A Simple Approach to Interest-Rate Option Pricing

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A simple introduction to contingent claim valuation of risky assets in a discrete time, stochastic interest-rate economy is provided. Taking the term structure of interest rates as exogenous, closed-form solutions are derived for European options written on (i) Treasury bills, (ii) interest-rate forward contracts, (iii) interest-rate futures contracts, (iv) Treasury bonds, (v) interest-rate caps, (vi) stock options, (vii) equity forward contracts, (viii) equity futures contracts, (ix) Eurodollar liabilities, and (x) foreign exchange contracts.

In this article, a relatively simple approach to pricing options in an economy with stochastic interest rates is described, taking the initial term structure as given. A general equilibrium model is used, and the work of Rubinstein (1976) and Brennan (1979), where it was assumed that interest rates were deterministic, is extended. The approach differs from the work of Heath, Jarrow, and Morton (1989a), which relies upon the assumption of dynamically complete markets. In Section 1, a general equilibrium model is described

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and an equivalent martingale measure defined. Sufficient structure is imposed to facilitate a direct comparison to the discrete time, one-factor models of Ho and Lee (1986) and Dybvig (1989), which assume that interest rates are normally distributed, and the continuous time model of Jamshidian (1989) and the examples in Heath, Jarrow, and Morton (1989a) which assume an Ornstein–Uhlenbeck process with mean reversion for forward rates. Closed-form solutions are derived in Section 2 for European options on Treasury bills, interest-rate forward and futures contracts, and Treasury bonds. The section concludes by considering the pricing of a call option written on an interest-rate cap. A closed-form solution is derived for this compound option.

The next two sections describe three extensions: the pricing of equity options in Section 3, and Eurodollar and foreign currency options in Section 4. The pricing of Eurodollar options necessitates the consideration of credit risk. The initial domestic and Eurodollar term structures are taken as exogenous, and the credit premium is modeled as a state variable. While this is similar to the work of Ramaswamy and Sundaresan (1986) on the pricing of corporate debt, it avoids two problems. First, in their analysis, the term structure of corporate debt is endogenous; thus, they misspecify the initial prices of corporate bonds. Second, they assume a form of the Local Expectations Hypothesis. This article shows that the three extensions have the same common structure. For equity, the one-period return can be decomposed into two components: the one-period risk-free rate of interest plus an equity risk premium. Similar decompositions apply for Eurodollar liabilities and foreign currency contracts. Under the equivalent martingale measure, the expected value of the premium is zero.

The assumption that the evolution of the term structure depends upon one factor, which implies that bond returns are (locally) perfectly correlated, is relaxed in Section 5. It is assumed that there are two factors, so that, in general, bond returns are not perfectly correlated. Closed-form solutions are derived for European Treasury bill and stock options. A summary is given in Section 6.

1. The Pricing of Risky Assets

A finite dated exchange economy is considered. A representative individual is assumed to exist who maximizes the expected utility of lifetime consumption. The utility function is assumed to be von Neumann–Morgenstern additively separable with constant proportional risk aversion. Let G_t denote the marginal rate of substitution from $t - 1$ to t, and $B_n(m)$ denote the value at $t = m$ of a pure discount bond that matures at $t = m + n$. Then from the first-order conditions[1]

$$B_n(m) = E\left(\prod_{j=m+1}^{l} G_j \middle| \Omega_m\right), \quad (1)$$

where Ω_m is the information set at $t = m$, and $l \equiv m + n$. In a more general model, the prices of the state contingent claims will depend upon many factors. For example, if claims are traded in nominal terms, prices will depend upon the marginal rate of substitution and the rate of inflation;[2] for a model that incorporates production, prices will also depend upon technology [see Milne and Turnbull (1989)].

To be able to derive relatively simple pricing expressions, three additional assumptions are made. First, it is assumed that there is only one source of risk or factor. The precise description of this factor is left unspecified. The assumption of only one factor is relaxed in Section 5. Second, if $Y_t = \ln G_t$, it is assumed that $\{Y_t\}$ are multivariate normally distributed, which implies that bond prices are lognormally distributed. Third, it is assumed that $y_t \equiv Y_t - \mu_t$ follows a first-order autoregressive process[3]

$$y_t = \theta y_{t-1} + e_t,$$

where $\mu_t = E(Y_t)$, θ is a constant, and (e_t) are multivariate normally distributed: $E(e_t) = 0$, $\mathrm{var}(e_t) = \sigma_t^2$, and $\mathrm{cov}(e_t, e_j) = 0$, $t \neq j$.

The price of a pure discount bond is given by

$$\begin{aligned} B_n(m) &= E\left[\exp\left(\sum_{j=m+1}^{l} Y_t\right) \middle| \Omega_m\right] \\ &= B_l(0)/B_m(0)\exp(\phi_m + x), \quad (2) \end{aligned}$$

where

$$x \equiv \theta\lambda(n-1)\sum_{j=1}^{m} \theta^{(m-j)} e_j,$$

$$\phi_m \equiv \sum_{j=1}^{m} [\lambda(m-j)^2 - \lambda(l-j)^2]\sigma_j^2/2,$$

$l = m + n$, and $\lambda(k) \equiv \Sigma_{j=0}^{k} \theta^j$. The one-period forward rate at $t = l$ implied by the term structure at $t = 0$ and defined by $B_{l+1}(0) \equiv$

[1] This is a standard result. See Huang and Litzenberger (1988, chap. 7).

[2] See Turnbull (1979) for a description of this type of model in discrete time, and in a continuous time framework see Richard (1978).

[3] A similar assumption is used in Campbell (1986).

$B_l(0)\exp[-f(0, l)]$, is

$$-f(0, l) = \mu_{l+1} + \frac{1}{2}\left[\sum_{j=1}^{l+1} \lambda(l + 1 - j)^2\sigma_j^2 - \sum_{j=1}^{l} \lambda(l - j)^2\sigma_j^2\right] + \theta^{l+1}y_0.$$

The change in the forward rate is

$$f(m, l) - f(0, l) = \sum_{j=1}^{m} [\lambda(l - j)^2 - \lambda(m - j)^2]\frac{\sigma_j^2}{2} - \theta\sum_{j=1}^{m} \theta^{(l-j)}e_j. \tag{3}$$

For the one-factor case, (3) has a similar structure to the special case considered by Heath, Jarrow, and Morton (1989a), who assume that forward rates are normally distributed and $\mathrm{var}[f(m, l) - f(0, l)] = m\sigma^2$, where σ^2 is a constant. These assumptions are implicit in the Ho–Lee (1986) model. In (3) if $\theta = 1$ and $\sigma_j^2 = \sigma^2$ for all j, then (3) collapses to the case considered by Ho and Lee.

The process for the spot rate, $r(m) = f(m, m)$, is derived from (3) and can be written in the form

$$r(m) - r(m - 1) = a_1 - a_2r(m - 1) - \theta e_m,$$

where

$$a_1 \equiv f(0, m) + \sum_{j=1}^{m} [\lambda(m + 1 - j)^2 - \lambda(m - j)^2]\frac{\sigma_j^2}{2} - \theta\left\{f(0, m - 1) + \sum_{j=1}^{m-1} [\lambda(m - j)^2 - \lambda(m - 1 - j)^2]\frac{\sigma_j^2}{2}\right\},$$

and $a_2 = 1 - \theta$. For $0 < \theta < 1$, $a_2 > 0$ and one would expect[4] $a_1 > 0$. Two points should be noted. First, the spot rate is perfectly correlated with the forward rate process. Second, in general, the spot rate follows a mean-reverting process. If $\theta = 1$, which implies no mean reversion, then the process corresponds to the Ho–Lee model.[5]

[4] Given the definition of a_1 and then simplifying, it can be shown that

$$a_1 = f(0, m) - \theta f(0, m - 1) + \theta\sigma_m^2 + \theta^2 \sum_{j=1}^{m} \theta^{2(m-j)}\sigma_j^2.$$

[5] Dybvig (1989) has generalized the Ho–Lee model to include the case of mean reversion.

1.1 Martingale pricing

Define the accumulation factor $A(m)$ to represent the amount generated by investing in a series of one-period bonds from $t = 0, 1, \ldots, m - 1$, $A(0) \equiv 1$. By rearranging the first-order conditions, it can be shown that there is an equivalent martingale measure such that with respect to this measure relative prices follow a martingale.[6] If P_0 represents the price of an asset at $t = 0$, which only has a final payoff at $t = m$ (there are no intermediate dividends), then there exists an equivalent martingale measure such that[7]

$$P_0/A(0) = E^{*}[P_m/A(m) \mid \Omega_0],$$

where P_m represents the payoff at $t = m$.

If the objective is to price only European options that mature at a particular point, say $t = m$, then it is advantageous to redefine the accumulation factor to be the amount earned by investing in an m-period pure discount bond. This implies a change in the martingale measure. Let $a(m)$—the accumulation factor—represent the amount generated by investing in an m-period pure discount bond, $a(0) = 1$. There exists another equivalent martingale measure such that[8,9]

$$\begin{aligned} P_0/a(0) &= E^{**}[P_m/a(m) \mid \Omega_0] \\ &= B_m(0) E^{**}[p_m \mid \Omega_0]. \end{aligned} \tag{4}$$

Given the form of this expression and the assumption that prices of bonds are lognormally distributed, option-pricing formulas will be similar in form to the traditional Black–Scholes model.

