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Pricing Interest Rate Options in a Two-Factor Cox–Ingersoll–Ross Model of the Term Structure

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Solutions are presented for prices on interest rate options in a two-factor version of the Cox–Ingersoll–Ross model of the term structure. Specific solutions are developed for caps on floating interest rates and for European options on discount bonds, coupon bonds, coupon bond futures, and Eurodollar futures. The solutions for the options are expressed as multivariate integrals, and we show how to reduce the calculations to univariate numerical integrations, which can be calculated very quickly. The two-factor model provides more flexibility in fitting observed term structures, and the fixed parameters of the model can be set to capture the variability of the term structure over time.

Markets for interest rate options represent some of the largest financial markets in the world as billions of dollars are traded daily in Treasury bond futures options, Eurodollar futures options, caps on floating

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interest rates, and swap options. A variety of models and techniques are used to value interest rate options.[1] In this article, we present solutions for bond and interest rate options in a two-factor Cox–Ingersoll–Ross model (hereafter the CIR model). The model is consistent with a general equilibrium for asset pricing, nonnegative interest rates, and no arbitrage opportunities. As in Brennan and Schwartz (1979), we use two factors to characterize the variation of the term structure. To capture the observed variability of interest rates, we set the parameters in this two-factor CIR model so that the first factor has strong mean reversion and the second factor has extremely slow mean reversion. The sum of the two factors determines the instantaneous interest rate, but most of the variation in short-term rates (short-term Treasury bills) is explained by the first factor. The second factor, with slow mean reversion, behaves like a random walk and plays the dominant role in the determination of long-term interest rates.[2] This model, in contrast to the Brennan–Schwartz model, has simple solutions for bond prices and bond futures prices, and we derive solutions for a variety of European options on bonds and interest rates. The general solutions for these European options are expressed as multivariate integrals, and we show how to reduce the calculations to univariate numerical integrations, which can be performed in a fraction of a second. The model analyzed in this article is the same special case of the two-factor CIR model that is analyzed by Longstaff and Schwartz (1992). They make the observation that this model can be interpreted as a stochastic volatility model because the volatility of the instantaneous interest rate is a function of the two factors. The advantage of the analysis in our article is the expression of various option prices in terms of univariate integrals. This approach reduces the computation time to a fraction of that required by the Longstaff and Schwartz procedure. The pricing model for interest rate options is presented in Section 1, and specific solutions are produced in Sections 2 and 3 for the following options: European options on default-free discount bonds, coupon bonds, and coupon bond futures; caps on floating interest rates; and European options on the Eurodollar futures contract.

[1] Examples of the various models and techniques can be found in Black, Derman, and Toy (1990), Chaplin (1987), Courtadon (1982), Cox, Ingersoll, and Ross (1985b), Dybvig (1989), Heath, Jarrow, and Morton (1990, 1992), Ho and Lee (1986), Hull (1989), Hull and White (1990b), Jamshidian (1989), and Turnbull and Milne (1991). Black's (1976) model for futures options is frequently used to value interest rate futures options, but the model assumes that the short-term interest rate is fixed, or at least deterministic.

[2] An analysis of this two-factor model of the term structure is contained in Chen and Scott (1991).

1. The Valuation Model for Interest Rate Options

The model for pricing interest rate options is a two-factor model of the term structure, set within the framework of CIR (1985a, 1985b). We use two independent factors, state variables, to determine the nominal instantaneous interest rate:

$$i_t = y_{1t} + y_{2t}. \tag{1}$$

The two factors are assumed to be generated by independent square root processes:

$$dy_i = \kappa_i(\theta_i - y_i)\, dt + \sigma_i \sqrt{y_i}\, dw_i, \qquad \text{for } i = 1,2, \tag{2}$$

where dw_1 and dw_2 are independent Wiener processes. The risk premiums associated with the two factors are $\lambda_1 y_1$ and $\lambda_2 y_2$, where λ_1 and λ_2 are constants. Two different approaches can be used to motivate the structure of this term structure model. The most direct method is to follow the suggestions in Section 6 of CIR (1985b) and let the means and variances of returns on the fundamental production processes be determined by the sum of the two state variables. In the case of log utility, the instantaneous real interest rate is a linear function of the two factors.[3] By ignoring the role of uncertain inflation, one can apply the model directly to the nominal instantaneous interest rate. Each risk premium is determined endogenously by the covariance of the state variable with marginal utility of real wealth.

An alternative approach is to use the nominal pricing model in Section 7 of CIR (1985b). Let one factor determine the means and variances of returns on fundamental production processes; this factor determines the real instantaneous interest rate. The second factor plays the role of expected inflation, and we use the following model from CIR for consumption price inflation:

$$\frac{dp}{p} = y_2\, dt + \sigma_p \sqrt{y_2}\, dz.$$

With log utility and no correlation between the real and nominal sectors of the economy, the nominal instantaneous interest rate is a linear combination of the two state variables. By rescaling the second state variable, we can express the nominal interest rate as the sum of two state variables, with each following a square root process. The risk premiums for the state variables are determined endogenously by covariances with marginal utility of nominal wealth, and they have

[3] Longstaff and Schwartz (1992) have followed this approach to derive a two-factor model of the term structure. In their model, one factor determines variances and both factors determine mean returns.

the following functional forms: $\lambda_i y_i$. The fundamental valuation equation for nominal prices, $C(X, Y, t)$, on nominal claims, which have no intermediate cash flows, follows as a special case of Equation (57) in CIR (1985b):

$$\frac{1}{2}\sigma_1^2 y_1 C_{11} + \frac{1}{2}\sigma_2^2 y_2 C_{22} + \sum_{i=1}^{2} (\kappa_i\theta_i - \kappa_i y_i - \lambda_i y_i) C_i + C_t - iC = 0. \quad (3)$$

To price discount bonds in the model, we take the risk-adjusted expectation of future interest rates:

$$\hat{E}_t\left(\exp\left\{-\int_t^s i\, du\right\}\right) = \hat{E}_t\left(\exp\left\{-\int_t^s (y_1 + y_2)\, du\right\}\right)$$
$$= A_1(t, s) A_2(t, s)\ e^{-B_1(t, s) y_{1t} - B_2(t, s) y_{2t}},$$

which is a special case of Equation (56) in CIR (1985b), with $\sigma_p = 0$. A and B are similar to the corresponding expressions in the one-factor CIR model:

$$A_i(t, s) = \left[\frac{2\gamma_i e^{(1/2)(\kappa_i + \lambda_i + \gamma_i)(s-t)}}{(\kappa_i + \lambda_i + \gamma_i)(e^{\gamma_i(s-t)} - 1) + 2\gamma_i}\right]^{2\kappa_i\theta_i/\sigma_i^2},$$

$$B_i(t, s) = \frac{2(e^{\gamma_i(s-t)} - 1)}{(\kappa_i + \lambda_i + \gamma_i)(e^{\gamma_i(s-t)} - 1) + 2\gamma_i},$$

where $\gamma_i = \sqrt{(\kappa_i + \lambda_i)^2 + 2\sigma_i^2}$. $\hat{E}_t$ is the risk-adjusted expectation operator that uses the following risk-adjusted processes for the state variables:

$$dy_i = (\kappa_i\theta_i - \kappa_i y_i - \lambda_i y_i)\, dt + \sigma_i\sqrt{y_i}\, dw_i.$$

