STAT22000 Summer 2020 Homework 6 Solutions

Problems to Turn In: due **midnight of Monday, July 6**, on Gradescope

1. Suppose that an IRS examiner correctly detects and flags 90% of all erroneous returns that he reviews. In addition, he mistakenly flags 2% of correct returns that he reviews. Suppose that about 20% of the tax returns he reviews contain errors.

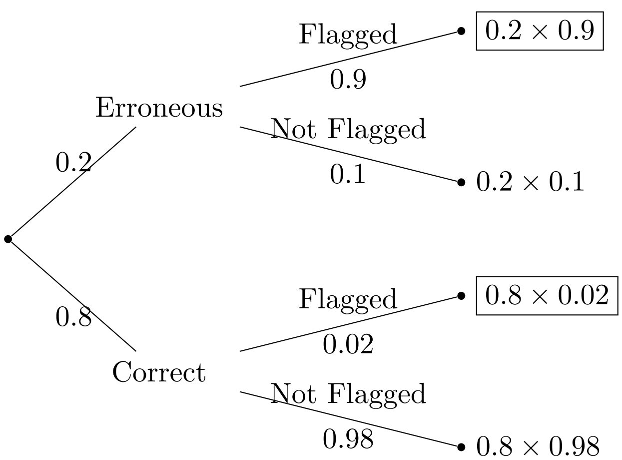
   (a) What proportion of tax returns he reviews are flagged?

   (b) What proportion of tax returns he flagged actually contain errors?

   (c) What proportion of tax returns not flagged actually contain errors?

   Hint: First draw a tree diagram of this scenario.

   [*6 points in total. Okay to omit the tree diagram if the student uses Bayes' theorem.*]



   (a) [*2pts*] $P(\text{flagged}) = 0.2 \times 0.9 + 0.8 \times 0.02 = \boxed{0.196}$

   (b) [*2pts*]

$$P(\text{erroneous} \mid \text{flagged}) = \frac{P(\text{erroneous and flagged})}{P(\text{flagged})} = \frac{0.2 \times 0.9}{0.2 \times 0.9 + 0.8 \times 0.02} = \frac{0.18}{0.196} \approx \boxed{0.92}$$

   (c) [*2pts*]

$$P(\text{erroneous} \,|\, \text{not flagged}) = \frac{P(\text{erroneous and not flagged})}{P(\text{not flagged})} = \frac{0.2 \times 0.1}{0.2 \times 0.1 + 0.8 \times 0.98} = \frac{0.02}{0.804} \approx \boxed{0.025}$$

2. In a game of 4-Spot Keno, the player picks 4 numbers from 1 to 80. The casino randomly selects 20 winning numbers from 1 to 80. If the 4 numbers the player picked are all among the 20 winning numbers, the player receives \$120. If three of the 4 numbers are winning numbers, the player receives \$3. If two of the 4 numbers are winning numbers, the player receives \$1. If only one or none are winning numbers, the player receives \$0. It costs \$1 to play the game. The (approximate) probabilities of picking 4, 3, or 2 winning numbers are given in the table below.

Number of winning numbers picked	0 or 1	2	3	4
Payout	\$0	\$1	\$3	\$120
Net profit = Payout −\$1	−\$1	\$0	\$2	\$119
Probability	?	0.21264	0.04325	0.00306

(a) What is the probability that the payout is \$0?

(b) Compute the expected value of the net profit.

(c) Compute the standard deviation of the net profit.

(d) If one plays the game 100 times, what is the expected total net profit?

(e) Continue the previous part. What is the standard deviation of the total net profit? Note the outcomes of different games are independent of each other as the winning numbers are selected independently each time.

(f) Explain why the probability that the 4 numbers picked are all among the 20 winning numbers is about 0.00306. Show how this is computed. Please note that the 4 numbers the player pick from 1 to 80 must be distinct.

Answer:

(a) *[1pt]* $1 - 0.21264 - 0.04325 - 0.00306 = \boxed{0.74105}$

(b) *[2pts]* The probability distribution for the net profit X is

Net profit X = Payout $-\$1$	$-\$1$	$\$0$	$\$2$	$\$119$
Probability	0.74105	0.21264	0.04325	0.00306

The expect next profit is

$$\mathrm{E}(X) = \sum_i x_i p_i = (-1) \times 0.74105 + 0 \times 0.21264 + 2 \times 0.04325 + 119 \times 0.00306 = \boxed{-\$0.29041}.$$

(c) *[2pts]* The variance of the net profit is

$$\begin{aligned} V(X) &= \sum_i (x_i - \mathrm{E}(X))^2 p_i \\ &= (-1 - (-0.29041))^2 \times 0.74105 + (0 - (-0.29041))^2 \times 0.21264 \\ &\quad + (2 - (-0.29041))^2 \times 0.04325 + (119 - (-0.29041))^2 \times 0.00306 \\ &\approx 0.373 + 0.018 + 0.227 + 43.544 = 44.162 \end{aligned}$$

The SD is hence $\sqrt{V(X)} \approx \sqrt{44.162} \approx \boxed{\$6.645}$. *[If the expected value is wrong, please give full credits for the SD as long as it is computed correctly based on the wrong expected value. The idea is not penalizing one mistake twice.]*

(d) *[2pts]* Let X_i be the net profit from the ith game. The total net profit from the 100 games is $X_1 + X_2 + \cdots + X_{100}$. All X_i's have the same distribution as the X in part (b), which implies $E(X_i) \approx -\$0.29041$, and $V(X_i) \approx 44.162$ for all i. The expected total net profit is hence:

$$\mathrm{E}(X_1 + X_2 + \cdots + X_{100}) = \mathrm{E}(X_1) + \mathrm{E}(X_2) + \cdots + \mathrm{E}(X_{100}) \approx 100 \times (-\$0.29041) = \boxed{-\$29.041}.$$

[Give full mark if the answer = (answer in part (b)) ×100.]

