Maximum Likelihood

Detail: $\log L\,(\theta|x) = \sum_{i=1}^{n} logp(x_i|\theta)$

Gaussian probability density function:

$$P(X = x|\mu, \sigma^2) = \sum_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\,(-\frac{1}{2\sigma^2}(x_i - \mu)^2)$$

<u>Step 1: Find the log-likelihood function of the Gaussian probability density function</u>

Log-likelihood function:

$$\sum_{i=1}^{n} \log(x|\mu, \sigma^2) = \log\,(\sum_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x_i - \mu)^2\right))$$

$= \log\,[\sum_{i=1}^{n}(2\pi\sigma^2)^{-0.5}] + \log\,[\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right)]$

$= \log\,[(2\pi\sigma^2)^{-0.5n}] + (\sum_{i=1}^{n} -\frac{1}{2\sigma^2}(x_i - \mu)^2)$

$= -0.5n * \log(2\pi\sigma^2) + (\sum_{i=1}^{n} -\frac{1}{2\sigma^2}(x_i - \mu)^2)$

<u>Step 2: Compute the first derivative of the log-likelihood function</u>

For parameter μ:

$$\frac{\partial(logf)}{\partial\mu} = (-0.5n)'(log2\pi\sigma^2) + (log2\pi\sigma^2)'(-0.5n) + \left(-\frac{1}{2\sigma^2}\right)'\left(\sum_{i=1}^{n}(x_i - \mu)^2\right) + \left(\sum_{i=1}^{n}(x_i - \mu)^2\right)'\left(-\frac{1}{2\sigma^2}\right)$$

$= 0 + 0 + \left(-\frac{1}{2\sigma^2}\right) * 2 * \sum_{i=1}^{n}(\mu - x_i)$

= $\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu)$

For parameter σ^2

Set a = σ^2

$$\frac{\partial(logf)}{\partial a} = (-0.5n)'(log2\pi a) + (log2\pi a)'(-0.5n) + \left(-\frac{1}{2a}\right)'\left(\sum_{i=1}^{n}(x_i - \mu)^2\right) + \left(\sum_{i=1}^{n}(x_i - \mu)^2\right)'\left(-\frac{1}{2a}\right)$$

= $\frac{-n}{2a} + \frac{1}{4a^2}\sum_{i=1}^{n}(x_i - \mu)^2$

= $\frac{1}{2a}(-n + \frac{1}{a}\sum_{i=1}^{n}(x_i - \mu)^2)$

Because a = σ^2, then we have

$\frac{\partial(logf)}{\partial a}$= $\frac{1}{2\sigma^2}(-n + \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2)$

<u>Step 3: solve the log-likelihood functions equals to 0 to find the MLE.</u>

Solve $\frac{\partial(logf)}{\partial \mu} = 0$

$\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu) = 0$

$$\sum_{i=1}^{n}(x_i - \mu) = 0$$

$$\sum_{i=1}^{n}\mu = \sum_{i=1}^{n}x_i$$

$$n * \mu = \sum_{i=1}^{n} x_i$$

$$\mu = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Solve $\frac{1}{2\sigma^2}(-n + \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2) = 0$

$$\frac{1}{2\sigma^2}(-n + \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2) = 0$$

$$(-n + \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2 = 0$$

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2 = n$$

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2 = \sigma^2$$

Therefore,

$$\mu_{MLE} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\sigma^2{}_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2$$