Task B

Zixia Zeng

2024-11-05

B.1

(1)

According to the background of B.1, the probability density function $p_\lambda(x)$ of a random variable X is:

$$p_\lambda(x) = \begin{cases} ae^{-\lambda(x-b)} & \text{if } x \geq b, \\ 0 & \text{if } x < b, \end{cases}$$

where: $b > 0$ is a known constant, $\lambda > 0$ is a parameter of the distribution, $ a $ is a constant to be determined in terms of λ and b.

According to the definition of probability density function, $p_\lambda(x)$ must integrate to 1 over its domain, so an equation can be written:

$$\int_{-\infty}^{\infty} p_\lambda(x)\, dx = 1$$

When $x < b$, $p_\lambda(x) = 0$, so we only need to calculate the integral from $x = b$ to $x = \infty$. To set up the integral, the equation can be written:

$$\int_{b}^{\infty} ae^{-\lambda(x-b)}\, dx = 1$$

a is a constant number, so it can be factored out and just calculate the remaining part:

$$a\int_{b}^{\infty} e^{-\lambda(x-b)}\, dx = 1$$

Let $\mu = x - b$, so domain changes to $\{0, \infty\}$ and the euqation should be transformed:

$$a\int_{0}^{\infty} e^{-\lambda\mu}\, d\mu = 1$$

Solve the integral:

$$\begin{aligned} a\int_{0}^{\infty} e^{-\lambda\mu} d\mu &= -\frac{1}{\lambda} \cdot e^{-\lambda\mu}|_0^\infty \\ &= a \cdot [\lim_{u\to\infty}(-\frac{1}{\lambda} \cdot e^{-\lambda\mu}) - \lim_{u\to 0}(-\frac{1}{\lambda} \cdot e^{-\lambda\mu})] \\ &= a \cdot [0 - (-\frac{1}{\lambda})] \\ &= a \cdot \frac{1}{\lambda} = 1 \end{aligned}$$

Therefore, it is obvious that $a = \lambda$

The answer of question(1) is $a = \lambda$

(2)

Mean:

The equation of mean of X is:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot p_\lambda(x)dx$$

Since $p_\lambda(x) = 0$, when $b < 0$, so the integral can be simplified to:

$$\mathbb{E}[X] = \int_{b}^{\infty} x \cdot p_\lambda(x)dx$$

Then transform x to make the integral easier to solve. Let: $\mu = x - b \Rightarrow x = u + b$, then: $dx = d\mu$ and when $x = b, \mu = 0$, when $x \to \infty, \mu \to \infty$.

Substituting, and can get:

$$\mathbb{E}[X] = \int_{0}^{\infty} (\mu + b) \cdot \lambda e^{-\lambda u} d\mu$$

Expand $(u + b)$ and calculate two integrals separately:

$$\mathbb{E}[X] = \int_{0}^{\infty} \lambda u e^{-\lambda u} du + \int_{0}^{\infty} \lambda b e^{-\lambda u} du.$$

For the first integral, use intergation by parts:

$$\begin{aligned}
\int_{0}^{\infty} \lambda u e^{-\lambda u} du &= -\mu^2 e^{-\lambda\mu}|_0^\infty + \int_{0}^{\infty} e^{-\lambda u} du \\
&= \lim_{u\to\infty}(-ue^{-\lambda u}) - \lim_{u\to 0}(-ue^{-\lambda u}) + (-\frac{1}{\lambda}e^{-\lambda\mu})|_0^\infty \\
&= 0 + \lim_{u\to\infty}(-\frac{1}{\lambda}e^{-\lambda\mu}) - \lim_{u\to 0}(-\frac{1}{\lambda}e^{-\lambda\mu}) \\
&= \frac{1}{\lambda}
\end{aligned}$$

For the second integral, this is similar to $\int_0^\infty e^{-\lambda u} du$, so:

$$\int_{0}^{\infty} \lambda b e^{-\lambda u} du = \lambda b \int_{0}^{\infty} e^{-\lambda u} du = \lambda b \cdot \frac{1}{\lambda} = b$$

Combine the first integral and the second integral, the mean is:

$$\mathbb{E}(X) = \frac{1}{\lambda} + b$$

Standard Deviation:

To calculate the standard deviation, calculate the variance first. The variance of X is:

$$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

$\mathbb{E}(X) = \frac{1}{\lambda} + b$ is already known, so this time just calculate $\mathbb{E}(X^2)$.

First, write the equation(already know that when $x < b$, $p_\lambda(x) = 0$):

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 p_\lambda(x)dx = \int_{b}^{\infty} \lambda x^2 e^{-\lambda(x-b)}dx$$

Let $\mu = x - b$, then re-write the equation(the transformation steps are the same as the previous proof):

$$\mathbb{E}(X^2) = \lambda \int_{0}^{\infty} (\mu + b)^2 e^{-\lambda\mu} d\mu$$

By expanding the $(\mu + b)^2$, the equation can be separated into three integrals:

$$\mathbb{E}(X^2) = \lambda \int_0^\infty u^2 e^{-\lambda u} du + 2b\lambda \int_0^\infty u e^{-\lambda u} du + b^2 \lambda \int_0^\infty e^{-\lambda u} du.$$

The second integral: $\lambda \int_0^\infty u e^{-\lambda u} du = \frac{1}{\lambda}$ and the third integral:$\lambda \int_0^\infty e^{-\lambda u} du = 1$ are solved in previous proof, so we can substitute these conclusions later.

Now we can only focus on the first integral: $\lambda \int_0^\infty u^2 e^{-\lambda u} du$.