1.2 Futures prices

Consider a futures contract that matures at $t = m$ written on an n-period pure discount bond. Following Cox, Ingersoll, and Ross (1981), the futures price at some time $t = q \le m$ is determined by

$$F_n(q, m) = E\{B_n(m) G_m \cdots G_{q+1}/[B_1(q) \cdots B_1(m-1)] \mid \Omega_q\},$$

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6 A general deviation of this result is given in Milne and Turnbull (1989), where it is shown that the same result can be derived via arbitrage arguments, assuming that we have at least as many distinct securities as branches leaving each node. See also Huang and Litzenberger (1988, chap. 8).

7 The relative bond price of $t = m$ of a pure discount bond that matures on $t = l$, is defined to be $Z(m, l) \equiv B_n(m)/A(m)$. Therefore, by substitution, $Z(m, l) = Z(m-1, l)X$, where $X \equiv B_n(m)B_1(m-1)/B_{n+1}(m-1)$. Under this measure let $E^{*}(e_m \mid \Omega_{m-1}) = \mu_m^{*}$ and $\text{var}^{*}(e_m \mid \Omega_{m-1}) = v_m^2$. Given that

$$X = \exp\{-[\theta\lambda(n-1)\sigma_m^2 + \theta^2\lambda(n-1)^2\sigma_m^2/2] + \theta\lambda(n-1)e_m\},$$

then $\mu_m^{*} = \sigma_m^2$. Under the martingale measure, it is necessary to adjust the mean of the underlying distribution; the variance remains unchanged under the new measure. This is a demonstration of a discrete time equivalent to Girsanov's theorem [Elliott (1982, p. 162)].

8 This result is obtained by rearranging the first-order condition. See Milne and Turnbull (1989).

9 Applying this to (2) gives $E^{**}(e_j \mid \Omega_0) = \lambda(m-j)\sigma_j^2$ and $\text{var}^{**}(e_j \mid \Omega_0) = \sigma_j^2$, $1 \le j \le m$.

or in terms of the equivalent martingale measure

$$F_n(q, m) = E^*[B_n(m) | \Omega_q] \tag{5}$$

[see Cox, Ingersoll, and Ross (1981) and Turnbull and Milne (1989)].

Lemma 1. *The futures price at $t = 0$ on a contract to purchase an n-period pure discount bond at $t = m$ is*

$$F_n(0, m) = h_n(0, m)\exp\left\{-\theta\lambda(n-1)\sum_{j=1}^{m}\theta^{m-j}[\lambda(m-j)-1]\sigma_j^2\right\}, \tag{6}$$

where $h_n(0, m)$ is the n forward price defined by $h_n(0, m) \equiv B_l(0)/B_m(0)$, $l \equiv m + n$. At $t = q < m$,

$$F_n(q, m) = F_n(0, m)\exp(\phi_q + x_q), \tag{7}$$

where

$$\phi_q = -\theta\lambda(n-1)\left[\sum_{j=1}^{q}\left(\theta^{m-j}\sigma_j^2 + \theta\lambda(n-1)\theta^{2(m-j)}\frac{\sigma_j^2}{2}\right)\right]$$

and

$$x_q \equiv \theta\lambda(n-1)\sum_{j=1}^{q}\theta^{m-j}e_j.$$

Proof. See Appendix A.

Three points should be noted from (6). First, if $\theta = 0$, which from (2) implies that the term structure is deterministic, then forward and future prices are equal [see Cox, Ingersoll, and Ross (1981) or Turnbull and Milne (1989)]. If θ is positive, the forward price is greater than the futures price.[10] If θ is negative, the forward price can be greater or less than the futures price. Equation (7) describes the price dynamics of the futures price. Given that the e_j are normally distributed, then the futures price follows a lognormal distribution.

2. Pricing Interest-Rate Options

The pricing of European options written on Treasury bills, interest-rate forward and futures contracts, and Treasury bonds is considered. Closed-form solutions are derived for these options and some of their

[10] If $\theta = 1$ and $\sigma_j = \sigma$, then $F_n(0, m) = h_n(0, m)\exp[-n\sigma^2 m(m-1)/2]$.

properties are discussed. The section is concluded with the derivation of a closed-form solution for the value of a call option on a cap.

Theorem 1. *European Treasury bill options.*

(a) The value of an m-period European call option that allows the holder to purchase an n-period Treasury bill is

$$c(0) = B_l(0)N(d) - KB_m(0)N(d - \sigma_2),$$

where K is the exercise price, $l \equiv m + n$, $B_T(0)$ *is the value of a T-period Treasury bill, and* $N(\cdot)$ *is the cumulative normal distribution function,*

$$d \equiv \{\ln[B_l(0)/KB_m(0)] + \sigma_{22}/2\}/\sigma_2,$$

and

$$\sigma_{22} \equiv \theta^2\lambda(n-1)^2 \sum_{j=1}^{m} \theta^{2(m-j)}\sigma_j^2.$$

(b) The value of an m-period European put option written on the same asset is

$$p(0) = KB_m(0)N(-d + \sigma_2) - B_l(0)N(-d).$$

Proof. Consider a call option that matures at $t = m$. The payoff of the option is $c(m) = \mathrm{Max}\{B_n(m) - K, 0\}$. From (2), $B_n(m) \equiv B_l(0)/B_m(0)\exp(\Phi_m + x)$, and $B_l(0) = B_m(0)E^{**}[B_n(m)\,|\,\Omega_0]$, which implies $\exp(\Phi_m + \mu^{**} + \sigma_{22}/2) = 1$, where

$$x \equiv \theta\lambda(n-1) \sum_{j=1}^{m} \theta^{m-j}e_j, \qquad \mu^{**} \equiv E^{**}(x\,|\,\Omega_0),$$

and

$$\sigma_{22} \equiv \theta^2\lambda(n-1)^2 \sum_{j=1}^{m} \theta^{2(m-j)}\sigma_j^2.$$

The option will be exercised if $x \geq \ln[B_m(0)K/B_l(0)] - \Phi_m \equiv k$. Thus, substituting $c(m)$ into (4) and using the lemma in Appendix B gives

$$c(0) = B_l(0)N(d) - KB_m(0)N(d - \sigma_2),$$

where

$$\sigma_2 d \equiv -k + \mu^{**} + \sigma_{22} = \ln[B_l(0)/KB_m(0)] - \sigma_{22}/2,$$

using (2) and the definition of k. The proof for part (b) is very similar so details are omitted. Q.E.D.

If $\theta = 1$, implying no mean reversion, then $\sigma_{22} = n^2 \sum_{j=1}^{m} \sigma_j^2$, and if variances are constant, $\sigma_{22} = n^2\sigma^2 m$, so that for this case the results given in Theorem 1 are identical to those derived by Heath, Jarrow, and Morton (1989a). As in Black–Scholes, the only term that cannot be directly observed is the variance of the forward rate process. Thus, the model provides a simple way of pricing interest-rate options, taking the initial term structure as exogenous. The expression for the option value is similar in form to Merton's (1973) stochastic interest-rate option pricing model.[11] The bond price $B_l(0)$ is analogous to the "stock," and $B_m(0)$ represents the bond. The replicating portfolio is thus the bond $B_l(0)$ and the short-term bond $B_1(0)$. The returns on all bonds are locally perfectly correlated.[12] It should be noted that put–call parity holds:

$$p(0) = c(0) + KB_m(0) - B_l(0).$$

Theorem 2. *Options on interest-rate forward contract.*

(a) Consider an m-period European call option to purchase an n-period forward contract. The forward contract is to purchase a q-period Treasury bill. The value of the call option is given by

$$c(0) = B_{l+q}(0)N(d) - KB_l(0)N(d - \sigma_2),$$

where k is the exercise price, $l \equiv m + n$, $B_T(0)$ is the value of a T-period Treasury bill;

$$d \equiv \{\ln[B_{l+q}(0)/KB_l(0)] + \sigma_{22}/2\}/\sigma_2;$$

and

$$\sigma_{22} = \theta^{2(n+1)}\lambda(q - 1)^2 \sum_{j=1}^{m} \theta^{2(m-j)}\sigma_j^2.$$

[11] Given the approximation $\ln[B_{n-1}(m + 1)/B_n(m)] = \Delta B/B_n(m)$, where ΔB is the change in bond price, we obtain, from (2),

$$\Delta B/B_n(m) = r(m) - [1 - \lambda(n - 1)^2]\sigma_{m+1}^2/2 + \theta\lambda(n - 2)e_{m+1}.$$

This discrete time form for changes in the bond price is analogous to the continuous time form used by Merton (1973).

[12] The call option formula can be written in the form

$$c(0) = B_m(0)[b_n(0, m)N(d) - KN(d - \sigma_2)],$$

where $d \equiv [\ln[b_n(0, m)/K] + \sigma_{22}/2]/\sigma_2$. If the call option is deep in-the-money, then $c(0) \simeq B_m(0)[b_n(0, m) - K]$. If the option could be exercised prematurely, its value would be $B_n(0) - K$. Thus, it may or may not be worthwhile to exercise such an option prematurely.

