CHAPTER 5

Factor Models

INTRODUCTION

The use of factor models in the financial industry has gained increased popularity recently. Much of this is due to the continued market turmoil and asset price uncertainty. Analysts are now turning to factor risk models to estimate asset returns and corresponding financial risk, rather than relying on their traditional approaches that all too often have proven unreliable.

The analyst's goal in using factor risk models is to determine a set of factors (e.g., explanatory variables) that will explain price movement. In these situations, analysts who are able to successfully forecast these factors or these factor returns will also be in a position to successfully forecast asset returns (the ultimate goal of financial management!). These factors models have also become increasingly popular for estimating asset volatility and (especially) covariance and correlation across asset returns. As we show ahead, the reason behind using factor risk models to estimate covariance and co-movement is that historical data alone is often unreliable and leads to incorrect estimates.

In this chapter, we focus our attention on the usage of factor risk models to estimate these covariance and correlation terms.

Data Limitations

Why do we need risk models at all when we can simply measure volatility, covariance, and correlation directly from the underlying asset returns data? Isn't it possible for analysts to use historical returns data and to determine co-movement across asset returns rather than implement a sophisticated risk modeling approach? The answer to these questions is simple and might even surprise some of the most seasoned practitioners. First, historical data is subject to a great deal of false relationships. For example, it is possible for two assets to move in the same direction (both assets increase in value or both assets decrease in value), but have a negative correlation measure, and it is possible for two assets to move in opposite directions (one asset increases in value and one asset decreases in value), but have a positive correlation measure. Second, in most situations we do not have enough historical data to determine a statistically significant correlation measure. In this case, there is a data limitation issue. Mathematicians refer to this as a degrees of freedom issue.

What effect can false relationships have in the process? Having incorrect false relationships in the data could have dire consequences on the portfolio management practices. Managers who uncover a false positive correlation relationship may implement a risk management approach to hold one asset long and short the other assets, believing that this will result in reduced risk. They believe that regardless of market movement, the gain on one asset will offset the loss on the other asset. For example, if the market increases the manager will lose on the asset they are short, but gain on the asset they are long, and thus have a net position that is unchanged (or at least a much smaller change). Similarly, if the market decreases, the manager will lose on the asset they are long, but earn a profit on the asset that they are short. Again, the net fund position will remain unchanged (or at least a much smaller change). Thus the portfolio has less potential for large price swings.

Managers who uncover a false negative correlation relationship between two assets may choose to hold both assets long in the portfolios. They would expect that regardless of market movement that loss that might be incurred in one asset will be offset by the gain in the other asset. A negative correlation is meant to imply that the stocks will move in opposite directions, but as we show ahead, this is not necessarily true. In both situations described previously, the manager could incur losses from both assets or gains from both assets, thus making their portfolio holdings much more risky then they may be led to believe.

Ahead we highlight two issues that may arise when relying on historical data for calculation of covariance and correlation across stocks using historical price returns. As explained in our examples, these issues can have dire consequences on the value of the portfolio. These are:

- False relationships
- Degrees of freedom

In the descriptions ahead we borrow from *The Science of Algorithmic Trading*, (Kissell, 2013), and we additionally provide further enhancement of these points through empirical examples and market observations.

False Relationships

It is possible for two stocks to move in same direction and have a negative calculated covariance and correlation measure, and it is possible for two stocks to move in the opposite direction and have a positive calculated covariance and correlation measure. Reliance on market data to compute covariance or correlation between stocks can result in false measures.

Following the mathematical definition of covariance and correlation we find that the covariance of price change between two stocks is really a measure of the co-movement of the "error terms" of each stock, not the co-movement of prices. For example, the statistical definition of covariance between two random variables x and y is:

$$\sigma_{xy} = E[(x-\bar{x})(y-\bar{y})]$$

It is quite possible for two stocks to have the same exact trend, but for their errors (noise term) to be on opposite sides of the trend lines. For example, if $\bar{x} = \bar{y} = z$, $x = d$, $y = -d$ and $d > z$, our covariance calculation is:

$$E[(x-\bar{x})(y-\bar{y})] = E[(d-z)(-d-z)] = E[-d^2+z^2]$$

Since $d > z$ we have $E[-d^2+z^2] < 0$ which is a negative measured covariance term indicating the stocks trend in opposite directions. But these two stocks move in exactly the same direction, namely, z.

It is also possible for two stocks to move in opposite directions, but have a positive covariance measure. For example, if $\bar{x} = z$ and $\bar{y} = -z$, $x = y = d$, and $d > z$, the covariance calculation is:

$$E[(x-\bar{x})(y-\bar{y})] = E[(d-z)(d--z)] = E[d^2-z^2]$$

Since $d > z$ we have $E[d^2-z^2] > 0$, which is a positive measured covariance term indicating the stocks trend in the same direction. But these two stocks move in the exact opposite direction.

The most important finding from the previous example is that when we compute covariance and correlation on a stock-by-stock basis using historical returns and price data, it is possible that the calculated measure is opposite what is happening in the market. These "false positive" and/or "false negative" relationships may be due to the error term about the trend rather than the trend, or possibly due to too few data points in our sample.

Example 5.1: False Negative Signal Calculations

Table 5.1 contains the data for two stocks A & B that are moving in the same direction. Figure 5.1a illustrates this movement over 24 periods. But when we calculate the covariance between these stocks, we get a negative correlation, rho = −0.71. How can stocks that move in the same direction have a negative covariance term? The answer is due to the excess terms being on opposite sides of the price trend (Figure 5.1b). Notice that these excess returns are now on opposite sides of the trend, which results in a negative covariance measure. The excess returns are indeed negatively correlated, but the direction of trend is positively correlated.

Example 5.2: False Positive Signal Calculation

Table 5.2 contains the data for two stocks C & D that are moving in opposite directions. Figure 5.2a illustrates this movement over 24 periods. But when

Table 5.1 False Negative Signals

Period	Market Prices		Period Returns		Excess Returns	
	A	B	A	B	A	B
0	10.00	20.00				
1	11.42	22.17	13.3%	10.3%	7.0%	5.3%
2	11.12	25.48	−2.6%	13.9%	−8.8%	8.9%
3	12.60	28.62	12.5%	11.6%	6.3%	6.6%
4	12.96	33.56	2.8%	15.9%	−3.4%	10.9%
5	16.91	30.59	26.6%	−9.3%	20.4%	−14.3%
6	17.63	33.58	4.2%	9.3%	−2.0%	4.3%
7	17.78	37.86	0.8%	12.0%	−5.4%	7.0%
8	19.93	38.93	11.4%	2.8%	5.2%	−2.2%
9	23.13	38.94	14.9%	0.0%	8.7%	−5.0%
10	24.21	39.64	4.6%	1.8%	−1.6%	−3.2%
11	23.39	46.32	−3.5%	15.6%	−9.7%	10.6%
12	23.92	49.59	2.3%	6.8%	−3.9%	1.8%
13	25.50	51.45	6.4%	3.7%	0.2%	−1.3%
14	23.97	56.96	−6.2%	10.2%	−12.4%	5.2%
15	27.35	56.60	13.2%	−0.6%	7.0%	−5.6%
16	31.27	57.37	13.4%	1.3%	7.2%	−3.7%
17	30.03	61.26	−4.0%	6.6%	−10.2%	1.6%
18	36.04	61.02	18.2%	−0.4%	12.0%	−5.4%
19	32.01	67.66	−11.9%	10.3%	−18.1%	5.3%
20	33.16	69.90	3.5%	3.3%	−2.7%	−1.7%
21	37.32	66.33	11.8%	−5.2%	5.6%	−10.2%
22	34.71	73.60	−7.3%	10.4%	−13.5%	5.4%
23	39.08	71.58	11.9%	−2.8%	5.7%	−7.8%
24	44.33	66.43	12.6%	−7.5%	6.4%	−12.5%
Avg:			6.2%	5.0%	0.0%	0.0%
Correl:			−0.71		−0.71	

Source: Science of Algorithmic Trading (Kissell, 2013)

we calculate the covariance between these stocks, we get a negative correlation, rho = +0.90. How can stocks that move in the same direction have a negative covariance term? The answer is due to the excess terms being on the same side of the price trend. Figure 5.2b illustrates the excess return in each time period. Notice that these excess returns are now on opposite sides of the trend, which results in a negative covariance measure. The excess returns are indeed positively correlated, but the direction of trend is negatively correlated.

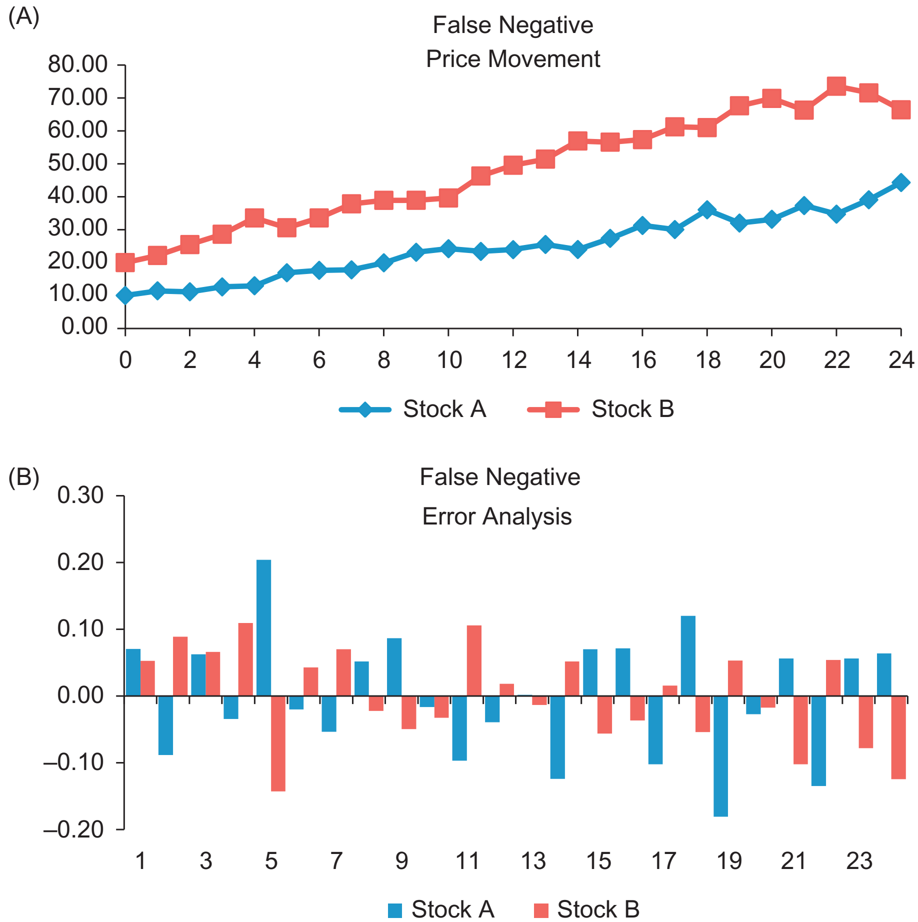

FIGURE 5.1 False Negative Signal

Empirical Evidence: False Signals

An interesting thing about financial theory is that many times in practice the empirical data does not conform to theory. Why is this so? There are many theories to explain why, and even these theories often break down. Many analysts' favorite saying to explain why things didn't happen the way they predicted is simply, "There was a regime shift." (I am still trying to figure out what that really means!). If regime shifts happen too often, it is more likely indicating issues with the risk-management practice or underlying data.