To solve:

$$\lambda \int_0^\infty u^2 e^{-\lambda u} du$$

we can apply integration by parts twice, which is similar to $\lambda \int_0^\infty u e^{-\lambda u} du$, so the result is:

$$\lambda \int_0^\infty u^2 e^{-\lambda u} du = \frac{2}{\lambda^2}.$$

Substitute these results back, we can get:

$$\mathbb{E}(X^2) = \frac{2}{\lambda^2} + 2b \cdot \frac{1}{\lambda} + b^2$$

Now we can calculate the variance. Substituting the results:

$$\begin{aligned} Var(X) &= \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \\ &= \frac{2}{\lambda^2} + 2b \cdot \frac{1}{\lambda} + b^2 - \left(\frac{1}{\lambda} + b\right)^2 \\ &= \frac{2}{\lambda^2} + 2b \cdot \frac{1}{\lambda} + b^2 - \left(\frac{1}{\lambda^2} + \frac{2b}{\lambda} + b^2\right) \\ &= \frac{1}{\lambda^2} \end{aligned}$$

The standard deviation is: $\sigma_X = \sqrt{\text{Var}(X)} = \frac{1}{\lambda}$

In conclusion, the results are: **Mean**: $\mathbb{E}[X] = \frac{1}{\lambda} + b$.

Standard Deviation: $\sigma\_X = \sqrt{\text{Var}(X)} = \frac{1}{\lambda}$.

(3)

cumulative distribution function(CDF):

Assume CDF of X is:

$$F_\lambda(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^x p_\lambda(t) dt$$

Case $x < b$: $P(X \leq x) = 0$ because the probability density function $p_\lambda(x) = 0$.

Therefore, we can get:

$$F_\lambda(x) = 0, \quad when\ x < b.$$

Case $x \geq b$: we need to integrate $p_\lambda(t)$ from $t = b$ to $t = x$:

$$F_\lambda(x) = \int_b^x \lambda e^{-\lambda(t-b)} dt.$$

Let $u = t - b$, so $t = u + b$ and $dt = du$ so, When $t = b$, $u = 0$, When $t = x$, $u = x - b$.

Substituting into the integral, we get:

$$F_\lambda(x) = \lambda \int_0^{x-b} e^{-\lambda u}\, du.$$

Now the integral can be solved:

$$\begin{aligned} F_\lambda(x) &= \lambda \cdot \left(-\frac{1}{\lambda} e^{-\lambda\mu} \right) |_0^{x-b} \\ &= \lambda \cdot \left(-\frac{1}{\lambda} e^{-\lambda(x-b)} + \frac{1}{\lambda} e^0 \right) \\ &= 1 - e^{-\lambda(x-b)} \end{aligned}$$

Combining the two cases, the cumulative distribution function is:

$$F_\lambda(x) = \begin{cases} 0 & \text{if } x < b, \\ 1 - e^{-\lambda(x-b)} & \text{if } x \geq b. \end{cases}$$

Quantile Function:

Quantile function is the inverse of cumulative distribution function(CDF), so for $x \geq b$, the CDF is:

$$F_\lambda(x) = 1 - e^{-\lambda(x-b)}$$

Set $F_{\lambda(x)} = p$ and find the relationship for x to p:

$$\begin{aligned} e^{-\lambda(x-b)} &= 1 - p \\ -\lambda(x-b) &= \ln(1-p) \\ x = b - \frac{1}{\lambda} \cdot \ln(1-p) \end{aligned}$$

Therefore, the quantile function of X is:

$$x = b - \frac{1}{\lambda} \cdot \ln(1-p),\ for\ p \in [0, 1)$$

In conlusion:

Cumulative distribution function: $F_\lambda(x) = \begin{cases} 0 & \text{if } x < b, \\ 1 - e^{-\lambda(x-b)} & \text{if } x \geq b. \end{cases}$

Quantile function:$x = b - \frac{1}{\lambda} \cdot \ln(1-p),\ for\ p \in [0, 1)$

(4)

The likelihood function is defined as:

$$L(\lambda) = \prod_{i=1}^{n} p_\lambda(X_i)$$

Since $\because p_\lambda(X_i) = 0$, when$X_i < b$, so for all $x < b$ the likelihood is zero. Thus, for $X \geq b$:

$$L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda(X_i - b)}$$

Expanding the product, we can get:

$$L(\lambda) = \lambda^n e^{-\lambda \sum_{i=1}^{n}(X_i - b)}$$

Then transfer it to log-likelihood function:

$$\ell(\lambda) = \ln L(\lambda) = n \ln(\lambda) - \lambda \sum_{i=1}^{n} (X_i - b)$$

To find the maximum likelihood estimator for λ, we need to differentiate the log-likelihood function and set it equals to zero.

$$\frac{d\ell(\lambda)}{d\lambda} = 0$$

Substituting the equation:

$$\frac{d(n \ln(\lambda) - \lambda \sum_{i=1}^{n} (X_i - b))}{d\lambda} = 0 n \cdot \frac{1}{\lambda} - \sum_{i=1}^{n} (X_i - b) = 0 \lambda = \frac{n}{\sum_{i=1}^{n} (X_i - b)}$$

The maximum likelihood estimate (MLE) for λ is:

$$\hat{\lambda}_{\text{MLE}} = \frac{n}{\sum_{i=1}^{n} (X_i - b)}.$$

Step 4: Solve for λ

Now, solve for λ by isolating it on one side of the equation:

$$\frac{n}{\lambda} = \sum_{i=1}^{n} (X_i - b).$$

Multiply both sides by λ and divide by $\sum_{i=1}^{n} (X_i - b)$:

$$\lambda = \frac{n}{\sum_{i=1}^{n} (X_i - b)}.$$

Final Answer

The maximum likelihood estimate (MLE) for λ is:

$$\hat{\lambda}_{\text{MLE}} = \frac{n}{\sum_{i=1}^{n} (X_i - b)}.$$