Mathematical Finance, Vol. 1, No. 4 (October 1991), 77–94

INTEREST RATE OPTION PRICING WITH POISSON-GAUSSIAN FORWARD RATE CURVE PROCESSES

HIROSHI SHIRAKAWA [1]

Institute of Human and Social Sciences, Tokyo Institute of Technology, Tokyo 152, Japan

We study a continuous trading bond model where the associated forward rate curve follows a multidimensional Poisson-Gaussian process. The bond market is complete, and the unique arbitrage-free interest rate call option price is explicitly derived.

KEYWORDS: Heath, Jarrow, Morton model, Poisson-Gaussian process, interest rate options

1. INTRODUCTION

This paper concerns the arbitrage-free pricing of interest rate options. The earliest bond pricing models used a finite number of state variables to express the bond price processes (see Ahn and Thompson 1988, Cox, Ingersoll, and Ross 1985a, 1985b, and Richard 1978). Vasicek (1977) developed an arbitrage-free bond pricing model based on one state variable, the short-term interest rate. Assuming the bond price is a functional of time, maturity, and the short-term interest rate, which follows a one-dimensional diffusion process, he derived a bond price function as a stochastic integral of the short-term interest rate process. Brennan and Schwartz (1979, 1982) extended Vasicek's approach to two state variables, the short- and long-term interest rates. Using a no-arbitrage argument, they derived a partial differential equation satisfied by the bond pricing function. Cox, Ingersoll, and Ross (1981) studied a general equilibrium model for an economy characterized by a finite number of state variables. In this model, the arbitrage-free expected rate of return for the bonds is given by a multifactor CAPM-type model. Interest rate options are priced by solving a partial differential equation in terms of preferences (Courtadon 1982). Unfortunately, this procedure requires an estimation of the unobservable risk premium processes.

For practical applications, it is important to utilize a model with only parameters that can be estimated from observable data. From this point of view, interest rate option pricing models in complete markets are proposed. Ho and Lee (1986) pioneered this approach with a discrete trading model which is consistent with any initial term structure. The main advantage of Ho and Lee's model is that it requires only two parameters. Heath, Jarrow, and Morton (1989a, 1989b) studied a continuous trading model where the associated forward rate curve is driven by a multidimensional Itô process. They characterized the set of conditions on the forward rate curve process necessary and sufficient for the existence of a unique equivalent martingale measure. Under these conditions, the risk premium processes are uniquely identified by the forward rate curve process, and unique arbitrage-

The author is grateful to the editor and the referees for their constructive comments and helpful suggestions.

[1]This research was supported by Grant-in-Aid for Scientific Research of the Ministry of Education, Science and Culture, Grant No. 63490010.

Manuscript received December 1990; final revision received June 1991.

free interest rate option prices are obtainable. For a simple model, they derived an analytic interest rate call option equation similar to the original Black and Scholes formula.

The purpose of this paper is to extend this approach to discontinuous forward rate dynamics. We let the forward rate curve satisfy a family of multidimensional Poisson and Gaussian processes. This assumption yields a simple model with only a few parameters to estimate. The structural advantage of this model is completeness of the corresponding bond market. Recent empirical studies by Brown and Dybvig (1986), Jarrow and Rosenfeld (1984), and Stambaugh (1988) suggest that the bond price processes do indeed follow a discontinuous sample path process with multifactors. Hence, this model may provide a better description of the actual bond price process.

This paper is organized as follows. In Section 2 we define some terminology and describe the bond market model. It is shown that the bond market is complete and conditions for the arbitrage-free forward rate curve are provided. In Section 3 the interest rate call option pricing formula and a related comparison theorem are derived. For the constant volatility case, arbitrage-free call option prices are numerically evaluated for various parameters. In Section 4 we summarize the results obtained and discuss further research.

2. THE BOND MARKET MODEL

Consider a default-free discount bond market where arbitrary maturity bonds are traded continuously within a time horizon $[0, T]$. Let $P(t, s)$ denote the price at time t of a discount bond maturing at time s, $t \leq s \leq T$, with maturity value $P(s, s) = 1$.

The forward rate $f(t, s)$ at time t for maturity s, $t \leq s \leq T$, is defined by

$$f(t, s) = -\frac{\partial}{\partial s} \log P(t, s). \tag{2.1}$$

We assume that the left side of (2.1) is well defined, almost surely. From definition (2.1), the bond prices $P(t, s)$, $t \leq s \leq T$, associated with the forward rate curve $\{f(t, s) \parallel t \leq s \leq T\}$ are given by

$$P(t, s) = \exp\left\{-\int_t^s f(t, u)\, du\right\}, \qquad t \leq s \leq T. \tag{2.2}$$

The forward rate $f(t, t)$ at time t for maturity t is called the spot rate $r(t)$ at time t; that is,

$$r(t) = f(t, t). \tag{2.3}$$

For convenience, we introduce as *numeraire*, $B(t)$, the value of a money market account starting with a dollar investment at time 0; that is,

$$B(t) = \exp\left\{\int_0^t r(s)\, ds\right\}. \tag{2.4}$$

In this paper we examine forward rate curves that belong to a specific family of multidimensional Poisson-Gaussian processes. The forward rate curve $\{f(t, s) \parallel t \leq s \leq T\}$ is assumed to follow the system of stochastic differential equations

$$df(t, s) = \alpha(s - t)\, dt + \delta(s - t)\{\beta\, dW(t) + \boldsymbol{\gamma}^{\top}(d\mathbf{Q}(t) - \boldsymbol{\lambda}\, dt)\}, \quad 0 \le t \le s \le T, \tag{2.5}$$

where

$\{f(0, s) \parallel 0 \le s \le T\}$ = given initial forward rates curve

$\alpha: [0, T] \to R_+$ = expected drift function

$\delta: [0, T] \to R_+$ = positive-valued volatility scaling function

β = positive constant volatility scalar for Gaussian risk

$\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_k)^{\top}$, a k-dimensional constant jump size vector for Poissonian risk ($\gamma_i \ne 0$, $\gamma_i \ne \gamma_j$ if $i \ne j$)

$\mathbf{W} = \{W(t) \parallel 0 \le t \le T\}$, a standard Gaussian process on complete probability space $(\Omega, \mathcal{F}, P)$

$\mathbf{Q} = \{\mathbf{Q}(t) = (Q_1(t), \ldots, Q_k(t))^{\top} \parallel 0 \le t \le T\}$, a k-dimensional independent Poisson process associated with a constant intensity vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)^{\top}$ on the complete probability space $(\Omega, \mathcal{F}, P)$

The vectors $\mathbf{W}$ and $\mathbf{Q}$ are mutually independent.

Equation (2.5) means that the stochastic behavior of the forward rate curve is driven by a multifactor Poisson-Gaussian process scaled by $\delta(\cdot)$. The scaling function $\delta(\cdot)$ is introduced to reflect the volatility of the forward rates associated with a common risk term. In practice, the long-term interest rate is less volatile than the short-term interest rate. Notice that any multifactor Gaussian process can be represented by a single Gaussian process whenever the quadratic variational process of any forward rate pair $f(t, u)$ and $f(t, v)$, $u \ne v$, depends only on $\delta(u - t)$ and $\delta(v - t)$.