The model is used to value a variety of interest rate options, as well as futures prices and prices on futures options. To distinguish between the option and futures expirations, we use the following notation: t for current time, T for the expiration of the option or futures option, T_f for the expiration of the futures, and s for the timing of the cash flows of the underlying spot bond that come after the expiration of either the option or the futures. For the various prices, we use $P(y_{1t}, y_{2t}, t, s)$ for the price of a default free bond that pays \$1 at time s, $C(y_{1t}, y_{2t}, t, T)$ for the price of a contingent claim, usually a European call option, K for the strike price for an option, $G(y_{1T}, y_{2T}, T)$ for the cash flow or payoff function at expiration of the claim, $f(y_{1t}, y_{2t}, t, T_f)$ for the futures price, and $F(y_{1t}, y_{2t}, t, T_f)$ for the forward price. $B^*(y_{1t}, y_{2t}, t)$ is used for the price of a coupon bond, with maturity and the level and timing of the cash flows to be specified; $f_i(z)$ is the probability density function for the state variable y_i; and $f^*(z)$ and $F^*(x)$

are used for the standard noncentral χ^2 density function and distribution function, respectively [see Johnson and Kotz (1970, chap. 28)]. The models for the prices of various contingent claims include the bivariate noncentral χ^2 distribution function.

In this version of the CIR model, the valuation function for a contingent claim must satisfy the partial differential equation (PDE), Equation (3). For a claim that has a single cash flow that occurs at a future date T, there is a boundary condition: as t approaches T, $C(y_{1t}, y_{2t}, t, T) = G(y_{1t}, y_{2t}, T)$. An example of such a claim is a European option on a bond. American options or claims with early exercise features have additional boundary conditions. The European options that we examine must satisfy the same PDE, but they differ in their boundary conditions, the terminal payoff functions. The general solution for the European option pricing function is[4]

$$\begin{aligned} C(y_{1t}, y_{2t}, t, T) &= P(y_{1t}, y_{2t}, t, T) \int_0^\infty \int_0^\infty G(z_1, z_2, T) f(z_1; \nu_1, \lambda_1^*) \\ &\qquad \times f(z_2; \nu_2, \lambda_2^*)\, dz_1\, dz_2, \end{aligned} \tag{4}$$

where

$$\begin{aligned} f(z_i; \nu_i, \lambda_i^*) &= (\phi_i + \psi_i) e^{-(\phi_i+\psi_i)z_i - (1/2)\lambda_i^*} \left(\frac{2(\phi_i + \psi_i) z_i}{\lambda_i^*} \right)^{(1/4)\nu_i - 1/2} \\ &\quad \times I_{(1/2)\nu_i - 1}\left(\sqrt{2\lambda_1^*(\phi_i + \psi_i) z_i}\right), \end{aligned}$$

which is the noncentral χ^2 density function. $I_q(x)$ is the modified Bessel function of the first kind, and

$$\nu_i = \frac{2\kappa_i\theta_i}{\sigma_i^2}, \qquad \lambda_i^* = \frac{2\phi_i^2 e^{\gamma_i(T-t)} y_{it}}{\phi_i + \psi_i},$$

$$\phi_i = \frac{2\gamma_i}{\sigma_i^2(e^{\gamma_i(T-t)} - 1)}, \qquad \psi_i = \frac{\kappa_i + \lambda_i + \gamma_i}{\sigma_i^2}.$$

A change of variable is required to convert this noncentral χ^2 density function to the standard form shown in Johnson and Kotz (1970). $f(z_i)$ is the density function for y_{iT}, conditional on y_{it}, where y_i is determined by the following diffusion process:

$$dy_i = [\kappa_i\theta_i - \kappa_i y_i - \lambda_i y_i - \sigma_i^2 B_i(t, T) y_i]\, dt + \sigma_i \sqrt{y_i}\, dw_i.$$

[4] This solution for the option pricing function can be verified by substituting the function and its partial derivatives into the PDE, Equation (3), and by checking the relevant boundary conditions. The details of the verification can be obtained from the authors.

The double integral in the pricing solution is the expectation of the cash flow, but the expectation is one in which two adjustments are made: first, the usual risk adjustment, $-\lambda_i y_i$, plus a second adjustment, $-\sigma_i^2 B_i(t, T) y_i$, because the discounting has been factored out of the original risk adjusted expectation.[5] The expected cash flow is multiplied by $P(y_{1t}, y_{2t}, t, T)$, which is the discount factor. For recent discussions on the separation of the discounting and the expectation of the cash flow, see Jamshidian (1987, 1989).

2. Options on Bonds

In this section, we use the option pricing model of Section 1 to derive solutions for options on discount bonds, coupon bonds, and coupon bond futures. The most active exchange market for bond options is the market for American options on Treasury bond futures. We present a solution for European options on coupon bond futures, which can be used as an initial approximation for the valuation of the American options.[6]

2.1 Options on discount bonds

Discount bond options do not trade actively, but they form a basis for pricing other options; one example is a floating rate cap, which can be treated as a portfolio of European puts on discount bonds. Solutions for discount bond options in one-factor models can be found in Jamshidian (1989) and CIR (1985b). Hull and White (1990b) show how to price these options in a two-factor version of the Vasicek (1977) model. In the two-factor CIR model, the payoff function at expiration is $\max[0, P(y_{1T}, y_{2T}, T, s) - K]$, where s is the maturity of the bond on which the option is written. The pricing function for this option is

$$\begin{aligned} C(y_{1t}, y_{2t}, t, T) &= P(y_{1t}, y_{2t}, t, T) \int^R \int [P(z_1, z_2, T, s) - K] \\ &\qquad \times f(z_1; \nu_1, \lambda_1^*) f(z_2; \nu_2, \lambda_2^*)\, dz_1\, dz_2. \end{aligned} \tag{5}$$

The region for the integration, R, is the range of values for y_1 and y_2 over which the option is in the money. This region is determined by

[5] In terms of the martingale pricing approach, these adjustments on the stochastic processes represent changes in the martingale measure used for valuation.

[6] The approximation formulas for American options discussed in Barone-Adesi and Whaley (1987) and in Barone-Adesi and Elliott (1989) use the valuation models for the corresponding European options.

the following linear equation:

$$B_1(T, s)y_1 + B_2(T, s)y_2 \leq C^* = \ln\left(\frac{A_1(T, s)A_2(T, s)}{K}\right).$$

In the two-dimensional space for y_1 and y_2, R is a triangle formed by the points $(0, 0)$, $(y_1^*, 0)$, and $(0, y_2^*)$, where

$$y_1^* = \frac{C^*}{B_1(T, s)} \qquad \text{and} \qquad y_2^* = \frac{C^*}{B_2(T, s)}.$$

A similar formula has been derived by Longstaff and Schwartz (1992) in a two-factor CIR model.