(b) The value of an m-period put option written on the same asset is

$$p(0) = KB_l(0)N(-d + \sigma_2) - B_{l+q}(0)N(-d).$$

Proof. For parts (a) and (b), see Appendix C. Q.E.D.

If it is assumed that $\theta = 1$, then $\sigma_{22} = q^2 \sum_{j=1}^{m} \sigma_j^2$, and if variances are constant, then $\sigma_{22} = q^2\sigma^2 m$. The q-period forward price at $t = 0$ is defined by $h_q(0, l) \equiv B_{l+q}(0)/B_l(0)$, so that the value of the call option can be written in the form

$$c(0) = B_l(0)[h_q(0, l)N(d) - KN(d - \sigma_2)],$$

where $d = \{\ln[h_q(0, l)/K] + \sigma_{22}/2\}/\sigma_2$. Written in this form, the above result is analogous to the Black (1976) model for call options on forward contracts. However, the Black model assumes that interest rates are constant.[13]

Theorem 3. *Options on interest-rate futures contracts.*

(a) Consider an m-period European call option to purchase a futures contract that matures at $t = m + n \equiv l$. The futures contract is written on a q-period Treasury bill. The value of the call option is given by

$$c(0) = B_m(0)[F_q(0, l)\exp(\sigma_{12})N(d) - KN(d - \sigma_2)],$$

where K is the exercise price, $F_q(0, l)$ is the futures price, $B_m(0)$ is the value of an m-period Treasury bill,

$$d = \{\ln[F_q(0, l)/K] + \sigma_{12} + \sigma_{22}/2\}/\sigma_2,$$

$$\sigma_{12} \equiv \theta^2\lambda(q - 1) \sum_{j=1}^{m-1} \theta^{(l-j)}\lambda(m - 1 - j)\sigma_j^2;$$

and

$$\sigma_{22} \equiv \theta^2\lambda(q - 1)^2 \sum_{j=1}^{m} \theta^{2(l-j)}\sigma_j^2.$$

(b) The value of an m-period put, written on the same futures contract, is

$$p(0) = B_m(0)[KN(-d + \sigma_2) - F_q(0, l)\exp(\sigma_{12})N(-d)].$$

[13] If the option is deep in-the-money, which for a call option implies $h_q(0, l) \gg K$, then the value of the option is $c(0) \approx B_l(0)[h_q(0, l) - K]$. If the option could be exercised immediately, then its value is $c(0) = B_l(0)[h_q(0, l) - K]$. Thus, it is not optimal to exercise an American forward rate option prematurely, a result first derived (in a more rigorous way) by Jarrow and Oldfield (1988).

Proof. See Appendix D.

These expressions differ in structure from the expressions for options on forward contracts because of the presence of the term σ_{12}. This term arises because of the covariance between the futures prices and the short-term interest rates over the life of the option. For $\theta = 1$ and constant variance ($\sigma_j^2 = \sigma^2$ for all j), $\sigma_{12} = q\sigma^2 m(m-1)/2$ and $\sigma_{22} = q^2\sigma^2 m$. The only term that cannot be directly observed is the variance of the forward rate process. Note that for this case the forward price $h_q(0, l)$ is greater than the futures price, and a European call option on the forward contract will have greater value than the equivalent option on the futures contract.[14]

2.1 Options on treasury bonds

Consider a European option that matures at time $t = m$. The option is written on a default-free bond, which will be referred to as a Treasury bond. Such a bond can always be viewed as a series of pure discount bonds. At $t = m$ the value of the bond is

$$B(m) \equiv \sum_{i=1}^{N} X_i B_{n_i}(m),$$

where X_i is the payment at time $t = m + n_i$, and $B_{n_i}(m)$ is the value of a pure discount bond that matures at $t = m + n_i$. Two points should be noted. First, the payments $\{X_i\}$, which may be coupon and/or principal, occur with certainty. Second, to determine the value of the option, it is only necessary to consider the payments that occur after the option matures. It is assumed that N such payments remain.

From (2),

$$B_{n_i}(m) = B_{l_i}(0)/B_m(0)\exp[\Phi_i + \theta\lambda(n_i - 1)x],$$

where

$$l_i \equiv m + n_i,$$

$$\Phi_i \equiv \sum_{j=1}^{m} [\lambda(m-j)^2 - \lambda(l_i - j)^2]\frac{\sigma_j^2}{2},$$

$$x = \sum_{j=1}^{m} \theta^{m-j} e_j.$$

[14] If a call option is deep in-the-money, which implies that $F_q(0, l) \gg K$, then the option value is $c(0) \approx B_m(0)[F_q(0, l)\exp(\sigma_{12}) - K]$. If the option could be exercised immediately, then its value is $c(0) = F_q(0, l) - K$. Given that $B_m(0) \leq 1$, then it may be optimal to exercise an American interest futures option prematurely, a result first derived by Jarrow and Oldfield (1988).

Consider a call option with exercise price K. Define $\bar{x}$ to be that value of x such that

$$B(m) - K = 0. \tag{8}$$

A solution to this equation is assumed to exist. Thus, for $\theta > 0$,

$$B(m) - K \geq 0 \qquad \text{as } x \geq \bar{x}.$$

Theorem 4. *European Treasury bond options.*

(a) The value of an m-period European call option, exercise price K is

$$c(0) = \sum_{j=1}^{N} X_i B_{l_i}(0)\ N(d_i) - KB_m(0)\ N(d),$$

where

$$\sigma d = -\bar{x} + \sum_{j=1}^{m} \theta^{(m-j)} \lambda(m - j)\sigma_j^2,$$

$$\sigma d_i = -\bar{x} + \sum_{j=1}^{m} \theta^{(m-j)} \lambda(l_i - j)\sigma_j^2, \qquad l_i \equiv m + n_i,$$

$$\sigma^2 = \sum_{j=1}^{m} \theta^{2(m-j)} \sigma_j^2$$

and $\bar{x}$ is defined by (8).

(b) The value of an m-period European put option written on the same asset is

$$p(0) = KB_m(0)N(-d) - \sum_{i=1}^{N} X_i B_{l_i}(0) N(-d_i).$$

Proof. Once $\bar{x}$ is defined, the proof is very similar to that for Theorem 1. See Appendix E.

If the bond makes only one payment ($n = 1$), then the results are identical to those in Theorem 1. It should be noted that in Theorem 4 the present values of the individual payments that occur after the option matures $\{X_i B_{l_i}(0)\}$ appear in the option formula and not the value of the bond $B(0)$. Theorem 4 provides an alternative derivation of a result given in Jamshidian (1989).[15]

[15] By repeated use of the result that relative prices follow a martingale, a binomial algorithm can be constructed to price American options, taking the initial term structure as given. See Heath, Jarrow, and Morton (1989b), Black, Derman, and Toy (1987), and Turnbull and Milne (1989).

2.2 Compound options

Many financial institutions are now selling options on caps—captions. In the absence of default risk, a cap can be viewed as a put option written on a Treasury bill [see Boyle and Turnbull (1989)]. Consider a put option written on a q-period Treasury bill. The option matures at $t = l\ (\equiv m + n)$. From Theorem 1 the value of this option at $t = m$ is given by

$$p(m) = KB_n(m)N(-d + \sigma) - B_{n+q}(m)N(-d),$$

where K is the exercise price, $B_j(m)$ is the value at $t = m$ of a j-period pure discount bond, and

$$\sigma d \equiv \ln[B_{n+q}(m)/KB_n(m)] + \sigma^2/2,$$

$$\sigma^2 \equiv \theta^2\lambda(q-1)^2 \sum_{j=m+1}^{l} \theta^{2(l-j)}\sigma_j^2.$$

Consider a call option written on this put option. If interest rates increase, the underlying put option will increase in value and so will a call option written on this put option. Let K_c denote the exercise price of the call option, which is assumed to mature at $t = m$. The value of the call option at maturity is

$$c(m) = \mathrm{Max}\{p(m) - K_c, 0\}.$$

Theorem 5. *Captions. Consider an m-period European call option to purchase a put option that matures at $t = m + n \equiv l$. The exercise price of the call option is K_c. The put option is written on a q-period Treasury bill. The exercise price of the put option is K. The value of the call option is given by*

$$c(0) = KB_l(0)N_2(\alpha_1, \beta_1; \rho) - B_{l+q}(0)N_2(\alpha_2, \beta_2; \rho) - K_cB_m(0)N(\alpha_3),$$

where $N_2(a, b; \rho)$ is the bivariate cumulative normal distribution function with upper limits a and b, and correlation coefficient ρ,

$$\alpha_1 \equiv \alpha_2 + \theta^{n+1}\lambda(q-1)\sigma_2,$$

$$\alpha_2 \equiv \frac{1}{\sigma^2}\left[\bar{x} - \sum_{j=1}^{m} \theta^{m-j}\lambda(l+q-j)\sigma_j^2\right],$$

$$\beta_1 \equiv \beta_2 + \sigma_1,$$

$$\beta_2 = -\{\ln[B_{l+q}(0)/KB_l(0)] + \sigma_1^2/2\}/\sigma_1,$$

$$\sigma_1^2 \equiv \theta^2\lambda(q-1)^2 \sum_{j=1}^{l} \theta^{2(l-j)}\sigma_j^2,$$

$$\sigma_2^2 \equiv \sum_{j=1}^{m} \theta^{2(m-j)}\sigma_j^2,$$

$$\alpha_3 \equiv \frac{1}{\sigma^2}\left[\bar{x} - \sum_{j=1}^{m} \theta^{(m-j)}\lambda(m - j)\sigma_j^2\right],$$

$$\rho \equiv \left[\left(\sum_{j=1}^{m} \theta^{2(l-j)}\sigma_j^2\right)\left(\sum_{j=1}^{l} \theta^{2(l-j)}\sigma_j^2\right)^{-1}\right]^{1/2}.$$

Proof. See Appendix F.