From (2.3) and (2.5), the forward rate $f(t, s)$ and the spot rate $r(t)$ are expressed as

$$\begin{aligned} f(t, s) = f(0, s) &+ \int_0^t \alpha(s - u)\, du + \beta \int_0^t \delta(s - u)\, dW(u) \\ &+ \boldsymbol{\gamma}^{\top} \int_0^t \delta(s - u)(d\mathbf{Q}(u) - \boldsymbol{\lambda}\, du) \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} r(t) = f(0, t) &+ \int_0^t \alpha(t - u)\, du + \beta \int_0^t \delta(t - u)\, dW(u) \\ &+ \boldsymbol{\gamma}^{\top} \int_0^t \delta(t - u)(d\mathbf{Q}(u) - \boldsymbol{\lambda}\, du). \end{aligned} \tag{2.7}$$

Thus whenever the forward rates curve process is modeled, we have both the bond price $P(t, s)$ and the discount factor $B(t)$ through (2.2) and (2.4).

From the generalized Itô lemma (Elliott 1982) we can show that $P(t, s)$ and $B(t)$ follow from the stochastic differential equations

$$\frac{dP(t, s)}{P(t-, s)} = [r(t) + \alpha(s - t)]\, dt - \beta\zeta(s - t)\, dW(t) + \sum_{j=1}^{k} (\eta_j(s - t) - 1)\, dQ_j(t), \tag{2.8}$$

where

$$\zeta(u) = \int_0^u \delta(v)\, dv, \tag{2.9}$$

$$\eta_j(u) = \exp\left\{-\gamma_j \int_0^u \delta(v)\, dv\right\},$$

$$a(u) = -\int_0^u \alpha(v)\, dv + \frac{1}{2}\zeta(u)^2\beta^2 + \zeta(u)\boldsymbol{\gamma}^{\mathsf{T}}\boldsymbol{\lambda},$$

and

$$dB(t)/B(t-) = r(t)\, dt. \tag{2.10}$$

Since the discount factor $B(t)$ is strictly positive, almost surely, the relative bond price $Z(t, s)$ normalized by $B(t)$ is well defined; that is,

$$Z(t, s) \equiv P(t, s)/B(t). \tag{2.11}$$

Then from (2.11) and the generalized Itô lemma, we have

$$\frac{dZ(t, s)}{Z(t-, s)} = \frac{dP(t, s)}{P(t-, s)} - \frac{dB(t)}{B(t-)}. \tag{2.12}$$

Hereafter we shall consider the completeness of the bond market associated with the forward rate curve process represented by (2.5). First we define the basic bonds which span the investment opportunities in the bond market at any time. Let $h_j(s)$, $1 \le j \le k + 1$, be the residual maturity defined by

$$h_j(s) = \inf\left\{u \,\|\, \zeta(u) \ge \frac{j}{k + 1}\zeta(s)\right\}. \tag{2.13}$$

Since $\zeta(\cdot)$ is continuous and strictly increasing, the $h_j(s)$ are well-defined and the risk coefficient matrix associated with the basic bonds is nonsingular.

LEMMA 2.1. *Suppose that the forward rate curve process satisfies* (2.5). *Then at any time t the set of investment opportunities in the bond market is spanned by the basic* *bonds associated with the maturities* $t + h_j(s - t)$, $1 \leq j \leq k + 1$, *for all* s, $t < s \leq T$; *that is,*

$$(2.14)\qquad \det \begin{pmatrix} -\beta \dfrac{\zeta(s-t)}{k+1} & \eta_1(h_1(s-t)) - 1 & \cdots & \eta_k(h_1(s-t)) - 1 \\ \vdots & & & \vdots \\ -\beta\zeta(s-t) & \eta_1(h_{k+1}(s-t)) - 1 & \cdots & \eta_k(h_{k+1}(s-t)) - 1 \end{pmatrix} \neq 0.$$

Proof. From the definition of $h_j(s)$, $\eta_i(h_j(s-t)) = \eta_1(h_j(s-t))^i$. Hence, we have

$$\det \begin{pmatrix} -\beta \dfrac{\zeta(s-t)}{k+1} & \eta_1(h_1(s-t)) - 1 & \cdots & \eta_k(h_1(s-t)) - 1 \\ \vdots & & & \vdots \\ -\beta\zeta(s-t) & \eta_1(h_1(s-t))^{k+1} - 1 & \cdots & \eta_k(h_k(s-t))^{k+1} - 1 \end{pmatrix}$$

$$= \det \begin{pmatrix} -\beta \dfrac{\zeta(s-t)}{k+1} & \eta_1(h_1(s-t)) - 1 & \cdots & \eta_k(h_1(s-t)) - 1 \\ 0 & (\eta_1(h_1(s-t)) - 1)^2 & \cdots & (\eta_k(h_1(s-t)) - 1)^2 \\ \vdots & & & \vdots \\ 0 & (\eta_1(h_1(s-t))^k - 1)(\eta_1(h_1(s-t)) - 1) & \cdots & (\eta_k(h_1(s-t))^k - 1)(\eta_k(h_1(s-t)) - 1) \end{pmatrix}$$

$$= -\beta \frac{\zeta(s-t)}{k+1} \prod_{i=1}^{k} (\eta_i(h_1(s-t)) - 1) \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & \eta_1(h_1(s-t)) - 1 & \cdots & \eta_k(h_1(s-t)) - 1 \\ \vdots & & & \vdots \\ 0 & \eta_1(h_1(s-t))^k - 1 & \cdots & \eta_k(h_1(s-t))^k - 1 \end{pmatrix}$$

$$= -\beta \frac{\zeta(s-t)}{k+1} \prod_{i=1}^{k} (\eta_i(h_1(s-t)) - 1)^2 \prod_{i=1}^{k-1} \prod_{j=i+1}^{k} (\eta_j(h_1(s-t)) - \eta_i(h_1(s-t))).$$

The last equation follows from Vandermonde's determinant. Since $\eta_i(h_1(s-t)) = \eta_1(h_1(s-t))^i < 1$, (2.14) follows. □

The basic bonds may exist even for more complicated forward rate curve models. In our modeling, however, the basic bonds are always identified by the functions $h_j(\cdot)$, $1 \leq j \leq k + 1$, for any pair (t, s) such that $0 \leq t \leq s \leq T$.

Next we shall characterize the drift function associated with the arbitrage-free forward rate curve process.