To determine if false positive relationships really occur in the market and how often they appear, we set out to test actual stock data. We compared computed covariance within a period to actual returns in the same period. It is important to note that we did not set out to test the accuracy of the covariance measure. We were much more conservative and only wanted to determine if the sign of the covariance measure was an indication of co-price movement. In other words, we

Table 5.2 False Positive Signals

Period	Market Prices		Period Returns		Excess Returns	
	C	D	C	D	C	D
0	60.00	50.00				
1	65.11	50.82	8.2%	1.6%	5.5%	5.1%
2	63.43	45.93	−2.6%	−10.1%	−5.3%	−6.6%
3	71.51	47.43	12.0%	3.2%	9.3%	6.7%
4	60.90	37.31	−16.1%	−24.0%	−18.7%	−20.5%
5	93.93	58.09	43.3%	44.3%	40.7%	47.8%
6	85.83	50.77	−9.0%	−13.5%	−11.7%	−10.0%
7	68.19	28.10	−23.0%	−59.2%	−25.7%	−55.7%
8	73.95	36.34	8.1%	25.7%	5.5%	29.2%
9	88.56	42.51	18.0%	15.7%	15.4%	19.2%
10	100.69	52.41	12.8%	20.9%	10.2%	24.4%
11	95.29	40.31	−5.5%	−26.3%	−8.2%	−22.8%
12	112.56	42.10	16.7%	4.3%	14.0%	7.8%
13	99.59	37.12	−12.2%	−12.6%	−14.9%	−9.1%
14	95.56	30.63	−4.1%	−19.2%	−6.8%	−15.7%
15	103.88	34.49	8.3%	11.9%	5.7%	15.4%
16	119.10	44.81	13.7%	26.2%	11.0%	29.7%
17	100.88	24.90	−16.6%	−58.7%	−19.3%	−55.3%
18	117.90	33.90	15.6%	30.9%	12.9%	34.3%
19	143.46	39.28	19.6%	14.7%	17.0%	18.2%
20	118.28	28.70	−19.3%	−31.4%	−22.0%	−27.9%
21	108.05	18.39	−9.0%	−44.5%	−11.7%	−41.0%
22	137.49	34.52	24.1%	63.0%	21.4%	66.5%
23	147.63	41.95	7.1%	19.5%	4.4%	23.0%
24	113.77	21.63	−26.1%	−66.2%	−28.7%	−62.7%
Avg:			2.7%	−3.5%	0.0%	0.0%
Correl:			0.90		0.90	

Source: Science of Algorithmic Trading (Kissell, 2013)

wanted to test whether or not, if the sign of the covariance measure was positive, the stock prices would move in the same direction (e.g., both stocks are up or both stocks are down), and, if the sign of the covariance measure was negative, if the stock prices would move in opposite directions (e.g., one stock increases and one stock decreases). The analysis performed is as follows.

Sample Universe

Our sample universe consisted of all stocks for which we had complete data over the full analysis period.

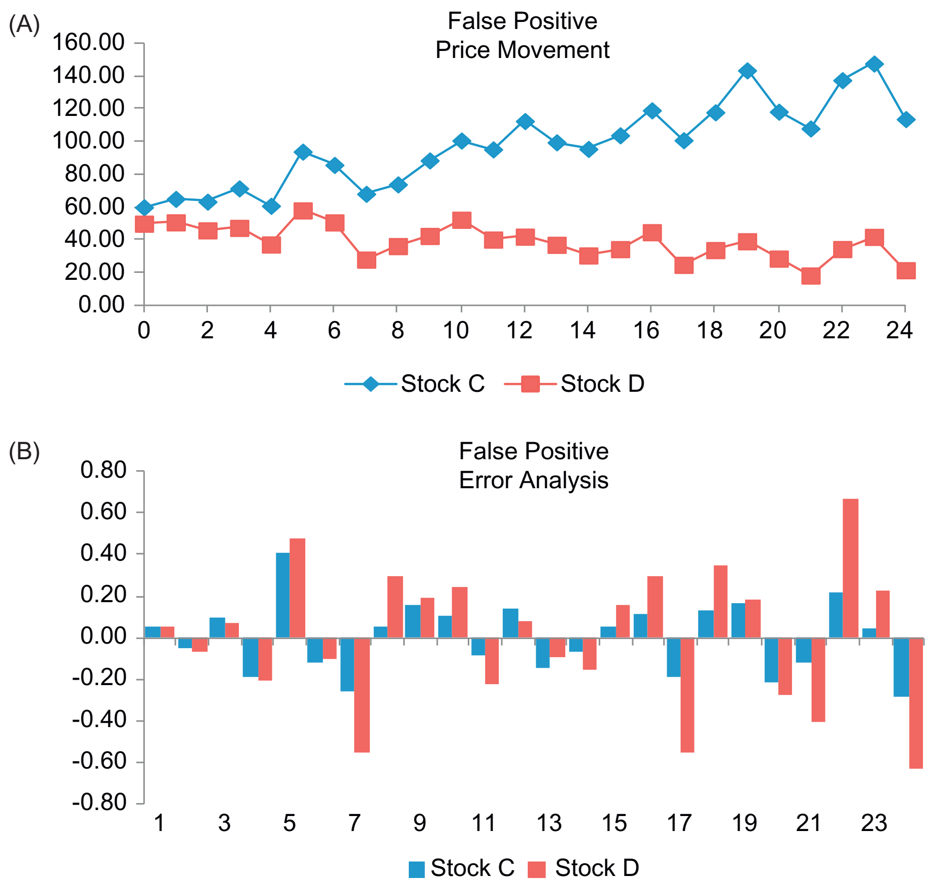

FIGURE 5.2 False Positive Signal

- SP500 Index (489 large-cap stocks)
- R2000 Index (1561 small-cap stocks

Date Period

Two years of data broken up over four six-month periods of data to determine stability of measures.

- January—June 2011
- July—December 2011
- January—June 2012
- July—December 2012

Calculation

- Covariance was computed using daily price change
- Return was computed as the total price movement over the period

Experiment

Our analysis consisted of computing the covariance across every pair of stock, and comparing whether the sign of the covariance metric was an indication of whether or not the stocks would move together in the same direction. Both metrics were computed from the same exact period and using the same exact data. For example, if the sign of the covariance between two stocks was positive, and the stocks both moved in the same direction, then the covariance metric provided a correct signal; but if the sign of the covariance measure was positive, and one stock was up and one stock was down, the covariance metric provided a false signal. The goal of the experiment was to determine how often we could get these negative signals.

Results

The results of our experiment were quite shocking. We found that using historical covariance computed from actual data would only provide a correct price movement signal for large-cap SP500 stocks 57% of the time (across all four periods). Thus, the historical covariance metric provided an incorrect signal 43% of time on average. And this is even using the same in-sample data! The results were very similar for small-cap R2000 stocks. The historical covariance metric only provided a correct price movement signal 54% of the time and provided an incorrect price movement signal 46% of the time! We leave it as an exercise for analysts to determine if these findings were specific for 2011−2012 and for our stock universe, or if they do indeed happen as often across different assets and time periods. We recommend analysts perform the same testing we performed previously for their specific financial instruments. Table 5.3 shows the actual price signal results for the SP500 and R2000 indexes for different periods of time.

Actual Stock Example

Figure 5.3 illustrates the previously experiment for two stocks ("A" and "ADM") over the period January 2011 through June 2011. The computed correlation of daily

Table 5.3 Actual Price Signal Results

Period	SP500 Correct	SP500 False	R2000 Correct	R2000 False
1/3/2011 – 6/30/2011	57%	43%	50%	50%
7/1/2011 – 12/30/2011	55%	45%	58%	42%
1/3/2012 – 6/29/2012	56%	44%	54%	46%
7/2/2012 – 6/30/2012	58%	42%	53%	47%
Avg	57%	43%	54%	46%

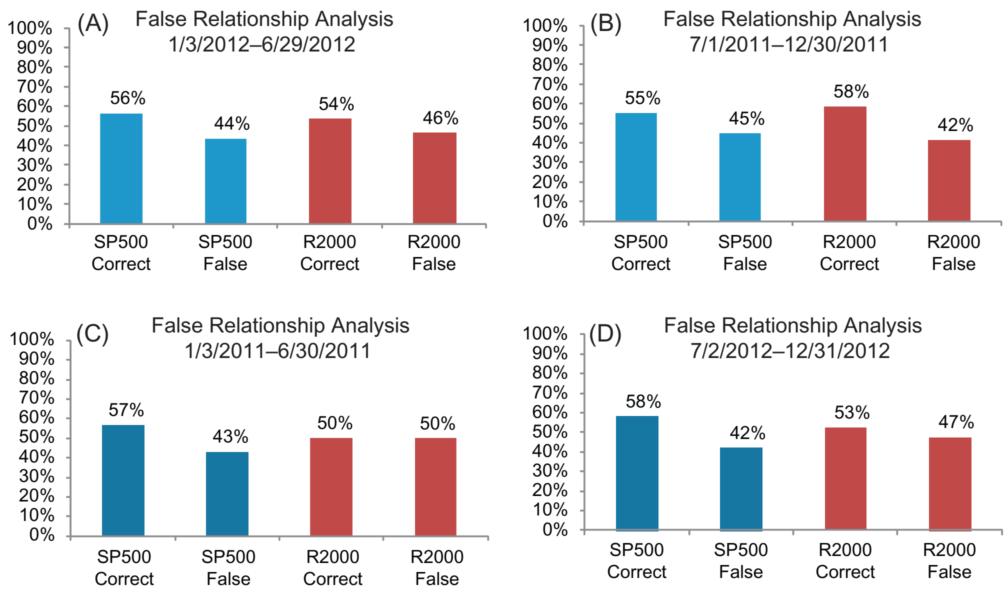

FIGURE 5.3 False Correlation Relationship Illustration for SP500 & R2000 Index

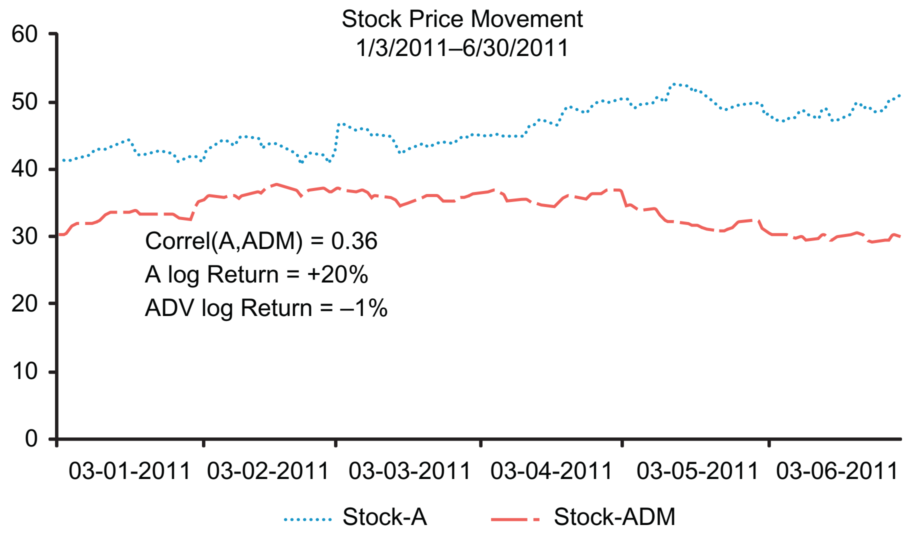

FIGURE 5.4 False Correlation Relationship for Stock A and Stock ADV (Jan 2011–Jun 2011)

log returns over this period was $\rho = 0.36$, which is often taken to be an indication that the stocks move up and down together. But the log return over the period for A was $+20\%$, and the log return for ADM over the period was -1%. Therefore, stock A and stock ADM moved in opposite directions over the period, but the calculated correlation was positive. Figure 5.4 illustrates price movement for each stock. Notice that these prices begin to deviate and move in opposite directions starting around April 2011, but the computed correlation is still positive.

