\vec{a})\sigma_i w = (a_1 \dots, a_i \wedge a_{i+1}, a_i, a_{i+2}, \dots, a_n)w, \tag{3}$$

where ε is the unity element.

Let us now try to colour all braid diagrams (not only positive). In order to do this, we shall have to introduce two more operations. Let us change the notation: denote * by \land and introduce new operations \lor and \circ . Then we can colour the braid diagram as shown in Fig. 10.7.

As before, we can express the labels of lower points by the labels of upper points; thus we define the operator f.

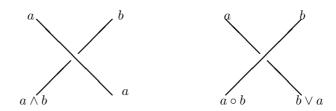


Figure 10.7. Defining lower labels at a negative crossing

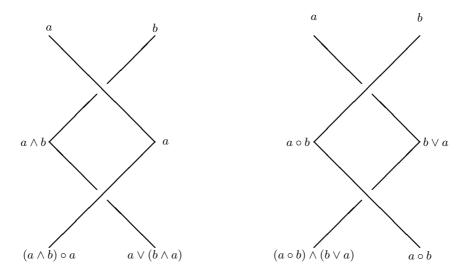


Figure 10.8. Invariance under Ω_2

Lemma 10.6. Let (M, \wedge, \circ, \vee) be a system with three binary operations. Then, the operator f defined above is invariant under isotopies generated by the third Reidemeister move if and only if the following relations hold in M:

$$\forall x,y \in M \qquad x \circ y = y, x \wedge (x \vee y) = x \vee (x \wedge y) = y.$$

Proof. Follows straightforwardly from Fig. 10.8.

Definition 10.12. An LD-system M (with respect to wedge) endowed with an extra operation \vee is said to be an LD-quasigroup if the relation

$$x \wedge (x \vee y) = x \vee (x \wedge y) = y$$

holds for arbitrary $x, y \in M$.

Remark 10.7. Unlike distributive grouppoid (quandle), LD-quasigroups do not require the idempotence relation.

Remark 10.8. The isotopy generated by Ω_2 preserves f, by definition. Thus, if M is an LD-quasigroup, then the operator f is invariant under all braid isotopies.

The following proposition can be checked straightforwardly.

Proposition 10.3. Let G be a group. Then the binary operations $x \wedge y = xyx^{-1}$, $x \vee y = x^{-1}yx$, $x \circ y = y$, define the LD-quasigroup structure on G.

Thus, for a given group G we can use the system (G, \wedge, \vee) (where $x \circ y = y \forall x, y$) for colouring braids. In particular, let FG_n be the free group generated by $\{x_1, \ldots, x_n\}$. For an n-strand braid word b, let us define \tilde{b} as the braid word obtained from b by reversing the order of letters. Thus, $\sigma_1 \sigma_2^{-1} \sigma_3 \sigma_1^{-1} \mapsto \sigma_1^{-1} \sigma_3 \sigma_2^{-1} \sigma_1$.

For given elements x_1, \ldots, x_n , define the elements y_1, \ldots, y_n according to the rule: $(y_1, \ldots, y_n) = (x_1, \ldots, x_n)\tilde{b}$. Let $\phi(b)$ be an automorphism of FG_n , mapping all x_i 's to y_i 's. Then ϕ is an endomorphism of the braid group Br(n) inside $Aut(FG_n)$ because $\phi(b^{-1})$ coincides with $\phi(b)^{-1}$ by construction.

Denote by FG_{∞} the limit of embedding $FG_1 \subset FG_2 \subset FG_3 \subset \dots$. In this way, we obtain an endomorphism $\phi: B_{\infty} \to Aut(FG_{\infty})$.

Denote $\phi(\sigma_i)$ by α_i .

By construction, we have:

$$\alpha_i(x_k) = \begin{cases} x_k & k < i \text{ or } k > i+1 \\ x_i x_{i+1} x_i^{-1}, & k = i \\ x_i, & k = i+1. \end{cases}$$

Remark 10.9. This action coincides with the action of the generator σ_i on the colours of arcs, see Fig. 10.7, cf. the definition of the Hurwitz action.

We have the operation sh on the generators: $sh(x_i) = x_{i+1}$. Now, let us define the shift sh on FG_{∞} , taking x_i to x_{i+1} for each i. Let us define the action of sh on $Aut(FG_{\infty})$ according to the following rule. For $\phi \in Aut(FG_{\infty})$, let $sh(\phi)(x_1) = x_1$ and $sh(\phi)(x_{i+1}) = sh(\phi(x_i))$ $i \geq 1$.

Then the operation \wedge is defined by

$$\phi \wedge \psi = \phi \circ sh(\psi) \circ \alpha_1 \circ sh(\phi^{-1}) \tag{*}$$

and ϕ is the homeomorphism from (B_{∞}, \wedge) to $(Aut(FG_{\infty}), \wedge)$. We are going to prove that the operation defined by (*) is left self-distributive.

Note that α_1 commutes with the image of the automorphism sh^2 . To complete the proof of self-distributivity of the operation \wedge , it remains to prove the following.

Proposition 10.4. Let G be a group, a be a fixed element of G, and s be an automorphism¹ of G. Then the formula $x \wedge y = xs(y)as(x^{-1})$ defines a left self-distributive operation if and only if the element a commutes with images of the map s^2 and the following relation holds

$$as(a)a = s(a)as(a). (5)$$

¹Later, this operation will play the role of shift in the free group.

Proof. Since s is an isomorphism, we obtain

$$x \wedge (y \wedge z) = xs(y)s^{2}(z)s(a)s^{2}(y^{-1})as(x^{-1})$$

$$(x \wedge y) \wedge (x \wedge z) = xs(y)as^2(z)s(a)s^2(x^{-1})as^2(x)s(a^{-1})s^2(y^{-1})s(x^{-1}).$$

If the operation \wedge is left self-distributive, then, taking x = y = z = 1, we get $s(a)a = as(a)as(a^{-1})$.

It is easy to check that the inverse statement holds as well.

Finally, formula (5) and the hypothesis that a commutes with $s^2(z)$ for all z implies the fact that the map f defined by $f(\sigma_i) = s^{i-1}(a)$ generates a homeomorphism from B_{∞} to G.

Thus, we have constructed a left self-distributive system on $Aut(FG_{\infty})$ with operation \wedge .

Theorem 10.4. This system $(Aut(FG_{\infty}), \wedge)$ is acyclic.

The proof follows from two auxiliary lemmas on free reductions that appear while calculating $\alpha_i(x)$.

Let us denote the set of words in the alphabet $\{x_1^{\pm 1}, x_2^{\pm 1}, \dots\}$ by W_{∞} .

We say that a word from W_{∞} is free reduced if it does not contain the following subwords: $x_i x_i^{-1}$ and $x_i^{-1} x_i$. For each $w \in W_{\infty}$, let us denote by red(w) the word obtained from w by means of consequent deleting of such words. Thus, we can identify the free group FG_{∞} with the set of all free reduced words; this set is endowed with the operation $u \cdot v = red(uv)$.

Definition 10.13. For a letter x from the alphabet $\{x_1^{\pm 1}, x_2^{\pm 1}, \dots\}$, denote by E(x) the subset of FG_{∞} , containing all reduced words whose final letter is x.

Let us now investigate the image of the set $E(x_1^{-1})$ with respect to the action of $\alpha_i^{\pm 1}$.

Lemma 10.7. Let ϕ be an arbitrary element from $Aut(FG_{\infty})$. Then the automorphism $sh(\phi)$ maps the set $E(x_1^{-1})$ to itself.

Proof. Let f be an automorphism W_{∞} mapping x_i to some reduced word $\phi(x_i)$ for each i. Let w be an arbitrary element from $E(x_1^{-1})$. Then $w = ux_1^{-1}$, where u is some reduced word that does not belong to $E(x_1)$. In this case we have $sh(f)(w) = sh(f)(u) \cdot sh(f)(x_1^{-1})$, i.e., $red(sh(f)(u)x_1^{-1})$. Assume that the latter does not belong to $E(x_1^{-1})$. This means that the last

Assume that the latter does not belong to $E(x_1^{-1})$. This means that the last letter x_1^{-1} is reduced with the letter x_1 that occurs in sh(f)(u). But the letter x_1 in sh(f)(u) can originate only from some x_1 in u. Thus, we should have a decomposition $u = u_1x_1u_2$ such that $red(sh(f)(u_2))$ is the empty word, i.e., $sh(f)(u_2) = 1$. Since sh(f) is an automorphism of FG_{∞} then $u_2 = 1$. This means $u_2 \in E(x_1)$. The contradiction to the initial hypothesis completes the proof.

Lemma 10.8. The automorphism α_1 maps $E(x_1^{-1})$ to itself.

Proof. Let w be an element from $E(x_1^{-1})$. By definition, $w = ux_1^{-1}$, where u is some reduced element that does not belong to $E(x_1)$, and $\alpha_1(w)$ is $red(a_1(w))$. More precisely,

$$\alpha_1(w) = red(a_1(u)x_1x_2^{-1}x_1^{-1}).$$

Suppose that the word in the right-hand side part does not belong to the set $E(x_1^{-1})$. This means that the final letter x_1^{-1} is reduced by some x_1 in the end of $a_1(u)$. This letter originates either from some x_2 , or from some $x_1^{\pm 1}$ in the word

In the first case, let us write down the letter x_2 that takes part in this reduction, and represent u as $u_1x_2u_2$, where u_2 is a reduced word whose initial letter is not x_2^{-1} . Thence,

$$\alpha_1(w) = red(a_1(u_1)x_1a_1(u_2)x_1x_2^{-1}x_1^{-1}),$$

and the hypothesis can be reformulated as follows: $red(a_1(u_2)x_1x_2^{-1})$ is the empty word (because $\alpha_1(u_2) = x_2 x_1^{-1}$).

Now, let α_1 be an automorphism, and $x_2x_1^{-1}$ be the image of $x_2^{-1}x_1$ with respect to α_1 . Thus, the only possible case is $u=x_2^{-1}x_1$. But, in this we assume the contradiction: u should not begin with x_2^{-1} .

In the second case, we analogously write $u = u_1 x_1^e u_2$ for $e = \pm 1$. In this case we obtain:

$$\alpha_1(w) = red(a_1(u_1)x_1x_2^ex_1^{-1}a_1(u_2)x_1x_2^{-1}x_1^{-1}).$$

Now our hypothesis is that $red(x_2^ex_1^{-1}a_1(u_2)x_1x_2^e)$ is the empty word. In this case, we conclude that $\alpha_1(u_2)=x_1x_2^{1-e}x_1^{-1}$. The latter word equals $\alpha_1(x_1^{1-e})$. Thus, u_2 should be equal to x_1^{1-e} . If e=+1 then the word u_2 is empty. If e=-1 then $u_2=x_1^2$. In both cases, u_2 belongs to $E(x_1)$ which is a contradiction.

Lemma 10.9. Suppose that ϕ is an automorphism of FG_{∞} , that can be expressed as a composition of images of sh and α_1 , whence the latter takes place at least once. Then $\phi(x_1)$ belongs to $E(x_1^{-1})$. In particular, the automorphism ϕ is not identical.

Proof. The condition of the lemma is that ϕ has the following representation:

$$\phi = sh(\phi_0) \circ \alpha_1 \circ sh(\phi_1) \circ \alpha_1 \circ \cdots \circ \alpha_1 \circ sh(\phi_p).$$

Then we have $sh(\phi_p)(x_1)=x_1$ and hence $\alpha_1(x_1)=x_1x_2x_1^{-1}$, i.e., $\alpha_1(x_1)\in E(x_1^{-1})$. In this case, each next map $sh(\phi_k)$ (as well as α_1) takes $E(x_1^{-1})$ to $E(x_1^{-1})$.

Let us now prove Theorem 10.4. We have to show that in $Aut(FG_{\infty})$ equalities like

$$\phi = (\dots(\phi \wedge \psi_1) \wedge \dots) \wedge \psi_p)$$

cannot hold.

By using the definition of \wedge , we get the representation of ϕ as

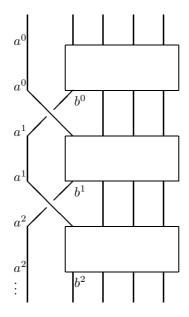


Figure 10.9. 1-positive braid is not trivial

 $\phi = \phi \circ \text{some mess}$

Here "some mess" contains α_1 and sh.

Thus, Id has a presentation by sh and α_1 where the latter occurs at least once. This is a contradiction to the last lemma.

Let us complete now the proof of Theorem 10.2. Let b be a 1-positive braid word. Consider the automorphism $\sigma(\phi)$. It satisfies the conditions of Lemma 10.9. Thus, it is not identical. Consequently, the braid ϕ is not trivial.

We can also present the following "intuitive" proof.

Suppose we have an LD-quasigroup Q that is an acyclic LD-system with the order operation <. Let us show that the existence of this quasigroup Q results in the claim of Theorem 10.2. Consider the elements a_1, a_2, \ldots, a_k , corresponding to lower arcs, corresponding to leftmost crossings, and the elements b_1, b_2, \ldots on the right hand from the a_i 's, see Fig. 10.9.

It is easy to see that $a_{i+1} = a_i \wedge b_i > a_i$. Thus, $a_k > a_1$. Because the operation < is acyclic, we have: $a_k \neq a_1$.

However, for the trivial braid, the elements of the set M, corresponding to upper points coincide with those corresponding to lower points. Thus we obtain a contradiction which completes the proof of the theorem.

One can easily prove the following corollaries.

Corollary 10.1. The braid that is inverse to the 1-positive braid is 1-negative; the reverse to a 1-neutral braid is 1-neutral.

Corollary 10.2. Each braid B belongs to no more than one of the three types: 1-positive, 1-negative, 1-neutral.

Proof. Suppose that B is simultaneously 1–positive and 1–negative. Then there exists a 1–positive braid word B' representing the reverse braid for B. Consequently, the unit braid BB' is 1–positive. Thus we obtain a contradiction.

The other cases can be proved analogously.

Corollary 10.3. The toric (p,q)-braid is not trivial.

As we know, no 1–positive or 1–negative braid–word can represent the trivial braid.

All 1-neutral braid words can be divided into 2-positive, 2-negative, and 2-neutral braids with respect to occurrences of σ_2 and σ_2^{-1} .

Analogously to Theorem 10.5, one can prove the following.

Proposition 10.5. Each 2-positive or 2-negative braid word represents a non-trivial braid.

Analogously to 1– and 2–positive (negative, neutral) braid words one can define k–positive (negative, neutral) braid words. Arguing as above, one proves that the first two types of braids do not contain the trivial braid.

Thus, according to our classification, there is only one n-strand (n-1)-neutral braid. Namely, it is the trivial braid.

Now, we have to show that all braids can be classified in this manner. To do this, we shall have to prove the following theorem.

Theorem 10.5. Each braid is either 1-positive, 1-negative or 1-neutral.

Let us first discuss this theorem for a while.

Suppose we have some braid word representing an n-strand braid K and we wish to use the relations

 $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i-j| \geq 2$, $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for removing either σ_1 or σ_1^{-1} .

Suppose that L is a subword of K looking like $\sigma_i^p w \sigma_i^{-p}$, where $p = \pm 1$, and the word w contains only generators $\sigma_i^{\pm 1}$ for i > i.

Definition 10.14. Such a word is called an 1-handle.

Geometrically, a 1-handle is shown as follows.

For such a handle, one can perform a *reduction*, i.e., a move, pulling the i-th strand over the nearest crossings as shown in Fig. 10.11

Proposition 10.6. In this case, the braid word $\sigma_i^e v \sigma_i^{-e}$ becomes the word $v^{'}$ that is obtained by replacing all occurrences of $\sigma_{i+1}^{\pm 1}$ in v with $\sigma_{i+1}^{-e} \sigma_i^{\pm 1} \sigma_{i+1}^e$.

Proof. Actually, let us consider the handle reduction shown in Fig. 10.11. All crossings lying on the right hand with respect to σ_2 will stay the same. The initial and the final crossings $\sigma_1^{\pm 1}$ will disappear. The crossings where the strand goes over are changed by the rule described in the formulation of the statement.

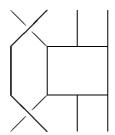


Figure 10.10. Handle

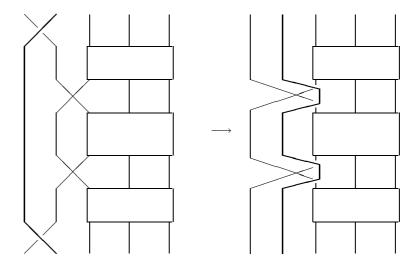


Figure 10.11. Handle reduction

Let us now consider the braid word K and consequently reduce all handles in it. If the process stops (i.e., we eliminate all handles) then the obtained braid word has either 1–positive, 1–negative or 1–neutral form.

Let us demonstrate now that this approach does not always work.

Convention. For the sake of convenience, let us write small Latin letters a,b,c,\ldots instead of generators σ_1,σ_2,\ldots and capital letters A,B,C,\ldots instead of $\sigma_1^{-1},\sigma_2^{-1},\ldots$

Example 10.2. Consider the word abcBA. It is a 1-handle. After applying the handle reduction, we obtain the braid word B(abcBA)b that contains the initial braid word (handle) as a subword. Thus, by applying handle reductions many times, we shall always have this handle and increase the length of the whole word.

The matter is that this 1-handle encapsulates a 2-handle; after reducing the 1-handle, the 2-handle goes out and becomes a 1-handle.

For this braid word, one can first reduce the "interior" handle and then the

"exterior" one. Then we get $a(bcB)A \rightarrow (aCbcA) \rightarrow CbABc$; thus, we conclude that this braid is 1-positive.

Fortunately, the existence of k-handles inside j handles (k > j) is the only obstruction for reducing the braid word to another word without 1-handles.

Let us prove the following lemma.

Lemma 10.10. If a j-handle has no (j + 1)-handle inside (as a subword) then after reducing this j-handle no new handle appears.

Proof. Without loss of generality, we can assume that j = 1. Let $u = \sigma_1^e v \sigma_1^{-e}$ be a 1-handle, $e = \pm 1$. Since v contains no 2-handles, we see that v contains either only positive exponents of σ_2 or only negative.

The same can be said about exponents of σ_1 in the word u' obtained from u by means of the handle reduction.

Consequently, the word u' contains no handles.

Definition 10.15. A handle containing other handles as subwords is called a *nest*.

Definition 10.16. An *i*-handle reduction not containing (i+1)-handles inside is called *proper*.

Now, let us describe the algorithm.

First, we reduce all interior handles (which are not nests), then we reduce the handles containing the handles that have already been reduced, and so on. Finally, we obtain a braid word that is either 1-positive or 1-negative or 1-neutral.

In the first two cases, everything is clear. In the third case, we forget about the first strand and repeat the same for all other strands.

Example 10.3. Consider the braid word ABacBCBaCbaa. Let us transform it to an equivalent braid word without handles.

ABacBCBaCbaa

b<u>ABcBCBa</u>Cbaa

bbABcbABCbABCbaa

bbABcbABCbAcBCaa

bbABcbABCbcbABCa

bbABcbABCbcbbABC

Here we underline the subword representing the handle to be reduced.

The obtained word is 1-negative. Thus, the braid is not trivial.

Thus, we have obtained a simple and effective algorithm for braid recognition.

Exercise 10.3. Write a computer program realizing this algorithm.

We have not yet proved that the Dehornoy algorithm stops within a finite number of steps. (The number of letters in the braid word grows and the thus we cannot guarantee the finite time of work.)

Exercise 10.4. Proof that the Dehornoy algorithm works directly in the case of 3-strand braids.

Below, we sketch the proof of the fact that the Dehornoy algorithm stops in a finite time. For a more detailed proof see [Deh].

10.2.1 Why the Dehornoy algorithm stops

Definition 10.17. A braid word is called *positive* (resp., *negative*) if it contains only σ_i 's (respectively, σ_i^{-1} 's).

It turns out that while performing the handle reductions $w_0 \to w_1 \to w_2 \to \dots$, each word w_k can be represented by drawing a path in a special finite labelled graph; starting from the word $w = w_0$, only a finite number of such words may occur.

Let us be more detailed.

We should give some definitions.

Definition 10.18. By a positive (negative) equivalence of two words we mean a relation where only σ_i in positive powers (respectively, negative) occurs.

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Example 10.4. \sigma_1 \sigma_2 \sigma_1 \rightarrow \sigma_2 \sigma_1 \sigma_2 is a positive equivalence; \sigma_1^{-1} \sigma_3^{-1} = \sigma_3^{-1} \sigma_1^{-1} is a negative equivalence; \sigma_1 \sigma_2 \sigma_1^{-1} = \sigma_2^{-1} \sigma_1 \sigma_2 is neither a positive nor a negative equivalence.
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Definition 10.19. By a word reversing we mean a simple equivalence $\sigma_i^{-1}\sigma_j \to \sigma_j\sigma_i^{-1}$ if $|i-j| \geq 2$ or $\sigma_i^{-1}\sigma_j \to \sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1}$ if |i-j| = 1 (right reversing). $\sigma_i^{-1}\sigma_i = e$. Analogously, one defines the left reversing.

It is easy to see that by reiterated left reversing, each braid word w can be transformed to a form $N_L D_R^{-1}$, where N_L and D_R are positive braid words. They are called the *left denominator* and *right denominator*.

Analogously, right-reversing can transform each braid word to a form $D_L^{-1}N_R$ of the left denominator D_L and right numerator N_R .

It is obvious that for each braid word u, $D_L(u)N_R(u) = N_L(u)D_R(u)$.

Definition 10.20. For a braid word w, the absolute value is $D_L(w)N_R(w)$. Notation: |w|.

Definition 10.21. For a positive braid X, the Cayley graph of X is the following graph: its vertices are the trivial braid e, the braid X and positive braids Y such that there exists a positive braid Z such that YZ = X. Two vertices P and Q are connected by an edge if there exists i such that $P = Q\sigma_i$ or $P = Q\sigma_i$. In this case, the edge is oriented from the "smaller" braid to the "bigger" braid.

Example 10.5. Consider the positive braid $\Delta_4 = \sigma_1 \sigma_2 \sigma_1 \sigma_3 \sigma_2 \sigma_1$. Then the Cayley graph for this braid is shown in Fig. 10.12.

Definition 10.22. Let u be a positive braid and let C(u) be the Cayley graph for u. Then each path (with the first and the last points chosen arbitrarily) on this graph can be expressed by generators of the braid group and their reverse elements; thus, each path generates a word. Such paths are called $traced\ words$ for u.

Now, we are going to formulate some auxiliary lemmas from which we can conclude that the Dehornoy algorithm stops within a finite type.

Lemma 10.11. Each proper reduction (without nested 2-handles) can be reduced to word-reversing, right equivalence, and left equivalence.

Lemma 10.12. For any braid word w, all words obtained from w by word reversing (right and left) and positive and negative equivalence are traced for |w|.

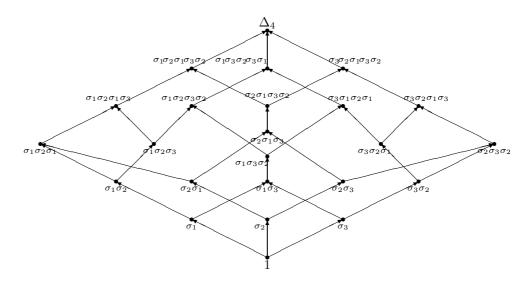


Figure 10.12. The Cayley graph for Δ_4

From these two lemmas we deduce the following lemma.

Lemma 10.13. Each word obtained from w by proper handle reductions is traced for |w|.

Definition 10.23. Let w be a braid word. The *height* of w is the maximal number of letters σ_i occurring in a word containing no σ_i^{-1} and traced in the Cayley graph of |w|.

Assume that $w_0 = w, w_1, w_2, \ldots$ is a sequence of handle reductions from w. The first point is that the number of σ_1 -handles on w_i is not larger than the number of σ_1 -handles in w

Definition 10.24. The p-th critical prefix $pref_p(w_k)$ is a braid represented by the prefix of w_k that ends with the first letter of the p-th σ_1 -handle of w_k .

The point is now that for every p, the prefix is "not increasing" from w_k to w_{k+1} and actually "increasing" if this handle is reduced. More precisely, the following lemma holds.

Lemma 10.14. Assume that the p-th σ_1 -handle is reduced from w_k to w_{k+1} . Then there exists a braid word $u_{p,k}$ traced in the Cayley graph of w from $pref_p(w_k)$ to $pref_p(w_{k+1})$ that contains one letter σ_1 and no letter σ_1 .

Now, it is not difficult to prove the following lemma.

Lemma 10.15. Assume that w is a braid word of length l and height h. Then the number of handle reductions from w is bounded by lh.

This lemma shows that any braid word can be reduced to a 1-positive, 1-negative or 1-neutral word in finite time. Thus, the Dehornoy algorithm stops in finite time.

10.3 Minimal word problem for Br(3)

Among other problems arising in braid theory we would like to note the problem of finding the minimal braid word representing braids from a given class. This

problem, of course, gives an algorithm for braid recognition since only the trivial braid has the braid word of length 0.

Here we give an algorithm solving this problem for the case of three strands. This algorithm is due to M. Berger, see [Ber].

For the braid group Br(3) there exists a natural automorphism that transposes σ_1 and σ_2 . Let T be a three–strand braid word; denote the braid obtained from T by applying this isomorphism by \hat{T} .

The *length* of a braid word is simply the number of characters in this word.

Definition 10.25. $\Delta = \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$

Note that Δ "almost commutes" with all braids. Namely, given a braid A, we have $A\Delta = \Delta \hat{A}$.

Definition 10.26. A wrap is any of the four words

$$\sigma_1 \sigma_2, \quad \sigma_2 \sigma_1, \quad \sigma_1^{-1} \sigma_2^{-1}, \quad \sigma_2^{-1} \sigma_1^{-1}.$$

The algorithm to be described consists of three steps.

The first step. Consider a braid word B_0 . Let us search for any occurrence of Δ or Δ^{-1} and take them to the left by using $A\Delta = \Delta \hat{A}$.

We proceed until B_0 has been converted to a word having the form $\Delta^n B_1$, where B_1 is free of Δ 's.

The second step clears away the wraps. Namely, finding a wrap in B_1 , we replace it as follows:

$$\sigma_1 \sigma_2 = \Delta \sigma_1^{-1},$$

$$\sigma_2 \sigma_1 = \Delta \sigma_2^{-1},$$

$$\sigma_1^{-1} \sigma_2^{-1} = \Delta^{-1} \sigma_1,$$

$$\sigma_2^{-1} \sigma_1^{-1} = \Delta^{-1} \sigma_2.$$

Thus we obtain the form $B = \Delta^p B_2$ or $B = \Delta^p \hat{B}_2$, where

$$B_2 = \sigma_1^{q_1} \sigma_2^{-r_1} \sigma_1^{q_2} \sigma_2^{-q_2} \dots \sigma_1^{q_m} \sigma_2^{-r_m},$$

where all q_i 's and r_i 's are some positive integers.

Finally, in the third step we partially reverse the second step. For definiteness, suppose that p > 0 and that $B = \Delta^p B_2$.

First, we take one Δ from the left and bring it to the right searching for σ_2^{-1} . We replace Δ with σ_2^{-1} by the wrap $\sigma_2\sigma_1$. Each time this is done, the length is reduced by 2.

Now we repeat this operation until there are no Δ 's remaining (if $p \leq r$) or no σ_2^{-1} remaining $(p \geq r)$. Denote the obtained braid word by B_{min} .

The main theorem of Berger's work is the following

Theorem 10.6. Given B, the word B_{min} has the minimum length among all braid words equivalent to B.

10.4 Spherical, cylindrical, and other braids

Actually, the invariant described above (by means of admissible systems of curves) admits generalisations for cases of braids in different spaces.

10.4.1 Spherical braids

We recall that a *spherical* braid on n strands is an element of $\pi_1(X_n)$, where X_n is the configuration space of non-ordered n-point sets on the standard sphere S^2 .

As in the case of ordinary braids, spherical braids admit a simple representation by n strands in the space $S_z^2 \times I_t$ coming downwards with respect to the coordinate t (height) and connecting fixed points $A_i \times \{1\}$ and $A_j \times \{0\}$, where A_1, \ldots, A_n are fixed points on S^2 . Like ordinary braids, spherical braids are considered up to natural isotopy: we decree isotopic braids to be the same; spherical braids form a group. Denote it by SB(n).

Without loss of generality one can assume that there exists a point $X \in S^2$ such that no strand of a given spherical braid contains $X \times t$ for any $t \in [0,1]$.

This means that each spherical braid comes from a (not necessarily unique) ordinary braid. More precisely, there exists a homomorphic map h from Br(n) onto SB(n) defined as follows: each braid b in $R^3 = R^2 \times R$ generates a spherical braid b' simply by compactifying R^3 by a point, thus by mapping R^3 to $S^2 \times R$. The homomorphic property of the braid group map follows in a straightforward way.

It is known (see, e.g.,[Fr]) that the kernel of h is generated by the only element $\Sigma_n = (\sigma_1 \dots \sigma_{n-1})^n$ for all $n \ge 2$, see [Fr].

Let us now prove this theorem explicitly. The proof that h commutes with the whole group Br(n) is obvious. Actually, to the braid Σ_n , one can attach a band such that the first and the last strands are parts of the boundary of this band and all the other strands divide the band into smaller bands, see Fig. 10.13.

Now, each generator σ_i of the braid group can be taken along the corresponding smaller band from the top to the bottom, as shown in Fig. 10.13.

This means that Σ_n really lies in the centre of the braid group.

The remaining part of the proof can also be expressed in the language of bands. To do it, one should use induction on the number of strands (starting from three strands). Here we should slightly modify the induction basis: each **pure braid** that commutes with the whole braid group is a power of Σ_n . We do it in order to be able to start from the case of two strands. For two strands, the induction basis is evident.

Then, the induction step can be proved in the following manner: we take our n-strand braid that commutes with anything. It should be pure. Thus, we can consider n pure braids obtained from this one by deleting some strand (one of n). By the induction hypothesis, each of these braids should be $\sum_{n=1}^{k}$ for some integer k. The remaining part of the proof is left to the reader.

Obviously, the invariant f (see page 130) distinguishes Σ_N and the trivial braid, thus it is not an invariant for Br(n). Moreover, the described kernel coincides with the centre of Br(n).

²Here I is a unit segment; z and t denote coordinates on S^2 and I, respectively.

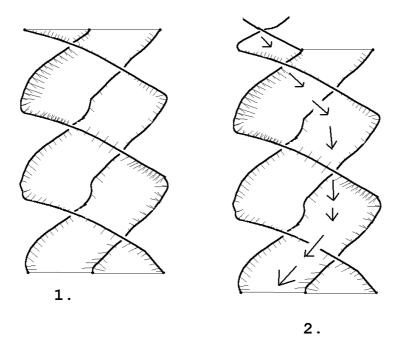


Figure 10.13. Σ_N commutes with generators

The main idea of the proof (see, e.g. [Fr]) is the following. Consider the trivial braid represented in the most natural way in $\mathbf{R}^3 \subset S^2 \times \mathbf{R}$. Let us attach a band to it in the simplest way. Now, while isotoping the braid in SB(n), one can observe what can happen with the band. The only thing that can happen is the twist of the band. This occurs when we pass through the compactification point $X \in S^2 = \mathbf{R}^2 \cup X$, see Fig. 10.14.

Now, it is evident that after a certain number of twists, our braid (in the sense of Br(n)) just becomes some power of Σ_n . Thus, we have proved that no other braids but powers of Σ_n lie in the kernel of the map $Br(n) \to SB(n)$. On the other hand, Σ_n really represents the trivial braid in SB(n) because of the same "twist" reasons

The aim of this subsection is to correct the invariant f for the case of spherical braids.

We shall do this in the following way. We take a spherical braid b and its (infinitely many) pre-images b_{α} with respect to h. Then we take their images $f(b_{\alpha})$, which are, certainly, different. Thus the aim is to construct a map acting on $f(\cdot)$ that should bring all $f(b_{\alpha})$ together. This is the way to construct a spherical braid invariant. We now construct a thin invariant that for any other braid b' and its pre-images b'_{α} does not glue $f(b_{\alpha})$ and $f(b'_{\alpha})$. Thus, the invariant to construct must be complete.

Let us introduce the sets E'_1, \ldots, E'_n by factorising E_i with respect to the relation $a_1 \ldots a_n = e$. Thus we get a map $\mathfrak{h} : (E_1, \ldots, E_n) \mapsto (E'_1, \ldots, E'_n)$.

Definition 10.27. A spherical n-system is a set of elements $e'_1 \in E'_1, \ldots, e'_n \in$

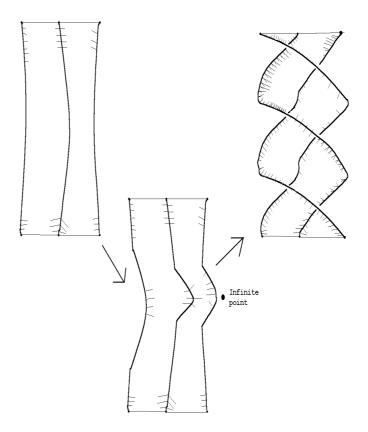


Figure 10.14. The twist of the band

 E'_n . An ordered spherical n-system is a spherical n-system together with a permutation from S_n .

Now, let us define the map f_S from ordinary braids to spherical n-systems as follows. For each braid b the ordered n-system f(b) consists of the permutation s corresponding to b and a set $e_i \in E_i, i = 1, ..., n$. Then the ordered spherical n-system $f_S(\beta)$ consists of the permutation s and the set $\mathfrak{h}(e_1), ..., \mathfrak{h}(e_i)$. Obviously, f is an ordinary braid invariant, and so is f_S .

Theorem 10.7. The function f_S is a complete invariant of spherical braids, i.e., two braids $b, b' \in B_n$ generate the same spherical braid if and only if $f_S(b) = f_S(b')$.

Proof. First, let us note that the statement of Lemma 10.1 is true for the invariant f_S as well. The proof is literally the same.

Thus, for any braid b we have $f_S(b) = f_S(\Sigma b)$, and, hence f_S is a braid invariant and Σ commutes with b, $f_S(b\Sigma) = f_S(\Sigma b) = f_S(b)$. Thus, if b and b' generate the same spherical braid then $f_S(b) = f_S(b')$. So, f_S is invariant.

Now, let us prove the reverse statement, i.e., that f_S is complete.

Indeed, we have to show that if β_1, β_2 are spherically equivalent braids, then $f_S(\beta_1) = f_S(\beta_2)$. By Lemma 10.1 for f_S we see that it suffices to show that f_S recognises the trivial spherical braid. Suppose β is an ordinary braid, and $h(\beta)$

is the spherical braid generated by β . Suppose that $f_S(h(\beta)) = e$. By definition, the value $f(\beta)$ is the following. The permutation of the braid is trivial and the n-system is $((a_1 \ldots a_n)^{k_1}, \ldots, (a_1 \ldots a_n)^{k_n})$ for some integer k_1, \ldots, k_n . Recall that the n-system comes from the admissible system of curves (non-intersecting). Thus we see that $k_1 = k_2 = \ldots k_n$. Let $k = k_1 = \cdots = k_n$.

The only thing to check is that if the *n*-system is $\{((a_1 \dots a_n)^k, \dots, (a_1 \dots a_n)^k)\}$ and the permutation is trivial then $\beta = \Sigma^k$.

But β represents the trivial spherical braid, i.e., $h(\beta)=e$. This completes the proof.

10.4.2 Cylindrical braids

Let C be the cylinder $S^1 \times I$.

Definition 10.28. A cylindrical n-strand braid is an element of $\pi_1(C_n)$, where C_n is the configuration space of non-ordered n-point sets on C. Cylindrical braids are considered up to natural isotopy. Like ordinary and spherical braids, cylindrical n-strand braids form a group. Denote this group by CB(n).

The construction of the invariant for cylindrical braids is even simpler than that for spherical braids. This simplicity results from the structure of C, which is the product of the interval I and the circle.

A cylindrical n-strand braid can be considered as a set of n curves in $C \times I_t$ coming downwards from t=2 to t=1 in such a way that the ends of the curves generate the set $\{Y_i \times \{1\}, Y_j \times \{2\}, i, j=1, \ldots, n, Y_i \in C\}$. The set $C = C \times I_t = S_{\varphi}^1 \times I_s \times I_t$ can be considered in $R^3 = Oxyz$: the coordinate t corresponds to $t \in [1, 2]$, and $t \in [0, 2\pi)$, $t \in [1, 2]$ form a polar coordinate system of the plane $t \in [0, 2\pi)$.

For each curve in \mathcal{C} we can consider its projection on the cylinder $S^1 \times I_t$. Thus, for a cylindrical braid β we have a system of curves on the cylinder $S^1 \times I_t$, with coordinate t decreasing from one to zero. In a general position these curves have only double transversal crossing points, lying on different levels of t. For each crossing we must indicate which curve has the greater coordinate X (forms an overcrossing); the other curve forms an undercrossing.

Fix a point $x \in S^1$. Now, a singular level is a value t such that $S^1 \times \{t\}$ contains either a crossing or an intersection of a braid strand with the line $x \times R$.

Let us require that no crossings lie in $x \times R$, all intersections of strands with $x \times R$ are transversal and each singular level contains either only one crossing or only one intersection point. Let us also require that no crossing lies on the intersection line.

Definition 10.29. Such a curve endowed with an undercrossing structure is called a diagram of a cylindrical braid.

Remark 10.10. Obviously, all ordinary braids generate cylindrical braids by embedding of \mathbf{R}^1 in S^1 and \mathbf{R}^2 in $S^1 \times \mathbf{R}^1$. The reverse statement, however, is not true: if a strand represents a non-trivial element of $\pi_1(S \times \mathbf{R}^1)$, then the braid does not come from an ordinary braid. For instance $CB(1) \cong \mathbf{Z}$.

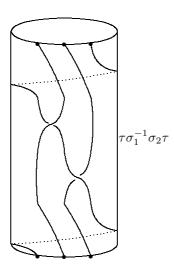


Figure 10.15. A cylindrical braid diagram and the corresponding word

Like the ordinary braid group, the cylindrical braid group CB(n) has a simple presentation by generators and relations.

Indeed, let β be a braid diagram on the cylinder $S^1 \times \mathbf{R}_t$. Then, having a cylindrical braid diagram β , we can write a word as follows. Denote the set $I \times \mathbf{R} = (S^1 \setminus x) \times \mathbf{R}$ by T. Each non-singular level of the braid β consists of n points (coming from the strand). So, each crossing can be given a number $\sigma_i^{\pm 1}$ as in the case of ordinary braids, $i = 1, \ldots, n$.

For the intersection point we write τ if while walking along the strand downwards we intersect the rightmost boundary of T and return from the left side, and τ^{-1} otherwise.

Here τ represents an additional generator (with respect to the ordinary braid group generator).

Obviously, the elements $\sigma_1 \dots, \sigma_{n-1}$ together with τ form a system of generators. An example of a braid word obtained from a cylindrical braid diagram is shown in Fig. 10.15.

As in the case of ordinary braids, the set of moves concerning cylindrical braids can be easily constructed. They are:

- 1. Moves of the diagram preserving the combinatorial structure of crossings (but, possibly, changing the height order of a crossing), see Fig. 10.16.
- 2. The second Reidemeister move, see Fig. 10.17.
- 3. The third Reidemeister move, see Fig. 10.18.

Besides the relations for the ordinary braid group $\{\sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| > 1\}$ and $\{\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\}$ we get more relations: $\tau\sigma_{i+1} = \sigma_i\tau$ for $i = 1, \ldots, n-2$ and $\sigma_{n-1}\tau^2 = \tau^2\sigma_1$.

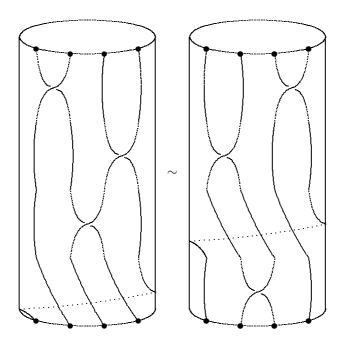


Figure 10.16. Transforming a cylindrical braid diagram

The geometric meaning of the additional relations is as follows. The first series represents the change of crossing numeration under the action of τ : σ_i becomes σ_{i+1} when the rightmost strand appears on the left flank. The second additional series (of one relation) means that the rightmost crossing is moved by one full turn together with the two strands, generating it.

It can easily be checked that this system of relations is complete.

Remark 10.11. It is easy to show that the additional relations place the element τ^n at the centre of the toric braid group.

Remark 10.12. Both the second and the third Reidemeister moves for cylindrical braids are considered in a part of cylinder, i.e., they are just the same as in the case of an ordinary braid.

Now, having a diagram β of a cylindrical braid b, let us construct the invariant $f_C(b) \equiv f(\beta)$ (in the sequel, we prove that it is well defined).

Consider the cylinder $S^1_{\varphi} \times R^1$.

Definition 10.30. An admissible cylindrical system of n curves is a family of n non-intersecting non-self-intersecting curves in the upper half-cylinder $S^1 \times R_+$ such that each curve connects a point with abscissa zero with a point with abscissa one, such that the coordinate φ for all curve ends runs through the set

$$\left\{0, \frac{2\pi}{n}, \dots, \frac{2(n-1)\pi}{n}\right\}$$

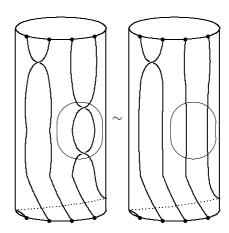


Figure 10.17. Applying the second Reidemeister move to a cylindrical braid diagram

All points

$$\left(\frac{2\pi j}{n},1\right),$$

where j = 1, ..., n are called *upper points*, and all points (j, 0) are called *lower points*.

Denote

$$S^1 \setminus \cup_i \left\{ \frac{2\pi i}{n} \right\}$$

by C_n .

Consider the diagram β on the cylinder. Now let us resolve all crossings of β starting from the upper one as in the case of the ordinary braid.

Thus we obtain an admissible cylindrical system of curves.

The next definition is similar to that for the case of ordinary braids.

Definition 10.31. Two admissible cylindrical systems of curves A and A' are called *equivalent* if there exists a homotopy between A and A' in the class of curves with fixed end points, lying in the upper half–cylinder, such that no interior point of any curve can coincide with an upper point.

Having a diagram β of a braid b, we obtain an admissible cylindrical system $A(\beta)$ of curves, corresponding to it. We can take an equivalence class of $A(\beta)$. Denote it by $f_C(\beta)$.

Now, let us prove the following theorem.

Theorem 10.8. 1. Map f_C is a braid invariant, i.e., for different β_1, β_2 representing the same braid b we have $f_C(\beta_1) = f_C(\beta_2)$. In this case we shall write simply f(b).

2. The invariant f_C is complete, i.e., $f_C(b_1) = f_C(b_2)$ implies $b_1 = b_2$.

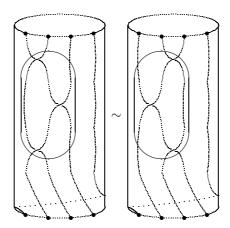


Figure 10.18. Applying the third Reidemeister move to a cylindrical braid diagram

Proof. To prove the first part of the theorem, we only have to check the invariance of f_C under the second and the third Reidemeister moves. The proof is the same as in the case of ordinary braids, see Figs. 10.3 and Fig. 10.4.

The proof of the second part is also analogous to the proof of completeness in the ordinary case. By an admissible cylindrical system of curves we restore a cylindrical braid (with lower points $r=1, \varphi=\frac{2\pi j}{n}$ and upper points $r=2, \varphi=\frac{2\pi k}{n}$) by parametrising each curve from the system from 1 to 2. Now, let us prove that equivalent admissible cylindrical systems of curves generate the same braid.

To do this, let us fix the permutation $s \in S_n$ (obviously, two admissible curves can be equivalent only if their permutations coincide). Then we choose two equivalent admissible systems A and A' of n curves and choose admissible parametrisations for them.

Like the ordinary braid invariant f, the invariant f_C is also easily recognisable. Indeed, instead of curves on P_n , we consider curves in C_n . So, our invariant can be completely encoded by the following object.

Let G_t be a free group with generators a_1, \ldots, a_n, t . Denote by G_i , $i = 1, \ldots, n$, the right residue class of G by a_i .

Definition 10.32. A cylindrical n-family is a set $g_1 \in G_1, \ldots, g_n \in G_n$ together with a permutation $s \in S_n$.

Obviously, values of the invariant f_C can be completely encoded by cylindrical n-families. The permutation is taken directly from the admissible system of curves, and elements g_i correspond to curve homotopy types in C_n with fixed points, where t stands for the element of C_n obtained by passing along the parallel of P_n .

Chapter 11

Markov's theorem. The Yang–Baxter equation

In his celebrated work [Mar'], A.A. Markov has described the theorem about necessary and sufficient conditions for braids to represent isotopic links. However, his proof did not contain all rigorous details. He left this problem to N.M. Weinberg, who died soon after his first publication on the subject [Wei]. The first published rigorous proof belongs to Joan Birman, [Bir]. The newest proofs of Markov's theorem can be found in [Tra] and in [BM].

We shall describe the proof according Hugh Morton [Mor], where a shorter (than Morton's one) proof is given.

After this, we shall give some precisions of Alexander's and Markov's theorems due to Makanin.

In the third part of the chapter, we shall discuss the Yang–Baxter equation, which is closely connected with braid groups and knot invariants.

11.1 Markov's theorem after Morton

11.1.1 Formulation. Definitions. Threadings.

The Markov theorem gives an answer to the question of when the closures of two braids represent isotopic links. However, this answer does not lead to an algorithm, i.e., it gives only a list of moves, necessary and sufficient to establish such an isotopy, but does not say how to use these moves and when to stop.

In his work, Morton uses the original idea of *threading* — an alternative way of representing a link as a closure of a braid (besides those proposed by Alexander and Vogel).

Remark 11.1. We shall consider each closure of braids as a set of curves inside the cylinder not intersecting its axis; the axis of the braid is the closure of the curve coinciding with the axis of the cylinder inside the cylinder.

We start with the definitions.

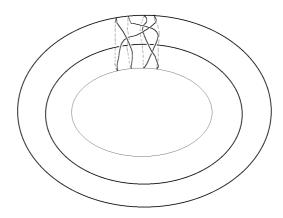


Figure 11.1. Representing a braid in a full torus

Definition 11.1. Let K be an oriented link in \mathbb{R}^3 . Let L be an unknotted curve. We say that K is braided with respect to L or $K \cup L$ is a braid-link, if K and L represent the closure of some braid and the axis of this braid, respectively (i.e., K lies inside the full torus $S^1 \times D$, where the coordinate of S^1 is increasing, and L is the axis of the full torus), see Fig. 11.1.

Having a planar diagram of some braid closure, the corresponding braid-link can be obtained from it by threading this diagram by a circle, see Fig. 11.2.

Let K be a planar diagram of some oriented link. Consider some curve L on the projection plane P of the link K such that the curve L intersects the projection of the link K transversely and does not pass through crossings of K.

Definition 11.2. A choice of overpasses for a link diagram K is a union of two sets $S = \{s_1, \ldots, s_k\}, F = \{f_1, \ldots, f_k\}$ of points at the edges of K (points should not coincide with crossings) such that while passing along the orientation of K, the points from S alternate with points from F; besides, each interval [s, f] does not contain undercrossings and each [f, s] does not contain overcrossings, i.e., [s, f] are arcs and [f, s] are lower arcs.

Definition 11.3. We say that a curve L whose projection on the plane of the link K is a simple curve threads K according to a given choice (S, F) of overpasses if the interval of K goes over L when it starts in a domain containing points from S, and it passes under L if it starts in a domain containing points from S, as it is shown in Fig. 11.3.

Remark 11.2. We do not require that this interval of L contain elements of the set S or F.

For a given link diagram K and a curve L on the projection plane, such that L separates points from S from points from F then we can arrange overcrossings and undercrossings at intersection points between K and L in such a way that the curve L' obtained from L threads the link K.

Let us now prove the following theorem.

Theorem 11.1. If L threads the link K then K is a braid with respect to L.

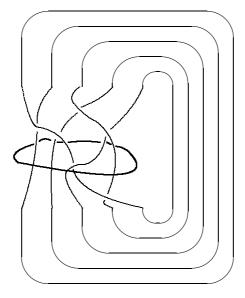


Figure 11.2. A threaded braid closure

Proof. Let us choose the overpasses (S, F) for the diagram of the link K (in an arbitrary way) and some curve L on the projection plane P of the diagram K such that L separates points from S from points belonging to F. Let us straighten the curve L in the plane P by a homeomorphism of P onto itself: we require that the transformed L is a straight line inside a domain D, containing K; L should be closed outside D (say, by a large half-circle). Consequently, points of S lie on one side of this line, and points of F lie on the other side. Such a transformation is shown in Fig. 11.4.

Without loss of generality, we can suppose that all undercrossings and overcrossings of the diagram L lie in two planes parallel to P (just over and under the images of the corresponding projections).

Now, let us change the point of view and think of P as the plane Oxz and L as the axis Oz that is closed far away from the origin of coordinates.

Let us consider the line L (without its "infinite" circular part) as the axis of cylindrical coordinates. Then the plane P is divided into two half–planes; one of them is given by the equation $\{\theta=0\}$ and the other one satisfies the equation $\{\theta=\pi\}$. Here the half–plane z=0,x>0 is thought to have coordinate $\theta=0$; points over this half-plane are thought to have positive coordinates.

Let us construct a link isotopic to K as follows. Place all lower arcs of K (i.e., all intervals [f,s]) on the half planes $\{\theta=-\varepsilon\}$ and $\{\theta=\pi+\varepsilon\}$, and all arcs on the half-planes $\{\theta=\varepsilon\}$ and $\{\theta=\pi-\varepsilon\}$, where ε is small enough. Herewith, we shall add small intervals over all points belonging to S or F such that each interval is projected to one point on Oxz.

Let us represent the arcs where K intersects with L by vertical arcs.

Thus, we have made the polar coordinate θ of the link K to be always constant or increasing.

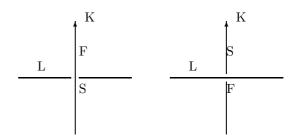


Figure 11.3. Crossings with L

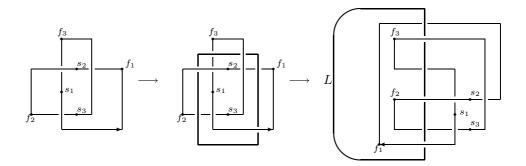


Figure 11.4. Straightening the curve L

In Figs. 11.5 and 11.6 we show how to construct a knot with non-decreasing polar coordinate. This knot is isotopic to the knot shown in Fig. 11.4.

After a small deformation of the obtained link, we can make this coordinate strictly monotonic.

Thus, the transformed link (which we shall also denote by K) will represent a braid with respect to L.

It follows from Theorem 11.1 that if some link K is a braid with respect to an unknotted curve L, then K is isotopic to a closure of some braid.

Theorem 11.2. Each closure K of any braid B admits a threading by some curve L in such a way that K is a braid with respect to L.

Proof. Let $D^2 \times I$ be a cylinder. Consider B as a braid connecting points lying on the upper base of the cylinder with points on the lower base of the same cylinder. Now, let us close the braid as follows. Connect the lower points with the upper ones by lines, going horizontally along the bases and vertically at some discrete moments, as shown in Fig. 11.7.

Let us apply the isotopy that straightens the strands and changes homomorphically the upper base of the cylinder. Thus we obtain a link that admits the simple

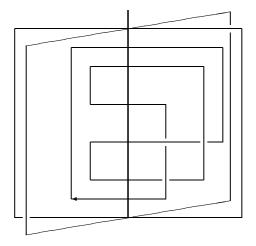


Figure 11.5. Link diagram intersecting L

threading that can be constructed as follows. Let h_0 be the height level of the lower base and h_1 be the level of the upper base. Denote the set of lower ends of the braid B by A_1 and the set of upper ones by A_2 .

One can assume that the levels h_0 and h_1 contain some additional sets of vertices B_1 and B_2 by means of which we are going to construct the closure of the braid. More precisely, the points from A_1 are connected by parallel lines with points from A_2 , and points from B_1 are connected by parallel lines with points from B_2 . Now, let us consider the circle lying on the plane at the level $h = \frac{h_1 + h_2}{2}$ and separating sets of lines A_1A_2 and B_1B_2 .

Let us project the diagram on the base of the cylinder and take the set A_1 as S and A_2 as F.

It is easy to see that in this case the projection of the circle is really a threading of the link. \Box

Theorems 11.1 and 11.2 imply the Alexander theorem; the proofs of these theorems give us a concrete algorithm (different from Alexander's and Vogel's methods) to represent any link as a closure of a braid.

Let us now recall the main theorem of this chapter. It has already been formulated in Chapter 8.

Theorem 11.3. The closures of braids A and B are isotopic if and only if B can be obtained from A by a sequence of the following moves (Markov moves):

- 1. conjugation $b \to a^{-1}ba$ by an arbitrary braid a with the same number of strands as b.
- 2. the move $b \to b\sigma_n^{\pm 1}$, where b is a braid on n strands and the obtained braid has n+1 strands,
- 3. the inverse transformation of 2.

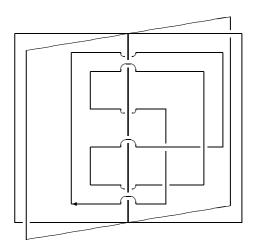


Figure 11.6. The polar coordinate is increasing while moving along the link

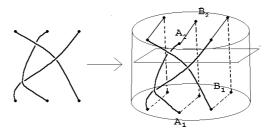


Figure 11.7. A braid and a braided link

The necessity of these two moves is evident. The isotopies between corresponding pairs of braid closures are shown in Fig. 11.8.

In Fig. 11.8.b the first Reidemeister move comes into play. This move did not take part in braid isotopies, so this kind of knot isotopy appears here.

11.1.2 Markov's theorem and threadings

Let us now reformulate the difficult part of the Markov theorem.

To do it, we shall need some definitions.

Definition 11.4. We say that two braided links $K \cup L$ and $K' \cup L'$ are *simply Markov equivalent* if there exists an isotopy of the second one, taking L' to L and K' to the link coinciding with K everywhere except one arc. The link K contains an arc α and K' contains a link α' with the same ends.

- 1. The polar coordinate is constant on the arc α and monotonically increasing on α' .
- 2. The arcs α, α' bound a disc intersecting L transversely at a unique point.

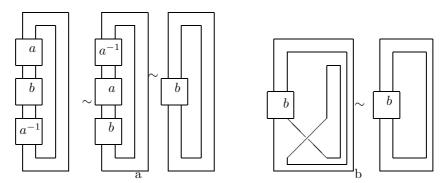


Figure 11.8. Two Markov moves represent isotopy

Definition 11.5. Two braided links are *Markov equivalent* if one of them can be transformed to the other by a sequence of isotopies and simple Markov equivalences.

Exercise 11.1. Show that closures of two n-strand braids are isotopic in the class of closures of n-strand braids if and only if these two braids are conjugated.

Lemma 11.1. If links $K \cup L$ and $K' \cup L'$ are simply Markov equivalent, then they represent threaded closures of braids, which are isotopic to some braids $\beta \in Br(n)$ and $\beta \sigma_n^{\pm 1} \in Br(n+1)$.