THEOREM 2.1. *Suppose that there are no arbitrage opportunities in the bond market associated with the forward rate curve process represented by* (2.5). *Then the expected drift function is expressible as*

$$(2.15)\qquad a(s) = -\beta\zeta(s)\phi - \sum_{j=1}^{k} (\eta_j(s) - 1)\psi_j,$$

where ϕ is a constant scalar and $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_k)^\top$ is a k-dimensional strictly positive constant vector. Equivalently,

$$\alpha(s) = \delta(s)\left(\beta\phi - \sum_{j=1}^{k} \gamma_j \eta_j(s)\psi_j + \beta^2\zeta(s) + \boldsymbol{\gamma}^\top\boldsymbol{\lambda}\right). \tag{2.16}$$

Proof. If there are no arbitrage opportunities, there exist risk premium processes $\phi = \{\phi(t) \parallel 0 \le t \le T\}$ and $\boldsymbol{\psi} = \{\boldsymbol{\psi}(t) = (\psi_1(t), \ldots, \psi_k(t))^\top \parallel 0 \le t \le T\}$ such that

$$\alpha(s-t) = -\beta\zeta(s-t)\phi(t) - \sum_{j=1}^{k} (\eta_j(s-t) - 1)\psi_j(t), \tag{2.17}$$

where $\boldsymbol{\psi}(t) > \mathbf{0}$, $0 \le t < s \le T$ (see Shirakawa 1990, Theorem 4.1). From Lemma 2.1, these risk premiums are uniquely determined by

$$\begin{pmatrix} \phi(t) \\ -\psi_1(t) \\ \vdots \\ -\psi_k(t) \end{pmatrix} = \tag{2.18}$$

$$\begin{pmatrix} -\beta\dfrac{\zeta(s-t)}{k+1} & \eta_1(h_1(s-t)) - 1 & \cdots & \eta_k(h_1(s-t)) - 1 \\ \vdots & & & \vdots \\ -\beta\zeta(s-t) & \eta_1(h_{k+1}(s-t)) - 1 & \cdots & \eta_k(h_{k+1}(s-t)) - 1 \end{pmatrix}^{-1} \begin{pmatrix} \alpha\dfrac{\zeta(s-t)}{k+1} \\ \vdots \\ \alpha(\zeta(s-t)) \end{pmatrix},$$

where $0 \le t < s \le T$. Since (2.18) holds for any pair (t, s) such that $s - t$ is constant, these risk premium processes are independent of time t. Therefore $a(\cdot)$ results in the form of (2.15). As (2.15) holds for any $0 \le s \le T$, we have

$$\begin{aligned} \frac{d\alpha(s)}{ds} &= -\alpha(s) + \delta(s)(\beta^2\zeta(s) + \boldsymbol{\gamma}^\top\boldsymbol{\lambda}) \\ &= \delta(s)\left(-\beta\phi + \sum_{j=1}^{k} \gamma_j\eta_j(s)\psi_j\right), \end{aligned}$$

for all $0 \le s \le T$. Thus (2.16) follows. $\square$

Theorem 2.1 means that the arbitrage-free forward rate curve process represented by (2.5) satisfies

$$df(t,s) = \delta(s-t)\left[\beta^2\zeta(s-t)\,dt + \beta\,dW^*(t) + \sum_{j=1}^{k}\gamma_j(dQ_j(t) - \eta_j(s-t)\psi_j\,dt)\right], \tag{2.19}$$

$$f(t,s) = f(0,s) + \frac{1}{2}\beta^2(\zeta^2(s) - \zeta^2(s-t)) + \sum_{j=1}^{k}(\eta_j(s) - \eta_j(s-t))\psi_j + \beta\int_0^t \delta(s-u)\,dW^*(u) + \boldsymbol{\gamma}^\top\int_0^t \delta(s-u)\,d\mathbf{Q}(u), \tag{2.20}$$

and

$$r(t) = f(0,t) + \frac{1}{2}\beta^2\zeta^2(t) + \sum_{j=1}^{k}(\eta_j(t) - 1)\psi_j + \beta\int_0^t \delta(t-u)\,dW^*(u) + \gamma^\top\int_0^t \delta(t-u)\,d\mathbf{Q}(u), \tag{2.21}$$

where $\mathbf{W}^* = \{W^*(t) = W(t) + \phi t \,\|\, 0 \le t \le T\}$. The arbitrage-free excess return process (2.12) is

$$\frac{dZ(t,s)}{Z(t-,s)} = -\beta\zeta(s-t)\,dW^*(t) + \sum_{j=1}^{k}(\eta_j(s-t) - 1)(dQ_j(t) - \psi_j\,dt). \tag{2.22}$$

From the generalized Itô lemma and (2.10), (2.12), and (2.22), we have

$$d\log P(t,s) = \left[r(t) - \frac{1}{2}\beta^2\zeta^2(s-t) - \sum_{j=1}^{k}(\eta_j(s-t) - 1)\psi_j\right]dt - \beta\zeta(s-t)\,dW^*(t) - \boldsymbol{\gamma}^\top\zeta(s-t)\,d\mathbf{Q}(t).$$

Hence, the arbitrage-free bond price process associated with (2.19) is expressible as

$$P(t,s) = \exp\left\{-\int_t^s f(0,u)\,du - \frac{1}{2}\beta^2\int_t^s(\zeta^2(u) - \zeta^2(s-u))\,du - \sum_{j=1}^{k}\int_t^s(\eta_j(u) - \eta_j(u-t))\,du\,\psi_j - \beta\int_0^t(\zeta(s-u) - \zeta(t-u))\,dW^*(u) - \boldsymbol{\gamma}^\top\int_0^t(\zeta(s-u) - \zeta(t-u))\,d\mathbf{Q}(u)\right\}. \tag{2.23}$$

The last equality follows from the generalized Fubini-type theorem for stochastic integrals (e.g., Ikeda and Watanabe 1989, Lemma 4.1, p. 116).

For a general security market model, Harrison and Pliska (1981, 1983) showed that the security market model is complete if and only if there exists a unique equivalent measure under which all the relative security price processes become martingales. Therefore, completeness of the bond market is proved when the unique existence of the martingale measure is established.

THEOREM 2.2. *Suppose there are no arbitrage opportunities in the bond market associated with the forward rates curve process represented by* (2.5). *Then there exists a unique equivalent measure P^* under which both the excess return process* (2.22) *and the relative bond price process $\mathbf{Z}(s) = \{Z(t, s) \parallel 0 \leq t \leq s\}$ become martingales for any maturity s, $0 \leq s \leq T$. Under this unique equivalent measure P^*, $\mathbf{W}^* = \{W^*(t) = W(t) + \phi t \parallel 0 \leq t \leq T\}$ is a standard Gaussian process and $\mathbf{Q} = \{\mathbf{Q}(t) = (Q_1(t), \ldots, Q_k(t))^\top \parallel 0 \leq t \leq T\}$ is a k-dimensional independent Poisson process associated with a constant intensity vector $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_k)^\top$ such that* $\mathbf{W}^*$ and $\mathbf{Q}$ *are mutually independent.*

Proof. (Existence) From Theorem 2.1, there exist the constant risk premium processes ϕ and $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_k)^\top$ which satisfy (2.16). Let P^* be a measure on $(\Omega, \mathcal{F}, P)$ defined by

$$P^*(A) = E\{1_A \cdot \rho\} \qquad \text{for } A \in \mathcal{F}, \tag{2.24}$$

where

$$\rho = \exp\left\{-\phi W(T) - \frac{1}{2}\phi^2 T + \sum_{j=1}^{k} (\lambda_j - \psi_j)T\right\} \prod_{1 \leq j \leq k} \left(\frac{\psi_j}{\lambda_j}\right)^{Q_j(T)}. \tag{2.25}$$