Conclusion

Our conclusion from this experiment is that relying on historical data alone, albeit daily price movement over a short horizon, has a very high likelihood of providing incorrect price movement signals. The mathematical theory previously discussed is not a rare event or a corner-case situation. This does indeed happen quite often. It is very important to further point out here that traders and portfolio managers are quite interested in short-term daily price movement, whether it be for stocks, bonds, FX, etc. Therefore, even relying on simple math formulas to determine a relationship between stocks is quite difficult.

To correct for covariance and correlation, it is advised to compare stock price movement based on a common trend (such as the market index) or a multi-factor model. Factor models are discussed further ahead.

Degrees of Freedom

A portfolio's covariance matrix consists of stock variances along the diagonal terms and covariance terms on the off diagonals. The covariance matrix is a symmetric matrix since the covariance between stock A and stock B is identical to the covariance between stock B and stock A.

If a portfolio consists of n-stocks, the covariance matrix will be $n \times n$ with n^2 total elements. The number of unique variance terms in the matrix is equal to the number of stocks n. The number of covariance terms is equal to $(n^2 - n)$, and the number of unique covariance terms is $(n^2 - n)/2 = n \cdot (n-1)/2$.

The number of unique covariance parameters can also be determined from:

$$Unique\ Covariances = \binom{n}{2} = \frac{n(n-1)}{2} \tag{5.1}$$

The number of total unique elements "k" in the $n \times n$ covariance matrix is equal to the total number of variances plus the total number of unique covariances. This is:

$$k = n + \frac{n(n-1)}{2} = \frac{n \cdot (n+1)}{2} \tag{5.2}$$

In order to estimate these total parameters, we need a large enough set of data observations to ensure that the number of degrees of freedom is at least positive (as a starting point!). For example, consider a system of m-equations and k-variables. In order to determine a solution for each variable, we need to have m≥k or m-k≥0. If $m < k$, then the set of equations is underdetermined and no unique solution exists, meaning that we cannot solve the system of equations exactly.

The number of data points "d" that we have in our historical sample period of time is equal to $d = n^* t$ since we have one data point for each stock n. If there are t days in our historical period, we will have n*t data points. Therefore, we need to ensure that the total number of data points "d" is greater than or equal to the number of unique parameters "k" in order to be able to solve for all the parameters in our covariance matrix. This is:

$$d \geq k$$
$$n \cdot t \geq \frac{n \cdot (n+1)}{2}$$
$$t \geq \frac{(n+1)}{2}$$

Therefore, for a 500-stock portfolio there will be 125,250 unique parameters. Since there are 500 data points per day, we need just over one year of data (250 trading days per year) just to calculate each parameter in the covariance matrix.

However, now the determination of each entry in the covariance matrix is further amplified because we are not solving for a deterministic set of equations. We are seeking to estimate the value of each parameter. A general rule of thumb is that there need to be at least 20 observations for each parameter to have statistically meaningful results.

The number of data points required is then:

$$d \geq 20 \cdot k$$
$$n \cdot t \geq 20 \cdot \frac{n \cdot (n+1)}{2}$$
$$t \geq 10 \cdot (n+1)$$

Therefore, for a 500-stock portfolio (the size of the market index) we need 5010 days of observations, which is equivalent to over 20 years of data! Even if we require only 10 data points per parameter, this still results in over 10 years of data!

Figure 5.5a shows the number of unique elements in the covariance matrix for various numbers of assets. Figure 5.5b shows the number of days of data required for statistically significant estimation of covariance parameters based on different numbers of observations required per each parameter.

It has been suggested by some industry pundits that it is possible to estimate all unique parameters of the covariance matrix using the same number of observations as there are unique parameters. However, these pundits also state that in order for this methodology to be statistically correct, we need to compute the covariance terms across the entire universe of stocks and not just for a subset of stocks. But even if this is true, the relationship across companies in the methodology needs to be stable. The reasoning is that if we do use the entire universe of stocks with enough data points, we will uncover the true intrarelationship across all sub-groups of stocks and have accurate variance and covariance measures.

In the United States, there are over 7000 stocks and thus over 24.5 million parameters. This would require over 14 years of data history! We are pretty confident in the last 14 years that many companies have changed main lines of products (e.g., Apple) and changed their corporate strategy (e.g., IBM), and thus

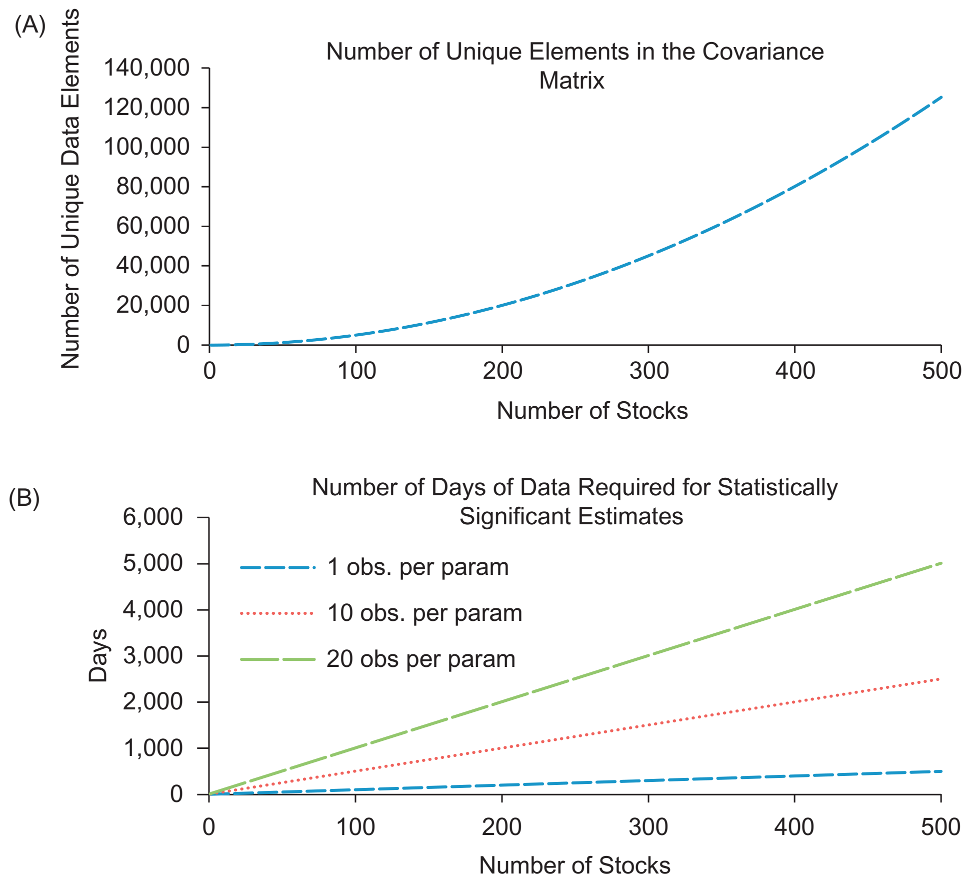

FIGURE 5.5 Data Requirements for Constructing a Statistically Significant Covariance Matrix

these relationships have changed. So even if we had enough data points, we know that companies do change, violating the requirements for this approach.

The last point to make is that for a global covariance matrix with a global universe of over 50,000 companies (at least 50,000!) there would be over 1.25 billion unique parameters and we would need a historical prices series of over 100 years! Think about how much has changed in just the last 10 years, let alone the last 100 years.

Mathematical Explanation

We can further illustrate this point mathematically using simple algebra. First, let us consider the following system of two equations and two unknowns (variables):

$$2x + 3y = 12$$
$$3x - 2y = 5$$

Here we can easily solve for our variables and we find $x = 3$ and $y = 2$. The reason we can solve for x and y is, first, that we have at least as many equations as variables, and, second, the columns are all independent. In matrix notation this is referred to as full rank. Now let us take a look at the following system of two equations with three unknowns:

$$x + y + z = 5$$
$$x - y + 2z = 6$$

Now the solution is not as simple; there is more than one solution. In fact, there are an infinite number of solutions for x, y, and z such that the system of equations previously shown will be correct.

In order to solve a system of equations, we need to have at least as many equations as variables. In finance, we need to have at least as many data points as we have parameters. The matrix needs to be full rank. Finance is no different in this respect than simple algebra.

Factor Models

Factor models address the two deficiencies we encountered when using historical market data to compute covariance and correlation. First, these models do not require the large quantity of historical observations that are needed for the sample covariance approach in order to provide accurate risk estimates. Second, factor models use a set of common explanatory factors across all stocks, and comparisons are made to these factors across all stocks. However, proper statistical analysis is still required to ensure accurate results. As with the previous sections, we follow the outline of risk models from *The Science of Algorithmic Trading and Portfolio Management* (Kissell, 2013) and enhance those explanations with empirical experiments and data analysis.

Factor models provide analysts with better insight into the overall covariance and correlation structure between stocks and across the market. Positive correlation means that the stocks will move in the same direction, and negative correlation means that stocks will move in opposite directions. A better way to word this is that factors models provide analysts with proper signals!

A factor model has the form:

$$r_t = \alpha_0 + f_{1t}b_1 + f_{2t}b_2 + \cdots + f_{kt}b_k + e_t \tag{5.3}$$

where,

r_t = stock return in period t
α_0 = constant term
f_{kt} = factor k value in period t
b_k = exposure of stock i to factor k—this is also referred to as beta, sensitivity, or factor loadings
e_t = noise for stock i in period t—this is the return not explained by the model

Parameters of the model are determined via ordinary least squares (OLS) regression analysis. Some analysts apply a weighting scheme so more recent observations have a higher weight in the regression analysis. These weighting schemes are often assigned using a smoothing function and "half-life" parameter. Various different weighting schemes for regression analysis can be found in Green (2000).