The correlation coefficient is positive, unlike the case considered by Whaley (1981), who derives a closed-form solution for an equity compound option, assuming interest rates are constant. The compound option considered by Whaley is a call option written on a call option. A caption is a call option written on a put option. Note that for $\theta = 1$ and constant variances ($\sigma_j^2 = \sigma^2$), $\rho = (m/l)^{1/2}$, which, apart from the sign, is identical to that derived by Whaley.

3. Equity Options

Up to the present point only interest-rate options have been considered. The analysis is now extended to the pricing of equity options. Closed-form solutions are derived for the pricing of European stock options and options on equity forward and futures contracts.

It is assumed that the one-period rate of return on equity can be decomposed into two components: the one-period rate of interest plus an equity risk premium. For simplicity it is assumed that the stock does not pay dividends over the period considered. Given the stock price S_{m-1} at $t = m - 1$, the price at $t = m$ is

$$S_m = S_{m-1}\exp[r(m - 1) + \zeta_m], \tag{9}$$

where ζ_m is the equity risk premium which is assumed to be normally distributed. It is assumed that ζ_m is contemporaneously correlated with e_m. In terms of the stock price at $t = 0$,

$$S_m = S_0 A(m)\exp\left[\sum_{j=1}^{m} \zeta_j\right]. \tag{10}$$

If the relative stock price is defined to be $z(m) \equiv S_m/A(m)$, then $z(m) = z(m - 1)\exp(\zeta_m)$. Given that there exists an equivalent martingale measure such that $z(m)$ follows a martingale, then this implies[16]

$$E^*[\exp(\zeta_m) \mid \Omega_{m-1}] = 1. \tag{11}$$

The condition is equivalent to the so-called risk-neutral pricing condition of Cox, Ross, and Rubinstein (1979), where the expected return on the stock is set equal to the risk-free rate of interest, implying that the expected equity premium is zero with respect to the risk-neutral distribution. For the following theorem it is more convenient to define the relative price to be $z(m) \equiv S_m/a(m)$, so that, from (4),

$$z(0) = E^{**}[z(m) \mid \Omega_0]. \tag{12}$$

Theorem 6. *European stock options.*

(a) The value of an m-period European call option is

$$c(0) = S_0 N(d) - KB_m(0)N(d - \sigma_2),$$

where K is the exercise price, $B_m(0)$ is value of an m-period Treasury bill,

$$d \equiv \{\ln[S_0/KB_m(0)] + \sigma_{22}/2\}/\sigma_2,$$

$$\sigma_{22} \equiv \sum_{j=1}^{m} \sigma_j(\zeta)^2 - 2\theta \sum_{j=1}^{m-1} \lambda(m - 1 - j)\mathrm{cov}(e_j, \xi_j)$$
$$+ \theta^2 \sum_{j=1}^{m-1} \lambda(m - 1 - j)^2 \sigma_j^2.$$

(b) The value of an m-period European put option is

$$p(0) = KB_m(0)N(-d + \sigma_2) - S_0 N(-d).$$

Proof. The value of a European call option at maturity is $c(m) = \mathrm{Max}\{S_m - K, 0\}$. From (10) the value of equity at $t = m$ in $S_m = [S_0/B_m(0)]\exp(\phi_m + x)$, where

$$\phi_m \equiv \theta \sum_{j=1}^{m-1} \lambda(m - 1 - j)\sigma_j^2 + \theta^2 \sum_{j=1}^{m-1} \lambda(m - 1 - j)\frac{\sigma_j^2}{2},$$

[16] Under the equivalent martingale measure, the expected equity premium is zero. See Equation (11). The equity is contemporaneously correlated with the short rate of interest. Thus, Equation (11) can be written in the form

$$E^*\{E^*[\exp(\xi_m) \mid r(m)] \mid \Omega_{m-1}\} = 1,$$

where $r(m)$ is the short-term rate of interest at $t = m$. By using a binomial distribution to represent changes in the short-term interest and equity premium, an algorithm similar to that described by Kishimoto (1989) can be constructed to price American equity options. See Turnbull and Milne (1989) for details.

and

$$x \equiv \sum_{j=1}^{m} \xi_j - \theta \sum_{j=1}^{m-1} \lambda(m - 1 - j) e_j.$$

Substituting the above expression for S_m into (12) implies $\exp(\phi_n + \mu^{**} + \sigma^2/2) = 1$, where $\mu^{**} \equiv E^{**}(x|\Omega_0)$ and $\sigma^2 \equiv \text{var}(x|\Omega_0)$. At maturity the option will be exercised if $x \geq \ln[KB_m(0)/S_0] - \phi_m \equiv k$. Hence, substituting $c(m)$ into (12) and using the results in Appendix B give

$$c(0) = S_0 N(d) - kB_m(0) N(d - \sigma_2),$$

where $\sigma_2 d = -k + \mu^{**} + \sigma^2 = \ln[S_0/KB_m(0)] + \sigma_{22}/2$.

The proof for a put option is similar, so details are omitted.

Q.E.D.

Two points should be noted. First, the option pricing equation has the same structure as the Black–Scholes option model. This is to be expected given Equation (12). Second, the variance term is the sum of the variances of the period-by-period rates of return on equity. The variance of the rate of return can be decomposed into three terms: the variance of the equity premium, the variance from the short-term interest rate, and the covariance between the equity premium and the short-term interest rate. For $\theta = 1$ and constant variances, then the above result is identical to that derived by Heath, Jarrow, and Morton (1989a).

3.1 Equity forwards

Let $h(0, i)$ denote the forward price at $t = 0$ of a contract which matures at $t = i$. At maturity the owner purchases one share of stock at the forward price. The forward price is set such that the initial value of the contract is zero. At time $m \leq i$, the forward price [see Cox, Ingersoll, and Ross (1981) and Jarrow and Oldfield (1981)] is $h(m, i) \equiv S_m/B_{i-m}(m)$. Using (10), the price dynamics of the forward price are described by

$$h(m, i) = h(0, i)\exp(\phi_m + x), \tag{13}$$

where

$$\phi_m \equiv \sum_{j=1}^{m} [\lambda(i - j)^2 - 1]\frac{\sigma_j^2}{2},$$

$$x \equiv -\theta \sum_{j=1}^{m} \lambda(i - 1 - j) e_j + \sum_{j=1}^{m} \zeta_j.$$

***Theorem* 7.** *European options on equity forward contracts. Consider options of maturity m written on an equity forward contract that expires at* $t = m + n \equiv l$, $n \geq 0$.

(a) *The value of a call option is*

$$c(0) = B_m(0)[h(0, l)\exp(\eta)N(d) - KN(d - \sigma_2)],$$

where K is the exercise price; $B_m(0)$ *is the value of an m-period Treasury bill,*

$$d = \{\ln[h(0, l)/K] + \eta + \sigma_{22}/2\}/\sigma_2,$$

$$\sigma_{22} \equiv \sum_{j=1}^{m} \sigma_j(\zeta)^2 - 2\theta \sum_{j=1}^{m} \lambda(l - 1 - j)\mathrm{cov}(e_j, \zeta_j)$$

$$+ \theta^2 \sum_{j=1}^{m} \lambda(l - 1 - j)\sigma_j^2,$$

$$\eta \equiv \theta^2\lambda(n - 1) \sum_{j=1}^{m} \theta^{m-j}\lambda(l - 1 - j)\sigma_j^2$$

$$- \theta\lambda(n - 1) \sum_{j=1}^{m} \theta^{m-j}\mathrm{cov}(e_j, \zeta_j).$$

(b) *The value of a put option is*

$$p(0) = B_m(0)[KN(-d + \sigma_2) - h(0, l)\exp(\eta)N(-d)].$$

Proof. See Appendix G.

If interest rates are deterministic ($\theta = 0$), then the above expressions simplify to the Black (1976) model. In general, the expressions differ from the Black model because of the presence of the extra term η, which is a measure of the covariance between the forward price and short-term interest rates over the life of the option. Jarrow (1989), in a continuous time model, has derived a similar result.