Since ϕ and ψ are positive constants, we can easily show that $E\{\rho\} = 1$ and $\rho > 1$, almost surely. Therefore, P^* is an equivalent probability measure to P. From Girsanov's theorem for the Gaussian process (Karatzas and Shreve 1987) and the Poisson process (Bremand 1981), $\mathbf{W}^*$ becomes a standard Gaussian process and $\mathbf{Q}$ becomes a k-dimensional independent Poisson processes with a constant intensity vector $\boldsymbol{\psi}$ under P^*. Furthermore, we can easily show that $\mathbf{W}^*$ and $\mathbf{Q}$ are mutually independent under P^* (see Shirakawa 1990, Theorem 4.4). Hence, both the excess return process (2.22) and the relative bond price process $\mathbf{Z}(s)$ become martingales, because $\zeta(\cdot)$ and $\eta_j(\cdot)$, $1 \leq j \leq k$, are uniformly bounded.

(Uniqueness) Let P' be an arbitrary equivalent martingale measure on $(\Omega, \mathcal{F}, P)$. Then from Theorem 4.4 of Shirakawa (1990) there exists a corresponding Radon-Nikodym derivative $\rho' = dP'/dP$ such that

$$\rho' = \exp\left\{-\int_0^T \phi'(t)\, dW(t) - \frac{1}{2}\int_0^T \phi'^2(t)\, dt + \sum_{j=1}^{k} \int_0^T (\lambda_j - \psi'_j(t))\, dt\right\} \times \prod_{1 \leq j \leq k, 0 \leq t \leq T} \left(\frac{\psi'_j(t)}{\lambda_j}\right)^{Q_j(t) - Q_j(t-)}. \tag{2.26}$$

Here $\boldsymbol{\phi}' = \{\phi'(t) \parallel 0 \le t \le T\}$ and $\boldsymbol{\psi}' = \{\psi'(t) = (\psi_1'(t), \ldots, \psi_k'(t))^{\mathsf{T}} \parallel 0 \le t \le T\}$ are risk premium processes which satisfy

$$\text{(2.27)} \qquad \alpha(s - t) = -\beta\zeta(s - t)\phi'(t) - \sum_{j=1}^{k} (\eta_j(s - t) - 1)\psi_j'(t),$$
$$0 \le t < s \le T.$$

Equating (2.16) and (2.27) and substituting $s = h_i(u - t) + t$ for $t < u \le T$, $1 \le i \le k + 1$, we have

$$\beta\zeta(h_i(u - t))(\phi'(t) - \phi) + \sum_{j=1}^{k} (\eta_j(h_i(u - t)) - 1)(\psi_j'(t) - \psi_j) = 0,$$
$$1 \le i \le k + 1,$$

for all $0 \le t < u \le T$. Then from Lemma 2.1, $\boldsymbol{\phi}'(t) = \phi$ and $\boldsymbol{\psi}'(t) = \boldsymbol{\psi}$ for $0 \le t \le T$, almost surely. Hence, $P'(= P^*)$ is unique. □

3. ARBITRAGE-FREE INTEREST RATE OPTION PRICING

In the last section, we showed that the bond market is complete. Here we evaluate arbitrage-free interest rate option prices. Let $\mathcal{F}_t$ be the augmented filtration of $\sigma(\{W(u), \mathbf{Q}(u) \parallel 0 \le u \le t\})$ under P. We use $E^*\{\cdot\}$ to denote the expectation operator with respect to P^*.

DEFINITION 3.1. An *interest rate option* $(\tau, \mathbf{f}, g)$ is a financial instrument consisting of (i) an expiration date $0 \le \tau \le T$, (ii) an $\{\mathcal{F}_t\}$-adapted positive dividend rate process $\mathbf{f} = \{f(t) \parallel 0 \le t \le \tau\}$, and (iii) an $\mathcal{F}_\tau$-measurable positive terminal payoff g at the exercise time τ with (iv) $E^*\{\int_0^\tau f(t)\,dt + g\} < \infty$.

In other words, an interest rate option is a contingent claim whose payoff and dividend at time t are determined by the history of the bond price processes until time t. The fourth condition is required for the martingale representation property under P^* (see Xue 1991, Theorem 3.1). Since the bond market is complete, the arbitrage-free option price is obtained using the unique equivalent martingale measure P^* (see Harrison and Pliska 1981, Proposition 3.31). That is, the arbitrage-free pricing for the interest rate option $(\tau, \mathbf{f}, g)$ at time t, $0 \le t \le \tau$, is

$$\text{(3.1)} \qquad E^*\left\{\int_t^\tau \frac{B(t)}{B(u)} f(u)\,du + \frac{B(t)}{B(\tau)} g \;\middle\|\; \mathcal{F}_t\right\}.$$

In this section, we examine only a call option, which is a financial instrument in the form of $(\tau, \mathbf{0}, (P(\tau, s) - K)^+)$, where K is the exercise price at expiration date τ and $E^*\{(P(\tau, s) - K)^+\} \le E^*\{P(\tau, s)\} < \infty$. The following call option formula is obtained from (3.1).

THEOREM 3.1. *The arbitrage-free price $C(t, P(t, s), K, \tau, s)$ of an interest rate call option $(\tau, \mathbf{0}, (P(\tau, s) - K)^+)$ at time t, $0 \le t \le \tau$, is given by*

$$(3.2)\quad C(t, P(t, s), K, \tau, s) = E^{\dagger}\{P(t, s)XN(D) - P(t, \tau)KN(D - \sigma(t, \tau, s)\sqrt{\tau - t}) \parallel \mathcal{F}_t\},$$

where

$E^{\dagger}\{\cdot\}$ = *expectation operator for an equivalent probability measure* $P^{\dagger}$ *on* $(\Omega, \mathcal{F}, P)$,

$N(\cdot)$ = *standard normal distribution function*,

$$(3.3)\quad X = \exp\left\{-\sum_{j=1}^{k}\int_t^{\tau}(\eta_j(s-u) - \eta_j(\tau-u))\,du\,\psi_j - \boldsymbol{\gamma}^{\top}\int_t^{\tau}(\zeta(s-u) - \zeta(\tau-u))\,d\mathbf{Q}(u)\right\},$$

$$D = \frac{\log(XP(t, s)/KP(t, \tau))}{\sigma(t, \tau, s)\sqrt{\tau - t}} + \frac{1}{2}\sigma(t, \tau, s)\sqrt{\tau - t},$$

$$\sigma(t, \tau, s) = \beta\left[\frac{\int_t^{\tau}(\zeta(s-u) - \zeta(\tau-u))^2\,du}{\tau - t}\right]^{1/2},$$

and $\mathbf{Q} = \{\mathbf{Q}(t) = (Q_1(t), \ldots, Q_k(t))^{\top} \parallel 0 \le t \le T\}$ *is a k-dimensional independent Poisson process associated with the intensity vector process* $\boldsymbol{\psi}^{\dagger} = \{\boldsymbol{\psi}^{\dagger}(t) = (\psi_1^{\dagger}(t), \ldots, \psi_k^{\dagger}(t))^{\top} \parallel \psi_j^{\dagger}(t) = \eta_j(\tau - t)\psi_j, 0 \le t \le \tau\}$ *under* $P^{\dagger}$.