Proof. Suppose the polar coordinate evaluated at points of the curve α equals θ_0 . Consider the arc α_0 . Without loss of generality, one can assume that the coordinate is almost everywhere equal to θ_0 and in some small neighbourhood of L' the arc α_0 makes a loop and this loop corresponds to the n-th (last) strand of the braid α . We can isotop the braids K and K' in the neighbourhood $\{\alpha = \theta_0 \pm \varepsilon\}$ in such a way that the final points of the arcs α and α_0 lie in a small neighbourhood of L'. The remaining part of the Lemma is now evident.

Lemma 11.1 together with Exercise 11.1 allows us to reformulate the difficult part of the Markov theorem as follows:

Theorem 11.4. Let β and γ be two braids whose closures B and Γ are isotopic as oriented links. Let us thread these closures and obtain some closures B' and Γ' . Then B' and Γ' are Markov equivalent.

To go further, we shall need some auxiliary lemmas and theorems.

Lemma 11.2. Consider an oriented link diagram K on the plane P and fix the choice of overpasses (S, F). Then the threadings of K by different curves L and L' separating the sets S and F are Markov equivalent.

Proof. The main idea of the proof is the following. First we consider the case when the curves L and L' are isotopic in the complement $P \setminus (S \cup F)$. In this case, one of them can be transformed to the other by means of moves in such a way that each of these moves is a Markov equivalence.

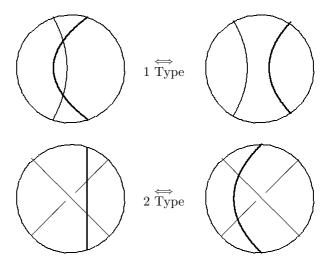


Figure 11.9. Composition of transformations of the two types

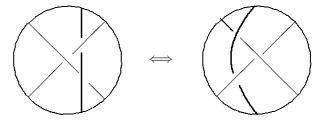


Figure 11.10. A Markov equivalence

In the common case we shall use one extra move when two branches of the line L pass through some point from S (or F). Such a move is a Markov equivalence as well (this will be clear from the definition).

Let us give the proof in more detail.

The case a).

Suppose L and L' are isotopic in $P \setminus (S \cup F)$. Then $K \cup L$ and $K \cup L'$ can be obtained from each other by a sequence of transformations of the first and the second type, shown in Fig. 11.9.

The first type is represented either by the second Reidemeister move or by the "hooking" move, that adds two crossings in alternating order. It will be shown below that this move is a simple Markov equivalence.

The second type of transformation is an isotopy in all cases except that shown in Fig. 11.10.

In this case, the first threading can be transformed to the second one by a sequence of moves of the first type and isotopies, see Fig. 11.11.

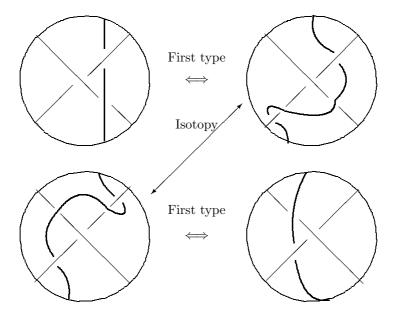


Figure 11.11. Composition of Markov equivalences

Here one should note that the passes of the diagram of K under the line L are alternating with passes of K over L while going along the link K. It remains to show that the two threadings obtained from each other by a transformation of the first type are simply Markov equivalent.

Thus, the part of the link K shown in Fig. 11.11 belongs either to an upper branch or to the lower branch. In the threading construction we can assume that both parts of the link K on the same side of the line L lie on one and the same level p_L (it might be either $\{\theta = \pi \pm \varepsilon\}$ or $\{\theta = \pm \varepsilon\}$ depending on the side of overcrossing).

Let us connect them by an arc as shown in Fig. 11.12.

Now, the arcs α and α' bound a disc. Thus we obtain a simple Markov equivalence of the two threadings.

The case b).

In the general case, note that if the curve M' of P that separates the set S from the set F can be isotoped to the curve M by means of moving the two arcs of the link K through some point of S or F (as shown in Fig. 11.13), then M and M' represent isotopic threadings.

Such an isotopy of the "curvilinear" line L (or, equivalently, motions of points from the sets S and F) is divided into several steps. Between these steps, we apply discrete moves changing the combinatorial type of the disposition for L with respect to the sets S and F. Thus, one can consider the discrete set of such dispositions, between two of each some elementary transformation takes place.

Without loss of generality, we may assume the following.

Let S and F consist of points $\{(-1, a_i)\}$ and $\{(1, a_i)\}$ for some $a_1, \ldots a_k$, respec-

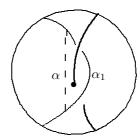


Figure 11.12. The curve passes through a disk

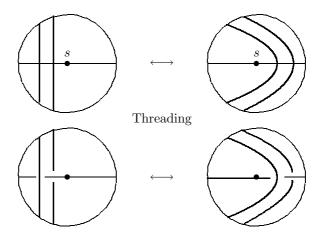


Figure 11.13. Two branches of the curve pass through a fixed point

tively. Let L be a part of Oy closed by a large half-circle in such a way that the interior of L contains F. We may assume that K is parallel to Ox near each of the points s_1, \ldots, s_k .

Let L' be any simple closed curve separating S from F and restricting a domain that contains the set F. Without loss of generality, we can suppose that the curve L' intersects the rays $y = a_i, x < -1$ transversely, see Fig.11.14. Let us enumerate these intersections according to the decreasing of the abscissa. For each of the rays, the number of intersections is even. For each ray, let us group them pairwise: $(1,2),(3,4),\ldots$ To do this, we first isotop L' inside $P\setminus S \cup F$ in such a way that between each pair of points there are no crossings of the diagram K.

Now, let us move the curve line L' to the right in such a way that after performing the operation all points lie on the left side of the curve L. Let us divide such a transformation into stages when L' does not contain points from S and moments when L' does. In the first case, such a transformation is a Markov equivalence as in the case a). In the second case, let us assume that the intersection points of L' with each ray disappear pairwise, i.e., the curve L' consequently passes two crossings

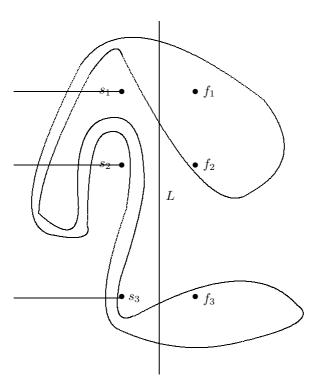


Figure 11.14. Moving fixed points to the left

with the same point s_i . This move is a Markov equivalence as well, see Fig. 11.13. Thus, the threading by means of L' is Markov equivalent to the threading by the curve lying on the right hand related from all s_i . This curve is isotopic to L inside the set $P\setminus (S \cup F)$, i.e., the threading by means of such curve is Markov equivalent to the threading by means of L, see a). Consequently, the threading by L' is Markov equivalent to the threading by L. This completes the proof of the lemma.

Lemma 11.3. Given a choice (S,F) of overpasses for a link diagram K, and a point s in K, not belonging to F, then there exists a choice of overpasses (\bar{S},\bar{F}) such that $s \in \bar{S}, S \subset \bar{S}, F \subset \bar{F}$.

Proof. The idea of the proof is pretty simple: we add elements of S or F where we want compensating them by corresponding elements of F or S. If s lies on an upper arc of (S,F) then one can choose f just before s with respect to the orientation of K; thus the interval $[f,s] \subset K$ contains no overcrossings. In the case when s lies on a lower arc, we can add f just after s; in this case $[s,f] \subset K$ contains no undercrossings.

Theorem 11.5. Each two threadings $K \cup L$ and $K \cup L'$ of the same diagram K are Markov equivalent.

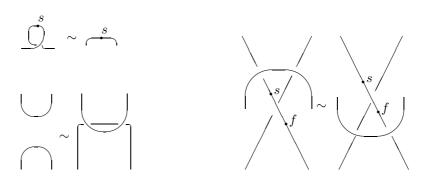


Figure 11.15. The three Reidemeister moves

Proof. Let us choose some overpasses (S,F) for the threading $K \cup L$ and (S',F') for the threading $K \cup L'$. According to Lemma 11.3, there exists a choice of overpasses (S'',F'') such that $(S,F),(S',F') \supset (S'',F'')$, and the two threadings with this choice of overpasses, the first of which is Markov equivalent to $K \cup L$ and the second is Markov equivalent to $K \cup L'$. By Lemma 11.2, these two threadings are Markov equivalent. Thus, the initial two threadings are Markov equivalent and this completes the proof.

Theorem 11.6. Any two planar diagrams of isotopic links have Markov-equivalent threadings.

Proof. To prove this theorem, we have to show how to construct Markov equivalent threadings for diagrams obtained from each other by using Reidemeister moves.

By Theorem 11.5, we can take any choice of overpasses for each of these diagrams. The idea is to be able to reconstruct the choice of overpasses together with L after each Reidemeister move.

Without loss of generality, for the first two Reidemeister moves we can choose the separating curve outside the small disc of the move (in the case of Ω_1 we choose one vertex s inside the disc, in the case of Ω_2 all vertices from (S, F) are outside the disc).

We have shown that the link diagrams obtained from each other by Ω_1 or Ω_2 obtain Markov equivalent threadings.

For Ω_3 we can take one vertex s and one vertex f inside the disc and all the other vertices outside the disc, see Fig. 11.15.

In Fig. 11.16 we show that the threading corresponding to diagrams obtained from each other by Ω_3 are isotopic and hence Markov equivalent (we show only one case, the other cases of Ω_3 , with orientation and disposition of L and K in the left picture are quite analogous).

In the upper part of this figure, we show how the line L can be transformed with respect to this move Ω_3 . In the lower part, we show how one concrete transformation is realised with overcrossings and undercrossings between L and K.

Now, we are ready to prove the difficult part of the Markov theorem.

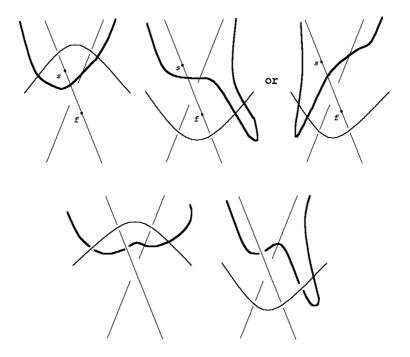


Figure 11.16. Isotopic threadings of diagrams differ by Ω_3

Proof of Theorem 11.4. Let $K \cup L$, $K' \cup L'$ be two braided links whence K and K' are isotopic as links.

By Theorem 11.2, the link $K \cup L$ is a threading of some diagram K and the link $K' \cup L'$ is a threading of some diagram K'. By Theorem 11.6, one can choose Markov–equivalent threadings for the first and the second diagram. By Theorem 11.5, the first one is Markov equivalent to $K \cup L$ and the second one is Markov equivalent to $K' \cup L'$.

Consequently, the threading $K \cup L$ is Markov equivalent to $K' \cup L'$, which completes the proof of Markov's theorem.

Let us now present an example of how to use the Markov theorem.

As we have proved before, for each two coprime numbers p and q, the toric knots of types (p,q) and (q,p) are isotopic. Let us demonstrate the Markov moves for the braids, whose closures represent trefoils: (2,3), (3,2).

Example 11.1. Actually, the first braid has two strands and is given by σ_1^{-3} ; the second one (which has three strands) is given by $\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}$. Let us write down a sequence of Markov moves transforming the first braid to the second one:

$$\sigma_1^{-3} \overset{\mathcal{Q}}{\longrightarrow} \overset{\textit{move.}}{\longrightarrow} \sigma_1^{-3} \sigma_2^{-1} \overset{\textit{conj.}}{\longrightarrow} \sigma_1^{-2} \sigma_2^{-1} \sigma_1^{-1}$$

$$=\sigma_1^{-1}(\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1})\stackrel{braid\ isotopy}{\longrightarrow}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}.$$

Exercise 11.2. Perform the analogous calculation for the case of toric knots T(2, 2n + 1) and T(2n + 1, 2) and for the knots T(3, 4), T(4, 3).

11.2 Makanin's generalisations. Unary braids

In his work [Mak], G.S. Makanin proposed a nice refinement of the Alexander and Markov theorems: he proved that all knots (not braids) can be obtained as closures of so-called unary braids. Besides, he proved that for any two unary braids representing the same knot, there is a change of Markov moves from one to the other that lies in the class of unary braids. Furthermore, Makanin also gave some generating system of (harmonic) braids, such that their adjoint action has unary braids as invariant set. For more details, see the original work [Mak].

Throughout this section, all knots are taken to be oriented.

Definition 11.6. An (n+1)-strand braid is called *unary* if the strand having ordinate one on the top has ordinate (n+1) on the bottom and, after deleting the first strand, we obtain the trivial n-strand braid.

Obviously, unary braids generate only knots (not links).

Theorem 11.7 ([Mak]). For each knot isotopy class K, there exists a unary braid B, such that Cl(B) is isotopic to K.

The first step here is to note that each knot K can be represented by a braid β with permutation $P = (1 \to 2 \to 3 \cdots \to n+1 \to 1)$.

The second step of the proof is to show that each braid (e.g. β with permutation P) is conjugated to some unary braid β' . Thus, by Markov's theorem, $Cl(\beta')$ is isotopic to K.

Here is the key lemma.

Lemma 11.4. Let K be a braid from Br(n+1) with permutation P. Then there exists a unary braid Y that is conjugated with K by means of a strand from Br(n).

This lemma follows from the construction of Artin [Art1] of so–called "reine Zöpfe." There is a nuance concerning mathematical terminology in German. "Reine Zöpfe" is literally "pure braids," but they have some other meaning in German rather than in English. The word for "pure braids" used in German is "gefärbte Zöpfe" which literally means "coloured braids."

For more details concerning these notions and the proof of the lemma, see the original work of Makanin [Mak].

Theorem 11.8 ([Mak]). Let K_1, K_2 be two isotopic knots. Let B, B' be two unary braids representing the knots K_1 and K_2 , respectively. Then there exists a chain of unary braids $B = B_1, B_2, \ldots, B_k = B'$ such that each N_i is obtained from B_{i-1} by a Markov move (for all $i = 2, \ldots, k$).

Proof. Our strategy is the following: first we find some chain "connecting" these braids by Markov's moves and then modify it by means of some additional Markov moves in order to obtain only unary braids.

Let K be a unary braid from Br(p+1) and L be a unary braid from Br(q+1). Thus, there exists a sequence of braids K, Q_1, Q_2, \ldots, Q_t where each braid Q_{i+1} can be obtained from the previous one by one Markov move. Without loss of generality, we can assume that no two conjugations are performed one after the other. Besides, between any two Markov moves of second type we can place a conjugation by the unit braid. Thus, we might assume that each move $Q_{2j-1} \to Q_{2j}$ is a conjugation, and $Q_{2j} \to Q_{2j+1}$ is the second Markov move (either addition or removal of a strand). The other case can be considered analogously.

Furthermore, one can easily see that when an m-strand braid is and a unary braid are conjugated, the braid for conjugation can be chosen from Br(m-1). This follows from the Lemma 11.4.

Now, let us construct our chain. For each braid Q_i having n_i strands, let Y_i be a unary braid conjugated with the braid Q_i by a $(n_i - 1)$ -strand braid.

Obviously, each Y_{2j-1} is conjugated with Y_{2j} by some braid from $Br(n_{j2})$. Besides, the braid $Y_{2j} \in Br(n_{2j})$ is conjugated with $Q_{wj} \in Br(n_{2j})$ by means of some braid δ from $Br(n_{2j}-1)$. In the case when the transformation $Q_{2j} \to Q_{2j+1}$ adds a strand, we can perform the same operation for Y_{2j} . We obtain a braid Y_{2j}^* . Obviously, the braids Y_{2j} and Y_{2j+1} are conjugated by means of δ . So, Y_{2j}^* and Y_{2j+1} are conjugated.

In the other case when $Q_{2j} \to Q_{2j+1}$ deletes a strand, the reverse move $Q_{2j+1} \to Q_{2j}$ adds a loop. Arguing as above, we see that the braids Y_{2j} and Y_{2j+1} are connected by some Markov moves involving only unary braids.

Thus, we have constructed a chain of unary braids, connecting B with B', which completes the proof.

The Makanin work allows us to encode all knots by using words in some finite alphabet. Namely, in order to set an n-strand braid, we should just describe the behaviour of the first strand of it, i.e., we must indicate when it goes to the right (to the left) and when it forms an over (under)crossing. To do this, it would be sufficient to use four brackets: \rightarrow , \rightarrow , $\stackrel{+}{\leftarrow}$, and $\stackrel{-}{\leftarrow}$.

The only condition for such a word to give a braid is that the total number of \rightarrow minus the total number of \leftarrow never exceeds n in all initial subwords, and equals precisely n for the whole word.

Another approach with some detailed encoding of knots and links by words in a finite alphabet (bracket calculus) will be described later in Part IV.

11.3 The Yang–Baxter equation, braid groups and link invariants

The Yang–Baxter equation (YBE) was first developed by physicists. However, these equations turn out to be very convenient in many areas of mathematics. In particular, they are quite well suited for describing braids and their representations. In our book we shall not touch on the connection between the YBE and physics, for more details see in, e.g., [Ka2].

Just after the revolutionary works by Jones [Jon1, Jon2], it became clear that the Jones polynomial and its generalisations (e.g. the HOMFLY polynomial) are in some sense a vast family of knot invariants coming from quantum representations – the quantum invariants; for the details see [Jon3, RT, Tur1].

Let V be the finite dimensional vector space with basis e_1, \ldots, e_n over some field F. Let $R: V \otimes V \to V \otimes V$ be some endomorphism of $V \times V$. Consider the endomorphism

$$R_i: Id \otimes \cdots \otimes Id \otimes R \otimes Id \otimes \cdots \otimes Id: V^{\otimes n} \to V^{\otimes n}$$

where $V^{\otimes n}$ means the *n*-th tensor power of V, Id is the identity map, and R_i acts on the product of spaces $V \otimes V$, which have numbers (i, i + 1).

Definition 11.7. An operator R is said to be an R-matrix if it satisfies the following conditions:

$$R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1}, i = 1, \dots, n-1$$

$$R_i R_i = R_i R_i, \ |i - j| \ge 2.$$

The first of these conditions is called the Yang-Baxter equation; the second one is just far commutativity for R_1, \ldots, R_{n-1} .

The YBE look quite similar to Artin's relations for the braid group: they differ just by replacing σ with R, and we obtain one equation from the others.

Now, we are going to show the mathematical connection between the YBE, braid group representations, and link invariants.

Having an R-matrix, one can construct a link invariant by using the following construction (proposed by V.G. Turaev in [Tur1]). Below, we just sketch this construction; for details see the original work.

First, we construct a representation of the braid group Br(n) to the tensor power $V^{\otimes n}$ as $\rho(\sigma_i) = R_i$, where σ_i are standard generators of the braid groups.

After this, for any given link L we find a braid b with closure Cl(b) that is isotopic to L.

One can set $T(L) = trace(\rho(b))$. In this case, T(L) is invariant under braid isotopies and the first Markov move (conjugation). The latter follows from the simple fact that for any square matrices A and B of the same size, we have $trace(A) = trace(BAB^{-1})$.

In some cases (see [Tur1]), the function T(L) is invariant under the second Reidemeister move as well. Thus, T(L) gives a link invariant. As we have shown before, one can also introduce some specific traces (namely, the Ocneanu trace) instead of the ordinary trace. Such traces behave quite well under both Markov's moves and lead to the Jones polynomial in two variables.

Remark 11.3. Numerical values of the Jones polynomial can be obtained by means of R-matrices.

In quantum mechanics, one also considers the quantum YBE that looks like

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. (*).$$

Here R_{ij} are obtained from some fixed matrix R generating an automorphism R: $V^{\otimes n^2} \to V^{\otimes n^2}$ (e.g. $R_{12} = R \otimes I_n$, where I_n is the $(n \times n)$ identity matrix. Each solution of (*) is called quantum R-matrix.

For the study of quantum invariants, we recommend the beautiful book by Ohtsuki, [Oht].

A review of classical R-matrices can be read in [Sem]. An interesting work concerning the problem of finding R-matrices was written by the well-known Dutch mathematician Michiel Hazewinkel [Haz].

Part III Vassiliev's invariants

Chapter 12

Definitions and basic notions of Vassiliev invariant theory

The Vassiliev knot invariants were first proposed around 1989 by Victor A. Vassiliev [Vas] while studying the topology of discriminant sets of smooth maps $S^1 \to \mathbf{R}^3$. A bit later, Mikhail N. Goussarov [Gus] independently found a combinatorial description of the same invariants.

12.1 Singular knots and the definition of finite type invariants

Throughout this part of the book, all knots are taken to be oriented, unless otherwise specified. Besides, we deal only with knots, not links. The analogous theory can be constructed straightforwardly for the case of links; the definitions are however, a bit more complicated.

As we know, each knot can be transformed to the unknot by switching some crossings. This switch can be thought of as performed in \mathbb{R}^3 .

Having a knot invariant f, one can consider its values on two knots that differ at only one crossing. Certainly, these two knots might not be isotopic; hence, these values might not coincide.

While switching the crossing continuously, the most interesting moment is the intersection moment: in this case we get what is called a $singular\ knot$. More precisely, a $singular\ knot$ of degree n is an immersion of S^1 in \mathbf{R}^3 with only n simple transverse intersection points (i.e., points where two branches intersect transversely).

Singular knots are considered up to isotopy. The isotopy of singular knots is defined quite analogously to that for the case of classical knots. The set of singular knots of degree n (for n = 0 the set \mathcal{X}_0 consists of the classical knots) is denoted by \mathcal{X}_n . The set of all singular knots (including \mathcal{X}_0) is denoted by \mathcal{X} .

So, while switching a crossing of a classical knot, at some moment we get a singular knot of order one.

Then, we can define the *derivative* f' of the invariant f according to the following relation:

$$f'(\langle X \rangle) = f(\langle X \rangle) - f(\langle X \rangle). \tag{1}$$

This relation holds for all triples of diagrams that differ only outside a small domain (two of them represent classical knots and represents the corresponding singular knot).

This relation is called the Vassiliev relation.

It is obvious that the invariant f' is a well-defined invariant of singular knots because with each singular knot and each vertex of it, we can associate the positive and the negative resolutions of it in \mathbb{R}^3 . If we isotop the singular knot, the resolutions are "isotoped" together with it.

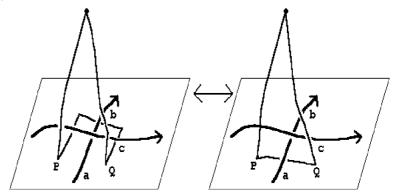
Having a knot invariant $f: \mathcal{X}_0 \to A$, one can define all its derivatives of higher orders. To do this, one should take the same formula for two singular knots of order n and one singular knot of order n+1 (n singular vertices of each of them lie outside of the "visible" part of the diagram) and then apply the Vassiliev relation (1).

Denote by \mathcal{V}_n the space of all Vassiliev knot invariants of order less than or equal to n.

Thus, we define some invariant on the set \mathcal{X} . This invariant is called the *extension of f for singular knots*.

Notation: $f^{(n)}$.

Example 12.1. Let us calculate the extension of the Jones polynomial evaluated on the simplest singular knot of order two. After applying the Vassiliev relation twice, we have:



Definition 12.1. An invariant $f: \mathcal{X}_0 \to A$ is said to be a (Vassiliev) invariant of order $\leq n$ if its extension for the set of all (n+1)-singular knots equals zero identically.

Definition 12.2. A Vassiliev invariant of order (type) $\leq n$ is said to have order n if it is not an invariant of order less or equal to n-1.

 $^{^{1}}A$ can be a ring or a field; we shall usually deal with the cases of \mathbf{Q}, \mathbf{R} and \mathbf{C} .

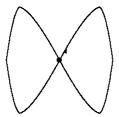


Figure 12.1. The simplest singular knot

12.2 Invariants of orders zero and one

The definition of the Vassiliev knot invariant shows us that if an invariant has degree zero then it has the same value on any two knots having diagrams with the same shadow that differ at precisely one crossing. Thus, it has the same value on all knots having the same shadow. Let K be a knot diagram, and S be the shadow of K. There is an unknot diagram with shadow S. So, the value of our invariant on K equals that evaluated on the unknot.

Thus, such an invariant is constant.

It turns out that the first order gives no new invariants (in comparison with 0-type invariants, which are constants).

Indeed, consider the simplest singular knot U shown in Fig. 12.1.

Let S be a shadow of a knot with a fixed vertex which is a singular point.

Exercise 12.1. Prove that one can arrange all other crossing types for S to get a singular knot isotopic to U.

It is easy to see that for each Vassiliev knot invariant I such that I'' = 0 we have I'(U) = 0. Indeed, $I'(U) = I(\bigcirc) - I(\bigcirc) = 0$.

Now, consider an invariant I of degree less than or equal to one. Let K be an oriented knot diagram. By switching some crossing types, the knot diagram K can be transformed to some unknot diagram. Thus, $I(K) = I(\bigcirc) + \sum \pm I'(K_i)$ where K_i are singular knots with one singular point. But, each K_i can be transformed to some diagram U by switching some crossing types. Thus, $I'(K_i) = I'(U) + \sum \pm I''(K_{ij})$, where K_{ij} are singular knots of second order. By definition, $I'' \equiv 0$, thus $I'(K_i) = 0$ and, consequently, $I(K) = I(\bigcirc)$. Thus, the invariant function I is a constant. So, there are no invariants of order one.

12.3 Examples of higher-order invariants

Consider the Conway polynomial C and its coefficients c_n .

Theorem 12.1. For each natural n, the function c_n is a knot invariant of degree less than or equal to n.

Proof. Indeed, we just have to compare the Vassiliev relation and the Conway skein relation:

$$c_n'((\bigotimes)) = c_n((\bigotimes)) - c_n((\bigotimes)) = x \cdot c_n((\bigotimes)).$$

Thus we see that the first derivative of C is divisible by x; analogously, the n-th derivative of C is divisible by x^n . Thus, after n+1 differentiations, c_n vanishes. \square

This gives us the first non-trivial example. The second coefficient c_2 of the Conway polynomial is the second-order invariant (one can easily check that it is not constant; namely, its value on the trefoil equals one).

However, this invariant does not distinguish the two trefoils because the Conway polynomial itself does not. In the next chapter, we shall show how an invariant of degree three can distinguish the two trefoils.

As will be shown in the future, all even coefficients of the Conway polynomial give us finite—order invariants of corresponding orders.

12.4 Symbols of Vassiliev's invariants coming from the Conway polynomial

As we have shown, each coefficient c_n of the Conway polynomial has order less than or equal to n.

Let v be a Vassiliev knot invariant of order n. By definition, $v^{(n+1)} = 0$. This means that if we take two singular knots K_1, K_2 of n—th order whose diagrams differ at only one crossing (one of them has the overcrossing and the other one has the undercrossing), then $v^{(n)}(K_1) = v^{(n)}(K_2)$. Thus, for singular knots of n—th order one can switch crossing types without changing the value of $v^{(n)}$. Hence, the value of $v^{(n)}$ does not depend on knottedness "that is generated" by classical crossings. It depends only on the order of passing singular points.

Definition 12.3. The function $v^{(n)}$ is called the *symbol* of v.

Definition 12.4. By a *chord diagram* we mean a finite cubic graph consisting of one oriented cycle (circle) and unoriented *chords* (edges connecting different points on this cycle). The *order* of a chord diagram is the number of its chords.

Remark 12.1. Chord diagrams are considered up to natural graph isomorphism taking chords to chords, circle to the circle and preserving the orientation of the circle.

Remark 12.2. We shall never indicate the orientation of the circle on a chord diagram, always assuming that it is oriented counterclockwise.

The above statements concerning singular knots can be put in formal diagrammatic language. Namely, with each singular knot one can associate a *chord diagram* that is obtained as follows. We think of a knot as the image of the standard oriented Euclidian S^1 in \mathbb{R}^3 and connect by chords the pre-images of the same point in \mathbb{R}^3 .

So, each invariant of order n generates a function on the set of chord diagrams with n chords. We can consider the formal linear space of chord diagrams with coefficients, say, in \mathbf{Q} , and then consider linear functions on this space generated by

symbols of n-th order Vassiliev invariants (together with the constant zero function that has order zero).

Now, it is clear that the space V_n/V_{n-1} is just the space of symbols that can be considered in the diagrammatic language.

We shall show that for even n, these invariants have order precisely n. Moreover, we shall calculate their symbols, according to [CDL].

Consider a chord diagram D of order n. Let us "double" each chord and erase small arcs between the ends of parallel chords. The constructed object (oriented circle without 2n small arcs but with n pairs of parallel chords) admits a way of walking along itself. Indeed, starting from an arbitrary point of the circle, we reach the beginning of some chord (after which we can see a "deleted small arc"), then we turn to the chord and move along it. After the end of the chord we again move to the arc (that we have not deleted), and so on. Obviously, we shall finally return to the initial points. Here we have two possibilities.

In the first case we pass all the object completely; in the second case we pass only a part of the object.

By performing a small perturbation in \mathbb{R}^3 we can make all chords non-intersecting. In this case our object becomes a manifold m(D). The first possibility described above corresponds to a connected manifold and the second one corresponds to a disconnected manifold.

Proposition 12.1 ([CDL]). The value of the n-th derivative of c_n on D equals one if m(D) has only one connected component and zero, otherwise.

Proof. Let L be a singular knot with chord diagram D. Let us resolve vertices of D by using the skein relation for the Conway polynomial and the Vassiliev relation:

$$C'(\langle \sum \rangle) = x \cdot C(\langle \sum \rangle).$$

Applying this relation n times, we see that the value of the n-th derivative of the invariant C on L (on D) equals the value of C on the diagram obtained from D by resolving all singular crossings, multiplied by x^n . Herewith, the coefficient c_n of the n-th derivative of the Conway polynomial for the case of the singular knot is equal to the coefficient c_0 evaluated at the "resolved" diagram.

This value does not depend on crossing types: it equals one on the unknot and zero on the unlink with more than one component. That completes the proof. \Box

It turns out that knots (as well as odd–component links) have only even–degree non-zero monomials of the Conway polynomial: $c_n \equiv 0$ for even n.

This fact can be proved by using the previous proposition. Let D be a chord diagram of even order n. Suppose that the curve m(D) corresponding to D has precisely one connected component. Let us attach a disc to this closed curve. Thus we obtain an orientable (prove it!) 2-manifold with disc cut. Thus, the Euler characteristic of this manifold should be even. On the other hand, the Euler characteristic equals V - E + S = 2n - 3n + 1 = -n + 1. Taking into account that n is even, we obtain a contradiction that completes the proof.

Obviously, for even n, there exist chord diagrams, where c_n does not vanish.

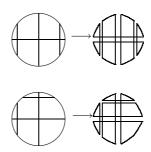


Figure 12.2. Transforming Chord diagrams into systems of curves

Example 12.2. The invariant c_4 evaluated at the diagram \bigoplus (see Fig. 12.2, upper picture) is equal to zero; c_4 evaluated at \bigoplus (see Fig. 12.2, lower picture), is equal to one.

Exercise 12.2. Show that for each even n the value of the n-th derivative of the invariant c_n evaluated on the diagram with all chords pairwise intersecting is equal to one.

This exercise shows the existence of Vassiliev invariants of arbitrary even orders.

Thus we have proved that the Conway polynomial is weaker than the Vassiliev knot invariants.

Thus, we can say the same about the Alexander polynomial that can be obtained from the Conway polynomial by a simple variable change.

12.5 Other polynomials and Vassiliev's invariants

If we try to apply formal differentiation to the coefficients of other polynomials, we might fail. Thus, for example, coefficients of the Jones polynomial themselves are not Vassiliev invariants. The main reason is that the Jones polynomial evaluated at some links might have negative powers of the variable q in such a way that after differentiation we shall still have negative degrees.

In [JJP] the authors give a criterion to detect whether the derivatives of knot polynomials are Vassiliev invariants. They also show how to construct a polynomial invariant by a given Vassiliev invariant.

Although other polynomials cannot be obtained from the Conway (Alexander) polynomial by means of a variable change, Vassiliev invariants are stronger than any of those polynomial invariants of knots (except possibly for the Khovanov polynomial). The results described here first arose in the work by Birman and Lin [BL] (the preprint of this work appeared in 1991), see also [BN].

First, let us consider the Jones polynomial. Recall that the Jones polynomial satisfies the following skein relation:

$$q^{-1}V((\sum))-qV((\sum))=(q^{\frac{1}{2}}-q^{-\frac{1}{2}})V((\sum))$$

Now, perform the variable change $q = e^x$. We get:

$$e^{-x}V(\langle \bigotimes \rangle) - e^xV(\langle \bigotimes \rangle) = (e^{\frac{x}{2}} - e^{-\frac{x}{2}})V(\langle \bigotimes \rangle).$$

Now let us write down the formal Taylor series in x of the expression above and take all members divisible by x explicitly to the right part.

In the right part we get a sum divisible by x and in the left part we obtain the derivative of the Jones polynomial plus something divisible by x:

$$V(\langle X \rangle) - V(\langle X \rangle) = x \langle \text{some mess} \rangle.$$

Arguing as above, we see that after the second differentiation, only members divisible by x^2 arise in the right part.

Consequently, after (n+1) differentiations, the n-th member of the series expressing the Jones polynomial in x, becomes zero. Thus, all members of this series are Vassiliev invariants. So, we obtain the following theorem.

Theorem 12.2. The Jones polynomial in one variable and the Kauffman polynomial in one variable are weaker than Vassiliev invariants.

One can do the same with the Jones polynomial (denoted by \mathcal{X})) in two variables. Let us write down the skein relation for it:

$$\frac{1}{\sqrt{\lambda}\sqrt{q}}\mathcal{X}(\sum) - \sqrt{\lambda}\sqrt{q}\mathcal{X}(\sum) = \frac{q-1}{\sqrt{q}}\mathcal{X}(\sum)$$

and let us make the variable change $\sqrt{q}=e^x, \sqrt{\lambda}=e^y$ and write down the Taylor series in x and y.

In the right part we get something divisible by x and in the left part something divisible by xy plus the derivative of the Jones polynomial.

Finally, we have

$$\mathcal{X}(\langle \sum \rangle) - \mathcal{X}(\langle \sum \rangle) = x \langle \text{some mess} \rangle.$$

Thus, after (n+1) differentiations, all members of degree $\leq n$ in x, vanish. Consequently, we get the following theorem.

Theorem 12.3. The Jones polynomial in two variables is weaker than Vassiliev invariants.

Since the HOMFLY polynomial is obtained from the Jones polynomial by a variable change, we see that the following theorem holds.

Theorem 12.4. The HOMFLY polynomial is weaker than Vassiliev invariants.

The most difficult and interesting case is the Kauffman 2-variable polynomial because this polynomial does not satisfy any Conway relations. This polynomial can be expressed in the terms of functions z, a, and $\frac{a-a^{-1}}{z}$. In order to represent the Kauffman polynomial as a series of Vassiliev invariants, we have to represent all these functions as series of positive powers of two variables.

We recall that the Kauffman polynomial in two variables is given by the formula²

$$Y(L) = a^{-w(L)}D(L),$$

where D is a function on the chord diagram that satisfies the following relations:

$$D(L) - D(L') = z(D(L_A) - D(L_B)); (1)$$

$$D(\bigcirc) = \left(1 + \frac{a - a^{-1}}{z}\right);\tag{2}$$

$$D(X\#(Q)) = aD(X), D(X\#(Q)) = a^{-1}D(X),$$
 (3)

where the diagrams L = (X), L' = (X), $L_A = (X)$, $L_B = (X)$ coincide outside a small neighbourhood of some vertex.

Let us rewrite (1) for Y. We get:

$$a^{-1}Y(\langle X \rangle) - aY(\langle X \rangle) = z(Y(\langle X \rangle) - Y(\langle X \rangle)) \cdot \langle \text{Power of } a \rangle. \tag{4}$$

Let us perform the variable change: $p = \ln(\frac{a-1}{z})$. Then, in terms of z and p, one can express $z, a, \frac{a-a^{-1}}{z}$ by using only positive powers and series. Actually, we have:

$$z = z,$$

$$a = ze^{p} + 1 = z(1 + p + \dots) + 1,$$

$$a^{-1} = 1 - z(1 + p + \dots) + z^{2}(1 + p + \dots)^{2} + \dots,$$

$$\frac{a - a^{-1}}{z} = a^{-1}(a + 1)e^{p}$$

Each of these right parts can evidently be represented as sequences of positive powers of p and z.

Thus, the value of the Kauffman polynomial in two variables on each knot is represented by positive powers of p and z. On the other hand, taking into account that $a=1+z\langle \text{some mess}_1\rangle$ and $a^{-1}=1+z\langle \text{some mess}_2\rangle$, we can deduce from (4) and (3) that

$$Y' = z \langle \text{some mess} \rangle$$
.

 $^{^2}$ Here we denote the oriented and the unoriented diagrams by the same letter L.

Herewith, all members of our double sequence having degree less than or equal to n in the variable z, vanish after the (n+1)-th differentiation. Thus, all these members are Vassiliev invariants.

Thus, we have proved the following theorem.

Theorem 12.5. The Kauffman polynomial in two variables is weaker than Vassiliev invariants.

Let us show how to calculate the derivative of products of two functions.

For any two functions f and g defined on knot diagrams one can formally define the derivatives f' and g' on diagrams of first-order singular knots just as we define the derivatives of the invariants. Analogously, one can define higher-order derivatives.

Consider the function $f \cdot g$ and consider a singular knot diagram K of order n. By a *splitting* is meant a choice of a subset of i singular vertices of n singular vertices belonging to K. Choose a splitting s. Let K_{1s} be the diagram obtained from K by resolving (n-i) unselected vertices of s negatively, and let K_{2s} be the knot diagram obtained by resolving i selected vertices positively.

Lemma 12.1. Let K be a chord diagram of degree n. Then the Leibniz formula holds:

$$(fg)^{(n)}(K) = \sum_{i=0}^{n} \sum_{s} f^{(i)}(K_{1s})g^{(n-i)}(K_{2s}).$$

Proof. We shall use induction on n.

First, let us establish the induction base (the case n=1). Given a singular knot of order one. Let us consider a diagram of it and the only singular vertex A of this diagram. Write down the Vassiliev relation for this vertex:

$$(fg)'((\bigotimes)) = f((\bigotimes))g((\bigotimes)) - f((\bigotimes))g((\bigotimes))$$

$$= g((\bigotimes))(f((\bigotimes)) - f((\bigotimes))) + f((\bigotimes))(g((\bigotimes)) - g((\bigotimes)))$$

$$= f'((\bigotimes))g((\bigotimes)) + g'((\bigotimes))f((\bigotimes)). \tag{5}$$

The equality (5) holds by definition of f' and g'. Thus, we have proved the claim of the theorem for n = 1. Note that we can apply the obtained formula for functions on singular (not ordinary) knots, when all singular points do not take part in the relation, i.e., lie outside the neighbourhood.

Now, for any given singular knot K of order n, let us fix a singular vertex A of the knot diagram K. The value of $(fg)^{(n)}$ on K equals the difference of $(fg)^{(n-1)}$ evaluated on two singular knots K^1 and K^2 ; these two diagrams of singular knots of order n-1 are obtained by positive and negative resolution of A, respectively.

By the induction hypothesis, we have:

$$(fg)^{(n-1)}(K^i) = \sum_{i=0}^{n-1} \sum_{s} f^{(i)}(K_{1s}^i) g^{(n-1-i)}(K_{2s}^i), \tag{6}$$

where s runs over the set of all splittings of order (n-1). We have:

$$(fg)^{(n)}(K) = (fg)^{(n-1)}(K^{1}) - (fg)^{(n-1)}(K^{2})$$

$$= \sum_{i=0}^{n-1} \sum_{s} \left[f^{(i)}(K_{1s}^{1})g^{(n-1-i)}(K_{2s}^{1}) - f^{(i)}(K_{1s}^{2})g^{(n-1-i)}(K_{2s}^{2}) \right]$$

$$= \sum_{i=0}^{n-1} \sum_{s} \left[f^{(i)}(K_{1s}^{1})g^{(n-1-i)}(K_{2s}^{1}) - f^{(i)}(K_{1s}^{2})g^{(n-1-i)}(K_{2s}^{1}) + f^{(i)}(K_{1s}^{2})g^{(n-1-i)}(K_{2s}^{1}) - f^{(i)}(K_{1s}^{2})g^{(n-1-i)}(K_{2s}^{2}) \right]$$

$$= \sum_{i=0}^{n-1} \sum_{s} \left[f^{(i+1)}(K_{1s})g^{(n-1-i)}(K_{2s}^{1}) - f^{(i)}(K_{1s}^{2})g^{(n-i)}(K_{2s}) \right]$$

$$= \sum_{i=0}^{n} \sum_{s} f^{(i)}(K_{1s})g^{(n-i)}(K_{2s}).$$

Lemma 12.1 implies the following corollary.

Corollary 12.1. Let f and g be two functions defined on the set of knot diagrams (not necessarily knot invariants) such that $f^{(n+1)} \equiv 0$, $g^{(k+1)} \equiv 0$. Then $(fg)^{(n+k+1)} \equiv 0$.

In particular, the product of Vassiliev invariants of orders n and k is a Vassiliev invariant of order than less or equal to (n + k).

12.6 An example of an infinite-order invariant

Until now, we have dealt only with invariants either having finite order or invariants that can be reduced to finite-order invariants. We have not yet given any proof that some knot invariant has infinite order.

Here we give an example of a knot invariant that has infinite order, [BL].

Definition 12.5. The unknotting number U(K) of an (oriented) link K is the minimal number $n \in \mathbf{Z}_+$ such that K can be transformed to the unlink by passing n times through singular links. In other words, n is the minimal number such that there exists a diagram of K that can be transformed to an unlink diagram by switching n crossings.

By definition, our invariant equals zero only for unlinks.

Theorem 12.6. The invariant U has infinite order.

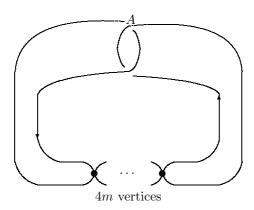


Figure 12.3. Singular knot, where $U^i \neq 0$.

Proof. Let us fix an arbitrary $i \in \mathbb{N}$. Now, we shall give an example of the singular knot for which $U^{(i)} \neq 0$. Fix an integer m > 0 and consider the knot K_{4m} with 4m singularity points which are shown in Fig. 12.3.

By definition of the derivative, the value of $U^{(4m)}$ on this knot is equal to the alternating sum of $2^{(4m)}$ summands; each of them is the value of U on a knot, obtained by somehow resolving all singular vertices of K_{4m} .

Note that for each such singular knot the value of U does not exceed one: by changing the crossing at the point A, we obtain the unknot. On the other hand, the knot obtained from K_{4m} by splitting all singular vertices is trivial if and only if the number of positive splittings equals the number of negative splittings (they are both equal to 2m).

The case of q positive and 4m-q negative crossings generates the sign $(-1)^q$. Thus we finally get that $U^{(4m)}(K_{4m})$ is equal to

$$U^{(4m)}(K_{4m}) = 2[C_{4m}^0 - C_{4m}^1 + \dots - C_{4m}^{2m-1}].$$

This sum is, obviously, negative: $U^{(4m)}(K_{4m}) \neq 0$. So, for $m > \frac{i}{4}$, we get $U^{(i)} \not\equiv 0$. Thus, the invariant U is not a finite type invariant of order less than or equal to i. Since i was chosen arbitrarily, the invariant U is not a finite type invariant.

Remark 12.3. We do not claim that U cannot be represented via finite type invariants.

The unknotting number is some "measure" of complexity for a knot. Thus, it would be natural to think that it is realised on minimal diagrams (i.e. the minimal diagram can be transformed to the unknot diagram by precisely j switchings if the unknotting number is equal to n). However, this is not true. The first results in this direction were obtained by Bleiler and Nakanishi [Bl, Nak]. Later, an infinite series of knots with this property was constructed by D.J. Garity [Gar].

Chapter 13

The chord diagram algebra

13.1 Basic structures

In the present chapter, we shall study the algebraic structure that arises on the set of Vassiliev knot invariants.

In the previous chapter, we defined symbols of the Vassiliev knot invariants in the language of chord diagrams.

Now, the main question is: Which functions on chord diagrams can play the role of symbols?

The simplest observation leads to the following fact. If we have a chord diagram $C=\bigcirc$ with a small solitary chord, then each symbol evaluated at this diagram equals zero. We have already discussed this in the language of singular knots.

This relation is called a 1*T*-relation (or one-term relation).

One can easily prove the generalised 1T-relation where we can take a diagram $C = \bigoplus$ with a chord that does not intersect any other chord. Then, each symbol of a Vassiliev knot invariant evaluated at the diagram C equals zero. The proof is left to the reader.

There exists another relation, consisting of four terms, the so-called 4T-relation. In fact, let us prove the following Theorem.

Theorem 13.1 (The four-term relation). For each symbol v^n of the invariant v the following relation holds:

$$v^n(\bigcirc) - v^n(\bigcirc) - v^n(\bigcirc) + v^n(\bigcirc) = 0.$$

This relation means that for any four diagrams having n chords, where (n-2) chords (not shown in the Figure) are the same for all diagrams and the other two look as shown above, the above equality takes place.

Proof. Consider four singular knots S_1, S_2, S_3, S_4 of the order n, whose diagrams coincide outside some small circle, and their fragments s_1, s_2, s_3, s_4 inside this circle look like this:

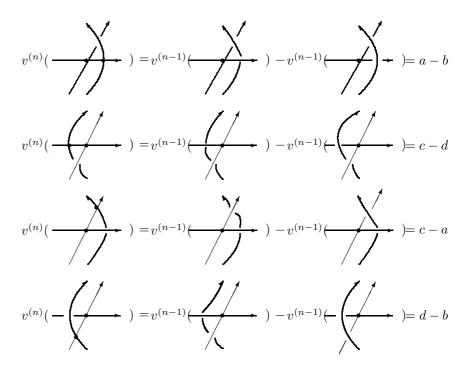


Figure 13.1. The same letters express $v^{(n-1)}$ for isotopic long knots

$$s_1 = 3, s_2 = 3, s_3 = 3, s_4 = 3.$$

Consider the invariant v of order n and the values of its symbol on these four knots. Vassiliev's relation implies the relations shown in Fig. 13.1.

Obviously,

$$(a-b) - (c-d) + (c-a) - (d-b) = 0.$$

In order to get singular knots, one should close the fragments s_1, s_2, s_3, s_4 . There are two possibilities to do this as shown in Fig. 13.2.

Thus, the diagrams S_1, S_2, S_3, S_4 satisfy the relation

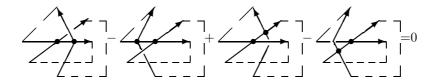
$$v^{(n)}(S_1) - v^{(n)}(S_2) + v^{(n)}(S_3) - v^{(n)}(S_4) = 0.$$
(1)

Each of the chord diagrams corresponding to S_1, S_2, S_3, S_4 has n chords; (n-2) chords are the same for all diagrams, and only two chords are different for these diagrams.

Since the order of v equals n, the symbol of v is correctly defined on chord diagrams of order n. Thus, the value of $v^{(n)}$ on diagrams corresponding to singular knots S_1, S_2, S_3, S_4 equals the value on the singular knots themselves.

Taking into account the formulae obtained above, and the arbitrariness of the remaining (n-2) singular vertices of the diagrams S_1, S_2, S_3, S_4 , we obtain the statement of the theorem.

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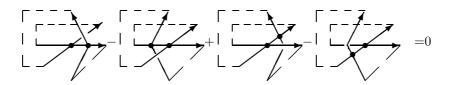


Figure 13.2. Closures of fragments imply 4T-relation

Both 1T- and 4T-relations can be considered for chord diagrams and on the dual space of linear functions on chord diagrams (since these two dual spaces can obviously be identified). For the sake of simplicity, we shall apply the terms 1Tand 4T-relation to both cases.

Definition 13.1. Each linear function on chord diagrams of order n, satisfying these relations, is said to be a weight system (of order n).

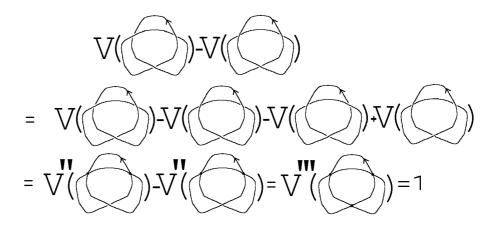
Notation: Denote the space of all weight systems of order n, by A_n or by Δ_n . In the last chapter, we considered invariants of orders less than or equal to two. The situation there is quite clear: there exists the unique non-trivial (modulo 1Trelation) chord diagram that gives the invariant of order two. As for dimension three, there are two diagrams: and . It turns out that they are linearly dependent. Namely, let us write the following 4T-relation (here the fixed chord is represented by the dotted line):

This means that $\bigcirc = 2 \bigcirc$. So, if there exists an invariant of order three, then its symbol is uniquely defined by a value on \bigcirc . Suppose we have such an invariant v and $V'''(\bigcirc) = 1$. Let us show that this invariant distinguishes the two trefoils. Namely, we have:

The existence of this invariant will be proved later.

Let us consider the formal space Δ_4 .

Exercise 13.1. Prove the following relations:



Exercise 13.2. Prove that $dim\Delta_4 = 3$ and that the following three diagrams can be chosen as a basis:

$$\left\{ \left(\begin{array}{c} \\ \\ \end{array} \right), \left(\begin{array}{c} \\ \\ \end{array} \right), \left(\begin{array}{c} \\ \\ \end{array} \right) \right\}.$$

It turns out that the chord diagrams factorised by the 4T-relation (with or without the 1T-relation) form an algebra. Namely, having two chord diagrams C_1 and C_2 , one can break them at points $c_1 \in C_1$ and $c_2 \in C_2$ (which are not ends of chords) and then attach the broken diagrams together according to the orientation. Thus we get a chord diagram. The obtained diagram can be considered as the product $C_1 \cdot C_2$. Obviously, this way of defining the product depends on the choice of the base points c_1 and c_2 , thus, different choices might generate different elements of \mathcal{A}^c . However, this is not the case since we have the 4T-relation.

Theorem 13.2. The product of chord diagrams in A^c is well defined, i.e., it does not depend on the choice of initial points.

To prove this theorem, we should consider $arc\ diagrams$ rather than chord diagrams.

Definition 13.2. By an *arc diagram* we mean a diagram consisting of one straight oriented line and several arcs connecting points of it in such a way that each arc connects two different points and each point on the line is incident to no more than one arc.

These diagrams are considered up to the natural equivalence, i.e., a mapping of the diagram, taking the line to the line (preserving the orientation of the line) and taking all arcs to arcs. Obviously, by breaking one and the same chord diagram at different points, we obtain different arc diagrams.

Now, we can consider the 4T-relation for the case of the arc diagrams, namely the relation obtained from a 4T-relation by breaking all four circles at the same point (which is not a chord end).

The point is that the two arc diagrams A_1 and A_2 obtained from the same chord diagram D by breaking this diagram at different points are equivalent modulo 4T-relation. This will be sufficient for proving Theorem 13.2. Obviously, one can obtain A_2 from A_1 by "moving a chord end through infinity." Thus, it suffices to prove the following lemma.

Lemma 13.1 (Kontsevich). Let A_1, A_2 be two arc diagrams that differ only at the chord: namely, the rightmost position of a chord end of A_2 corresponds to the leftmost position of the corresponding chord end of A_1 ; the other chord ends of A_1 and A_2 are on the same places. Then A_1 and A_2 are equivalent modulo the four-term relation.

Proof. Suppose that each of the diagrams A_1 and A_2 have n arcs. Denote the common arc ends A_1 and A_2 by $X_1, X_2, \ldots, X_{2n-1}$ enumerated from the left to the right. They divide the line into 2n intervals I_1, \ldots, I_{2n} (from the left to the right). Denote by D_j the arc diagram having the same "fixed" arc ends as A_1 and A_2 and one "mobile" arc end at I_j . Thus, $A_1 = D_1, A_2 = D_{2n}$. Suppose that the second end of the "mobile" arc is X_k . Then, obviously, $A_k = A_{k+1}$.

Now, consider the following expression

$$A_{2n} - A_1 = A_{2n} - A_{2n-1} + A_{2n-1} - A_{2n-2} + \dots + A_{k+2} - A_{k+1} + A_k - A_{k-1} + \dots + A_{2n-k} - A_{2n-1} + A_{2n-1} + A_{2n-2} + \dots + A_{2n-k} + A_{$$

Here we have 4n-4 summands. It is easy to see that they can be divided into n-1 groups, each of which forms the 4T-relation concerning one immobile chord and the mobile chord.

Thus, $A_{2n} = A_1$. This completes the proof of the theorem.

13.2 Bialgebra structure of algebras A^c and A^t . Chord diagrams and Feynman diagrams

The chord diagram algebra \mathcal{A}^c has, however, very sophisticated structures. It is indeed a bialgebra. The coalgebra structure of \mathcal{A}^c can be introduced as follows.

Let C be a chord diagram with n chords. Denote the set of all chords of the diagram C by \mathcal{X} . Let $\Delta(C)$ be

$$\sum_{s \in 2^X} C_s \otimes C_{x-s},$$

where the sum is taken over all subsets s of X, and C_y denotes the chord diagram consisting of all chords of C belonging to the set y. Now, let us extend the coproduct Δ linearly.

Now we should check that this operation is well defined. Namely, for each four diagrams $A = \bigcirc$, $B = \bigcirc$, $C = \bigcirc$, $D = \bigcirc$ such that A - B + C - D = 0 is the 4T-relation, one must check that $\Delta(A) - \Delta(B) + \Delta(C) - \Delta(D) = 0$.

Actually, let A, B, C, D be four such diagrams (A differs from B only by a crossing of two chords, and D differs from C in the same way). Let us consider the comultiplication Δ . We see that when the two "principal" chords are in different parts of X, then we have no difference between A, B as well as between C, D. Thus, such subsets of X give no impact. And when we take both chords into the same part for all A, B, C, D, we obtain just the A-relation in one part and the same diagram at the other part. Thus, we have proved that Δ is well-defined.

Now, let us give the formal definition of the bialgebra.¹

Definition 13.3. An algebra A with algebraic operation μ and unit map e and with coalgebraic operation Δ and counit map ε is called a bialgebra if

- 1. e is an algebra isomorphism;
- 2. ε is an algebra isomorphism;
- 3. Δ is an algebra isomorphism.

Definition 13.3. An element x of a bialgebra B is called *primitive* if $\Delta(x) = x \otimes 1 + 1 \otimes x$.

Obviously, for the case of \mathcal{A}^c with natural e, ε and endowed with the product and coproduct Δ , e and ε are isomorphic. The map Δ is isomorphic: it has the empty kernel because for each $x \neq 0$, $\Delta(x)$ contains the summand $x \otimes 1$.

Thus, \mathcal{A}^c is a bialgebra.

There is another interesting algebra \mathcal{A}^t that is in fact isomorphic to \mathcal{A}^c .

Definition 13.4. A Feynman diagram² is a finite connected graph of valency three at each vertex with an oriented cycle (circle)³ on it. All vertices not lying on the circle are called *interior* vertices. Those lying on the circle are *exterior* vertices. Each interior vertex should be endowed with a cyclic order of outgoing edges.