3.2 Equity futures

Consider a futures contract written on the stock. The futures contract matures at $t = i$. From (10) the stock price at $t = i$ can be written, after substituting for $A(m)$, in the form[17]

[17] The relationship

$$A(i)^{-1} = B_i(0)\exp\left\{\sum_{j=1}^{i} [1 - \lambda(i - j)^2]\frac{\sigma_j^2}{2} + \theta \sum_{j=1}^{i-1} \lambda(i - 1 - j)e_j\right\}$$

is used. See Appendix C. Note that

$$\sum_{j=1}^{i} [\lambda(i - j)^2 - 1]\frac{\sigma_j^2}{2} = \theta \sum_{j=1}^{i-1} \lambda(i - 1 - j)\sigma_j^2 + \theta^2 \sum_{j=1}^{i-1} \lambda(i - 1 - j)^2 \frac{\sigma_j^2}{2}.$$

$$S_i = \left[\frac{S_0}{B_i(0)}\right]\exp\left\{\sum_{j=1}^{i} [\lambda(i-j)^2 - 1]\frac{\sigma_j^2}{2} - \theta \sum_{j=1}^{i-1} \lambda(i-1-j)e_j + \sum_{j=1}^{i} \zeta_j\right\}. \tag{14}$$

The futures price is given by $F(0, i) = E^*(S_i|\Omega_0)$, so that after simplification

$$F(0, i) = \left[\frac{S_0}{B_i(0)}\right]\exp\left[\theta^2 \sum_{j=1}^{i-1} \lambda(i-1-j)^2\sigma_j^2 - \theta \sum_{j=1}^{i-1} \lambda(i-1-j)\operatorname{cov}(e_j, \zeta_j)\right]. \tag{15}$$

The term $S_0/B_i(0)$ represents the forward price. The other terms represent the variance of the short-term interest rates and the covariance between short-term interest rates and the equity risk premium over the life of the futures contract. Equation (15) is a special case of a result first derived by Cox, Ingersoll, and Ross (1981). The futures price at some time $m < i$ can be written

$$F(m, i) = F(0, i)\exp\left[\phi_m - \theta \sum_{j=1}^{m} \lambda(i-1-j)e_j + \sum_{j=1}^{m} \zeta_j\right], \tag{16}$$

where

$$\phi_m \equiv \theta \sum_{j=1}^{m} \lambda(i-1-j)\sigma_j^2 - \theta^2 \sum_{j=1}^{m} \lambda(i-1-j)^2 \frac{\sigma_j^2}{2} + \theta \sum_{j=1}^{m} \lambda(i-1-j)\operatorname{cov}(e_j, \zeta_j).$$

Note that the price dynamics for the futures price are endogenous to the model. Given the approximation $\ln[F(m+1, i)/F(m, i)] = \Delta F/F(m, i)$, where ΔF is the change in the futures price, then

$$\begin{aligned}\Delta F/F(m, i) = {} & \theta\lambda(i-m)\sigma_j^2 - \theta^2\lambda(i-m)^2\sigma_j^2/2 \\ & + \theta\lambda(i-m)\operatorname{cov}(e_{m+1}, \zeta_{m+1}) \\ & - \theta\lambda(i-m)e_{m+1} + \zeta_{m+1}.\end{aligned}$$

Changes in the futures price arise from contemporaneous changes in the short-term interest rate and the equity price, and are serially uncorrelated.

Theorem 8. *European options on equity futures contract.*

(a) Consider an m-period European call option to purchase a futures contract which matures at $t = m + n \equiv l$. The futures contract allows the holder to purchase one share of stock. The value of the call option is given by

$$c(0) = B_m(0)[F(0, l)\exp(\sigma_{12})N(d) - KN(d - \sigma_2)],$$

where K is the exercise price, $F(0, l)$ is the futures price, $B_m(0)$ the value of an m-period Treasury bill,

$$d = \{\ln[F(0, l)/K] + \sigma_{12} + \sigma_{22}/2\}/\sigma_2,$$

$$\sigma_{12} \equiv \theta \sum_{j=1}^{m-1} \lambda(m - 1 - j)\text{cov}(e_j, \zeta_j)$$

$$- \theta^2 \sum_{j=1}^{m-1} \lambda(l - 1 - j)\lambda(m - 1 - j)\sigma_j^2,$$

and

$$\sigma_{22} \equiv \sum_{j=1}^{m} \sigma_j(\zeta)^2 - 2\lambda(l - 1 - j)\text{cov}(e_j, \zeta_j) + \theta^2\lambda(l - 1 - j)^2\sigma_j^2.$$

(b) The value of an m-period put option written on the same contract is

$$p(0) = B_m(0)[KN(-d + \sigma_2) - F(0, l)\exp(\sigma_{12})N(-d)].$$

Proof. See Appendix H.

From the expression for options on forward contracts, the term η can be written in the form

$$\eta = \theta^2 \sum_{j=1}^{m} \lambda(l - 1 - j)^2\sigma_j^2 - \theta \sum_{j=1}^{m} \lambda(l - 1 - j)\text{cov}(e_j, \zeta_j) + \sigma_{12},$$

where σ_{12} is defined in Theorem 8. If the covariance between the short-term interest rate and the equity risk premium is negative, then a call option on the futures will be worth more than a call option on the forward. Jarrow (1989), in a continuous time model, has derived a similar result.

4. Extensions

Two extensions are described. The first extension considers the pricing of Eurodollar options. The risk of these options arise from changes in the domestic term structure and changes in the so-called TED spread.[18] A closed-form solution is derived. The second extension considers the pricing of European foreign currency options. The two examples have the same structure as the pricing of equity options.

4.1 Eurodollar options

Options on Eurodollar interest rates are now actively traded. To price such options, the same approach used to price Treasury bill options can be applied. It is necessary to describe the term structure of Eurodollar interest rates and its evolution. Eurodollar interest rates can be decomposed into two components: the domestic dollar interest rate plus a spread. This spread reflects the credit risk associated with the financial institutions issuing Eurodollar liabilities. Consequently, there are two sources of risk which affect the pricing of Eurodollar options: changes in domestic interest rates and changes in the credit spread.

It is assumed that firms belong to different risk classes. For a particular risk class, the yields on zero coupon Eurodollar bonds issued by firms within the risk class is taken as exogenous. Given this term structure of risky yields and the term structure of default-free debt, it is possible to define a set of risk premia $\{\xi(n, m)\}$, where $\xi(n, m)$ is the risk premium at $t = m$ on an n-period zero coupon bond. Ramaswamy and Sundaresan (1986) define a short-term risk premium and assume a mean-reverting square root diffusion process. In their model, the initial term structure of corporate debt is endogenous, and thus they misspecify the initial prices of corporate bonds. They also assume a form of the local expectations hypothesis. Such an assumption is not made in this article.

Let $L_n(m)$ denote the dollar value at time $t = m$ of an n-period zero coupon Eurodollar bond, with a face value of \$1. The value of the bond at $t = 0$ is $L_l(0)$, $l \equiv m + n$, which is taken as exogenous. Given the bond price at $t = m - 1$, it is assumed that the price at $t = m$ is given by

$$L_n(m) = L_{n+1}(m - 1)\exp[r(m - 1) + \xi(n, m)],$$

where $r(m - 1)$ is the default-free domestic one-period rate of interest at $t = m - 1$, and $\xi(n, m)$ is the credit spread at $t = m$ on an

[18] The spread is sometimes called the TED spread. Normally, the TED spread is defined as a spread established by the simultaneous purchase or sale of futures contracts on domestic Treasury bills minus the opposite transaction in the Eurodollar futures contract.

n-period zero coupon Eurodollar bond, and is assumed to be normally distributed. It is assumed that $\xi(n, m)$ is contemporaneously correlated with e_m and thus $r(m)$. Note that the credit spread is assumed to depend upon the maturity of the bond. In terms of the initial bond price

$$L_n(m) = L_l(0)A(m)\exp\left[\sum_{j=1}^{m} \xi(l - j, j)\right]. \tag{17}$$

Equation (17) is isomorphic to Equation (10).

Theorem 9. *European Eurodollar options.*

(a) Consider a European call option, which matures at $t = m$, to purchase an n-period Eurodollar zero coupon bond. The value of the call option is given by

$$c(0) = L_l(0)N(d) - KB_m(0)N(d - \sigma_2),$$

where K is the exercise price, $B_m(0)$ is the value of an m-period Treasury bill,

$$d \equiv [\ln[L_l(0)/KB_m(0)] + \sigma_{22}/2]\sigma_2;$$

and

$$\sigma_{22} \equiv \sum_{j=1}^{m} \sigma_{l-j,j}(\xi)^2 - 2\theta \sum_{j=1}^{m-1} \lambda(m - 1 - j)\mathrm{cov}[e_j, \xi(l - j, j)]$$
$$+ \theta \sum_{j=1}^{m-1} \lambda(m - 1 - j)^2\sigma_j^2.$$

(b) The value of an m-period European put option is

$$p(0) = KB_m(0)N(-\mathrm{d} + \sigma_2) - L_l(0)N(-d).$$

Proof. The proof is identical to that given for Theorem 6. Q.E.D.

Two points should be noted. First, the above result takes the initial term structure of domestic and Eurodollar interest rates as given and allows the variance of the spread to change (in a deterministic manner) over the life of the option. Second, the approach can be extended to price options on corporate bonds and swaps, given the presence of default risk.[19]

[19] Independently, Robert Jarrow has used a similar approach for pricing options on corporate debt.