Equation (5) can be solved directly by using bivariate numerical integration as Longstaff and Schwarz suggest; however, the problem can be reduced to a univariate numerical integration. The first step is to split the integral in (5) as follows:

$$\begin{aligned} &C(y_{1t}, y_{2t}, t, T) \\ &\quad = P(y_{1t}, y_{2t}, t, s) \int^R \int g(z_1; \nu_1, \lambda_1^\circ) g(z_2; \nu_2, \lambda_2^\circ)\, dz_1\, dz_2 \\ &\qquad - P(y_{1t}, y_{2t}, t, T)K \int^R \int f(z_1; \nu_1, \lambda_1^*) f(z_2; \nu_2, \lambda_2^*)\, dz_1\, dz_2, \quad (6) \end{aligned}$$

where

$$\lambda_1^\circ = \frac{2\phi_1^2 e^{\gamma_1(T-t)} y_{1t}}{\phi_1 + \psi_1 + B_1(T, s)}, \qquad \lambda_2^\circ = \frac{2\phi_2^2 e^{\gamma_2(T-t)} y_{2t}}{\phi_2 + \psi_2 + B_2(T, s)},$$

and $g(z_i)$ is the noncentral χ^2 density function with $\phi_i + \psi_i + B_i(T, s)$ in place of $\phi_i + \psi_i$. The region is the same for both integrations. We now apply the change of variable technique to express these probability functions in terms of the standard noncentral χ^2 distribution:

$$\begin{aligned} C(y_{1t}, y_{2t}, t, T) &= P(y_{1t}, y_{2t}, t, s)\, \chi^2(L_1^*, L_2^*; \nu_1, \nu_2, \lambda_1^\circ, \lambda_2^\circ) \\ &\quad - KP(y_{1t}, y_{2t}, t, T)\, \chi^2(L_1, L_2; \nu_1, \nu_2, \lambda_1^*, \lambda_2^*), \quad (7) \end{aligned}$$

where

$$\begin{aligned} \chi^2(L_1, L_2; \nu_1, \nu_2, \lambda_1^*, \lambda_2^*) = \int_0^{L_2} &F^*\left(L_1 - \frac{L_1}{L_2} x_2; \nu_1, \lambda_1^*\right) \\ &\times f^*(x_2; \nu_2, \lambda_2^*)\, dx_2, \end{aligned}$$

$$f^*(x;\, \nu,\, \lambda^*) = \tfrac{1}{2} e^{-(1/2)(x+\lambda^*)} (x/\lambda^*)^{(1/4)\nu - 1/2} I_{(1/2)\nu - 1}(\sqrt{\lambda^* x}),$$

$$F^*(x;\, \nu,\, \lambda^*) = \int_0^x f^*(z;\, \nu,\, \lambda^*)\; dz.$$

f^* and F^* are the standard noncentral χ^2 density function and probability distribution function, and

$$L_1 = 2(\phi_1 + \psi_1) y_1^*, \qquad L_2 = 2(\phi_2 + \psi_2) y_2^*,$$

$$L_1^* = 2(\phi_1 + \psi_1 + B_1(T, s)) y_1^*, \qquad L_2^* = 2(\phi_2 + \psi_2 + B_2(T, s)) y_2^*.$$

The numerical routine for computing F^*, the noncentral χ^2 distribution function, and a method for computing the univariate numerical integration are described in Appendix A.

The discount bond option formula has the same form as the Jamshidian model and the CIR one-factor model: the option price is the price of the bond on which the option is written multiplied by a probability function minus the discounted strike price multiplied by a second probability function.[7] Here the relevant probability functions are bivariate noncentral χ^2 functions, and we have reduced the calculation to a univariate numerical integration. The numerical integration can be done very quickly and accurately on a computer, and it is faster and more accurate than the various numerical approximation techniques like the finite difference method, the Monte Carlo simulation, or the binomial approach. In Table 1, we compare the computing times for our solution in Equation (7) with the computing times for the bivariate integration in (5) and the finite difference method applied to the PDE, Equation (3). The solution in Equation (7) can produce accurate option prices in a fraction of the time required by either the bivariate integration in Equation (5) or the finite difference method. We should also note that the finite difference method requires substantially more computing time than either of the two integral solutions. Solutions like Equation (7) can be differentiated with respect to the state variables, so that one can develop dynamic trading strategies for hedging against changes in the term structure. European puts on discount bonds can be valued by applying the put–call parity relationship: Put = Call + $P(y_{1t}, y_{2t}, t, T)K - P(y_{1t}, y_{2t}, t, s)$. As noted by CIR, the European call formula also applies to

[7] The martingale pricing approach to valuation can be used to show that this is a very general form for European call option pricing models. The two probability functions are probabilities that the option will be in the money, but different martingale measures are used.

Table 1
Alternative methods for calculating discount bond options

4 call options on a discount bond   Time to maturity, $T = 6$ months $s - T = 3$ months

Current price on a 3-month discount bond = 98.238
6-month forward price on 3-month discount bond = 97.863

A. Univariate numerical integration, Equation (7)

Grid, M = No. of intervals into which the range of the integration is divided

	Option prices			
Strike price	$M = 10$	$M = 15$	$M = 25$	$M = 400$
96.884	0.9423	0.9439	0.9439	0.9439
97.373	0.4921	0.4924	0.4924	0.4924
97.863	0.1437	0.1437	0.1437	0.1437
98.352	0.0112	0.0112	0.0112	0.0112
CPU time in seconds (4 options)	0.04	0.07	0.11	1.75

B. Bivariate numerical integration of the joint density function, Equation (5)

Grid, M = No. of intervals in each direction for integration

	Option prices			
Strike price	$M = 25$	$M = 50$	$M = 100$	$M = 200$
96.884	0.9439	0.9439	0.9439	0.9439
97.373	0.4921	0.4923	0.4924	0.4924
97.863	0.1432	0.1436	0.1436	0.1437
98.352	0.0111	0.0111	0.0112	0.0112
CPU time in seconds (4 options)	0.22	0.83	3.23	12.86

C. Finite difference method, explicit method of Hull and White (1990a)

N = No. of steps in time direction, $\Delta t = T/N$

	Option prices			
Strike price	$N = 25$	$N = 50$	$N = 100$	$N = 200$
96.884	0.9390	0.9415	0.9427	0.9433
97.373	0.4884	0.4905	0.4914	0.4919
97.863	0.1422	0.1430	0.1433	0.1435
98.352	0.0112	0.0112	0.0112	0.0112
CPU time in seconds (4 options)	0.80	6.00	43.70	315.49

The parameter values have been set at $\kappa_1 = 1.8341$, $\theta_1 = 0.05148$, $\sigma_1 = 0.1543$, $\lambda_1 = -0.1253$, $\kappa_2 = 0.005212$, $\theta_2 = 0.03083$, $\sigma_2 = 0.06689$, and $\lambda_2 = -0.06650$. The values for the state variables have been set at $y_1 = 0.02516$ and $y_2 = 0.040016$; these values generate an upward-sloping term structure with a three-month yield of 7.11% and a 20-year yield of 10.76%. The set of parameter values is taken from Chen and Scott (1991) and it generates term structure variability that reflects the experience of the 1980s. Options on three-month Treasury bills do not trade actively, but there is an option on three-month Treasury-bill futures traded at the Chicago Mercantile Exchange. The minimum price fluctuation on this option is 1 basis point. The prices shown in this table have not been annualized, and 1 basis point corresponds to 0.0025. The values for the probability functions range from .086 to .990. The Black–Scholes model and the one-factor CIR model were used to value the same set of four discount bond options, and the computing times were 0.005 and 0.007 seconds, respectively. All of the calculations were done on an IBM ES9000 mainframe computer.