Remark 13.1. Feynman diagrams on the plane are taken to have the counterclockwise orientation of the circle and counterclockwise cyclic order of outgoing edged at each interior vertex.

Definition 13.6. The *degree* of a Feynman diagram is half the number of its vertices.

Obviously, all chord diagram are Feynman diagrams; in this case the two definitions of the degree coincide.

Consider the formal linear space of all Feynman diagrams of degree n. Let us factorize this space by the STU-relation that is shown in Fig. 13.3.

¹In [O] this is also called a *Hopf algebra* One usually requires more constructions for the algebra to be a Hopf algebra, see e.g. [Cas]. However, the bialgebras of chord and Feynman diagrams that we are going to consider are indeed Hopf algebras: the antipode map is defined to be one on primitive elements and zero on other elements of the basis. We shall not use the antipode and its properties.

²Also called Chinese diagram or circular diagram.

 $^{^3}$ This circle is also called the Wilson loop; we shall not use this term.

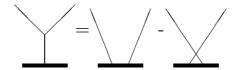


Figure 13.3. STU-relation

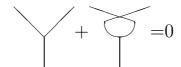


Figure 13.4. The antisymmetry relation

Denote this space by \mathcal{A}_n^t .

Theorem 13.3. There exists a natural isomorphism $f: \mathcal{A}_n^t \to \mathcal{A}_n^c$ which is identical on \mathcal{A}_n^c . Moreover, the STU-relation implies the following relations for \mathcal{A}^t :

- 1. Antisymmetry, see Fig. 13.4.
- 2. IHX-relation, see Fig. 13.5.

Proof. First, let us prove that the algebras \mathcal{A}^t and \mathcal{A}^s are isomorphic. Obviously, the STU-relation implies the 4T-relation for the elements from \mathcal{A}^c . Let us construct now the isomorphism f. For all elements from $\mathcal{A}^c \subset \mathcal{A}^t$ we decree f to be the identity map. To define f on all Feynman diagrams, we shall use induction on the number x of interior vertices. For x = 0, there is nothing to prove.

Suppose f is well defined for all Feynman diagrams of degree d. Let K be a Feynman diagram of degree d+1.

Obviously, there exists an interior vertex V of K that is adjacent to some exterior vertex by an edge v. Thus, we can apply the STU-relation to this vertex and obtain two diagrams of degree d. However, this operation is not well defined: it depends on the choice of such a vertex V and the edge v. Suppose there are two such pairs

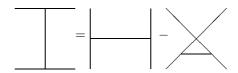


Figure 13.5. The IHX-relation

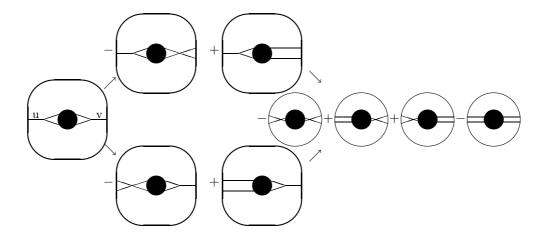


Figure 13.6. Applying the STU-relation twice

V, v and U, u, where $V \neq U, v \neq u$. In this case, we can prove that our operation is well-defined by applying the STU-relation twice, see Fig. 13.6.

In the case when U=V and $u\neq v$, we can try to find another pair. Namely, the pair W,w, where W is a vertex adjacent to an exterior vertex and w is an edge connecting this vertex with the circle. Then we prove that the result for U,u equals that for W,w and then it equals that for U,v.

Finally, we should consider the case when U = V, $u \neq v$, and U is the only interior vertex adjacent to the circle. In this case, we are going to show that our diagram is equivalent to zero modulo the STU–relation.

In this case, we can indicate some domain containing all interior vertices except one. This domain has only one connection with exterior vertices, namely the connection via the vertex U and one of the chords u, v. In this case, the two possible splittings are equals because the product on Feynman diagrams is well defined. By the induction hypothesis, we see that the product of two Feynman diagrams of total degree d is well defined and commutative. By using this commutativity, we can move the vertex with the small domain from one point to the other one. Thus, we see that each of both splittings give us zero. The concrete calculations are shown in Fig. 13.7.

Let us prove now that STU implies the antisymmetry relation.

Applying the STU-relations many times, one can reduce the antisymmetry relation to the case when all chords outgoing from the given interior points finish at exterior points. In this case, the antisymmetry relation follows straightforwardly, see Fig. 13.8.

The proof of the fact that the IHX-relation holds can be reduced to the case when one of the four vertices (say, lower left) is an interior one. This can be done by taking the lower left vertex for all diagrams that have to satisfy the IHX-relation and then splitting all interior vertices between this vertex and the circle in the same manner for all diagrams. Then we repeat this procedure for all obtained diagrams.

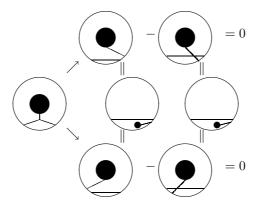


Figure 13.7. The diagram of a special type equals zero

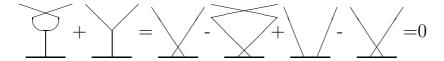


Figure 13.8. STU-relation implies antisymmetry

Finally, we get many triples of diagrams for each of which we have to check the IHX relation. For each of them, we have to consider only the partial case. The last step is shown in Fig. 13.9.

Remark 13.2. Note that the 1-term relation does not spoil the bialgebra structure; the corresponding bialgebra is obtained by a simple factorisation.

13.3 Lie algebra representations, chord diagrams, and the four colour theorem

There is a beautiful idea connecting the representation theory of Lie algebras, and knot theory. It was popularised in [BN]; for further developments see e.g., [BN5, CV].

The motto is: to contract trivalent tensors along graphs.

Remark 13.3. Within this section, we do not take into account the 1T-relation. We work only with the 4T-relation (or STU-relation for Feynman diagrams).

In more detail, having a trivalent graph and a trivalent tensor, we can set this tensor to each vertex of the graph, and then contract the tensors along the edges of

Figure 13.9. The STU-relation implies the IHX relation

the graph. Clearly, we need some metric; besides we must be able to switch indices. All these conditions obviously hold for the case of semisimple Lie algebras: we can take the structure constant tensor C_{ijk} (with all lower indices) and the metric g_{ij} that is not degenerate. As a trivalent graph, one can take a Feynman diagram together with its rotation structure at vertices.

The beautiful observation is that the STU-relation (as well as the IHX-relation) for Feynman diagrams represents the Jacobi identity for Lie algebras. Thus, the constructed numbers (obtained after all contractions) are indeed invariant under the STU-relation.

This construction is the simplest case of the general construction; it deals only with adjoint representations of Lie algebras. In the general case, one should fix the circle of the Feynman diagram and the representation R of the Lie algebra G. Then, along the circle, we put elements of the representation space such that any two adjacent elements are obtained from each other by the action of the Lie algebra element associated with the edge outgoing from the point connecting these two arcs.

Consider the adjoint representation of SO(3). In this case, it is very easy to calculate the contractions. Namely, we have three elements a, b, c and the following contraction law: [a, b] = c; [b, c] = a; [c, a] = b.

Definition 13.7. By a *planar map* we mean a cubic graph embedded in \mathbb{R}^2 (this graph divides the plane into cells which are called *regions*).

Now, suppose we wish to colour the map with four colours. Let us take them from the palette $\mathbb{Z}_2 \oplus \mathbb{Z}_2$: they are (0,0), (0,1), (1,0) and (1,1).

Definition 13.8. The map is *four colourable* if one can associate one of the four colours to each region in such a way that no two adjacent regions have the same colour

The four colour theorem claims that every planar map (without loops) is four colourable.

It was remaining unsolved for a long time. Its first solution [AH] is very technically complicated and contains numerous combinatorial constructions to work with. Below, we give some sufficient condition for a map to be four colourable [BN5].

Suppose we have some colouring of some map. Then, we can colour each edge

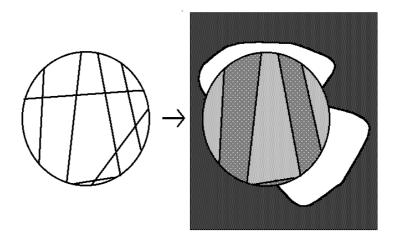


Figure 13.10. Each map coming from d-diagram is four colourable

by an element from $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ which is a sum of the elements associated with the adjacent regions.

Now, it is obvious that the map is four colourable if the edges of it can be coloured only with the colours (0,1),(1,0), and (1,1) in such a way that no two adjacent edges have the same colour. So, the edges should be *three colourable*. The reverse statement is also true: if edges are three–colourable then the map is four–colourable.

Now, we can think of each map M as a Feynmann diagram and associate a number to it with respect to the adjoint representation of SO(3). Denote this number by $I_3(N)$.

Theorem 13.4 (BN5). If $I_3(M) \neq 0$ then M is four colourable.

Proof. Indeed, we can use the basis a, b, c for calculating $I_3(M)$. Since $I_3(M) \neq 0$, there exists at least one contraction that gives a non-zero element. Taking into account the law for SO(n), we see that the edges are three colourable (triples with at least two equal elements give zero!). Thus, M is four colourable.

Suppose we have a d-diagram embedded in \mathbb{R}^2 . Then, the corresponding map is four colourable. Actually, our plane is divided into two parts by the circle. A simple observation shows that each of these parts (interior and exterior) is two colorable, see Fig. 13.10. Thus, the whole picture is four colourable.

Thus, one can ask the following question. Consider a map (or planar Feynman diagram). Can one recognise a d-diagram in it? In other words, can one select some subset edges that compose a cycle in such a way that all other edges connect points of this cycle?

This problem is very famous and is still unsolved. It was first stated by W.R.Hamilton, and the cycle we are looking for is called a *Hamiltonian cycle*. To date, only some (positive) solutions for some classes of maps are known.

One can easily see that the unsolved problem on Hamiltonian curves leads to the positive solution of the four colour problem via d-diagrams.

However, the way proposed by Bar–Natan is not a criterion. Actually, having a Feynmann diagram D, for which $I_3(D)=0$, the corresponding map can have a Hamiltonian curve and thus be four colourable. The point is that while calculating $I_3(D)$, some members (corresponding to proper four colourings) give a positive contribution and the others give a negative contribution and the total number can be equal to zero.

13.4 Dimension estimates for A_d . A table of known dimensions.

We are going to talk about the lower and upper bounds for the dimensions of spaces Δ_n . Later, we shall prove that Δ_n is precisely $\mathcal{V}_n/\mathcal{V}_{n-1}$.

13.4.1 Historical development

A priori it is obvious that the cardinality of the set of all chord diagrams on d chords does not exceed $(2d-1)!! = 1 \cdot 3 \cdots (2d-1)$.

This gives the first evident upper bound. After this, the following results appeared (results listed according to [CDBook]).

- 1. (1993) Chmutov and Duzhin in [CD1] proved that $dim \mathcal{A}_d < (d-1)!$.
- 2. (1995) K. Ng in [Ng] replaced (d-1)! by $\frac{(d-2)!}{2}$
- 3. (1996) A. Stoimenov [St] proved that $dim \mathcal{A}_d$ grows slower than $\frac{d!}{a^d}$, where a = 1.1.
- 4. (2000) B. Bollobás and O. Riordan [BR] obtained the asymptotical bound $\frac{d!}{(2 \ln{(2)} + o(1))^d}$ (approximately $\frac{d!}{1.38^d}$).
- 5. (2001) D. Zagier in [Zag] improved the result to $\frac{6^d \sqrt{d} \cdot d!}{\pi^{2d}}$, which is asymptotically smaller than $\frac{d!}{a^d}$ for any constant $a < \frac{\pi^2}{6} = 1.644...$

The history of lower bounds was developing as follows.

- 1. (1994) Chmutov, Duzhin and Lando [CDL] gave a lower approximation for the number of primitive elements \mathcal{P}_n ("forest elements"): $dim\mathcal{P}_d \geq 1$ for d > 1.
- 2. (1995) $dim\mathcal{P}_d \geq \left[\frac{d}{2}\right]$ (see Melvin–Morton [MeM] and Chmutov–Varchenko [CV]).
- 3. (1996) $dim\mathcal{P}_d \stackrel{>}{\sim} \frac{d^2}{96}$, see Duzhin [Du1].
- 4. (1997) $dim \mathcal{P}_d \stackrel{>}{\sim} d^{logd}$, see Chmutov-Duzhin, [CD2].
- 5. (1997) Kontsevich $dim \mathcal{P}_n > e^{\pi \sqrt{n/3}}$.

6. (1997) $dim \mathcal{P}_n > e^{C\sqrt{n}}$ for any constant $C < \pi \sqrt{2/3}$ (Dasbach, [Da]).

Below, we are going to prove the simplest upper bound from [CD1] and give an idea that leads to the lower bound estimates from [CD2]. The ideas of [Da] generalise the techniques from [CD2] by adding some more low–dimensional topology. For more details, see [CDBook] or the original works (for all the other estimates).

13.4.2 An upper bound

First, let us discuss the upper bound [CD1]. They state that $dim\Delta_n \leq n-1!$. Namely, they present a set of (n-1)! generating elements for Δ_n .

Definition 13.9. A chord diagram is called a *spine* if it contains a chord of it intersecting all other chords.

Theorem 13.5. The set of all spine chord diagrams generates Δ_n .

Rather than proving this theorem, we shall divide it into small steps (according to [CD2]); each of these steps can be easily proved by the reader as a simple exercise. While performing these steps, we shall use induction on some parameters.

Definition 13.10. Given a chord diagram D, let d be a chord of diagram D. The *degree* of d is the number of chords intersecting d. The degree of the chord diagram D is the maximal degree of its chords.

Notation: deg(d), deg(D).

Fix a diagram D. By definition, if D has n chords and deg(D) = n - 1 then D is a spine diagram.

We shall use induction on the degree of D and prove that if deg(D) < n-1 then D can be represented as a linear combination of diagrams of smaller degrees.

Suppose deg(D) = f. Choose a chord d such that deg(d) = f. After a rotation, we can assume that the chord f is vertical and the right part of D (with respect to d) contains more chord ends than the left one.

Then there exists a chord d' with both ends lying in the right part. The top end of it lies at some distance from the top end of f. Denote the number of points between them by k.

Exercise 13.3. If k > 0 then the diagram D can be represented as a linear combination of chord diagrams of smaller degree and diagrams of the same degree with smaller k.

Thus we can assume that k = 0.

Now let us consider the lower end of d'.

Definition 13.11. The lower end of any chord which intersects d' and does not intersect d is a *bound*. A point that lies on the lower arc between d and d' and is not a bound will be referred as *loose*.

Let l be the number of loose points in a chord diagram, and b be the number of bound points between the lower end of d' and the first loose point. The index of the chord diagram is the pair (l,b). The index will be used as the induction parameter with respect to the following lexicographical ordering: $(l_1,b_1) > (l_2,b_2)$ if and only if either $l_1 > l_2$ or $l_1 = l_2, b_1 > b_2$.

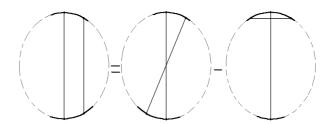


Figure 13.11. A linear combination of diagrams of smaller degree

Now we are going to show that each non-zero degree diagram can be represented as a linear combination of diagrams of the same degree and lower index and some diagrams of lower degree.

This consists of two induction steps, both of which are left to the reader. Each of them follows straightforwardly from applying the 4T-relation.

The first step.

Exercise 13.4. If l > 0, b = 0, then the diagram can be represented as a linear combination of diagrams of smaller degrees and diagrams of the same degree and smaller l.

The second step.

Exercise 13.5. If l > 0, b > 0 then the diagram can be represented as a linear combination of chord diagrams of smaller degree or the same degree and smaller index.

Now, if $l = 0, b \neq 0$ then each chord intersecting d intersects d' as well and there are chords intersecting d' but not d. Thus, the degree of the diagram is indeed greater than d, so this is not the case.

If l=0, b=0, then we have two "parallel chords" and the following 4T-relation (together with a 1T-relation that we do not illustrate on the picture) completes the proof of the theorem, see Fig. 13.11.

13.4.3 A lower bound

We are going to present a lower bound for the dimension of \mathcal{A}_n according to [CD2]. As the authors say, "the story of lower bounds for the Vassiliev invariants is more enigmatic" [than that of upper bounds]. The first estimate was proposed by Bar–Natan in [BN] as follows: $\dim \mathcal{V}_n > e^{c\sqrt{n}}, n \to \infty$. This estimate comes from the connection between the Vassiliev knot invariants and Lie algebras that will be discussed in the last part of this chapter.

Definition 13.12. By a *Jacobi* diagram we mean the same as a Feynman diagram but without the oriented circle; namely, a graph all vertices of which have valency one or three; each vertex of valency three should be endowed with a cyclic order of outgoing edges. The degree of a Jacobi diagram is half of the total number of its vertices.

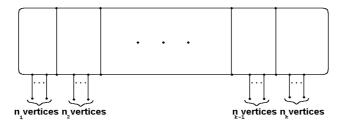


Figure 13.12. The diagram $B_{n_1,...,n_k}$

For Jacobi diagrams, we can consider the IHX relation and the antisymmetry relations. Obviously, they are both homogeneous with respect to the graduation described above. Like Feynman diagrams, Jacobi diagrams have multiplication and comultiplication, which are defined even more simple than that for Feynman diagrams: we have no oriented circle, so the multiplication is just the disconnected sum, and the comultiplication is defined by splittings into connected components.

Thus, we know which Jacobi diagrams are primitive: they are just connected uni–trivalent graphs for which each trivalent vertex is endowed with a cyclic order.

To go on, we shall need to introduce some notions. Consider the space of all primitive Jacobi diagrams. Each primitive Feynman diagram has an even number of vertices. Denote half of this number by d. Both IHX and antisymmetry relations are homogeneous with respect to d. The space \mathcal{C} is thus bigraded: $\mathcal{C} = \oplus \mathcal{C}_{d,n}$, where $\mathcal{C}_{d,n}$ is a subspace of the space \mathcal{C} generated by primitive Jacobi diagrams with a total of 2d vertices, precisely n of which are univalent.

Let us define a family of Baguette diagrams.

Definition 13.13. A Baguette diagram $B_{n_1,...,n_k}$ is a Jacobi diagram shown in Fig. 13.12.

The baguette diagram B_{n_1,\ldots,n_k} has $2(n_1+\cdots+n_k-k-1)$ vertices, out of which $n_1+\cdots+n_k$ are univalent.

Theorem 13.6 (Main theorem,[CD2]). Let $n = n_1 + \cdots + n_k$ and d = n + k - 1. The elements $B_{n_1...n_k}$ defined as above are linearly independent in $C_{d,n}$ if n_1, \ldots, n_k are all even and satisfy the following conditions:

$$n_1 < n_2$$

$$n_1 + n_2 < n_3$$

$$n_1 + n_2 + n_3 < n_4$$

$$\cdots$$

$$n_1 + n_2 + \cdots + n_{k-2} > n_{k-1}$$

$$n_1 + n_2 + \cdots + n_{k-2} + n_{k-1} < \frac{n}{3}$$

The proof of this theorem involves the techniques of Bar–Natan. Namely, instead of chord diagrams (Feynman diagrams) we consider the corresponding polynomials coming from the natural representation of SL(n). Thus, we obtain a polynomial in N. The polynomials corresponding to baguette diagrams are linearly independent; thus, so are the diagrams themselves. In addition, one can consider uni–trivalent diagrams, which are in natural correspondence with linear combinations of Feynman diagrams, see [CDBook].

Theorem 13.7. ([CD2]) For any fixed value of k = d - n + 1 we have the following asymptotic inequality as d tends to ∞ :

$$dim \mathcal{C}_{d,n} \gtrsim \frac{1}{2^{\frac{k(k-1)}{2}} 3^{k-1} (k-1)!} (d-k+1)^{k-1}.$$

Proof. This theorem follows from Theorem 13.6. Actually, we have to count the number of integer points with even coordinates belonging to the body in \mathbf{R}^{k-1} described by the set of inequalities above. Asymptotically, the number of such points is equal to the volume of the body divided by 2^{k-1} .

To find this volume, let us note that the condition $n_1 + \cdots + n_{k-1} < \frac{n}{3}$ specifies the interior part of a (k-1)-simplex in \mathbf{R}^{k-1} that has (k-1) sides of length $\frac{n}{3}$ and all right angles between sides. Obviously, its volume is equal to $\frac{(n/3)^{k-1}}{(k-1)!}$. The inequality $n_1 < n_2$ cuts exactly one half of this body, the next equality cuts a quarter of the obtained half, and so on, and the last one cuts $\frac{1}{2^{k-2}}$ -th part of the result obtained at the previous step. Summarising the results above, we obtain the statement of the theorem.

13.4.4 A table of dimensions

The first precise calculation for the dimensions $\dim \mathcal{A}_n$ were made by Bar–Natan. To implement his algorithm and to calculate the dimensions of \mathcal{A}_n up to n=9 he borrowed extra RAM for his computer. The program had been working for several days.

Below, we give the table of dimensions up to n = 12.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$dim \mathcal{P}_n$	0	0	1	1	2	3	5	8	12	18	27	39	55
$dim \mathcal{A}_n$	1	0	1	1	3	4	9	14	27	44	80	132	232
$dim V_n$	1	1	2	3	6	10	19	33	60	104	184	316	548

The answer for n = 10, 11, 12 was obtained by using a thin technique using special structures on the set of primitive diagrams.

Chapter 14

The Kontsevich integral and formulae for the Vassiliev invariants

The Kontsevich integral was first invented by M.L. Kontsevich [Kon] in 1992. It was based on a remarkable construction of the product integral, better known as Chen construction or iterated integration formula. Kontsevich used the integration in the way proposed by Knizhnik and Zamolodchikov [KZ].

After Kontsevich's original proof, some other sympathetic (mostly combinatorial) constructions describing the same knot invariant arose; see the works of Cartier and Piunikhin [Car93, Piu93]. The work by Le and Murakami [LM] proposes a concrete method of calculation of the Kontsevich integral. See also [La, CD].

A very fundamental approach to Kontsevich's integral is presented in the book by Chmutov and Duzhin [CDBook].

First, recall some definitions. Given a Vassiliev invariant V of degree n, then its (n+1)-th derivative equals zero. The value of the n-th derivative of the invariant V (the symbol of V) depends only on the passing order of singular points; thus it can be considered as a function on chord diagrams. It was shown that each such function satisfies the one–term and the four–term relations (such functions are called weight systems).

Theorem 14.1. (1) (V.A. Vassiliev) Each symbol of an invariant of degree n comes from some element of graduation n of the chord diagram algebra (with 1T and 4T-relations).

(2) (M.L. Kontsevich) All elements of Δ_n are symbols of the Vassiliev knot invariants of degree n.

The first part of the theorem follows from Vassiliev's works [Vas, Vas2]. We have already proved it previously.

The main goal of this chapter is to prove the second part of this theorem.

14.1 Preliminary Kontsevich integral

Definition 14.1. The completion $\overline{\Delta}$ of $\Delta = \bigoplus_{m=0}^{\infty} \Delta_m$ is the set of all formal series $\sum_m c_m a_m$, where $c_m \in \mathbf{C}$ are numeric coefficients, and $a_m \in \Delta_m$ are elements of the space of degree m chord diagrams.

Let us think of the space \mathbb{R}^3 as a Cartesian product of \mathbb{C}^1 with the coordinate z and \mathbb{R}^1 with the coordinate t.

Given an oriented knot K in $\mathbf{R}^3 = \mathbf{C}_z \times \mathbf{R}_t$, by a small motion in \mathbf{R}^3 (without changing the knot isotopy type), we can make the coordinate t a *simple Morse function* on the knot K. This means that all critical points of t on the knot K are regular and all critical points have different critical values.

Remark 14.1. Later on, such embeddings will be called Morse knots.

Definition 14.2. The *preliminary Kontsevich integral* of a knot K is the following element of $\overline{\Delta}$:

$$Z(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int_{\substack{c_{min} < t_1 < \dots < t_m < c_{max} \\ t_j \text{ non-critical}}} \sum_{P = \{(z_j, z_j')\}} (-1)^{\downarrow} D_P \bigwedge_{j=1}^m \frac{dz_j - dz_j'}{z_j - z_j'} \quad (1)$$

We decree the coefficient of the "empty" chord diagram to be equal to one.

Let us discuss the formula (1) in more detail.

The real numbers c_{min} and c_{max} are maximal and minimal values of the function t on the knot K.

The integration domain is an n-simplex $c_{min} < t_1 < \cdots < t_m < c_{max}$. This domain is divided into connected components. Herewith, z_i and dz_i should be understood as functions of the corresponding t_i . For instance, for the unknot shown in Fig. 14.1 and m=2 the integration domain consists of n components and looks as shown in Fig. 14.1.

The number of summands is constant for each connected component, but it can vary when passing from one component to another. The part of the knot lying inside the margin between two adjacent critical levels is a set of curves; each of these curves is uniquely parametrised by t.

Let us fix m and choose m horizontal planes $\{t=t_i\}, i=1,\ldots,m$, each of which does not contain critical points and lies between the minimal and the maximal levels. Later, we shall take the sum over all natural m. At each plane $\{t=t_i\} \subset \mathbf{R}^3$, let us choose an unordered pair $(z_i,t_i), (z_i',t_i)$ of different points lying on K. Denote by $P = \{(z_i,z_i')\}$ the system of m such pairings. Fix a pairing P. If we think of a knot as a circle and then connect the points of the circle corresponding to z_i,z_i' of the same pair (according to P) we obtain a chord diagram. Denote this diagram by D_P .

Now, under the integral we have the sum of such diagrams corresponding to different pairings P. The coefficients are obtained in the following way. Choosing any arbitrary connected component, the choice of P means that for each t_i , some pair of knot branches is taken. Thus, choosing m planes, we get m pairs of points.

As a matter of fact, after we have chosen all pairings, the diagram D_P is defined; thus we should integrate not chord diagrams, but only the form $(-1)^{\downarrow} \wedge_{j=1}^{m} \frac{dz_j - dz'_j}{z_j - z'_j}$.

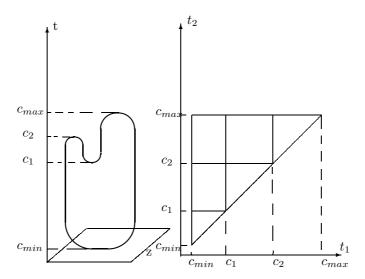


Figure 14.1. Integration domains for the Kontsevich integral

The obtained integral will give us the coefficient of our chord diagram D_P . Later, we shall collect similar terms.

In the example shown above, the connected component $\{c_{min} < t_1 < c_1, c_2 < t_2 < c_{max}\}$ corresponds to the a unique pair of points at the levels $\{t=t_1\}$ and $\{t=t_2\}$. In this case, the desired sum consists of a unique summand. For the component $\{c_{min} < t_1 < c_1, c_1 < t_2 < c_2\}$, we have a unique choice at the level $\{t=t_1\}$, but the plane $\{t=t_2\}$ intersects the knot at four points; thus we have $C_4^2=6$ possible pairings (z_2,z_2') , and the total number of summands equals six. For the component $\{c_1 < t_1, t_2 < c_2\}$ we have 36 summands, among them the most interesting case of (z_1, z_2) appears. In each part of the figure, we choose exactly one pairing and show the corresponding chord diagram.

It is easy to see that in all cases except $\{c_1 < t_1 < t_2 < c_2\}$ we obtain the chord diagram \bigcirc with two non-intersecting chords. These diagrams are equal to zero modulo one–term relation. Thus, the integration can be reduced to the small simplex $\{c_1 < t_1 < t_2 < c_2\}$.

The symbol \downarrow for a given set choice of P denotes the number of points (z,t_i) or (z',t_i) of P, where the coordinate t is decreasing while moving along the knot according to its orientation. In Fig. 14.2, the diagrams corresponding to different integration domains are shown.

Now we have the following questions to answer.

- 1. Do the coefficients of $\bar{\Delta}$ in the formula (1) converge?
- 2. Is the obtained element a knot invariant?
- 3. How is it related to the Vassiliev invariants?
- 4. How do we calculate this integral?

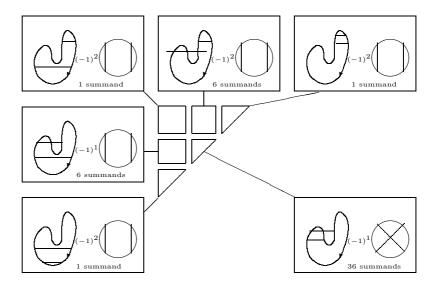


Figure 14.2. Integration domain and chord diagrams

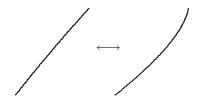


Figure 14.3. Horizontal deformation

Theorem 14.2 ([Kon], see also [BN]). All coefficients of (1) are finite.

Definition 14.3. A horizontal deformation is an isotopy of a Morse embedding of a curve in \mathbb{R}^3 that does not change the setup of singular points.

The horizontal deformation can be expressed as a composition of moves shown in Fig. 14.3.

Theorem 14.3 ([BN]). The function Z(K) is invariant under horizontal deformations of a knot and under the transformation shown in Fig. 14.4, but not invariant under the transformation (*), shown in Fig. 14.5.

Denote the knot representing the closure of the arc shown in Fig. 14.5 by A.

We can consider the simplest realisation of the unknot (with one minimum and one maximum) and the realisation given by ∞ , see Fig. 14.6. It is easy to see that Z(K) for the simplest realisation is equal to one (i.e., the series consisting of the only diagram without chords with coefficient one). Moreover, $Z(\infty)$ is not equal to $1 = \bigcirc$.

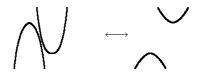


Figure 14.4. Moving critical values

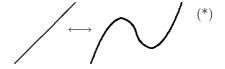


Figure 14.5. Forbidden transformation

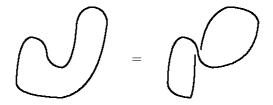


Figure 14.6. The " ∞ " knot

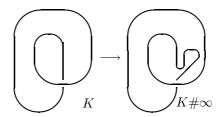


Figure 14.7. Adding the ∞ -knot

Thus we see that Z is not a knot invariant. On the other hand, one can prove the following theorem.

Theorem 14.4. If the knot K' is obtained from K by using (*) then $Z(K') = Z(K) \cdot Z(\infty)$.

Proof. First, let us note that ∞ is obtained from the knot A by using allowed moves, thus $Z(\infty) = Z(A)$.

Let us consider now the connected sum of K with a "small" knot ∞ in such a way that the interval of the coordinate t, corresponding to the knot A, has no critical points of the knot K. In this case, just two new critical points — one maximum and one minimum are added to this knot, see Fig. 14.7.

By virtue of the previous theorems, the Kontsevich integral of the obtained knot coincides with the Kontsevich integral for K' that is obtained from $K\#\infty$ by using horizontal deformations. Comparing the Kontsevich integral for the initial knot and for the knot K, we see that each member for the integral of the knot K corresponds to the same member multiplied by the Kontsevich integral for K (in the integral of K'). Consequently, K (in the integral of K').

14.2 $Z(\infty)$ and the normalisation

Thus, the change of the preliminary integral $Z(\cdot)$ under (*) is not difficult: the value is just multiplied by $Z(\infty)$. Now let K be a Morse embedding of S^1 in \mathbf{R}^3 , and c be the number of critical points of t on K.

Let us consider now the preliminary Kontsevich integral as a formal series. Hence this series consists of elements of a graded algebra and its initial element is the unit element of this algebra. Then one can inverse such rows by

$$(1+a)^{-1} = 1 - a + a^2 - a^3 + \dots,$$

where a^i is the formal series for the *i*-th power of the series a. Furthermore, one can formally multiply such series.

Definition 14.4.

The universal Vassiliev–Kontsevich invariant of a knot K is the following element of the completion of the chord diagram algebra:

$$I(K) = \frac{Z(K)}{Z(\infty)^{\frac{c}{2}-1}}. (2)$$

Remark 14.2. Here the degree $(\frac{c}{2}-1)$ is taken for the following majors. In the case of the simplest embedding representing the unknot we wish to have $I(\bigcirc) = 1$. For one maximum and one minimum we have $\frac{c}{2} - 1 = 0$.

Remark 14.3. Obviously, if (1) converges, then (2) makes sense: it is just the fraction of two series.

Thus we obtain the following theorem.

Theorem 14.5. The Kontsevich integral $I(\cdot)$ is a knot invariant.

Proof. Indeed, by virtue of Theorem 14.4 we see that Z(K) depends not on the configuration of critical points but only on their quantity.

It is easy to check that two Morse embeddings represent the same knot if and only if one can be transformed to the other by means of moves not changing the setting of critical points and moves shown in Figs. 14.4 and 14.5.

Taking into account the invariance of Z under all moves but the last one, we obtain the statement of the theorem.

The invariant $I(\cdot)$ is called the universal Vassiliev-Kontsevich invariant.

Now it remains to formulate and to prove the most important theorem.

Let W be a weight system of degree m. Decree that W(d) = 0 for all diagrams d with the number of chords not equal to m.

Theorem 14.6 (Kontsevich, see also [BN]). The invariant $W(I(\cdot))$ is a Vassiliev invariant with symbol W, i.e.,

$$V(W)(K) = W(I(K))$$

for each knot K.

This theorem implies the second (difficult) part of the Vassiliev–Kontsevich theorem about the existence of Vassiliev invariants corresponding to any given weight system.

We shall prove Theorems 14.2, 14.3 and 14.6 later.

14.3 Coproduct for Feynman diagrams

Now, let us return to the algebras \mathcal{A}^c and \mathcal{A}^t of chord and Feynman diagrams.

Remark 14.4. Within this section, we consider the algebras not factorised by the 1T-relation.

As shown above, these algebras are isomorphic. Thus, \mathcal{A}^t has a Hopf algebra structure as well. Let us describe this structure explicitly.

Let D be a Feynman diagram and let V(D) be connected components of the diagram, i.e., connected components of the graph obtained from D by deleting a

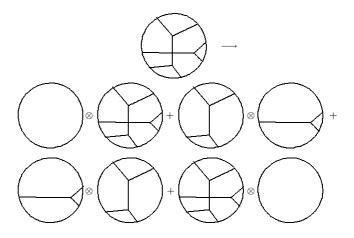


Figure 14.8. Coproduct of a Feynman diagram

circle. Let $J \subset V(D)$ be a subset of V(D). This subset defined a Feynman diagram c_J , whose connected components lie in V(D), i.e., the Feynman diagram consisting of the circle of the diagram D and those connected components of the graph V(D) belonging to V(C).

Let us define the coproduct $\tilde{\mu}(D)$ as

$$\tilde{\mu}(D) = \sum_{J \subset V(c)} C_J \otimes C_{V(c) \setminus J}.$$

Example 14.1. In Fig. 14.8 we illustrate the coproduct operation for a Feynman diagram.

Theorem 14.7. The coproduct defined below coincides with that for A^c , i.e. $\mu \equiv \tilde{\mu}$.

Proof. We have to show that for each Feynman diagram D, its coproduct coincides with the linear combination of coproducts of chord diagrams that D can be decomposed into.

We shall use induction on the number k of interior vertices of the diagram. For k = 0, there is nothing to prove.

Suppose that the statement is true for all Feynman diagrams with n interior vertices. Consider a Feynman diagram D with (n+1) interior vertices. We have to show that $\mu(D) = \tilde{\mu}(D)$. According to the STU-relation, the diagram D can be represented as a difference $D_+ - D_-$, as shown in Fig. 14.9; in this case the connected component of D corresponds to a pair of connected components for each of D_1, D_2 .

Let us choose the components (a_+, b_+) and (a_-, b_-) for the diagrams D_+ and D_- . These components are obtained by resolving a point of D. Each of D_\pm has n interior vertices. Thus, the claim of the theorem is true for them: $\mu(D_\pm) = \tilde{\mu}(D_\pm)$. Let us now write $\mu(D) = \mu(D_+) - \mu(D_-)$. In each of these coproducts we have only those members where the components (a_+, b_+) (respectively, (a_-, b_-)) lie on

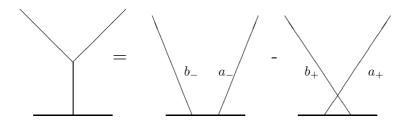


Figure 14.9. Applying STU-relation

the same side with respect to the \otimes sign. Obviously, these members collected together give $\tilde{\mu}(D)$ (in the previous sense of the coproduct). It is easy to see that the remaining members give us zero. In fact, suppose we have a splitting of the Feynman diagram D_+ into two diagrams, where a_+ belongs to one of them and b_+ belongs to the other. Then if we divide D_- in just the same way as we did with D_+ with respect to all other connected component and take a_- to be the first multiplicator of the tensor product and b_2 to be the second one, we obtain two coinciding tensor products.

Collecting all previous statements together, we obtain the statement of the theorem.

14.4 Invariance of the Kontsevich integral

We are now going to prove now theorems 14.2, 14.3 and 14.6.

Remark 14.5. By $Z_m(K)$ and $I_m(K)$ we mean the m-th graded summand of Z(K) and I(K), respectively.

First, let us prove Theorem 14.2 which states that the series for each coefficient at each member of (1) converges.

Proof. Consider a Morse knot K in \mathbb{R}^3 . Let us fix $m \in \mathbb{N}$ and choose some m planes not intersecting K at critical points.

Choose some chord diagram D and consider the coefficient at this diagram. It is obtained by integrating the form

$$\bigwedge_{j=1}^{m} \frac{dz_j'(t_j) - dz_j(t_j)}{z_j'(t_j) - z_j(t_j)}$$

over the part of the simplex $\{c_{min} < t_1 < \cdots < t_m < c_{max}\}$ corresponding to the chord diagram D.

Let us consider the singular points of the form, namely, those where the condition $z_j = z'_j$ holds for some j. The integral of the form might diverge only in the

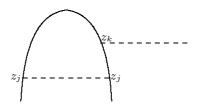


Figure 14.10.

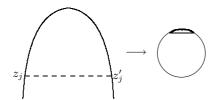


Figure 14.11.

neighbourhood of these points. Consider such pairs of points z_j, z_j' closed to the singular position.

Then we have the two possibilities:

- 1. The arc between z_j and z'_j contains other ends of chords (as shown in Fig. 14.10). Then the integration domain (where we integrate $z_j z'_j$) has smallness of higher order than $z_j z'_j$ because the singular point is not degenerate. Consequently, this part of (2) gives no divergence.
- 2. The arc between z_j and z'_j has no other chord ends; see Fig. 14.11. Then the chord $z_j z'_j$ of the diagram D is isolated; thus, the diagram D equals zero modulo 1T–relation.

This completes the proof of the theorem.

14.4.1 Integrating holonomies

In order to prove the remaining two theorems, we shall have to integrate holonomies and introduce the so called *Knizhnik–Zamolodchikov connection*.

First, let us recall some constructions.

Definition 14.5. Let X be a smooth manifold and let \mathcal{U} be an associative topological algebra with the unit element (considered over \mathbf{R} or \mathbf{C}).

Then a \mathcal{U} -connection Ω on the manifold X is a 1-form Ω on X with coefficients from \mathcal{U} .

The *curvature* of the connection Ω is the 2-form

$$F_{\Omega} = d\Omega + \Omega \wedge \Omega.$$

The connection is *flat* if its curvature equals zero.

Definition 14.6. Let $B: I \to X$ be a smooth mapping of the interval [a,b] to the space X. Let Ω be a \mathcal{U} -connection on X.

Let us define the holonomy $h_{B,\Omega}$ of the form Ω along the path B as the solution of the differential equation $\frac{\partial}{\partial t}h_{B,\Omega}(t) = \Omega(B'(t)) \cdot h_{B,\Omega}(t), t \in I$, with the initial condition $h_{B,\Omega}(a) = 1$.

Remark 14.6. It is easy to show that if the connection Ω is flat, then the holonomy is defined only by the ends of a path and the homotopy type of this path.

This is quite analogous to the Gauss-Ostrogradsky formula in the commutative case. Then the multiplicative integral is just the exponent of the Riemmanian integral for the logarithmic function. In the non-commutative case the extra member $\Omega \wedge \Omega$ arises.

The product integration

The solution of such an equation (holonomy) often exists. It is called the *product* or *multiplicative* integral of the form Ω . In many cases, the holonomy can be calculated according to the following *iterated formula*:

$$h_{B,\Omega}(t) = 1 + \sum_{m=1}^{\infty} \int_{a \le t_1 \le t_2 \le \dots \le t_m \le t} (B^*\Omega)(t_m) \dots (B^*\Omega)(t_1).$$
 (3)

In order to clarify the situation, let us consider the following simple construction.

Example 14.2. Let

$$Y' = AY$$

be a differential equation with the initial condition Y(0) = 1, say, in $n \times n$ matrices.¹ Obviously, its solution Y(t) is the product of "infinitely many" elements "infinitely close to the unit element." This is naturally called the product integral of A and denoted by

$$Y(x) = \int_0^{x \cap} (E + A(t)dt).$$

Note that in order to calculate Y(x), one can use the following formula

$$Y(x) = E + \int_0^x A(t_1)dt_1 + \int_0^x A(t_1) \int_0^{t_1} A(t_2)dt_2dt_1 + \cdots$$
 (4)

if the series (4) converges.

Remark 14.7. In all "normal" cases this series actually converges.

¹Or in any other topological algebra: all we need here is linear operations over the main field, multiplication, and an intrinsic topology (in order to consider limits of sequences).

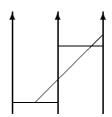


Figure 14.12. An element \mathcal{A}^{KZ}

Actually, while integrating the series, each next member becomes equal to the previous one multiplied by A.

Each member of the iterated integral (4) can be considered as an integral over some simplex.

The formula (3) is completely analogous to the formula (4).

The theory of product integration is well described in [DF, Man1, ManMar].

Remark 14.8. In the normal (convergent) case it is obvious that the formula actually gives a solution of the differential equation. The initial condition evidently holds. The derivative of the m-th integral gives the (m-1)-th integral with coefficient $\Omega(\dot{B})$.

The Knizhnik-Zamolodchikov connection

Denote by \mathcal{D}_n^{KZ} the set of all diagrams consisting of n ascending infinite arrows (in Fig. 14.12 they are shown by thick lines) and a finite number of edges such that:

- 1. each end point of each edge either lies on the arrow or is a trivalent vertex (with two other ends of edges);
- 2. one point on the arrow is incident to no more than one interval (only one end of this edge can coincide with this point).

Such diagrams are considered up to combinatorial equivalence.

Let C be the main field. Consider the set $\mathcal{A}_n^{KZ} = span(\mathcal{D}_n^{KZ})/\{STU\text{-relations}\}$. The STU-relation means the same as for the Feynman diagrams (by "multiplication" of all "partial" integrals), where we consider a part of an arrow instead of part of an oriented circle. Note that the STU-relation is local. When we finally close the "arrow" diagrams in order to obtain the Feynman diagram, we get the STU-relation as well.

For a fixed n, the set \mathcal{A}_n^{KZ} admits an algebraic structure: the product means the juxtaposition of one diagram over the other.

Example 14.3. For n = 3 such a multiplication for A_n^{KZ} is shown in Fig. 14.13.

For a fixed n, the algebra \mathcal{A}_n^{KZ} is graded: the order of an element is equal to half of the total number of vertices.

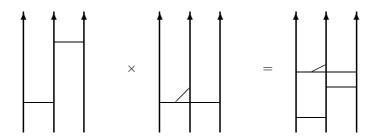


Figure 14.13. Multiplication in \mathcal{A}^{KZ}

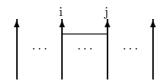


Figure 14.14. The element Ω_{ij}

For $1 \leq i, j \leq n$, let us define $\Omega_{ij} \in \mathcal{A}_n^{KZ}$ as the element with only one edge connecting the arrows i and j.

Remark 14.9. It is easy to see that if $\{i, j\} \cap \{k, l\} = \emptyset$ then Ω_{ij} and Ω_{kl} commute.

Let X_n be the configuration space of n pairwise different points on \mathbb{C}^1 . Let ω_{ij} be the following 1-form on X_n :

$$\omega_{ij} = d(\ln(z_i - z_j)) = \frac{dz_i - dz_j}{z_i - z_j}.$$

Let us define the formal Knizhnik–Zamolodchikov connection Ω_n with coefficients in \mathcal{A}_n^{KZ} as $\Omega_n = \sum_{1 \leq i < j \leq n} \Omega_{ij} \omega_{ij}$ on X_n .

Theorem 14.8. This connection is flat. More precisely $\Omega_n \wedge \Omega_n = 0$ and $d\Omega_n = 0$.

Proof. The last statement is evident. Indeed, $d\omega_{ij} = d^2(\ln(z_i - z_j))$ and this vanishes by definition of d.

Let us prove the first statement. Consider the element

$$\Omega_n \wedge \Omega_n = \sum_{i < j; k < l} \Omega_{ij} \Omega_{kl} \omega_{ij} \omega_{kl} \tag{5}$$

and the set $\{i,j,k,l\}$. If this set consists of two or four elements then the corresponding member of the sum equals zero (this case is commutative). Consequently, the desired sum equals the sum along all i,j,k,l, where the set $\{i,j,k,l\}$ consists of three elements. Consider, e.g., the set $\{i,j,k,l\} = \{1,2,3\}$ and all corresponding members in the sum (5). In this case we get:

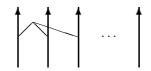


Figure 14.15. The element Ω_{123}

$$\sum_{\{i,j,k,l\}=\{1,2,3\}} \Omega_{ij} \Omega_{kl} \omega_{ij} \omega_{kl} = (\Omega_{12} \Omega_{23} - \Omega_{23} \Omega_{12}) \omega_{12} \wedge \omega_{23} + \langle \text{cyclic permutations} \rangle.$$

By using the STU-relation, we see that the desired sum equals

$$-\Omega_{123}(\omega_{12} \wedge \omega_{23} + \omega_{23} \wedge \omega_{31} + \omega_{31} \wedge \omega_{12}),$$

where Ω_{123} is the element shown in Fig. 14.15.

Exercise 14.1. (V.I. Arnold's identity.) Show that $\omega_{12} \wedge \omega_{23} + \omega_{23} \wedge \omega_{31} + \omega_{31} \wedge \omega_{12} = 0$.

Remark 14.10. This identity appeared in [Arn2] when Arnold studied the cohomologies of the pure braid group.

Thus if some set $\{i, j, k, l\}$ consists of precisely three different members, it gives no contribution. We have considered all possible cases. Thus, $\Omega_n \wedge \Omega_n = 0$.

Remark 14.11. The connection Ω_n can be slightly modified for the case of the algebra \mathcal{A}_{nn}^{KZ} . This algebra is generated by arrow diagrams with 2n arrows (the first n arrows oriented upwards and the last n arrows oriented downwards). The STU-relation for such diagrams depends on the direction of the chord that the relation has to be applied to.

Let
$$\Omega_{nn} = \sum_{1 \leq i < j \leq n} s_i s_j \Omega_{ij} \omega_{ij}$$
, where s_i equals 1 for $i \leq n$ and -1 for $i > n$.

Exercise 14.2. Show that the connection Ω_{nn} is flat.

Now, let us prove the invariance theorem (Theorem 14.3).

Proof. First, let us prove that the preliminary Kontsevich integral Z(K) is invariant under the transformation preserving the critical points.

The point is that the Kontsevich integral for the whole knot can be decomposed into a product of similar integrals for parts of this knot; each of these parts represents an element of some Knizhnik–Zamolodchikov algebra with ascending and descending arrows; being connected together, they constitute a normal chord diagram. Thus, the product of elements in some A^{KZ} is thought to be an element of $\bar{\Delta}$.

Let $c_{min} \leq a < b \leq c_{max}$. Let us define Z(K, [a, b]) just as was done in (1), but taking the integration domain to be $\{a < t_1 < \cdots < t_n < b\}$, and replacing the chord diagrams with elements of the Knizhnik–Zamolodchikov algebra.

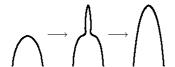


Figure 14.16. Moving a maximum

Although Z(K, [a, b]) does not belong to $\overline{A^c}$, the corresponding series (evaluated at a knot) converges for the same reasons as Z. Since the interval (a, b) has no critical points, the intersection of the knot with the margin $\mathbb{C} \times (a, b)$ is a set of oriented curves without horizontal tangent lines. Suppose that the number of such curves equals 2n. Obviously, n of them are ascending and the other n are descending. Let us fix the lower points a_1, \ldots, a_{2n} and the corresponding upper points b_1, \ldots, b_{2n} , where the first n coordinates correspond to ascending curves and the other ones correspond to descending curves. The convergence of the integral can be proved in the same manner as before. One should, however, introduce an analogue of the one—term relation taking all diagrams with a "solitary" chord (with one end on an ascending chord and one end on a descending arc) to zero.

Now, the integral Z(K[a,b]) can be represented as the holonomy of the connection Ω_{nn} along the path from (a_1,\ldots,a_{2n}) to (b_1,\ldots,b_{2n}) by virtue of the iteration formula (3). Actually, the m-th member of the iteration formula for Ω_{nn} corresponds to the m-th member of the Kontsevich integral because in both cases we integrate the form

$$\sum_{P=\{(z_j, z_j')\}} (-1)^{\downarrow} \Omega_{jj'} \bigwedge_{j=1}^m \frac{dz_j - dz_j'}{z_j - z_j'}, \tag{6}$$

where $(-1)^{\downarrow}$ corresponds to the sign of the product $s_j s'_j$.

Recall that the STU-relation for Feynmann diagrams is "the same" as the 4T-relation for chord diagrams. This is just the place when we use the 4T-relation (in its STU-form).

Since the curvature of the connection Ω_{nn} is zero, the integral (6) is invariant under homotopies of the integration path with fixed endpoints, i.e., under horizontal isotopies of the part of the knot lying inside $t \in (a, b)$.

It is not difficult to show that for arbitrary a < b < c (possibly, critical), we have $Z(K, [a, c]) = Z(K, [a, b]) \cdot Z(K, [b, c])$. Thus we conclude that the integral Z(K) which is a product $Z(K, [c_i, c_{i+1}])$, where c_i, c_{i+1} are all pairs of "adjacent" critical points, is invariant under horizontal deformation in the intervals not containing critical points.

Now, let us consider the cases when critical points are moving during the knot isotopy.

1. The critical point is moving, but the disposition of all critical points stays the same, see Fig. 14.16.



Figure 14.17. Changing disposition of critical points

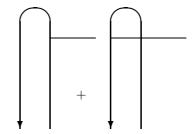


Figure 14.18. Cancellation of two terms

2. The order of applicates of two critical points changes, see Fig. 14.17.

As shown in Figs. 14.16 and 14.17, one can first perform the transformation that does not change Z(K) to obtain a knot with a thin "needle." Let us show that the removal of this needle changes the m-th graduation member of the Kontsevich integral by some infinitely small ε depending on the diameter of the needle.

Actually, let K be a knot and let K' be the knot obtained from the knot K by means of adding a vertical needle somewhere.

Obviously, the difference Z(K) - Z(K') contains only the members corresponding to the diagrams with ends lying inside the needle. Suppose that the width of the needle equals ε . Let us show that $Z_m(K) - Z_m(K') = O(\varepsilon)$.

Actually, consider all the chords incident to the needle. If the upper chord has both ends on the needle then the chord diagram equals zero modulo 1T-relation. If there are no chords with all ends lying on the margin, the situation is quite simple as well: the member shown in Fig. 14.18 should have smallness of order ε : while integrating the left and the right part, the numbers \uparrow have difference 1, thus we obtain a contraction because for each member there exists a "mirror" member, see Fig. 14.18.

Thus, we only have to consider the case when the upper chord (z_i, z'_i) has one end lying on the needle, and there are k chords lying under this with both ends on the needle. Suppose the lowest one is (z_{j_1}, z'_{j_1}) and the upper one is (z_{j_k}, z'_{j_k}) , see Fig. 14.19.

We may assume that (z_i, z'_i) is the only chord such that one end of it lies on the needle. If we delete such chords, we multiply the final integral by some number

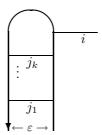


Figure 14.19. A knot having diagram with a solitary chord

bounded from zero and the infinity.

Let $\delta_{\alpha} = |z_{j_{\alpha}} - z_{j'_{\alpha}}|$. Then the difference Z(K') - Z(K) is bounded by some constant multiplied by

$$\int_0^{\varepsilon} \frac{d\delta_1}{\delta_1} \int_0^{\delta_1} \frac{d\delta_2}{\delta_2} \dots \int_0^{\delta_{k-1}} \frac{d\delta_k}{\delta_k} \int_{z_{j_k}}^{z_{j'_k}} \frac{dz_i - dz'_i}{z_i - z'_i}.$$

The integral has smallness of the order $\tilde{\varepsilon}$. Actually, the last integral has smallness of the order of δ_k . Consequently, the member δ_k is reduced in the penultimate integral, so this integral has smallness of δ_{k-1} , and so on. Finally, the total integral has smallness of $\delta_1 \sim \varepsilon$.

Since ε is arbitrary small, we conclude the desired invariance.

Thus, we have proved that $I(\cdot)$ is a knot invariant.

Now, let us prove Theorem 14.6 that for each weight system W, the function $W(I(\cdot))$ generates a Vassiliev invariant with symbol W.

Proof. Without loss of generality, we might assume that our knots are not only Morse embedded in \mathbb{R}^3 but their projections on some vertical plane (say, Oxz) represent planar knot diagrams (in the ordinary sense). Let W be a weight system of order m. In order to prove the theorem, we have to show that if D is a chord diagram of degree m and K_D is a Morse embedding of the singular curve (curve with intersection) in $\mathbb{C}_z \times \mathbb{R}_t$ (the singular knot corresponds to D) then we have

$$I(K_D) = \bar{D} + \langle \text{members of order } \geq m \rangle,$$

where \bar{D} is the equivalence class of the chord diagram D and $I(K_D)$ is defined to be the alternating sum of I evaluated at 2^m knots generating the singular knot K_D .

If two Morse knots K_1 and K_2 in $\mathbf{C}_z \times \mathbf{R}_t$ coincide everywhere except for a small part, where the branches of K_1 form an overcrossing (with respect to the projection on a vertical plane) and those of K_2 form an undercrossing, then the values $Z(K_2)$ and $Z(K_1)$ differ only in those chord diagrams, for which some point(s) on this branches is (are) paired with other point(s).

By virtue of Vassiliev's relation, the singular knot K_D is an alternating sum of 2^m knots that differ in small neighbourhoods of m points. Note that the sign of this alternating sum is regulated by the multiplicator $(-1)^{\downarrow}$ in (1).

Arguing as above, we conclude that $Z(K_D)$ has non-zero coefficients only at those chord diagrams obtained by pairing points for each neighbourhood. Thus, chord diagrams with non-zero coefficients must have at least m chords.

For chord diagrams of degree m this coefficient is not equal to zero only for the diagram K_D .

Let us calculate this coefficient.

At each of m vertices we obtain the difference of the integrals of the differential form $\frac{dz_i - dz'_i}{z_i - z'}$.