To perform a statistically correct regression analysis, the regression model is required to have the following properties. See Green (2000), Kennedy (1998), Mittelhammer (2000), etc.

Regression properties:

1. $E(e_t) = 0$
2. $Var(e_t) = E(e'e) = \sigma_e^2$
3. $Var(f_k) = E[(f_k - \overline{f}_k)^2] = \sigma_{fk}^2$
4. $E(ef_k) = 0$
5. $E(f_{kt}, f_{lt}) = 0$
6. $E(e_t e_{t-j}) = 0$
7. $E(e_{it} e_{jt}) = 0$

Property 1 states that the error term has a mean of zero. This will always be true for a regression model that includes a constant term $\hat{b}_{0k}$ or for a model using excess returns $E(r_{it}) = 0$. Property 2 states that the variance of the error term for each stock is σ_{ei}^2. Properties 1–2 are direct byproducts of a properly specified regression model. Property 3 states that the variance of each factor is σ_{fk}^2 and is true by definition. Property 4 states that the error term (residual) and each factor are independent. Analysts need to test to ensure this property is satisfied. Property 5 states that the explanatory factors are independent. Analysts need to properly select factors that are independent, or make adjustments to ensure that they are independent. If the factors are not truly independent, the sensitivities to these factors will be suspect. Property 6 states that the error terms are independent for all lagged time periods, e.g., no serial correlation or correlation of any lags across the error terms. Property 7 states that the error terms across all stocks are independent, e.g., the series of all error terms are uncorrelated. Since the error term in a factor model indicates company-specific returns or noise that is not due to any particular market force, these terms need to be independent across companies. If there are stocks with statistically significant correlated error terms, then it is likely that there is some market force or some other explanatory variable that is driving returns that we have not accounted for in the model. In this case, although the sensitivities to the selected variables may be correct, some of our risk calculations may be suspect because we have not fully identified all sources of risk. For example, company-specific risk, covariance and correlation, and portfolio risk may be suspect due to an incomplete model and may provide incorrect correlation calculations.

When constructing factor models, analysts need to test and ensure that all properties are satisfied.

Matrix Notation

In matrix notation our single stock factor model is:

$$r_i = \alpha_i + Fb_i + e_i \tag{5.4}$$

where,

$$r_i = \begin{bmatrix} r_{i1} \\ r_{i2} \\ \vdots \\ r_{in} \end{bmatrix}, \quad \alpha_i = \begin{bmatrix} \alpha_{i1} \\ \alpha_{i2} \\ \vdots \\ \alpha_{in} \end{bmatrix}, \quad F = \begin{bmatrix} f_{11} & f_{21} & \cdots & f_{k1} \\ f_{12} & f_{22} & \cdots & f_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ f_{1n} & f_{2n} & \cdots & f_{kn} \end{bmatrix}, \quad b_k = \begin{bmatrix} b_{i1} \\ b_{i2} \\ \vdots \\ b_{ik} \end{bmatrix}, \quad e_i = \begin{bmatrix} e_{i1} \\ e_{i2} \\ \vdots \\ e_{in} \end{bmatrix}$$

r_i = vector of stock returns for stock i
r_{it} = return of stock i in period t
α_i = vector of the constant terms
F = column matrix of factor returns
f_{jt} = factor j in period t
b_i = vector of risk exposures
b_{ij} = risk sensitivity of stock i to factor j
e_i = vector of errors (unexplained return)
e_{it} = error term of stock i in period t
n = total number of time periods
m = total number of stocks
k = total number of factors

Constructing Factor Independence

Real-world data often results in factors that are not independent, which violates regression property 5. This makes it extremely difficult to determine accurate risk exposures to these factors. In these situations, analysts can transform the set of dependent original factors into a new set of factors that are linearly independent (Kennedy, 1988).

This process is described as follows:

Let, F, G, and H represent three explanatory factors that are correlated.
First, sort the factors by explanatory power. Let F be the primary driver of risk and return, let G be the secondary driver, and let H be the tertiary driver.
Second, remove the correlation between F and G. This is accomplished by regressing the secondary factor G on the primary factor F as follows:

$$G = \tilde{\nu}_0 + \tilde{\nu}_1 F + e_G$$

The error term in this regression e_G is the residual factor G that is not explained by the regression model, and by definition (P4) is independent of F. Then let that $\tilde{G}$ is simply e_G from the regression. That is:

$$\tilde{G} = G - \tilde{\nu}_0 - \tilde{\nu}_1 F$$

Third, remove the correlation between factor H and factor F and the new secondary factor $\tilde{G}$. This is accomplished by regressing H on F and $\tilde{G}$ as follows:

$$H = \hat{\gamma}_0 + \hat{\gamma}_1 F + \hat{\gamma}_2 \tilde{G} + e_H$$

The error term in this regression e_H is the residual factor H that is not explained by the regression model, and by definition (P4) is independent of F and G. This process can be repeated for as many factors as is present.

The factor model with uncorrelated factors is finally rewritten as:

$$r = \alpha_o + Fb_f + \tilde{G}b_{\tilde{g}} + \tilde{H}b_{\tilde{h}} + \varepsilon \tag{5.5}$$

This representation now provides analysts with a methodology to calculate accurate risk exposures to a group of pre-defined factors which are now independent.

Estimating Covariance Using a Factor Model

A factor model across a universe of stocks can be written as:

$$R = \alpha + F\beta + \varepsilon \tag{5.6}$$

where

$$R = \begin{bmatrix} r_{11} & r_{21} & \cdots & r_{m1} \\ r_{12} & r_{22} & \cdots & r_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ r_{1n} & r_{2n} & \cdots & r_{mn} \end{bmatrix} \quad F = \begin{bmatrix} f_{11} & f_{21} & \cdots & f_{k1} \\ f_{12} & f_{22} & \cdots & f_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ f_{1n} & f_{2n} & \cdots & f_{kn} \end{bmatrix}, \quad \alpha' = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$\beta = \begin{bmatrix} b_{11} & b_{21} & \cdots & b_{m1} \\ b_{12} & b_{22} & \cdots & b_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1k} & b_{2k} & \cdots & b_{mk} \end{bmatrix} \quad \varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{21} & \cdots & \varepsilon_{m1} \\ \varepsilon_{12} & \varepsilon_{22} & \cdots & \varepsilon_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{1n} & \varepsilon_{2n} & \cdots & \varepsilon_{mn} \end{bmatrix}$$

This formulation allows us to compute the covariance across all stock without the issues that come up when using historical market data. This process is described following Elton & Gruber (1995) as follows:

The covariance matrix of returns C is calculated as:

$$C = E[(R - E[R])'(R - E[R])]$$

From our factor model relationship we have

$$R = \alpha + F\beta + \varepsilon$$

The expected value of returns is:

$$E[R] = \alpha + \overline{F}\beta$$

Now we can determine the excess returns as:

$$R - E[R] = (F - \overline{F})\beta + \varepsilon$$

Now substituting in the previous result, we have:

$$\begin{aligned} C &= E[((F-\overline{F})^2\beta + \varepsilon)'((F - \overline{F})\beta + \varepsilon)] \\ &= E[\beta'(F-\overline{F})^2\beta + 2\beta'(F - \overline{F})\varepsilon + \varepsilon'\varepsilon] \\ &= \beta' E[(F-\overline{F})^2]\beta + 2\beta' E[(F - \overline{F})\varepsilon] + E[\varepsilon'\varepsilon] \end{aligned}$$

By property 4:

$$E[2\beta'(F - \overline{F})\varepsilon] = 0$$

By property 2 and property 7 we have

$$E[\varepsilon'\varepsilon] = \begin{bmatrix} \sigma_{e1}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{e2}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{en}^2 \end{bmatrix} = \Lambda$$

which is the idiosyncratic variance matrix and is a diagonal matrix consisting of the variance of the regression term for each stock.

By property 3 and property 5, the factor covariance matrix is:

$$E[(F - \overline{F})^2] = \begin{bmatrix} \sigma_{f1}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{f2}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{fk}^2 \end{bmatrix} = \mathrm{cov}(F)$$

The factor covariance matrix will be a diagonal matrix of factor variances. In certain situations there may be some correlation across factors. When this occurs, the off-diagonal entries will be the covariance between the factors. Additionally, the beta sensitivities may be suspect, meaning that we may not know the true exposures to each factor, and we will have some difficulty determining how much that particular factor contributes to returns. However, the covariance calculation will be correct, providing that we include the true factor covariance matrix.

Finally, we have our covariance matrix derived from the factor model:

$$C = \beta' \mathrm{cov}(F)\beta + \Lambda \tag{5.7}$$

This matrix can be decomposed into the systematic and idiosyncratic components. Systematic risk component refers to the risk and returns that is explained by the factors. It is also commonly called market risk or factor risk. The idiosyncratic risk component refers to the risk and returns that are not explained by the factors. This component is also commonly called stock-specific risk, company-specific risk, or diversifiable risk. This is shown as:

$$C = \underbrace{\beta' \operatorname{cov}[F]\beta}_{\text{Systematic Risk}} + \underbrace{\Lambda}_{\text{Idiosyncratic Risk}} \tag{5.8}$$

Types of Factor Models

Factor models can be divided into four categories of models: index models, macroeconomic models, cross-sectional or fundamental data models, and statistical factor models. These are described ahead.

Index Model

There are two forms of the index model commonly used in the industry: single-index and multi-index model. The single-index model is based on a single major market index such as the SP500. The same index is used as the input factor across all stocks. The multi-index model commonly incorporates the general market index, the stock's sector index, and additionally, the stock's industry index. The market index will be the same for all stocks, but the sector index and industry index will be different based on the company's economic grouping. All stocks in the same sector will use the same sector index, and all stocks in the same industry will use the same industry index.

Single-Index Model

The simplest of all the multi-factor models is the single-index model. This model formulates a relationship between stock returns and market movement. In most situations, the SP500 index or some other broad market index is used as a proxy for the whole market.

In matrix notation, the single factor model has the general form:

$$r_i = \alpha_i + \hat{b}_i R_m + e_i \tag{5.9}$$

r_i = column vector of stock returns for stock i
R_m = column vector of market returns
e_i = column vector of random noise for stock i
$\hat{b}_i$ = stock return sensitivity to market returns

In the single-index model, we need to estimate the risk exposure $\hat{b}_i$ to the general index R_m. In situations where the index used in the single-index model is the broad market index and the constant term is the risk-free rate, the single-index model is known as the CAPM model (Sharpe, 1964), and the risk exposure $\hat{b}_i$ is the stock beta β.

Empirical Analysis

We analyzed daily stock returns as a function of the SP500 index returns for 2011 and 2012. We computed daily betas for stocks in the SP500 index and stocks in the R2000 index.