American calls on discount bonds because there is no intermediate cash flow on the discount bond. The comparative statics for this two-factor option pricing model are similar to those for the one-factor CIR model.

2.2 Options on coupon bonds

For single-factor models, Jamshidian (1989) has shown that an option on a coupon bond can be priced by valuing a portfolio of options on discount bonds. This approach does not work in multifactor models, and we must take a direct approach to value coupon bond options in the two-factor model. Consider a European call option on a coupon bond that will have N periods of cash flows after the expiration of the option. The cash flows, coupon payments and principal, occur at periods $s_1, \ldots, s_N$. The value of the stream of cash flows is

$$B^*(y_{1t}, y_{2t}, t) = \sum_{j=1}^{N} c_j P(y_{1t}, y_{2t}, t, s_j),$$

where c_j is the cash flow in period s_j. The payoff function for this option is $\max[0, B^*(y_{1T}, y_{2T}, T) - K]$.

For the value of the option, we take the relevant expectation of this cash flow by integrating over the exercise region. In the case of the discount bond option, the determination of the exercise region is a solution to a linear equation. In the coupon bond option, the determination of this exercise region requires solutions to a nonlinear equation: (y_1, y_2) such that $B^*(y_1, y_2, T) \geq K$. The critical values for the boundary of this exercise region now form a curve instead of a straight line. From the solution for bond prices, we determine the set of points $Y = (y_1, y_2)$ such that $B^*(y_1, y_2, T) = K$. Our approach here is to express the model in terms of probability functions that can be computed by a univariate numerical integration similar to the method employed in Equation (7).

The model for the coupon bond option is

$$\begin{aligned} C(y_{1t}, y_{2t}, t, T) = B^*(y_{1t}, y_{2t}, t) &\left\{ \sum_{j=1}^{N} w_j \chi^{2*}(Y_j^*; \nu_1, \nu_2, \lambda_{1j}^*, \lambda_{2j}^*) \right\} \\ &- P(y_{1t}, y_{2t}, t, T) K \chi^{2*}(Y^*; \nu_1, \nu_2, \lambda_1^*, \lambda_2^*), \end{aligned} \tag{8}$$

where

$$w_j = \frac{c_j P(y_{1t}, y_{2t}, t, s_j)}{B^*(y_{1t}, y_{2t}, t)}.$$

The probability functions are given in Appendix B, where we show

how to reduce the calculations to univariate numerical integrations. $B^*(y_{1t}, y_{2t}, t)$ is the value of the underlying coupon bond minus the value of coupon payments to be made before the option expires, and the weights, the w_j's, sum to unity. This model requires a weighted average of probability functions; the corresponding call formula in a two-factor Vasicek model requires a weighted average of normal probability functions, because bond prices in the Vasicek model are also exponential functions of the state variables, and coupon bonds are valued as sums of discount bond prices. It may be optimal to exercise American calls on coupon bonds early, depending upon the treatment of coupons and accrued interest. Again, the European puts can be valued via the put–call parity relation.

2.3 Options on coupon bond futures

We now consider a European call on a coupon bond futures where T is the option expiration, T_f is the futures expiration, and $s_1,\ldots,s_N$ represent the timing of the cash flows on the underlying bond that come after T_f. Because $G = \max[0, f(y_{1T}, y_{2T}, T, T_f) - K]$, we need the futures price at time T:[8]

$$
\begin{aligned}
f(y_{1T}, y_{2T}, T, T_f) &= \hat{E}_T\{B^*(y_{1T_f}, y_{2T_f}, T_f)\} \\
&= \hat{E}_T\left\{\sum_{j=1}^{N} c_j A_1(T_f, s_j) A_2(T_f, s_j)\, e^{-B_1(T_f,s_j)y_{1T_f} - B_2(T_f,s_j)y_{2T_f}}\right\}.
\end{aligned}
$$

This risk-adjusted expectation can be evaluated by using the moment generating function or the Laplace transform for the noncentral χ^2 distribution. The result is

$$
\begin{aligned}
f(y_{1T}, y_{2T}, T, T_f) = \sum_{j=1}^{N} & c_j H_1(T, T_f, s_j) \\
& \times H_2(T, T_f, s_j)\, e^{-D_1(T,T_f,s_j)y_{1T} - D_2(T,T_f,s_j)y_{2T}},
\end{aligned}
$$

where

$$
d_1(T, T_f) = \frac{2(\kappa_1 + \lambda_1)}{\sigma_1^2(1 - e^{-(\kappa_1+\lambda_1)(T_f - T)})},
$$

$$
d_2(T, T_f) = \frac{2(\kappa_2 + \lambda_2)}{\sigma_2^2(1 - e^{-(\kappa_2+\lambda_2)(T_f - T)})},
$$

[8] See Cox, Ingersoll, and Ross (1981) on the pricing of futures and forward contracts. They show that the futures price in continuous-time models is equal to the risk-adjusted expectation of the spot price at delivery.

$$H_1(T, T_f, s_j) = A_1(T_f, s_j)\left[\frac{d_1(T, T_f)}{d_1(T, T_f) + B_1(T_f, s_j)}\right]^{2\kappa_1\theta_1/\sigma_1^2},$$

$$H_2(T, T_f, s_j) = A_2(T_f, s_j)\left[\frac{d_2(T, T_f)}{d_2(T, T_f) + B_2(T_f, s_j)}\right]^{2\kappa_2\theta_2/\sigma_2^2},$$

$$D_1(T, T_f, s_j) = \frac{d_1(T, T_f)e^{-(\kappa_1+\lambda_1)(T_f-T)}B_1(T_f, s_j)}{d_1(T, T_f) + B_1(T_f, s_j)},$$

$$D_2(T, T_f, s_j) = \frac{d_2(T, T_f)e^{-(\kappa_2+\lambda_2)(T_f-T)}B_2(T_f, s_j)}{d_2(T, T_f) + B_2(T_f, s_j)}.$$

The futures price has the same exponential form as the bond price.