Proof. Substitution of $f(u) = 0$ and $g = (P(\tau, s) - K)^+$ into (3.1) generates

$$(3.4)\quad C(t, P(t, s), K, \tau) = E^*\left\{\left(\frac{B(t)}{B(\tau)}P(\tau, s) - \frac{B(t)}{B(\tau)}K\right)^+ \Bigg\| F_t\right\}.$$

From (2.21) and (2.23), we can easily show that

$$(3.5)\quad \frac{B(t)}{B(\tau)} = \exp\left\{-\int_t^{\tau} i(u)\,du\right\} = P(t, \tau)A_1\exp(Y_1 + Y_3),$$

and

$$(3.6)\quad \frac{B(t)}{B(\tau)}P(\tau, s) = P(t, s)A_1A_2\exp(Y_2 + Y_4),$$

where

$$A_1 = \exp\left\{-\frac{1}{2}\beta^2\int_t^{\tau}\zeta^2(\tau-u)\,du - \sum_{j=1}^{k}\int_t^{\tau}(\eta_j(u-t) - 1)\,du\,\psi_j\right\},$$

$$A_2 = \exp\left\{-\frac{1}{2}\beta^2\int_t^{\tau}(\zeta^2(s-u)\,du - \zeta^2(\tau-u))\,du - \sum_{j=1}^{k}\int_t^{\tau}(\eta_j(s-u) - \eta_j(\tau-u))\,du\,\psi_j\right\},$$

$$(3.7)\qquad Y_1 = -\beta \int_t^\tau \zeta(\tau - u)\, dW^*(u),$$

$$Y_2 = -\beta \int_t^\tau \zeta(s - u)\, dW^*(u),$$

$$Y_3 = -\boldsymbol{\gamma}^\top \int_t^\tau \zeta(\tau - u)\, d\mathbf{Q}(u),$$

$$Y_4 = -\boldsymbol{\gamma}^\top \int_t^\tau \zeta(s - u)\, d\mathbf{Q}(u).$$

Then the right side of (3.4) equals

$$(3.8)\qquad A_1 E^*\{(P(t, s)A_2 \exp(Y_2 + Y_4) - P(t, \tau)K \exp(Y_1 + Y_3))^+ \,\|\, \mathcal{F}_t\}.$$

Since $P(t, s)A_2 \exp(Y_2 + Y_4) \geq P(t, \tau)K \exp(Y_1 + Y_3)$ if and only if $Y_1 - Y_2 \leq \log(P(t, s)A_2/P(t, \tau)K) - Y_3 + Y_4$, the right side of (3.8) equals

$$(3.9)\qquad A_1 E^*\left\{\int_{(y_1, y_2)\in R(Y_3, Y_4)} (P(t, s)A_2 \exp(y_2 + Y_4) - P(t, \tau)K \exp(y_1 + Y_3)) \times f(y_1, y_2)\, dy_1\, dy_2 \,\|\, \mathcal{F}_t\right\}$$

$$= P(t, s)A_1A_2E^*\left\{\exp(Y_4) \int_{(y_1, y_2)\in R(Y_3, Y_4)} \exp(y_2) f(y_1, y_2)\, dy_1\, dy_2 \,\|\, \mathcal{F}_t\right\}$$

$$- P(t, \tau)KA_1E^*\left\{\exp(Y_3) \int_{(y_1, y_2)\in R(Y_3, Y_4)} \exp(y_1) f(y_1, y_2)\, dy_1\, dy_2 \,\|\, \mathcal{F}_t\right\},$$

where

$$(3.10)\qquad R(Y_3, Y_4) = \{(y_1, y_2) \,\|\, y_1 - y_2 \leq D(Y_3, Y_4)\},$$

$$D(Y_3, Y_4) = \log\left(\frac{P(t, s)A_2}{P(t, \tau)K}\right) - Y_3 + Y_4,$$

$$f(y_1, y_2) = \frac{1}{\sqrt{(2\pi)^2\sigma_1^2\sigma_2^2(1 - \rho^2)}} \exp\left\{\frac{-1}{2(1 - \rho^2)} \left(\frac{y_1^2}{\sigma_1^2} - 2\rho \frac{y_1}{\sigma_1}\frac{y_2}{\sigma_2} + \frac{y_2^2}{\sigma_2^2}\right)\right\},$$

$$\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} \beta^2 \int_t^\tau \zeta^2(\tau - u)\, du & \beta^2 \int_t^\tau \zeta(\tau - u)\zeta(s - u)\, du \\ \beta^2 \int_t^\tau \zeta(\tau - u)\zeta(s - u)\, du & \beta^2 \int_t^\tau \zeta^2(s - u)\, du \end{pmatrix}.$$

The second integral term of (3.9) is evaluated as

$$(3.11)\quad \int_{(y_1, y_2)\in R(Y_3, Y_4)} \exp(y_1) f(y_1, y_2)\, dy_1\, dy_2$$

$$= \int_{(y_1, y_2)\in R(Y_3, Y_4)} \frac{1}{\sqrt{(2\pi)^2\sigma_1^2\sigma_2^2(1-\rho^2)}}$$

$$\times \exp\left\{y_1 + \frac{-1}{2(1-\rho^2)}\left(\frac{y_1^2}{\sigma_1^2} - 2\rho\frac{y_1}{\sigma_1}\frac{y_2}{\sigma_2} + \frac{y_2^2}{\sigma_2^2}\right)\right\} dy_1\, dy_2$$

$$= \int_{(y_1, y_2)\in R(Y_3, Y_4)} \frac{1}{\sqrt{2(\pi)^2\sigma_1^2\sigma_2^2(1-\rho^2)}}$$

$$\times \exp\left\{\frac{-1}{2(1-\rho^2)}\left(\left(\frac{y_1-\sigma_1^2}{\sigma_1}\right)^2 - 2\rho\left(\frac{y_1-\sigma_1^2}{\sigma_1}\right)\left(\frac{y_2-\rho\sigma_1\sigma_2}{\sigma_2}\right)\right.\right.$$

$$\left.\left. + \left(\frac{y_2-\rho\sigma_1\sigma_2}{\sigma_2}\right)^2\right) + \frac{\sigma_1^2}{2}\right\} dy_1\, dy_2$$

$$= \exp\left(\frac{\sigma_1^2}{2}\right) N\left(\frac{D(Y_3, Y_4) - \sigma_1^2 + \rho\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}}\right).$$

Similarly,

$$(3.12)\quad \int_{(y_1, y_2)\in R(Y_3, Y_4)} \exp(y_2) f(y_1, y_2)\, dy_1\, dy_2$$

$$= \exp\left(\frac{\sigma_2^2}{2}\right) N\left(\frac{D(Y_3, Y_4) + \sigma_2^2 - \rho\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}}\right).$$