This difference equals the integral of $\frac{dz}{z}$ along the circuit passing once around zero. According to Cauchy's theorem, this integral equals $2\pi i$. Because the number of such contours equals m, the coefficients should be multiplied. Thus we obtain the multiplication factor $(2\pi i)^m$ that is cancelled by the denominator of (1). This means that

$$Z(K_D) = \bar{D} + \langle \text{members of order } \geq m \rangle.$$

Taking into account $I(K) = \frac{Z(K)}{Z(\infty)^{\frac{c}{2}-1}}$, we have

$$I(K_D) = \bar{D} + \langle \text{members of order } \geq m \rangle.$$

Consequently, $W(I(K_D)) = W(D)$, and the Vassiliev invariant $W(I(\cdot))$ of order m has the symbol W. This completes the proof.

The calculation of the Kontsevich integral is, however, very difficult. For instance, it was quite a complicated problem to calculate the integral (preliminary) of ∞ . The form (in the Feynman diagram) of the integral was conjectured by Bar–Natan, Garoufalidis, Rozansky, Thurston [BGRT] and finally proved in [BN2].

The formula is represented in terms of Feynman diagrams. It looks like

$$I(\infty) = \exp \sum_{n=0}^{\infty} b_{2n} w_{2n} = 1 + \left(\sum_{n=0}^{\infty} b_{2n} w_{2n}\right) + \frac{1}{2} \left(\sum_{n=0}^{\infty} b_{2n} w_{2n}\right)^{2} + \dots$$

Here b_{2n} are modified Bernoulli numbers, i.e., the coefficients of the Taylor series:

$$\sum_{n=0}^{\infty} b_{2n} x^{2n} = \frac{1}{2} \ln \frac{e^{x/2} - e^{-x/2}}{x/2},$$

and w_{2n} are wheels.

Each wheel w_{2n} is $\frac{1}{(2n)!}$ multiplied by the sum of (2n!) Feynman diagrams. Each of these diagrams consists of one exterior circle, one interior circle (treated just as a circular set of interior edges), and 2n chords connecting fixed n points on the first one with fixed 2n points on the second one. These points can be connected according to arbitrary permutation from S_{2n} . Thus, we have (2n)! summands and take their average.

For instance, if we consider w_4 , we see that eight summands represent the diagram D_x shown in the left part Fig. 14.20 and another sixteen summands represent the diagram D_y , see the right part of Fig. 14.20.

In the terms of chord diagrams w_4 can be represented as follows:

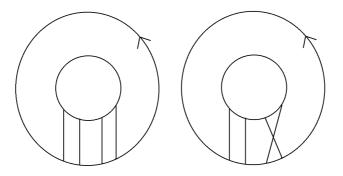


Figure 14.20. Elements D_x and D_y

$$w_4 = -\frac{10}{3} + \frac{4}{3}$$
.

Analogously (in fact, even more easily) one can find the expression for w_2 and w_2^2 .

Exercise 14.3. Prove the formulae above.

The first members of the final result look like:

$$I(\infty) = 1 + \frac{1}{48}w_2 + \frac{1}{4608}w_2^2 - \frac{1}{5760}w_4 + \dots$$

or, in terms of chord diagrams.

$$I(\infty) = 1 - \frac{1}{24} \left(\sum_{i=1}^{4} - \frac{1}{5760} \left(\sum_{i=1}^{4} + \frac{1}{1152} \left(\sum_{i=1}^{4} + \frac{1}{2880} \left(\sum_{i=$$

Besides this, Le and Murakami [LM] constructed a generalisation of the Kontsevich integral for the case of so–called tangles — one–dimensional manifolds lying between two horizontal planes and incident to these planes only at a finite number of points. A tangle is a common generalisation of both knots and braids, and the computation of the Kontsevich integral for the case of braids is much easier. In fact, tangles appeared indirectly in the text while calculating Z[a,b] for some interval [a,b]. By using their own techniques, they calculated $Z(\infty)$. Later, S.D. Tyurina calculated such integrals for various knots, see [Tyu1, Tyu2].

14.5 Vassiliev's module

It is not known whether the Vassiliev knot invariants distinguish all isotopy classes of knots and link. This is conjectured and known as *Vassiliev's conjecture*.

Let us introduce the *Vassiliev module* where two knots are taken to be different if they are distinguished by some Vassiliev invariant having order not higher than some fixed order. Besides, each knot can be decomposed into a finite sum of generators of the module.

Let us give now the precise definition.

Definition 14.7. The *Vassiliev module* of order n is the module over \mathbf{Z} (or \mathbf{Q}) generated by isotopy classes of oriented knots and singular knots modulo the following relations:

- 1. $\bigcirc = 0$, where \bigcirc is the unknot.
- 2. The Vassiliev relation.
- 3. $K_m = 0$ for m > n, where K_m is an arbitrary singular knot of order m.

The following theorem holds.

Theorem 14.9 (Decomposition theorem). In the Vassiliev module of order n, each knot K has the following decomposition:

$$K = \sum_{i=1}^{r+s} v_i(K)K_i,$$

where r is the dimension of the set of Vassiliev invariants having order less than or equal to (n-1), and s is the dimension of weight systems of order n; all v_i 's are Vassiliev's invariants of order less than or equal to n, and K_i are some fixed basic knots independent of the knot K.

This theorem follows straightforwardly from the definitions.

Definition 14.8. For any n, the *actuality table* of a Vassiliev invariant of type n is the set of its values on all basic knots.

The simplest decomposition in Vassiliev's module [La] of order two is the following:

$$K = V_2(K) \bigodot$$
,

where V_2 is the second coefficient of the Conway polynomial.

For more details see, e.g. [CDBook, Tyu1, Tyu2].

14.6 Goussarov's theorem

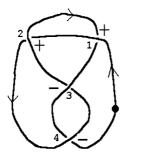
14.6.1 Gauss diagrams

Oriented knot planar diagrams are characterised by their *Gauss diagram*. A Gauss diagram consists of a circle together with the pre-images of each double point of the immersion connected by a chord. To take the information about overcrossings and undercrossings, the chords are oriented from the upper branch to the lower one. Furthermore, each chord is equipped with the local writhe number of the corresponding crossing. An example is shown in Fig. 14.21.

Denote the set of Gauss diagrams by \mathcal{D} .

One of the most important achievements in the combinatorial theory of Vas-siliev's invariant is Goussarov's theorem, see [GPV].

Before we formulate this theorem, we should introduce the notion of arrow diagram that appears also in the study of finite type invariants of virtual knots.



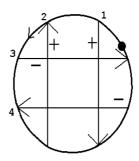


Figure 14.21. A diagram of the figure eight knot and its corresponding Gauss diagram

$$\mathbb{I}: \left(\begin{array}{c} \varepsilon \\ \end{array} \right) \mapsto \left(- \begin{array}{c} - \\ \end{array} \right) + \left(\begin{array}{c} \end{array} \right)$$

Now, let us turn to so-called *arrow diagrams*, i.e. formal diagrams consisting of an oriented circle and some arrows labelled by signs \pm (considered as a graph with a special structure). Denote the set of all arrow diagrams by A.

Let G be an Abelian group (say, \mathbf{Q} or \mathbf{R}) and let D be a Gauss diagram. Define the map I by the following symbolic formula:

where ε is a ± 1 sign.

The above formula means that each arrow should be decomposed in this way.

Exercise 14.4. Prove that $I: \mathbf{Z}[\mathcal{D}] \to \mathcal{A}$ is an isomorphism, where the reverse map $I^{-1}: \mathcal{A} \to \mathbf{Z}[\mathcal{D}]$) is given by

$$I^{-1}(A) = \sum_{A' \subset A} (-1)^{|A-A'|} i^{-1}(A'),$$

where A is an arrow diagram, the sum is taken over all subdiagrams A' of A and |A - A'| denotes the number of chords belonging to A but not to A'.

Since the set \mathcal{A} has a distinguished basis consisting of arrow diagrams, there exists a natural orthonormal product on \mathcal{A} . Namely, we put (D_1, D_2) to be one if $D_1 = D_2$ and zero, otherwise; then we extend (\cdot, \cdot) bilinearly.

This allows us to define a pairing $\langle \cdot, \cdot \rangle : \mathcal{A} \times \mathcal{D} \to \mathbf{Z}$ in the following way. For any $D \in \mathcal{D}$ and $A \in \mathcal{A}$ we put

$$\langle A,D\rangle=(A,I(D))=(A,\sum_{D'\subset D}i(D')),$$

where i is the natural map from \mathcal{D} to \mathcal{A} making all arrows dashed.

Theorem 14.10 (Goussarov). Let ν be a G-valued invariant of degree n of classical virtual knots. Then there exists a function $\pi: A \to G$ such that $\nu = \pi \circ I$ and such that π vanishes on any arrow diagram with more than n arrows.

Corollary 14.1. Any integer-valued finite type invariant of degree n can be represented as $\langle A, \cdot \rangle$, where A is a linear combination of arrow diagrams on a line with n arrows.

The main idea of the Goussarov theorem is the following. The standard method for calculation of a Vassiliev invariant ν goes as follows [Vas]. One picks a basic set of singular knots, which span the Vassiliev module. The invariant ν (and its derivatives) are determined by the actuality table, i.e., by the values on this set. Given a knot diagram, one unknots it, making it descending by a sequence of crossing switchings. Under each switching, the value of the invariant jumps. By Vassiliev's relation, the value of this jump is equal to the value of ν on the corresponding singular knot.

Then each singular knot is deformed to a knot with a single double point from the actuality table by an isotopy and a sequence of crossing switches. The jumps of ν correspond to knots with two double points. They are again deformed to the knots from the actuality table. The process eventually stops when the number of double points exceed the degree of ν .

Part IV $\label{eq:Atoms} \mbox{Atoms and d--diagrams}$

Chapter 15

Atoms, height atoms and knots

In the present chapter, we shall talk about an alternative way for encoding knots and links (different from planar diagrams and closures of braids). Namely, all knots can be encoded by so-called "atoms" and d-diagrams.

Atoms are combinatorial objects that arose several years ago in [F] for purposes of classification of integrable Hamiltonian systems of low complexity. d-diagrams are special chord diagrams closely connected with atoms.

By using this approach, we are going to prove several theorem on knots and curves: Kauffman–Murasugi's theorem on alternating links, the criterion for embeddability of special graphs, etc. We shall also describe a way of encoding knots by words in a finite alphabet via d-diagrams ("bracket calculus"). For a review on the bracket calculus see [Ma3].

15.1 Atoms and height atoms

Let us start with definitions and introduce the notation.

Definition 15.1. An *atom* is a pair: a connected 2-manifold M^2 without boundary and a graph $\Gamma \subset M^2$ such that $M^2 \setminus \Gamma$ is a disconnected union of cells that admit a chessboard colouring (with black and white colours).

The graph Γ is said to be the *frame* of the atom. The *genus* (respectively, *Euler characteristic*) of the atom is that of its first component.

The complexity of the atom is the number of vertices of its frame.

Atoms are considered up to natural isomorphism: two atoms are called *isomorphic* if there exists a one–to–one map of their first components taking frame to frame and black cells to black cells.

Atoms can be generated by Morse functions on 2–surfaces: an atom's frame is just the critical level with several critical point on it.

Definition 15.2. An atom is called a *height* (or a *vertical*) atom if it is isomorphic to an atom obtained by the third projection function on some closed 2-manifold embedded in \mathbb{R}^3 .

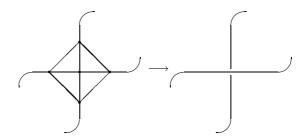


Figure 15.1. A part of a knot diagram constructed by a frame embedding

Each atom (more precisely, its equivalence class) can be completely restored from the following combinatorial structure:

- 1. the frame (four-valent graph);
- 2. the A-structure (dividing the outgoing half-edges into two pairs according to their disposition on the surface); and the
- 3. *B*-structure (for each vertex, we indicate some two pairs of adjacent half-edges (also: two angles) that constitute a part of the boundary of black cells).

In [Ma'1] the following criterion is proved.

Theorem 15.1. An atom V is a height atom if and only if its frame Γ is embeddable in \mathbb{R}^2 with respect to the A-structure (i.e., the intrinsic A-structure on the surface coincides with that induced from the plane).

It turns out that height atoms are closely connected with knots unlike the non-height ones. Having a height atom V, one can construct a knot diagram as follows. Consider the frame Γ of V and let us embed Γ in the plane with respect to the A-structure on V. Then, the B-structure of this atom can be illustrated on the plane: if a pair of edges outgoing from a vertex is adjacent in V, it remains so on the plane. Thus, one can locally indicate the structure of supercritical levels on the plane.

Thus, we have a four–valent graph on the plane with endowed B–structure. This B–structure allows us to construct a link diagram as shown in Fig. 15.1.

Remark 15.1. In Fig. 15.1, the angles of the supercritical level (in the left part) are marked by additional thick lines.

For each vertex A we set the crossing type in such a way that while turning inside the supercritical angle clockwise, one passes from the undercrossing to the overcrossing.

Thus, having an embedding of the frame of an atom with respect to the A-structure, one can construct a knot (link) diagram.

15.2 Theorem on atoms and knots

It turns out that all knots can be encoded (not uniquely) by height atoms. In fact, consider a height atom V. Let us embed its frame in \mathbb{R}^2 while preserving the A-structure of the atom.

Furthermore, the following theorem holds.

Theorem 15.2 (Theorem on atoms and knots, [Ma'1]). Let V be an atom. Then the planar link diagrams obtained from V by using the algorithm above generate diagrams representing the same link isotopy type.

Thus, we can say that an atom generates a knot (link).

The proof of the theorem on atoms and knots follows from a well–known theorem:

Theorem 15.3. If two planar graphs are isomorphic, then their embeddings in one and the same plane \mathbb{R}^2 are homeomorphic in \mathbb{R}^3 .

This theorem allows to restore the homeomorphism of knots in the ambient space from embeddings of the atom's frame. For more details see, e.g., [Ma'1].

15.3 Encoding of knots by d-diagrams

We begin with a theorem from [Ma'3].

Theorem 15.4. For each link isotopy class L, there exists a height atom V that encodes a link from this class and has only one supercritical circle.

Proof. Consider an arbitrary diagram of link L and the corresponding height atom V_1 . Suppose that the atom V_1 has k supercritical circles. If k = 1 then there is nothing to prove.

If k > 1 then there exists a vertex A of V_1 such that the two supercritical angles of this vertex correspond to two arcs of different supercritical circles.

Let us apply the move Ω_2 to the initial diagram, as shown in Fig. 15.2.

It is easy to see that after such transformation, the two circles shown in Fig. 15.2 are transformed into one circle. Thus, we decrease the number of supercritical circles by one without changing the link isotopy class. Reiterating this operation many times, we obtain a diagram with precisely one supercritical circle.

Consequently, d-diagrams encode all knot and link isotopy classes.

Let us give some examples. Consider the simplest planar diagram of the left trefoil knot. The corresponding height atom has two supercritical circles. Thus, by applying one move Ω_2 , we can obtain a diagram of the same trefoil for which the corresponding atom has one supercritical circle. The corresponding d-diagram is shown in Fig. 15.3.

Exercise 15.1. Show that d-diagrams shown in Figs. 15.4.a, 15.4.b encode the right trefoil and the figure eight knot.

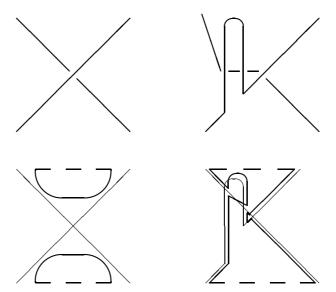


Figure 15.2. Link diagram above and supercritical circles below

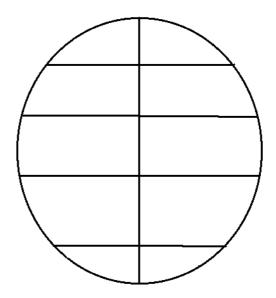


Figure 15.3. d-Diagram

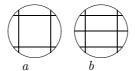


Figure 15.4. d-Diagrams of trefoil and figure eight knot

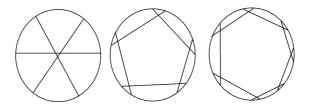


Figure 15.5. Chord diagrams which are not d-diagrams

From the arguments described above, we conclude that if a link L has a planar diagram with n crossings such that the corresponding height atom V(L) has k supercritical circles, then L can be encoded by a d-diagram having n+2(k-1) chords.

It is easy to see that the total number of subcritical and supercritical circles of V(L) does not exceed $\chi(V)-n+(2n) \leq n+2$, where $\chi(V)$ is the Euler characteristic of V(L). Since V has at least one subcritical circle then k does not exceed (n+1). So, we obtain an upper bound for the minimal number of chords of the d-diagram corresponding to our knot: it does not exceed 3n.

Exercise 15.2. Show that a chord diagram is a d-diagram if and only if it does not contain the subdiagrams shown in Fig. 15.5 (2n + 1-gons).

We have constructed the map from the set of all d-diagrams to the set of all link isotopy classes. This map is not injective (for instance, by adding a solitary chord to a d-diagram we do not change the link isotopy type). In fact, d-diagrams encode not planar diagrams of links but only those corresponding to atoms with a unique supercritical circle. To simplify the situation, let us generalise the notion of d-diagram as follows.

Definition 15.3. A marked or labelled d-diagram is a d-diagram where each chord is endowed with a label "+" or "-". Unlabelled d-diagrams are taken to have all labels positive.

Having a d-diagram C, one can construct a link diagram as follows. Let us split chords of the diagram into two families of non-intersecting chords. Then, let us embed C into the plane: chords of the first family are embedded inside the circle; chords of the second family are embedded outside the circle. Then we replace all chords together with small pieces of arcs by crossings, as shown in Fig. 15.6.

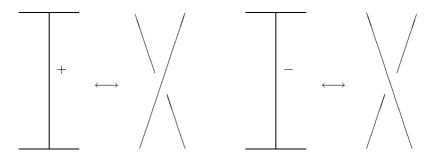


Figure 15.6. Crossings corresponding to chords

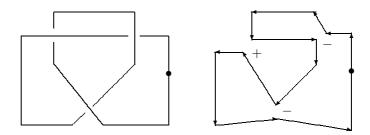


Figure 15.7. A circuit for the trefoil

It is easy to see that this definition coincides with the old one in the case of an unlabelled d-diagram.

Exercise 15.3. Show that the link isotopy class constructed in this way (for a labelled d-diagram) does not depend on the splitting of chords into two families.

Theorem 15.5 ([Ma0]). Each planar link diagram can be obtained from some labelled d-diagram in the way described above.

Proof. Let L be a planar diagram of some link. Let us construct a circuit of this diagram as follows. Let us choose a vertex V and an edge e outgoing from this vertex. Then, let us move along e. When we meet a vertex, we turn to one of the two possible directions (not opposite to the direction where we have come from).

Exercise 15.4. Show that one can choose the directions of our turns in such a way that we return to V after passing each edge once and each vertex twice.

Such a circuit generates some chord diagram. Actually, it represents a circle together with a rule for identifying points on it: we identify pre-images of vertices. Denote this diagram by C. Obviously, C is a d-diagram (there is a natural splitting of chords into interior and exterior ones). To each chord of C, there corresponds a crossing of L.

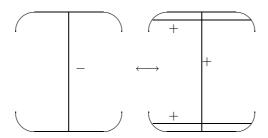


Figure 15.8. Twice cut positive chord and negative chord

Here, the circuit at vertices looks as shown in the left part of Fig. 15.6. We set the positive label in this case, and the negative label otherwise.

By construction, the obtained labelled d-diagram encodes the link diagram L.

Definition 15.4. A chord a is said to be cut if one "small" positive chord intersecting only a is added to it.

A chord is said to be *cut twice* if it is cut from both ends.

Exercise 15.5. Show that if we replace a negative chord of a marked d-diagram with a positive chord cut twice (see Fig. 15.8), we obtain a d-diagram representing the same link isotopy type.

Thus, we give one more way for constructing an unmarked d-diagram representing the given link. Namely, consider a link L and an arbitrary planar diagram P of it. There exists a marked d-diagram D_M corresponding to P. Replacing each negative chord of D_M with a twice cut positive chord, we obtain a d-diagram representing a link isotopic to L.

15.4 *d*-Diagrams and chord diagrams. Criterion of embeddability for a curve in terms of chord diagrams

The method of encoding links by using d-diagrams can be considered for a simpler object, namely, on smooth curves immersed in \mathbf{R}^2 where only double transverse intersection points are available. Having a d-diagram, we construct such a curve just like a knot diagram: we put an intersection instead of crossing.

In terms of d-diagrams, one can easily solve the realisability problem for a Gauss diagram. One solution is given in [Burm].

The main features of our d-diagram method of recognition realisability are the following. We look at a diagram of a curve (disposition of its crossings) from two points of view: Gaussian (when we go along the curve transversely) or d-diagram

(when we should always turn left or right). In the second point of view, only d-diagrams represent realisable curves. So, we just have to translate Gauss diagrams into the language of d-diagrams and see what happens there.

First, let us solve this problem for the case of regularly immersed curves. To do this, we shall not pay attention to crossing types. In this way, each d-diagram encodes not a link but one or several immersed curves (immersion is thought to be regular if it has no tangencies and no intersections of multiplicity more than two). Moreover, instead of Gauss diagrams, we deal just with chord diagrams corresponding to such immersions.

Definition 15.5. Such immersion are called *proper*; the corresponding chord diagrams are called *realisable*.

Suppose G is a chord diagram of an immersed curve. Then we can construct a circuit of this diagram according to "d-diagram rules." Namely, let us choose an arc a of G and let us go along this arc in an arbitrary direction. When we get to some vertex V of the diagram G, we have to turn right or left (in terms of planar diagrams). In the language of Gauss diagrams, this means the following. Suppose the vertex V is incident to the chord X of G, the two arcs incident to V are G and G incident to the other end of the chord G are G and G are must choose one of the two arcs: G or G are passed once and finally we come to the point we started from.

In this way, we can construct a chord diagram (that should be in fact a d-diagram). Namely, we just give a new enumeration for the edges according to our circuit and compose a circle of them. Then, if a quadruple of edges (arcs of the diagram to be constructed) p,q,r,s corresponds to a chord (say p,q are incident to one end of it, and r,s are incident to the other end), then the same quadruple will give us a couple of points for the new diagram (say, p,r and q,s). These points must be connected by a chord.

This is the algorithm of translating a chord diagram to the "d-diagram language." By construction, we have the following statement.

Statement 15.1. If G is a realisable chord diagram then the corresponding diagram is a d-diagram.

The condition above is, however, not sufficient. In fact, for the diagram it does not hold.

If we perform the algorithm above, we obtain the same diagram. However, this is not a realisable Gauss diagram. The reason is obvious. Let us call this algorithm the A-algorithm.

Remark 15.2. Note that this algorithm is not uniquely defined.

Definition 15.6. A chord d of a chord diagram D is called *even* if the number of chord ends lying in the arcs between ends of this chord is even. Otherwise, the chord is called odd.

Obviously, this is well defined (does not depend on the choice of one of the two arcs between two points).

Thus, for each realisable chord diagram each chord is even.

Exercise 15.6. Prove this fact.

It turns out that the two conditions described above are sufficient. In fact, the following theorem is true.

Theorem 15.6. Let G be a chord diagram. Let D be a diagram obtained from G by applying the A-algorithm. Then G is realisable if and only if each chord of G is even and D is a d-diagram.

Before proving this theorem, we formulate a corollary from it.

Corollary 15.1. If G is a chord diagram with all chords even, then diagrams that can be obtained from G by applying the A-algorithm are either all d-diagrams, or $not\ d$ -diagrams.

In order to prove Theorem 15.6, we shall construct the reversed algorithm (how to obtain the chord diagram of a curve from its d-diagram).

Let D be a d-diagram. Then it generates some curve K, which is standardly immersed in \mathbb{R}^2 . In order to construct the chord diagram corresponding to K, we must find some *unicursal* circuit for D (as in the case of the A-algorithm). More precisely, let us choose some vertex V_1 of D and some edge a outgoing from it (say, in the clockwise direction). Then we come to some other vertex V_2 . After it, our algorithm is uniquely defined: we have only the possibility to go forward (not right or left). This means that for the chord c incident to V_2 we go to the opposite side, and then proceed moving in the opposite direction of our circuit (i.e., counterclockwise, if the initial moving was clockwise). After moving to each next vertex, we jump to the other side of the chord and change the direction. We can choose our circuit in such a way that finally we successfully return to the vertex V_1 with the initial direction after we shall have passed all arcs precisely once.

Let us call this algorithm the B-algorithm.

If we have a realisable chord diagram C then, for each diagram D obtained from C by applying the A-algorithm, the diagram obtained from D by the Balgorithm will be just the diagram C. However, for some diagrams this is not so. For instance, if we take the diagram \bigcirc then the A-algorithm will give us \bigcirc , and the B-algorithm applied to the "new \bigcirc " will give \bigcirc .

The only thing that remains to prove is that it is not the case for diagrams all

chords of which are even.

The reason is the following. Consider a chord diagram C, and a vertex V of it. There are four arcs incident to V and the opposite (by edge) vertex. Denote them by a, b, c, d. Suppose that a is opposite to c (i.e., next on the diagram c). Then, for the d-diagram D(C) obtained from C by the A-algorithm a is adjacent either to b or to d. Without loss of generality, suppose a is adjacent to b. Then there are two hypothetical possibilities for the B-algorithm to restore the opposite chord for a: it will be either c (as it must be) or d, see Fig. 15.9.

The B-algorithm is uniquely defined. Thus, we need to find a condition for the initial diagram C such that this algorithm always restores the opposite edge correctly (in our case c for a).

Now, suppose that C is a chord diagram, all edges of which are even. Then, for any circuit of C constructed according to the A-algorithm, orient all edges of the

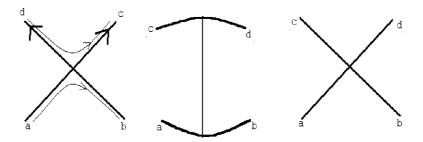


Figure 15.9. The "bad" case

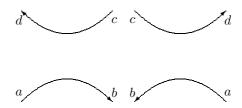


Figure 15.10. Local structure of edges and arcs

diagram according to this circuit. A vertex is said to be good if it is either a source (both incident edges are outgoing) or a strain (both are incoming). It is easy to see that for C having all even edges, all vertices are good.

The case of good edges is just what we wanted: in this case, the algorithm B will restore the initial diagram. Let us consider this fact in detail.

Let C, V, a, b, c, d be defined as above. Consider the d-diagram D(C) obtained from the diagram C by applying the A-algorithm. Without loss of generality, assume that in the diagram C, the arc c goes after a, and the arc d goes after b. Suppose we have chosen a way of applying the A-algorithm such that in the diagram D(C), the arc b is adjacent to a and d is adjacent to c. Moreover, suppose that b follows a in our circuit (in the d-diagram D). Thus, the vertex V is a strain. The vertex V' connected with V by a chord (in C) is thus a source. Thus we conclude that the chord d follows c. So, when we draw our d-diagram D on the plane, we see the picture shown in Fig. 15.10 (in the right part or in the left part).

This means that the arc c is opposite to the arc a (and the analogous situation is true at all vertices), and after applying the B-algorithm, we obtain the initial diagram C. This completes the proof of the theorem.

One can easily modify this algorithm for the case of knots and Gaussian curves. Obviously, it is sufficient to consider only the case of connected Gauss diagrams.

First, having a Gauss diagram G one should forget about its labels and arrows and consider the chord diagram C. If it is not realisable then G is not realisable either.

In the case when C is realisable, one should apply the first algorithm to it and obtain the d-diagram D. It is not difficult to consider all embeddings of D in the

plane. Then one should try to set all crossings according to the labels on the arcs and check carefully whether there is no contradiction with the directions of these arcs (the arcs can be set with respect to the initial point, thus one should consider several cases). If there are no contradictions then the diagram G is realisable.

15.5 A new proof of the Kauffman–Murasugi theorem

In order to prove the Kauffman–Murasugi theorem, one should appreciate the length of the Jones polynomial. Obviously, it is just the same as the length of the Kauffman bracket divided by four.

Consider formulae (6.4) and (6.5) for the definition of the Jones–Kauffman polynomial. We are interested in the states that give the maximal and the minimal possible degree of monomials in the sum (6.4).

Definition 15.7. Now, let the *minimal state* of L be the state where all crossings are resolved positively, and the *maximal state* be the state of L where all crossings are resolved negatively.

In order to estimate these degrees, we shall use atoms. Namely, consider a diagram L of a link (all link diagrams are thought to be connected). It has an intrinsic A-structure as any four-valent graph embedded in the plane. It also has a B-structure of the atom that can be restored from it.

Suppose the diagram L is prime. The remaining case will be considered later.

It is easy to see that the maximal possible monomial degree in formula (6.4) corresponds to the maximal state and the minimal possible degree corresponds to the minimal state. These easy facts are left for the reader as exercises.

Denote these states by s_{max} and s_{min} and the corresponding numbers of circles by γ_{max} and γ_{min} , respectively.

Let us calculate the desired maximal and minimal monomial degrees. We have: $n+2(\gamma_{max}-1)$ and $-n-2(\gamma_{min}-1)$. Here "2" and "minus 2" come from the exponent $a^{-2(\gamma(s)-1)}$.

The difference (the upper bound) equals

$$2n + 2(\gamma_{min} + \gamma_{max}) - 4.$$

Now, let us return to the atom V corresponding to L. Its Euler characteristic obviously equals $-n + \gamma_{min} + \gamma_{max}$. Taking into account that this is less than or equal to two, we conclude that our upper bound does not exceed 4n.

Thus, we have proved the first part of the Kauffman–Murasugi theorem.

Now, the question is: when can we get this upper bound? First, the atom should be a spherical one.

So, we must present a B-structure of a spherical atom with respect to the A-structure of the shadow of L in order to obtain a spherical atom. There are two such structures corresponding to the two alternating diagrams with the same shadow.

But, since L is prime, there are no other B-structures creating spherical atoms. Thus, the diagram L is alternating.

Now, let us check that the only obstruction for X(L) to have length 4n is the existence of splitting points. Obviously, if a diagram has a splitting point, it can be represented as a connected sum of two diagrams having a smaller number of crossings. Thus, its length cannot be equal to 4n.

If the length of the Kauffman bracket for an alternating link diagram with n crossings is less than 4n, then either the leading or the lowest coefficient coming from γ_{max} is cancelled by some other member. Without loss of generality, assume that the first one is the case. Then, there exists a state different from s_{max} that gives the same exponent of a: $n+2\gamma_{max}-1$. Suppose it differs from s_{max} at some crossings. Let us choose one of them and denote it by X. It is obvious that the state s'_{max} that differs from s_{max} at the only crossing X also gives the power $n+2\gamma_{max}-1$. So, $\gamma(s'_{max})=\gamma(s_{max})+1$. This means that when we change the state s_{max} at the vertex X, one circle is divided into two circles. Taking into account that the initial diagram is alternating, we conclude that X is a splitting point.

Finally, if L is not a prime diagram, we can decompose it into prime components. Taking into account the multiplicativity of the Jones (or Kauffman) polynomial, we see that they all are alternating diagrams without splitting points. This completes the proof of the Kauffman–Murasugi theorem.

Chapter 16

The bracket semigroup of knots

16.1 Representation of long links by words in a finite alphabet

As shown above, all links can be represented by d-diagrams. One can view a d-diagram as follows. First, fix the way of splitting chords of d-diagrams into two families. Choose a point of a d-diagram different from any chord end.

d–diagrams with a marked point admit a simple combinatorial representation by words in the four-bracket alphabet. Indeed, while "reading" the chord diagram starting from the given point, we can write down a round bracket when encountering an end of chord belonging to the first family and a square bracket when we meet an end of chord from the second family. Thus we get what is called a "balanced bibracket structure."

Definition 16.1. A balanced bibracket structure is a word in the alphabet (,) , [,] such that:

- 1. In each initial subword a' of the word a the number of ")" does not exceed that of "(", and the number of "]" does not exceed that of "[";
- 2. In the word a the number of "(" equals the number of ")", and that of "[" equals that of "]".

It is obvious that the d-diagram with initial point can be uniquely restored from the corresponding balanced bibracket structure.

d-diagrams encode links. Thus, d-diagrams with a fixed point encode links with a fixed point on the oriented component (or, what is just the same) long links.

Definition 16.2. By a *long link* is meant a smooth compact 1–manifold with boundary, embedded in \mathbb{R}^3 coinciding with Ox outside some ball centred at 0 and isotopic to the disconnected sum of one line and several (possibly, none) circles.

A long link consisting of one component is called a *long knot*.

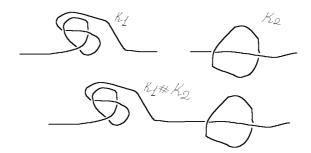


Figure 16.1. Product of two long links

Long links are considered up to natural isotopy. Now, let is define the semigroup K of long links as follows.

- 1. Elements of K are isotopy classes of long links.
- 2. The unit of K is the equivalence class of the manifold, given by $\{y = 0, z = 0\}$. This equivalence class is called the *long unknot* or the *long trivial knot*.
- 3. The product of two elements $L_1, L_2 \in G$ is defined as follows. First, we choose representatives of these classes: some links K_1 and K_2 . They coincide with Ox outside the balls centred at zero; the radii of these balls are some R_1 and R_2 , respectively. Then we construct a long link consisting of the three following parts. One part of it lies in $\{x > 2R_2\}$ and $\{x < -2R_1\}$ and coincides with Ox there. Another part lies inside $B_{R_2}(R_2,0,0)$ and coincides there with the shift of $K_2 \cap B_{R_2}(0,0,0)$ along the vector $(R_2,0,0)$. The third part lies inside $B_{R_1}(-R_1,0,0)$ and coincides there with the shift of $K_1 \cap B_{R_1}(0,0,0)$ along the vector $(-R_1,0,0)$. See Fig. 16.1. The isotopy class of the constructed long link is decreed to be the product of $L_1 \cdot L_2$. Obviously, this isotopy class depends neither on the randomness of K_1 and K_2 nor on the randomness of the radii R_1 and R_2 . Thus, the product in K is well defined.

It is obvious that the concatenation of balanced bibracket structures corresponds to the connected sum operation for long links.

Thus, one can say that the long link semigroup can be encoded in terms of balanced bibracket structures. This is called the *bracket calculus*.

All balanced bibracket structures themselves form a semigroup, where the empty word plays the role of the unit element, and the multiplication is expressed by concatenation. Denote this group by G.

The main problem of the bracket calculus is to describe the equivalence of long links in terms of bibracket structures. This was done in [Ma'3].

The main idea is to describe the "elementary" isotopy moves in terms of bracket structures.

These isotopies originate from Reidemeister moves and one special move that corresponds to the "circuit change." They are completely enumerated in [Ma3].

The semigroup of balanced bibracket structures factorised by these relations is thus isomorphic to the semigroup K.

Each of these relations is an identity A=B, where A,B are some (possibly, non-balanced) words in the bracket alphabet. Such a relation means that any for any balanced bibracket structure xAy, the structure xBy is balanced as well, and represents the same long link isotopy class (the same statement is true for xBy, xAy). We factorise the group G by all relations xAy = xBy for all x,y such that xAy is balanced which are taken with respect to the relation A=B taken from the given list. Some parts of this list are given below.

First Reidemeister move:

Second Reidemeister move:

$$([(]A[)]) = A$$

 $[(](A)[)] = A,$

In each of the two relations above, A has balanced round–bracket structure.

$$[([)A(])] = A$$
$$([)[A](]) = A.$$

Here A has balanced square–bracket structure.

Third Reidemeister move:

$$(([[]([]([]([])([])])([]) = ([[([]([])[])]([[])[]))$$

$$([[](([])([])([])[])] = [([[])(([])[(]]).$$

For the change of circuits we use the following four moves:

$$[A(]C) = [(]C([)[)]A(]),$$

$$([)A((])C) = (C([))A(]),$$

$$[A[(]]C[)] = [(]C[[)]A],$$

$$([)A[(](])C[)] = (C[)A].$$

In each of these relations, A should have balanced square bracket structure and C has balanced round bracket structure.

For more details and all proofs, see [Ma3].

16.2 Representation of links by quasitoric braids

In the present section, we are going to describe how knots can be represented by closures of a small class of braids, and the class of d-diagrams, generating these braids, see [Ma4].

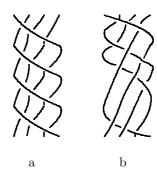


Figure 16.2. Toric and quasitoric braids

16.2.1 Definition of quasitoric braids

We recall that toric braids (depending on the two parameters p and q, where p is the number of strands) are given by the following formula:

$$T(p,q) = (\sigma_1 \dots \sigma_{p-1})^q$$

and have an intuitive interpretation, see Fig. 16.2.a.

Definition 16.3. A braid β is said to be *quasitoric* of type (p,q) if it can be expressed as $\beta_1 \ldots, \beta_q$, where for each $\beta_j = \sigma_1^{e_{j1}} \ldots \sigma_{p-1}^{e_{j,p-1}}$, each e_{jk} is either 1 or -1. In other words, a quasitoric braid of type (p,q) is a braid obtained from the standard diagram of the toric (p,q) braid by switching some crossing types, see Fig. 16.2.b.

It is easy to see that the product of quasitoric n-strand braids is a quasitoric n-strand braid.

In fact, a more precise statement can be made.

Proposition 16.1. For every $p \in \mathbb{N}$, p-strand quasitoric braids make a subgroup in B_p .

To prove this result, we only have to prove the following lemma.

Lemma 16.1. For every p, the inverse of a p-strand quasitoric braid is quasitoric.

Proof. We have to prove that for the braid $\delta = \sigma_1^{e_1} \dots \sigma_{p-1}^{e_{p-1}}$, where each e_i equals 1 or -1, the braid δ^{-1} is a quasitoric braid. In this case, the proof of the lemma follows straightforwardly, since each quasitoric braid is just a product of positive powers of such braids.

In other words, we have to prove that there exists a quasitoric braid η , such that $\eta \cdot \delta$ is the trivial braid.

We are going to prove that there exist braids $\delta_1, \ldots \delta_{p-1}$, where $\delta_i = \sigma_1^{e_{i1}} \ldots \sigma_{p-1}^{e_{i,p-1}}$, such that for $\eta = \delta_1, \ldots, \delta_{p-1}$ we have $\eta \delta = e$ is the trivial braid.

Let us consider the shadow S of the standard toric braid diagram of type (p, p) that will be the shadow of our diagram $\delta_1 \dots \delta_{p-1} \delta$. The lower part of this diagram has crossing types coming from δ . So we only have to set the rest of the crossings (i.e. to define $\delta_1 \dots \delta_{p-1}$ in order to get the trivial braid $\delta_1 \dots \delta_{p-1} \delta$). The lower

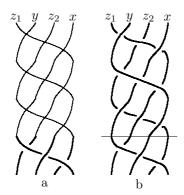


Figure 16.3. Inverting a quasitoric braid

part of the shadow S consists of n strands; one of them (denote it by x) intersects all other strands once; other strands do not intersect each other.

If we set all crossing types for the lower part of S as in δ then we see that some strands in the lower part come over x, and the others come under x, see Fig. 16.3.a.

Let us denote strands from the first set of strands by $y_1, \ldots y_p$, and strands from the second set by $z_1, \ldots z_q$.

Let us say that for a pure r-strand braid β_1 with strands $a_i, i = 1 \dots r$, the order of strands is $a_1 > a_2 > \dots > a_r$ if at each crossing X involving $a_i, a_j, i < j$, the strand a_i comes over a_j . It is obvious that in this case the braid β_1 is trivial.

Now we can easily set all crossing types for the upper part of S in such a way that for the braid $\delta_1\delta_2\ldots\delta_{p-1}\delta$ the order of strands is $y_1>y_2>y_3\ldots y_p>x>z_1>\cdots>z_q$, see Fig. 16.3.b. So the braid $\eta\delta$ is trivial, which completes the proof of the lemma.

Thus, we have the *group of quasitoric braids*. Now, let us state the main theorem of this section.

Theorem 16.1. Each knot isotopy class can be obtained as a closure of some quasitoric braid.

16.2.2 Pure braids are quasitoric

First, note that, by Alexander's theorem, for a given knot K there exists a braid β , whose closure is isotopic to K. Our goal is to transform β in a proper way in order to obtain a quasitoric braid.

Let us prove the following lemma.

Lemma 16.2. Every braid β is Markov–equivalent to an r-strand braid whose permutation is a power of the cyclic permutation (1, 2, ..., r) for some r.

Proof. Suppose β has n strands. Consider the permutation α corresponding to it, and orbits of the action of α on the set $(1, \ldots, n)$.

These orbits might contain different numbers of elements. Now, let us apply Markov's move for transforming these orbits. The first Markov move conjugates

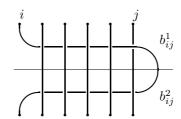


Figure 16.4. Generator b_{ij} of the pure braid group

the braid, thus, it conjugates the corresponding permutation. So, the number of elements in orbits does not change, but elements in orbits permute. The second Markov move increases the number of strands by one, adds the element (n+1) to the orbit, containing the element n and does not change other orbits. Thus, by using Markov's moves, one can re–enumerate elements in such a way that the smallest orbit contains n, and then increase the number of elements in this orbit by one. Reiterating this operation many times, we finally obtain the same number of elements for all orbits. Suppose the permutation corresponding to the obtained km-strand braid β acts on km elements in such a way that each of k orbits of the permutation contains m elements. By conjugating β , we can get the corresponding permutation equal to m-th power of the cyclic permutation $(12 \dots km)$, $k, m \in \mathbb{N}$.

Denote the obtained braid by γ . As shown in Lemma 16.2, γ is Markov–equivalent to β .

The next step is to prove that γ is a quasitoric braid.

So, let us change the braid diagram of γ , without changing its isotopy type. To complete the proof of the theorem, we have to prove the following lemma.

Lemma 16.3. An r-strand braid whose permutation is a power of the cyclic permutation $(12 \dots r)$ is quasitoric.

Proof. Let γ be a r-strand braid, having the permutation $(1 \ 2 \dots r)^s$. Consider the braid $\gamma' = \gamma \cdot T(r, 1)^{-s}$.

Then, γ' is a pure braid. Besides, the braid γ' is quasitoric if and only if γ is quasitoric (hence $B(r,1)^s$ is a quasitoric braid).

Thus, it remains to prove the following

Lemma 16.4. Every pure braid is quasitoric.

Recall that one can choose generators $b_{i,j}, 1 \le i < j \le r$, of the pure r-strand braid group, as shown in Fig. 16.4.

Now, we only have to show that all generators b_{ij} are quasitoric braids.

Actually, for all i and j between 1 and n, i < j, the braid b_{ij} is a product of the two braids $b_{ij}^1 \cdot b_{ij}^2$ (they are shown in Fig. 16.4 above and below the horizontal line), where the first braid $b_{ij}^1 = \sigma_i^{-1} \dots \sigma_{j-2}^{-1} \sigma_{j-1}$ has ascending order of generators, and $b_{ij}^2 = \sigma_{j-1}\sigma_{j-2}\dots\sigma_i$ has descending order of generators, see Fig. 16.4. Now, consider only strands from numbered from i-th to j-th. Then, we can introduce

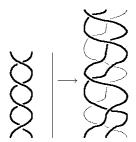


Figure 16.5. Adding a "thin" strand to a quasitoric braid

the analogous definition of quasitoric braids on strands from the *i*-th to *j*-th (i.e. with other strands vertical). It is evident that both b_{ij}^1 and b_{ij}^2 are (i,j)-quasitoric braids on the strands from *i*-th to *j*-th (for b_{ij}^1 it is clear by definition, and b_{ij}^2 is the reverse to a quasitoric braid).

Definition 16.4. For $1 \le i < j \le n$ an *n*-strand braid ζ is said to be (i, j)-quasitoric if it has a diagram with strands of it except those numbered from *i*-th to *j*-th going vertically and unlinked with the other strands, and strands from *i* to *j* forming a quasitoric braid (in the standard sense). Such a diagram is called a standard diagram of an (i, j)-quasitoric braid.

To complete the proof of Lemma 16.4 and the main theorem, it suffices to prove the following lemma.

Lemma 16.5. Assume $1 \le i < j \le r$. Then every (i, j)-quasitoric pure r-strand braid is quasitoric.

Proof. We use induction on r - (j - i + 1). We have to show that by adding a separate vertical strand to the standard diagram of a quasitoric braid, we obtain a diagram of a quasitoric braid.

Consider a standard quasitoric q-strand braid diagram ρ and add a separate strand on the right hand (the case of a left–handed strand can be considered analogously). Let the initial braid diagram be obtained from the toric braid (q,ql) by switching some crossings.

Consider the standard diagram of the pure toric braid (q+1,(q+1)l) and the first q strands of it. Obviously, they form the toric braid diagram of type (q,ql). Let us set the crossing types of these strands as in the case of the diagram ρ , and let us arrange the additional strand under all the others. Obviously, we get a diagram, isotopic to that obtained from ρ by adding a separate strand on the right hand, see Fig. 16.5.

Thus, the standard generators of the pure braid group P_n for arbitrary n are quasitoric, hence, by Proposition 16.1, so is every pure braid.

So, by using Markov's moves and braid diagram isotopies to the initial braid diagram, we obtain a quasitoric braid ζ , whose closure is isotopic to K and this completes the proof of the main theorem.

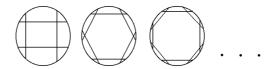


Figure 16.6. Quasitoric d-diagrams for q=3

16.2.3 *d*-diagrams of quasitoric braids

Toric (and quasitoric) braids in their natural representation allow us to consider two ways of encoding links: by braids and by d-diagrams together.

Namely, the following statement is true.

Statement 16.1. Standard diagrams of closures of quasitoric braids (with odd p) are the only link diagrams that can be obtained from labelled d-diagrams and braided around the centre of the corresponding circle.

More precisely, we require that the circle is standardly embedded in \mathbf{R}^2 : $x^2+y^2=1$, and that ends of chords are uniformly distributed along the circle; chords of one (interior) family of a d-diagram are taken to be straight lines (thus we require the absence of diametral chords), and chords of the other family are the images of straight lines inverted in the circle. One should also make one more correction. Namely, let D be a d-diagram, and let a and b be some two intersecting chords of D belonging to different families, such that one end a_1 of a and one end b of b_1 are adjacent vertices. Let us choose points P, Q on the chords a, b and a point R on the arc a_1b_1 . Denote the corresponding unit tangent vectors (at these points) by t(P), t(Q), t(R). Then the vectors $OP \times t(P)$ and $OQ \times t(Q)$ (where O is the centre of the circle) are collinear, and the vector $OR \times t(R)$ has opposite direction.

So, we shall delete such an arc, i.e., construct the link diagram by a d-diagram in such a way that one half of a_1b_1 is deleted together with the chord a, and the other is deleted together with the chord b. Within this chapter, we accept these corrections.

To check the Statement 16.1, one should only study the property of shadows of such link diagrams. The proof of the statement is left for the reader.

The d-diagram corresponding to the toric braid T(p,q) is constructed as follows. Let p=2m+1. Let us mark the 4mq points on the sphere, split into 2q groups of 2m adjacent points in each group. Enumerate the points in each group by numbers from 1 to 2m; in "even groups" we enumerate clockwise; in odd groups we enumerate counterclockwise.

Each marked point is connected with a point from an adjacent group having the same number. The adjacent group is chosen according to the following rules:

- Points from the same group having the same parity have the same adjacent groups; points from the same group having different parity have different adjacent groups.
- 2. Point number one is never connected with the adjacent point (on the circle).

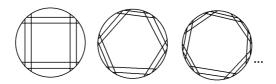


Figure 16.7. Quasitoric d-diagrams for q = 5

Examples of these d–diagrams are shown in Fig. 16.6 for p=3 and Fig. 16.7 for p=5.

Obviously, "quasitoric" d-diagrams with odd q are obtained from these "toric" ones by marking some chords as "negative."

${f Part\ V}$ ${f Virtual\ knots}$

Chapter 17

Virtual knots. Basic definitions and motivation

Virtual knot theory was proposed by Kauffman [Kau]. This theory arises from the theory of knots in thickened surfaces $S_g \times I$, first studied by Kauffman, Jaeger, and Saleur, see [JKS]. Virtual knots (and links) appear by projecting knots and links in S_g to \mathbf{R}^2 and hence, $S_g \times \mathbf{R}$ onto \mathbf{R}^3 . By projecting link diagrams (i.e., graphs of valency four with overcrossing and undercrossing structures at vertices) in S_g onto \mathbf{R}^2 , one obtains diagrams on the plane. Virtual crossings arise as artefacts of such projection, i.e., intersection points of images of arcs, non-intersecting in S_g and classical crossings appear just as projections of crossings.

In the very beginning of this theory, the creators have proposed generalisations of some basic knot invariants: the knot quandle, the fundamental group, the Jones polynomial [Kau]. For further developments see [Ma2, Ma5, Ma6, KaV2, KaV3, Saw, SW] On the other hand, see [GPV], virtual knots arise from non-realisable Gauss diagrams: having a non-realisable (by embedding) diagram, one can "realise it" by means of immersion; the "new" intersection points are marked by virtual crossings. We recall that realisability of Gauss diagrams was described in chapter 15.

17.1 Combinatorial definition

Let us start with the definitions and introduce the notation.

Definition 17.1. A virtual link diagram is a planar graph of valency four endowed with the following structure: each vertex either has an overcrossing and undercrossing or is marked by a virtual crossing, (such a crossing is shown in Fig. 17.1).

All crossings except virtual ones are said to be classical.

Two diagrams of virtual links (or, simply, virtual diagrams) are said to be equiv-



Figure 17.1. Virtual crossing

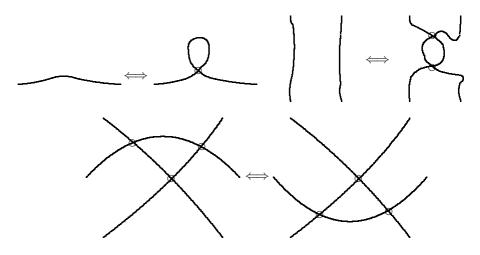


Figure 17.2. Moves $\Omega'_1, \Omega'_2, \Omega'_3$

alent if there exists a sequence of generalised Reidemeister moves, transforming one diagram to the other one.

As in the classical case, all moves are thought to be performed inside a small domain; outside this domain the diagram does not change.

Definition 17.2. Here we give the list of generalised Reidemeister moves:

- 1. Classical Reidemeister moves related to classical vertices.
- 2. Virtual versions $\Omega_1', \Omega_2', \Omega_3'$ of Reidemeister moves, see Fig. 17.2.
- 3. The "semivirtual" version of the third Reidemeister move, see Fig. 17.3,

Remark 17.1. The two similar versions of the third move shown in Fig. 17.4 are forbidden, i.e., they are not in the list of generalised moves and cannot be expressed via these moves.

Definition 17.3. A *virtual link* is an equivalence class of virtual diagrams modulo generalised Reidemeister moves.

One can easily calculate the number of components of a virtual link. A $virtual\ knot$ is a one–component virtual link.

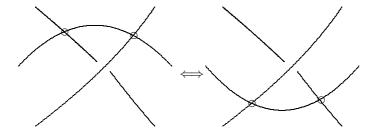


Figure 17.3. The semivirtual move Ω_3''

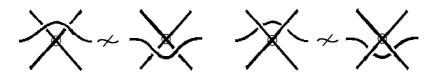


Figure 17.4. The forbidden move

Exercise 17.1. Show that any virtual link having a diagram without classical crossings is equivalent to a classical unlink.

Remark 17.2. Formally, virtual Reidemeister moves give the new equivalence relation for classical links: there exist two isotopies for classical links, the classical one that we are used to working with and the virtual one. Later we shall show that this is not the case.

Remark 17.3. Actually, the forbidden move is a very strong one. Each virtual knot can be transformed to other one by using all generalised Reidemeister moves and the forbidden move. This was proved by Sam Nelson in [Nel] by using Gauss diagrams of virtual links.

The idea was that one can make each pair intersecting chords of Gauss diagrams of a virtual knot having adjacent ends non-intersecting.

If we allow only the forbidden move shown in the left part of Fig. 17.4, we obtain what are called welded knots. Some initial information on this theory can be found in [Kam].

Definition 17.4. By a mirror image of a virtual link diagram we mean a diagram obtained from the initial one by switching all types of **classical** crossings (all virtual crossings stay on the same positions).

17.2 Projections from handle bodies

The choice of generalised Reidemeister moves is very natural. Namely, it is the complete list of moves that may occur while considering the projection of $S_g \times I$ to $\mathbf{R} \times I$ (or, equivalently, \mathbf{R}^3). Obviously, all classical Reidemeister move can be

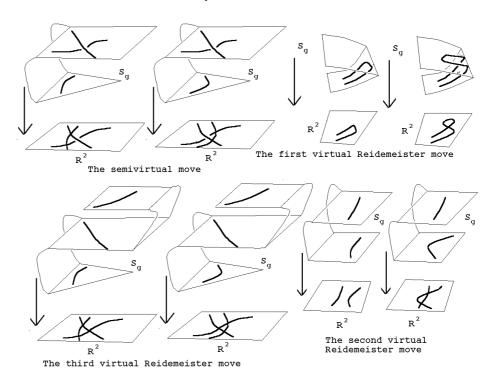


Figure 17.5. Generalised Reidemeister moves and thickened surfaces

realised on a small part of any S_g that is homeomorphic to \mathbf{S}^3 . The other moves, namely, the semivirtual move and purely virtual moves, are shown in Fig. 17.5 together with the corresponding moves in handle bodies.

There exists a more intuitive topological interpretation for virtual knot theory in terms of embeddings of links in thickened surfaces [KaVi, KaV2]. Regard each virtual crossing as a shorthand for a detour of one of the arcs in the crossing through a 1-handle that has been attached to the 2-sphere of the original diagram. The two choices for the 1-handle detour are homeomorphic to each other (as abstract manifolds with boundary). By interpreting each virtual crossing in such a way, we obtain an embedding of a collection of circles into a thickened surface $S_g \times \mathbf{R}$, where g is the number of virtual crossings in the original diagram L and S_g is the orientable 2-manifold homeomorphic to the sphere with g handles. Thus, to each virtual diagram L we obtain an embedding $s(L) \to S_{g(L)} \times \mathbf{R}$, where g(L) is the number of virtual crossings of L and s(L) is a disjoint union of circles. We say that two such stable surface embeddings are stably equivalent if one can be obtained from the other by isotopy in the thickened surface, homeomorphisms of surfaces, and the addition of substraction or handles not incident to images of curves.

Theorem 17.1. Two virtual link diagrams generate equivalent (isotopic) virtual links if and only if their corresponding surface embeddings are stably equivalent.

This result was sketched in [KaV]. The complete proof appears in [KaV3].

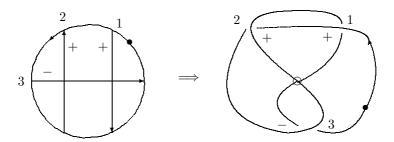


Figure 17.6. A virtual knot and its Gauss diagram

A hint to this prove is demonstrated in Fig. 17.5.

Here we wish to emphasise the following important circumstance.

Definition 17.5. A virtual link diagram is *minimal* if no handles can be removed after a sequence of Reidemeister moves.

An important Theorem by Kuperberg [Kup] says the following.