To value the European call on this futures contract, we again take the relevant expectation by integrating over the exercise region. As in the case of the coupon bond option, we need to determine the nonlinear boundary for the exercise region: $Y = (y_1, y_2)$ so that $f(y_1, y_2, T) = K$. The resulting model is

$$\begin{aligned} C(y_{1t}, y_{2t}, t, T) &= P(y_{1t}, y_{2t}, t, T)F(y_{1t}, y_{2t}, t, T, T_f) \\ &\quad \times \left(\sum_{j=1}^{N} w_j \chi^{2*}(Y_j^*; \nu_1, \nu_2, \lambda_{1j}^*, \lambda_{2j}^*)\right) \\ &\quad - KP(y_{1t}, y_{2t}, t, T)\chi^{2*}(Y^*; \nu_1, \nu_2, \lambda_1^*, \lambda_2^*), \end{aligned} \tag{9}$$

where

$$w_j = \frac{c_j F(y_{1t}, y_{2t}, t, T; T_f, s_j)}{F(y_{1t}, y_{2t}, t, T; T_f)}.$$

The probability functions are described in Appendix B. $F(y_{1t}, y_{2t}, t, T; T_f)$ represents the forward price for an asset whose value at time T will be the coupon bond futures price:

$$F(y_{1t}, y_{2t}, t, T; T_f) = \sum_{j=1}^{N} c_j F(y_{1t}, y_{2t}, t, T; T_f, s_j).$$

Each term, $F(y_{1t}, y_{2t}, t, T; T_f, s_j)$, is a forward price, specifically a forward price on an asset whose value at time T will be equal to the futures price for a discount bond with maturity s_j to be delivered at T_f. This forward price can be interpreted as a forward price on the futures price. The pricing model for each forward price is

$$\begin{aligned} &F(y_{1t}, y_{2t}, t, T_f, s_j) \\ &\quad = H_1(T, T_f, s_j)H_2(T, T_f, s_j)\left(\frac{\phi_1 + \psi_1}{\phi_1 + \psi_1 + D_1(T, T_f, s_j)}\right)^{2\kappa_1\theta_1/\sigma_1^2} \end{aligned}$$

$$\times\left(\frac{\phi_2+\psi_2}{\phi_2+\psi_2+D_2(T,T_f,s_j)}\right)^{2\kappa_2\theta_2/\sigma_2^2}$$
$$\times\exp\left\{-\frac{\phi_1^2 e^{\gamma_1(T-t)}D_1(T,T_f,s_j)y_{1t}}{(\phi_1+\psi_1)(\phi_1+\psi_1+D_1(T,T_f,s_j))}\right.$$
$$\left.-\frac{\phi_2^2 e^{\gamma_2(T-t)}D_2(T,T_f,s_j)y_{2t}}{(\phi_2+\psi_2)(\phi_2+\psi_2+D_2(T,T_f,s_j))}\right\}.$$

This model for the coupon bond futures option uses a weighted average of noncentral χ^2 probability functions as in the coupon bond option. One of the differences between this model and Black's model is the use of a current forward price in place of the current futures price. Consider the case of a futures call option with a strike price of zero. The valuation problem becomes the solution to the following expectation:

$$\hat{E}_t\left\{\exp\left(-\int_t^T i\,du\right)f(y_{1T},y_{2T},T,T_f)\right\},$$

which, from the results of CIR (1981), is the discount bond price multiplied by a forward price. If the interest rates were uncorrelated with the futures price in this expectation, then the forward price would be equal to the futures price. The forward price in the option pricing solution accounts for the correlation between the instantaneous interest rate and the futures price on which the option is written. Futures options in Black's model depend on the current futures price, but in that model there is no difference between futures and forward prices because the interest rate is nonstochastic. A similar result has been derived by Chen (1992) for bond futures options in a one-factor Vasicek model. Turnbull and Milne (1991), in a model with normally distributed interest rates, find that they need to multiply the futures price by an adjustment term to account for the correlation between interest rates and futures prices. We find that for plausible parameter values and short-time horizons (less than one year), the differences between the forward and futures prices are extremely small For short-term options, one could use the current futures price in place of this forward price and the difference would be negligible.

3. Options on Short-term Interest Rates

The most actively traded short-term interest rate option is the Eurodollar futures option traded at the Chicago Mercantile Exchange (CME). When these options are exercised, the buyer and the seller of the option take positions in Eurodollar futures. The Eurodollar futures is a cash-settled futures contract and the final price at delivery is equal to 100 minus the London interbank offer rate (LIBOR); the exchange uses an average from several London banks. LIBOR is a simple interest rate, quoted on an annual basis, and three-month LIBOR is used for the Eurodollar futures contract traded at the CME. In addition to the Eurodollar futures option, there is a very active over-the-counter market in floating rate caps. Most caps are written on LIBOR, and these caps are effectively portfolios of European options on LIBOR. To price these interest rate options, we need a model for future changes in LIBOR. Several approaches have been used to model LIBOR in continuous-time models.[9] One approach has been to assume a diffusion process for the interest rate, but by doing this one loses all of the relevant information in the current term structure for Treasury rates. Our approach is to model LIBOR as a combination of the corresponding risk-free (Treasury) rate and a spread or markup over the risk-free rate: $(1 + R_L) = (1 + R_f)(1 + S)$, where R_f is the simple interest rate on a Treasury security with the same time to maturity as LIBOR, and S is the spread variable. This spread reflects the default risk in the Eurodollar market, and it can be different for the various maturities. The most commonly used maturities are three months and six months. In most of our examples, the maturity, $s - t$, is set at .25 years or three months. We use the simple interest rate for LIBOR, which is the quoted annualized rate divided by 4, for the three-month LIBOR. The simple interest rate on the corresponding Treasury bond is $R_f = 1/P(y_{1t}, y_{2t}, t, s) - 1$ in our notation. A random spread variable adds a third state variable to the models for Eurodollar futures options and caps and increases the dimension of the solutions. Our model for the random spread variable is $1 + S_t = \exp(y_{3t})$, where y_{3t} is also a square root diffusion process:

$$dy_3 = \kappa_3(\theta_3 - y_3)\,dt + \sigma_3\sqrt{y_3}\,dw_3. \tag{10}$$

dw_3 is independent of dw_1 and dw_2 and the risk premium is $\lambda_3 y_3$. The option pricing function for a claim that depends on LIBOR with a random spread is a risk-adjusted expectation where the integration is over a trivariate noncentral χ^2 density function.

3.1 The Eurodollar futures option

We consider first a European call on the Eurodollar futures. The exchange traded option matures when the futures matures so that $T = T_f$, and the final futures settlement is based on the three-month LIBOR. For our analysis, we follow the convention of using $100 for

[9] For examples, see Hull (1989), Boyle and Turnbull (1989), and Turnbull and Milne (1991).

the face amount. For the futures price at delivery, we have

$$f(y_{1T_f}, y_{2T_f}, y_{3T_f}, T_f, T_f) = 100(1 - R_{LT_f}) = 100(2 - (1 + R_{LT_f}))$$
$$= 100\left(2 - \frac{1 + S_{T_f}}{P(y_{1T_f}, y_{2T_f}, T_f, T_f + .25)}\right),$$

where R_{Lt} is the LIBOR at time t. The model for three-month LIBOR with a random spread is

$$1 + R_{Lt} = \frac{1 + S_t}{P(y_{1t}, y_{2t}, t, t + .25)}$$
$$= A_1^{-1}(t, t + .25)A_2^{-1}(t, t + .25)$$
$$\times \exp\{B_1(t, t + .25)y_{1t} + B_2(t, t + .25)y_{2t} + y_{3t}\}.$$

To price the option, we need to price first the futures contract. The three-factor model for the futures price prior to delivery is[10]

$$f(y_{1t}, y_{2t}, y_{3t}, t, T_f) = 100[2 - H_1^{-1}(t, T_f)H_2^{-1}(t, T_f)H_3^{-1}(t, T_f)$$
$$\times e^{D_1(t, T_f)y_{1t} + D_2(t, T_f)y_{2t} + D_3(t, T_f)y_{3t}}]$$
$$= 100 - R^*(y_{1t}, y_{2t}, y_{3t}, t, T_f),$$

where

$$d_3(t, T_f) = \frac{2(\kappa_3 + \lambda_3)}{\sigma_3^2(1 - e^{-(\kappa_3+\lambda_3)(T_f - t)})},$$

$$D_3(t, T_f) = \frac{d_3(t, T_f)e^{-(\kappa_3+\lambda_3)(T_f - t)}}{d_3(t, T_f) - 1},$$

$$H_3(t, T_f) = \left[\frac{d_3(t, T_f) - 1}{d_3(t, T_f)}\right]^{2\kappa_3\theta_3/\sigma_3^2}$$

R^* is the corresponding futures rate.