Substitution of (3.11) and (3.12) into (3.9) yields the right side of (3.9):

$$(3.13)\quad E^*\{(P(t, s)XN(D) - P(t, \tau)KN(D - \sigma(t, \tau, s)\sqrt{\tau - t}))\rho^\dagger(t) \parallel \mathcal{F}_t\},$$

where

$$(3.14)\quad \rho^\dagger(t) = \prod_{j=1}^{k}\left\{\exp\left(\int_t^\tau (\psi_j - \eta_j(\tau-u)\psi_j)\, du\right)\right.$$

$$\left.\times \prod_{t\le u\le \tau}\left(\frac{\eta_j(\tau-u)\psi_j}{\psi_j}\right)^{Q_j(u)-Q_j(u-)}\right\}.$$

On the other hand, the Radon-Nikodym derivative $\rho^\dagger = dP^\dagger/dP^*$ is equal to $\rho^\dagger(0)$ defined by (3.14) (Bremand 1981, Theorem 3, p. 166). Therefore,

$$\text{(3.15)}\quad E^{\dagger}\{P(t, s)XN(D) - P(t, \tau)KN(D - \sigma(t, \tau, s)\sqrt{\tau - t}) \parallel \mathcal{F}_t\}$$
$$= \frac{E^{*}\{(P(t, s)XN(D) - P(t, \tau)KN(D - \sigma(t, \tau, s)\sqrt{\tau - t}))\rho^{\dagger}(0) \parallel \mathcal{F}_t\}}{E^{*}\{\rho^{\dagger}(0) \parallel \mathcal{F}_t\}}$$
$$= E^{*}\{(P(t, s)XN(D) - P(t, \tau)KN(D - \sigma(t, \tau, s)\sqrt{\tau - t}))\rho^{\dagger}(t) \parallel \mathcal{F}_t\}.$$

Here the first equality follows from Bayes' rule (Bremand 1981, Lemma 5, p. 171). From (3.13) and (3.15), (3.2) follows. □

Theorem 3.1 shows that the arbitrage-free call option price is obtained as the expectation of the Black and Scholes call option pricing formula with the initial bond price given by $XP(t, s)$. Since the probability distribution of X is so complicated, a more explicit representation of (3.2) is unavailable. However, in special cases this simplifies considerably.

3.1. Gaussian Case

Suppose the stochastic behavior of the forward rate curve is driven only by a single Gaussian process. Then

$$\text{(3.16)}\quad C_0(t, P(t, s), K, \tau, s) = P(t, s)N(D_0) - P(t, \tau)KN(D_0 - \sigma(t, \tau, s)\sqrt{\tau - t}),$$

where

$$\text{(3.17)}\quad D_0 = \frac{\log(P(t, s)/KP(t, \tau))}{\sigma(t, \tau, s)\sqrt{\tau - t}} + \frac{1}{2}\sigma(t, \tau, s)\sqrt{\tau - t},$$
$$\sigma(t, \tau, s) = \beta\left[\frac{\int_t^{\tau} (\zeta(s - u) - \zeta(\tau - u))^2\, du}{\tau - t}\right]^{1/2}.$$

In particular, when the volatility scaling function is exponential type, say $\delta(u) = \exp\{-\lambda u\}$, then

$$\text{(3.8)}\quad \sigma(t, \tau, s) = \begin{cases} \beta\left(\dfrac{1 - \exp(-\lambda(s - \tau))}{\lambda}\right)\sqrt{\dfrac{1 - \exp(-2\lambda(\tau - t))}{2\lambda(\tau - t)}}, & \text{if } \lambda \neq 0, \\ \beta(s - \tau), & \text{if } \lambda = 0. \end{cases}$$

□

Heath, Jarrow, and Morton (1989a, eqn. 40, p. 28) derived the interest rate call option pricing formula when $\lambda = 0$.

3.2. Constant Volatility Case

Suppose that the volatility scaling function is constant, say $\delta(u) \equiv 1$. Then we have

$$(3.19)\qquad X = \exp\left\{\sum_{j=1}^{k}(1-\eta_j(\tau-t))(1-\eta_j(s-\tau))\frac{\psi_j}{\gamma_j} - (s-\tau)\boldsymbol{\gamma}^{\top}(\mathbf{Q}(\tau)-\mathbf{Q}(t))\right\}.$$

Here $Q_j(\tau) - Q_j(t)$, $1 \le j \le k$, are k independent Poisson random variables with mean $(1-\eta_j(\tau-t))\psi_j/\gamma_j$ under $P^{\dagger}$. Here (3.2) can be evaluated given the probability distribution associated with $\mathbf{Q}(\tau) - \mathbf{Q}(t)$, especially if the Poissonian risk is restricted to be binomial by setting $k = 2$ and $\gamma_1 = -\gamma_2 = \gamma > 0$. That is,

$$(3.20)\qquad C(t, P(t,s), K, \tau, s) = \sum_{n=-\infty}^{\infty}\left(\frac{\psi_1\eta(\tau-t)}{\psi_2}\right)^{n/2} \times \exp\left\{\frac{1-\eta(\tau-t)}{\gamma}\left(\psi_1 + \frac{\psi_2}{\eta(\tau-t)}\right)\right\} \times I_n\left(\frac{2(1-\eta(\tau-t))}{\gamma}\sqrt{\frac{\psi_1\psi_2}{\eta(\tau-t)}}\right) \times C_0(t, Me^{-(s-\tau)\gamma n}P(t,s), K, \tau, s),$$

where

(3.21)

$$\eta(u) = \exp\{-\gamma u\},$$

$$I_n(x) = \sum_{k\ge 0}\frac{(x/2)^{n+2k}}{k!(k+n)!},\quad \text{a modified Bessel function of the first kind of order } n,$$

$$C_0(\cdot) = \text{call option pricing formula defined by (3.16)},$$

$$M = \exp\left\{\frac{(1-\eta(\tau-t))(1-\eta(s-\tau))}{\gamma}\left(\psi_1 - \frac{\psi_2}{\eta(\tau-t)\eta(s-\tau)}\right)\right\}. \qquad \square$$

Next we analyze the effect of a discontinuous change in forward rates on the interest rate option's price. Intuition suggests that the discontinuous change of the forward rate curve increases the arbitrage-free call option price. This intuition is supported by the following theorem.

THEOREM 3.2 (Comparison Theorem). *Let $C(t, P(t,s), K, \tau, s)$ ($C'(t, P(t,s), K, \tau, s)$, respectively) denote the arbitrage-free call option price associated with the Poissonian risk premiums $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_k)^{\top}$ ($\boldsymbol{\psi}' = (\psi_1', \ldots, \psi_k')^{\top}$). If $\boldsymbol{\psi} \ge \boldsymbol{\psi}'$, then*

$$(3.22)\qquad C(t, P(t,s), K, \tau, s) \ge C'(t, P(t,s), K, \tau, s).$$

In particular, when $\boldsymbol{\psi}' = \mathbf{0}$,

$$C(t, P(t, s), K, \tau, s) \geq C_0(t, P(t, s), K, \tau, s), \tag{3.23}$$

where $C_0(\cdot)$ *is given by* (3.16). *Each inequality in* (3.22) *and* (3.23) *holds under equality if and only if* $K = 0$ *or* $\boldsymbol{\psi} = \boldsymbol{\psi}'$.

