STAT22000 Summer 2020 Homework 5 Solutions

Problems to Turn In: due **midnight** on **Sunday, July 5, on Gradescope**.

1. Suppose in a certain state, 60% of the people have a Visa credit card, 40% have a MasterCard, and 24% have both. Let V be the event that a randomly selected individual has a Visa credit card, and M be the analogous event for a MasterCard. So $P(V) = 0.60$, $P(M) = 0.40$, and $P(V \text{ and } M) = 0.24$.

   Please **show your work** to get full credit for all the parts below. *Hint: Draw a Venn Diagram.*

   (a) What is the probability that the randomly selected individual has at least one of the two types of cards?

   (b) What is the probability that the randomly selected individual has neither type of credit card?

   (c) What is the probability that the randomly selected individual has a MasterCard but no Visa credit card?

   (d) If the randomly selected individual is known to have a MasterCard, what is probability that he/she also owns a Visa card?

   (e) If the randomly selected individual is known to have no MasterCard, what is probability that he/she owns a Visa card?

   (f) Are the events V and M independent? Explain briefly.

   (g) Are the events V^c and M^c disjoint? Explain briefly.

Answer:

(a) *[2pts]* Using the General Addition Rule:

$$\begin{aligned} P(V \text{ or } M) &= P(V) + P(M) - P(V \text{ and } M) \\ &= 0.60 + 0.40 - 0.24 = \boxed{0.76} \end{aligned}$$

(b) *[2pts]* Note that the complement of (A and B) is (A^c or B^c). One can draw a Venn Diagram to verify it. By the complementation rule, we have

$$\begin{aligned} P(\text{no Visa and no MasterCard}) &= P(V^c \text{ and } M^c) \\ &= P((V \text{ or } M)^c) \\ &= 1 - P(V \text{ or } M) = 1 - 0.76 = \boxed{0.24}. \end{aligned}$$

where $P(V \text{ or } M) = 0.76$ is calculated in (a).

(c) *[2pt]* $P(M \text{ and } V^c) = P(M) - P(V \text{ and } M) = 0.40 - 0.24 = \boxed{0.16}$.
(Draw a Venn Diagram, from which we can see that excluding the percentage of those who have both V and M from the percentage of those who have M we can get the percentage of those who have M but not V.)

(d) *[2pts]*

$$P(V|M) = \frac{P(V \text{ and } M)}{P(M)} = \frac{0.24}{0.40} = \boxed{0.6}$$

(e) *[2pts]* By definition,

$$P(V|M^c) = \frac{P(V \text{ and } M^c)}{P(M^c)},$$

in which

$$P(V \text{ and } M^c) = P(V) - P(V \text{ and } M) = 0.60 - 0.24 = 0.36$$
$$P(M^c) = 1 - P(M) = 1 - 0.40 = 0.6.$$

So

$$P(V|M^c) = \frac{P(V \text{ and } M^c)}{P(M^c)} = \frac{0.36}{0.6} = \boxed{0.6}.$$

(f) [3pts] Yes, they are independent. The reason can be the equality of any two of the following:

- $P(V) = 0.60$
- $P(V|M) = 0.60$
- $P(V|M^c) = 0.60$

Knowing the individual has a MasterCard doesn't affect the probability that he/she has a Visa card. Alternatively, one can also argue that $P(V \text{ and } M) = 0.24 = P(V)P(M)$ to prove the independence.

(g) [1pt] No, as calculated in the part (b), $P(V^c \text{ and } M^c) = 0.24 > 0$. The intersection of V^c and M^c is not empty, and hence V^c and M^c are not disjoint.

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2. A 6-face die is rolled 4 times. The six faces of the die have 1, 2, 3, 4, 5, 6 spots respectively. The die is fair meaning that all six faces are equally likely to come up. Find the probability of the following outcomes:

(a) All 4 rolls show 3 or more spots

(b) None of the 4 rolls show 3 or more spots. That is, all 4 rolls show 1 or 2 spots.

(c) Six-spot comes up at least once in 4 rolls

Please **show your work** to get full credit.

Answer: *[6 points in total, 2 points each]*

(a) As the die is fair, the probability of getting 3 or more spots in one roll is $4/6 = 2/3$. As the rolls are independent,

$$\begin{aligned} P(3 \text{ or more in all 4 rolls}) &= P(3 \text{ or more, } 3 \text{ or more, } 3 \text{ or more, } 3 \text{ or more}) \\ &= P(3 \text{ or more})P(3 \text{ or more})P(3 \text{ or more})P(3 \text{ or more}) \\ &= (2/3)(2/3)(2/3)(2/3) = (2/3)^4 \approx 0.1975 \end{aligned}$$

(b) The probability of getting 1 or 2 spots in one roll is $2/6 = 1/3$. As the rolls are independent,

$$\begin{aligned} & P(1 \text{ or } 2 \text{ spots in all 4 rolls}) \\ =& P(1 \text{ or } 2 \text{ spots, } 1 \text{ or } 2 \text{ spots, } 1 \text{ or } 2 \text{ spots, } 1 \text{ or } 2 \text{ spots}) \\ =& P(1 \text{ or } 2 \text{ spots})P(1 \text{ or } 2 \text{ spots})P(1 \text{ or } 2 \text{ spots})P(1 \text{ or } 2 \text{ spots}) \\ =& (1/3)^4 \approx 0.01234568. \end{aligned}$$

(c) P(at least 1 six spot) $= 1 - P(\text{no six spots}) = 1 - (5/6)^4 \approx 1 - 0.4823 = 0.5177$.

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3. Five cards are drawn at random without replacement from a well-shuffled deck of poker cards

(a) Find the probability that the 5th card is a queen, given that the first 4 cards are 3 Kings and a Queen.

(b) Find the probability that the first 3 cards are Kings and the next two cards are the Queens. You may leave the answer as a product of fractions.

Answer:

(a) [*2pts*] When drawing the 5th card, there are $52 - 4 = 48$ cards remain in the deck. As the first four cards are 3 Kings and a Queen, there are 1 King and 3 Queens remains in the deck. So the probability is $3/48 = 1/16$.

(b) [*3pts*] Let K_i be the event that the ith card dealt is a King, $i = 1, 2, 3$ and Q_i be the event that the ith card is a Queen, $i = 4, 5$.

- $P(K_1) = P(\text{1st card is a King}) = 4/52$
- $P(K_2|K_1) = P(\text{2nd is a King}|\text{1st card is a King}) = 3/51$.
- $P(K_3|K_1K_2) = P(\text{3rd card is a King}|\text{the first two cards are Kings}) = 2/50$.
- $P(Q_4|K_1K_2K_3) = P(\text{4th card is a Queen}|\text{the first 3 cards are Kings}) = 4/49$.
- $P(Q_5|K_1K_2K_3Q_4) = P(\text{4th card is a Queen}|\text{the first 4 cards are 3Ks and 1Q}) = 3/48$ is found in part (a)
- By the General Multiplication Rule,

$$\begin{aligned} P(K_1K_2K_3Q_4Q_5) &= P(K_1)P(K_2|K_1)P(K_3|K_1K_2)P(Q_4|K_1K_2K_3)P(Q_5|K_1K_2K_3Q_4) \\ &= \frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} \times \frac{4}{49} \times \frac{3}{48} \end{aligned}$$