The European call formula now requires a trivariate noncentral χ^2 distribution function:

$$C(y_{1t}, y_{2t}, t, T) = P(y_t, y_{2t}, t, T)$$
$$\times \int\int\int^{R} [f(z_1, z_2, z_3, T, T_f) - K]f(z_1; \nu_1, \lambda_1^*)$$
$$\times f(z_2; \nu_2, \lambda_2^*)f(z_3; \nu_3, \lambda_3^*)\, dz_1\, dz_2\, dz_3.$$

[10] The futures price is equal to the risk-adjusted expectation of the futures price at delivery, which can be evaluated by using the moment generating function.

The boundary for this exercise region is a plane in a three-dimensional space: $B_1(.25)y_1 + B_2(.25)y_2 + y_3 = C^*$, where $C^* = \ln[(200 - K)A_1(.25)A_2(.25)/100]$. The resulting model for the European futures call is

$$\begin{aligned}
&C(y_{1t}, y_{2t}, y_{3t}, t, T) \\
&\quad = P(y_{1t}, y_{2t}, t, T)(100 - K)\chi^2(L_1, L_2, L_3) \\
&\qquad - P(y_{1t}, y_{2t}, y_{3t}, t, T)R(y_{1t}, y_{2t}, y_{3t}, t, T)\chi^2(L_1^*)\chi^2(L_1^*, L_2^*, L_3^*) \\
&\qquad + 100P(y_{1t}, y_{2t}, y_{3t}, t, T)[\chi^2(L_1, L_2, L_3) - \chi^2(L_1^*, L_2^*, L_3^*)], \quad (11)
\end{aligned}$$

where $\chi^2(L_1, L_2, L_3)$ and $\chi^2(L_1^*, L_2^*, L_3^*)$ are trivariate noncentral χ^2 probability functions which are described in Appendix B. $R^*(y_{1t}, y_{2t}, y_{3t}, t, T)$ is forward LIBOR and the model for this forward rate is

$$\begin{aligned}
R(y_{1t}, y_{2t}, y_{3t}, t, T) = -100 &+ 100A_1^{-1}(.25)A_2^{-1}(.25)H_3^{-1}(t, T) \\
&\times e^{D_3(t, T)y_{3t}}\left[\frac{\phi_1 + \psi_1}{\phi_1 + \psi_1 - B_1(.25)}\right]^{2\kappa_1\theta_1/\sigma_1^2} \\
&\times \left[\frac{\phi_2 + \psi_2}{\phi_2 + \psi_2 - B_2(.25)}\right]^{2\kappa_2\theta_2/\sigma_2^2} \\
&\times \exp\left\{\frac{\phi_1^2 e^{\gamma_1(T-t)}B_1(.25)y_{1t}}{(\phi_1 + \psi_1)(\phi_1 + \psi_1 - B_1(.25))}\right. \\
&\qquad \left. + \frac{\phi_2^2 e^{\gamma_2(T-t)}B_2(.25)y_{2t}}{(\phi_2 + \psi_2)(\phi_2 + \psi_2 - B_2(.25))}\right\}.
\end{aligned}$$

For short-term options, this forward rate will be very close to the futures rate. The option pricing model picks up a third term (the last of the three terms above), because of the manner in which the interest rate calculations are made. As in the case of the coupon bond futures option, we use the forward rate to value the option, but now we use a trivariate noncentral χ^2 probability function because there are three factors that determine LIBOR in the model. If we use the univariate noncentral χ^2 distribution function for one state variable, we can reduce the calculations of the probability functions to bivariate numerical integrations.[11]

[11] By using the Fourier inversion formula for distribution functions [see Feller (1971, p. 511)], one can reduce the calculations for these probability functions in the three-factor model to a univariate numerical integration.

3.2 Interest rate caps

Interest rate caps are effectively call options on interest rates, and these instruments can be priced as portfolios of European options. Caps are sold in an over-the-counter market and the terms can vary: the interest rate can be a Treasury rate or LIBOR, the payments can be made at the beginning or end of each reset period, and the life of a cap can be only a few years or as long as 10 years. A typical cap is written on LIBOR with discounted payments made at the beginning of each period. For each period, the potential payoff on the cap is the face amount times $\max[0, R_{Lt} - K]$, where K is the cap rate. If the payment is to be made at the beginning of the period, it is discounted by dividing by $(1 + R_{Lt})$. For the valuation of the cap, we have

$$V(\text{Cap}) = \sum_{j=1}^{N-1} (\$100)\hat{E}_t\left\{\exp\left(-\int_t^{T_j} i\,du\right)\max\left(0, \frac{R_{LT_j} - K}{1 + R_{LT_j}}\right)\right\}.$$

The face amount is set at \$100. The cap is a European call option on the interest rate. As Hull and White (1990b) and others have shown, the cap, for valuation purposes, can be transformed into a portfolio of European puts on discount bonds, and the model for pricing discount bond options can be applied. Take each option, or each potential cash flow in the valuation of the cap:

$$\begin{aligned}
&\max\left[0, \frac{R_{LT_j} - K}{1 + R_{LT_j}}\right] \\
&\quad = \max\left[0, 1 - \frac{1 + K}{1 + R_{LT_j}}\right] \\
&\quad = \max[0, 1 - (1 + K)A_1A_2\exp\{B_1y_{1t} - B_2y_{2t} - y_{3t}\}],
\end{aligned}$$

where A_1, A_2, B_1, and B_2 are computed with $s - t = \Delta t$. The valuation formula for each European put is

$$\begin{aligned}
C_p(y_{1t}, y_{2t}, y_{3t}, t, T_j) = {}& P(y_{1t}, y_{2t}, t, T_j)[1 - \chi^2(L_{1j}, L_{2j}, L_{3j})] \\
& - (1 + K)\hat{E}_t\left(\frac{1}{1 + S_{T_j}}\right)P(y_{1t}, y_{2t}, t, T_j + \Delta t) \\
& \times [1 - \chi^2(L_{1j}^*, L_{2j}^*, L_{3j}^*)], \qquad (12)
\end{aligned}$$

where

$$\hat{E}_t\left(\frac{1}{1 + S_{T_j}}\right) = \hat{E}_t(e^{-y_{3T_j}})$$

$$= \left[\frac{d_3(t, T_j)}{d_3(t, T_j) + 1}\right]^{2\kappa_3\theta_3/\sigma_3^2} \exp\left\{-\frac{d_3(t, T_j)}{d_3(t, T_j) + 1} e^{-(\kappa_3+\lambda_3)(T_j - t)} y_{3t}\right\}.$$