Proof. For $i = 1, 2$, let

$$X_i = \exp\left\{-\sum_{j=1}^{k} \int_t^{\tau} (\eta_j(s - u) - \eta_j(\tau - u))\, du\, \psi_{i,j} - \boldsymbol{\gamma}^{\top} \int_t^{\tau} (\zeta(s - u) - \zeta(\tau - u))\, d\mathbf{Q}_i(u)\right\}, \tag{3.24}$$

where $\boldsymbol{\psi}_i = (\psi_{i,1}, \ldots, \psi_{i,k})^{\top} \geq \mathbf{0}$ and $\mathbf{Q}_i = \{\mathbf{Q}_i(t) = (Q_{i,1}(t), \ldots, Q_{i,k}(t))^{\top} \parallel 0 \leq t \leq T\}$ is a k-dimensional independent Poisson process associated with the intensity vector process $\boldsymbol{\psi}_i^{\dagger} = \{\boldsymbol{\psi}_i^{\dagger}(t) = (\psi_{i,1}^{\dagger}(t), \ldots, \psi_{i,k}^{\dagger}(t))^{\top} \parallel \psi_{i,j}^{\dagger}(t) = \eta_j(\tau - t)\psi_{i,j}, 0 \leq t \leq \tau\}$ under $P^{\dagger}$. We assume that $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are mutually independent under $P^{\dagger}$. Then $\mathbf{Q} \equiv \mathbf{Q}_1 + \mathbf{Q}_2$ becomes a k-dimensional independent Poisson process associated with the intensity vector process $\boldsymbol{\psi}^{\dagger} \equiv \boldsymbol{\psi}_1^{\dagger} + \boldsymbol{\psi}_2^{\dagger}$. Let $C(\cdot)$ ($C_0(\cdot)$, $C_1(\cdot)$, respectively) be the arbitrage-free interest rate call option price associated with the Poissonian risk premiums $\boldsymbol{\psi} \equiv \boldsymbol{\psi}_1 + \boldsymbol{\psi}_2(\mathbf{0}, \boldsymbol{\psi}_1)$. Since $C_0(\cdot)$ is the Black–Scholes-type call option pricing formula, it is convex (strictly convex, respectively) with respect to the initial bond price (when $K > 0$). Then we have

$$\begin{aligned} C(t, P(t, s), K, \tau, s) &= E^{\dagger}\{C_0(t, X_1 X_2 P(t, s), K, \tau, s) \parallel \mathcal{F}_t\} \\ &= E^{\dagger}\{E^{\dagger}\{C_0(t, X_1 X_2 P(t, s), K, \tau, s) \parallel \mathcal{F}_t, \sigma(X_1)\} \parallel \mathcal{F}_t\} \\ &\geq E^{\dagger}\{C_0(t, E^{\dagger}\{X_1 X_2 P(t, s) \parallel \mathcal{F}_t, \sigma(X_1)\}, K, \tau, s) \parallel \mathcal{F}_t\}, \end{aligned} \tag{3.25}$$

where $\sigma(X_1)$ denotes the smallest σ-field generated by X_1. Inequality follows from Jensen's inequality. It holds under equality if and only if $K = 0$ or $X_2 \parallel_{\mathcal{F}_t, \sigma(X_1)} = 1$, almost surely (i.e., $\boldsymbol{\psi}_2 = \mathbf{0}$). On the other hand, from (3.24),

$$\begin{aligned} E^{\dagger}\{X_2 \parallel \mathcal{F}_t, \sigma(X_1)\} &= E^{\dagger}\{X_2 \parallel \mathcal{F}_t\} \\ &= \prod_{1 \leq j \leq k} \exp\left\{-\int_t^{\tau} (\eta_j(s - u) - \eta_j(\tau - u))\, du\, \psi_{2,j}\right\} \\ &\quad \times E^{\dagger}\left\{\exp\left(-\gamma_j \int_t^{\tau} (\zeta(s - u) - \zeta(\tau - u)) \times dQ_{2,j}(u)\right) \parallel \mathcal{F}_t\right\}. \end{aligned} \tag{3.26}$$

The first equality follows from the independence of X_1 and X_2 under $P^\dagger$. Now,

$$
\begin{aligned}
(3.27)\quad & E^\dagger\left\{\exp\left(-\gamma_j \int_t^\tau (\zeta(s-u)-\zeta(\tau-u))\, dQ_{2,j}(u)\right) \,\|\, \mathcal{F}_t\right\} \\
&= \sum_{n\geq 0} E^\dagger\left\{\exp\left(-\gamma_j \int_t^\tau (\zeta(s-u)-\zeta(\tau-u))\right.\right. \\
&\qquad \left.\left. \times\, dQ_{2,j}(u)\right) \,\|\, \mathcal{F}_t,\, Q_{2,j}(\tau) - Q_{2,j}(t) = n\right\} \\
&\quad \times P^\dagger\{Q_{2,j}(\tau) - Q_{2,j}(t) = n \,\|\, \mathcal{F}_t\} \\
&= \sum_{n\geq 0} \left(\frac{\int_t^\tau \eta_j(s-u)\,du}{\int_t^\tau \eta_j(\tau-u)\,du}\right)^n \exp\left\{-\int_t^\tau \eta_j(\tau-u)\psi_{2,j}\,du\right\} \\
&\quad \times \frac{1}{n!}\left(\int_t^\tau \eta_j(\tau-u)\psi_{2,j}\,du\right)^n \\
&= \exp\left\{\int_t^\tau (\eta_j(s-u)-\eta_j(\tau-u))\,du\,\psi_{2,j}\right\}.
\end{aligned}
$$

Substituting (3.27) into (3.26) generates $E^\dagger\{X_2 \,\|\, \mathcal{F}_t, \sigma(X_1)\} = 1$. This together with (3.25) results in

$$
\begin{aligned}
C(t, P(t,s), K, \tau, s) &\geq E^\dagger\{C_0(t, X_1 P(t,s), K, \tau, s) \,\|\, \mathcal{F}_t\} \\
&= C_1(t, P(t,s), K, \tau, s). \qquad \square
\end{aligned}
$$

Inequality (3.23) means that the arbitrage-free call option price is bounded below by the simple Gaussian case. In practical applications, (3.23) is useful as a lower bound on the call option price.