4.2 European foreign currency options

Consider a foreign currency option written on sterling. Let $S(m)$ denote the exchange rate (\$/£) at time $t = m$, and $X_n(m)$ the £ value of an n-period default-free pure discount bond with face value of £1. The dollar value of this bond is $V_m \equiv S(m)X_n(m)$. Assuming interest rate parity holds, then $V_m = F_n(m)B_n(m)$, where $F_n(m)$ is the forward exchange rate at $t = m$ for an n-period contract. At maturity, the forward exchange rate equals the spot exchange. For a European call option, maturity l and exercise price K, the value at maturity is

$$c(l) = \mathrm{Max}\{V_l - K, 0\}.$$

Given that $V_m = F_n(m)B_n(m)$, it is assumed that changes in the value of the asset V_m can be described by

$$V_m = V_{m-1}\exp[r(m-1) + \xi(n, m)], \tag{18}$$

where $\xi(n, m)$ is a premium, which is assumed to be normally distributed. It is assumed that $\xi(n, m)$ is contemporaneously correlated with e_m and thus $r(m)$. Equation (18) is isomorphic to Equation (10).

Theorem 10. *European foreign currency options.*

(a) Consider a European call option, which matures at $t = m$, to purchase one unit of foreign currency. The value of the call option is given by

$$c(0) = F_m(0)B_m(0)N(d) - KB_m(0)N(d - \sigma_2),$$

where K is the exercise price, $B_m(0)$ is the dollar value of an m-period Treasury bill, $F_m(0)$ is the forward exchange rate for a contract that matures at $t = m$,

$$d = \{\ln[F_m(0)/K] + \sigma_{22}/2\}\sigma_2,$$

and

$$\sigma_{22} = \sum_{j=1}^{m} \sigma_{m-j,j}(\xi)^2 - 2\theta \sum_{j=1}^{m-1} \lambda(m-1-j)\mathrm{cov}[e_j, \xi(m-j, j)] + \theta^2 \sum_{j=1}^{m-1} \lambda(m-j-j)^2\sigma_j^2.$$

(b) The value of an m-period European put option is

$$p(0) = KB_m(0)N(-d + \sigma_2) - F_m(0)B_m(0)N(-d).$$

Proof. See Theorem 6.

A similar result in a continuous time framework has been derived by Amin and Jarrow (1989).

5. Multiple Sources of Risk

Up to the present point it has been assumed that there is only one source of risk, which implies that all bond returns are locally perfectly correlated. In a more general model the prices of state-contingent claims will depend upon many factors. The precise description of these factors will be left unspecified. It will be assumed that there are two factors; the extension to an arbitrary number is straightforward. The assumption of two factors implies that the evolution of the term structure depends upon these two factors. The price of a T-period pure discount bond is given by

$$B_T(0) = E\left[\prod_{t=1}^{T} \exp(Y_{1t} + Y_{2t})\right],$$

where Y_{it} is the ith factor at time t, $i = 1, 2$. It is assumed that $\{Y_{1t}, Y_{2t}\}$ are multivariate normally distributed and that $y_{it} \equiv Y_{it} - \mu_{it}$ follows a first-order autoregressive process of the form

$$y_{it} = \theta_i y_{it-1} + e_{it},$$

where $\mu_{it} = E(Y_{1t})$; θ_i is a constant, the e_{it} are multivariate normally distributed, $E(e_{it}) = 0$, $\operatorname{var}(e_{it}) = \sigma^2_{it}$, $\operatorname{cov}(e_{it}, e_{ij}) = 0$, $t \neq j$, $i = 1, 2$, and $\operatorname{cov}(e_{1t}, e_{2t}) = 0$. Note that it is assumed that the two factors are independent. This assumption can easily be relaxed.

Equation (3), which describes the forward rate dynamics, now becomes

$$f(m, l) - f(0, l) = \sum_i \left\{\sum_{j=1}^{m} [\lambda_i(l + 1 - j)^2 - \lambda_i(l - j)^2] \frac{\sigma_{ij}}{2} - \theta_i \sum_{j=1}^{m} \theta_i^{(m-j)} e_{ij}\right\},$$

where $\lambda_i(m) \equiv \Sigma_{j=0}^{m} \theta_i^j$. Heath, Jarrow, and Morton (1989a) and Brenner and Jarrow (1989), as a special case, assume that for a forward rate $f(m, l)$ there are two sources of risk: a long-run factor with a constant variance and a short-run factor with a variance of the form $\sigma^2 \exp[-\eta(l - m)]$, where η is a positive constant. Similar assumptions can be imposed upon the variance specifications of the $\{e_{ij}\}$.

The value at $t = m$ of an n-period pure discount bond can be written in the form

$$B_n(m) = [B_l(0)/B_m(0)]\exp(\phi_m + x),$$

where

$$\phi_m \equiv \sum_i \sum_{j=1}^{m} [\lambda_i(m-j)^2 - \lambda_i(l-j)^2]\frac{\sigma_{ij}^2}{2}$$

and

$$x \equiv \sum_i \theta_i \lambda_i (n-1) \sum_{j=1}^{m} \theta_i^{(m-j)} e_{ij}.$$

The specification of x implies that bond returns will no longer be perfectly correlated.

For deriving closed-form solutions for European options that mature at a given time, say $t = m$, the martingale measure described by (4) will be used. This implies $E^{**}(e_{ij}|\Omega_0) = \lambda_i(m-j)\sigma_{ij}^2$, with variances remaining unchanged.

European Treasury bill options (Theorem 1 extended).

(a) The value of an m-period European call option, which allows the holder to purchase an n-period Treasury bill, is

$$c(0) = B_l(0)N(d) - KB_m(0)N(d - \sigma_2),$$

where K is the exercise price, $l \equiv m + n$, $B_T(0)$ is the dollar value of a T-period Treasury bill,

$$d \equiv \{\ln[B_l(0)/KB_m(0) + \sigma_{22}/2\};$$

and

$$\sigma_{22} = \sum_i \theta_i \lambda_i (n-1)^2 \sum_{j=1}^{m} \theta_i^{2(m-j)} \sigma_{ij}^2.$$

(b) The value of an m-period European put option written on the same asset is

$$p(0) = KB_m(0)N(-d + \sigma_2) - B_l(0)N(-d).$$

Proof. The proof involves a minor extension of that given in Theorem 1 so details are omitted. Q.E.D.

The analysis can readily be extended to equity options. From Section 3, the value of equity at $t = m$ is given by

$$S_m = [S_0/B_m(0)]\exp(\Phi_m + x),$$

where

$$\Phi_m \equiv \sum_i \sum_{j=1}^{m} [\lambda_i(m-j)^2 - 1]\frac{\sigma_{ij}^2}{2}$$

and

$$x \equiv \sum_{j=1} \xi_j - \sum_i \theta_i \sum_{j=1}^{m-1} \lambda_i(m-1-j)e_{ij}.$$

Again the presence of a second factor will affect the specification of the option variance. Note that e_{1t} and e_{2t} are contemporaneously correlated with ξ_t.

European stock options (Theorem 6 extended).

(a) The value of an m-period European call option is

$$c(0) = S_0N(d) - KB_m(0)N(d - \sigma_2),$$

where K is the exercise price, $B_m(0)$ is the dollar value of an m-period Treasury bill,

$$d \equiv \{\ln[S_0/KB_m(0)] + \sigma_{22}/2\}/\sigma_2;$$

and

$$\sigma_{22} \equiv \sum_{j=1}^{m} \sigma_j(\xi)^2 - 2\sum_i \theta_i \sum_{j=1}^{m-1} \lambda_i(m-1-j)\operatorname{cov}(\xi_j, e_{ij})$$
$$+ \sum_i \theta_i^2 \sum_{j=1}^{m-1} \lambda_i(m-1-j)^2\sigma_{ij}^2.$$

(b) The value of an m-period European put option is

$$p(0) = KB_m(0)N(-d + \sigma_2) - S_0N(-d).$$

Proof. The proof involves a minor extension of that given in Theorem 6, so details are omitted. Q.E.D.

6. Summary

The purpose of this article has been to describe a simple approach toward the pricing of interest-rate option pricing, taking the existing term structure as given. Closed-form solutions are derived for European options written on Treasury bills, Treasury bonds, and interest-rate forward and futures contracts. Closed form solutions are derived for equity options, options on equity forward and futures contracts, captions, Eurodollar options, and foreign currency options. Finally, it is shown that the results can be extended in a direct way to incorporate the assumption that the term structure of interest rates depends upon two or more factors. Such an assumption implies that bond returns are not perfectly correlated. All of these formulas can be empirically tested.