The probability functions, which are also trivariate noncentral χ^2 distribution functions, are described in Appendix B. As in the case of the three-factor Eurodollar option pricing model, the calculations for these probability functions can be reduced to bivariate numerical integrations. The value of the cap is the sum of the values of these put options:

$$V(\text{Cap}) = (\$100) \sum_{j=1}^{N-1} C_p(y_{1t}, y_{2t}, y_{3t}, t, T_j).$$

To price a cap when the payments are made at the end of each period without discounting, we modify the put formula as follows:

$$\begin{aligned} &C_p(y_{1t}, y_{2t}, y_{3t}, t, T_j) \\ &\quad = \hat{E}_t(1 + S_{T_j})P(y_{1t}, y_{2t}, t, T_j)[1 - \chi^2(L_{1j}, L_{2j}, L_{3j})] \\ &\qquad - (1 + K)P(y_{1t}, y_{2t}, t, T_j + \Delta t)[1 - \chi^2(L_{1j}^*, L_{2j}^*, L_{3j}^*)], \qquad (13) \end{aligned}$$

where

$$\begin{aligned} \hat{E}_t(1 + S_{T_j}) &= \hat{E}_t(e^{y_{3T_j}}) \\ &= \left[\frac{d_3(t, T_j)}{d_3(t, T_j) - 1}\right]^{2\kappa_3\theta_3/\sigma_3^2} \exp\left\{\frac{d_3(t, T_j)}{d_3(t, T_j) - 1} e^{-(\kappa_3+\lambda_3)(T_j - t)} y_{3t}\right\}. \end{aligned}$$

These models for interest rate options are easy to implement. The only requirement is a program for the numerical integration of the noncentral χ^2 distribution functions, and we have shown how to reduce the calculations to fast univariate numerical integrations. If a three-factor model is used for LIBOR, then the calculations can be reduced to bivariate numerical integrations. The two factors for the term structure, the state variables y_1 and y_2, can be set so that the bond pricing function matches bond prices at the short and long ends of the yield curve. The fixed parameters in the stochastic processes for the state variables should be set to reflect the variability of the term structure over time. One approach is to estimate these parameters from a sample of bond prices over time. Small adjustments in the parameter values should be sufficient for matching the model to the current term structure.

4. Summary

Valuation models for caps and European options on discount bonds, coupon bonds, coupon bond futures, and Eurodollar futures have been presented in Sections 2 and 3. The general solutions for these interest rate option pricing models require the evaluation of multivariate integrals, but the calculations can be reduced to univariate numerical integrations or, in the case of the three-factor model for LIBOR, to bivariate numerical integrations. These solutions are faster and easier to use than alternative numerical techniques such as the finite difference method, the Monte Carlo simulation, or the binomial model. We have compared prices from these two-factor CIR models with prices from Black's model for a variety of options across different shapes for the term structure. The prices from the two models are close for short-term futures options when we use the expected volatility of the futures price in Black's model.[12] Large, significant differences between the two models occur when longer-term options, such as three- to five-year coupon bond options or interest rate caps, are valued.

Our analysis of interest rate options has uncovered several subtle observations on valuation in multifactor models. For European options on coupon bonds and coupon bond futures, the exercise region is nonlinear because the logarithm of the bond price is not a linear function of the state variables of the models. In one-factor models, such as the Vasicek model or the one-factor CIR model, one can analyze coupon bond options as a portfolio of options on the discount bonds that comprise the coupon bond. If the discount bond options are structured properly, then the exercise of the European options in the portfolio will be an all-or-none decision. This approach does not work in multifactor models, and one must trace out the boundary of the exercise region to value European options. This observation also applies to multifactor versions of Vasicek's model. Our second observation concerns the valuation of futures options. In Black's model, the current futures price is the relevant price; in our models, the relevant price is a forward price, or a forward rate as in Eurodollar futures options. The forward prices account for the correlation between interest rates and prices on interest rate futures. If the interest rates are fixed as in Black's model, then there is no difference between the forward and futures prices. For short-term options, the differences are very small, but they could be significant for long-term futures options.

Appendix A: Calculation of the Noncentral χ^2 Distribution Function

The option pricing functions require numerical integration to calculate either a bivariate or a trivariate noncentral χ^2 probability dis-

[12] The futures prices and expected volatilities are calculated from the two-factor CIR model.

tribution function. The numerical integration is performed on products of the distribution function and the probability density functions. The following algorithm is used to compute the noncentral χ^2 distribution function:

$$F^*(x; \nu, \lambda) = \sum_{j=0}^{\infty} \frac{e^{-(1/2)\lambda}(\frac{1}{2}\lambda)^j}{j!} \Pr\{\chi^2_{\nu+2j} \leq x\},$$

where ν is the degrees of freedom and λ is now the noncentrality parameter. The noncentral χ^2 distribution can be interpreted as a mixture of distributions: a χ^2 distribution with the degrees of freedom determined randomly by a Poisson distribution. The χ^2 distribution satisfies a recursive relation:

$$\Pr\{\chi^2_{\nu+2j+2} \leq x\} = \Pr\{\chi^2_{\nu+2j} \leq x\} - \frac{(\frac{1}{2}x)^{(1/2)\nu+j}e^{-(1/2)x}}{\Gamma(\frac{1}{2}\nu + j + 1)}.$$

The χ^2 distribution function is calculated for $\Pr\{\chi^2 \leq x\}$ when $j = 0$, and the recursion is used for all of the terms for $j \geq 1$. The gamma function also satisfies a recursive relation so that only one call to a gamma function routine is required. The summation for F^* continues for a minimum number of terms and then stops when the next term is less than machine precision for the computer. For our calculations, we have used the subroutine DCSNDF in the International Mathematical and Statistical Library (IMSL). In the case of the bivariate noncentral χ^2 distribution function, the integration is reduced to a univariate numerical integration. The density function in the integration includes a modified Bessel function of the first kind, which can also be calculated quickly. For large values of the argument, we use the formula (9.7.1) of Abramowitz and Stegun (1972, p. 377), and for small values we use the subroutine DBSIES in IMSL. We use the Euler–Maclaurin summation formula, with first derivatives, to perform the numerical integration. This method is more accurate than the trapezoidal rule, which assumes that the function is linear between grid points. See formulas (25.4.4) and (25.4.7) in Abramowitz and Stegun. In the three-factor models for Eurodollar futures options and caps, which incorporate a random spread between LIBOR and Treasury-bill rates, we reduce the trivariate integral to a bivariate numerical integration.