For SP500 stocks the average beta in 2011 was 1.12 (± 0.37), and in 2012 it was 1.10 (± 0.43). This is shown in Table 5.4, and Figure 5.6a shows the daily beta across stocks and over time. The question that naturally arises in these analyses is, why isn't the average beta equal to $B = 1$? There are a few reasons why. First, we are using an equal-weighted average of beta over time, rather than a market-cap-weight average as is used to compute the SP500 index returns. Larger stocks typically have betas closer to one. Second, our sample universe consists of stocks that were in the SP500 at the end of 2012 and for which we had complete data over the full period 1/2011 through 12/2012. Finally, and most importantly, the stocks in our sample were the stocks that were always in the index or were added to the index some time during the two year window. Stocks that were deleted from the index are not included in the index. Very often, stocks that are underperforming the index are removed from the index, while stocks that are outperforming the index are added into the index. This results in an equal-weighted beta greater than one.

We next investigated the predictability of beta from one year to the next. Quite often we see in publications that stock betas tend to revert to one over time. If this were true, we should see stocks with a beta greater than one decreasing and stocks with beta less than one increasing. Our regression analysis, however, did not find evidence of this relationship with large-cap stocks. The slope of the

Table 5.4 SP500 Index Stocks

Daily Beta Calculation Statistics		
	2011	**2012**
Avg:	1.12	1.10
Stdev:	0.37	0.43

2012 Beta as function of 2011 Beta				
	Value	**Std Err**	**t-stat**	**t-stat2**
Slope	1.04	0.03	40.14	1.57
Intercept	−0.06	0.03	−2.12	
R2	0.77			
SE	0.21			

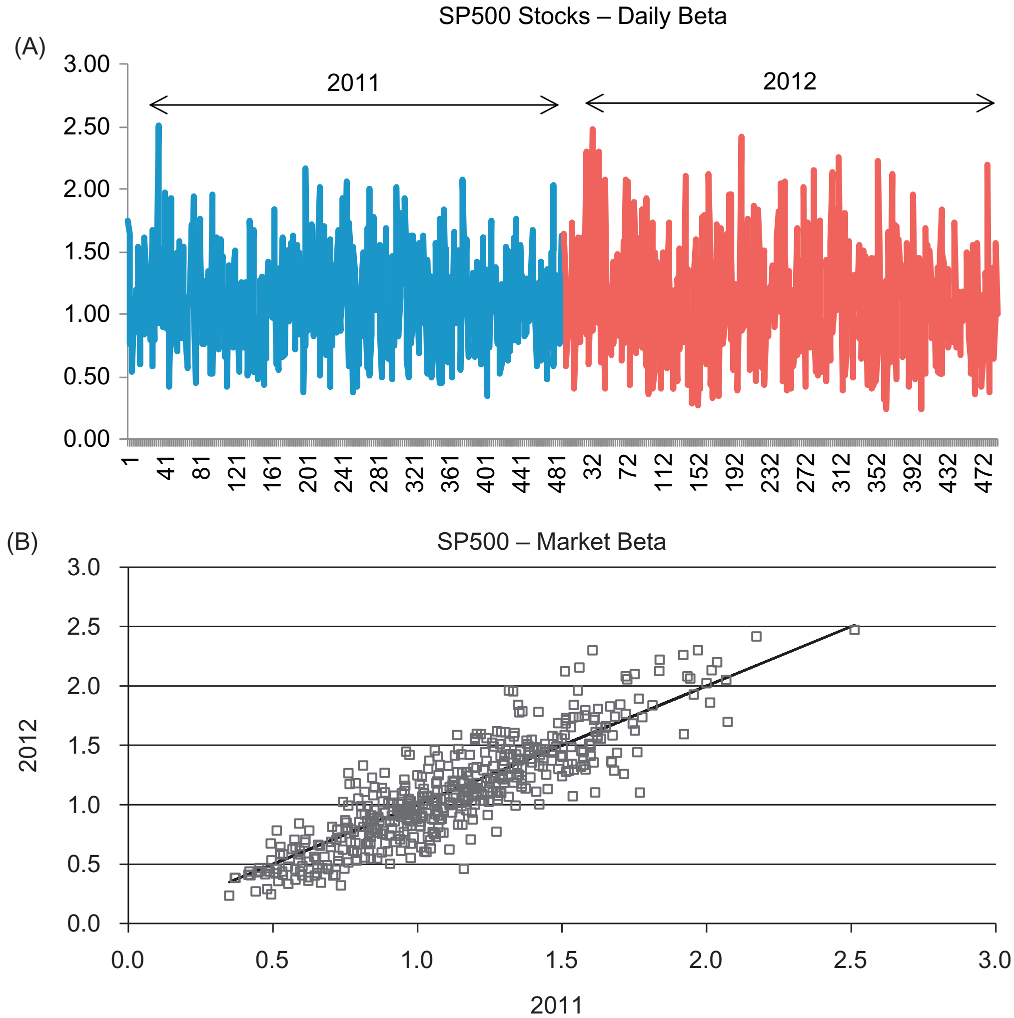

FIGURE 5.6 Evaluation of Market Beta for SP500 Index Stocks

regression line was 1.04, but this value was not significantly different from 1.0. Therefore, we conclude that large-cap stocks do not exhibit a mean reversion pattern (at least during 2011–2012). It could possibly be because these stocks are large-cap mature companies already. Figure 5.6b shows an x-y plot of 2012 beta as a function of 2011 beta.

For R2000 stocks, the average beta in 2011 was 1.40 ($\pm$0.38), and in 2012 it was 1.26($\pm$0.46). Since small-cap stocks for the most part are more risky and have higher beta, it is interesting that these stocks had beta that moved closer to one in the second year (Table 5.5). There appears to be some evidence that beta reversion did occur for the small-cap stocks. Figure 5.7a plots small-cap stock betas over the period 1/2011 through 12/2012. Visual inspection of the data indicates that beta may be mean reverting. To test this hypothesis, we ran a regression of 2012 beta as a function of 2011 beta and found

Table 5.5 R2000 Index Stocks

Daily Beta Calculation Statistics		
	2011	**2012**
Avg:	1.40	1.26
Stdev:	0.38	0.46

R2000 Stocks: 2012 beta as a function of 2011 beta				
	Value	**Std Err**	**t-stat**	**t-stat2**
Slope	0.85	0.02	39.15	– 6.98
Intercept	0.07	0.03	2.14	
R2	0.50			
SE	0.33			

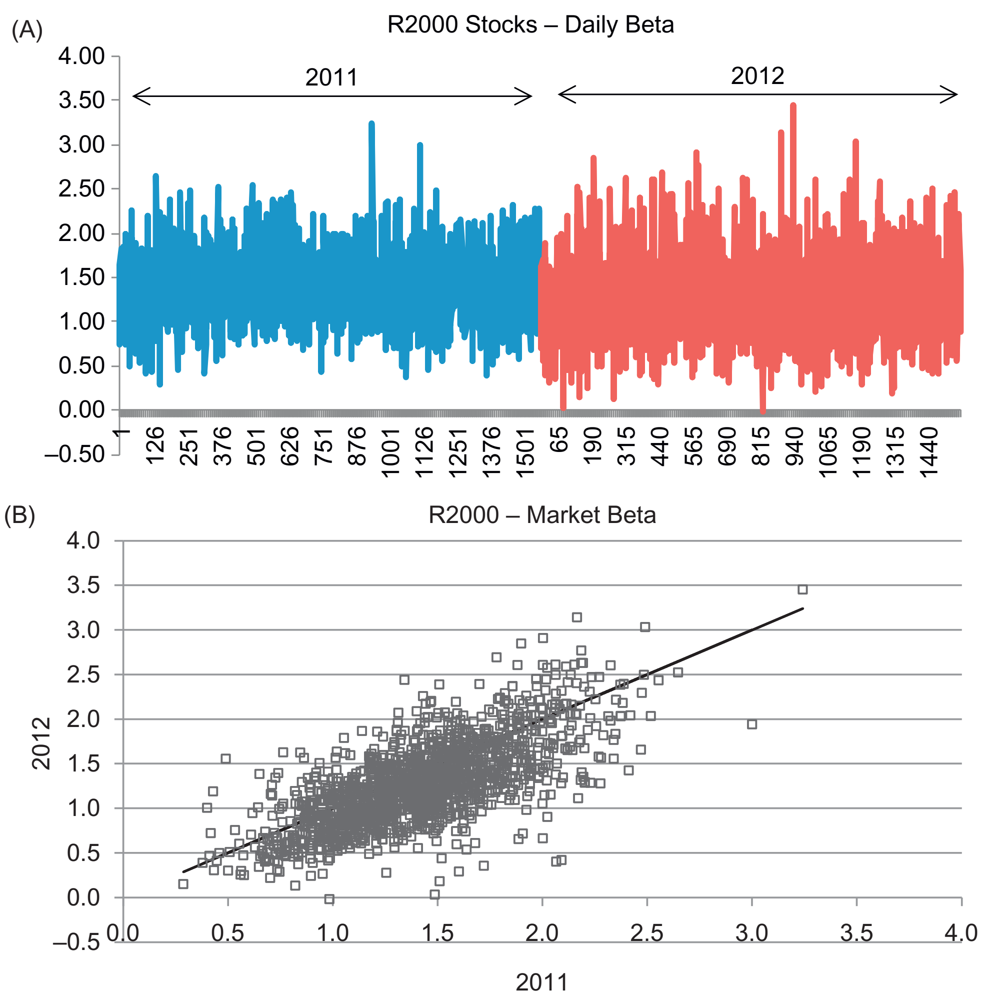

FIGURE 5.7 Evaluation of Market Beta for R2000 Index Stocks

statistical evidence (Table 5.5). The slope of this regression line was m = 0.85. Statistical tests show that this value is different from one. Hence, since small-cap stocks do tend to have beta > 1, there is statistical evidence suggesting beta mean reversion (at least for small-cap companies). Figure 5.7b is an x-y plot of 2012 beta as a function of 2011 beta. Notice that there is a higher concentration of data points below the y = x lines for stocks with a higher beta (e.g., beta >1).

Multi-Index Models

The multi-index factor model is an extension of the single-index model that captures additional relationships between price returns and corresponding sectors and industries. There have been numerous studies showing that the excess returns (error) from the single-index model are correlated across stocks in the same sector, and with further incremental correlation across stocks in the same industry (see Elton and Gruber, 1995).

Let R_m = market returns, S_k = the stock's sector returns, and I_k = the stock's industry return. Then the linear relationship is:

$$r_i = \alpha_i + b_{im}R_m + b_{ik}S_k + b_{il}I_i + e_i$$

where b_{im} is the stock's sensitivity to the general market movement, b_{ik} is the stock's sensitivity to its sector movement, and b_{il} is the stock's sensitivity to its industry movement.

There is a large degree of correlation, however, across the general market, sectors, and industry. These factors are not independent, and analysts need to make appropriate adjustment following the process outlined previously.