Appendix A: Futures Price

The futures contract is written on a q-period pure discount bond, so that when the contract matures at $t = i$, the futures price is

$$F_q(i, i) = B_q(i) = [B_l(0)/B_i(0)]\exp(\Phi + x),$$

where

$$l \equiv i + q,$$

$$\Phi \equiv \sum_{j=1}^{i} [\lambda(i - j)^2 - \lambda(l - j)^2]\frac{\sigma_j^2}{2},$$

and

$$x \equiv \theta\lambda(q - 1) \sum_{j=1}^{i} \theta^{i-j} e_j.$$

From (5),

$$\begin{aligned} F_q(0, i) &= E^*[F_q(i, i) \mid \Omega_0] \\ &= h_q(0, i)\exp\Bigg[\Phi + \theta\lambda(q - 1) \sum_{j=1}^{i} \theta^{(i-j)}\sigma_j^2 + \theta^2\lambda(q - 1)^2 \\ &\quad \times \sum_{j=1}^{i} \theta^{2(i-1)}\sigma_j^2/2\Bigg], \end{aligned}$$

where the $h_q(0, i)$ is the q-period forward price. Substituting for Φ and simplifying gives the required result:

$$F_q(0, i) = h_q(0, i)\exp\left\{-\theta\lambda(q - 1) \sum_{j=1}^{i} \theta^{i-j}[\lambda(i - j) - 1]\sigma_j^2\right\}.$$

At $t = m$, $F_q(m, i) = E^*[F_q(i, i) \mid \Omega_m]$ and, from above,

$$F_q(i, i) = h_q(0, i)\exp\left[\Phi + \theta\lambda(q - 1) \sum_{j=1}^{m} \theta^{(i-j)} e_j + x\right],$$

where

$$x \equiv \theta\lambda(q-1)\sum_{j=m+1}^{i}\theta^{(i-j)}e_j.$$

Evaluating the conditional expectation, substituting the expression for $F_q(0, i)$, and simplifying give

$$F_q(m, i) = F_q(0, i)\exp\left\{\theta\lambda(q-1)\sum_{j=1}^{m}\theta^{i-j}e_j - \theta\lambda(q-1) \times \left[\sum_{j=1}^{m}\theta^{i-j}\sigma_j^2 + \theta\lambda(q-1)\theta^{2(i-j)}\frac{\sigma_j^2}{2}\right]\right\}.$$

Appendix B: Lemma

Lemma. *If x_1 and x_2 are bivariate normally distributed, $E(x_j) = m_j$, $cov(x_i, x_j) = \sigma_{ij}$, $i, j = 1, 2$, and c and k are constants, then*

(a) $$E[\exp(cx_1)\,|\,x_2 \geq k] = \exp(cm_1 + c^2\sigma_{11}/2)N(d),$$
$$\sigma_2 d = -k + m_2 + c\sigma_{12}.$$

(b)
$$E[\exp(x_1 + x_2)\,|\,x_2 \geq k] = \exp[m_1 + m_2 + (\sigma_{11} + 2\sigma_{12} + \sigma_{22})/2]N(d_1),$$
$$\sigma_2 d_1 = -k + m_2 + \sigma_{12} + \sigma_{22}.$$

Proof. See Huang and Litzenberger (1988, chap. 6).

Appendix C: Options on Forward Contracts

At time $t = m$, the q-period forward price applicable at $t = m + n$ is by definition $h_q(m, l) \equiv B_{n+q}(m)/B_n(m)$, $l = m + n$. Using (2) gives $h_q(m, l) = \exp(\phi + x_2)$, where

$$\phi \equiv \left\{\ln\left[\frac{B_{l+q}(0)}{B_{l(0)}}\right] - \lambda(q-1)\sum_{j=1}^{m}\theta^{l-j+1}\lambda(l-j)\sigma_j^2 - \lambda(q-1)^2\sum_{j=1}^{m}\theta^{2(l-j+1)}\frac{\sigma_j^2}{2}\right\}$$

and

$$x_2 \equiv \theta^{n+1}\lambda(q-1)\sum_{j=1}^{m}\theta^{(m-j)}e_j,$$

so that

$$m_2 \equiv E^*(x_2 | \Omega_0) = \theta^{n+1}\lambda(q-1)\sum_{j=1}^{m}\theta^{m-j}\sigma_j^2$$

and

$$\sigma_{22} = \theta^{2(n+1)}\lambda(q-1)^2\sum_{j=1}^{m}\theta^{2(m-j)}\sigma_j^2.$$

The option will be exercised if $x_2 > \ln K - \phi \equiv k$.

When the option matures, the option holder can purchase a forward contract with forward price K. The value of this contract is $[h_q(m, l) - K]B_n(m)$, so that the value of the option is $c(m) = \text{Max}\{[h_q(m, l) - K]B_n(m), 0\}$, and, at $t = 0$,

$$\begin{aligned} c(0) &= E^*[c(m)/A(m) | \Omega_0] \\ &\equiv L_1 - KL_2. \end{aligned}$$

Now

$$\frac{B_{n+q}(m)}{A(m)} = B_{l+q}(0)\exp\left\{\sum_{j=1}^{m}[1 - \lambda(l+q-j)^2]\frac{\sigma_j^2}{2} + x_1\right\},$$

where

$$x_1 \equiv \theta\sum_{j=1}^{m}\lambda(l+q-1-j)e_j$$

and

$$\sigma_{12} = \theta^{n+2}\lambda(q-1)\sum_{j=1}^{m}\theta^{m-j}\lambda(l+q-1-j)\sigma_j^2.$$

Recognizing that $B_{n+q}(m) = B_n(m)h_q(m, l)$ and $B_{l+q}(0) = E^*[B_{n+q}(m)/A(m)]$, then

$$L_1 = B_{l+q}(0)N(d_1),$$

where

$$\sigma_2 d_1 = -k + m_2 + \sigma_{12} = \ln[B_{l+q}(0)/KB_l(0)] + \sigma_{22}/2.$$

In a similar manner,

$$\frac{B_n(m)}{A(m)} = B_l(0)\exp\left\{\sum_{j=1}^{m}[1 - \lambda(l-j)^2]\frac{\sigma_j^2}{2} + x_1\right\},$$

where

$$x \equiv \theta\sum_{j=1}^{m}\lambda(l-1-j)e_j$$

and

$$\sigma_{12} = \theta^{n+2}\lambda(q-1)\sum_{j=1}^{m}\sigma^{m-j}\lambda(l-1-j)\sigma_j^2.$$

Given that $B_l(0) = E^*[B_n(m)/A(m)]$, then $L_2 = B_l(0)N(d_2)$, where $\sigma_2 d_2 = -k + m_2 + \sigma_{12} = \ln[B_{l+q}(0)/KB_l(0)] - \sigma_{22}/2$.

The deriviation for the put option is similar, so details are omitted.

Appendix D: Options on Futures

The option contract, which matures at $t = m$, allows the option holder to purchase a futures contract that matures at $t = m + n \equiv i$. The futures contract is written on a q-period pure discount bond. From Appendix A,

$$F_q(m, i) = F_q(0, i)\exp(\Phi_m + x_2),$$

where

$$\Phi_m \equiv -\theta\lambda(q-1)\sum_{j=1}^{m}\theta^{(i-j)}\sigma_j^2 - \theta^2\lambda(q-1)^2\sum_{j=1}^{m}\sigma^{2(i-j)}\frac{\sigma_j^2}{2}$$

and

$$x_2 \equiv \theta\lambda(q-1)\sum_{j=1}^{m}\theta^{(i-j)}e_j.$$

Now

$$F_q(0, i) = E^*[F_q(m, i)\,|\,\Omega_0],$$

which implies

$$\exp(\Phi_m + m_2 + \sigma_{22}/2) = 1,$$

where

$$m_2 \equiv E^*(x_2\,|\,\Omega_0)$$

and

$$\sigma_{22} = \text{var}(x_2\,|\,\Omega_0) = \theta^2\lambda(q-1)^2\sum_{j=1}^{m}\theta^{2(i-j)}\sigma_j^2.$$

The value of a call option at maturity is $c(m) = \text{Max}\{F_q(m, i) - K, 0\}$. The option will be exercised if $x_2 \geq \ln K/F_q(0, i) - \Phi_m \equiv k$. The value of the option at $t = 0$ is

$$c(0) = E^*[c(m)/A(m)\,|\,\Omega_0]$$
$$= L_1 - KL_2.$$

Now $A(m)^{-1} = B_m(0)\exp(\Phi + x_1)$, where $\Phi \equiv \Sigma_{j=1}^{m} [1 - \lambda(m - j)^2]\sigma_j^2/2$, and $x_1 \equiv \theta\ \Sigma_{j=1}^{m-1} \lambda(m - 1 - j)e_j$. Given that $B_m(0) = E^*[A(m)^{-1}\,|\,\Omega_0]$, then $\exp(\Phi + m_1 + \sigma_{11}/2) = 1$, where $m_1 \equiv E^*(x_1\,|\,\Omega_0)$ and $\sigma_{11} = \mathrm{var}(x_1\,|\,\Omega_0)$.

Consider

$$L_1 \equiv E^*[F_q(m, i)/A(m)\,|\,x_2 \geq k]$$
$$= B_m(0)F_q(0, i)E^*[\exp(\Phi_m + x_2 + \Phi + x_1)\,|\,x_2 \geq k].$$

Using the lemma in Appendix B and simplifying gives $L_1 = B_m(0)F_q(0, i)\exp(\sigma_{12})N(d_1)$, where

$$\sigma_2 d_1 = -k + m_2 + \sigma_{12} + \sigma_{22}$$
$$= \ln[F_q(0, i)/K] + \sigma_{12} + \sigma_{22}/2,$$

and

$$\sigma_{12} \equiv \theta^2\lambda(q - 1) \sum_{j=1}^{m-1} \theta^{(i-j)}\lambda(m - 1 - j)\sigma_j^2.$$

Consider

$$L_2 \equiv E^*[A(m)^{-1}\,|\,x_2 \geq k] = B_m(0)E^*[\exp(\Phi + x_1)\,|\,x_2 \geq k].$$

Using the lemma in Appendix B and simplifying gives $L_2 = B_m(0)\,N(d_1 - \sigma_2)$.

The proof for the put option is similar, so details are omitted.