Theorem 17.2. For a virtual knot diagram K there exists a unique minimal surface in which an I-neighbourhood of an equivalent diagram embeds and the embedding type of the surface is unique.

17.3 Gauss diagram approach

Definition 17.6. A Gauss diagram of a (virtual) knot diagram K is an oriented circle (with a fixed point) where pre-images of overcrossing and undercrossing of each crossing are connected by a chord. Pre-image of each **classical** crossing are connected by an arrow, directed from the pre-image of the overcrossing to the pre-image of the undercrossing. The sign of each arrow equals the local writhe number of the vertex. The signs of chords are defined as in the classical case. Note that arrows (chords) correspond only to classical crossings.

Remark 17.4. For classical knots this definition is just the same as before.

Given a Gauss diagram with labelled arrows, if this diagram is realizable then it (uniquely) represents some classical knot diagram. Otherwise one cannot get any classical knot diagram.

Herewith, the four–valent graph represented by this Gauss diagram and not embeddable in \mathbb{R}^2 can be *immersed* to \mathbb{R}^2 . Certainly, we shall consider only "good" immersions without triple points and tangencies.

Having such an immersion, let us associate virtual crossings with intersections of edge images, and classical crossings at images of crossing, see Fig. 17.6.

Thus, by a given Gauss diagram we have constructed (not uniquely) a virtual knot diagram.

Theorem 17.3 ([GPV]). The virtual knot isotopy class is uniquely defined by this Gauss diagram.

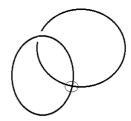


Figure 17.7. A virtual link that is not isotopic to any classical link

Exercise 17.2. Prove this fact.

Hint 17.1. Show that purely virtual moves and the semivirtual move are just the moves that do not change the Gauss diagram at all.

17.4 Virtual knots and links and their simplest invariants

There exist a lot of simple combinatorial ways for constructing virtual knot and link invariants.

Consider the virtual link shown in Fig. 17.7.

It is intuitively clear that this link cannot be isotopic to any classical one because of "the linking number."

Exercise 17.3. Define accurately the linking number for virtual links and prove that the link shown in Fig. 17.7 is not isotopic to any virtual link.

The next simplest invariant is the colouring invariant: we take the three colours (as before) and associate a colour with each (long) arc (i.e., a part of the diagram going from one undercrossing to the next undercrossing; this part might contain virtual crossings). Then we calculate the number of proper colourings. This invariant will be considered in more detail together with the quandle and fundamental group.

Exercise 17.4. Prove the invariance of the colouring invariant.

17.5 Invariants coming from the virtual quandle

Later, we shall define the quandle for virtual knots in many ways. In the present sections, we are going to construct invariants "coming from this quandle"; however, we are going to describe them independently.

17.5.1 Fundamental groups

Though virtual knots are not embeddings in \mathbb{R}^3 , one can easily construct a generalisation of the knot complement fundamental group (or, simply, the *knot group*) for

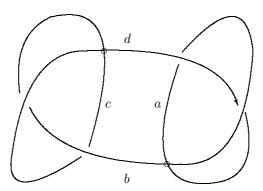


Figure 17.8. A virtual knot with different upper and lower groups

virtual knots. Namely, one can modify the Wirtinger representation for virtual diagrams. Consider a diagram \bar{L} of a virtual link L. Instead of arcs we shall consider long arcs of \bar{L} . We take these arcs as generators of the group to be constructed. After this, we shall write down the relations at classical crossings just as in the classical case: if two long arcs a and c are divided by a long arc b, whence a lies on the right hand with respect to the orientation of b, then we write the relation $c = bab^{-1}$.

The invariance of this group under classical Reidemeister moves can be checked straightforwardly: the combinatorial proof of this fact works both for virtual and classical knots (see Exercise 4.7). For the semivirtual move and purely virtual moves there is nothing to prove: we shall get the same presentation.

However, this invariance results from a stronger result: invariance of the virtual knot quandle, which will be discussed later.

Definition 17.7. The group defined as above is called the group of the link L. Obviously, the analogue of the colouring invariant Lemma 4.6 is true for virtual knots. Its formulation and proof literally coincide with the formulation and proof of Lemma 4.6.

17.5.2 Strange properties of virtual knots

Some virtual links may have properties that do not occur in the classical case. For instance, both in the classical and the virtual case one can define "upper" and "lower" presentations of the knot group (the first is as above, the second is just the same for the knot (or link) where all classical types are switched). In the classical case, these two presentations give the same group (for geometric reasons). In the virtual case it is however not so. The example first given in [GPV] is as follows.

In fact, taking the arcs a,b,c,d shown in Fig. 17.8 as generators, we obtain the following relations:

$$b = dad^{-1}$$
, $a = bdb^{-1}$, $d = bcb^{-1}$, $c = dbd^{-1}$.

Thus, a and c can be expressed in the terms of b and d. So, we obtain a presentation $\langle b, d|bdb = dbd \rangle$. So, this group is isomorphic to the trefoil group.

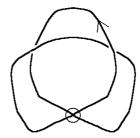


Figure 17.9. The virtual trefoil

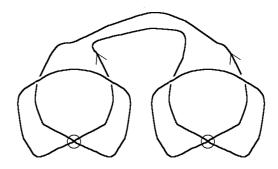


Figure 17.10. Non-trivial connected sum of two unknots

Exercise 17.5. Show that the group of the mirror virtual knot is isomorphic to Z.

This example shows us that the knot shown above is not a classical knot. Moreover, it is a good example of the existence of a non-trivial virtual knot with group \mathbf{Z} (the same as that for the unknot). The latter cannot happen in the classical case.

The simplest example of the virtual knot with group \mathbf{Z} is the virtual trefoil, see Fig. 17.9.

The fact that the virtual trefoil is not the unknot will be proved later.

Besides this example, one encounters the following strange example: the connected sum of two (virtual) unknots is not trivial, see Fig. 17.10. This example belongs to Se-Goo Kim [Kim]. We shall discuss this problem later, while speaking about long virtual knots.

It is well known that the complement of each classical knot is an Eilenberg–McLane space $K(\pi,1)$, for which all cohomology groups starting from the second group, are trivial. However, this is not the case for virtual knots: if we calculate the second cohomology the $K(\pi,1)$ space where π is some virtual knot group, we might have some torsion. In [Kim] one can find a detailed description of such torsions.

Chapter 18

Invariant polynomials of virtual links

We have already considered some generalisations of basic knot invariants: the colouring invariant, the fundamental group and the Jones–Kauffman polynomial [KaVi].

All the generalisations described above were constructed by using the following idea: one thinks of a virtual link diagram as a set of classical crossings provided with the information about how they are connected on the plane and one does not pay attention to virtual crossings. Thus, for instance, the generators of the fundamental group that correspond to arcs of the diagram may pass through virtual crossings, and all relations are taken only at classical crossings.

In the present chapter, we shall describe the invariants proposed in the author's papers [Ma2, Ma5, Ma6] (for short versions see in [Ma'4, Ma'5]). Interested readers may also read the excellent review of Kauffman [KaV2] and his work with Radford [KR] about so–called "biquandles." Polynomial invariants of virtual knots and links were also constructed in [Saw, SW].

In the present chapter, we are going to modify these invariants in the following manner: we find a way how a virtual crossing can have an impact on the constructed object (e.g., for the case of the fundamental group and what it does with the generator) and then prove its invariance.

The main results present here can be found in [Ma2].

The main idea of this construction is the following: while constructing the invariant of the virtual link (or braid), we have taken into consideration that virtual crossings can change the corresponding element, say, by multiplying one of them by q, and the other one will be denoted by q^{-1} (if we deal with some group structures). This idea of adding a new "variable" will be the main one in this article.

It turns out that in some cases (e.g., the Alexander module) this new variable plays a significant role and allows us to construct a virtual knot invariant that is not a generalisation of any classical knot invariant.

Throughout the present chapter, all knots and links (virtual or classical) are thought to be oriented, unless otherwise specified.

By a homomorphism of two objects \mathcal{O}_1 and \mathcal{O}_2 both endowed with a set of

operations (o_1, \ldots, o_m) we mean a map from \mathcal{O}_1 to \mathcal{O}_2 with respect to all these operations.

18.1 The virtual grouppoid (quandle)

We recall that the notion of quandle (also known as a distributive grouppoid) first appeared in the pioneering works of Matveev, [Mat] and Joyce [Joy]. They have proved that it is a complete invariant of knots.

We recall that a quandle is a set M together with the operation \circ that is

- 1. idempotent: $\forall a \in M : a \circ a = a$,
- 2. right self-distributive:

$$\forall a, b, c \in M : (a \circ b) \circ c = (a \circ c) \circ (b \circ c), \tag{1}$$

and

3. left–invertible: $\forall a, b \in M : \exists ! x \in M : x \circ a = b$. This element is denoted by b/a.

All these conditions are necessary and sufficient for the constructed quandle to be invariant under the Reidemeister moves. More precisely, for each knot (link) one can construct the knot (link) quandle.

Let L be an oriented virtual link diagram.

Definition 18.1. A Kauffman arc or long arc of this diagram is an oriented interval (piece of a curve) between two adjacent undercrossings (i.e., while walking along this arc, we make only overcrossings or virtual crossings).

With each arc a_i , i = 1, ..., n, we associate an element x_i of the quandle to be constructed. First, we take the free quandle generated by $a_1, ..., a_n$. Then, if three arcs a_1, a_2, a_3 meet each other at a classical crossing as shown in Fig. 18.1, we write down the relation

$$a_{i_1} \circ a_{i_2} = a_{i_3}. \tag{2}$$

Definition 18.2. The *Kauffman quandle* of L is the formal quandle, generated by $a_i, i = 1, ..., n$, and all relations (2) for all classical vertices.

More precisely, elements of such a quandle are equivalence classes of words obtained from a_i by means of \circ and /, where equivalence is defined by crossing relations.

The invariance of this quandle under purely virtual moves and the semivirtual move comes straightforwardly: the representation stays the same.

Remark 18.1. The invariance of this quandle under the classical Reidemeister moves can be checked straightforwardly just as in the classical case.

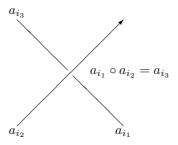


Figure 18.1. Crossing relation for the quandle

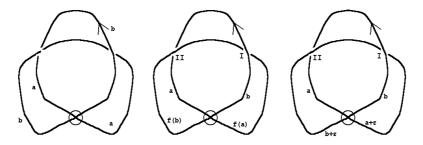


Figure 18.2. The virtual trefoil and its labelings

Denote the obtained quandle by $Q_K(L)$.

Obviously, for the unknot U we have $Q_K(L) = \{a\}$.

Thus, we have constructed a link invariant. By definition, it coincides with the classical quandle [Mat, Joy] on the classical links.

It might seem that for the classical links there are two equivalences: the classical one and the virtual one. However, this is not the case.

Theorem 18.1. [GPV] Let L and L' be two (oriented) classical link diagrams such that L and L' are equivalent under generalised Reidemeister moves. Then L and L' are equivalent under classical Reidemeister moves.

Proof. Note that longitudes (see definition on page 26) are preserved under virtual moves (adding a virtual crossing to the diagram does not change the expression for a longitude). Thus, an isomorphism for $Q_K(L)$ and $Q_K(L')$ induced by generalised Reidemeister moves preserves longitudes. Since the isomorphism class of the quandle plus longitudes classifies classical knots, we conclude that L and L' are classically equivalent.

However, unlike the classical case, this quandle is rather weak in the virtual sense: there are different simple knots that cannot be recognised by it.

Indeed, consider the knot diagram K shown in the left part of Fig. 18.2. The quandle Q_K corresponding to this diagram has two generators a, b and two relations $a \circ b = b$ and $b \circ b = a$. Obviously, they imply a = b. Thus this quandle is the same as that for the unknot.

Later, we shall see that this non-trivial knot is recognised by the "virtual quandle" to be constructed and thus, it is not trivial.

Besides the virtual quandle to be constructed, there is another generalisation of the quandle: the so-called biquandle (due to Kauffman and Radford, [KR]). Unlike our "virtual quandle" approach, where we add a new operation at a virtual crossing, Kauffman and Radford extend the algebraic structure at a classical crossing: the algebraic element associated to a part of an arc "before" the (under)crossing differs from that associated to the part "after" the undercrossing. The comparison of these two approaches seems to be quite interesting for further investigations.

Now, let us construct the modified (virtual) quandle (we shall denote it just by Q unlike Kauffman's quandle Q_K).

Definition 18.3. A virtual quandle is a quandle (M, \circ) endowed with a unary operation f such that:

- 1. f is invertible; the inverse operation is denoted by f^{-1} ;
- 2. \circ is distributive with respect to f:

$$\forall a, b \in M : f(a) \circ f(b) = f(a \circ b). \tag{3}$$

Remark 18.2. The equation (3) easily implies for all $a, b \in M$:

$$f^{-1}(a) \circ f^{-1}(b) = f^{-1}(a \circ b),$$

$$f(a)/f(b) = f(a/b),$$

and

$$f^{-1}(a)/f^{-1}(b) = f^{-1}(a/b).$$

For a given virtual link diagram L, let us construct its virtual quandle $\mathcal{Q}(L)$ as follows.

First, let us choose a diagram L' in such a way that it can be divided into long arcs in a proper way. Such diagrams are called *proper*. Obviously, a proper diagram with m crossings has m long arcs.

More precisely, we need the result that each long arc has two different final crossing points. For some diagrams this is not true. However, this can easily be done by slight deformations of the diagram, see Fig. 18.3.

Exercise 18.1. Show that equivalent proper diagrams of virtual links can be transformed to each other by generalised Reidemeister moves in the class of proper diagrams.

Remark 18.3. Later in this chapter, all diagrams are taken to be proper, unless otherwise specified.

Let L' be a virtual diagram; let us think of its undercrossings as broken (disconnected) lines as they are drawn on the plane. Let \hat{L}' be the set obtained from L' by removing all virtual crossings (vertices).

Definition 18.4. An arc of L' is a connected component of \hat{L}' .

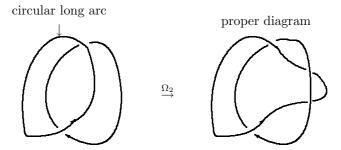


Figure 18.3. Reconstructing a link diagram in a proper way

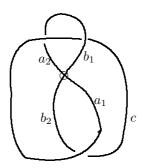


Figure 18.4. A knot diagram and its arcs

Exercise 18.2. The knot shown in Fig. 18.4 has three classical crossings, three arcs $(a_1 \text{ and } a_2; b_1 \text{ and } b_2; c)$, and five virtual arcs (a_1, a_2, b_1, b_2, c) .

Remark 18.4. The knot in Fig. 18.2 has thus four arcs (see the middle picture): each of the two "former" arcs a and b is now divided into two arcs by the virtual crossing.

The invariant Q(L) is now constructed as follows. Consider all arcs $a_i, i = 1, \ldots, n$, of the diagram L'. Consider the set of formal words X(L') obtained inductively from a_i by using $\circ, /, f, f^{-1}$. In order to construct Q(L') we shall factorise X(L') by some equivalence relations.

First of all, for each $a, b, c \in X(L')$ we identify:

$$f^{-1}(f(a)) \sim f(f^{-1}(a)) \sim a;$$

$$(a \circ b)/b \sim a;$$

$$(a/b) \circ b \sim a;$$

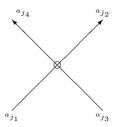


Figure 18.5. Relation for a virtual crossing

 $a \sim a \circ a$;

$$(a \circ b) \circ c \sim (a \circ c) \circ (b \circ c);$$

$$f(a \circ b) \sim f(a) \circ f(b)$$
.

Thus, we get the "free" quandle with generators a_1, \ldots, a_n . The following factorisation will be done with respect to the structure of the diagram L'.

For each classical crossing we write down the relation (2) just as in the classical case. For each virtual crossing V we also write relations. Let $a_{j_1}, a_{j_2}, a_{j_3}, a_{j_4}$ be the four arcs incident to V as it is shown in Fig. 18.5.

Then, let us write the relations:

$$a_{j_2} = f(a_{j_1}) (4)$$

and

$$a_{i_3} = f(a_{i_4}) (5)$$

So, the virtual quandle Q(L) is the quandle generated by all arcs $a_i, i = 1, ..., n$, all relations (2) at classical vertices and all relations (4) (5) at virtual crossings.

Theorem 18.2. The virtual quantile Q(L) is a virtual link invariant.

Proof. The invariance under classical Reidemeister moves is just the same as in the classical case (cf. Remark 18.1).

First, let us note that two proper diagrams generate isotopic virtual links if and only if one of them can be deformed to the other by using a sequence of virtual Reidemeister moves. Indeed, if a circular long link occurs during the isotopy, then we can modify the isotopy by applying the first classical Reidemeister move to this long arc and subdividing it into two parts.

Now, we have to show that by applying a Reidemeister move to a link diagram, we transform our quandle to an isomorphic one.

Consider some virtual Reidemeister move. Let \bar{L} and \bar{L}' be two virtual link diagrams obtained from each other by applying this move. Since this move is performed inside a small circle C, all arcs of L and L' can be split into three sets:

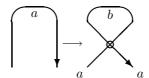


Figure 18.6. Invariance of Q under the first virtual move

the common set E of exterior arcs belonging to both L and L', the common set S of arcs intersecting the circle C, and the sets I, I' of interior arcs belonging to L and L', respectively. Hence, the quandle $\mathcal{M}(L)$ has the following generators and relations.

First, we have the relations to be denoted by ε (distributivity and idempotence) and E, S, I, and $\Gamma(L')$ is generated by ε and E, S, I'.

Relations (crossings) for diagrams L, L' are also divided into two parts: exterior R_E which are common for L and L' and interior R_I, R'_I related to L and L', respectively. Besides them, each quandle has common quandle relations (left–invertibility and right self–distributivity). Later in the proof, by relation we shall mean only those relations that come from crossings (not idempotence or distributivity).

Now, it is easy to see that for each concrete generalised Reidemeister move, by using R_I one can remove the generators I by expressing them in terms of S. Actually, this will add some "interior" relations R_S for S. The same can be done for I'. Denote these relations by R'_S . So, we transform both quandles $\Gamma(L)$ and $\Gamma(L')$ into isomorphic quandles $\tilde{\Gamma}(L)$ and $\tilde{\Gamma}(L')$. The latter ones are generated only by E, S (and ε). They have a common set of exterior relations R_E .

The only thing to show is that relations R_S and R'_S determine the same equivalence on S (by means of ε).

Let us perform it for concrete versions of Reidemeister moves; the other cases are completely analogous to those to be described.

We have to show that Q(L) is invariant under virtual Reidemeister moves.

The invariance of Q under all classical moves is checked in the same way as that of Q.

Let us now check the invariance of Q under purely virtual Reidemeister moves. The first virtual Reidemeister move is shown in Fig. 18.6. In the initial local picture we have one local generator a. Here we just add a new generator b and two coinciding relations: $b = f^{-1}(a)$. Thus, it does not change the virtual quandle at all.

The case of inverse orientation at the crossings gives us b = f(a) which does not change the situation.

For each next relation, we shall check only one case of arc orientation.

The second Reidemeister move (see Fig. 18.7) adds two generators c and d and two pairs of coinciding relations: $c = f(a), d = f^{-1}(b)$. Thus, the quandle Q stays the same.

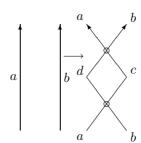


Figure 18.7. Invariance of Q under the second virtual move

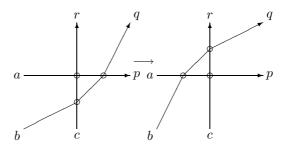


Figure 18.8. Invariance of Q under the third virtual move

In the case of the third Reidemeister move we have six "exterior arcs": three incoming (a,b,c) and three outgoing (p,q,r), see Fig. 18.8. In both cases we have $p=f^2(a), q=b, r=f^{-2}(c)$. The three interior arcs are expressed in a,b,c, and give no other relations.

Finally, let us check the mixed move. We are going to check the only version of it, see Fig. 18.9

In both pictures we have three incoming edges a,b,c and three outgoing edges p,q,r. In the first case we have relations: $p=f(a),q=b,r=f^{-1}(c)\circ a$. In the second case we have: $p=f(a),q=b,r=f^{-1}(c\circ f(a))$.

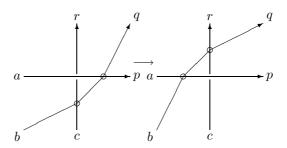


Figure 18.9. Invariance of Q under the mixed move

The distributivity relation $f(x \circ y) = f(x) \circ f(y)$ implies the relation $f^{-1}(c) \circ a = f^{-1}(c \circ f(a))$. Hence, two virtual quandles before the mixed move and after the mixed move coincide.

The other cases of the mixed move lead to other relations all equivalent to $f(x \circ y) = f(x) \circ f(y)$.

This completes the proof of the theorem.

Remark 18.5. Note that for the classical links, Q_K can be easily restored from Q. In the case of virtual links Q is, indeed, stronger: having Q, one easily obtains Q_K by putting $\forall x \in Qf(x) \equiv x$.

Example 18.1. Consider the virtual knot K (middle part) shown in Fig. 18.2. It has four arcs: $a_1 = a, b_1 = f(b)$ (before the virtual crossing) and a_2, b_2 (after the virtual crossing).

The relations in this quandle are: $a_2 = f(a_1), b_1 = f(b_2), a_2 \circ b_2 = b_1$ and $b_2 \circ b_1 = a_1$. Rewriting the last two relations for the two generators a_1, b_2 , we get: $f(a_1) \circ b_2 = f(b_2)$ and $b_2 \circ f(b_2) = a_1$.

Obviously, two quandles are not easy to compare. Of course, neither are virtual quandles. And we cannot show just now why $V(K) \neq \{a\}$. However, we shall soon describe some simplifications of Q which are weaker, but easier to compare. They will show us that $V(K) \neq \{a\}$, and consequently K is indeed knotted.

18.2 The Jones–Kauffman polynomial

The Kauffman construction for the Jones polynomial for virtual knots [KaVi] works just as well as in the case of classical knots. Namely, we first consider an oriented link L and the corresponding unoriented link |L|. After this, we smooth all classical crossings of |L| just as before (obtaining *states* of the diagram). In this way, we obtain a diagram without classical crossings, which is an unlink diagram. The number of components of this diagram (for a state s) is denoted by $\gamma(s)$. Then we define the Kauffman polynomial by the same formula

$$X(L) = \sum_{s} (-a)^{3w(L)} a^{\alpha(s) - \beta(s)} (-a^2 - a^{-2})^{\gamma(s) - 1}, \tag{1}$$

where w(L) is the writhe number taken over all classical crossings of L, and $\alpha(s)$, $\beta(s)$ are defined as in the classical case.

The invariance proof for this polynomial under classical Reidemeister moves is just the same as in the classical case; under purely virtual and semivirtual moves it is clearly invariant term—by—term.

However, this invariant has a disadvantage [KaVi]: invariance under a move that might not be an equivalence.

Exercise 18.3. Prove that the polynomial X is invariant under the following local moves twist move or virtual switch move, see Fig. 18.10.

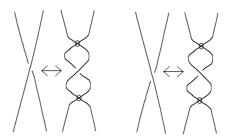


Figure 18.10. The two variants of the twist move

18.3 Presentations of the quandle

The quandle admits some presentations such as the fundamental group, the Alexander polynomial, and the colouring invariant. Here we shall show how to construct analogous presentations for the quandle Q.

18.3.1 The fundamental group

We recall that the fundamental group G of the complement to an oriented link L is obtained from its quandle $Q_K(L)$ as follows. Instead of elements a_i of the quandle Q we write elements of the formal group G (to be constructed), and instead of the operation \circ we write the conjugation operation: $x \circ y$ becomes yxy^{-1} . It is easy to check that this presentation of the operation \circ preserves the idempotence property and the relation (2). Besides, it has an evident inverse operation, namely: in the group a/b is going to be $b^{-1}ab$.

Thus, having written all relations for all classical vertices of the link l, we get a group G(L) that is called the *fundamental group of the complement to* L. In the case of classical knots, this group has a real geometric sense.

Obviously, we can do just the same for the case of virtual knots. In this case we also get a virtual link invariant, called the *Kauffman fundamental group of a virtual link*.

Actually, each such presentation makes the initial invariant weaker. Thus, Kauffman's virtual fundamental group does not distinguish the "virtual trefoil" knot K and the unknot.

Now, we construct the presentation of the invariant Q in the category of groups.

We have already seen that the conjugation plays role of the operation \circ . So, we only have to find an appropriate operation to present $f(\cdot)$.

This operation can be taken as follows: we just add a new generator q and say that $f(a) = qaq^{-1}$.

These two operations together make a grouppoid from each group. In fact, the following lemma holds.

Lemma 18.1. For each group G, the group $G*\{q\}$ (free product) with the two operations \circ , $f(\cdot)$ defined as $a \circ b = bab^{-1}$, $f(c) = qcq^{-1}$ for all $a, b, c \in G*\{q\}$, is a virtual grouppoid.

Proof. Indeed, we just have to show that $f(a \circ b) = f(a) \circ f(b)$. Actually, $f(a \circ b) = f(bab^{-1}) = qbab^{-1}q^{-1} = qbq^{-1}(qaq^{-1})qb^{-1}q^{-1} = f(b)f(a)f(b^{-1}) = f(a) \circ f(b)$. \square

Now, let us construct the fundamental group G(L) of a virtual link diagram L. Let us enumerate all arcs of L by $a_i, i = 1, \ldots, n$. So, G is the group generated by a_1, \ldots, a_n, q with the relations obtained from (2), (4), (5) by putting $f(x) = qxq^{-1}, y \circ z = zyz^{-1}$.

Thus, we obtain the following important theorem.

Theorem 18.3. The group G(L) is an invariant of virtual links.

Proof. The proof follows immediately from Theorem 18.2 and Lemma 18.1. \Box

Obviously, for the unknot U we have $G(U)=\langle a,q|\rangle$ is a free group with two generators.

Exercise 18.4. So, we can calculate the fundamental group G(K) of the virtual trefoil (together with the element q in it) and prove that this group together with q distinguishes K from the unknot.

Thus, the knot K is not trivial.

Besides the fundamental group, for each knot one can construct another invariant group by using the following relations:

$$x \circ y = y^p x y^{-p}, \ f(x) = q x q^{-1}$$

where p is a fixed integer, and q is the fixed group element. The proof is quite analogous to the previous one.

Another way to construct an invariant group is the following:

$$x \circ y = yx^{-1}y, f(x) = qxq^{-1}$$

or

$$x \circ y = yx^{-1}y, f(x) = qx^{-1}q.$$

18.3.2 The colouring invariant

The idea of coloring invariant is very simple. We take a presentation of some finite quandle Q' (say, obtained from a finite group G') by generators a_1, \ldots, a_k and relations.

More precisely, the following lemma holds.

Lemma 18.2. Let Q' be a virtual quandle. Then the set of homeomorphisms $Q(L) \to Q'$ is an invariant of link L.

This claim is obvious.

Now, let us prove the following lemma.

Lemma 18.3. For each finite virtual quandle Q' the number of homeomorphisms $Q(L) \to Q'$ is finite for each link L.

Proof. Indeed, consider a link L and a proper diagram \bar{L} of it. In order to construct a homeomorphism $h: Q(L) \to Q'$ we only have to define the images $h(a_i)$ of those elements of Q(L) that correspond to arcs. Since the number of arcs is finite, the desired number of homomorphism is finite.

The two lemmas proved above imply the following theorem.

Theorem 18.4. Let Q' be a finite virtual quandle. Then the number of homomorphisms $Q(L) \to Q'$ is an integer-valued invariant of L.

The sense of this invariant is pretty simple: it is just the proper coloring number of arcs of L by elements of Q'; the coloring is proper if and only if it satisfies the virtual quandle condition.

How do we construct finite virtual quandles? Let us generalise the ideas for ordinary quandle construction from [Mat] for the virtual case. Here are some examples.

Let G be a finite group, $g \in G$ be a fixed element of it, and n be an integer number. Then the set of elements $x \in G$ equipped with the operation $x \circ y = y^n x y^{-n}$, $f(x) = gxg^{-1}$ is a virtual grouppoid.

Another way for constructing virtual grouppoids by using groups is as follows: for a group G with a fixed element $g \in G$, we set $x \circ y = yx^{-1}y$, $f(x) = gxg^{-1}$.

These examples give two series of integer-valued virtual link invariants.

18.4 The VA-polynomial

In the present section we give a generalisation of the Alexander module. This module leads to the construction of the so-called VA-polynomial that has no analogue in the classical case.

Consider the ring R of Laurent polynomials in the variable t over \mathbf{Q} .

Remark 18.6. Here we take the field \mathbf{Q} (instead of the ring \mathbf{Z} as in the classical case) in order to get a graded Euclidean ring of polynomials. In our case degP = length(P) where length means the difference between the leading degree and the lowest degree.

For any two elements of this ring we can define the operation \circ as follows:

$$a \circ b = ta + (1 - t) b. \tag{6}$$

Obviously, this operation is invertible; the inverse operation (denoted by "/") is given by the formula

$$a/b = \frac{1}{t}a + \left(1 - \frac{1}{t}\right)b\tag{7}.$$

Clearly, $a \circ a = a$. The self-distributivity of (6) can be easily checked.

Thus, having a classical link diagram we can define a module over this ring by the following rule: the generator system of this module consists of elements a_i , corresponding to arcs of the diagram; at each classical crossing we write down a relation (2), where the operation \circ is taken from (6).

Remark 18.7. In the sequel, all modules are right-left modules; i.e., one can multiply elements of the module by an element of the ring on the right or left hand; thus one obtains the same result.

Thus, the defined module is a link invariant. However, this module allows us to extract a more visible invariant, called *the Alexander polynomial*. This can be done as follows.

For a diagram of a classical link, the system of relations defining this module is a linear systems of n equations on n variables $a_i, i = 1, ..., n$. Thus, we get an $n \times n$ matrix of relations. It is also called the Alexander matrix M(L). Hence for each equation the sum of the coefficients equals zero, the rows of this matrix are linearly-dependent, and thus the determinant of this matrix equals zero.

It is not difficult to prove that all minors of order n-1 of this matrix are the same up to multiplying by $\pm t^k$. The Alexander polynomial is just this minor (defined up to $\pm t^k$). The complete proof of invariance for the classical Alexander polynomial can be read, in e.g., [Ma0].

Remark 18.8. Note that $\pm t^k$ are precisely all invertible elements in the ring of Laurent polynomials.

Now, let us generalise the Alexander approach for the case of virtual knots. We shall use the same ring of Laurent polynomials over t. Fortunately, this is quite easy. Indeed, to define the *virtual Alexander module*, we need to find a "good" presentation for the function f, such that

$$f(a \circ b) = f(ta + (1 - t)b) = tf(a) + (1 - t)f(b).$$
(8)

Here we can just put $f(a) \equiv a + \varepsilon$, where ε is a new vector (it is the fixed vector in the new module). More precisely, consider the ring R of polynomials of t, t^{-1} (say, with rational coefficients) and set

$$\forall a, b, c \in R: \quad a \circ b = ta + (1 - t)b, f(c) = c + \varepsilon. \tag{9}$$

In this case, the formula (9) follows straightforwardly.

Thus, having a virtual link L diagram, we can define the virtual diagram Alexander module M(L) over R taking arcs of the diagrams as generators and (2), (4), (5) as relations (in the form (9)). In this module there exists a fixed element denoted by ε .

This definition together with Theorem 18.2 implies the following Theorem.

Theorem 18.5. The pair consisting of the virtual Alexander module together with the fixed element $(M(L), \varepsilon)$ is a virtual link invariant.

Now, let us see what happens with the Alexander polynomial.

Remark 18.9. For the sake of simplicity, we shall think of arcs of the diagram as elements of the Alexander module (i.e. we shall not introduce any other letters).

Let L be a proper virtual link diagram with m crossings. Thus, it has precisely m long arcs; each of the long arcs is divided into several arcs. According to the

operation $f(\cdot)$, we see that if two arcs p and q belong to the same long arc, they satisfy the relation

$$p = q + r\varepsilon$$
.

For each long arc, choose an arc of it. Since we have m long arcs, we can denote chosen arcs by b_1, \ldots, b_m . All other arcs are, hence, equal to $b_i + p_{ij}\varepsilon$, where $p \in \mathbf{Z}$.

Now, we can write down the relations of the Alexander polynomial. For each classical crossing v we just write the relation (2) in the form (6). Thus, we obtain an $n \times n$ matrix, where strings correspond to classical crossings, and columns correspond to long arcs. Thus, the element of the j-th column and i-th string is something like $P(t) \cdot (b_j + p_{ji}\varepsilon)$, where P(t) is a polynomial in t.

Now, we can take all members containing ε and move them to the right part. Thus we obtain an equation

$$(A) b = c\varepsilon, \tag{10}$$

where b is the column $(b_1, \ldots, b_m)^*$, c is the column $(c_1, \ldots c_m)^*$ of coefficients for ε , and A is a matrix.

The system (10) completely defines the virtual Alexander module M together with the element $\varepsilon \in M$.

As in the classical case, A is called the Alexander matrix of the diagram L. It is easy to see that in both the classical and in the virtual case A is degenerate: for the vector $x = (1, ..., 1)^*$ we have

$$Ax = 0$$
.

However, unlike the classical case, here we have a non-homogeneous system (10) of equations.

Since A is degenerated, it has rank at most m-1. Thus, the equation (10) implies some condition on ε : since strings of A are linearly-dependent, so are $c_1\varepsilon,\ldots,c_m\varepsilon$. This means that our Alexander module might have a relation $V(t)\varepsilon=0$, where $V(t)\varepsilon=0$. Here we write "might have" because it can happen that $V(t)\varepsilon=0$ implies V(t)=0.

The set of all $i \in R$ such that $i\varepsilon = 0$ forms an ideal $I \subset R$. The ring R is Euclidean, thus the ideal I is a principal ideal. It is characterised by its minimal polynomial $VA \in I$ that is defined up to invertible elements of R.

Definition 18.5. The minimal polynomial $VA(t) \in R$ of the ideal I is called the VA-polynomial of the virtual link diagram L.

Remark 18.10. Obviously, VA is defined up to invertible elements of R, i.e., up to $\pm t^k$.

The polynomial VA(L) depends only on the module M(L) with the selected element ε .

Thus, we obtain the following theorem.

Theorem 18.6. The polynomial VA (defined up to invertible elements of R) is a virtual link invariant.

Now, let us calculate the value of the VA-polynomial on some virtual knots.

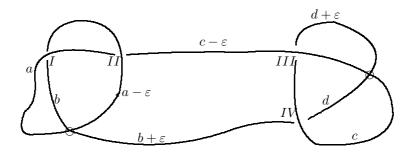


Figure 18.11. One way to construct K # K

Exercise 18.5. Consider the "virtual trefoil knot." Let us calculate the virtual quandle of it. In Fig. 18.2, rightmost picture, we have two classical vertices: I and II. They give us a system of two relations:

$$I: (a+\varepsilon)t + b(1-t) = b + \varepsilon$$

$$II: bt + (b+\varepsilon)(1-t) = a,$$

or:

$$at - bt = \varepsilon(1 - t)$$

 $b - a = \varepsilon(t - 1)$.

Multiplying the second equation by t and adding it to the first one, we get: $0 = \varepsilon(t-1)(1-t)$. Thus, $VA(K) = (1-t)^2$.

Having two oriented virtual knots K_1 and K_2 , one can define the *connected sum* of these knots as follows. We just take their diagrams, break each of them at two points close to each other and connect them together according to the orientation. This construction is well known in the classical case. In the classical case, the product is well defined (i.e., it does not depend on the broken point).

Let us see what happens in the virtual case. For example, let us take the two copies of one and the same knot K (shown in Fig. 18.2) and attach them together in two different ways as shown in Figs. 18.11 and 18.12.

We have two systems of equations. The first knot gives:

$$I: a-b=t\varepsilon$$

$$II: -ta+tc=\varepsilon$$

$$III: -tc+td=(1-2t)\varepsilon$$

$$IV: tb+(1-t)c-d=-t\varepsilon.$$

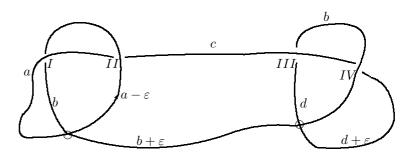


Figure 18.12. Another way to construct K # K

The zero linear combination is $t^2(I) + t(II) + (III) + t(IV) = 0$. Thus, the VA-polynomial equals $t^2(t) + t(1) + (1-2t) + t(-t) = (t-1)^2(t+1)$. The second knot gives:

$$I: a-b=t\varepsilon$$

$$II: -ta+tc=(1-t)\varepsilon$$

$$III: tb+(1-t)c-d=0$$

$$IV: (1-t)b-c+dt=-t\varepsilon.$$

The linear combination is $(t^3 - t^2 + t)(I) + (t^2 - t + 1)(II) + t^2(III) + t(IV) = 0$. Thus, the VA-polynomial equals $(t-1)^2(t^2+1)$.

These two polynomials are not proportional with invertible coefficient λt^k . Thus, the two connected sums are not equivalent.

Definition 18.6. By an oriented long virtual knot diagram we mean an immersion of the oriented line \mathbb{R}^1 into \mathbb{R}^2 with double crossing points, endowed with crossing structure at each intersection point (classical or virtual). We also require that outside an interval the image coincides with the line Ox where the abscissa increases while walking along the line according to its orientation.

Definition 18.7. By an *oriented long virtual knot* we mean an equivalence class of oriented long virtual knot diagrams modulo Reidemeister moves.

A long virtual knot can be obtained from an ordinary virtual knot by breaking it at a point and taking the free ends to infinity (say, $+\infty$ and $-\infty$ along Ox). It is well known that the theory of classical long knots is isomorphic to that of classical ordinary knots, i.e. the long knot isotopy class does not depend on the choice of the break point.

The two examples shown above demonstrate that there exist two long virtual knots K_1 , K_2 shown in Fig. 18.13 and obtained from the same virtual knot K_1 which are not isotopic. This fact was first mentioned in [GPV].

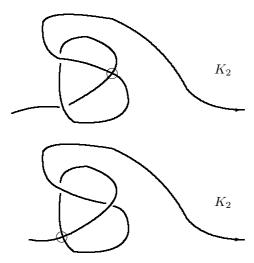


Figure 18.13. Two different long virtual knots coming from the same knot

Indeed, if K_1 and K_2 were isotopic, the virtual knot shown in Fig. 18.11 would be isotopic to that shown in Fig. 18.12. The latter claim is however not true.

Thus, long virtual knot theory differs from ordinary virtual knot theory.

18.4.1 Properties of the VA-polynomial

Theorem 18.7. For each virtual knot K the polynomial VA(K) is divisible by $(t-1)^2$.

Proof. Let K be a virtual knot diagram. Choose a classical crossing V_1 of it. Let X be a long arc outgoing from V_1 , and let x be the first arc of X incident to V_1 . Denote this arc by a_1 . By construction, all arcs belonging to X are associated with $a_1 + k\varepsilon$, $k \in \mathbb{N}$. Let the last arc of X be marked by $a_1 + k_1\varepsilon$. Denote the final point of it by V_2 . Now, let us take the first arc outgoing from V_2 and associate $a_2 + k_1\varepsilon$ to it. Then, we set the labels $a_2 + k\varepsilon$ for all arcs belonging to the same long arc. Let the last arc have the label $a_2 + k_2\varepsilon$ and have the final point at V_3 . Then, we associate $a_3 + k_2$ with the first arc outgoing from V_3 , and so on.

Finally, we shall come to V_1 . Let us show that the process converges, i.e. the label of the arc coming in V_1 has the label $a_{j+1} + 0 \cdot \varepsilon$, where j is the total number of long arcs.

Indeed, let us see the ε -part of labels while walking along the diagram from V_1 to V_1 . In the very beginning, it is equal to zero by construction. Then, while passing through each virtual crossing, it is increased (or decreased) by one. But each virtual crossing is passed twice, thus each $+\varepsilon$ is compensated by $-\varepsilon$ and vice versa. Thus, finally we come to V_1 with $0 \cdot \varepsilon$.

Note that the process converges if we do the same, starting from any arc with arbitrary integer number as a label.

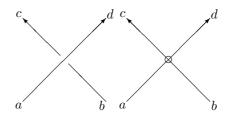


Figure 18.14. Labels of \bar{K} and \bar{K}'

In this case, each relation of the virtual Alexander module has the right part divisible by $\varepsilon(t-1)$: the relation

$$(a_i + p\varepsilon)t + (a_i + q\varepsilon)(1 - t) = (a_k + p\varepsilon)$$

is equivalent to

$$a_i t + a_j (1 - t) - a_k = (t - 1)(q - p)\varepsilon. \tag{*}$$

This proves that VA(K) is divisible by (t-1).

Denote the summands for the i-th vertex in the right part of (*) by q_i and p_i , respectively.

Let us seek relations on rows of the virtual Alexander matrix. Each relation holds for arbitrary t, hence for t = 1.

Denote rows of M by M_i . So, if for the matrix M we have $\sum_{i=1}^n c_i M_i = 0$ then $\sum_{i=1}^n c_i|_{t=1} M_i|_{t=1} = 0$.

The matrix $M(\bar{K})|_{t=1}$ is very simple. Each row of it (as well as each column of it) consists of 1 and -1 and zeros. The relation for rows of this matrix is obvious: one should just take the sum of these rows that is equal to zero. So, $\forall i, j = 1 \dots, n : c_i|_{t=1} = c_j|_{t=1}$.

Each relation for ε looks like $(\sum_{i=1}^{n} c_i(q_i - p_i))(t-1) = 0$.

Since we are interested in whether this expression is divisible by $(t-1)^2$, we can easily replace c_i by $c_i|_{t=1}$. Thus, it remains to prove that $\sum_{i=1}^n (q_i - p_i) = 0$ for the given diagram \bar{K} .

Let us prove it by induction on the number n of classical crossings.

For n = 0, there is nothing to prove.

Now, let \bar{K} be a diagram with n classical crossings, and \bar{K}' be a diagram obtained from \bar{K} by replacing a classical crossing by a virtual one.

Consider the case of the positive classical crossing X (the "negative" case is completely analogous to this one), see Fig. 18.14.

Denote the lower–left arc of both diagrams by a, and other arcs by b and c, d (for \bar{K} we have a=d), see Fig. 18.14. Assign the label 0 to the arc a of both diagrams. Let us calculate $\sum (q_i - p_i)$ for \bar{K}' and \bar{K} . By the induction hypothesis, for \bar{K}' this sum equals 0.

Denote the label of b for the first diagram by l_{b1} and that for the second diagram by l_{b2} .

The crossing X of \bar{K} has $q = l_{b1}, p = 0$, thus, its impact is equal to l_{b1} .

The other crossings of \bar{K} (classical or virtual) are in one-to-one correspondence with those of \bar{K}' . Let us calculate what the difference is between the p's and q's for these two diagrams. The difference comes from classical crossings. Their labels differ only in the part of the diagram from d to b. While walking from d to b, we encounter classical and virtual crossing. The total algebraic number is equal to zero. The algebraic number of virtual crossings equals -q. Thus the algebraic number of classical crossings equals q. Each of them impacts -1 to the difference between \bar{K} and \bar{K}' . Thus, we have q-q=0 which completes the induction step and hence, the theorem.

The following theorem can be proved straightforwardly.

Theorem 18.8. The invariant VA is additive. More precisely, for any connected sum $K = K_1 \# K_2$ of the two links K_1 and K_2 there exist invertible elements $\lambda, \mu \in R : VA(K) = \lambda VA(K_1) + \mu VA(K_1)$.

One should also mention that the VA-polynomial can be defined more precisely (up to $\pm t^k$) if we consider the ring $\mathbf{Z}[t, t^{-1}]$. However, this approach works for knots when the corresponding ideal over $\mathbf{Z}[t, t^{-1}]$ is a principal ideal.

18.5 Multiplicative approach

18.5.1 Introduction

Here we are going to use a construction quite analogous to the previous ones; however, instead of an extra "additive" element ε added to the module we shall add some "multiplicative" elements to the basic ring.

Throughout the section, we deal only with oriented links. In the sequel, we deal only with proper diagrams.

18.5.2 The two-variable polynomial

Below, we construct two invariant polynomials of virtual links ([Ma5], for a short version see [Ma'5]). In this section, the first one (in two variables) will be constructed. The second one, which will be described in subsection 18.5.3, deals with coloured links: with each n-component coloured link we associate an invariant polynomial in n+1 variables. The first invariant can be easily obtained from the second one by a simple variable change.

Let \bar{L} be a proper diagram of a virtual link L with n classical crossings.

Let us construct an $n \times n$ -matrix $M(\bar{L})$ with elements from $\mathbf{Z}[t, t^{-1}, s, s^{-1}]$ as follows.

First, let us enumerate all classical crossings of L by integer numbers from one to n and associate with each crossing the outgoing long arc. Each long arc starts with a (short) arc. Let us associate the label one with this arc. All other arcs of the long arc will be marked by exponents $s^k, k \in \mathbf{Z}$, as follows, see Fig. 18.15. While passing through the virtual crossing, we multiply the label by s if we pass from the left to the right (assuming that the arc we pass by is oriented upwards) or by s^{-1} otherwise.

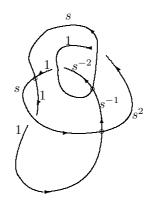


Figure 18.15. Symbols s^k on arcs

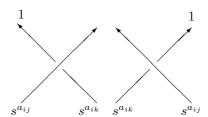


Figure 18.16. Arcs incident to classical crossings

Since the diagram is proper, our labelling is well defined. Consider a classical crossing V_i with number i. It is incident to some three arcs p, q, r, belonging to long arcs with numbers i, j, k; whence the number j belongs to the arc passing through V_i . Denote the exponent of s of the label corresponding to q by a_{ij} , and that of the label corresponding to r by a_{ik} , see Fig. 18.16.

Let us define the i-th row of the matrix $M(\bar{L})$ as the sum of the following three rows y_1, y_2, y_3 of length n. Each of these rows has only one non-zero element. The i-th element of the row y_1 is equal to one. If the crossing is positive, we set

$$y_{2k} = -s^{a_{ik}}t, y_{3j} = (t-1)s^{a_{ij}}; (1)$$

otherwise we put

$$y_{2k} = -s^{a_{ik}}t^{-1}, y_{3j} = (t^{-1} - 1)s^{a_{ij}}.$$
 (2)

Let $\zeta(\bar{L})$ be equal to $det M(\bar{L})$. Obviously, $\zeta(\bar{L})$ does not depend on the enumeration of rows of the matrix.

Theorem 18.9. For each two diagrams \bar{L} and \bar{L}' of the same virtual link L we have $\zeta(\bar{L}) = t^l \zeta(\bar{L}')$ for some $l \in \mathbf{Z}$.

Proof. Note that purely virtual moves do not change the matrix $M(\bar{L})$ and hence ζ .

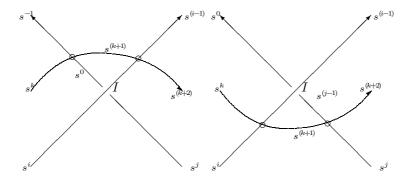


Figure 18.17. Invariance under the semivirtual move

While applying the semivirtual third Reidemeister move, we multiply one row of the matrix by $s^{\pm 1}$ and one column of the matrix by $s^{\mp 1}$. Indeed, let i be the number of classical crossings, the semivirtual move is applied to. Then we have the following two diagrams, L and L', see Fig. 18.17.

For the sake of simplicity, let us assume that arcs labelled by s^i, s^j and s^k correspond to long arcs numbered 2, 3, and 4.

In order to compare the elements of the matrices M(L) and M(L') in the p-th row and q-th column, one should consider the labels of (short) arcs of the q-th long arc incident to the p-th crossing. There are four different cases. In the simplest case $p \neq 1$ and $q \neq 1$, the labels of the two arcs are the same. If p = 1 and q = 1 they are the same by definition: they both equal s^0 . The first row $p = 1, q \neq 1$ can be considered straightforwardly: we have at most three non-zero elements in this row. Finally, if $q = 1, p \neq 1$, then we deal with the long arc outgoing from the first crossing. Consider the domain D of the semivirtual Reidemeister move, shown in Fig. 18.17. In the case of L, the first long arc leaves this domain with label s^{-1} on some (short) arc, and in the case of L' it leaves this domain with label s^0 . So, all further labels of this long arc (e.g. all those containing crossings except the first one) will be different. Namely, the label for L' will be equal to that for L multiplied by s.

This allows us to conclude the following:

The two matrices M(L) and M(L') will both have 1 on the place (1,1) and the same elements except for (1,p) or (q,1) for $p \neq 1$ and $q \neq 1$.

In the case $p \neq 1$ we have $M(L')_{p1} = s \cdot M(L)_{p1}$ and for $q \neq 1$ we have $M(L')_{1q} = s^{-1}M(L)_{1q}$.

Thus, the semivirtual move does not change the determinant either.

Now let us consider classical Reidemeister moves. We begin with the first move Ω_1 . Suppose we add a loop dividing some arc labelled by s^i of the first long arc into two long arcs. One of them lies "before" the loop of the move, the other one lies "after" the loop. These parts correspond to some columns in the matrix M. Denote these columns by A and B, respectively.

After performing this move, we obtain one more row and one more column in the matrix. The row will correspond to the added vertex. Let us renumber vertices

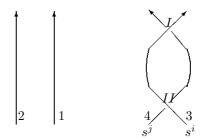


Figure 18.18. The second move

by $j \to j+1$, and associate the number one with the added vertex. This row will contain only two non-zero elements on places 1 and 2. Besides this, instead of columns A and B we have columns A and $s^{-i} \cdot B$ because, according to our rules, the part corresponding to B will start not from s^i but from $s^0 = 1$.

So, in each of the four cases of the first Reidemeister move, our transformation will look like this:

$$\left(\begin{array}{cc} A+B & * \end{array}\right) \to \left(\begin{array}{cc} x & s^i y & 0 \\ Bs^{-i} & A & * \end{array}\right).$$

Here x and y are some functions depending only on t. Indeed, consider the two arcs of the diagram with curl. If the first arc is incident to the first crossing once, then the element x equals 1, and y equals -1: it will be the sum of two elements -t and (t-1) or $-\frac{1}{t}$ and $(\frac{1}{t}-1)$. If the first arc is incident twice, we may have two different possibilities: x=t, y=-t, or $x=\frac{1}{t}, y=-\frac{1}{t}$.

Now, it can be checked straightforwardly that in each of the four cases the determinant will either stay the same or be multiplied by $t^{\pm 1}$.

Let us now consider the move Ω_2 . We shall perform all calculations just for the one case shown in Fig. 18.18.

This move adds two new crossings (they are numbered by one and two, and all other numbers are increased by two). Let us look at what happens with the matrix.

Assume that the initial diagram has n crossings. Denote the first two columns of the first matrix by A and B+C, where B and C have the following geometric sense. The column B+C corresponds to the long arc, i.e., to all crossings of the arc. After performing the second Reidemeister move, this long arc breaks at some interval, thus, all its incidences with crossings can be divided into two parts: those before and those after. Accordingly, the column will be decomposed into the sum of two columns which are denoted by B and C.

Thus, the first matrix looks like:

$$(A \quad B + C \quad *).$$

In right part of Fig. 18.18, we have a $(n+2) \times (n+2)$ -matrix. The matrix will look like:

$$\begin{pmatrix} 1 & -t & (t-1)s^i & 0 & 0 \dots 0 \\ 0 & 1 & \left(\frac{1}{t} - 1\right)s^i & -\frac{s^j}{t} & 0 \dots 0 \\ Cs^{-j} & 0 & A & B & * \end{pmatrix}.$$

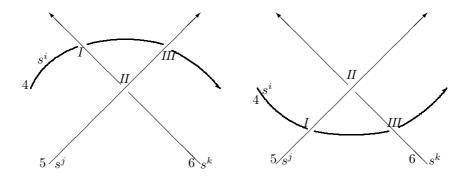


Figure 18.19. The third Reidemeister move

We only have to show that the initial and the transformed matrices have equal determinants.

We shall do this in the following way, transforming the second matrix. First, we add the second column multiplied by $s^i\left(\frac{t-1}{t}\right)$ to the third one. Thus, the elements (1,3) and (2,3) vanish. We get:

$$\begin{pmatrix} 1 & -t & 0 & 0 & 0 \dots 0 \\ 0 & 1 & 0 & -\frac{s^j}{t} & 0 \dots 0 \\ Cs^{-j} & 0 & A & B & * \end{pmatrix}.$$

Now, let us add the first column multiplied by s^{j} to the fourth one. We get:

$$\begin{pmatrix} 1 & -t & 0 & s^j & 0 \dots 0 \\ 0 & 1 & 0 & -\frac{s^j}{t} & 0 \dots 0 \\ Cs^{-j} & 0 & A & B+C & * \end{pmatrix}.$$

Finally, we add the second column multiplied by $\frac{s^j}{t}$ to the fourth one. We obtain the matrix

$$\begin{pmatrix} 1 & -t & 0 & 0 & 0 \dots 0 \\ 0 & 1 & 0 & 0 & 0 \dots 0 \\ Cs^{-j} & 0 & A & B+C & * \end{pmatrix}.$$

The determinant of this matrix obviously coincides with that of the first matrix.

Now, it remains to prove the invariance of ζ under Ω_3 . As before, we are going to consider only one case. We shall perform explicit calculation for the case, shown in Fig. 18.19.

The three crossings are marked by roman numbers I, II, III, so one can uniquely restore the numbers of outgoing arcs and their labels. All the other long arcs are numbered 4, 5, 6 with labels s^i, s^j, s^k at their last arcs.

We have two matrices M_1 and M_2 :

$$M_{1} = \begin{pmatrix} 1 & \left(\frac{1}{t} - 1\right) & 0 & -\frac{s^{i}}{t} & 0 & 0 & 0 \dots 0 \\ 0 & 1 & 0 & 0 & s^{j}(t-1) & -ts^{k} & 0 \dots 0 \\ -\frac{1}{t} & 0 & 1 & 0 & s^{j}\left(\frac{1}{t} - 1\right) & 0 & 0 \dots 0 \\ 0 & & & & & & \\ \vdots & & & * & & & & \\ 0 & & & & & & & \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & -\frac{s^i}{t} & s^j \left(\frac{1}{t} - 1\right) & 0 & 0 \dots 0 \\ 0 & 1 & 0 & 0 & s^j (t - 1) & -ts^k & 0 \dots 0 \\ -\frac{1}{t} & 0 & 1 & 0 & 0 & s^k \left(\frac{1}{t} - 1\right) & 0 \dots 0 \\ 0 & & & & & \\ \vdots & & * & & & \\ 0 & & & & & \end{pmatrix}.$$

To show that these two matrices have equal determinants, we shall perform the following operations with rows and columns (by using only the first three rows having zeros at positions ≥ 7) and only the six columns (the first of them has zeros at positions ≥ 4).

