SOFR Option Pricing

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1 Pricing SOFR option prices

1.1 Option pricing formula

The option price can be expressed as the discounted contract payoff at the option's expiration date. Assuming that the underlying future price follows a continuous distribution similar to a normal distribution, denoted as $S(\mu_1, \sigma_1, ..., \mu_n, \sigma_n)$, with a probability distribution function $f(x)$, the payoff can be written as follows:

$$\begin{aligned} P &= \mathbb{E}[e^{-\int_{t_0}^{T} r_s \mathrm{ds}}(F_T - K)^+] \\ &= e^{-\int_{t_0}^{T} r_s \mathrm{ds}} \int \{(x-K)^+ f(x) | x = F_T\} \mathrm{dx} \end{aligned} \tag{1}$$

- t_0: calendar date
- T: expiration date
- F_t: future price

1.2 Forward rate forluma

Whenever the Federal Bank adjusts the interest rate at the altitude θ_k, we make the assumption that the rate hike will persist until the end of the contract. Consequently, the final price undergoes a proportional change equal to the remaining date between the meeting and the contract's expiration, divided by the length of the contract.

Therefore, the final SOFR can be expressed as the realized SOFR rate, taking into account the sum of all proportional rate changes that occur after the meeting has taken place. The following formula illustrates the discrete forward rate estimation:

$$f_t = r_t + \sum^{k} \theta_k \frac{d_k}{D} 1\{t > M(t)\} \quad , t_0 < t < T \tag{2}$$

- d_k: k^{th} meeting date to mature
- D: contract length
- θ_k: FOMC rate change at the k^{th} meeting
- $M(k)$: the date of k^{th} FOMC meeting
- r_t: realized SOFR rate

1.3 Assumptions for future price distribution

F_T can be calculated by the formula below:

$$F_T = 100 - \frac{1}{D}(\sum_{t<=t_0} r_{t,realized} + \sum_{t>t_0} f_t) \tag{3}$$

We assume f_T follows a continuous multi-modal distribution with the number of mode equals to n(number of scenarios). Each mode has a mean of μ_i and a standard deviation of σ_i. As shown in Moments of mixtures, let

$$\begin{aligned} f(x) &= \sum^n p_i g_i(x) \\ &= \sum^n p_i g_i(x, \mu_i; \sigma_i) \quad \text{, where } \sum p_i = 1 \end{aligned} \tag{4}$$

where g_i is a probability distribution and p_i is the mixing parameter.

$$p_i = \prod^{\kappa} p_k \tag{5}$$

$$\mu_i = f_{ti} = r_t + \sum^{k} \theta_{ki} \frac{d_k}{D} 1\{t > M(t)\} \quad , t_0 < t < T \tag{6}$$

- n: number of scenarios
- κ: number of meetings

Under this assumption, we can calculate the implied volatility for each option.

Let's make the assumption that $\sigma_i = \sigma(t)$, where $t = T - t_0$. In this scenario, where sigma is solely dependent on the time to maturity, and $\hat{f}_t$ represents an unbiased estimation of μ_i under various assumptions, we can calculate option prices using formula (4).