The general multi-index model after multicollinearity now has the form:

$$r_i = \alpha_i + \hat{b}_{im}R_m + \hat{b}^*_{isk}\tilde{S}_k + \hat{b}^*_{iIl}\tilde{I}_l + \varepsilon \tag{5.10}$$

Empirical Analysis

We analyzed daily stock returns and market risk using a variation of the multi-index model; but instead of using sector or industry returns, we incorporated a small-cap stock index (R2000 index). We incorporated the small-cap stock index to determine if there were any market forces specific to the small-cap universe that are driving stock returns.

The first step in the analysis is to construct small-cap index (R2000) returns that are uncorrelated with the main large-cap index (SP500) following procedures shown previously. The second step is to estimate daily stock returns based on these two indexes. The regression model we used is:

$$r_k = \hat{\alpha}_k + \hat{\beta}_{k1}R_m + \hat{\beta}_{k2}\tilde{I}_m + \varepsilon_k$$

where $\tilde{I}_m$ is the vector of small-cap index returns that are uncorrelated with SP500 index returns. The data period covered 2011–2012.

Following this model, risk can be decomposed into large-cap market risk, small-cap market risk, and idiosyncratic risk using the variance risk metric. These are:

$$LC\ market\ risk = \hat{\beta}_{k1}^2 \sigma_{Rm}^2$$

$$SC\ market\ risk = \hat{\beta}_{k2}^2 \sigma_{Im}^2$$

$$idiosyncratic = \sigma_{\varepsilon}^2$$

where total risk is:

$$\sigma_{rk}^2 = \hat{\beta}_{k1}^2 \sigma_{Rm}^2 + \hat{\beta}_{k2}^2 \sigma_{Im}^2 + \sigma_{\varepsilon}^2$$

The percentage of risk from each of these factors is simply the variance corresponding to those factors divided by the total variance for the stock. This is as follows:

$$LC\ market\ risk\ Pct = \frac{\hat{\beta}_{k1}^2 \sigma_{Rm}^2}{\sigma_{rk}^2}$$

$$SC\ market\ risk\ Pct = \frac{\hat{\beta}_{k2}^2 \sigma_{Im}^2}{\sigma_{rk}^2}$$

$$idiosyncratic = \frac{\sigma_{\varepsilon}^2}{\sigma_{rk}^2}$$

For our large-cap stock universe, we find that the market index represents 49% of total stock risk. The small-cap index only accounts for 1% of risk. Thus there is some small capitalization occurring in the market. Idiosyncratic market risk accounts for 50% of daily stock risk. Figure 5.8a shows risk decomposition across all of our stocks in the universe and Figure 5.8b shows the average quantity of risk corresponding to the indexes.

For our small-cap stock universe, we find that the market index represents 37% of total stock risk, which is much lower than observed for large-cap stocks. The small-cap index accounted for 5% of risk. While this is still small, it is still a much higher quantity than for large-cap stocks. Thus, further evidence of a small capitalization phenomenon effecting prices. Idiosyncratic market risk accounts for 59% of daily stock risk. Figure 5.9a shows risk decomposition across all of our stocks in the universe, and Figure 5.9b shows the average quantity of risk corresponding to the indexes.

The risk decomposition on a daily basis will often associate a higher percentage to idiosyncratic (company specific) risk. This is due primarily to actual company-specific risk, but it is also due to buying and selling pressure in the market. Over time, buying and selling pressure tends to net out.

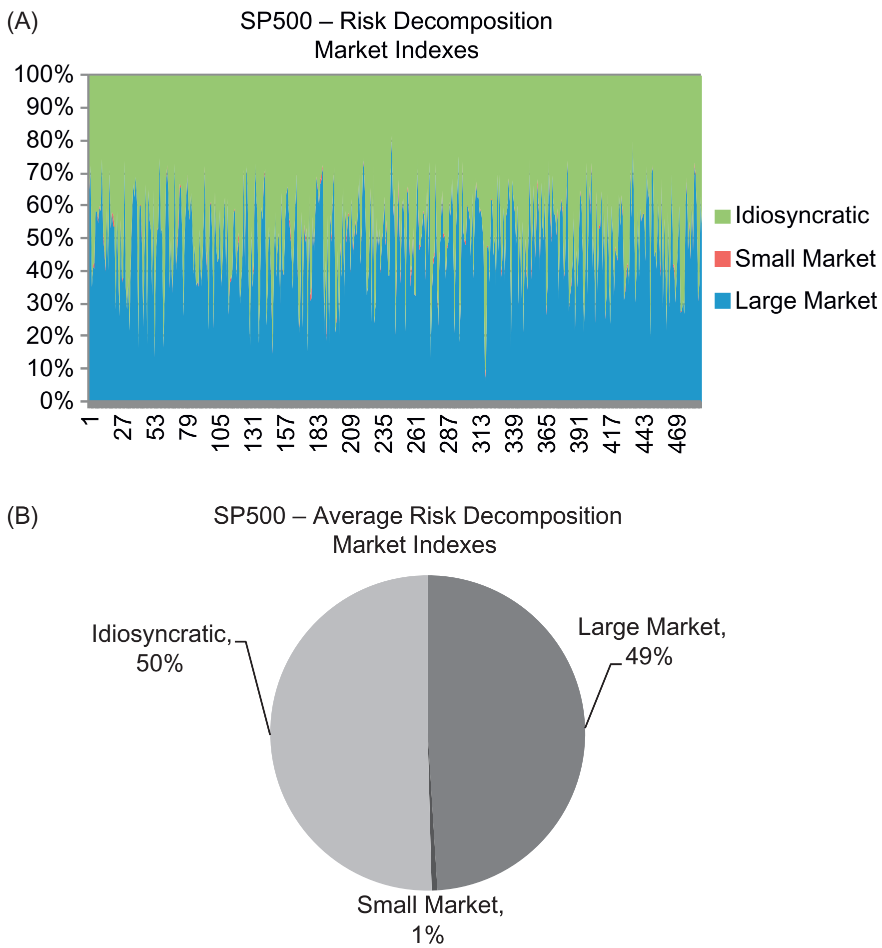

FIGURE 5.8 Risk Decomposition for SP500 Index Stocks Using CAPM

Macroeconomic Factor Models

A macroeconomic multifactor model defines a relationship between stock returns and a set of macroeconomic variables such as GDP, inflation, industrial production, bond yields, etc. The appeal of using macroeconomic data as the explanatory factors in the returns model is that these variables are readily measurable and have real economic meaning.

While macroeconomic models offer key insight into the general state of the economy, they may not sufficiently capture the most accurate correlation structure of price movement across stocks. Additionally, macroeconomic models may not do a good job capturing the covariance of price movement across stocks in "new economies" or a "shifting regime," such as the sudden arrival of the financial crisis beginning in September 2008.

Chen, Roll, and Ross (1986) identified the following four macroeconomic factors as having significant explanatory power with stock return. These strong relationships still hold today and are:

- Unanticipated changes in inflation

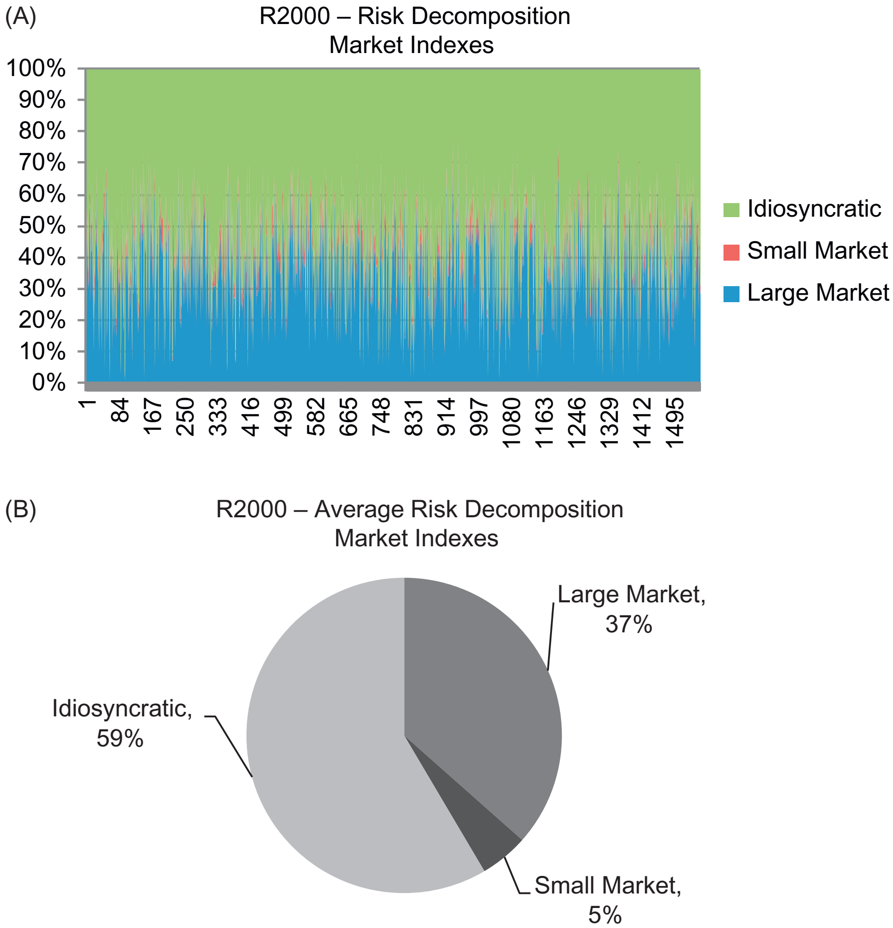

FIGURE 5.9 Risk Decomposition for R2000 Index Stocks Using CAPM

- Unanticipated changes in industrial production
- Unanticipated changes in the yield between high-grade and low-grade corporate bonds
- Unanticipated changes in the yield between long-term government bonds and t-bills; this is the slope of the term structure

Other macroeconomic factors have also been incorporated into these models, including change in interest rates, growth rates, GDP, capital investment, unemployment, oil prices, housing starts, exchange rates, etc. The parameters are determined via regression analysis using monthly data over a five-year period, e.g., 60 observations.

It is often assumed that the macroeconomic factors used in the model are uncorrelated, and analysts do not make any adjustment for correlation across returns. But improvements can be made to the model following the adjustment process described previously.

A k-factor macroeconomic model has the form:

$$r_i = \alpha_{i0} + \hat{b}_{i1}f_1 + \hat{b}_{i2}f_2 + \cdots + \hat{b}_{ik}f_k + e_i \tag{5.11}$$

Analysts need to estimate the risk exposures $b'_{ik}s$ to these macroeconomic factors.

Cross-Sectional Multi-Factor Model

Cross-sectional models estimate stock returns from a set of variables that are specific to each company, rather than through factors that are common across all stocks. Cross-sectional models use stock-specific factors that are based on fundamental and technical data. The fundamental data consists of company characteristics and balance sheet information. The technical data (also called market-driven data) consists of trading activity metrics such as average daily trading volume, price momentum, size, etc.

Because of the reliance on fundamental data, many authors use the term "fundamental model" instead of cross-sectional model. The rationale behind the cross-sectional models is similar to the rationale behind the macroeconomic model. Since managers and decision makers incorporate fundamental and technical analysis into their stock selection process, it is only reasonable that these factors provide insight into the return and risk of those stocks. Otherwise, why would they be used?

