Task B

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2024-11-05

B.1

(1)

According to the background of B.1, the probability density function $p_\lambda(x)$ of a random variable X is:

$$p_\lambda(x) = \begin{cases} ae^{-\lambda(x-b)} & \text{if } x \geq b, \\ 0 & \text{if } x < b, \end{cases}$$

where: \$ b > 0 \$ is a known constant, $\lambda > 0$ is a parameter of the distribution, \$ a \$ is a constant to be determined in terms of λ and b.

According to the definition of probability density function, $p_\lambda(x)$ must integrate to 1 over its domain, so an equation can be written:

$$\int_{-\infty}^{\infty} p_\lambda(x)\, dx = 1.$$

when $x < b$, $p_\lambda(x) = 0$, so we only need to calculate the integral from $x = b$ to $x = \infty$. To set up the integral, the equation can be written:

$$\int_{b}^{\infty} ae^{-\lambda(x-b)}\, dx = 1.$$

a is a constant number, so it can be factored out and just calculate the remaining part:

$$a\int_{b}^{\infty} e^{-\lambda(x-b)}\, dx = 1.$$

Let $\mu = x - b$, so domain changes to $\{0, \infty\}$ and the euqation should be transformed:

$$a\int_{0}^{\infty} e^{-\lambda\mu}\, d\mu = 1.$$

Solve the integral:

$$\begin{aligned} a\int_{0}^{\infty} e^{-\lambda\mu} d\mu &= -\frac{1}{\lambda} \cdot e^{-\lambda\mu}|_0^\infty \\ &= a \cdot [\lim_{u\to\infty}(-\frac{1}{\lambda} \cdot e^{-\lambda\mu}) - \lim_{u\to 0}(-\frac{1}{\lambda} \cdot e^{-\lambda\mu})] \\ &= a \cdot [0 - (-\frac{1}{\lambda})] \\ &= a \cdot \frac{1}{\lambda} = 1. \end{aligned}$$

so it is obvious that

$$a = \lambda$$

The answer of question(1) is $a = \lambda$

(2)

Mean: The equation of mean of X is:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot p_\lambda(x) dx$$

because $p_\lambda(x) = 0$, when $b < 0$, so the integral can be simplified to:

$$\mathbb{E}[X] = \int_{b}^{\infty} x \cdot p_\lambda(x) dx$$

Then transform x to make the integral easier to solve. Let: $\mu = x - b \Rightarrow x = u + b$, then: $dx = d\mu$ and when $x = b, \mu = 0$, when $x \to \infty, \mu \to \infty$. Substituting, and can get:

$$\mathbb{E}[X] = \int_{0}^{\infty} (\mu + b) \cdot \lambda e^{-\lambda u} d\mu$$

Expand $(u + b)$ and calculate two integrals separately:

$$\mathbb{E}[X] = \int_{0}^{\infty} \lambda u e^{-\lambda u} du + \int_{0}^{\infty} \lambda b e^{-\lambda u} du.$$

For the first integral, use intergation by parts:

$$\begin{aligned}
\int_{0}^{\infty} \lambda u e^{-\lambda u} du &= -\mu^2 e^{-\lambda \mu}|_0^\infty + \int_{0}^{\infty} e^{-\lambda u} du \\
&= \lim_{u \to \infty} (-u e^{-\lambda u}) - \lim_{u \to 0} (-u e^{-\lambda u}) + (-\frac{1}{\lambda} e^{-\lambda \mu})|_0^\infty \\
&= 0 + \lim_{u \to \infty} (-\frac{1}{\lambda} e^{-\lambda \mu}) - \lim_{u \to 0} (-\frac{1}{\lambda} e^{-\lambda \mu}) \\
&= \frac{1}{\lambda}
\end{aligned}$$

For the second integral, this is similar to $\int_0^\infty e^{-\lambda u} du$, so:

$$\int_{0}^{\infty} \lambda b e^{-\lambda u} du = \lambda b \int_{0}^{\infty} e^{-\lambda u} du = \lambda b \cdot \frac{1}{\lambda} = b$$

Combine the first integral and the second integral, the mean is:

$$\mathbb{E}(X) = \frac{1}{\lambda} + b$$

Standard Deviation: To calculate the standard deviation, calculate the variance first. The variance of X is:

$$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

and $\mathbb{E}(X) = \frac{1}{\lambda} + b$ is already known, so this time just calculate $\mathbb{E}(X^2)$. First, write the equation(already know that when $x < b$, $p_\lambda(x) = 0$):

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 p_\lambda(x) dx = \int_{b}^{\infty} \lambda x^2 e^{-\lambda(x-b)} dx$$

Let $\mu = x - b$, then re-write the equation(the transformation steps are the same as the previous proof):

$$\mathbb{E}(X^2) = \lambda \int_{0}^{\infty} (\mu + b)^2 e^{-\lambda \mu} d\mu$$

By expanding the $(\mu + b)^2$, the equation can be separated into three integrals:

$$\mathbb{E}(X^2) = \lambda \int_0^{\infty} u^2 e^{-\lambda u} du + 2b\lambda \int_0^{\infty} u e^{-\lambda u} du + b^2 \lambda \int_0^{\infty} e^{-\lambda u} du.$$

the second integral: $\lambda \int_0^{\infty} u e^{-\lambda u} du = \frac{1}{\lambda}$ and the third integral:$\lambda \int_0^{\infty} e^{-\lambda u} du = 1$ are solved in previous proof, so we can substitute the conclusion later. Now we can only focus on the first integral: $\lambda \int_0^{\infty} u^2 e^{-\lambda u} du$. To solve:

$$\lambda \int_0^{\infty} u^2 e^{-\lambda u} du$$

we can apply integration by parts twice, which is similar to $\lambda \int_0^{\infty} u e^{-\lambda u} du$, so the result is:

$$\lambda \int_0^{\infty} u^2 e^{-\lambda u} du = \frac{2}{\lambda^2}.$$

Substitute these results back, we can get:

$$\mathbb{E}(X^2) = \frac{2}{\lambda^2} + 2b \cdot \frac{1}{\lambda} + b^2$$

Now we can calculate the variance. Substitute the results:

$$\begin{aligned}
Var(X) &= \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \\
&= \frac{2}{\lambda^2} + 2b \cdot \frac{1}{\lambda} + b^2 - (\frac{1}{\lambda} + b)^2 \\
&= \frac{2}{\lambda^2} + 2b \cdot \frac{1}{\lambda} + b^2 - (\frac{1}{\lambda^2} + \frac{2b}{\lambda} + b^2) \\
&= \frac{1}{\lambda^2}
\end{aligned}$$

The standard deviation is: $\sigma_X = \sqrt{\text{Var}(X)} = \frac{1}{\lambda}$

In conclusion, the results are: **Mean**: $\mathbb{E}[X] = \frac{1}{\lambda} + b$. **Standard Deviation**: $\sigma\_X = \sqrt{\text{Var}(X)} = \frac{1}{\lambda}$.