(e) *[2pts]* From part (c), we know the variance of net profit in one game is $V(X_i) \approx 44.162$ Since the games are independent, the variance of the total winnings is the sum of the variance for each game,

$$V(X_1 + X_2 + \cdots + X_{100}) = V(X_1) + V(X_2) + \cdots + V(X_{100}) = 100 \times 44.162 = 4416.2$$

So the SD of the total winning is $\sqrt{4416.2} = \boxed{\$66.4545}$.

[Give full mark if the answer = (SD in part (c)) $\times\sqrt{100}$.]

(f) [2pts] Let W_i be the event that the ith number picked is a winning number. When selecting the first number, there are 80 numbers available and 20 of them are winning numbers, so $P(W_1) = 20/80$.
Given the first number picked is a winning number, 19 of the 79 remaining numbers are winning, so $P(W_2|W_1) = 19/79$.
Given the first two numbers picked are both winning, 18 of the 78 remaining numbers are winning, so $P(W_3|W_1W_2) = 18/78$.
Given the first three numbers picked are all winning, 17 of the 77 remaining numbers are winning, so $P(W_4|W_1W_2W_3) = 18/78$.
By the multiplication rule,

$$\begin{aligned} P(W_1W_2W_3W_4) &= P(W_1)P(W_2|W_1)P(W_3|W_1W_2)P(W_4|W_1W_2W_3) \\ &= \frac{20}{80} \times \frac{19}{79} \times \frac{18}{78} \times \frac{17}{77} \qquad = \frac{969}{316316} \approx 0.00306 \end{aligned}$$

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3. There are many ways to bet your money on the outcome the Nevada roulette. In addition to the popular Red-and-Black bet, here are two more bets.

- *Single Number*: If you bet a dollar on a *single number* at Nevada roulette, and that number comes up, you get the \$1 back together with winnings of \$35. If any other number comes up, you lose the dollar. Gamblers say that a single number pays 35 to 1, and there are 1 chances in 38 to win.
- *Split* If you bet a dollar on a *split* at Nevada roulette. (A *split* is two adjacent numbers, like 11 and 12). If either number comes up, you gets the dollar back, together with winnings of \$17. If neither number comes up, you loses the dollar. So a split pays 17 to I, and there are 2 chances in 38 to win.

A gambler thinks 7 is very likely to come up, so he puts a dollar at 7. He also think 11 and 12 look promising, so he bets another dollar on 11 and 12 as a split. Let X be his net profit from the bet at a single number 17, and Y be his net profit from the bet at the split 11 and 12. The probability distributions of X and Y are as follows.

Outcome	7	Other 37 numbers
Value of X	35	-1
probability	1/38	37/38

Outcome	11 or 12	Other 36 numbers
Value of Y	17	-1
probability	2/38	36/38

(a) Find the expected value and the variance of X.

Answer: *[2pts = 1pt for the expected value + 1pt for the variance]*

$$E(X) = 35 \times (1/38) + (-1) \times (37/38) = -2/38 = \boxed{-1/19}$$

$$V(X) = [35 - (-\frac{1}{19})]^2 \times \frac{1}{38} + [-1 - (-\frac{1}{19})]^2 \times \frac{37}{38} = \frac{11988}{361} = \frac{18^2 \times 37}{19^2} \approx \boxed{33.21}$$

(b) Find the expected value and the variance of Y.

Answer: *[2pts = 1pt for the expected value + 1pt for the variance]*

$$E(Y) = 17 \times (2/38) + (-1) \times (36/38) = -2/38 = \boxed{-1/19}$$

$$V(Y) = [17 - (-\frac{1}{19})]^2 \times \frac{2}{38} + [-1 - (-\frac{1}{19})]^2 \times \frac{36}{38} = \frac{2 \times 18^2}{19^2} = \frac{5832}{361} \approx \boxed{16.155}$$

(c) If the gambler put the two bets in one spinning, let $T = X + Y$ be his total net profit from the two bets. The probability distribution of T can be found as follows.

Outcome	7	11 or 12	the other 35 numbers
value of X	35	-1	-1
value of Y	-1	17	-1
value of $T = X + Y$	34	16	-2
Probability	1/38	2/38	35/38

Find the expected value and the variance of T.

Answer: [*2pts = 1pt for the expected value + 1pt for the variance*]

$$E(T) = 34 \times (1/38) + 16 \times (2/38) + (-2) \times (35/38) = -4/38 = \boxed{-2/19}$$

$$V(T) = [34 - (-\frac{2}{19})]^2 \times \frac{1}{38} + [16 - (-\frac{2}{19})]^2 \times \frac{2}{38}[-2 - (-\frac{2}{19})]^2 \times \frac{35}{38}$$

$$= \frac{17172}{361} \approx \boxed{47.568}$$

(d) Are the events $\{X = 35\}$ and $\{X = 17\}$ independent? Explain briefly.

Answer: [*1pt*] They are NOT independent since $P(X = 35|X = 17) = 0 \neq P(X = 35) = 1/38$. Or simply speaking, one cannot with both the single bet and the split bet at the same time because he bet on different numbers.

(e) Is the variance of T equal to the sum of the variance of X and the variance of Y? Explain briefly.

Answer: [*1pt*] $V(X) + V(Y) = 33.21 + 16.155 = 49.365 \neq 47.568 = V(T)$. This is because X and Y are not independent as they are bets in one spinning. One cannot win both bets.