NUMERICAL EXAMPLE (Constant Volatility Case). Consider an arbitrage-free forward rate curve process with $\delta(t) \equiv 1$, $k = 2$, and $\beta = 0.2$. For this model, we compute the call option price at $t = 0$, provided that $\tau = 0.5$, $s = 1.0$, $K = 0.95$, and $P(0, 0.5) = \exp\{-0.05 \times 0.5\} \cong 0.975$. Table 3.1 shows the computational results for various initial bond prices $P(0, 1.0)$, jump sizes $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$ ($\gamma_1 > 0$, $\gamma_2 < 0$), and Poissonian risk premiums $\boldsymbol{\psi} = (\psi_1, \psi_2)$. As observed explicitly in Table 3.1, the call option price increases as the absolute jump size $|\gamma_1|$, $|\gamma_2|$ or the Poissonian risk premiums ψ_1, ψ_2 increase.

4. CONCLUDING REMARKS

We analyzed a family of forward rate curve processes satisfying a multifactor Poisson-Gaussian process scaled by $\delta(\cdot)$. We characterized the arbitrage-free drift term function in Theorem 2.1, and the bond market associated with this process is complete (Theorem 2.2). For interest rate call options, an arbitrage-free call option pricing formula is presented and a related comparison theorem is deduced in Theorems 3.1 and 3.2.

TABLE 3.1.
Call Option Price for Various Jump Sizes $\gamma = (\gamma_1, \gamma_2)$ and Risk Premiums $\psi = (\psi_1, \psi_2)$

$P(0,1.0)$	$\gamma=(0.1,-0.1)$ $\psi=(1.0,1.0)$	$\psi=(1.0,1.0)$ $\gamma=(0.2,-0.1)$	$\psi=(1.0,1.0)$ $\gamma=(0.1,-0.2)$	$\gamma=(0.1,-0.1)$ $\psi=(2.0,1.0)$	$\gamma=(0.1,-0.1)$ $\psi=(1.0,2.0)$
0.80	0.00146	0.00235	0.00510	0.00189	0.00232
0.81	0.00203	0.00319	0.00622	0.00258	0.00307
0.82	0.00278	0.00425	0.00753	0.00347	0.00402
0.83	0.00375	0.00558	0.00906	0.00460	0.00520
0.84	0.00497	0.00721	0.01084	0.00599	0.00663
0.85	0.00649	0.00918	0.01287	0.00770	0.00836
0.86	0.00835	0.01152	0.01520	0.00975	0.01042
0.87	0.01059	0.01425	0.01784	0.01219	0.01284
0.88	0.01324	0.01741	0.02080	0.01504	0.01566
0.89	0.01635	0.02101	0.02413	0.01832	0.01889
0.90	0.01993	0.02505	0.02782	0.02206	0.02256
0.91	0.02401	0.02955	0.03189	0.02627	0.02669
0.92	0.02859	0.03449	0.03636	0.03096	0.03127
0.93	0.03368	0.03987	0.04124	0.03612	0.03633
0.94	0.03926	0.04566	0.04652	0.04174	0.04185
0.95	0.04534	0.05186	0.05221	0.04781	0.04782

Although the forward rate curve process examined is simple, this model seems suitable for practical applications. This simplicity enables us to derive an arbitrage-free call option price analytically. Unfortunately, this model allows forward rates to become negative with a small positive probability. We can model a more complicated forward rate curve process which satisfies both the arbitrage-free condition and the positivity (see Heath, Jarrow, and Morton 1989a, Proposition 6), although it is hard to compute the option price for such a complicated process.

REFERENCES

AHN, C. M., and H. E. THOMPSON (1988): "Jump-Diffusion Processes and the Term Structure of Interest Rates," *J. Finance*, **43**, 155–174.

BREMAUD, P. (1981): *Point Processes and Queues: Martingale Dynamics*. New York: Springer-Verlag.

BRENNAN, M. J., and E. S. SCHWARTZ (1979): "A Continuous Time Approach to the Pricing of Bonds," *J. Bank. Finance*, **3**, 133–155.

BRENNAN, M. J., and E. S. SCHWARTZ (1982): "An Equilibrium Model of Bond Pricing and a Test of Market Efficiency," *J. Finance Quant. Anal.*, **17**, 303–329.

BROWN, S., and P. DYBVIG (1986): "The Empirical Implications of the Cox, Ingersoll, Ross Theory of the Term Structure of Interest Rates," *J. Finance*, **41**, 617–630.

COURTADON, G. (1982): "The Pricing of Options on Default-Free Bonds," *J. Finance Quant. Anal.*, **17**, 75–100.

COX, J. C., J. E. INGERSOLL, Jr., and S. A. ROSS (1981): "A Re-examination of Traditional Hypotheses about the Term Structure of Interest Rates," *J. Finance*, **36**, 769–799.

COX, J. C., J. E. INGERSOLL, Jr., and S. A. ROSS (1985a): "An Intertemporal General Equilibrium Model of Asset Prices," *Econometrica*, **53**, 363–384.

COX, J. C., J. E. INGERSOLL, Jr., and S. A. ROSS (1985b): "A Theory of the Term Structure of Interest Rates," *Econometrica*, **53**, 385–407.

ELLIOTT, R. J. (1982): *Stochastic Calculus and Applications.* New York: Springer-Verlag.

HARRISON, J. M., and S. R. PLISKA (1981): "Martingales and Stochastic Integrals in the Theory of Continuous Trading," *Stoch. Process. Appl.*, **11**, 215–260.

HARRISON, J. M., and S. R. PLISKA (1983): "A Stochastic Calculus Model of Continuous Trading: Complete Markets," *Stoch. Process. Appl.*, **15**, 313–316.

HEATH, D., R. JARROW, and A. MORTON (1989a): "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claim Valuation," Technical Report, Cornell University.

HEATH, D., R. JARROW, and A. MORTON (1989b): "Contingent Claim Valuation with a Random Evaluation of Interest Rates," Technical Report, Cornell University.

HO, T. S., and S. B. LEE (1986): "Term Structure Movements and Pricing Interest Rate Contingent Claims," *J. Finance,* **41**, 1011–1029.

HULL, J. (1989): *Options, Futures and Other Derivative Securities.* Englewood Cliffs, NJ: Prentice Hall.

IKEDA, N., and S. WATANABE (1989): *Stochastic Differential Equations and Diffusion Processes.* Amsterdam: North-Holland.

JARROW, R. A., and E. R. ROSENFELD (1984): "Jump Risks and the Intertemporal Capital Asset Pricing Model," *J. Business,* **57**, 337–351.

KARATZAS, I., and S. E. SHREVE (1987): *Brownian Motion and Stochastic Calculus.* New York: Springer Verlag.

RICHARD, S. F. (1978): "An Arbitrage Model of the Term Structure of Interest Rates," *J. Financial Econom.*, **6**, 33–57.

SHIRAKAWA, H. (1990): "Security Market Model with Poisson and Diffusion Type Return Process," Technical Report, IHSS-18, Tokyo Institute of Technology.

STAMBAUGH, R. F. (1988): "The Information in Forward Rates: Implications for Models of the Term Structure," *J. Financial Econom.*, **21**, 41–70.

VASICEK, O. (1977): "An Equilibrium Characterization of the Term Structure," *J. Financial Econom.*, **5**, 177–188.

XUE, X. X. (1991): "The Martingale Representation Theorem for a Class of Lévy Processes, and Its Applications," Doctoral Thesis, Department of Statistics, Columbia University.