Appendix B: The Probability Functions

In this appendix, we describe the calculations of the probability functions for options on coupon bonds, coupon bond futures, and Eurodollar futures and the probability functions for the put options in the valuation model for caps. The model for options on coupon bonds requires the computation of a number of bivariate noncentral χ^2 probability functions,

$$\chi^{2*}(Y^*; \nu_1, \nu_2, \lambda_1^*, \lambda_2^*) = \int_0^L F^*(x_1^*(x_2); \nu_1, \lambda_1^*) f^*(x_2; \nu_2, \lambda_2^*)\; dx_2,$$

where Y^* is the set of transformed values $[2(\phi_1 + \psi_1)y_1, 2(\phi_2 + \psi_2)y_2]$ and L is the largest value for $2(\phi_2 + \psi_2)y_2$. The values $x_1^*(x_2)$ are determined as follows:

1. Calculate $y_2 = x_2/2(\phi_2 + \psi_2)$ for the given value x_2.
2. Find y_1^* such that $B^*(y_1^*, y_2, T) = K$ for this value of y_2.
3. Compute $x_1^*(x_2) = 2(\phi_1 + \psi_1)y_1^*$.

F^* and f^* are the standard noncentral χ^2 distribution and density functions. The probability function is a double integral over the nonlinear exercise region for the option, but we eliminate integration in one direction by using the noncentral χ^2 distribution function. The other probability functions are computed as follows:

$$\chi^{2*}(Y_j^*; \nu_1, \nu_2, \lambda_{1j}^*, \lambda_{2j}^*) = \int_0^{L_j} F^*(x_{1j}^*(x_2); \nu_1, \lambda_{1j}^*) f^*(x_2; \nu_2, \lambda_{2j}^*)\, dx_2,$$

where

$$\lambda_{1j}^* = \frac{2\phi_1^2 e^{\gamma_1(T-t)} y_{1t}}{\phi_1 + \psi_1 + B_1(T, s_j)}, \qquad \lambda_{2j}^* = \frac{2\phi_2^2 e^{\gamma_2(T-t)} y_{2t}}{\phi_2 + \psi_2 + B_2(T, s_j)}.$$

Y_j^* is the set of values $[2(\phi_1 + \psi_1 + B_1(T, s_j))y_1, 2(\phi_2 + \psi_2 + B_2(T, s_j))y_2]$, and L_j is the largest value for $2(\phi_2 + \psi_2 + B_2(T, s_j))y_2$. x_{1j}^* is determined from the set Y_j^* by the procedure described above for x_1^*.

In the model for coupon bond futures options, the probability functions are calculated as follows:

$$\chi^{2*}(Y^*; \nu_1, \nu_2, \lambda_1^*, \lambda_2^*) = \int_0^L F^*(x_1^*(x_2); \nu_1, \lambda_1^*) f^*(x_2; \nu_2, \lambda_2^*)\; dx_2$$

and

$$\chi^2(Y_j^*; \nu_1, \nu_2, \lambda_{1j}^*, \lambda_{2j}^*) = \int_0^{L_j} F^*(x_{1j}^*(x_2); \nu_1, \lambda_{1j}^*) f^*(x_2; \nu_2, \lambda_{2j}^*)\; dx_2,$$

where

$$\lambda_{1j}^* = \frac{2\phi_1^2 e^{\gamma_1(T-t)} y_{1t}}{\phi_1 + \psi_1 + D_1(T, T_f, s_j)}, \qquad \lambda_{2j}^* = \frac{2\phi_2^2 e^{\gamma_2(T-t)} y_{2t}}{\phi_2 + \psi_2 + D_2(T, T_f, s_j)}.$$

Y^* is the set of values $[2(\phi_1 + \psi_1)y_1, 2(\phi_2 + \psi_2)y_2]$ and Y_j^* is the set of values $[2(\phi_1 + \psi_1 + D_1(T, T_f, s_j))y_1, 2(\phi_2 + \psi_2 + D_2(T, T_f, s_j))y_2]$. $x_1^*(x_2)$ and $x_{1j}^*(x_2)$ are calculated using the boundary of the exercise region.

In the model for Eurodollar futures options, the probability functions are calculated as follows:

$$\chi^2(L_1, L_2, L_3) = \int_0^{L_3} \int_0^{(L_2-(L_2/L_3)x_3)} F^*\left(L_1 - \frac{L_1x_2}{L_2} - \frac{L_1x_3}{L_3}; \nu_1, \lambda_1^*\right) \times f^*(x_2; \nu_2, \lambda_2^*) \times f^*(x_3; \nu_3, \lambda_3^*)\, dx_2\, dx_3,$$

$$\chi^2(L_1^*, L_2^*, L_3^*) = \int_0^{L_3^*} \int_0^{(L_3^*-(L_2^*/L_3^*)x_3)} F^*\left(L_1^* - \frac{L_1^*x_2}{L_2^*} - \frac{L_1^*x_3}{L_3^*}; \nu_1, \lambda_1^\circ\right) \times f^*(x_2; \nu_2, \lambda_2^\circ) \times f^*(x_3; \nu_3, \lambda_3^\circ)\, dx_2\, dx_3,$$

with

$$L_3 = 2d_3(t, T)C^*, \qquad L_3^* = 2(d_3(t, T) - 1)C^*,$$

$$\lambda_3^* = 2d_3(t, T)e^{-(\kappa_3+\lambda_3)(T-t)}y_{3t}, \qquad \lambda_3^\circ = \frac{2d_3^2(t, T)e^{-(\kappa_3+\lambda_3)(T-t)}y_{3t}}{(d_3(t, T) - 1)}.$$

The probability functions for the cap model are calculated as follows:

$$\chi^2(L_{1j}, L_{2j}, L_{3j}) = \int_0^{L_{3j}} \int_0^{(L_{2j}-(L_{2j}/L_{3j})x_3)} F^*\left(L_{1j} - \frac{L_{1j}}{L_{2j}}x_2 - \frac{L_{1j}}{L_{3j}}x_3; \nu_1, \lambda_{1j}^*\right) \times f^*(x_2; \nu_2, \lambda_{2j}^*)f^*(x_3; \nu_3, \lambda_{3j}^*)\, dx_2\, dx_3,$$

where

$$C^* = \ln[A_1A_2(1 + K)],$$

$$L_{1j} = \frac{2(\phi_{1j} + \psi_1)C^*}{B_1},$$

$$L_{2j} = \frac{2(\phi_{2j} + \psi_2)C^*}{B_2},$$

$$L_{3j} = 2d_3(t, T_j)C^*,$$

$$L_{1j}^* = \frac{2(\phi_{1j} + \psi_1 + B_1)C^*}{B_1},$$

$$L_{2j}^* = \frac{2(\phi_{2j} + \psi_2 + B_2)C^*}{B_2},$$

$$L_{3j}^* = 2(d_3(t, T_j) + 1)C^*,$$

$$\lambda_{3j}^* = 2d_3(t, T_j)e^{-(\kappa_3+\lambda_3)(T_j-t)}y_{3t},$$

$$\lambda_{3j}^\circ = \frac{2d_3^2(t, T_j)e^{-(\kappa_3+\lambda_3)(T_j-t)}y_{3t}}{d_3(t, T_j) + 1}.$$

The probability functions $\chi^2(L_{1j}^*, L_{2j}^*, L_{3j}^*)$ are calculated with λ_{ij}° in place of λ_{ij}^*. ϕ_{1j}, ϕ_{2j}, λ_{1j}^*, λ_{2j}^*, λ_{1j}°, and λ_{2j}° are computed with T_j in place of T for each put option.

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