First, let us transform the first matrix as follows. Add the first column multiplied by $s^{j}(1-t)$ to the fifth column, and the first column multiplied by $s^{k}(t-1)$ to the sixth column. We get the matrix

$$M_1' = \begin{pmatrix} 1 & \left(\frac{1}{t} - 1\right) & 0 & -\frac{s^i}{t} & s^j \left(1 - t\right) & s^k \left(t - 1\right) & 0 \dots 0 \\ 0 & 1 & 0 & 0 & s^j \left(t - 1\right) & -ts^k & 0 \dots 0 \\ -\frac{1}{t} & 0 & 1 & 0 & 0 & s^k \left(\frac{1}{t} - 1\right) & 0 \dots 0 \\ 0 & & & & & & \\ \vdots & & & * & & & \\ 0 & & & & & & \\ \end{pmatrix}.$$

We see that all elements of M'_1 and M_2 , except for those lying in the first row, coincide. Now, it can be easily checked, that if we add the second row of M_2 multiplied by $(\frac{1}{t}-1)$ to the first row of M_2 , we obtain just the first row of M'_1 . Thus, $det M_1 = det M'_1 = det M_2$, which completes the proof.

Let m,M be the leading and the lowest powers of t in monomials of $\zeta(L)$. Define $\xi(L)=t^{-\left(\frac{m+M}{2}\right)}\zeta(L)$. By construction, ξ is a virtual link invariant.

The following properties of ξ hold.

Theorem 18.10. For a virtual link L isotopic to a classical one, we have $\xi(L) = 0$.

Proof. For a diagram \bar{L} of L having no virtual crossings, arcs coincide with long arcs; hence all labels equal $s^0=1$. Thus, the matrix M has the eigenvector $(1,\ldots,1)$

with zero eigenvalue. Thus $det M(\bar{L}) = 0$.

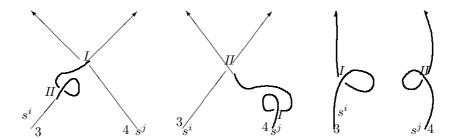


Figure 18.20. A modified Conway triple

Theorem 18.11. For any Conway triple L_+, L_-, L_0 there exist $p, q \in \mathbf{Z}$ such that $t^p \xi(L_+) - t^q \xi(L_-) = (1-t)\xi(L_0)$.

Proof. In view of Theorem 18.9, we can slightly modify the Conway triple by performing the first Reidemeister move, and then check the conditions of the theorem for triples of diagrams locally looking as shown in Fig. 18.20.

Here we enumerate crossings by roman letters. We also mark by labels and numbers the arcs not starting from selected crossings. Such a "coordinated" enumeration is possible always when all arcs represented at the picture are different. This is the main case when each diagonal element of the matrix is equal to one. In the other case, the labelling shown in Fig. 18.20 may not apply; the statement, however remains true.

Denote the matrices corresponding to the three diagrams in Fig. 18.20 by M_+, M_-, M_0 . As before, we shall restrict our calculations to small parts of the matrices. In this case, these matrices differ only in rows 1 and 2. In these rows, all elements but those numbered 1, 2, 3, and 4 are equal to zero. So, we shall perform calculations concerning 2×4 parts of M_+, M_-, M_0 .

Initially, these parts are as follows:

$$M_{+} \to \begin{pmatrix} 1 & t-1 & 0 & -ts^{j} \\ 0 & 1 & -s^{i} & 0 \end{pmatrix};$$

$$M_{-} \to \begin{pmatrix} 1 & 0 & 0 & -s^{j} \\ \frac{1}{t} - 1 & 1 & -\frac{s^{i}}{t} & 0 \end{pmatrix};$$

$$M_{0} \to \begin{pmatrix} 1 & 0 & -s^{i} & 0 \\ 0 & 1 & 0 & -s^{j} \end{pmatrix}.$$

Let us add the second row of the first matrix to the first row of it. For the second matrix, let us multiply the first row by t and then add the second row multiplied by t to the (modified) first row. For the third matrix, let us add the second row multiplied by t to the first row.

After performing these operations, we obtain three matrices M'_+ , M'_- , M'_0 with common first rows. The submatrices 2×4 we work with are as follows:

$$M'_+ \rightarrow \left(\begin{array}{ccc} 1 & t & -s^i & -ts^j \\ 0 & 1 & -s^i & 0 \end{array}\right);$$

$$M'_{-} \to \begin{pmatrix} 1 & t & -s^{i} & -ts^{j} \\ \frac{1}{t} - 1 & 1 & -\frac{s^{i}}{t} & 0 \end{pmatrix};$$

$$M'_{0} \to \begin{pmatrix} 1 & t & -s^{i} & -ts^{j} \\ 0 & 1 & 0 & -s^{j} \end{pmatrix}.$$

Obviously, $det(M'_{+}) = det(M_{+}), det(M'_{0}) = det(M_{0}), det(M'_{-}) = t \cdot det(M_{-}).$

Now, for the matrix M'_{-} , let us add the first row multiplied by $(1-\frac{1}{t})$, to the second row. We obtain the matrix M''_{-} such that $det M''_{-} = det M'_{-}$. The matrices M'_{+}, M''_{-}, M''_{0} differ only in the second row. Their second rows look like:

$$p = (0, 1, -s^{i}, 0, 0, \dots, 0),$$

$$q = (0, t, -s^{i}, (1 - t)s^{j}, 0, \dots, 0),$$

$$r = (0, 1, 0, -s^{j}, 0, \dots, 0).$$

Taking into account that p-q=(1-t)r, one obtains the claim of the theorem. \square

The polynomial ξ allows us to distinguish some virtual links that cannot be recognised by the Jones polynomial V introduced in [Kau] (e.g., the trivial two-component link and the closure of the two-strand virtual braid $\sigma_1\zeta_1\sigma_1^{-1}\zeta_1$) and the VA-polynomial (the disconnected sum of the "virtual trefoil" with itself and with the unknot).

18.5.3 The multivariable polynomial

The multivariable polynomial is constructed quite analogously to the previous one. Let L be a k-component link. Let \bar{L} be a proper link diagram with n classical crossings representing L. Let us associate with each component K_i , $i=1,\ldots,k$, of L the letter s_i . Consider a component \bar{K}_i of \bar{L} and let us mark its arcs by monomials which are products of $s_1,\ldots,s_k,s_1^{-1},\ldots,s_k^{-1}$. As above, each long arc of K_i starts with a (short) arc. Let us associate the label 1 with the latter. All other arcs of the long arc will be marked by monomials as follows. While passing through the virtual crossing with j-th component we multiply the label by s_j if we pass from the left to the right or by s_j^{-1} otherwise.

As before, we construct the $n \times n$ matrix according to the same rule. In this case, elements of the matrix belong to $\mathbf{Z}[t, t^{-1}, s_1, \dots, s_k, s_1^{-1}, \dots, s_k^{-1}]$.

The matrix M will be constructed just as in the case of the 2-variable polynomial. To define it, we just modify formulae (1) and (2) by replacing exponents of s by monomials in s_i . Let us define $\chi(\bar{L}) = det M(\bar{L})$. Obviously, $\chi(\bar{L})$ does not depend on the enumeration of rows of the matrix.

Theorem 18.12. The polynomial χ is invariant under all generalised Reidemeister moves but the first classical one. The first classical Reidemeister move either does not change the value of χ or multiplies it by $t^{\pm 1}$.

Proof. First, it is evident that purely virtual moves do not change the matrix at all. Let us consider the case of the semivirtual move shown in Fig. 18.17. Let m be the number of the component that takes part in the semivirtual move and has two virtual crossings with the other components.

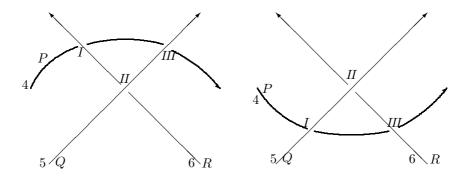


Figure 18.21. The third move in the multivariable case

Thus, after applying the semivirtual move, the first row of the matrix is multiplied by $s_m^{\pm 1}$, and the first column is multiplied by $s_m^{\mp 1}$. The proof of this fact is quite analogous to that in the case of Theorem 18.9. One should just look at Fig. 18.17 and consider the labels shown in it. In the multivariable case, the arbitrary arcs will have labels P, Q, R instead of s^i, s^j, s^k , where all P, Q, R are some monomials in s_1, s_2, \ldots, s_k .

The remaining part of the proof (of the invariance under classical Reidemeister moves) repeats that of Theorem 18.9. One should just replace arbitrary powers of s by some monomials T_p in many variables s_i .

Here we consider only the most interesting case, i.e., the third Reidemeister move. Let us consider the move shown in Fig. 18.21.

In this case, we have the following two matrices:

$$\begin{pmatrix} 1 & \left(\frac{1}{t} - 1\right) & 0 & -\frac{P}{t} & 0 & 0 & 0 \dots 0 \\ 0 & 1 & 0 & 0 & Q(t-1) & -Rt & 0 \dots 0 \\ -\frac{1}{t} & 0 & 1 & 0 & Q\left(\frac{1}{t} - 1\right) & 0 & 0 \dots 0 \\ 0 & & & & & & \\ \vdots & & & * & & & \\ 0 & & & & & & \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{P}{t} & Q\left(\frac{1}{t}-1\right) & 0 & 0 \dots 0 \\ 0 & 1 & 0 & 0 & Q(t-1) & -Rt & 0 \dots 0 \\ -\frac{1}{t} & 0 & 1 & 0 & 0 & R\left(\frac{1}{t}-1\right) & 0 \dots 0 \\ 0 & & & & & \\ \vdots & & * & & & \\ 0 & & & & & \end{pmatrix}.$$

Consider the first matrix. Adding the first column multiplied by Q(1-t) to the fifth one, and the first column multiplied by R(t-1) to the sixth one, we get

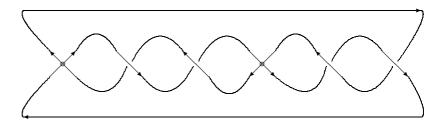


Figure 18.22. A link for which $\xi(L) = 0$.

The same matrix can be obtained if we replace the first row of the second matrix with the sum of the first row and the second row of the second matrix by $(\frac{1}{t} - 1)$ to the first row of it.

One can also prove the analogue of Theorem 18.10 for the polynomial χ . Also, the normalization for χ can be done in the same manner as that for ζ . Namely, let m, M be the leading and the lowest powers of t in monomials of $\chi(L)$. Define $\eta(L) = t^{-\left(\frac{m+M}{2}\right)}\chi(L)$. By construction, η is a virtual link invariant.

Theorem 18.13. For a virtual link L isotopic to a classical one, we have $\eta(L) = 0$.

The following statement follows from the construction.

Statement 18.1. For any k-component link L, we have

$$\eta(L)|_{s=s_1=\cdots=s_k}=\xi(L).$$

This shows that $\eta(L)$ is at least not weaker than ξ . In fact, it is even stronger. Consider the link L shown in Fig. 18.22.

It is not difficult to calculate that for this link L, the polynomial $\eta(L)$ is divisible by $(s_2 - s_1)$ and is not equal to zero. Thus, $\xi(L) = 0$, so η is strictly stronger than the invariant ξ .

Chapter 19

Generalised Jones–Kauffman polynomial

In the present chapter, we are going to give a generalisation of the Jones–Kauffman polynomial for virtual knots by adding some "extra information" to it, namely, some objects connected with curves in 2–surfaces (for a short version see [Ma'7]). In the second part of the chapter, we are going to consider the minimality aspects in virtual knot theory and give a proof of the generalised Murasugi theorem (short version in [Ma'6]).

19.1 Introduction. Basic definitions

Virtual equivalence and classical equivalence for classical knots coincide [GPV] and the set of all classical knots is a subset of the set of all virtual knots. Thus, each invariant of virtual links generates some invariant of classical links. In the previous chapter, we described some virtual link polynomials vanishing on classical links.

Below, we shall construct an invariant polynomial of virtual links that equals the classical Jones–Kauffman polynomial on classical links.

Let us first recall how one defines the classical Jones–Kauffman polynomial [Kau] for the case of virtual links. Let L be an oriented virtual link diagram with n classical crossings. Denote by |L| the diagram obtained from L by "forgetting" the orientation.

Just as in the classical case, for the non-oriented virtual link diagram |L|, one can "smooth" each classical crossing of |L| in two possible ways, called $A: \bigotimes \to \bigotimes$ and $B: \bigotimes \to \bigotimes$.

After such a smoothing of all classical crossings, one obtains a non-oriented diagram that does not contain classical crossings. Hence, this diagram generates the trivial virtual link.

Recall that the *state* of |L| is a choice of smoothing type for each classical crossing of |L|. Thus, |L| has 2^n states. Each state s has the following three important characteristics: the number $\alpha(s)$ of smoothings of type A, the number

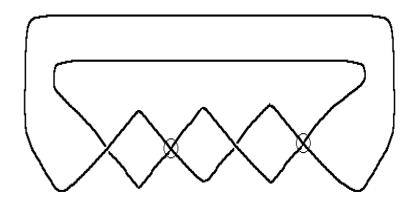


Figure 19.1. A virtual knot with trivial Jones-Kauffman polynomial

 $\beta(s) = n - \alpha(s)$ of smoothings of type B, and the number $\gamma(s)$ of link components of the smoothed diagram.

The Jones-Kauffman polynomial for virtual links [Kau] is given by

$$X(L) = (-a)^{-3w(L)} \sum_{s} a^{\alpha(s)-\beta(s)} (-a^2 - a^{-2})^{\gamma(s)-1}.$$
 (1)

Here the sum is taken over all states of |L|; w(s) is the writhe number of L.

In [Kau], Kauffman shows the invariance of X under generalised Reidemeister moves. However, he indicates a significant disadvantage of X: this polynomial is invariant under the move shown in Fig. 18.10 which is not an isotopy.

Thus, the Jones–Kauffman polynomial does not distinguish the trivial two-component virtual link and the virtual link Λ shown in Fig. 19.1.

Let us construct now a modification of the Jones–Kauffman invariant. Let S be the set of all pairs (M, γ) where M is a smooth orientable surface without boundary (possibly, not connected) and γ is an unordered finite system of unoriented closed curves immersed in M.

Let us define the equivalence on S by means of the following elementary equivalences:

- 1. Two pairs (M, g), (M', g') are equivalent if there exists a homeomorphism $M \to M'$ identifying g with g'.
- 2. For a fixed manifold M, if the set γ is homotopic¹ to the set γ' in M then the pairs (M, γ) and (M, γ') are equivalent.
- 3. Two pairs (M, γ) and (M, γ') are said to be equivalent if γ' is obtained from γ by adding a curve bounding a disk and not intersecting all other curves from γ' .
- 4. Pairs (M, γ) and (N, γ) should be equivalent, if N is a manifold obtained from M by cutting two disks not intersecting the curves from γ , and attaching a handle to boundaries of these disks.
- 5. Finally, for any closed compact orientable 2–manifold N, pairs (M, γ) and $(M \sqcup N, \gamma)$ are equivalent.

¹Here we mean a regular homotopy without fixed points.

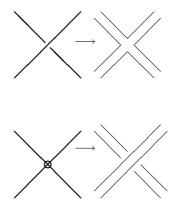


Figure 19.2. Local structure of the surface M'

Here \sqcup means the disjoint sum of M (with all curves of γ lying in it) and N without curves.

Denote the set of equivalence classes on S by \mathfrak{S} . There are several algorithms to distinguish elements of this set; the first follows from B. L Reinhart's work [Rein].

The basic idea of this invariant is the construction of a $\mathfrak{S}\mathbf{Z}[a,a^{-1}]$ -valued invariant function on the set of virtual links; values of this function should be linear combinations of elements from \mathfrak{S} with coefficients from $\mathbf{Z}[a,a^{-1}]$.

Let L be a virtual link diagram. Let us construct a 2-manifold M' as follows. At each classical crossing of the diagram we draw a cross (the upper picture of Fig. 19.2), and at each virtual crossing we set two non-intersecting bands (the lower picture). Connecting these crosses and bands by bands going along link arcs, we obtain a 2-manifold with boundary. This manifold is obviously orientable.

One can naturally project the diagram of L to M' in such a way that arcs of the diagram are projected to middle lines of bands; herewith classical crossings generate crossings in "crosses." Thus, we obtain a set of curves $\gamma \subset M'$. Attaching discs to boundary components of M', one obtains an orientable manifold M = M(L) together with the set γ of circles immersed in it.

Now, each state of the diagram L can be considered directly on M because to each local neighbourhood of a classical crossing of L, there corresponds an intersection point of one or two curves from γ . Thus, to each state s of L there corresponds the set $\gamma(s)$ of "smoothed" curves in M. The manifold M with all curves belonging to $\gamma \sqcup \gamma(s)$ generates some element of $p(s) \in \mathfrak{S}$.

Now, let us define $\Xi(L)$ as follows.

$$\Xi(L) = (-a)^{-3w(L)} \sum_{s} p(s) a^{\alpha(s) - \beta(s)} (-a^2 - a^{-2})^{\gamma(s) - 1}.$$
 (*)

Theorem 19.1. The function $\Xi(L)$ is invariant under generalised Reidemeister moves; hence, it is a virtual link invariant.

Proof. It is obvious that purely virtual Reidemeister moves and the semivirtual move applied to L do not change $\Xi(L)$ at all: by construction, all members of (*) stay the same.

The proof of the invariance of $\Xi(L)$ under the first and the third classical Reidemeister moves is quite analogous to the same procedure for the classical Jones–Kauffman polynomial; one should accurately check that the corresponding elements of \mathfrak{S} coincide.

In fact, if L and L' are two diagrams obtained one from the other by some first or third Reidemeister move, then for the diagrams |L| and |L'|, the corresponding surfaces M' are homeomorphic, and the behaviour of the system of curves γ for M(L) and M(L') differs only inside the small domain where the Reidemeister moves take place.

For the first move, the two situations (corresponding to the twisted curls with local writhe number +1 or -1) are considered quite analogously. Let L be a diagram and L' be the diagram obtained from L by adding such a curl. To each state s of |L| there naturally corresponds two states of |L'|. Fix one of them and denote it by s'. Let $L \sqcup \bigcirc$ be the disconnected sum of L and a small circle. Then we have:

$$p(s) = p(s').$$

Indeed, both surfaces for |L| and |L'| are the same and the only possible difference between corresponding curve systems is one added circle (elementary equivalence No. 3, see page 294). So, we have to compare members with the same coefficients from \mathfrak{S} . The comparison procedure coincides with that for the classical Jones–Kauffman polynomial.

Now, if we consider two diagrams L and L' obtained one from the other by using the third Reidemeister move, we see again that their surfaces M coincide. Let us select the three vertices P, Q, R of the diagram L and the corresponding vertices P', Q', R' of the diagram L', as shown in the upper part of Fig. 19.3.

So, both diagrams L, L' differ only inside a small disc D in the plane. The same can be said about the system of curves corresponding to some states of them: they differ only inside a small disc D_M in M. Thus, one can indicate six points on the boundary ∂D such that all diagrams of smoothings (in M) of L, L' pass through these and only these points of ∂D .

Consider the three possibilities X, Y, Z of connecting these points shown in the lower part of Fig. 19.3. In fact, there are other possibilities to do it but only these will play a significant role in the future calculations.

Let $|L_X|, |L_Y|, |L_Z|$ be the three planar diagrams of unoriented links coinciding with |L| outside D and coinciding with X, Y, and Z inside D, respectively.

We shall need the following three elements from \mathfrak{S} represented by K_X, K_Y , and K_Z , see Fig. 19.4. The element K_X contains the three lines of the third Reidemeister move (with fixed six endpoints) inside D_M . It also contains X. Analogously, K_Y contains the three lines and Y, and K_Z contains the three lines and Z. The only thing we need to know about the behaviours of K_X, K_Y , and K_Z outside D_M is that they coincide.

We have to prove that $\Xi(L) = \Xi(L')$. Obviously, we have w(L) = w(L'). So, we have to compare the members of (*) for |L| and |L'|. With each state of L, one can naturally associate a state for L'. For each state of |L| having the crossing P in position A, the corresponding state of |L'| gives just the same contribution to (*) as |L| since diagrams |L| and |L'| after smoothing P in position A coincide.

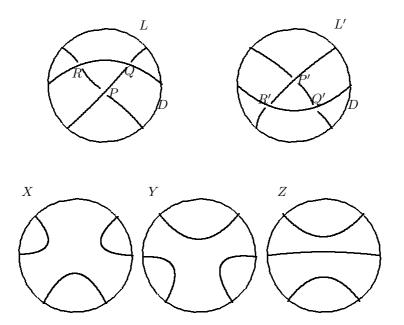


Figure 19.3. Diagrams and lines after smoothings

So, we have to compare all members of (*) corresponding to the smoothing of P in position B. We shall combine these members (for |L| and |L'|) in fours that differ only in the way of smoothing the vertices R and S. Now, let us fix the way of smoothing for A and A' outside D and compare the corresponding four members. If we delete the interior of the disc D and insert there X,Y, or Z, we obtain a system of curves in the plane. Denote the numbers of curves in these three systems by ν_X, ν_Y , and ν_Z , respectively.

Now, the four members for |L| give us the following:

$$aK_X(-a^2 - a^{-2})^{(\nu_X - 1)} + a^{-1}(K_Z(-a^2 - a^{-2})^{(\nu_Z - 1)} + K_X(-a^2 - a^{-2})^{\nu_X}) + a^{-3}K_X(-a^2 - a^{-2})^{(\nu_X - 1)} =$$

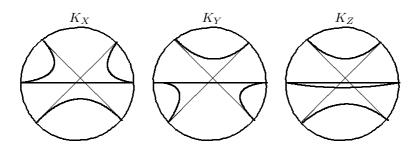


Figure 19.4. Parts of diagrams K_X, K_Y, K_Z

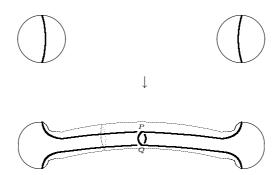


Figure 19.5. Adding a handle while performing Ω_2

$$= a^{-1}(-a^2 - a^{-2})^{(\nu_Z - 1)}K_Z.$$

Analogously, for |L'| we have a similar formula with members containing K_Z and K_Y . The latter members are reduced, so we obtain the same expression:

$$a^{-1}(-a^2-a^{-2})^{(\nu_Z-1)}K_Z.$$

Let us now check the invariance of Ξ under the second classical Reidemeister move. Let L' be the diagram obtained from L by applying the second classical Reidemeister move adding two classical crossings. Obviously, w(L) = w(L').

Consider the manifold M(L). The image of L divides it into connected components. We have two possibilities. In one of them, the Reidemeister move is applied to one and the same connected component. Then M(L') is homeomorphic to M(L), and curves from the set γ get two more crossings. In this case the proof of the equality $\Xi(L) = \Xi(L')$ is just the same as in the classical case (the reduction here treats not polynomials but elements from $\mathfrak S$ with polynomial coefficients). Moreover, the proof is even simpler than that for the third move: we have to consider the sum of four summands for |L| and |L'|. In each case, three of them vanish, and the remaining ones (one for |L| and one for |L'|) coincide. Taking into consideration that w(L) = w(L'), we get the desired result.

Finally, let us consider the case of the second Reidemeister move, where M(L') is obtained from M(L) by adding a handle. On this handle, two extra points P and Q appear, see Fig. 19.5.

Consider all states of the diagram |L'|. They can be split into four types depending on smoothing types of the crossings P and Q. Thus, each state s of |L| generates four states s_{++}, s_{--}, s_{-+} , and s_{+-} of |L'|. Note that $p(s) = p(s_{+-})$ (this follows from handle removal, see Fig. 19.5), and $p(s_{++}) = p(s_{--}) = p(s_{-+})$.

Besides, for each s, we have the following equalities:

$$\alpha(s) - \beta(s) = \alpha(s_{+-}) - \beta(s_{+-}), \gamma(s) = \gamma(s_{+-}),$$
$$\gamma(s_{++}) = \gamma(s_{--}) = \gamma(s_{-+}) - 1.$$

Thus, all members of (*) for L' corresponding to s_{--}, s_{++} , and s_{-+} will be reduced because of the identity $a^2 + a^{-2} + (-a^2 - a^{-2}) = 0$. The members corresponding to s_{+-} give just the same as (*) for L.

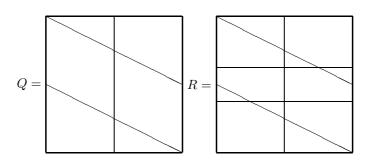


Figure 19.6. Two elements from S

19.2 An example

Let $P \in \mathfrak{S}$ be the element represented by the sphere without curves. It is obvious that for each classical link L, $\Xi(L) = P \cdot V(L)$. So, for the two-component unlink L we have $\Xi(L) = P \cdot (-a^2 - a^{-2})$.

It is known that the two–component unlink L and the closure Λ shown in Fig. 19.1 have the same Jones polynomial.

Consider the following two elements from \mathcal{S} (for the sake of simplicity, we shall draw the elements of S), see Fig. 19.6. Here we consider the torus as the square with identified opposite sides.

The element $Q \in \mathfrak{S}$ is initially represented by the same diagram shown in Fig. 19.6 with two additional circles that can be removed by equivalence No. 3, see page 294

Let us show that $Q \neq P, R \neq P$ and $Q \neq R$ in \mathfrak{S} . Actually, $Q \neq P$ because Q has two curves with non-zero intersection (+2 or -2 according to the orientation); thus, none of these curves can be removed by the equivalences described above. So, $R \neq P$ either. Besides, $R \neq Q$ because R contains three different curves on the torus (in coordinates from Fig. 19.6 they are (0,1),(1,0), and (2,1)); each two of them has non-zero intersection. Thus, none of them can be removed. So, the simplest diagram of [R] in S cannot have less than three curves.

Now, for the link Λ , we have

$$\Xi(\Lambda) = Qa^2 + 2R(-a^2 - a^{-2}) + Qa^{-2} = (2R - Q)(-a^2 - a^{-2}).$$

Thus, $\Xi(\Lambda) \neq \Xi(L)$.

19.3 Atoms and virtual knots. Minimality problems

In this section, we shall not distinguish virtual diagrams that can be obtained from each other by using only purely virtual and semivirtual moves. Such diagrams are called *strongly equivalent*; furthermore, equivalent diagrams are thought to be

different if in order to show their equivalence one needs some classical Reidemeister move. In this sense, a virtual knot can be completely generated by the setup of its classical crossings and lines connecting them.

Like classical links, virtual links may or may not be oriented. The *complexity* of a virtual diagram is the number of its classical crossings. Obviously, diagrams of complexity zero generate unlinks. We are interested in the minimality (in the sense of the absence of diagrams with smaller complexity) of diagrams realizing the given link.

In Chapter 7 we formulated the Kauffman-Murasugi theorem that was proved in Chapter 15. We are going to generalise this result for the case of virtual links. The notions of primitivity and splitting point are well known in the classical case. Their virtual analogues will be defined later.

In order to prove Theorem 7.5, K. Murasugi used some properties of the Jones polynomial. Let K be a virtual diagram and let X be a crossing of K. Now, $\rho_X(K)$ is the diagram where the small neighborhood of the crossing X is transformed as shown in Fig. 18.10 (this transformation is called the *twist move*). It is easy to see that if we apply the twist move twice to the same crossing, we get the initial diagram (in the sense that the obtained diagram is strongly equivalent to the initial one). Denote the set of all diagrams obtained from K by arbitrary twists ρ , by [K].

Definition 19.1. By a *splitting point* of a virtual diagram L we mean a classical crossing X of it such that for any diagram L' strongly equivalent to L, the removal of the small neighborhood of the corresponding crossing X' divides the diagram.

Definition 19.2. A virtual diagram L is said to be *non-primitive* if for some diagram L' equivalent to it, there exists a closed simple curve separating some non-empty set of classical crossings of L' from the other non-empty set of classical crossings, and intersecting the diagram L' precisely in two points. In this case the virtual diagram L' can be represented as a *connected sum of two* virtual diagrams.

Analogously, a virtual diagram L is called *disconnected* if there exists a diagram L' that is strongly equivalent to L such that L' can be divided into two parts $L'_1 \sqcup L'_2$ such that L'_1 and L'_2 lie inside two open non-intersecting sets on the plane. All diagrams we shall deal with, are thought to be connected.

There are exactly two (up to combinatorial equivalence) ways for embedding a primitive diagram of a classical link into the sphere with respect to the opposite outgoing edge structure. These two embeddings coincide up to the orientation of the sphere.

A virtual link L' is called *quasi-alternationg* if there exists a classical alternating diagram L such that $L' \in [L]$.

The main result of the present work is the following

Theorem 19.2. Any quasi-alternating diagram without splitting points is minimal.

An analogue of the minimality theorem was proved for the case of knots in \mathbb{RP}^3 , see [Dro].

Let us first prove the analogue of the Murasugi theorem (on the Jones polynomial) for the case of virtual links. In the proof, we use the techniques proposed in [Ma0] (which differ from the original Murasugi techniques, [Mur1]).

Theorem 19.3. Let L be a connected virtual diagram of complexity n. Then $span(V(L)) \leq n$. Moreover, the equality span(V(L)) = n holds only for virtual diagrams representing a connected sum of some quasi-alternating diagrams without splitting points.

Consider formula (1). We are interested in the states that give the maximal and the minimal possible degree of monomials in the sum (1). It is easy to check the fact that the maximal state gives the maximal possible degree, and the minimal state gives the minimal possible degree.

In order to estimate these degrees, we shall need the notions of atom and d-diagram. Recall that an atom is a two-dimensional connected closed manifold without boundary together with an embedded graph of valency four (frame) that divides the manifold into cells that admit a chessboard colouring. Atoms are considered up to the natural equivalence. An atom (more precisely, its equivalence class) can be completely restored from the following combinatorial structure: the framework (four-valent graph), the A-structure (dividing the outgoing half-edges into two pairs according to their disposition on the surface), and the B-structure (for each vertex, we indicate two pairs of adjacent half-edges that constitute a part of the boundary of black cells). A height (vertical) atom (according to [Ma'2]) is an atom whose frame is embeddable in \mathbb{R}^2 with respect to the A-structure.

In the case of an arbitrary atom one should replace embeddings by regular immersions. There might be many immersions for a given frame. The point is that having some A-structure (1,3),(2,4) at some vertex, there can be two different dispositions of this order on the place: 1, 2, 3, 4 or 1, 4, 3, 2 (counterclockwise).

Obviously, the obtained knot diagrams are defined up to strong equivalence and (possibly) twist moves at some classical crossings.

First, let us consider the case of primitive diagrams. Consider the maximal and the minimal states of the diagram \bar{L} . Let us define the numbers of link components corresponding to them by γ_{max} and γ_{min} , respectively. Thus, the length of the Kauffman bracket $\langle \bar{L} \rangle$ is going to be $A=2n+2(\gamma_{max}+\gamma_{min}-2)$. The diagram \bar{L} has intrinsic A-structure of some atoms: having it, one can construct an atom with γ_{max} black cells, γ_{min} white cells thinking of the circles of minimal and maximal states to be boundaries of the 2-cells to be attached. The Euler characteristic of the constructed manifold equals $n-2n+\gamma_{min}+\gamma_{max}$. Since it does not exceed two, we see that $A\leq 4n$.

Equality can take place only in the case when the obtained surface is a sphere. In this case the B-structure of \bar{L} corresponds to some planar atom.

These structures correspond to an embedding of the frame. These structures correspond to embeddings of the frame in S^2 with respect to the A-structure. There are only two such embeddings; they correspond to alternating diagrams. Thus, the diagram \bar{L} has one of these two B-structures. So, it can differ from an alternating diagram only by its C-structure and hence, it is quasi-alternating.

If the diagram is not primitive, it is sufficient to decompose it into a connected sum and apply the multiplicativity of the Jones polynomial.

Thus we have proved Theorem 19.3. Theorem 19.2 is just a simple corollary of it.

Chapter 20

Long virtual knots and their invariants

20.1 Introduction

Long virtual knots and their invariants first appeared in [GPV]. The present chapter consists of the author's results. The two main arguments that can be taken into account in the theory of "long" virtual knots and could not be used before, are the following:

- 1. One can indicate the initial and the final *arcs* (which are not compact) of the quandle; the elements corresponding to them are invariant under generalised Reidemeister moves.
- 2. One can take two different quandle–like structures of the same type at vertices depending on which arc is "before" and which is "after" according to the orientation of a long knot.

As shown in the previous chapter, the procedure of breaking a virtual knot is not well defined: breaking the same knot diagram at different points, we obtain different long knots. Moreover, a "virtual" unknot diagram broken at some point can generate a non-trivial long knot diagram. The aim of this chapter is to construct invariants of long virtual knots that feel "the breaking point."

Remark 20.1. Throughout the chapter, we deal only with long virtual knots, not links.

Remark 20.2. We shall never indicate the orientation of the long knot, assuming it to be oriented from the left to the right.

Throughout this section, R will denote the field of rational functions in one (real) variable t: $R = \mathbf{Q}(t)$.

Let us recall the definitions of virtual long link.

Definition 20.1. By a *long virtual knot diagram* we mean a smooth immersion f of the oriented line $L_x, x \in (-\infty, +\infty)$ in \mathbf{R}^2 , such that:

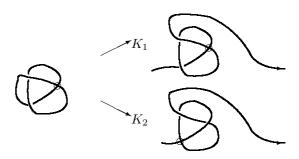


Figure 20.1. Transforming a diagram into a long diagram

- 1. Outside some big circle, we have f(t) = (t, 0);
- 2. Each intersection point is double and transverse;
- Each intersection point is endowed with a classical or virtual crossing structure.

Definition 20.2. A *long virtual knot* is an equivalence class of long virtual knot diagrams modulo generalised Reidemeister moves.

Long virtual knots admit a well–defined concatenation operation: for $K_1 \# K_2$ we just put a diagram of K_2 after a diagram of K_1 .

Thus, we can define the $semigroup \mathfrak{W}$ of virtual knots where the long unknot plays the role of the unit element.

An *arc* and a *long arc* of a long virtual knot diagram are just the same as in the ordinary case.

Obviously, having a virtual knot diagram, we can break it at some "interior" point in order to get a long virtual knot diagram, see Fig. 20.1.

It is known that in the ordinary case the result (i.e., the isotopy class of the obtained long knot) does not depend on the choice of the break point.

We shall give one more proof that in the virtual case this is not so. We are going to present an invariant of long virtual knots by using the ideas of the previous paragraph.

20.2 The long quandle

Definition 20.3. A long quandle is a set Q equipped with two binary operations \circ and * and one unary operation $f(\cdot)$ such that (Q, \circ, f) is a virtual quandle and (Q, *, f) is a virtual quandle and the following two relations hold: The reverse operation for \circ is / and the reverse operation for * is //.

$$\forall a, b, c \in Q : (a \circ b) * c = (a * c) \circ (b * c),$$

$$\forall a, bc \in Q : (a * b) \circ c = (a \circ c) * (b \circ c)$$

(new distributivity relations) and

$$\forall x, a, b \in Q : x\alpha(a \circ b) = x\alpha(a * b)$$

$$\forall x, a, b \in Q : x\beta(a/b) = x\beta(a/b),$$

where α and β are some operations from the list $\circ, *, /, //$.

Remark 20.3. It might seem that the last two relations hold only in the case when \circ coincides with *. However, the equation $(a \circ b) = c$ has the only solution in a, not in b!

Consider a diagram \bar{K} of a virtual knot and arcs of it. Let us fix the initial arc a and the final arc b.

Now, we construct the long quandle of it by the following rule. First, we take all arcs of it including a and b and consider the *free long quandle*, just by using formal operations 0, *, /, //, f factorised only by the quandle relations (together with the new relations).

After this, we factorise by relations at crossings. At each virtual crossing, we do just the same as in the case of a virtual quandle. At each classical crossing we write the relation either with \circ or with *, namely, if the overcrossing is passed **before** the undercrossing (with respect to the orientation of the knot) then we use the operation \circ (respectively, /); otherwise we use * (respectively, //).

After this factorisation, we obtain an algebraic object M equipped with the five operations \circ , /, *, //, and f and two selected elements a and b.

Definition 20.4. Denote the obtained object by $Q_L(\bar{K})$.

Let \overline{K} be a diagram of a long knot K. Call $Q_L(\overline{K})$ the long quantile of K.

Obviously, for the long unknot U (represented by a line without crossings) we have for $a, b \in Q_L(U)$: a = b.

Theorem 20.1. The quandle Q_L together with selected elements a, b is invariant with respect to generalised Reidemeister moves.

Proof. The proof is quite analogous to the invariance proof of the virtual quandle. Thus, the details will be sketched. The invariance under purely virtual moves and the semivirtual move goes as in the classical case: we deal only with f and one of the operations * or \circ . Only one of *, \circ appears when applying the first or the second classical Reidemeister move.

So, the most interesting case is the third classical Reidemeister move. In fact, it is sufficient to consider the following four cases shown in Fig. 20.2 (a, b, c, d).

In each of the four cases everything is OK with p and q (p does not change and q is affected by p in the same manner on the right hand and on the left hand). So, one should only check the transformation for r.

In each picture, at each crossing we put some operation α, β or γ . This means one of the operations $\circ, *, /, //$ (that will be applied to the arc below to obtain the corresponding arc above).

Consider the case a. We have: each α, β, γ is a multiplications \circ or *.

Thus, at the upper left corner we shall have: $(r\gamma q)\alpha p$ in the left picture and $(r\alpha p)\gamma(q\beta p)$. But, by definition, $(r\gamma q)\alpha p=(r\alpha p)\gamma(q\alpha p)$. The latter expression

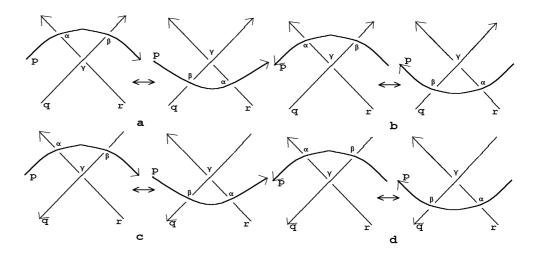


Figure 20.2. Checking the move Ω_3

equals $(r\alpha p)\gamma(q\beta p)$ according to the "new relation" (because both β and α are multiplications).

Now, let us turn to the case b. Here γ is multiplications and α, β are divisions. Thus, the same equality holds: $(r\gamma q)\alpha p = (r\alpha p)\gamma(q\alpha p) = (r\alpha p)\gamma(q\beta p)$.

The same equation is true for the cases shown in pictures c and d: the only important thing is that α and β are either both multiplications (as in the case c) or both divisions (as in the case d). The remaining part of the statement follows straightforwardly.

20.3 Colouring invariant

Let us consider one example: the colouring function. Namely, let $Q_L(K, a_1, a_2)$ be the long virtual quandle of the long K, with operations $f, \circ, *, /, //$, where a_1 and a_2 are the elements of Q corresponding to the initial and the final arc, respectively.

Let G be a finite virtual quandle. Let g_1, g_2 be two elements of G. Then the following theorem holds.

Theorem 20.2. The number of homomorphisms from Q(K) to G such that $Q(g_1) = a_1$ and $Q(g_2) = a_2$ is finite; besides, it is an invariant of the long knot K.

The proof of this theorem is obvious. However, it allows us to emphasise the following effect: for long links, each finite virtual quandle G generates not only one colouring function, but a matrix of colouring functions. Namely, we enumerate elements of G by integers $1, \ldots, n$ and set M_{ij} to be the total number of proper colourings such that the initial arc has colour i and the final arc has colour j. Denote the obtained matrix for a long virtual knot K by M(K).

The following theorem is obvious by construction.

Theorem 20.3. For any two long virtual knots K_1, K_2 we have $M(K_1 \# K_2) = M(K_1) \cdot M(K_2)$.

This means that each finite virtual quandle defines a representation of the semi-group $\mathfrak{W}.$

20.4 The \mathfrak{V} -rational function

Two arcs of each diagram are special: those containing the two infinite points.

Consider a diagram \bar{K} of a long knot K. Let us construct the virtual Alexander module of it. For the sake of simplicity, we shall preserve the previous notation. This module (which is now a linear space over R) will be denoted by \mathcal{M} .

Suppose we have n+1 long arcs (this case corresponds to n long arcs in the classical case). Each of the two infinite long arcs has one infinite arc. Denote the arc containing $-\infty$ by a_1 , and that containing $+\infty$ by a_{n+1} .

For each of the remaining n-1 long arcs, choose an arc of it. Denote these chosen arcs by a_2, \ldots, a_n .

Now, let us construct the linear space over R. First, consider the (n+2)-dimensional space S generated by $a_1, \ldots, a_{n+1}, \varepsilon$.

Now, let us define \mathcal{M} as the factor space obtained from S by factorizing it by relations just as in the classical case. We get the invariant triple $\mathcal{M}, a_1, a_{n+1}$. Obviously, $a_{n+1} - a_1 = k \cdot \varepsilon$. By definition, this k is a long knot invariant. Denote it by $\mathfrak{V}(K)$.

Obviously, the \mathfrak{V} polynomial has the following property.

Theorem 20.4. For any long classical knot L we have $\mathfrak{V}(L) = 0$.

Now, let us consider the following example. In [Ma2] it was shown that if we break the virtual trefoil in two different ways (see Fig 20.1), we obtain two different long virtual knots. This fact was proved by using the VA polynomial of connected sums of virtual knots. Let us prove this fact now by using \mathfrak{V} .

For the knot K_1 we have:

$$a_1(-t) + a_2t - a_3 = \varepsilon(-2t)$$

$$a_1t + a_3(-t) - a_2 = \varepsilon t.$$

Multiplying the second equation by t and adding it to the first one, we get

$$\mathfrak{V}(K_1) = a_3 - a_1 = -\frac{t^2 - t + 1}{(t - 1)^2}.$$

For K_2 we have:

$$ta_1 + (-t)(a_3 + \varepsilon) - a_2 = 0$$

$$t(a_2 + \varepsilon) + (-t)a_3 - (a_3 + \varepsilon) = 0.$$

Multiplying the first equation by t and adding it to the second equation, we get



Figure 20.3. Two long virtual knots obtained by breaking the unknot

$$\mathfrak{V}(K_2) = -\frac{t^2}{(t-1)^2}.$$

Thus, we see that $\mathfrak V$ distinguishes long virtual knots, corresponding to one and the same (ordinary) virtual knot.

The connected sum # of long virtual knots is well defined. Obviously, the following theorem is true.

Theorem 20.5. For any two long virtual knots K_1 and K_2 we have $\mathfrak{V}(K_1 \# K_2) = \mathfrak{V}(K_1) + \mathfrak{V}(K_2)$.

20.5 Virtual knots versus long virtual knots

We have already given some examples that show that when breaking the same virtual knot diagram at different points, we obtain different (non-equivalent) long virtual knots. The simplest and, probably, most interesting example is that by Se-Goo Kim (Fig. 17.10). The (non-trivial) virtual knot represented there is the connected sum of two unknots. In particular, this means that the corresponding long virtual knots are not trivial.

Consider the unknots shown in Fig. 20.3, a and b. Let us show that they are not isotopic to the trivial knot. To do this, we shall use the presentation of the long virtual quandle to the module over \mathbf{Z}_{16} by:

$$a \circ b = 5a - 4b, \ a * b = 9a - 8b$$

$$f(x) = 3 \cdot x.$$

It can be readily checked that these relations satisfy all axioms of the long quandle. Let us show that for none of these two knots a = b. Indeed, for the first knot (Fig. 20.3.a), denote by c the next arc after a. Then we have:

$$9a - 8 \cdot (3c) = c, 5b - 4 \cdot (3c) = c \Longrightarrow b = 9a.$$

For the second knot (Fig. 20.3.b), denote by c the upper (shortest) arc. We have:

$$5 \cdot (3b) - 4a = c, 9 \cdot (3a) - 8b = c \Longrightarrow b = 9a.$$

As we can see, in none of these cases does a=b. Besides, the expressions of b via a are different. Thus, none of the two long knots shown in Fig. 20.3.a and Fig. 20.3.b are trivial.

Chapter 21

Virtual braids

Just as classical knots can be obtained as closures of classical braids, virtual knots can be similarly obtained by closing *virtual braids*. Virtual braids were proposed by Vladimir V. Vershinin, [Ver].

21.1 Definitions of virtual braids

As well as virtual knots, virtual braids have a purely combinatorial definition. Namely, one takes virtual braid diagrams and factorises them by virtual Reidemeister moves (all moves with the exception of the first classical and the first virtual moves; the latter moves do not occur).

Definition 21.1. A virtual braid diagram on n strands is a graph lying in $[1,n] \times [0,1] \subset \mathbf{R}^2$ with vertices of valency one (there should be precisely 2n such vertices with coordinates (i,0) and (i,1) for $i=1,\ldots,n$) and a finite number vertices of valency four. The graph is a union of n smooth curves without vertical tangent lines connecting points on the line $\{y=1\}$ with those on the line $\{y=0\}$, their intersection makes crossings (four-valent vertices). Each crossing should be either endowed with a structure of overcrossing or undercrossing (as in the case of classical braids) or marked as a virtual one (by encircling it).

Definition 21.2. A *virtual braid* is an equivalence class of virtual braid diagrams by planar isotopies and all virtual Reidemeister moves except the first classical move and the first virtual move.

Like classical braids, virtual braids form a group (with respect to juxtaposition and rescaling the vertical coordinate). The generators of this group are:

 $\sigma_1, \ldots, \sigma_{n-1}$ (for classical crossings) and $\zeta_1, \ldots, \zeta_{n-1}$ (for virtual crossings).

The reverse elements for the σ 's are defined as in the classical case. Obviously, for each $i=1,\ldots,n-1$ we have $\zeta_i^2=e$ (this follows from the second virtual Reidemeister move).

One can show that the following set of relations [Ver] is sufficient to generate this group:

1. (Braid group relations):

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

for $|i-j| \geq 2$;

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1};$$

2. (Permutation group relations):

$$\zeta_i \zeta_j = \zeta_j \zeta_i$$

for $|i-j| \geq 2$;

$$\zeta_i \zeta_{i+1} \zeta_i = \zeta_{i+1} \zeta_i \zeta_{i+1};$$
$$\zeta_i^2 = e;$$

3. (Mixed relations):

$$\sigma_i \zeta_{i+1} \zeta_i = \zeta_{i+1} \zeta_i \sigma_{i+1};$$

$$\sigma_i \zeta_i = \zeta_i \sigma_i$$

for
$$|i - j| \ge 2$$
.

The proof of this fact is left to the reader.

21.2 Burau representation and its generalisations

In [Ver], the following generalisation of the Burau representation is given. The virtual braid group VB(n) is represented by $n \times n$ matrices where the generators σ_i, ζ_i are represented by block-diagonal matrices with the only nontrivial block on lines and columns (n, n-1). The block for σ_i 's is just as in chapter 8. For ζ_i we use simply permutations, namely, the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
.

The proof that it really gives a representation is left to the reader as an exercise.

However, this representation is rather weak. It is easy to check that for the non-trivial virtual two–strand braid represented by the word $b = (\sigma_1^2 \zeta_1 \sigma_1^{-1} \zeta_1 \sigma_1^{-1} \zeta_1)^2$ we have f(b) = f(e).

The generalisation of this Burau representation is the following: we take polynomial matrices in two variables, t and q, and construct the following 2×2 blocks: the same for σ and

$$\left(\begin{array}{cc} 0 & q \\ q^{-1} & 0 \end{array}\right)$$

for ζ .

Denote the map, defined above on generators of the braid group, by R.

Theorem 21.1. The map R can be generated as a representation of the braid group.

Proof. Obviously, the matrix $R(\sigma_i)$ is invertible, and for the matrix $R(\zeta_i)$ we have $(R(\zeta_i))^2 = e$.

Furthermore, the relations of the braid group for the σ 's can be easily checked as in the case of the "weaker" Burau representation.

So, we only have to check the relations $R(\zeta_i\zeta_{i+1}\zeta_i) = R(\zeta_{i+1}\zeta_i\zeta_{i+1})$ and $R(\zeta_i\zeta_{i+1}\sigma_i) = R(\sigma_{i+1}\zeta_i\zeta_{i+1})$.

They can be checked straightforwardly by direct calculation with 3×3 matrices.

Now, we can prove the following theorem.

Theorem 21.2. The group Br(3) is naturally embedded in the virtual braid group VB(3).

Proof. Actually, let β_1, β_2 be some braid—words written in $\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}$. Suppose they represent the same braid in VB(n). Then their Burau matrices coincide. Hence the Burau representation of the classical braid group is faithful for the case of three strands, and we conclude that β_1 and β_2 represent the same word in Br(n).

21.3 Invariants of virtual braids

In this section, we are going to present an invariant of virtual braids proposed by the author in [Ma'8] and show that the classical braid group is a subgroup of the virtual one. More precisely, we give a generalisation of the complete braid invariant described before for the case of virtual braids. The new "virtual invariant" is very strong: it is stronger than the Burau representation, the Jones–Kauffman polynomial. A simple computer program written by the author recognises all virtual braids on three and four crossings, given by the author. So, the question of whether the invariant is complete remains open so far.

21.3.1 Introduction

A virtual braid diagram is called regular if any two different intersection points have different ordinates.

Let us start with basic definitions and introduce the notation.

Remark 21.1. In the sequel, the number of strands for a virtual braid diagram is denoted by n, unless otherwise specified.

Remark 21.2. In the sequel, regular (virtual) braid diagrams and corresponding braid words (see definition below) will be denoted by Greek letters (possibly, with indices). Virtual braids will be denoted by Latin letters (with indices, maybe).

Remark 21.3. We shall also treat braid words and braids familiarly, saying, e.g., "a strand of a braid word" and meaning "a strand of the corresponding braid".

Let us describe the construction of the word by a given regular virtual braid diagrams as follows. Let us walk along the axis Oy from the point (0,1) to the point (0,0) and watch all those levels $y=t\in[0,1]$ having crossings. Each such

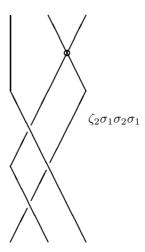


Figure 21.1. A virtual braid diagram and the corresponding braid word

crossing permutes strands #i and #(i+1) for some i = 1, ..., n-1. If the crossing is virtual, we write the letter ζ_i , if not, we write σ_i if overcrossing is the "northwest–southeast" strand, and σ_i^{-1} otherwise.

Thus, we have got a braid word by a given proper virtual braid diagram, see Fig. 21.1.

Thus the main question is the word problem for the virtual braid group: **How** to recognise whether two different (proper) virtual braid diagrams β_1 and β_2 represent the same braid b^1 . One can apply the virtual braid group relations to one diagram without getting the other, and one does not know whether he has to stop and say that they are not isomorphic or he has to continue.

A partial answer to this question is the construction of a virtual braid group invariant, i.e., a function on virtual braid diagrams (or braid words) that is invariant under all virtual braid group relations. In this case, if for an invariant f we have $f(\beta_1) \neq f(\beta_2)$ then β_1 and β_2 represent two different braids.

21.3.2 The construction of the main invariant

Here we give the generalisation of the complete classical braid group invariant, described in Chapter 8 for the case of virtual braids.

Let G be the free group with generators $a_1 \ldots, a_n, t$. Let E_i be the quotient set of right residue classes $\{a_i\}\setminus G$ for $i=1,\ldots,n$.

Definition 21.3. A virtual n-system is a set of elements $\{e_1 \in E_1, e_2 \in E_2, \dots, e_n \in E_n\}$.

The aim of this subsection is to construct an invariant map (non-homomorphic) from the set of all virtual n-strand braids to the set of virtual n-systems.

¹To the best of the author's knowledge, there exists no algorithm for virtual braid recognition.

Let β be a braid word. Let us construct the corresponding virtual n-system $f(\beta)$ step-by-step. Namely, we shall reconstruct the function $f(\beta\psi)$ from the function $f(\beta)$, where ψ is σ_i or σ_i^{-1} or ζ_i .

First, let us take n residue classes of the unit element of $G: \langle e, e, \dots, e \rangle$. This means that we have defined

$$f(e) = \langle e, e, \dots, e \rangle.$$

Now, let us read the word β . If the first letter is ζ_i then all words but e_i, e_{i+1} in the n-systems stay the same, e_i becomes equal to t and e_{i+1} becomes t^{-1} (here and in the sequel, we mean, of course, residue classes, e.g., [t] and $[t^{-1}]$. But we write just t and t^{-1} for the sake of simplicity).

Now, if the first letter of our braid word is σ_i , then all classes but e_{i+1} stay the same, and e_{i+1} becomes a_i^{-1} . Finally, if the first letter is σ_i^{-1} then the only changing element is e_i : it becomes a_{i+1} .

The procedure for each next letter (generator) is the following. Denote the index of this letter (the generator or its inverse) by i. Assume that the left strand of this crossing originates from the point (p, 1), and the right one from the point (q, 1). Let $e_p = P, e_q = Q$, where P, Q are some words representing the corresponding residue classes. After the crossing all residue classes but e_p, e_q should stay the same.

Then if the letter is ζ_i then e_p becomes $P \cdot t$, and e_q becomes $Q \cdot t^{-1}$. If the letter is σ_i then e_p stays the same, and e_q becomes $QP^{-1}a_p^{-1}P$. Finally, if the letter is σ_i^{-1} then e_q stays the same, e_p becomes $PQ^{-1}a_qQ$. Note that this operation is well

Actually, if we take the words $a_p^l P, a_q^m Q$ instead of the words P, Q, we get: in the first case $a_p^l P t \sim P t$, $a_q^m Q t^{-1} \sim Q t^{-1}$, and in the second case we obtain $a_p^l P \sim P$, $a_q^m Q P^{-1} a_p^{-1} a_p^l P = a_q^m Q P^{-1} a_p^{-1} P \sim Q P^{-1} a_p^{-1} P$. In the third case we obtain $a_p^l P Q^{-1} a_q^{-m} a_q^{-1} a_q^m P = a_p^l P Q^{-1} a_q^{-1} Q \sim P Q^{-1} a_q^{-1} Q$, $a_q^m Q \sim Q$. Thus, we have defined the map f from the set of all virtual braid diagrams to

the set of virtual n-systems.

Theorem 21.3. The function f, defined above, is a braid invariant. Namely, if β_1 and β_2 represent the same braid β then $f(\beta_1) = f(\beta_2)$.

Proof. We have to demonstrate that the function f defined on virtual braid diagrams is invariant under all virtual braid group relations. It suffices to prove that, for the words $\beta_1 = \beta \gamma_1$ and $\beta_2 = \beta \gamma_2$ where $\gamma_1 = \gamma_2$ is a relation we have proved, we can also prove $f(\beta_1) = f(\beta_2)$. During the proof of the theorem, we shall call it the A-statement.

Indeed, having proved this claim, we also have $f(\beta_1 \delta) = f(\beta_1 \delta)$ for arbitrary δ because the invariant $f(\beta_1 \delta)$ (as well as $f(\beta_2 \delta)$) is constructed step-by-step, i.e., knowing the value $f(\beta_1)$ and the braid word δ , we easily obtain the value of $f(\beta_1\delta)$. Hence, for braid words β, δ and for each braid group relation $\gamma_1 = \gamma_2$ we prove that $f(\beta \gamma_1 \delta) = f(\beta \gamma_2 \delta)$. This completes the proof of the theorem.

Now, let us return to the A-statement.

To prove the A-statement, we must consider all virtual braid group relations. The commutation relation $\sigma_i \sigma_j = \sigma_j \sigma_i$ for "far" i, j is obvious: all four strands involved in this relation are different, so the order of applying the operation does not affect on the final result. The same can be said about the other commutation relations, involving one σ and one ζ or two ζ 's.