Appendix E: European Treasury Bond Options

For a call option the value at maturity is $c(m) = \mathrm{Max}\{B(m) - K, 0\}$. This option will be exercised if $x > \bar{x}$, for $\theta > 0$. From (4), the value of the option at $t = 0$ is

$$c(0) = B_m(0)E^{**}[c(m)\,|\,\Omega_0] \equiv \sum_{i=1}^{N} B_{l_i}(0)X_i L_i - KB_m(0)L,$$

where

$$m = E^{**}(x\,|\,\Omega_0) = \sum_{j=1}^{m} \theta^{m-j}\lambda(m - j)\sigma_j^2$$

and

$$\sigma^2 \equiv \operatorname{var}(x|\Omega_0) = \sum_{j=1}^{m} \theta^{2(m-j)}\sigma_j^2.$$

Using the lemma in Appendix B and simplifying, one obtains

$$\begin{aligned} L_i &\equiv E^{**}\{\exp[\Phi_i + \theta\lambda(n_i - 1)x]|x > \bar{x}\} \\ &= N(d_i), \end{aligned}$$

where $\sigma d_i = -\bar{x} + m + \theta\lambda(n_i - 1)\sigma^2$. Substituting for m and σ^2 gives

$$\sigma d_i = -\bar{x} + \sum_{j=1}^{m} \theta^{(m-j)}\lambda(l_i - j)\sigma_j^2.$$

In a similar manner $L = N(d)$, where

$$\sigma d = -\bar{x} + m = -\bar{x} + \sum_{j=1}^{m} \theta^{(m-j)}\lambda(m - j)\sigma_j^2.$$

The proof for put options is very similar, so details are omitted.

Appendix F: Captions

From the definitions of $B_n(m)$ and d,

$$\begin{aligned} \sigma d = &-\ln[B_{l+q}(0)] + \sum^{m} [\lambda(l - j)^2 - \lambda(l + q - j)^2]\frac{\sigma_j^2}{2} + \frac{\sigma^2}{2} \\ &+ \theta^{n+1}\lambda(q - 1)x, \end{aligned}$$

where

$$\sigma^2 \equiv \theta^2\lambda(q - 1)^2 \sum_{j=m+1}^{l} \theta^{2(l-j)}\sigma_j^2$$

and

$$x \equiv \sum_{j=1}^{m} \theta^{(m-j)}e_j.$$

Let $-d + \sigma \equiv a_1 + b_1x$ and $-d \equiv a_2 + b_2x$. The value of the caption at $t = 0$ is $c(0) = B_m(0)E^{**}[c(m)|\Omega_0]$. Substituting for $c(m)$ gives

$$\begin{aligned} c(0) = &KB_l(0)E^{**}\{\exp[\phi_n + \theta\lambda(n - 1)x]N(a_1 + b_1x)|x \le \bar{x}\} \\ &- B_{l+q}(0)E^{**}\{\exp[\phi_{n+q} + \theta\lambda(n + q - 1)x] \\ &\times N(a_2 + b_2x)|x \le \bar{x}\} - B_m(0)E^{**}(K_c|x \le \bar{x}), \end{aligned}$$

where

$$\phi_n \equiv \sum_{j=1}^{m} [\lambda(m-j)^2 - \lambda(l-j)^2] \frac{\sigma_j^2}{2}$$

and

$$\phi_{n+q} \equiv \sum_{j=1}^{m} [\lambda(m-j)^2 - \lambda(l+q-j)^2] \frac{\sigma_j^2}{2}.$$

Using the results given in Geske (1979) and Whaley (1981), and after much simplification,

$$c(0) = KB_l(0)N_2(\alpha_1, \beta_1; \rho) - B_{l+q}(0)N_2(\alpha_2, \beta_2; \rho) - K_c B_m(0)N(\alpha_3).$$

Appendix G: Options on Equity Forward Contracts

A call option will be exercised if $x_2 \geq \ln[K/b(0, l)] - \phi_m \equiv k$, where

$$x_2 \equiv \sum_{j=1}^{m} \zeta_j - \theta \sum_{j=1}^{m} \lambda(l-1-j) e_j,$$

$$m_2 \equiv E^*(x_2 | \Omega_0) = -\sum_{j=1}^{m} \frac{\sigma_j(\zeta)^2}{2} - \theta \sum_{j=1}^{m} \lambda(l-1-j)\sigma_j^2,$$

$$\sigma_{22} = \sum_{j=1}^{m} \sigma_j(\zeta)^2 - 2\theta\lambda(l-1-j)\text{cov}(e_j, \zeta_j) + \theta^2\lambda(l-1-j)^2\sigma_j^2.$$

From Appendix D, $A(m)^{-1} = B_m(0)\exp(\Phi + x_1)$ and $\Phi + m_1 + \sigma_{11}/2 = 0$, where

$$x_1 \equiv \theta \sum_{j=1}^{m-1} \lambda(m-1-j) e_j,$$

$$m_1 \equiv E^*(x_1 | \Omega_0), \qquad \sigma_{11} = \theta^2 \sum_{j=1}^{m-1} \lambda(m-1-j)^2\sigma_j^2,$$

and

$$\sigma_{12} = \theta \sum_{j=1}^{m-1} \lambda(m-1-j)\text{cov}(e_j, \zeta_j) - \theta^2 \sum_{j=1}^{m-1} \lambda(l-1-j)\lambda(m-1-j)\sigma_j^2.$$

The value of the option at $t = 0$ is given by

$$c(0) = E^*[c(m)/A(m) \mid \Omega_0] \\ \equiv L_1 - KL_2.$$

Consider first

$$L_1 \equiv E^*[h(0, l)\exp(\Phi_m + x_2)/A(m) \mid x_2 \geq k].$$

Using the lemma in Appendix B and simplifying gives

$$L_1 = h(0, l)B_m(0)\exp(\eta)N(d),$$

where

$$\sigma_2 d = -k + m_2 + \sigma_{12} + \sigma_{22} = \ln[h(0, l)/K] + \eta + \sigma_{22}/2,$$

and

$$\eta = \theta^2\lambda(n-1)\sum_{j=1}^{m}\theta^{m-j}\lambda(l-1-j)\sigma_j^2 \\ - \theta\lambda(n-1)\sum_{j=1}^{m}\theta^{m-j}\mathrm{cov}(e_j, \zeta_j).$$

Next consider $L_2 \equiv E^*[1/A(m) \mid x_2 \geq k] = B_m(0)N(d_2)$, where $\sigma_2 d_2 \equiv -k + m_2 + \sigma_{12}$, implying that $d_2 = d_1 - \sigma_2$.

The proof for the put options is very similar, so details are omitted.

Appendix H: Options on Equity Futures

From (15), the futures price $F(m, i) = F(0, i)\exp(\Phi_m + x_2)$ and, from (16), $F(0, i) = E^*[F(m, i) \mid \Omega_0]$, which implies $\Phi_m + m_2 + \sigma_{22}/2 = 0$, where

$$x_2 \equiv \sum_{j=1}^{m}\zeta_j - \theta\sum_{j=1}^{m}\lambda(i-1-j)e_j, \qquad m_2 \equiv E^*(x_2 \mid \Omega_0),$$

and

$$\sigma_{22} = \sum_{j=1}^{m}\sigma_j(\xi)^2 - 2\theta\lambda(i-1-j)\mathrm{cov}(e_j, \xi_j) + \theta^2\lambda(i-1-j)^2\sigma_j^2.$$

From Appendix D, $A(m)^{-1} = B_m(0)\exp(\Phi + x_1)$ and $\Phi + m_1 + \sigma_{11}/2 = 0$, where

$$x_1 \equiv \theta\sum_{j=1}^{m-1}\lambda(m-1-j)e_j, \qquad m_1 \equiv E^*(x_1 \mid \Omega_0),$$

$$\sigma_{11} = \theta^2 \sum_{j=1}^{m-1} \lambda(m - 1 - j)^2 \sigma_j^2,$$

and

$$\sigma_{12} = \theta \sum_{j=1}^{m-1} \lambda(m - 1 - j)\mathrm{cov}(e_j, \xi_j) - \theta^2 \sum_{j=1}^{m-1} \lambda(i - 1 - j)\lambda(m - 1 - j)\sigma_j^2.$$

A call option will be exercised if $x_2 \geq \ln K/F(0, i) - \Phi_m \equiv k$. The value of the call option at $t = 0$ is given by

$$\begin{aligned} c(0) &\equiv E^*[c(m)/A(m) \mid \Omega_0] \\ &\equiv L_1 - KL_2. \end{aligned}$$

Consider first

$$L_1 \equiv F(0, i)B_m(0)E^*[\exp(\Phi_m + x_2 + \Phi + x_1) \mid x_2 \geq k].$$

Using the lemma in Appendix B and simplifying gives

$$L_1 = F(0, i)B_m(0)\exp(\sigma_{12})N(d_1),$$

where

$$\sigma_2 d_1 \equiv -k + m_2 + \sigma_{12} + \sigma_{22} = \ln[F(0, i)/K] + \sigma_{12} + \sigma_{22}/2.$$

Next consider

$$L_2 \equiv E^*[1/A(m) \mid x_2 \geq k] = B_m(0)N(d_2),$$

where $\sigma_2 d_2 = -k + m_2 + \sigma_{12}$, implying that $d_2 = d_1 - \sigma_2$.

The proof for put options is similar, so details are omitted.

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