Fama and French (1992) found that three factors consisting of 1) market returns, 2) company size (market capitalization), and 3) book-to-market ratio have considerable explanatory power. While the exact measure of these variables remains a topic of much discussion in academia, notice that the last two factors in the Fama-French model are company-specific fundamental data.

While many may find it intuitive to incorporate cross-sectional data into multi-factor models, these models have some limitations. First, data requirements are cumbersome, requiring analysts to develop models using company-specific data (each company has its own set of factors). Second, it is often difficult to find a consistent set of robust factors across stocks that provide strong explanatory power. Ross and Roll had difficulty determining a set of factors that provided more explanatory power than the macroeconomic models without introducing excessive multicollinearity into the data.

The cross-sectional model is derived from company-specific variables referred to as company factor loadings. The parameters are typically determined via regression analysis using monthly data over a longer period of time, e.g., a five-year period, with 60 monthly observations.

The cross-sectional model is written as:

$$r_{it} = x^*_{i1}\hat{f}_{1t} + x^*_{i2}\hat{f}_{2t} + \cdots + x^*_{ik}\hat{f}_{kt} + e_{it} \tag{5.12}$$

where x^*_{ij} is the normalized factor loading of company i to factor j. For example,

$$x^*_{kl} = \frac{x_{kl} - E(x_k)}{\sigma(x_k)}$$

where $E(x_k)$ is the mean of x_k across all stocks, and $\sigma(x_k)$ is the standard deviation of x_k across all stocks.

And unlike the previous models in which the factors were known in advance, and we estimated the risk sensitivities, here we know the factor loadings (from company data) and we need to estimate the factors.

Empirical Evidence

We analyzed daily stock risk for our large-cap and small-cap indexes using the Fama-French factors[1]. We revised these factors so that each is independent, following approaches shown previously. The correlation matrix of factor is shown in Table 5.6.

The factors of this model are Mkt-Rf (market returns minus risk-free rate), SBM (small minus big), and HML (high book-to-market minus low book-to-market).

Surprisingly, the results of our risk decomposition were similar to our single-index and multi-index model approach. Figure 5.10a shows the percentage of risk corresponding to each of the Fama-French factors for large-cap stocks. Figure 5.10b shows the average values across all stocks for the large-cap universe. For large cap, the market is driving about 49% consistent with our multi-index model. HML is explaining 1% of risk, and we did not find SMB associated with any market risk. Therefore, idiosyncratic risk is 49%. Notice that this is slightly lower than for the multi-index model in which 50% of the risk was denoted as company specific.

Figure 5.11a shows the percentage of risk corresponding to each of the Fama-French factors for small-cap stocks. Figure 5.11b shows the average values across all stocks for the small-cap universe. For small-cap stocks, approximately 38% corresponds to the market (consistent with the multi-index model), HML corresponds to 1% of risk, SMB corresponds to 4% of market risk, and 58% of the risk is company specific.

Statistical Factor Models

Statistical factor models are also referred to as implicit factor models and principal component analysis (PCA). In these models, neither the explanatory factors

Table 5.6 Correlation across Fama-French Factors

	Mkt-Rf	SMB	HML
Mkt-Rf	1.000	− 0.001	− 0.003
SMB	− 0.001	1.000	− 0.001
HML	− 0.003	− 0.001	1.000

[1]Source: http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

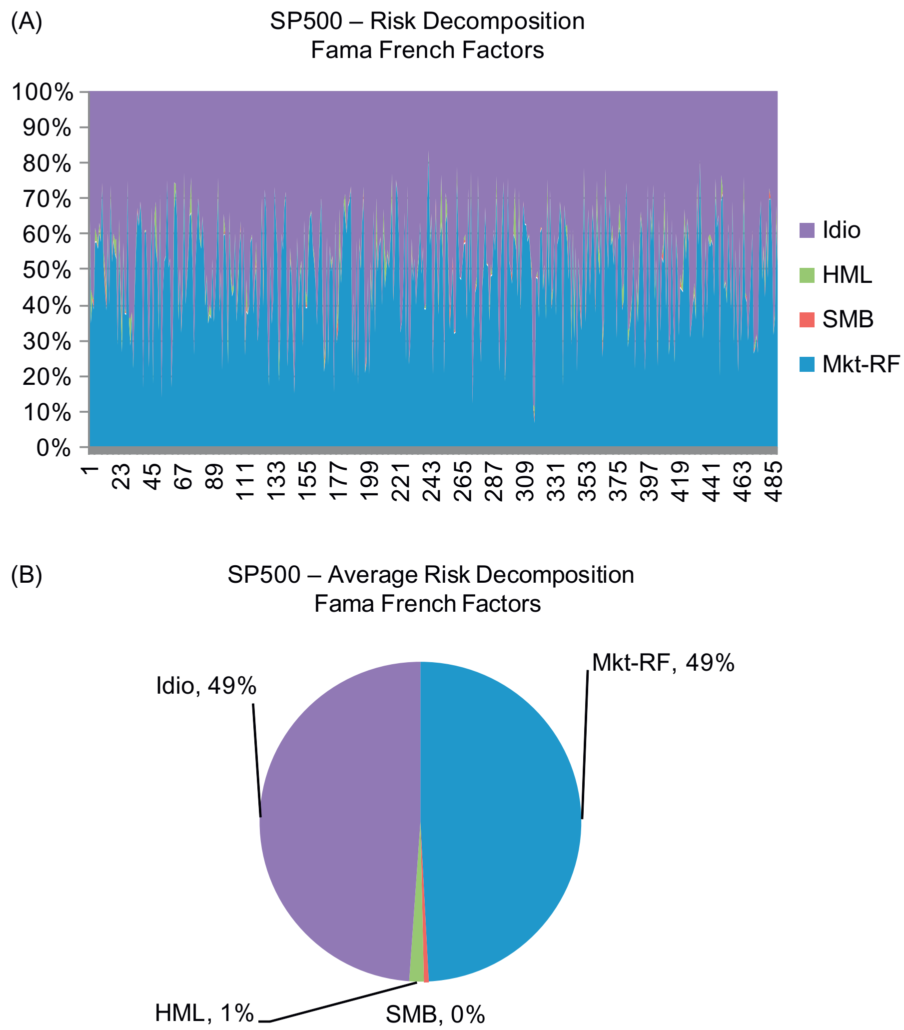

FIGURE 5.10 Risk Decomposition for SP500 Index Stocks Using Fama-French Factors

nor sensitivities to these factors are known in advance, and they are not readily observed in the market. However, both the statistical factors and sensitivities can be derived from historical data.

There are three common techniques used in statistical factor models: eigenvalue-eigenvector decomposition, singular value decomposition, and factor analysis. Eigenvalue-eigenvector is based on a factoring scheme of the sample covariance matrix, and singular value decomposition is based on a factoring scheme of the returns matrix of returns (see Pearson, 2002). Factor analysis (not to be confused with factor models) is based on a maximum likelihood estimate of the correlations across stocks. In this section we discuss the eigenvalue-eigenvector decomposition technique.

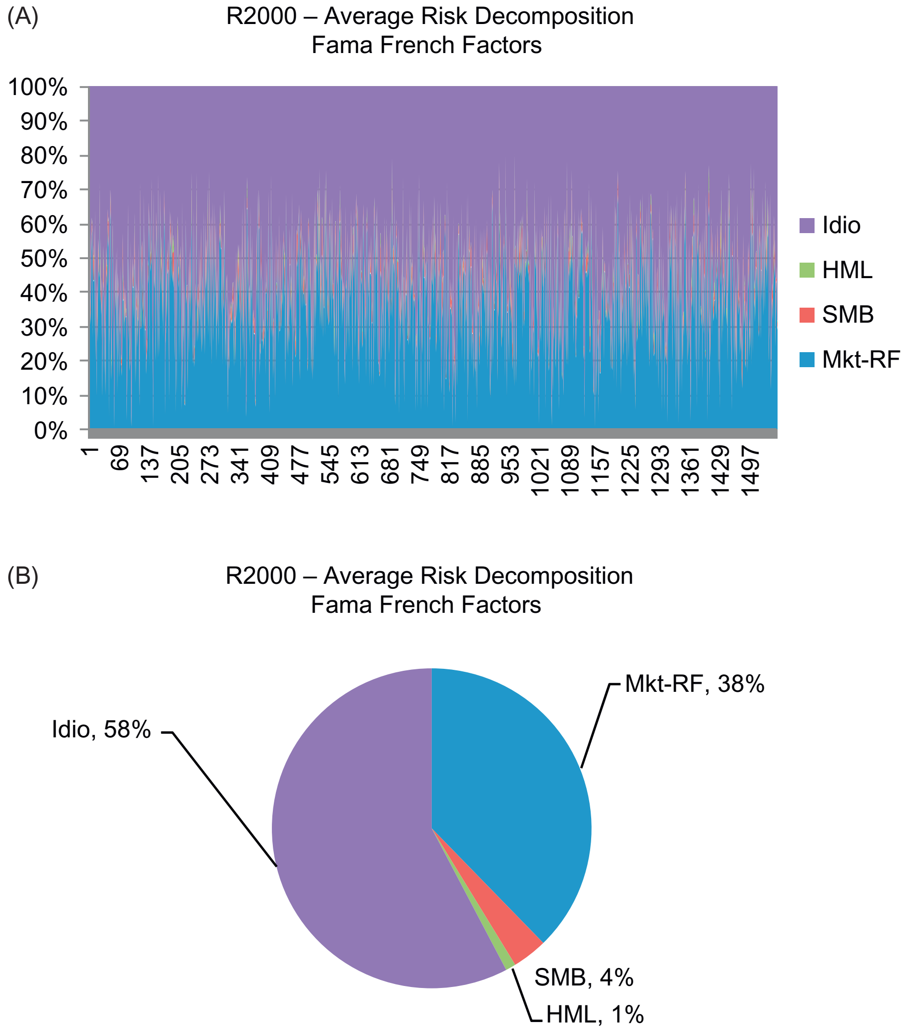

FIGURE 5.11 Risk Decomposition for R2000 Index Stocks Using Fama-French Factors

The statistical factor models differs from the previously mentioned models in that analysts estimate both the factors (F_ks) and the sensitivities to the factors (b_{ik}s) from a series of historical returns. This model does not make any prior assumptions regarding the appropriate set of explanatory factors or force any preconceived relationship into the model.

This approach is in contrast to the explicit modeling approaches in which analysts must specify either a set of explanatory factors or a set of company-specific factor loadings. In the explicit approaches, analysts begin with either a set of specified factors and estimate sensitivities to those factors (i.e., index models and macroeconomic factor model) or begin with the factor loadings (fundamental data) and estimate the set of explanatory factors (cross-sectional model).

The advantage of statistical factor models over the previously described explicit approaches is that it provides risk managers with a process to uncover accurate covariance and correlation relationships of returns without making any assumptions regarding what is driving the returns. Any preconceived bias is removed from the model. The disadvantage of these statistical approaches is that it does not provide portfolio managers with a set of factors to easily determine what is driving returns since the statistical factors do not have any real-world meaning.