Now let us consider the relation $\zeta_i^2 = e$ which is pretty simple too.

Actually, let us consider a braid word β , and let the word β_1 be defined as $\beta \zeta_i^2$ for some i. Let $f(\beta) = (P_1, \ldots, P_n), f(\beta_1) = (P'_1, \ldots, P'_n)$. Let p and q be the numbers of strands coming to the crossing from the left side and from the right side. Obviously, for $j \neq p, q$ we have $P_j = P'_j$. Besides, $P'_p = (P_p \cdot t) \cdot t^{-1} = P_p, P'_q = P'_j$ $(P_q \cdot t^{-1}) \cdot t = P_q.$

Now let us consider the case $\beta_1 = \beta \cdot \sigma_i \cdot \sigma_i^{-1}$ (obviously, the case $b_1 = \beta \sigma_i^{-1} \sigma_i$ is quite analogous to this one).

As before, denote $f(\beta)$ by $(\ldots P_i \ldots)$, and $f(\beta_1)$ by $(\ldots P_i' \ldots)$, and the corresponding numbers of strands by p and q. Again, we have: for $j \neq p, q : P'_j = P_j$. Moreover, $P_p = P'_p$ by definition of f (since the p-th strand makes an overcrossing twice, and $P'_q = (P_q P_p^{-1} a_p^{-1} P_p) P_p^{-1} a_p P_p = P_q$. Now let us check the invariance under the third Reidemeister move. Let β be a

braid word, $\beta_1 = \beta \zeta_i \zeta_{i+1} \zeta_i$, and $\beta_2 = \beta \zeta_{i+1} \zeta_i \zeta_{i+1}$. Let p, q, r be the global numbers of strands occupying positions n, n + 1, n + 2 at the bottom of b.

or strands occupying positions n, n+1, n+2 at the bottom of b. Denote $f(\beta)$ by (P_1, \ldots, P_n) , $f(\beta_1)$ by (P_1^1, \ldots, P_n^1) , and $f(\beta_2)$ by P_1^2, \ldots, P_n^2 . Obviously, $\forall i \neq p, q, r$ we have $P_i = P_i^1 = P_i^2$. Direct calculations show that $P_p^1 = P_p^2 = P_p \cdot t^2, P_q^1 = P_q^2 = P_q$ and $P_r^1 = P_r^2 = P_r \cdot t^{-2}$. Now, let us consider the mixed move by using the same notation: $\beta_1 = \beta \zeta_i \zeta_{i+1} \sigma_i$, $\beta_2 = \sigma_{i+1} \zeta_i \zeta_{i+1}$. As before, $P_j^1 = P_j^2 = P_j$ for all $j \neq p, q, r$. Now, direct calculation shows that $P_p^1 = P_p t^2, P_q^1 = P_q t^{-1}, P_r^1 = P_r t^{-1} (P_q t^{-1})^{-1} a_q^{-1} (P_q t^{-1}) = P_r P_q a_q^{-1} P_q t^{-1}$ and $P_p^2 = P_p t^2, P_q^2 = P_q t^{-1}, P_r^2 = P_r P_q^{-1} a_q^{-1} P_q$. Finally, consider the "classical" case $\beta_1 = \beta \sigma_i \sigma_{i+1} \sigma_i, \beta_2 = \beta \sigma_{i+1} \sigma_i \sigma_{i+1}$; the notation is the same. Again $\forall j \neq p, q, r : P_j^1 = P_j^2 = P_j$. Besides this, since the p-th strand forms two overcrossings in both cases then $P^1 = P^2 - P$. Then

As we see, the final results coincide and this completes the proof of the theorem.

Thus, we have proved that f is a virtual braid invariant, i.e., for a given braid the value of f does not depend on the diagram representing b. So, we can write simply f(b).

Remark 21.4. In fact, we can think of f as a function valued not in (E_1, \ldots, E_n) , but in n copies of G: all these invariances were proved for the general case of (G,\ldots,G) . The present construction of (E_1,\ldots,E_n) is considered for the sake of simplicity.

First fruits 21.3.3

As well as classical knots, classical braids (i.e., braids without virtual crossings) can be considered up to two equivalences: classical (modulo only classical moves) and virtual (modulo all moves). Now, we prove that they are the same (as in the case of classical knots). This fact is not new. It follows from [FRR].

Theorem 21.4. Two virtually equal classical braids b_1 and b_2 are classically equal.

Proof. Since b_1 is virtually equal to b_2 , we have $f(b_1) = f(b_2)$. Now, taking into account that f is a complete invariant on the set of classical braids, we have $b_1 = b_2$ (in the classical sense).

As in the case of virtual knots, in the case of virtual braids there exists a forbidden move, namely, $X = \sigma_i \sigma_{i+1} \zeta_i = \zeta_{i+1} \sigma_i \sigma_{i+1} = Y$. Now, we are going to show that it cannot be represented by a finite sequence of the virtual braid group relations.

Theorem 21.5. A forbidden move (relation) cannot be represented by a finite sequence of legal moves (relations).

Proof. Actually, let us calculate the values $f(\sigma_1\sigma_2\zeta_1)$ and $f(\zeta_2\sigma_1\sigma_2)$. In the first case we have:

$$(e, e, e) \to (e, a_1^{-1}, e) \to (e, a_1^{-1}, a_1^{-1}) \to (e, a_1^{-1}t, a_1^{-1}t^{-1}).$$

In the second case we have:

$$(e,e,e) \to (e,t,t^{-1}) \to (e,t,t^{-1}a_1^{-1}) \to (e,ta_1^{-1},t^{-1}a_1^{-1}).$$

As we see, the final results are not the same (i.e., they represent different virtual n-systems); thus, the forbidden move changes the virtual braid.

Remark 21.5. If we put t = 1, the results f(X) and f(Y) become the same. Thus that is the variable t that feels the forbidden move.

Definition 21.4. For a given braid diagram β and two numbers $1 \le i < j \le 2$ let us define the linking coefficient (see, e.g., [GPV]) $l_{i,j}(\beta)$ as follows. Let us watch all those crossings where the i-th strand is the undercrossing, and the j-th strands is the overcrossing, and take the algebraic sum of all these crossings (-1 if the crossing is negative, and 1 if it is positive).

We shall show now that this function can be calculated by using only $f(\beta)$, thus, it is a braid invariant.

Now consider a braid b and the value f(b). It consists of n members (e_1, \ldots, e_n) . For $1 \leq p \neq q \leq n$ let f_{pq} be the algebraic number of entrances of a_q in e_p . All numbers f_{pq} are well-defined from f: each element e_i is defined up to multiplication by a_i from the left side; such multiplication does not change f_{ij} for $j \neq i$.

Thus, $\forall 1 \leq p \neq q \leq n$ the function f_{pq} is a virtual braid invariant.

Theorem 21.6. The invariant f_{pq} coincides with the linking coefficient $-l_{pq}$ of strands p and q.

Proof. Actually, let us consider a braid word β and let us construct $f(\beta)$ step—by—step. Note that f_{pq} demonstrates the "commutativisation" of the invariant f: instead of multiplication of generators, we just add them and watch the corresponding coefficients. Let us study the subject more precisely. Now we begin to prove the statement of the theorem using induction on the length of β . For the case of

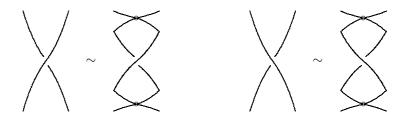


Figure 21.2. Pairs of diagrams not distinguished by the Kauffman polynomial

zero crossings everything is obvious. Now let us add a new crossing and see what happens.

The case of a virtual crossing does not change the linking coefficient (which is constructed by taking the algebraic sum for classical crossings). This letter (crossing) does not change f_{ij} either: two elements e_i, e_j are multiplied by $t^{\pm 1}$, but the number of entrances of $a_k, k=1\ldots,n$, stays the same. By adding some letter σ_i , we change $f_{kl}, l=1,\ldots,n$, as follows. Let p be the number of strands coming to this crossing from the right side, and q be the number of the strand coming from the left side. Then the only thing changing here is e_q . It is to be multiplied by $e_p^{-1}p^{-1}e_p$. While calculating the algebraic number of entrances of some letters, e_p on the right hand cancels the effect of e_p^{-1} on the left hand. Thus, f_{qp} becomes $f_{qp}-1$, the coefficient l_{qp} becomes $l_{qp}+1$, and all other coefficients l_{xy} , f_{xy} stay the same. The same thing happens with the linking coefficient l_{pq} .

The case of σ_i^{-1} can be considered analogously.

We have proved the induction step and, thus, the theorem is proved.

So, we have shown that f is stronger than a well–known invariant, called the linking coefficient.

Here we give two more examples showing the advantages of the invariant f.

Consider the 3–strand braid $b=\zeta_2\sigma_2^{-1}\zeta_2\sigma_1\sigma_2\zeta_1\sigma_1\zeta_1\sigma_2^{-1}\sigma_1^{-1}$. A computer shows that for this braid $f(b)\neq f(e)$. However, this braid is not distinguished by the virtual Jones–Kauffman polynomial proposed in [KaVi]. More precisely, consider the link L(b) obtained as a closure of b and the Kauffman polynomial K(L(b)) of this link. It is well known that the Kauffman polynomial does not distinguish links that differ as shown in Fig. 21.2.

Thus, it is easy to see that if for some braid we substitute $\sigma_i^{\pm 1}$ for $\zeta_i \sigma_i^{\pm 1} \zeta_i$ then the closures of both braids will have the same Kauffman polynomial.

So,

$$K(L(b)) = K(L(\sigma_2^{-1}\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1})).$$

The transformed braid is trivial, so K(L(b)) = K(L(e)).

Now, we give another example of the strength of the invariant f. Consider the Burau representation of the virtual braid group VB(n), see, e.g. [Ver], more precisely, the representation of VB(2), generated by two matrices:

$$R(\sigma_1) = \begin{pmatrix} 1-t & t \\ 0 & 1 \end{pmatrix}, \ R(\zeta_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is easy to see that the matrix $R(\sigma_1)$ has the following eigenvalues: 1 and -t. More precisely,

$$CR(\sigma_1)C^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1-t \end{pmatrix}$$

for

$$C = \left(\begin{array}{cc} 0 & 1 \\ 1 & -1 \end{array}\right).$$

In this case,

$$CR(\zeta_1)C^{-1} = \left(\begin{array}{cc} 1 & t-1 \\ 0 & 1 \end{array} \right).$$

Now, let us write simply: ζ instead of $CR(\zeta_1)C^{-1}$ and σ instead of $CR(\sigma_1)C^{-1}$.

Thus we have: $F(k,l,m) = \sigma^k \zeta \sigma^l \zeta \sigma^m \zeta$ is an upper–triangular matrix with 1 and -1 on the main diagonal if k+l+m=0. Assume k=2, l=-1, m=-1. Then $F(2,-1,-1)^2=e$.

It is easy to check that for the non-trivial virtual braid $b = (\sigma_1^2 \zeta_1 \sigma_1^{-1} \zeta_1 \sigma_1^{-1} \zeta_1)^2$ we have $f(b) \neq f(e)$.

In this sense, the invariant f is stronger than the Burau representation even for the case of two strands.

21.3.4 How strong is the invariant f?

As we have shown above, the new invariant is stronger than link coefficients, sometimes it recognises virtual braids, which cannot be recognised by the Jones–Kauffman polynomial or by the Burau representation.

Besides this, the restriction of the invariant f for the case of classical braids (also denoted by f) coincides with the complete classical braid group invariant, described in Chapter 8.

The invariant f gives us an example of a map from one algebraic object (braid group) to another algebraic object (n copies of a free group or n residue classes in free groups). However, this map is not homomorphic.

Thus, in order to understand the strength of the invariant f, we are going to establish some properties of this map.

Fortunately, there are some properties that make f similar to a homomorphic map. Namely, the following lemma holds.

Lemma 21.1. If $f(b_1) = f(b_2)$ for some braids b_1, b_2 then for any two braids a and c we have $f(ab_1c) = f(ab_2c)$ (all braids are taken to have the same number of strands).

Proof. We shall prove the lemma in two steps. The first step is to prove that $f(ab_1) = f(ab_2)$. The second step is to prove that if $f(a_1) = f(a_2)$ then $f(a_1c) = f(a_2c)$. If we prove both statements, then, substituting ab_1 for a_1 and ab_2 for a_2 , we obtain the statement of the theorem.

The second step is obvious, and it was already proved while proving Theorem 21.3 as the A-statement.

So, let us prove that if $f(b_1) = f(b_2)$ then $f(ab_1) = f(ab_2)$. Let us consider some words α , β_1 , and β_2 representing the braids a, b_1 , and b_2 , respectively. We are going to apply the induction method on the length of α . If α has length zero, then a = e, and then $f(ab_1) = f(ab_2)$ by the main assumption.

Now, let us consider the case when α has length one, i.e., it is just a letter.

Suppose $\alpha = \zeta_i$. Then instead of the system of generators (a_1, \ldots, a_n, t) of the group G we can consider the system $a_1, \ldots, a_{i-1}, ta_{i+1}t^{-1}, t^{-1}a_it, a_{i+2}, \ldots, a_n$. Obviously, these n generators are independent and they generate the same group G. Denote $P(\beta_j)$ by P_1^j, \ldots, P_n^j for j = 1, 2 (P without upper index concerns β without lower index).

Obviously, each P_j^i depends on "the old generators": $P_j^i = P_j^i(a_1, \ldots, a_n, t)$. Now let us, for a given function $X(a_1, \ldots, a_n)$, define X' as the value $X(a_1, \ldots, a_{i-1}, ta_{i+1}t^{-1}, t^{-1}a_it, a_{i+2}, \ldots, a_n$, i.e., just by substituting the new generators for the old ones. Now, we state that $\forall \beta \ f(\alpha\beta) = (P_1', \ldots, P_{i-1}', tP_i', t^{-1}P_{i+1}', \ldots, P_n')$.

This can be easily checked by using the induction method on the length of β . But here the set of P'_j depends only on the set of P_j and can be uniquely restored from it (strictly speaking, one should also check that multiplying some P_j by a^l_j on the left side, all residue classes of P_k stay the same, but this can be checked straightforwardly). This shows that the map $f(\beta) \to f(\alpha\beta)$ is well–defined and injective.

Thus, if $f(\beta_1) = f(\beta_2)$ then $f(\alpha\beta_1) = f(\alpha\beta_2)$.

The same reasons are true in the cases when $\alpha = \sigma_i^{\pm 1}$. Here we just indicate the way of transforming the P_i 's.

In the case of $\alpha = \sigma_i$ the generators are: $a_1, \ldots, a_{i-1}, a_i a_{i+1} a_i^{-1}, a_i, a_{i+2}, \ldots, a_n,$ $f(\alpha\beta) = (P'_1, \ldots, P'_{i-1}, P'_i, a_i^{-1} P'_{i+1}, \ldots, P'_n)$ (here and later by P'_i are meant the result of substituting the new generators for the old ones).

In the case of $\alpha = \sigma_i^{-1}$ the generators are: $a_1, \ldots, a_{i-1}, a_{i+1}, a_{i+1}^{-1} a_i a_{i+1}, a_{i+2}, a_{i+3}, \ldots, a_n, f(\alpha\beta) = (P'_1, \ldots, P'_{i-1}, a_{i+1} P'_i, P'_{i+1}, \ldots, P'_n)$

Thus, in the three cases described above the word $f(\alpha\beta)$ can be uniquely restored from α and $f(\beta)$. Therefore $f(\alpha\beta_1) = f(\alpha\beta_2)$.

So, we have established the induction basis. Suppose the statement is true for any word with length less than k for some given $k \ge 1$. Let α be a word of length k. Then $\alpha = \alpha' \psi$, where ψ is the last letter of α and α' has length k - 1. Let $\beta'_1 = \psi \beta_1, \beta'_2 = \psi \beta_2$. By the induction hypothesis, $f(\beta'_1) = f(\beta'_2)$. Applying again the induction hypothesis, we get $f(\alpha\beta_1) = f(\alpha'\beta'_1) = f(\alpha'\beta'_2) = f(\alpha\beta_2)$.

This completes the proof of the first step and the lemma. Combining it with the second step (already proved), we obtain the desired result. \Box

Corollary 21.1. If for some braid a we have f(a) = f(e) = (e, ..., e) then for any braid b: $f(b^{-1}ab) = f(e)$.

The next step is now to describe all possible values of the invariant f. In the general case this problem is very difficult; we restrict ourselves only to the case of n=2 strands. We shall consider an even simpler problem, concerning a simpler

Notation change: instead of generators a_1, a_2 we shall write a, b; instead of σ_1, ζ_1 we write simply σ, ζ .

Let \mathcal{F} be an invariant, obtained from f by putting t=1. Denote the free group with generators a, b by G', and let $E'_1 = \{a\} \setminus G, E'_2 = \{b\} \setminus G$.

In the case of two strands, \mathcal{F} is a map from VB(2) to (E'_1, E'_2) or, simply, to (G',G').

For a braid α , denote $\mathcal{F}(\alpha)$ by $(P(\alpha), Q(\alpha))$.

First, let us consider some examples of virtual two-strand braid words and values of \mathcal{F} on them:

- 1. For the trivial word we have (e, e);
- 2. For σ we have $(1, a^{-1})$;
- 3. For σ^{-1} we have (b, 1):
- 4. For ζ we have t, t^{-1} .

It is not difficult to prove the following:

Theorem 21.7. Let β be a braid word. Then $P(\beta)Q(\beta)^{-1} = a^k b^l$ for some k, l.

Proof. We shall use the induction method on the number of crossings. For zero crossings there is nothing to prove. Now, let β be a braid with n crossings, $\beta' = \beta \alpha$, whence $\alpha = \zeta$, σ or σ^{-1} . Let $P(\beta)Q(\beta)^{-1} = a^n b^m$.

For $\alpha = \zeta$ we have $P(\beta') = P(\beta)t$, $Q(\beta') = Q(\beta)t^{-1}$, thus $P(\beta')\overline{Q}(\beta')^{-1} = a^n b^m$. For $\alpha = \sigma$, the word β is even, and we have: $P \mapsto P, Q \mapsto QP^{-1}a^{-1}P, PQ^{-1} \mapsto a^{-1}PQ^{-1}$; for odd β : $Q \mapsto Q, P \mapsto PQ^{-1}b^{-1}Q, PQ^{-1} \mapsto PQ^{-1}b^{-1}$. For $\alpha = \sigma^{-1}$, β is even: $Q \mapsto Q, P \mapsto PQ^{-1}bQ, PQ^{-1} \mapsto PQ^{-1}b$, for odd β :

 $P \mapsto P, Q \mapsto QP^{-1}aP, PQ^{-1} \mapsto aPQ^{-1}.$

Thus, we have made the induction step that completes the proof of the theorem.

The condition on PQ^{-1} is, indeed, quite natural. It means that $\exists g \in G' : g \in [P] \in E'_1$ and $g \in [Q] \in E'_2$. Obviously, this element g is unique. Thus, g can be considered as an invariant of the group VB(2).

Indeed, the situation in the group VB(2) is quite simple.

Obviously, for any braid b we have $\mathcal{F}(b) = \mathcal{F}(b\zeta)$. Besides, for each even virtual braid b in VB(2) there exist the unique braid $b\zeta$, corresponding to it. Thus, it is actual to consider only the even subgroup EVB(2) of the group VB(2).

Theorem 21.8. The invariant g (as well as the invariant \mathcal{F}) of the virtual braid group EVB(2) is complete.

It suffices to prove that g is complete. To prove this theorem, we shall need an auxiliary lemma.

Lemma 21.2. For any even two-strand braid words π , ρ we have $g(\pi \rho) = g(\rho)g(\pi)$ and $g(\pi)^{-1} = g(\pi^{-1})$, thus g is an antihomomorphism.

Proof. First, let us note that the group VB(2) is a free group with two generators $\alpha = \zeta \sigma$ and $\beta = \zeta \sigma^{-1}$.

It can easily be checked that g(e)=e, $g(\alpha)=a,$ $g(\beta)=b^{-1},$ $g(\alpha^{-1})=a^{-1},$ $g(\beta^{-1})=b.$

It can also be checked straightforwardly that $\forall \rho : g(\rho \alpha) = g(\alpha) f(\rho), g(\rho \beta) = g(\beta)g(\rho), g(\rho \alpha^{-1}) = g(\alpha^{-1})g(\rho), g(\rho \beta^{-1}) = g(\beta^{-1})g(\rho).$

Let us first prove that $g(\pi \rho) = g(\rho)g(\pi)$. We shall do it by using the induction method on the length of ρ (by "length" we mean here the minimal number of $\alpha, \beta, \alpha^{-1}$ and β^{-1} in the decomposition of ρ). For $\rho = e$ there is nothing to prove.

For the word ρ having length one, it can be checked straightforwardly that:

$$\forall \alpha : g(\rho \alpha) = g(\alpha)g(\rho), g(\rho \beta) = g(\beta)g(\rho),$$

$$g(\rho\alpha^{-1})=g(\alpha^{-1})g(\rho), g(\rho\beta^{-1})=g(\beta^{-1})g(\rho).$$

Now, assume that the word ρ has length k+1>1, and for each word ρ' having length $\leq k$ we have $g(\pi\rho')=g(\rho')g(\pi)$. So, let $\rho=\rho_1\rho_2$, where ρ_1 has length 1 and ρ_2 has length k.

Then,

$$g(\pi \rho) = g(\pi \rho_1 \rho_2) = g((\pi \rho_1) \rho_2)$$

by the induction hypothesis for ρ_2

$$= q(\rho_2)q(\pi\rho_1) =$$

again by the induction hypothesis for ρ_1

$$= g(\rho_2)g(\rho_1)g(\pi) =$$

and again by induction hypothesis for ρ_1

$$= g(\rho_1 \rho_2)g(\pi) = g(\rho)g(p),$$

Q.E.D.

Now, since g(e) = e, we have:

$$e = g = e = g(\rho \rho^{-1}) = g(\rho)^{-1}g(\rho),$$

and thus we obtain the second statement of the lemma.

Proof of the Theorem. The lemma shows that g is an antihomomorphic map mapping the free group EVB(2) to the free group with generators a, b. This map maps generators α, β to generators a, b^{-1} . Thus, it has no kernel. So, g is a complete invariant of EVB(2), and so is \mathcal{F} .

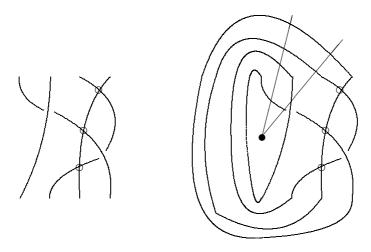


Figure 21.3. Closure of a virtual braid

Certainly, f is a complete invariant of the group EVB(2) too. Besides, this invariant "feels" multiplication by ζ on the right side, thus f recognises all elements of VB(2) as well. In order to recognise whether a pair of elements $(e_1 \in E_1, e_1 \in E_1)$ is a value of the invariant f on some braid, we just factorise them by t, take the pre-image b of the obtained couple (e'_1, e'_2) under \mathcal{F} , and see whether $f(b) = (e_1, e_2)$ or $f(b\zeta) = (e_1, e_2)$.

Certainly, the group VB(2) is simple to recognise: it is just a free product of **Z** (generator σ) and **Z**₂ (generator ζ).

So, the simplest example of $(e_1 \in E_1, e_2 \in E_2)$ that is not a value of f on a virtual braid is (b, a). In this case $PQ^{-1} = ba^{-1}$ which is not equal to $a^k b^l$ for any integer numbers k, l.

21.4 Virtual links as closures of virtual braids

Analogously to classical braids, virtual braids admit *closures* as well, see Fig. 21.3. The obtained virtual link diagram will be braided with respect to some point A.

The closure of a virtual braid is a virtual link diagram. Obviously, isotopic virtual braids generate isotopic virtual links. Furthermore, all virtual link isotopy classes can be represented by closures of virtual braids.

Exercise 21.1. Construct the analogues of the Alexander and Vogel algorithms.

21.5 An analogue of Markov's theorem

In [Kam], Seiichi Kamada proved an analogue of Markov's theorem for the case of virtual braids. Namely, he proved the following.

Theorem 21.9. Two virtual braid diagrams have equivalent (isotopic) closures as

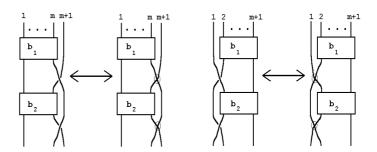


Figure 21.4. The virtual exchange move

virtual links if and only if they are related by a finite sequence of the following moves (VM0)-(VM3).

(VM0) braid equivalence;

(VM1) a conjugation (in the virtual braid group);

(VM2) a right stabilisation (adding a strand with additional positive, negative or virtual crossing) and its inverse operation;

(VM3) a right/left virtual exchange move, see Fig. 21.4.

The moves (VM0)–(VM2) are analogous to those in the classical case. The "new" move has two variants: the right one and the left one.

The necessity of the moves listed above is obvious; it is left for the reader as a simple exercise. For sufficiency, we refer the reader to the original work [Kam].

Part VI Other theories

Chapter 22

3-Manifolds and knots in 3-manifolds

In the present chapter, we shall give an introduction to the theory of three–manifolds. The aim of this chapter is to show how different branches of low–dimensional topology can interact and give very strong results. In the first part, we shall describe a sympathetic theory of knots in $\mathbb{R}P^3$. The second part will be devoted to the deep construction of Witten invariants of three–manifolds. We are also going to describe some fundamental constructions of three–dimensional topology, such as the Heegaard decomposition and the Kirby moves. On one hand, they are necessary for constructing the Witten theory; on the other hand, they have their own remarkable interest. The theory of Witten invariants, which is based on the Kauffman bracket on one hand and the Kirby theory on the other, is very deep. In fact, it leads to the theory of invariants of knots in three–manifolds. Many proofs will be sketched or omitted (e.g. the Kirby theorem). For the study of three-manifolds, we recommend Matveev's book [Matv4].

22.1 Knots in $\mathbb{R}P^3$

Here we are going to present a method of encoding knots and links in $\mathbb{R}P^3$ and a generalisation of the Jones polynomial for the case of $\mathbb{R}P^3$ proposed by Yu.V. Drobotukhina, see [Dro].

The projective space $\mathbb{R}P^3$ can be defined as the sphere S^3 with identified opposite points. The sphere S^3 consists of two half–spheres; each of them is homeomorphic to the ball D^3 . Thus, the space $\mathbb{R}P^3$ can be obtained by identifying the opposite points of the boundary $S^2 = \partial D^3$.

Consequently, any link in $\mathbb{R}P^3$ can be defined as a set of closed curves and arcs in D^3 such that the set of endpoints of arcs lies in ∂D^3 ; and this set is centrosymmetrical in D^3 . Without loss of generality, one can think of these points as lying on the "equator" of the ball. Thus, a link in $\mathbb{R}P^3$ can be represented by a diagram in D^2 ; the points of these diagrams lying at the boundary $S^1 = \partial D^2$, have to be centrosymmetrical.

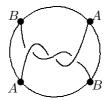


Figure 22.1. A diagram of a link in $\mathbb{R}P^3$

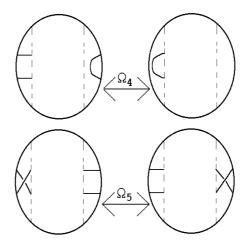


Figure 22.2. Moves $\Omega_{4,5}$

An example of such a diagram is shown in Fig. 22.1.

Exercise 22.1. Consider an arbitrary link diagram in $\mathbb{R}P^3$. Describe the pre-image of this diagram with respect to the natural covering $p: S^3 \to \mathbb{R}P^3$.

We shall consider only diagrams for $\mathbf{R}P^3$ in a general position. This means that the edges of the diagram are smooth, all crossings are double and transverse, and no crossings are available at the boundary of the circle, and no branch of the diagram is tangent to the circle.

We recall that the singularity conditions for the case of links in \mathbb{R}^3 (or S^3) lead to the moves Ω_2 , Ω_3 . Analogously, two new moves for diagrams of links in $\mathbb{R}P^3$ generate two new moves called Ω_4 and Ω_5 , see Fig. 22.2.

Theorem 22.1. Two link diagrams generate isotopic links in $\mathbb{R}P^3$ if and only if one can be transformed to the other by means of isotopies of D^2 , classical Reidemeister moves $\Omega_1, \Omega_2, \Omega_3$, and moves Ω_4 Ω_5 .

Proof. The proof of this theorem is quite analogous to that of the classical Reidemeister theorem. It is left to the reader as an exercise. \Box

Now we are ready to define the analogue of the Jones–Kauffman polynomial for the case of oriented links in $\mathbb{R}P^3$.

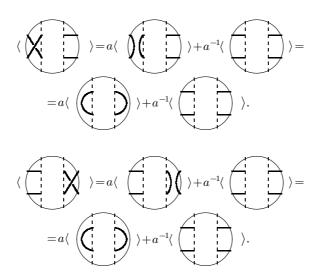


Figure 22.3. Invariance of bracket under Ω_5

First, for a given diagram of an unoriented link $L = \bigotimes$ in $\mathbb{R}P^3$, let us define an analogue of the Kauffman bracket satisfying the following axioms:

$$\langle L \rangle = a \langle \langle \rangle \rangle + a^{-1} \langle \langle \rangle \langle \rangle \rangle; \tag{1}$$

$$\langle L \sqcup \bigcirc \rangle = (-a^2 - a^{-2})\langle L \rangle;$$
 (2)

$$\langle \bigcirc \rangle = 1, \tag{3}$$

where () means a separated unknotted circle.

The proof of the existence and uniqueness of $\langle L \rangle$ is completely analogous to that for the Kauffman bracket defined on links in \mathbf{R}^3 (or S^3).

Also, we can analogously prove the invariance of $\langle L \rangle$ under Ω_2, Ω_3 .

To prove the invariance of $\langle L \rangle$ under Ω_4 , let us use the following formula

$$\langle D \rangle = \sum_{s} a^{\alpha(s) - \beta(s)} (-a^2 - a^{-2})^{\gamma(s) - 1},$$
 (4),

where $\gamma(s)$ is defined as the number of circles of L in the state s.

Actually, for each state s of the diagram L, the numbers $\alpha(s)$, $\beta(s)$ and $\gamma(s)$ are invariant under Ω_4 .

The invariance of the bracket $\langle L \rangle$ under Ω_5 follows from the equalities shown in Fig. 22.3.

As in the case of links in \mathbb{R}^3 , the bracket $\langle L \rangle$ is not invariant under Ω_1 ; this move multiplies the bracket by $(-a)^{\pm 3}$.

Now, for a diagram of an oriented link L, let us define w(L) as in the usual case.

Definition 22.1. The *Drobotukhina polynomial* of the oriented link L in $\mathbb{R}P^3$ is defined as

$$X(L) = (-a)^{-3w(L)} \langle |L| \rangle,$$

where |L| is obtained from L by "forgetting" the orientation.

Theorem 22.2. The polynomial X(L) is an invariant of oriented links in $\mathbb{R}P^3$.

Proof. The proof is quite analogous to the invariance proof of the Jones–Kauffman polynomial for links in S^3 . It follows immediately from Theorem 22.1, the definition, invariance of the bracket $\langle \cdot \rangle$ under the moves $\Omega_2 - \Omega_5$ and its behaviour under Ω_1 coordinated with the behaviour of $w(\cdot)$.

By using this polynomial, Yu.V. Drobotukhina classified all links in $\mathbb{R}P^3$ having diagrams with no more than six crossings up to isotopy and proved an analogue of the Murasugi theorem for links in $\mathbb{R}P^3$.

22.2 An introduction to the Kirby theory

Kirby theory is a very interesting way for encoding three—manifolds. In the present book, we shall give an introduction to the theory of the Witten invariants via Kirby theory. However, Kirby theory is interesting in itself. It can be used for constructing other invariants of three—manifolds (Witten, Viro, Reshetikhin, Turaev et al.)

22.2.1 The Heegaard theorem

Below, we shall use the result of Moise [Moi] that each three–manifold admits a triangulation.

Definition 22.2. The *Heegaard decomposition* of an orientable closed compact manifold M^3 without boundary is a decomposition of M^3 into the union of some two handle bodies — interiors of two copies of S_g attached to each other according to some homeomorphism of the boundary.

There are different ways to attach a handle body to another handle body (or to its boundary — a sphere with handles). However, it is obvious that the way of attaching each handle body can be characterised by a system of meridians — contractible curves in the handle body.

So, in order to construct a three-manifold, one can take two handle bodies M_1 and M_2 with the same number g of handles and the abstract manifold N that is homeomorphic to ∂M_1 (and hence, to ∂M_2). This manifold N should be endowed with some standard system of meridians m_1, \ldots, m_g (say, it is embedded in \mathbb{R}^3 and meridians are taken to be in the most natural sense), see Fig. 22.4.

After this, we fix two systems of meridians u_1, \ldots, u_g for M_1 and v_1, \ldots, v_g for M_2 and two maps f_1, f_2 mapping $u_i \mapsto m_i$ and $v_i \mapsto m_i$, respectively.

Definition 22.3. The system of curves $\{f_1(u_i)\}$ and $\{f_2(v_i)\}$ is said to be the Heegaard diagram for this attachment.

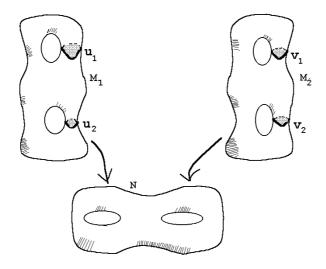


Figure 22.4. A Heegaard diagram

Each three—manifold obtained by such an attachment has some Heegaard diagram. While reconstructing the manifold by a Heegaard diagram, we have some arbitrariness: the images of meridians do not define the map completely.

However, the following theorem takes place.

Theorem 22.3. If two three-dimensional manifolds M and M' have the same Heegaard diagram then they are homeomorphic.

Proof. Without loss of generality, suppose the manifold M consists of M_1 and M_2 and M' consists of M'_1 and $M'_2 = M_2$.

Let $f_i: \partial M_i \to N, i = 1, 2$, be the glueing homeomorphisms for the manifold M; for M' we shall use $f'_i, i = 1, 2$.

Since M and M' have the same Heegaard diagram, then f and f' are homeomorphisms from ∂M and $\partial M'$ to N such that the images of meridians u_i and u_i' coincide for all $i=1,\ldots,g$. Let us show that in this case the identical homeomorphism $h_2:M_2\to M_2'$ can be extended to a homeomorphism $h:M\to M'$. The homeomorphism $(f_1')^{-1}\circ f_1:\partial M_1\to\partial M_1'$ takes each meridian of the manifold M_1 to the corresponding meridian of the manifold M_1' . Since each homeomorphism of one circle to another one can be extended to a homeomorphism of whole discs (along radii), then the homeomorphism of meridians can be extended to the homeomorphism of discs bounded by these meridians in M_1 and M_1' . After cutting M_1 and M_1' along these discs, we obtain manifolds D_1 and D_1' both homeomorphic to the three-ball.

Because each homeomorphism between two 2–spheres can be extended to the homeomorphism between bounded balls (along the radii), then the homeomorphism between boundaries of D_1 and D'_1 can be extended to the homeomorphism between the balls.

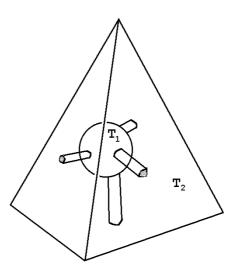


Figure 22.5. Dividing tetrahedra

Thus, we have constructed the homeomorphism between M and M'. This completes the proof of the theorem.

Now we are ready to formulate the Heegaard theorem

Theorem 22.4. Each orientable three–manifold M without boundary admits a Heegaard decomposition.

Below, we give just a sketchy idea of the proof. It is well known (see, e.g. [Moi]) that each three–manifold can be triangulated. So, let us triangulate the manifold M. Now, we construct M_1 and M_2 from parts of tetrahedra representing the constructed triangulation. Each tetrahedron T will be divided into two parts T_1 and T_2 as shown in Fig. 22.5.

Now, the manifold M_1 will be constructed of T_1 's, and M_2 will be constructed of T_2 's. It is easy to see that each of these manifolds is a handle body (the orientability follows immediately from that of M). The number of handles of M_1 and M_2 is the same because they have a common boundary.

Obviously, the only manifold that can be obtained by attaching two spheres is S^3 . The spaces that can be obtained from two tori are S^3 , $\mathbb{R}P^3$, and so-called lens spaces. They are closely connected with toric braids. For their description see [BZ].

22.2.2 Constructing manifolds by framed links

In the present section, we shall give a very sketchy introduction to the basic concepts of Kirby theory — how to encode three—manifolds by means of knots (more precisely, by framed links).

Definition 22.4. A *framed* link is a link in \mathbb{R}^3 to each component of which an integer number is associated.

Each framed link can be represented as a band: for each link component we construct a band in its neighbourhood in such a way that the linking coefficient between the boundaries equals the framing of the component. One boundary component of the band should lie in the toric neighbourhood of the second one and intersect each meridional disc of the latter only once.

Definition 22.5. Two framed links are called *isotopic* if the corresponding bands are isotopic.

Having a framed link L, one can construct a three–manifold as follows. For each component K of L, we consider its framing n(K). Now, we can construct a band: we take a knot K' collinear to K such that the linking coefficient between K and K' equals n. Note that this choice is well defined (up to isotopy): while changing the orientations of K and K' simultaneously, we do not change the linking coefficient.

Thus, we have chosen a curve on (each) link component.

The next step of the construction is the following. We cut all full tori, and then attach new ones such that their meridians are mapped to the selected curves which are called *longitudes*.

Theorem 22.5. For each three–manifold M, there exists a manifold M' homeomorphic to M that can be obtained from a framed link as described above.

The idea of the proof is the following: by using the Heegaard decomposition, we can use handle bodies with arbitrarily many handles; each homeomorphism of the S_g can be considered locally and be reduced to "primitive homeomorphisms"; the latter can be realised by using toric transformations as above. Here one should mention the following important theorem.

Theorem 22.6 (Dehn–Lickorish). Each orientation–preserving homeomorphism of S_g can be represented as a composition of Dehn twistings and homeomorphisms homotopic to the identity.

The first proof of this theorem (with some gaps) was proposed by Dehn in [Dehn2]. The first rigorous proof was found by Lickorish [Lic2].

22.2.3 How to draw bands

The framed link can be easily encoded by planar link diagrams: here the framing is taken to be the self-linking coefficient of the component (it can be set by means of moves Ω_1 ; each such move changes the framing by ± 1). If we want to consider links together with framing, we must admit the following moves on the set of planar diagrams: the moves Ω_2 , Ω_3 , and the double twist move Ω'_1 , see Fig. 22.6.

22.2.4 The Kirby moves

Obviously, different links may encode the same three–manifolds (up to a diffeomorphism). It turns out that there exist two moves that do not change the three–manifold.

The first Kirby move is an addition (removal) of a solitary circle with framing ± 1 , see Fig. 22.7.

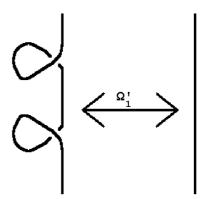


Figure 22.6. The double twist

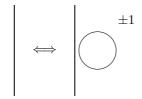


Figure 22.7. The first Kirby move

The second move is shown in Fig. 22.8. Let us be more detailed. While performing this move, we pull one component with framing k along the other one (with framing l that stays the same). The transformed component would have framing k+l.

Remark 22.1. Note that the component might be linked or not.

The planar diagram formalism allows us to describe this move simply without framing numbers.

Theorem 22.7 (The Kirby theorem). Two framed links generate one and the same diagram if one can be transformed to the other by a sequence of Kirby moves and isotopies.

The theorem says that these two moves are necessary and sufficient. The necessity of the first Kirby move is obvious: we cut one full torus off and attach a new torus almost in the same manner. The necessity of the second Kirby move can also be checked straightforwardly: one should just look at what happens in the neighbourhood of the second component and compare the obtained manifolds.

For the sake of convenience, one usually takes another approach: one uses the unique necessary and sufficient Fenn–Rourke move instead of the two Kirby moves. This move is shown in Fig. 22.9.

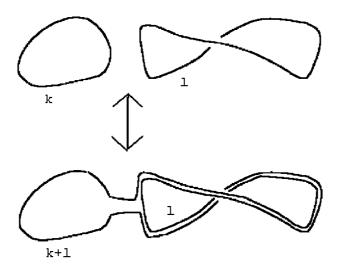


Figure 22.8. The second Kirby move

The Fenn–Rourke theorem states that in order to establish that two three–manifolds given by planar diagrams of framed links are diffeomorphic, it is necessary and sufficient to construct a chain of Fenn–Rourke moves transforming one diagram to the other one.

The necessity can be easily checked as in the case of the Kirby theorem.

The sufficiency can be reduced to Kirby moves. Namely, all Kirby moves can be represented as combinations of Fenn–Rourke moves and vice versa. The proof of this equivalence can be found, e.g., in [PS].

The simplest rigorous proof of the sufficiency of Fenn–Rourke moves can be found in [Lu].

22.3 The Witten invariants

In the present section, we are going to give a sketchy introduction to the famous theory of Witten invariants. The first article by Witten on the subject [Wit2] was devoted to the construction of invariants of links in three—manifolds. However, mathematicians were not completely satisfied by the strictness level of this work. The mathematical foundations of this theory are due to Viro, Turaev and Reshetikhin. Here we follow the work of Lickorish [Lic], where he simplified the work of these three authors for the case of three—manifolds (without links embedded in them).

The basic idea of the description is to use the Kauffman bracket (which is invariant under band isotopies) and apply it to some sophisticated combinations of links in such a way that the obtained result is invariant under the Kirby moves.

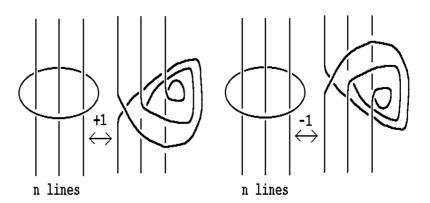


Figure 22.9. The Fenn–Rourke move

22.3.1 The Temperley–Lieb algebra

The Temperley–Lieb algebra is a classical object in operator algebra theory. It has much in common with the Hecke algebra. However, it is realised as a *skein algebra* (the name has come from skein relations). Here we are going to describe how the Kauffman bracket can be used for the construction of the three–manifold invariants via the Temperley–Lieb algebra.

Let M^3 be a manifold represented by means of a framed link in S^3 . We shall use bands on the plane (or, simply, planar diagrams) in order to define the framing. The isotopies of these diagrams are considered up to Ω_2, Ω_3 and the double twist Ω'_1 , shown in Fig. 22.6.

The Temperley–Lieb algebra is a partial case of so–called *skein spaces*. Below, we shall give some examples that will be useful in the future.

To each diagram D of an oriented link, there corresponds the Kauffman bracket $\langle D \rangle$ in the variable a. For the concrete value $a_0 \in \mathbf{C}$ of a, the value of $\langle D \rangle$ can be calculated just as the unnormalised Kauffman bracket evaluated at a_0 (with the condition that $\langle \bigcirc \rangle$ is equal to one and not to $-a_0^2 - a_0^{-2}$. Let V be the linear space over \mathbf{C} that consists of finite linear combinations of the D_i 's. In the space V, consider the subspace V_0 generated by vectors of the type

$$\{(\sum) -a_0(\sum) -a_0^{-1}(\sum), D \sqcup \bigcirc + (a_0^{-2} + a_0^2)D\}.$$

Put $S = V/V_0$. The set S is called the skein space for \mathbf{R}^2 . Under the natural projection $p: V \to V/V_0 = S$, the element D is mapped to λe_1 , where λ is the value of the polynomial $\langle D \rangle$ at a_0 , and $e_1 \in S$ is the image of the diagram consisting of one circle.

In fact, let E_i be the diagram consisting of i pairwise non-intersecting circles. It follows from the construction of $\langle D \rangle$ that $D = \sum \lambda_i E_i + w_1$ and $\sum \lambda_i E_i = \lambda E_1 + w_2$, where $w_1, w_2 \in V_0$, and λ is the value of $\langle D \rangle$ at a_0 .

Now, let e be a non-zero element of S. Then, p(D) = f(D)e, where $f(D) \in \mathbb{C}$. For e, it is convenient to choose the element e_0 , corresponding to the empty diagram.

Such a choice of the basic element will correspond to multiplication of f with respect to the disconnected sum operation.

Exercise 22.2. Prove this fact.

Consequently,

$$f(D_1 \cup D_2) = \sum_{i,j} (c^i)^{\lambda_i} \times (c^j)^{\mu_j}$$
$$= \sum_i (c^i)^{\lambda_i} \times \sum_j (c^j)^{\mu_j} = f(D_1)f(D_2),$$

which completes the proof.

The construction of S admits the following generalisation. Let D be an oriented 2–surface with boundary ∂F (possibly, $\partial F = \emptyset$). Fix 2n points on ∂F (in the case of empty ∂F , n = 0).

By a diagram on F we mean a set (possibly, disconnected or containing circles) Γ on F such that $P \cap F$ consists precisely of 2n chosen points; these points are the only vertices of Γ of valency one; all the other vertices have valency four and are endowed with crossing structures (as in the case of knot diagrams). These diagrams are considered up to isotopy (not changing the combinatorial structure and fixing the endpoints).

Fix $a_0 \in \mathbb{C}$, and consider the vector space V(F, 2n) over \mathbb{C} consisting of finite linear combinations of diagram isotopy classes. Let $V_0(F, 2n)$ be the subset of V(F, 2n), generated by vectors of the type

$$\{(\sum)-a_0(\sum)-a_0^{-1}(\sum),\ D\sqcup\bigcirc+(a_0^{-2}+a_0^2)D\}.$$

Put $S(F, 2n) = V(F, 2n)/V_0(F, 2n)$. For the sake of simplicity, denote S(F, 0) by S(F).

Remark 22.2. It is essential to consider oriented surfaces in order to be able to distinguish crossings \bigotimes and \bigotimes .

Theorem 22.8. The image P(D) of the diagram D under the natural projection on S(F,2n) is invariant under $\Omega'_1,\Omega_2,\Omega_3$.

The proof of this theorem is completely analogous to the invariance proof for the Kauffman bracket. It is left for the reader as a simple exercise.

Theorem 22.9. The basis of S(F,2n) consists of images (under the natural isomorphism) of all isotopy classes of diagrams D_i containing neither crossings nor compressible curves.

The proof is left to the reader.

The following proposition is obvious.

Proposition 22.1. $S(S^2) \cong S(\mathbf{R}^2) \cong \mathbf{C}$.

Proposition 22.2. The space $S(I \times S^1)$ has the natural algebra structure over \mathbb{C} . This algebra is isomorphic to $\mathbb{C}[\alpha]$.

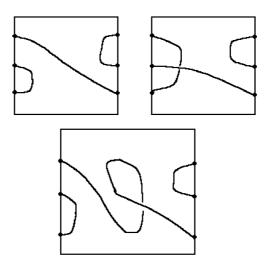


Figure 22.10. Product in the Temperley–Lieb algebra

Proof. By Theorem 22.9, as a basis one can take the family of diagrams where each of these diagrams consists of n circles parallel to the base of the cylinder $I \times S^1$. The multiplication is defined by attaching the lower base of one cylinder to the upper base of the other cylinder and rescaling the height of the cylinder. Let α be the image (under the natural projection) of the diagram consisting of one circle going around the parallel of the cylinder. The remaining part of the proposition is evident.

Now, let us consider the space $S(D^2, 2n)$. By Theorem 22.9, the basis of this space consists of pairwise non-intersecting arcs with endpoints on ∂D .

The number c_n is the n-th Catalan number. These numbers have the following properties:

1.
$$c_{n+1} = \sum_{i=0}^{n} c_i c_{n-i};$$

2.
$$c_n = \frac{1}{n+1}C_{2n}^n$$
.

Exercise 22.3. Prove these two statements.

The space $S(D^2, 2n)$ has an algebra structure, however, this structure is not canonical. To define this structure, one should consider the disc D^2 with fixed 2n points as a square having n fixed points on one side and the other points on the other side. After this, the multiplication is obtained just by attaching one square to the other and rescaling, see Fig. 22.10.

Definition 22.6. The algebra defined above is called the *Temperley–Lieb algebra*.

Notation: TL_n .

Let e_i be the element of TL_n corresponding to the diagram shown in Fig. 22.11.

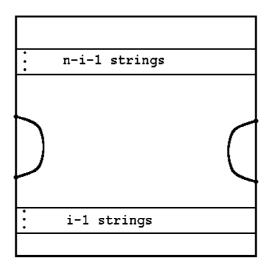


Figure 22.11. The element e_i

Then, let 1_n be the element of TL_n corresponding to the diagram with n parallel arcs

Theorem 22.10. The element 1_n is the unity of TL_n . The elements e_1, \ldots, e_{n-1} represent a multiplicative basis of the algebra TL_n .

The proof is evident.

It is also easy to see that the following set of relations is a sufficient generating set for TL_n :

1.
$$e_i^2 = e^i(-a_0^2 - a_0^{-2});$$

2. $e_i e_{i+1} e_i = e_i$.

The proof is left for the reader.

22.3.2 The Jones–Wenzl idempotent

In order to go on, one should introduce an idempotent element $f^{(n)}$ in the Temperley-Lieb algebra TL_n that is called the *Jones-Wenzl idempotent*.

Let A_n be the subalgebra in TL_n generated by e_1, \ldots, e_n (without 1_n). Then the element $f^{(n)}$ is defined by the following relations:

$$f^{(n)}A_n = A_n f^{(n)} = 0; (5)$$

$$1_n - f^{(n)} \in A_n. \tag{6}$$

Such an element exists only if we make some restrictions for $a_0 \in \mathbb{C}$.

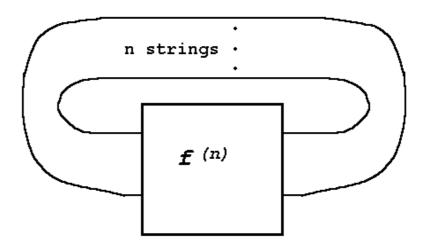


Figure 22.12. Closure of $f^{(n)}$

Theorem 22.11. Let for each k = 1, ..., n - 1, $a_0^{4k} \neq 1$. Then there exists a unique element $f^{(n)} \in TL_n$ satisfying the properties described above. This element is idempotent: $f^{(n)} \cdot f^{(n)} = f^{(n)}$.

Exercise 22.4. Calculate $f^{(2)}$.

Note that the *existence* of $f^{(n)}$ yields uniqueness and idempotence. Namely, the uniqueness means the uniqueness of the unit element $1_n - f^{(n)}$ in A_n . Moreover, since $1_n - f^{(n)}$ is a unit element of A_n , we have $(1 - f^{(n)})(1 - f^{(n)}) = (1 - f^{(n)})$. From this equation we easily deduce the idempotence of $f^{(n)}$.

The construction of $f^{(n)}$ uses induction of n.

We shall need one more construction and some properties of it. Define the element $\Delta_n \in S(\mathbf{R}^2)$ as the *closure* of $f^{(n)}$, see Fig. 22.12. More precisely, $f^{(n)}$ is a linear combination of some diagrams; one should take their closures as shown in Fig. 22.12 and then take the corresponding linear combination.

Analogously, one defines the element $S_n(\alpha) \in S(S^1 \times I)$: this is a "closure" of $f^{(n)}$ in the ring, see Fig. 22.13. Here α denotes the generator of the polynomial ring $S(S^1 \times I)$. Obviously, Δ_n is obtained from S_n by natural projecting $S(S^1 \times I) \to S(\mathbf{R}^2)$ (glueing the interior circle by a disc).

The elements Δ_n can be defined inductively as $\Delta_{n+1}=(-a_0^{-2}-a_0^2)\Delta_n-\Delta_{n-1}$. Obviously, $\Delta_1=-a_0^2-a_0^{-2}$. Besides, if $a_0^{4(n+1)}\neq 1$ then $\Delta_n\neq 0$; more precisely, $\Delta_n=\frac{(-1)^n(a_0^{2(n+1)}-a_0^{-2(n+1)})}{a_0^2-a_0^{-2}}$

We shall only give a concrete inductive formula for $f^{(n)}$:

$$f^{(n+1)} = f_1^{(n)} - \frac{\Delta_{n-1}}{\Delta_n} f_1^n e_n f_1^{(n)},$$

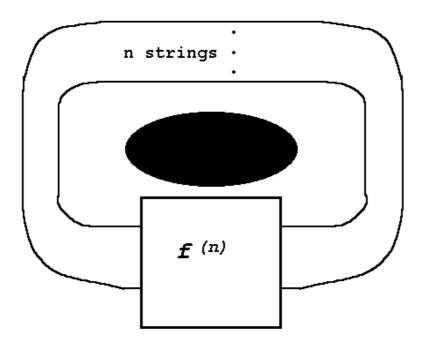


Figure 22.13. The element $S_n(\alpha)$

where $f_1^{(n)}$ means the element of A_n obtained from $f^{(1)}$ (all summands with coefficients) by adding (n-1) horizontal lines.

The elements $S_n(\alpha)$ can be constructed inductively by using the following formula:

$$S_{n+1}(\alpha) = \alpha S_n - S_{n-1}.$$

22.3.3 The main construction

Now, we are ready to define the Witten invariant of three–dimensional manifolds. Associate to each link diagram D with components K_1, \ldots, K_n a polylinear map

$$\langle \cdot, \cdots, \cdot \rangle_D : S_1 \times \cdots \times S_n \to S(\mathbf{R}^2),$$

where $S_i \cong S(I \times S^1)$. In order to define this map, it is sufficient to define the elements $(\alpha_1^k, \ldots, \alpha_n^{k_n})_D \in S(\mathbf{R}^2)$, where α_i is the generator of S_i corresponding to the generator α of the algebra $S(I \times S^1)$.

We deal with planar diagrams of unoriented links.

In order to obtain the diagram (and, finally, the number) corresponding to the element $\langle \alpha_1^{k_1}, \dots, \alpha_n^{k_n} \rangle_D$, we consider the knots B_i corresponding to K_i , and on each knot B_i , we draw k_i copies of the non-intersecting curves parallel to its boundary.

Suppose framed diagrams D and D' are equivalent by means of $\Omega'_1, \Omega_2, \Omega_3$. Then the diagrams $\langle \alpha_1^{k_1}, \ldots, \alpha_n^{k_n} \rangle_D$ and $\langle \alpha_1^{k_1}, \ldots, \alpha_n^{k_n} \rangle_{D'}$ can also be obtained from each other by means of Ω'_1 and Ω_2, Ω_3 . Thus, the images of these two diagrams in $S(\mathbf{R}^2)$ coincide

So, the polylinear map constructed above is a framed link invariant.

Now, we are going to construct the further invariant by using $\langle \cdot, \cdots, \cdot \rangle$. But since we are constructing an invariant of three–manifolds, we should also care about the Kirby moves.

To obtain the invariance under the second Kirby move, we shall use the element

$$\omega = \sum_{n=0}^{r-2} \Delta_n S_n(\alpha) \in S(I \times S^1),$$

where $r \geq 3$ is an integer.

Theorem 22.12. Suppose a_0 is such that a_0^4 is the primitive root of unity of degree r. Suppose diagrams D and D' are obtained from each other by the second Kirby move. Then we have:

$$\langle \omega, \omega, \dots, \omega \rangle_D = \langle \omega, \omega, \dots, \omega \rangle_{D'}.$$

The main idea is the following: while performing the second Kirby move, the difference between the obtained elements contains a linear combination of only those members of $S(S^1 \times I)$ containing $f^{(r-1)}$ as a subdiagram. Since $a_0^{4r} = 1$, we have $\Delta_{r-1} = 0$ and all these members vanish.

Now, let us consider the first Kirby move. We shall need the $linking\ coefficient\ matrix.$

For any n-component link diagram L, one can construct a symmetric $(n \times n)$ matrix of linking coefficients (previously, we orient all link components somehow).
For the diagonal elements, we take the self-linking coefficient. Denote the obtained
matrix by B. Since B is symmetric, all eigenvalues of this matrix are real. Let b_+ be the number of positive eigenvalues of B, and b_- be the number of negative
eigenvalues of B. For constructing the link invariants, we shall need not the matrix B itself, but only b_+, b_- .

It is easy to see that the change of the orientation for some components of B leads to a transformation $B \to B' = X^T B X$ for some orthogonal X, thus it does not change b_+ and b_- .

Let us see now that b_+ and b_- are invariant under the second Kirby move. To do this, it is sufficient to prove the following theorem.

Theorem 22.13. Under the second Kirby move, the matrix B is changed as follows: $B' = X^T B X$, where X is some non-degenerate real matrix.

This Theorem is left to the reader as an exercise.

Consider the three standard framed diagrams U_+, U_-, U_0 shown in Fig. 22.14. They represent the unknotted curves with framings 1,0, and -1, respectively. In the case when $\langle \omega \rangle_{U_+} \langle \omega \rangle_{U_-} \neq 0$, for each diagram D, one can consider the following complex number:

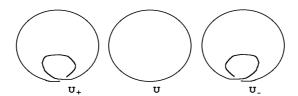


Figure 22.14. Diagrams U_+, U_0, U_-

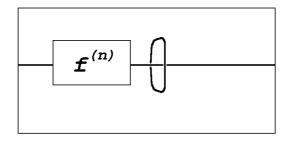


Figure 22.15. The diagram D_n

$$I(D) = \langle \omega, \dots, \omega \rangle_D \langle \omega \rangle_{U_+}^{-b_+} \langle \omega \rangle_U^{-b_-}.$$

Proposition 22.3. The complex number I(D) is a topological invariant of the three–manifold defined by D if $\langle \omega \rangle_{U_{+}} \langle \omega \rangle_{U_{-}} \neq 0$.

To satisfy those conditions, one should make the following restrictions on a_0 . Namely, the conditions hold if a_0 is a primitive root of unity of degree 4r or a primitive root of unity of degree 2r for odd r. In both cases, a_0^4 is a primitive root of unity (so that it satisfies the condition of Theorem 22.12).

The proof of Proposition 22.3 is very complicated and consists of many steps. Here, we are going to give the only step that occurs in this proof.

Lemma 22.1. Suppose a_0 is a complex number as above. Then we have:

- 1. for $1 \le n \le r 3$ each diagram containing D_n (see Fig. 22.15) equals zero;
- 2. for n = r 2 in the case when a_0 is a primitive root of degree 4r, each diagram containing D_n , equals zero as well, and in the other case (degree 2r for odd r) the diagram D_n can be replaced with $\langle \omega \rangle_U f^{(n)}$.

(Here by "diagrams" we mean their images in $S(\mathbf{R}^2)$.)

Collecting the results of Theorems 22.12 and 22.13 and Proposition 22.3, we obtain the main theorem.

Theorem 22.14. If $a_0 \in \mathbb{C}$ is either the primitive root of unity of degree 4r or the primitive root of unity of degree 2r for odd r and $r \geq 3$ then

$$W(M^3) = I(D) = \langle \omega, \dots, \omega \rangle_D \langle \omega \rangle_{U_+}^{-b_+} \langle \omega \rangle_{U_-}^{-b_-}$$

is a topological invariant of any compact three-manifold without boundary (here D is a framed link diagram representing M^3).

This invariant is called the Witten invariant of three-manifolds.

Examples 22.1. The sphere S^3 can be represented by the empty diagram; thus $I(S^3) = 1$.

The manifold $S^1 \times S^2$ is represented by the diagram U. The linking coefficient matrix for U is $B = (b_{11})$, where $b_{11} = 0$. Thus, $b_+ = b_- = 0$. Consequently, $I(S^1 \times S^2) = \langle \omega \rangle_{\ell} U$.

22.4 Invariants of links in three–manifolds

The construction of invariants of three—manifolds proposed in the present chapter can also be used for more sophisticated objects, knots in three—manifolds. We shall describe these invariants following [PS].

Consider a compact orientable manifold M^3 without boundary. The manifold M^3 can be obtained by reconstructing S^3 along a framed link. This means that there exists a homeomorphism

$$f: S^3 \backslash L_M \to M^3 \backslash \tilde{L}_M$$

where \tilde{L}_M is a link in M^3 . Without loss of generality, we can assume that the link \tilde{L}_M does not intersect the given link L (this can be done by a small perturbation). Let $L_L = f^{-1}(L)$ be the pre-image of $L \subset M^3$ in the sphere S^3 . Suppose if L is framed then L_L is framed as well. Thus, a framed link L in M^3 can be given by a pair of framed links (L_L, L_M) in the sphere S^3 ; the number of components of L_L coincides with that of L.

Let us discuss the following question: when do two couples (L_L, L_M) and (L'_L, L'_M) generate the same framed link L in M^3 ? The reconstructions of the sphere along framed links L_M and L'_M must generate the same manifold M^3 ; thus, L'_M can be obtained from L_M by Kirby's moves and isotopies. Moreover, during the isotopy, the link L_M should not intersect L_L . The point is that the isotopy that takes L_M to L'_M induces the homeomorphism

$$f: S^3 \backslash L_M \to M^3 \backslash \tilde{L}_M$$
.

It is clear that in a general position the Kirby move does not touch a small neighbourhood of the link L_L . After we have performed the necessary Kirby moves and isotopies, we may assume $L'_M = L_M$. After this, we can apply Kirby moves and isotopies to $L_L \cup L_M$, whence the link L_L can only be isotoped. Thus, we have to clarify the connection between L_L and L'_L in the case when $L_M = L'_M$. In this case, after reconstructing S^3 along L_M , we obtain a manifold M^3 , where the images of the links L_L and L'_L are isotopic. During this isotopy, the images of L_L and L'_L may intersect the image of L_M . Thus, the desired reconstruction is not reduced to the isotopy of L_L and L_M in the sphere S^3 . More precisely, if in M^3 , a component K_K of the image of K_L intersects a component K_M of the image of L_M , then the sphere S^3 undergoes the second Kirby move: namely, a band parallel to K_M is added to the band K_L .

Let us point out the following two important circumstances:

- 1. The numbers of components of L_L and L'_L are equal;
- 2. Under second Kirby moves, we never add bands which are parallel to components of L.

Now, with the framed link $L_L \cup L_M$ we can associate a framed diagram $D_L \cup D_M$. For the link L_M , consider the matrix B of link coefficients. Let b_+ and b_- be the numbers of positive and negative eigenvalues of this matrix; let n be the number of components of L. Let us fix the elements

$$p_1(\alpha), \ldots, p_n(\alpha) \in S(I \times S^1) \cong \mathbf{C}[\alpha]$$

(here we use the previous notation).

Theorem 22.15. Let $r \geq 3$ be an integer, a be a complex number and a_0 be such that: either a_0 is the primitive root of unity of degree 4r or r is odd and a_0 is the primitive root of unity of degree 2r.

Then the complex number

$$W(M^3, L) = \langle p_1(\alpha), \dots, p_m(\alpha), \omega, \dots \omega \rangle_{D_L \sqcup D_M} \langle \omega \rangle_{U_+}^{-b_-} \langle \omega \rangle_{U_-}^{-b_-}$$
 (7)

is an isotopy invariant of the framed link L in M^3 .

Proof. First, note that n is invariant: it is the number of components of L. The proof of invariance of the number (7) under admissible transformations of the link $L_L \cup L_M$ is almost the same as the proof of invariance of Witten's invariant. The only additional argument is the following. While performing second Kirby moves, one should never add bands parallel to L_L . This allows us to use alternative marking: they can be marked as $p_1(\alpha), \ldots p_n(\alpha) \in S(I \times S^1)$, not only by ω .

Remark 22.3. The formula (7) defines not a unique invariant, but an infinite series of invariants with the following parameters: $\alpha_0, p_1, p_2, \ldots, p_n$; the number α_0 should satisfy the conditions above.

22.5 Virtual 3-manifolds and their invariants

Just recently, L. Kauffman and H. Dye [DK, Kau3], see also H. Dye's thesis [Dye] generalised the Kirby theory for the case of virtual knots and constructed so–called "virtual three–manifolds" and generalised the invariants described here for the virtual case.

The main idea is that all the constructions describes above generalise straightforwardly.

In order to define a virtual three–manifold, one considers virtual link diagrams (just as in the classical case). After this, one allows the following equivalence between them: two diagrams are called *equivalent* if one can be obtained from the other by a sequence of generalised Reidemeister moves (all but the first classical one, Ω_1) and the two Kirby move. As for the first Kirby move, the situation is quite clear: we are just adding a new circle with framing one.

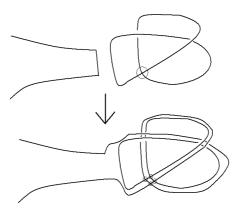


Figure 22.16. Virtual handle slide move

As for the second Kirby move, the only thing we should do accurately is to define it for the component having virtual crossing. This move is called *the virtual handle slide move*; it is shown in Fig. 22.16.

These equivalence classes generate *virtual three–manifolds*. After such a definition, the following question arises immediately (stated by the author of this book).

Now, quite an interesting point arises.

First, the Poincaré conjecture evidently fails in this sense.

Furthermore, given two classical diagrams K_1 and K_2 representing the equivalent manifolds (in the sense of virtual Kirby theory). Do they actually represent the same classical three–manifolds in the ordinary sense?

Now, what we have to do is to define the Witten invariant for virtual knots. In fact, we already have all for the definition of it: namely, we have to use the normalised Kauffman bracket and the Jones-Wenzl idempotent, see theorem 22.14 and the formula therein. For more details, see original work [Dye].

Chapter 23

Legendrian knots and their invariants

The theory of Legendrian knots first introduced by Dmitry Fuchs and Serge Tabachnikov [FT], lies at the juncture of knot theory, the theory of wave fronts and contact geometry.

Legendrian knots in \mathbb{R}^3 represent the one–dimensional case of Legendrian manifolds (in the general case, a Legendrian manifold is a k-dimensional submanifold in a (2k+1)-dimensional manifold satisfying some conditions). The Legendrian knot theory is interesting because it allows us to introduce a new equivalence for knots: besides topological isotopy, one can consider an isotopy in the space of Legendrian knots.

23.1 Legendrian manifolds and Legendrian curves

One of the main questions of the theory of differential equations is to find an enveloping curve for the family of straight lines on the plane. It is well known that in the smooth case, this problem has a solution according to the existence and uniqueness theorem.

If we consider the field of, say, planes in \mathbb{R}^3 , or, more generally, hyperplanes in odd–dimensional space \mathbb{R}^{2n+1} , the enveloping surface does not always exist.

In the general position called "the maximally non-integrable case," the maximal dimension of a surface tangent to these hyperplanes at each point equals n.

For the case n = 1 we obtain just curves in \mathbb{R}^3 , i.e., knots (which can be tangent to the given family of planes).

Maximally non-integrable fields of hyperplanes are closely connected with contact structures.

23.1.1 Contact structures

Let us first introduce the notion of contact structure, see, e.g., [Arn, Thu].



Figure 23.1. A cusp of the front projection

Definition 23.1. A contact structure (form) on an odd-dimensional manifold M^{2n+1} is a smooth 1-form ω on M such that

$$\omega \wedge \underbrace{d\omega \wedge \cdots \wedge d\omega}_{n}$$

is a non-degenerate form.

Having a contact form, one obtains a hyperplane at each point: the plane of vectors v such that $\omega(v) = 0$.

Let us consider the case of \mathbf{R}^3 and the form $\omega = -xdy + dz$. Obviously, at each point (x, y, z) the incident plane is generated by the two vectors (1, 0, 0) and (0, 1, x). Denote this field of planes by τ .

Definition 23.2. A Legendrian link is a set of non-intersecting oriented curves in \mathbb{R}^3 such that each link at each point is tangent to τ .

Because a Legendrian link is a link, one can consider its projection onto planes, i.e., planar diagrams. It turns out that projections on different planes have interesting properties.

23.1.2 Planar projections of Legendrian links

First, let us consider a projection of a Legendrian link L to the Oyz plane. Let γ be the projection of one component of L. Consider $\dot{\gamma}=(0,\dot{y},\dot{z})$. By definition $x\dot{y}=\dot{z}$, and we conclude that the coordinate x of the Legendrian curve L equals the fraction $\frac{\partial z}{\partial y}$, or, in other words, the abscissa equals the tangens of the tangent line angle.

The only inconvenience here is that \dot{y} cannot be equal to zero.

This effect can be avoided by allowing x to be equal to ∞ , i.e., by considering $\mathbf{R}^2 \times S^1$ instead of \mathbf{R}^3 . Indeed, in this case there arises a beautiful theory of Legendrian links (as well as in any three–manifold that is a bundle over two–surface with fiber S^1). However, here we consider knots and links in \mathbf{R}^3 , thus we must take the restriction $y \neq 0$.

This means that on the plane Oyz our curve has no "vertical" tangent lines. So, the only possibility to change the sign of \dot{y} is the existence of a cusp, see Fig. 23.1.

Generically, the cusp has the form of a semicubic parabola: the curve $(\frac{3\bar{t}}{2}, t^2, t^3)$ is a typical example of such a curve (the cusp takes place at (0,0,0)).

Having any piecewise–smooth oriented curve (smooth everywhere except cusps where \dot{y} changes the sign), we can easily reconstruct a Legendrian curve in \mathbb{R}^3 by

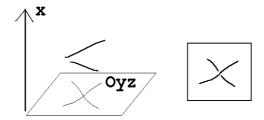


Figure 23.2. Restoring crossing types from the front projection

putting

$$x = \frac{\dot{z}}{\dot{y}}.$$

Obviously, taking a curve γ in a general position (with only double transverse intersection points), one gets a link L having projection γ . So, in order to construct a shadow of a link isotopic to L, one should just smooth all cusps.

Besides this, these transverse intersection points define uniquely the crossing structure of the link. Namely, the x-coordinate is greater for the piece of curve where the tangens is greater.

One can slightly deform the projection γ (without changing the isotopy type of the corresponding Legendrian link) such that the two intersecting pieces of the curve $\tilde{\gamma}$ have directions northwest–southeast and northeast–southwest, see Fig. 23.2.

In this case, the line northeast–southwest will form an overcrossing.

Hence, we know how to construct planar link diagrams from diagrams of Legendrian links projections to Oyz.

The reverse procedure is described in the proof of the following:

Theorem 23.1. For each link isotopy class there exists a Legendrian link L representing this class.

Now, let us look at what happens if we take the projection to Oxy.

It is more convenient in this case to consider each component separately.

Let γ be a curve of projection. If we take an interval of this curve starting from a point A and finishing at a point B, we deduce from $\dot{z} = x\dot{y}$ that $z_A - z_B = \int_A x dy$. If we take an integral along all the curve γ (from A to A), we see that

$$\oint_{\gamma} x dy = 0,$$

or by the Gauss-Ostrogradsky theorem,

$$S(M_{\gamma}) = 0, \tag{1}$$

where S means oriented area, and M_{γ} is the domain bounded by γ (with signs).

Equation (1) is the unique necessary and sufficient condition for the Legendrian curve to be closed.

If we want to get some crossing information for a Legendrian knot, we should take a crossing P and take an integral along γ from P lying on one branch of γ to

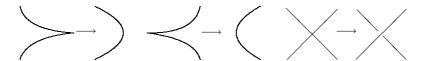


Figure 23.3. Restoring a diagram from a front

P on the other branch of γ . If the value of this integral is positive then the first branch is an undercrossing, otherwise it is an overcrossing.

If we consider an n-component link, one can easily establish the crossings for each of its components. Then one has n-1 degrees of freedom in posing the components.

So, for the case of a link, the crossing structure cannot be restored uniquely.

23.2 Definition, basic notions, and theorems

Here we shall use the notation from [Che2]. We are going to give definitions only for the case of knots; analogous constructions for links can be presented likewise.

Consider a smooth knot in the standard contact space $\mathbf{R}^3 = \{q, p, u\}$ with the contact form $\alpha = du - pdq$ (we introduce the new coordinates u, p, q instead of z, x, y respectively).

Definition 23.3. A smooth knot is called *Legendrian* if the restriction of α to L vanishes.

Definition 23.4. Two Legendrian knots are called *Legendrian isotopic* if one can be sent to the other by a diffeomorphism g of \mathbf{R}^3 such that $g^*\alpha = \phi\alpha$, where $\phi > 0$.

There are two convenient ways of representing Legendrian knots by projecting them on different planes. The projection $\pi: \mathbf{R}^3 \to \mathbf{R}^2$, $(q, p, u) \to (q, p)$ is called the Lagrangian projection, and the projection $\sigma: (p, q, u) \to (q, u)$ is called the front projection.

The front projection

Having a front projection, we can restore the Legendrian knot as follows. We just smooth all cusps, and set the crossings according to the following rule: the upper crossing is the branch having greater tangency. Since the front has no vertical tangent lines, the choice of crossing is well defined, see Fig. 23.3.

The front projection is called σ -generic if all self-intersections of it have different q coordinates.

The reverse procedure can be done as follows. After a small perturbation we can make a diagram having no vertical tangents at crossings. Now, let us replace all neighbourhoods of points with vertical tangents by cusps. All "good" crossings are just replaced by intersections. Besides, all double points with "bad" crossings are to be replaced just as shown in Fig. 23.4.



Figure 23.4. Constructing a front by a diagram

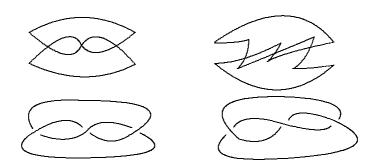


Figure 23.5. Left trefoil and right trefoil

Thus, we have proved that each knot can be represented by a front, i.e., **each knot has a Legendrian representative**. Thus, we have proved Theorem 23.1.

Below, we show diagrams and fronts for the two trefoils.

The assymmetricity of these two diagrams follows from the convention concerning the "good" crossings. It is quite analogous to the assymetricity of the simplest d-diagrams of the trefoils.

The Lagrangian projection

In the normal case, the Lagrangian projection is smooth (has no cusps or other singularities unlike the front projection). As in the case of front projection, the Lagrangian projection allows us to restore the crossing structure, and hence, knot (topological).

More precisely, a Lagrangian projection is called π -generic if all self-intersections of it are transverse double points.

Namely, having a planar diagram L of a knot, let us fix some point P of it different from a crossing and fix p(P) = 0. Now, we are able to restore the coordinate p for all the points of L. Taking into account that du = pdq along the curve, we see that the difference $u_A - u_B$ equals the oriented area of the domain restricted by the curve.

Thus, if we have some projection P (combinatorial knot diagram) and we want it to be Lagrangian, we should check the following condition: if we go along the knot from some point A to itself, we obtain an equation on areas of domains cut by P. All crossing types are regulated by equations.

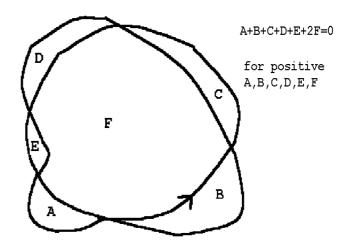


Figure 23.6. A braided diagram is not Lagrangian

Exercise 23.1. Write these equations explicitly.

This shows that some projections cannot be realised as Lagrangian ones.

For instance, having a braided diagram (see Fig. 23.6), we cannot realise it as Lagrangian because the equation to hold will consist only of positive numbers whose sum with positive coefficients should be equal to zero: we go around each area a positive number of times and each area is positive. Thus, it cannot be equal to zero.

In this sense, Lagrangian diagrams are opposite to braided diagrams.

Remark 23.1. In fact, the third projection $(p,q,u) \rightarrow (p,q)$ is not interesting.

A Legendrian knot $L \subset \mathbf{R}^3$ is said to be π -generic if all self intersections of the immersed curve $\pi(L)$ are transverse double points. In this case, this projection endowed with overcrossing and undercrossing structure represents a knot diagram that is called the *Lagrangian diagram*.

Of course, not every abstract knot diagram in \mathbb{R}^2 is a diagram of a Legendrian link, or is oriented diffeomorphic to such.

For a given Legendrian knot $L \subset \mathbf{R}^3$, its σ -projection or front $\sigma(L) \subset \mathbf{R}^3$ is a singular curve with no vertical tangent vectors.

23.3 Fuchs–Tabachnikov moves

Legendrian knots and links in their frontal projection admit a combinatorial interpretation like ordinary knots and links. Namely, there exists a set of elementary moves transforming one frontal projection of a Legendrian link to each other projection of the same link.

In fact, the following theorem holds.

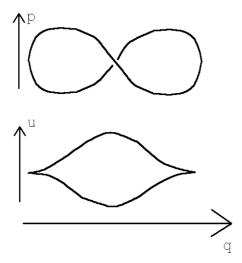


Figure 23.7. Lagrangian projection and front projection

Theorem 23.2 (Fuchs, Tabachnikov [FT]). Two fronts represent Legendrian-equivalent links if and only if one can be transformed to the other by a sequence of moves 1–3 shown in Fig. 23.8.

By admitting the 4th move, we obtain the ordinary (topological) equivalence of links.

Note that the tangency move (when two lines pass through each other) is not a Legendrian isotopy: this tangency in \mathbb{R}^2 means an intersection in \mathbb{R}^3 (the third coordinate is defined from the tangent line). Thus, this move changes the Legendrian knot isotopy type; it is not the second Reidemeister move.

23.4 Maslov and Bennequin numbers

The invariants named in the section title can be defined as follows. The Bennequin number (also called the Thurston–Bennequin number) $\beta(L)$ of L is the linking number between L and s(L), where s is a small shift along the u direction.

The Maslov number m(L) is twice the rotation number of the projection of L to the (q, p) plane.

The change of orientation on L changes the sign of m(L) and preserves $\beta(L)$.

Both these invariants can be defined combinatorially by using the front projection of the Legendrian knot. Namely, the Maslov number is half of the difference between the numbers of positive cusps and negative cusps, see Fig. 23.9.

Exercise 23.2. Prove the equivalence of the two definitions of the Maslov number.

Definition 23.5. A crossing is called *positive* if the orientation of two branches of the front have directions in different half–planes, and *negative* otherwise, see Fig. 23.10.

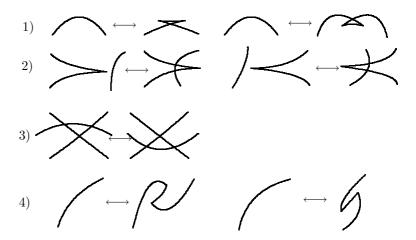


Figure 23.8. Fuchs-Tabachnikov moves



Figure 23.9. Positive cusps

Negative cusps

The Bennequin number is $\frac{1}{2}$ (# cusps)+ (# positive crossings)-(# negative crossings).

Exercise 23.3. Prove the equivalence of the two definitions of the Bennequin number.

23.5 Finite-type invariants of Legendrian knots

By definition, each Legendrian knot is a knot; besides, two equivalent Legendrian knots are (topologically) equivalent knots. Thus, each knot invariant represents an invariant of Legendrian knots. So are finite type invariants of knots. Moreover, one can easily define the finite type invariants of Legendrian knots and show that all finite type invariants coming from "topological knots" have finite-type in the Legendrian sense.

Besides, it is not difficult to show that the Maslov number and the Bennequin number are finite type invariants: the first of them has order zero and the second one has order one.

The most important achievement (classification) of the finite type invariants is described by the following:

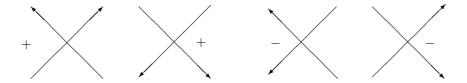


Figure 23.10. Positive crossings

Negative crossings

Theorem 23.3 (Fuchs, Tabachnikov). All Vassiliev invariants of Legendrian knots can be obtained from topological finite type invariants and Maslov and Bennequin numbers.

This means that the theory of finite—type invariants for Legendrian links is not so rich. As we are going to show, there are stronger invariants that cannot be represented in terms of finite order invariants.

23.6 The differential graded algebra (DGA) of a Legendrian knot

In the present section, we shall speak about the differential graded algebra (free associative algebra with the unit element) of Legendrian knots, proposed by Chekanov. It turns out that all homologies of this algebra are invariants of Legendrian knots. Now, we are going to work with Lagrangian diagrams of Legendrian knots.

We associate with every π -generic Legendrian knot K a DGA (A, ∂) over \mathbb{Z}_2 (see [Che]).

Remark 23.2. A similar construction was given by Eliashberg, Givental and Hofer, see [EGH]).

Let L be a Lagrangian diagram of a Legendrian link. Denote crossings of this diagram by $\{a_1, \ldots, a_n\}$.

We are going to denote a tensor algebra $T(a_1, \ldots, a_n)$ with generators a_1, \ldots, a_n . This algebra is going to be a $\mathbf{Z}_{m(L)}$ -graded algebra (free, associative and with unity).

First of all, let us define the grading for this algebra. Let a_j be a crossing of L. Let z_+, z_- be the two pre-images of a_j in \mathbf{R}^3 under the Lagrangian projection, whence z_+ has greater u-coordinate than z_- .

Without loss of generality, one might assume that the two branches of the Lagrangian projection at a_j are perpendicular.

These points divide the diagram L into two pieces, γ_1 and γ_2 , and we orient each of these pieces from z_+ to z_- .

Now, for $\varepsilon \in \{1, 2\}$, the rotation number of the curve $\pi(\gamma_{\varepsilon})$ has the form $\frac{N_{\varepsilon}}{2} + \frac{1}{4}$, where $N_{\varepsilon} \in \mathbf{Z}$. Clearly, $N_1 - N_2$ is equal to $\pm m(L)$. Thus N_1 and N_2 represent one and the same element of the group $\mathbf{Z}_{m(L)}$, which we define to be the degree of a_i .

Now, we are going to define the differential ∂ . For every natural k, let us fix a curved convex k-gon $\Pi_k \subset \mathbf{R}^2$ whose vertices x_0^k, \ldots, x_{k-1}^k are numbered counterclockwise.

The form $dq \wedge dp$ defines an orientation on the plane. Denote by $W_k(Y)$ the collection of smooth orientation–preserving immersions $f: \Pi_k \to \mathbf{R}^2$ such that $f(\partial \Pi_k) \subset Y$. Note that $f \in W_k(Y)$ implies that $f(x_i^k) \in \{a_1, \ldots, a_n\}$.

Let us consider these immersions up to combinatorial equivalence (parametrisation) and denote the quotient set by $\tilde{W}_k(Y)$. The diagram Y divides a neighbourhood of each of its crossings into four sectors. Two of them are marked as positive (opposite to the way used while we defined Kauffman's bracket) and the other two are taken to be negative. For each vertex x_i^k of the polygon Π_k , a smooth immersion $f \in \tilde{W}_k(Y)$ maps its neighbourhood in Π_k to either a positive or negative sector; in these cases, we shall call x_i^k positive or negative, respectively.

Define the set $W_k^+(Y)$ to consist of immersions $f \in W_k(Y)$ such that the vertex x_0^k is the only positive vertex for f; all other vertices are to be negative. Let $W_k^+(Y,a_j) = \{f \in W_k^+(Y) | f(x_0^k) = a_j\}$. Let $A_1 = \{a_1,\ldots,a_n\} \otimes \mathbf{Z}_2 \subset A, A_k = A_1^{\otimes k}$. Then $A = \bigoplus_{l=0}^{\infty} A_l$.

Let $\partial = \sum_{k\geq 0} \stackrel{\sim}{\partial k}$, where $\partial_k(A_i) \in A_{i+k-1}$ and

$$\partial_k(a_j) = \sum_{f \in W_{k+1}^+(Y, a_j)} f(x_1) \cdots f(x_k).$$

Now, we can extend this differential for the algebra A by linearity and the Leibnitz rule. Now, the following theorem says that (A, ∂) is indeed a DGA.

Theorem 23.4. The differential ∂ is well defined. We have $deg\partial = -1$ and $\partial^2 = 0$.

The main theorem on this invariant [Che], see also [Che2], is the following:

Theorem 23.5. Let $(A, \partial), (A', \partial')$ be the DGA's of $(\pi$ -generic) Legendrian knot diagrams L, L'. If L and L' are Legendrian isotopic then (A, ∂) and (A', ∂') are stable tame isomorphic. In particular, the homology rings $H(A, \partial)$ and $H(A', \partial')$ are isomorphic.

23.7 Chekanov–Pushkar' invariants

The invariants described in this section are purely combinatorial. They are described in the terms of front projection. Though they are combinatorial and the proof of their invariance can be easily obtained just by checking all Reidemeister moves, they have deep homological foundations.

Given a σ -generic oriented Legendrian knot L, let us denote by C(L) the set of its points corresponding to cusps of σ_L . The Maslov index $\mu: L/C(L) \to \Gamma = \mathbf{Z_m}$ is a locally constant function uniquely defined up to an additive constant by the following rule: the value of μ jumps at points of C(L) by ± 1 as shown in Fig. 23.11. A crossing is called a Maslov crossing if μ takes the same value on both its branches.

Assume that $\Sigma = \sigma(L)$ is a union of closed curves X_1, \ldots, X_n that have finitely many self–intersections and meet each other at finitely many points. Then we call the unordered collection $\{X_1, \ldots, X_n\}$ a decomposition of Σ .

Now, a decomposition is called *admissible* if it satisfies some conditions.

1. Each curve X_i bounds a topologically embedded disc: $X_i = \partial B_j$.

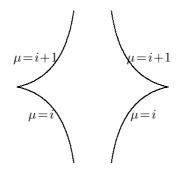


Figure 23.11. Jumps of the Maslov index near cusps



Figure 23.12. Three types of switchings

- 2. Each B_i has exactly two cusps points, thus the number of cusps is equal to 2n.
- 3. If $(q_0, u) \in X_i \cap X_j$ for some $i \neq j$ then for each $q \neq qq_0$ sufficiently close to q, the set $B_i(q) \cap B_j(q)$ either coincides with $B_i(q)$ or with $B_j(q)$, or is empty.
 - 4. Every switching crossing is Maslov.

Definition 23.6. A decomposition is called *admissible* if it satisfies conditions 1–3 and *graded admissible* if it also satisfies condition 4.

Note that there are three types of switchings, shown in Fig. 23.12

The first type of switching shown in Fig. 23.12.a is automatically ruled out by conditions 1, 2.

The second type of decomposition (see Fig. 23.12.b) is called *switching*. The third type (Fig. 23.12.c) is called *non-switching*.

Denote by $Adm(\Sigma)$ (respectively, $Adm_+(\Sigma)$) the set of admissible (respectively, graded admissible) decompositions of Σ . Given $D \in Adm(\Sigma)$, denote by Sw(D) the set of its switching points. Define $\Theta(D) = \#(D) - \#Sw(D)$.

Now, we are ready to formulate the main theorem on the Chekanov–Pushkar' invariants.

Theorem 23.6. If the σ -generic Legendrian knots $L, L' \subset \mathbf{R}^3$ are Legendrian isotopic then there exists a one-to-one mapping

$$g: Adm(\sigma(L)) \to Adm(\sigma(L'))$$

such that

$$g(Adm_{+}(\sigma(L))) = Adm_{+}(\sigma(L'))$$

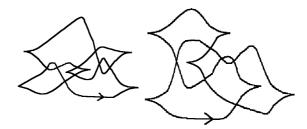


Figure 23.13. Chekanov's pair

and $\Theta(g(D)) = \Theta(D)$ for each $D \in Adm(\sigma(L))$.

In particular, the numbers $\#(Adm(\sigma(L)))$ and $\#(Adm_+(\sigma(L)))$ are invariants of Legendrian isotopy.

23.8 Basic examples

Both Chekanov and Chekanov–Pushkar' invariants cannot be expressed in terms of finite type invariants.

To show this (in view of the Fuchs–Tabachnikov theorem) it suffices to present a couple of Legendrian knots which represent the same topological knot (thus have all equal topological finite order invariants) and have equal Maslov and Bennequin numbers. This couple of knots is called a *Chekanov pair*. Their front projections are shown in Fig. 23.13.

Appendix A

Independence of Reidemeister moves

Let us show that each of the three moves Ω_1 , Ω_2 , Ω_3 is necessary for establishing knot isotopy, i.e. that for each move there exist two diagrams of the same classical link which cannot be transformed to each other by using the two remaining Reidemeister moves (without the chosen one).

Example A.1. The first Reidemeister move is the only move that changes the parity of the number of crossings. Thus, each unknot diagram with an odd number of crossings cannot be transformed to the diagram without crossings by using only Ω_2, Ω_3 .

Example A.2. Let K_1, K_2, K_3, K_4 be some diagrams of different prime (non-trivial) knots. Consider the diagrams $L = K_1 \# K_2 \# K_3 \# K_4$ and $M = K_1 \# K_3 \# K_2 \# K_4$, shown in Fig. A.1.a,b. These diagrams represent the same knot.

Let us show that there is no isotopy transformation from L to M involving all Reidemeister moves but Ω_2 . Actually, consider the subdiagrams K_i , i=1,2,3,4 inside L. Their order is such that between the knots K_1 and K_3 there are trivial knots at both sides. Let us show that this property remains true under Ω_1, Ω_3 . Actually, while performing these moves, different K_i do not meet; thus one can always indicate each of these knots on the diagram. During the isotopy their order remains the same. However, in M, the knots K_1 and K_3 are adjacent. Thus L cannot be transformed to M by using only Ω_1, Ω_3 and planar isotopy.

Example A.3. Now, let us consider the shadow of the standard Borromean rings diagram and construct a link diagram as shown in Fig. A.2.a. Denote this diagram by L_1 .

Since the link represented by L_1 is trivial, it has a diagram L_2 , shown in Fig. A.2.b.

Let us show that the diagram L_1 cannot be transformed to L_2 by using a sequence of Ω_1 and Ω_2 . Actually, let us consider an arbitrary planar diagram of the three-component unlink and assign a certain element of \mathbf{Z}_2 to it as follows. Define the

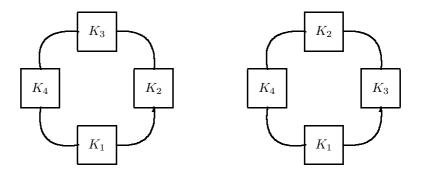


Figure A.1. Counterexample for $\Omega_1, \Omega_3 \Longrightarrow \Omega_2$

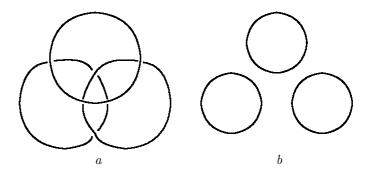


Figure A.2. Counterexample for $\Omega_1, \Omega_2 \Longrightarrow \Omega_3$

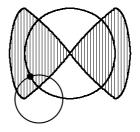


Figure A.3. The domain U is shaded.

domain U, restricted by one link component. Let us fix this component l. It tiles the plane (sphere) into cells which admit a checkboard colouring. Let us use the colouring where the cell containing the infinite point is white. Denote by U(l) the set of all black cells, see Fig. A.3.

Let us now consider crossings of the two components different from l, select those lying inside U and calculate the parity of their number.

Let us do the same for the second and the third component. Thus, we get three elements of \mathbb{Z}_2 . It is easy to see that for L_1 all these three numbers are equal to one and for L_2 they all are equal to zero.

Then this (non-ordered) triple of numbers is invariant under Ω_1, Ω_2 .

For Ω_1 this statement is evident. In the case of Ω_2 the move is applied to arcs of two different circles. It suffices to see that either both crossings lie outside U(l) or they both lie inside l.

Thus, L_1 cannot be transformed to L_2 by using only Ω_1, Ω_2 .

Let us note that analogous statements hold place in ornament theory, see chapter 6 of [Va] and [Bj, BW].

Appendix B

Vassiliev's invariants for virtual links

There are two approaches to the finite type invariants of virtual knots: the one proposed by Goussarov, Polyak and Viro [GPV] and the one proposed by Kauffman [KaV]. They both seem to be natural because they originate from the formal Vassiliev relation but the invariants proposed in [GPV] are not so strong.

Below, we shall give the basic definitions and some examples.

B.1 The Goussarov-Viro-Polyak approach

First, we shall give the definitions we are going to work with. We introduce the *semivirtual crossing*. This crossing still has overpasses and underpasses. In a diagram, a semivirtual crossing is shown as a classical one but encircled. Semivirtual crossings are related to the crossings of other types by the following formal relation:

Let D be a virtual knot diagram and let $\{d_1, d_2, \ldots, d_n\}$ be different classical crossings of it. For an n-tuple $\{\sigma_1, \ldots, \sigma_1\}$ of zeros and ones, let D_{σ} be the diagram obtained from D by switching all d_i 's with $\sigma_i = 1$ to virtual crossings. Denote by $|\sigma|$ the number of ones in σ . The formal alternating sum

$$\sum_{\sigma} (-1)^{|\sigma|} D_{\sigma}$$

is called a diagram with n semivirtual crossing.

Denote by K the set of all virtual knots. Let $\nu: K \to G$ be an invariant of virtual knots with values in an abelian group G. Extend this invariant to $\mathbf{Z}[K]$ linearly. We say that ν is an invariant of finite type if for some $n \in \mathbb{N}$, it vanishes on any virtual knot K with more than n semivirtual crossing. The minimal such n is called the degree of K.

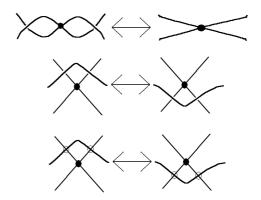


Figure B.1. Rigid vertex isotopy moves for virtual knots

The formal Vassiliev relation in the form $(\bigotimes) = (\bigotimes) - (\bigotimes)$ together with (1) implies the relation

Remark B.1. Note that singular knots are not considered here as independent objects; just as linear combinations of simpler objects.

It is obvious that for any finite type invariant of the virtual theory, its restriction for the case of classical knots is a finite type invariant in the ordinary sense.

However, not every classical finite type invariant can be extended to a finite type invariant in this virtual sense. For instance, there are no invariants of order two for virtual knots.

Starting from (1) and (2), Polyak constructed the Polyak algebra [GPV] that gives formulae for all finite type invariants of virtual knots. Besides this, they give explicit diagrammatic formulae for some of them and also construct some finite type invariants for long virtual knots.

B.2 The Kauffman approach

Kauffman starts from the formal definition of a singular virtual knot (link).

Definition B.1. A *singular virtual link diagram* is a four–valent graph in the plane endowed with orientations of unicursal curves and crossing structure: each crossing should be either classical, virtual, or singular.

Definition B.2. A *singular virtual knot* is an equivalence class of virtual knot diagrams by generalised Reidemeister moves and *rigid vertex isotopy*, shown in Fig. B.1

Now, the definition of the Vassiliev knot invariants is literally the same as in the classical case. For each invariant f of virtual links, one defines its formal derivative

f' by Vassiliev's rule $f'((\sum)) = f((\sum)) - f((\sum))$ and says that the invariant f has order less than or equal to n if $f^{(n+1)} \equiv 0$.

B.2.1 Some observations

The space of finite type invariants of virtual knots (by Kauffman) is much more complicated than that of classical knots. For instance, the space of invariants of order zero is infinite—dimensional because there are infinitely many classes of virtual knots that can not be obtained from each other by using isotopy and classical crossing switches (one can separately define the value of an invariant on each of these equivalence classes). The structure of higher order invariants is even more complicated.

The Jones–Kauffman invariant in the form [KaV] is weaker than the finite type invariants. Namely, one can transform it (by the exponential variable change) into a series with finite type invariant coefficient just as was done in Chapter 12.

It is easy to see that many such invariants are not of finite type in the sense of [GPV]. This observation is due to Kauffman [KaVi].

Appendix C

Energy of a knot

There is an interesting approach to the study of knots, the knot energy. Such energies were first investigated by Moffat [Mof]. We shall describe the approach proposed by Jun O'Hara. In his work [OH] (see also [OH2, OH3]) he proposed studying the *energy of the knot*. First, it was thought to be an analogue of the Gauss electromagnetic function for links, but it has quite different properties.

An energy is a function on knots that has some interesting properties and some invariance (not invariant under all isotopies!) However, these properties are worth studying because some of them lead to some invariants of knots; besides, they allow us to understand better the structure of the space of knots. Here we shall formulate some theorems and state some heuristic conjectures.

In the present appendix, we shall give a sketchy introduction to the best-known energy, the *Möbius energy*.

Definition C.1. Let K be a knot parametrised by a natural parameter $r: S^1 \to \mathbf{R}^3$, where S^1 is the standard unit circle in \mathbf{R}^2 . Then the Möbius energy of the knot r is given by:

$$E_f(K) = \int \int_{S^1 \times S^1} f(|r(t_1) - r(t_2)|, D(t_1, t_2)) dt_1 dt_2,$$

where $f(\cdot, \cdot)$ is some function on the plane (one can take various functions, in these cases different properties of the energy may arise), $D(\cdot, \cdot)$ is the function on $S^1 \times S^1$ representing the distance between the two points along the circle (one takes the minimal length), and the integration is taken along the direct product of the knot by itself with two parameters t_1, t_2 .

Physical matters (for example, a uniformly charged knot) lead to the formula with $f(x,y) = \frac{1}{x}$. However, such a function does not always converge. Thus, other cases are worth studying.

One often uses the function $f(x,y) = \frac{y^2}{x}$.

In this case, the Möbius energy has the following properties. The main property is that the integral converges for smooth knots.

Actually, the natural parametrisation is not always convenient to use. In fact, we shall study some transformations of knots where the natural parametrisation becomes unnatural.

So, it is convenient to define the Möbius energy for the arbitrary parametrisations u, v as follows:

$$E(r) = \int \int \left(\frac{1}{|r(t_1) - r(t_2)|^2} - \frac{1}{D^2(t_1, t_2)} \right) \cdot |\dot{t}_1(u)| \cdot |\dot{t}_2(v)| du dv.$$

- 1. The Möbius energy is invariant under homothetic transformations.
- 2. The Möbius energy tends to infinity while the knot is closing to a singular knot.
- 3. The Möbius energy is strictly positive.
- 4. The Möbius energy is invariant under Möbius transformations, namely, under inversions in spheres not centered at a point of the knot. If we invert in a sphere centred at a point of the knot K then we obtain a long knot K'; one can define the Möbius energy in the same way. In this case, E(K) = E(K') + 4.
- 5. The absolute minimum of the Möbius energy is realised on the standard circle.
- 6. The Möbius energy is smooth with respect to smooth deformations of the knot.

Let us construct these properties in more detail and prove some of them.

The first property easily follows from the form of the invariant: we have a double integration that is cancelled (while performing homothety) by the second power in the denominator. The invariance under shifts and orthogonal moves is evident.

The second property is obvious: one obtains a denominator that tends to infinity while the integration domain and coefficients remain separated from zero.

The third property follows from the fact that the distance between two points in \mathbb{R}^3 does not exceed the distance along the arc of the circle for the corresponding parametrising points (in the natural parametrisation case).

The fourth and the sixth property follow from a straightforward check.

Let us now discuss the fifth property in more detail and establish some more properties of the Möbius energy. First, let us prove the "long link property": if K is a knot and X is a point on K then for any sphere centred at X, for the long knot K' obtained from K by inversion in this sphere, we have E(K) = E(K') + 4.

Rather than proving this property explicitly, we shall prove that the energy of the circle equals four. Then, from some reasonings this property will follow immediately.

In fact, for the circle in the natural parametrisation, the length along the circle is $(t_1 - t_2)$ if $0 \le (t_1 - t_2) \le \pi$. The distance is then $2\sin(\frac{t_1 - t_2}{2})$.

After a suitable variable change $u = t_1 - t_2$, the integral is reduced to the single integral multiplied by 2π . Besides, for symmetry reasons, is sufficient to consider u from zero to π , and then double the obtained result. Then we get:

$$4\pi \int_0^{\pi} \left(\frac{1}{4\sin^2(u/2)} - \frac{1}{u^2} \right) du.$$

The latter equals

$$4\pi(-\frac{1}{2}\cot(u/2) + \frac{1}{u})|_0^{2\pi-\varepsilon},$$

which is equal to four.

Now, after a small perturbation, each knot can be considered as a knot having a small piece of a straight line with arbitrarily small change of energy.

Taking into account that for any long knot different from the straight line, we see that the only knot having energy four is the circle; all other knots have greater energies. This proves the fifth property.

It is easy to see that for long knots the energy is non-negative as well. Thus, we immediately see that E(K) > 4 for ordinary knots.

Furthermore, for the straight line, the energy equals zero, whence for any other long knot it is strictly greater than zero. So, the energy of the circle equals four; this realises the minimum of the energy for all (classical) knots.

This property is very interesting. Thus one can consider the knot energy as the starting point of *Morse theory* for the space of knots. One considers the space of all (smooth knots) and studies the energy function and its properties. For instance, for each knot there exists its own knot theory with minima, maxima and other critical points.

The property described above shows that for the unknot there exists only one minimum, namely, four. This minimum is realised by the circle. The circle is considered up to moves of the space \mathbf{R}^3 (orthogonal and shifts) and homotheties. So, in some sense one can say that there exists *only one minimum* of the energy function on the space of knots.

So, if we consider knots up to orthogonal moves of \mathbb{R}^3 , shifts, and homotheties, then the circle is the unique minimum for the unknot.

We see that the energy of each long knot is positive, so the energy of each closed knot is greater than or equal to four.

One can ask the question whether for each knot there exists a *normal form*, i.e., a representative with the minimal energy. The natural questions (conjectures) are:

- 1. Does such a normal form (realising the minimum) exist?
- 2. Is this unique?

Both these questions were stated by Freedman, and now none of them has any satisfactory solution except for the case of the unknot. Moreover, these conjectures are not strictly stated: one should find the class of deformation that define "the same knot."

However, the set of minima and minimal values of the energy function can be considered as a knot invariant. Since we have many energies (for different functions f), these invariant seem to be quite strong, so that they seem to differ for all non-isotopic knots. This is another conjecture.

The only thing that can be said about all knot isotopy classes is that there exists only a *countable number of energy minima*.

The absolute minimum of any unknot has not yet been calculated. However, the following theorem holds.

Theorem C.1 (Freedman [FHW]). If $E(K) \leq 6\pi + 4$, then K is the unknot.

There is another interesting result on the subject, namely, the existence theorem:

Theorem C.2. Let K be a prime knot isotopy class. Then there exists a minimal representative K_{γ} of K such that for each other representative K', we have $E(K') \geq E(K_g)$.

Note that the analogous statement for knots that are not prime is not proved.

Appendix D

Unsolved problems in knot theory

Below, we give a list of unsolved problems. While compiling the list, we mostly referred to the Robion Kirby homepage, www.math.berkeley.edu/~kirby, (his problem book on low–dimensional topology). We could not place here the whole list of problems listed there. We have chosen the problems with possibly easier formulations but having a great importance in modern knot theory. Besides, we used some "old" problems formulated in [CF] and still not solved, and problems from [CD]. As for virtual knot theory, we mostly used private communications with Louis H. Kauffman and Roger A. Fenn. A more complete modern list of problems will appear in our joint work: "Virtual Knots: Unsolved Problems."

If the problem was formulated by some author, we usually indicate the author's name. If the problem belongs to the author of the present book, we write (V.M).

- 1. How do we define whether a knot is invertible?
- 2. Let C(n) be the number of simple unoriented knots in S^3 with minimal number of crossings equal to n. Describe the behaviour of C(n) as n tends to ∞ . The first values of C(n) (starting from n=3) are: 1,1,2,3,7,21,49,165,552,2167,9998.
- 3. Similarly, let U(n) be the number of prime knots K for which the unknotting number equals n. What is the asymptotic behaviour of U(n)?; of C(n)/U(n)?
- 4. Is it true that the minimal number of crossings is additive with respect to the connected sum operation: $c(K_1 \# K_2) = c(K_1) + c(K_2)$?
- 5. (de Souza) Does the connected sum of n knots (not unknots) have unknotting number at least n.
 - The positive answer to this question would follow from the positive answer to the following question:
- 6. Is the unknotting number additive with respect to the connected sum: $u(K_1 \# K_2) = u(K_1) + (K_2)$?

- 7. (Milnor) Is the unknotting number of the (p,q)-torus knot equal to $\frac{(p-1)(q-1)}{2}$? The answer is yes for many partial cases.
- 8. The counterexample to the third Tait conjecture (see Chapter 7) was found in [HTW]. It has 15 crossings.
 - Are there amphicheiral knots with arbitrary minimal crossing number ≥ 15 ?
- 9. Are there (prime, alternating) amphicheiral knots with every possible even minimal crossing number?
- 10. (Jones) Is there a non-trivial knot for which the Jones polynomial equals 1? Bigelow proved that this problem is equivalent to the existence of a kernel for the Burau representation of Br(4).
 - (Kauffman, "virtual reformulation") There are different for constructing virtual knots with the trivial Jones polynomial. Are there classical knots among them?
- 11. Are there infinitely many different knots with the same Kauffman two-variable polynomial?
 - In fact, it is shown in [DH] that if such an example exists, the number of its crossings should be at least 18.
- 12. Find an upper bound on the number of Reidemeister moves transforming a diagram of a knot with n crossings to another diagram of the same knot with m crossing (a function of m, n).
- 13. Find a criterion for whether a knot is prime in terms of its planar diagram or d-diagram.
- 14. Is the Burau representation of Br(4) faithful?
- 15. (Vassiliev) Are the Vassiliev knot invariants complete?
- 16. (Partial case of the previous problem) Is it true that the Vassiliev knot invariants distinguish inverse knots. In other words, is it true that there exists inverse knots K and K' and a Vassiliev invariant v such that $v(K) \neq v(K')$?
- 17. Is there a faithful representation of the virtual braid group VB(n) for arbitrary n?
- 18. (Fox) A knot is called a *slice* if it can be represented as an intersection of some $S^2 \subset \mathbf{R}^4$ and some three-dimensional hyperplane.

A knot is called a *ribbon* if it is a boundary of a ribbon disk. Obviously, each ribbon knot is a slice.

If K is a slice knot, is K a ribbon knot?

19. Two knots given by maps $f_1, f_2: S^1 \to S^3$ are called *concordant* if the maps $f_{1,2}$ can be extended to an embedding $F: S^1 \times I \to S^3$.

Problem (Akbulut and Kirby) If 0–frame surgeries on two knots give the same 3–manifold then the knots are concordant.



Remark D.1. The theory of knots, cobordisms and concordance is represented very well in the works [COT] by Cochran, Orr and Teichner and [Tei] by Teichner.

20. (Przytycki) We say that an oriented diagram D is matched if there is a pairing of crossings in D so that each pair looks like one of those shown below (notice that the orientations are chosen to be antiparallel).

Conjecture. There are oriented knots without a matched diagram.

21. (Jones and Przytycki) A *Lissajous knot* is a knot in \mathbb{R}^3 given by the following parametric equations:

$$x = \cos(\eta_x t + \phi_x)$$

$$y = cos(\eta_y t + \phi_y)$$

$$x = \cos(\eta_z t)$$

for integer numbers η_x, η_y, η_z . Which knots are Lissajous?

- 22. (V.M) Find an analogue of combinatorial formulae for the Vassiliev invariants for **knots and links** in terms of *d*–diagrams.
- 23. Does every non-trivial knot K have property P, that is, does Dehn surgery on K always give a non-simply connected manifold?

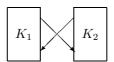
To date, this problem has been solved positively in many partial cases.

- 24. (Kauffman) Find a purely combinatorial proof that any two virtually equivalent classical knots are classically equivalent.
- 25. (V.M) Can the forbidden move be used for constructing a system of axioms in order to obtain virtual knot invariants (like skein relations and Conway algebras)?
- 26. Is there an algorithm for recognising virtual knots?
- 27. Is there an algorithm for solution of the word problem for the virtual braid group? In particular, is the invariant of virtual braids proposed in this book complete?
- 28. Is there an algorithm that distinguishes whether a virtual knot is isotopic to a classical one?

29. Crossing number problems (R. Fenn, L. Kauffman, V. M).

For each virtual link L, there are three crossing numbers: the minimal number C of classical crossings, the minimal number V of virtual crossings, and the minimal total number T of crossings for representatives of L.

- 1) What is the relationship between the least number of virtual crossings and the least genus in a surface representation of the virtual knot?
- 2) Is it true that T = V + C?
- 3) Are there some (non-trivial) upper and lower bounds for T, V, C coming from virtual knot polynomials?
- 30. Let U_1, U_2 be some two virtual knots. Is it true that the triviality of $U_1 \# U_2$ implies the triviality of one of them, where by $U_1 \# U_2$ we mean
 - 1) Any connected sum;
 - 2) Arbitrary connected sum.
 - 3) The same question about long virtual knots.
- 31. (X.–S. Lin) Suppose that oriented knots K_+ and K_- differ at exactly one crossing at which K_+ is positive and K_- is negative. If $K_+ = K_-$, does it follow that K_+ equals



where either K_1 or K_2 could be the unknot.

Appendix E

A knot table

Below, we give a list of prime knots (up to mirror image) with less than 10 crossings and corresponding d-diagrams. For the corresponding mirror images we write a "bar" over our code. Thus, $\overline{3}_1$ means the right trefoil knot.

Since the composition of each knot into prime components is unique, it is sufficient to tabulate all prime knots to know all knots.

Prime knots are tabulated as follows. First we enumerate knots having diagrams with no more than three crossings, then knots having a diagram with four crossings, and so on. Within each group, the numbering is arbitrary. In the table below, we use the enumeration established in [Rol].

However, Rolfsen's table [Rol] still contains one error. It is the famous Perko pair 10_{161} and 10_{162} two equivalent knots that were thought to be different for 75 years; their equivalence was established by Perko [Per] in 1973.



In the list given below, we remove the "old" knot 10_{162} and change the numeration.

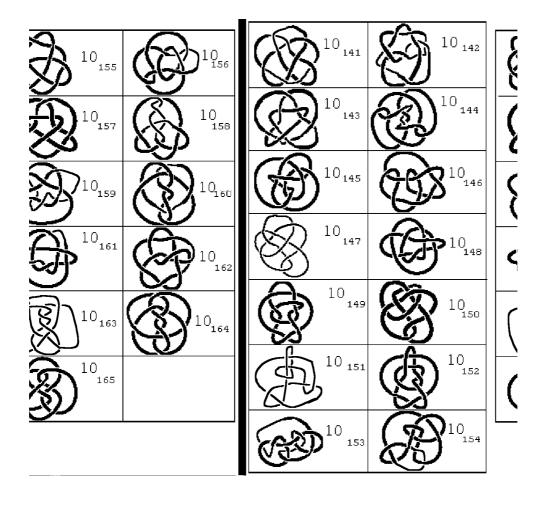
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