To the extent that analysts are only interested in uncovering covariance and correlation relationships for risk management purposes, PCA has proven to be a viable alternative to the traditional explicit modeling approaches. Additionally, with the recent growth of exchange traded funds (ETFs), many managers have begun correlating their statistical factors to these ETFs in much the same way Ross and Roll did with economic data to better understand these statistical factors.

The process to derive the statistical model is as follows:

Step 1: Compute the sample covariance matrix by definition from historical data. This matrix will likely suffer from spurious relationships due the data limitations (not enough degrees of freedom and potential false relationships). But these will be resolved via principal component analysis.

Let $\overline{C}$ represent the sample covariance matrix.

Step 2: Factor the sample covariance matrix. We based the factorization scheme on eigenvalue-eigenvector decomposition. This is:

$$\overline{C} = VDV' \tag{5.13}$$

where D is the diagonal matrix of eigenvalues sorted from largest to smallest, $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ and V is the corresponding matrix of eigenvectors and. These matrices are as follows:

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \quad V = \begin{bmatrix} v_{11} & v_{21} & \cdots & v_{n1} \\ v_{12} & v_{22} & \cdots & v_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ v_{1n} & v_{2n} & \cdots & v_{nn} \end{bmatrix}$$

Since D is a diagonal matrix we have $D = D^{1/2}D^{1/2}$, $D = D'$, and $D^{1/2} = (D^{1/2})'$

Then, our covariance matrix C can be written as:

$$\overline{C} = VDV' = VD^{1/2}D^{1/2}V' = VD^{1/2}(VD^{1/2})'$$

Step 3: Compute β in terms of the eigenvalues and eigenvectors:

$$\beta = (VD^{1/2})'$$

Then the full sample covariance matrix expressed in terms of β is:

$$\beta'\beta = VD^{1/2}(VD^{1/2})' \tag{5.14}$$

Step 4: Remove spurious relationships due to data limitation

To remove the potential spurious relationships, we only use the eigenvalues and eigenvectors with the strongest predictive power.

How many factors should be selected? In our eigenvalue-eigenvector decomposition, each eigenvalue λ_k of the sample covariance matrix explains exactly $\lambda_k / \sum \lambda$ percent of the total variance. Since the eigenvalues are sorted from highest to lowest, a plot of the percentage of variance explained will show how quickly the predictive power of the factors declines. If the covariance matrix is generated by, say, 10 factors, then the first 10 eigenvalues should explain the large majority of the total variance.

There are many ways to determine how many factors should be selected to model returns. For example, some analysts will select the minimum number of factors that explain a pre-specified amount of variance, and some will select the number of factors up to where there is a break-point or fall-off in explanatory power. Others may select factors so that the variance > 1/n, assuming that each factor should explain at least 1/n of the total. Readers can refer to Dowd (1998) for further techniques.

If it is determined that there are k-factors that sufficiently explain returns, the risk exposures are determined from the first k risk exposures for each stock since our eigenvalues are sorted from highest predictive power to lowest.

$$\beta = \begin{bmatrix} \beta_{11} & \beta_{21} & \cdots & \beta_{m1} \\ \beta_{12} & \beta_{22} & \cdots & \beta_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{1k} & \beta_{2k} & \cdots & \beta_{mk} \end{bmatrix}$$

The estimated covariance matrix is then:

$$\boldsymbol{C} = \underset{nxk}{\beta'} \underset{kXn}{\beta} + \underset{nXn}{\Lambda} \tag{5.15}$$

In this case the idiosyncratic matrixΛ is the diagonal matrix consisting of the difference between the sample covariance matrix and $\beta'\beta$. That is,

$$\Lambda = diag(\overline{C} - \beta'\beta) \tag{5.16}$$

It is important to note that in the previous expression $\overline{C} - \beta'\beta$, the off-diagonal terms will often be nonzero. This difference is considered to be the spurious relationship caused by the data limitation and degrees of freedom issue stated previously. Selection of an appropriate number of factors determined via eigenvalue decomposition will help eliminate these false relationships.

Empirical Evidence

We performed an analysis of our sample covariance matrix for large-cap and small-cap stocks using eigenvalue-eigenvector decomposition over our historical period (2011–2012). Figure 5.12a depicts the explanatory power from the top 25 eigenvalues. Notice how quickly the explanatory power of the eigenvalues decreases after the first eigenvalue. Additionally, notice how the largest explanatory eigenvalue for large-cap stocks accounts for 48% of total variance, and for small-caps the first eigenvalue accounts for approximately 34% of total variance. These percentages are consistent with our findings using the multi-index and Fama French approaches. Figure 5.12b illustrates the cumulative explanatory power through the first 25 eigenvalues for both samples. The first 25 eigenvalues account for approximately 70% of total variance for large-cap stocks. This shows that there may be something else happening in the markets that is driving returns, perhaps a sector effect (see Elton and Gruber) or an industry effect (see Barra). The top 25 eigenvalues for small-cap stocks explain approximately 50% of total variance. This is consistent with the large-cap findings, especially after considering that small-cap stocks have a higher quantity of company-specific risk.

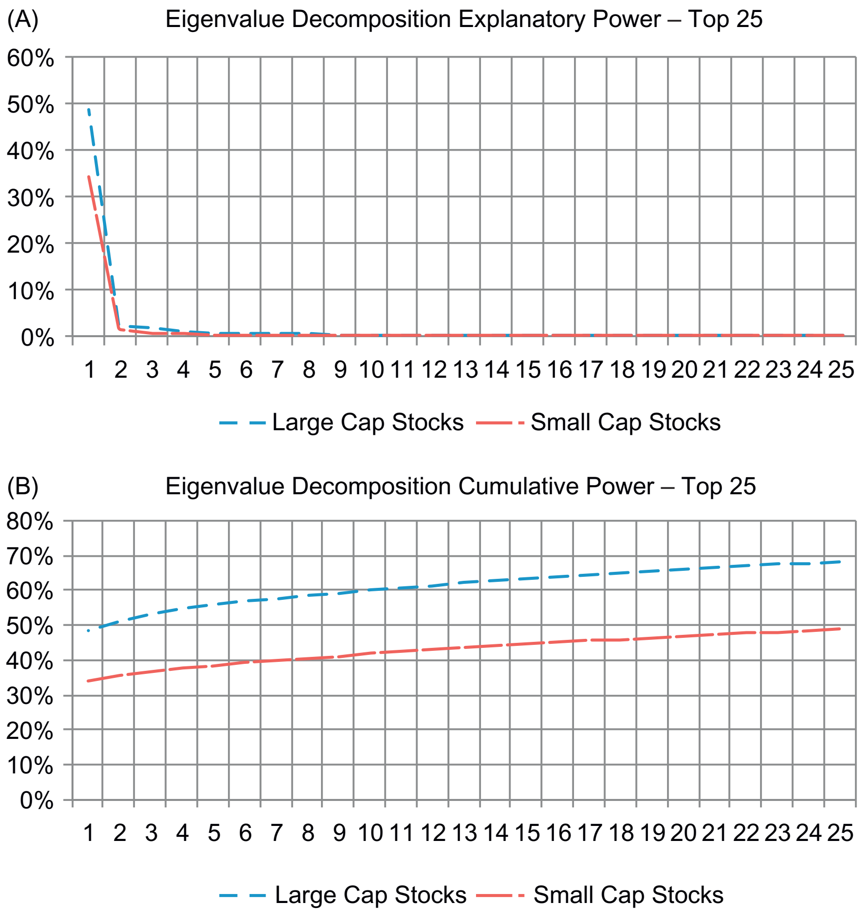

FIGURE 5.12 Eigenvalue Decomposition of Daily Stock Returns

Table 5.7 Eigenvalue Decomposition of Daily Stock Returns (2011–2012)

Rank	Explanatory Power		Cumulative Power	
	SP500	R2000	SP500	R2000
1	48.6%	34.2%	48.6%	34.2%
2	2.5%	1.7%	51.1%	35.8%
3	2.1%	1.0%	53.2%	36.8%
4	1.6%	0.8%	54.8%	37.7%
5	1.1%	0.8%	55.9%	38.5%
6	1.0%	0.8%	56.9%	39.2%
7	0.9%	0.7%	57.8%	39.9%
8	0.8%	0.7%	58.6%	40.6%
9	0.8%	0.6%	59.4%	41.3%
10	0.7%	0.6%	60.1%	41.9%
11	0.7%	0.6%	60.8%	42.5%
12	0.7%	0.6%	61.5%	43.1%
13	0.6%	0.5%	62.1%	43.6%
14	0.6%	0.5%	62.7%	44.2%
15	0.6%	0.5%	63.3%	44.7%
16	0.6%	0.5%	63.9%	45.1%
17	0.6%	0.5%	64.5%	45.6%
18	0.5%	0.5%	65.0%	46.1%
19	0.5%	0.4%	65.5%	46.5%
20	0.5%	0.4%	66.0%	46.9%
21	0.5%	0.4%	66.5%	47.4%
22	0.5%	0.4%	67.0%	47.8%
23	0.5%	0.4%	67.4%	48.2%
24	0.4%	0.4%	67.9%	48.6%
25	0.4%	0.4%	68.3%	49.0%

Table 5.7 illustrates the percentage of daily price variability explained by the primary eigenvalues over the period 2011 through 2012.

CONCLUSION

In this chapter, we provided readers with an overview of the different types of factor risk models, including single-index, multi-index, macroeconomic, cross-sectional, and statistical or principal component analysis (PCA) models. The chapter started with a discussion of why risk models are needed in the industry. Analysts who rely on historical data alone to compute covariance could get incorrect results due false relationships resulting from uncommon trend lines across the assets or data limitation issues in which there are not enough observations to compute statistically significant covariance across all the stocks.

We then performed a thorough analysis of the daily stocks returns for a large-cap sample (SP500 stocks) and a small-cap sample (R2000 stocks) over the period 2011−2012. Key findings of our analysis are that when relying on historical data as opposed to risk models, analysts may uncover false relationships that appear to occur about 40% of the time. Risk models are needed to better understand co-movement of asset returns. We also uncovered supporting evidence that shows beta tends towards one. Specifically, we found that stocks with a beta higher than one appeared to move closer to one in the following year. But we did not find as much supporting evidence for stocks with a beta much lower than one to increase towards one in the following year. Some of this may be explained by survivorship bias in our data. We concluded the chapter by decomposition risk for both the large-cap and small-cap universes using our multi-index modeling approach, the Fama-French factor model, and the eigenvalue-eigenvector statistical approach, with all the methodologies uncovering similar results. One of the biggest findings from these different models is that on a daily basis about 50% of stock risk (variance) is company-specific risk (e.g., risk that is not explained by any of the models). This is a larger percentage than many of the models find using a longer horizon such as weekly or monthly returns. However, it is essential that portfolio managers consider the high degree of idiosyncratic risk when developing a hedging strategy and when managers need to liquidate a position on a given day.

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