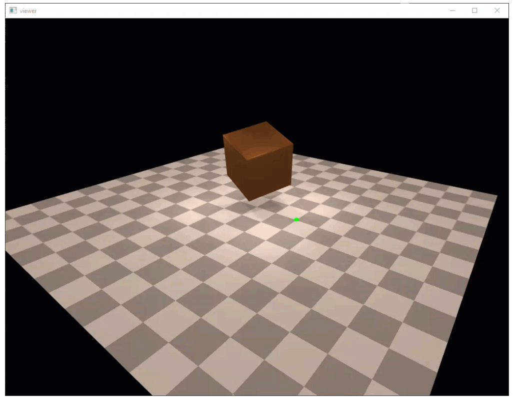
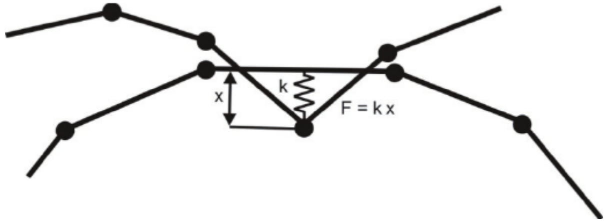
RESOLUTION OF COLLISION

Linear Complementarity Programming (LCP)



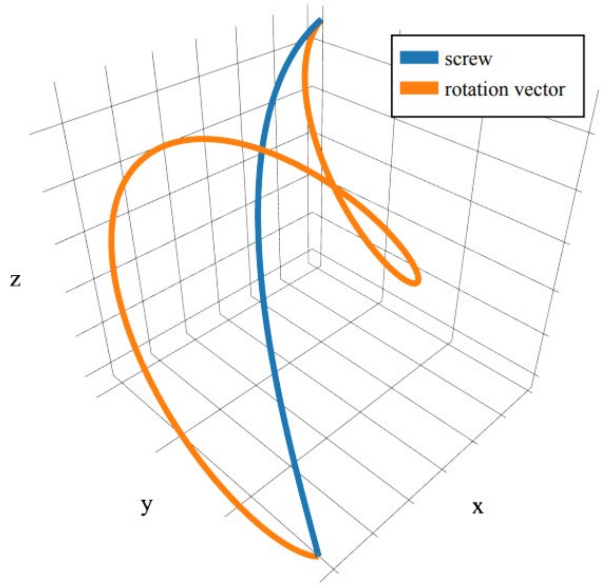
- The resulting system remains non-convex and challenging to solve with accuracy for complex scenes
- The required constraint linearization will cause intersections

Penalties



- Fails for fast-moving models or large time steps
- Manual tuning of stiffness parameters
- Interpenetration

RESOLUTION OF COLLISION



Rigid-IPC
[Ferguson et al. 2021]

- Guaranteed intersection-free collision resolution
- Faster than LCP-based solutions for complicated contacts
- More robust than regular penalty methods

Benefit from IPC

- The trajectory in Rigid-IPC becomes curved due to strictly rigidity motion

Using smooth approximation
to substitute rigidity constraints

Affine Body Dynamics

ABD KINEMATICS

For each d body b ,there is a time-varying linear transform $A_b(t) \in \mathbb{R}^{3\times3}$, and a translation $p_b(t) \in \mathbb{R}^3$.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

Then store per-body configuration in the vector form of:

$$q = \begin{pmatrix} p_1 & p_2 & p_3 & a_{11} & a_{12} & a_{13} & a_{21} & a_{22} & a_{23} & a_{31} & a_{32} & a_{33} \end{pmatrix}^{T} \in \mathbb{R}^{12}$$

Each material point k in body b has a body frame (equivalently rest) position $\bar{x}_k$, corresponding world frame coordinates given by the affine map:

$$x_k = A_b \bar{x}_k + p_b = J(\bar{x}_k) q,$$

$$\dot{x}_k = \dot{A}_b \bar{x}_k + \dot{p}_b = J(\bar{x}_k) \dot{q}.$$

Here note that $J(\bar{x}) = [I_3, I_3 \otimes \bar{x}]$ is constant across all configuration changes.

KINETIC ENERGY

Given a mass density distribution, ρ, over the body domain, Ω, the kinetic energy of each affine body is then

$$
\begin{aligned}
\frac{1}{2}\int_{\Omega} \rho \dot{x}^T \dot{x}\, \mathrm{d}\Omega &= \frac{1}{2}\int_{\Omega} \rho(\dot{\mathrm{A}}\bar{x} + \dot{p})^T (\dot{\mathrm{A}}\bar{x} + \dot{p})\, \mathrm{d}\Omega \\
&= \frac{1}{2}\dot{q}^T \boxed{\left(\int_{\Omega} \rho\, \mathrm{J}(\bar{x})^T \mathrm{J}(\bar{x})\, \mathrm{d}\Omega\right)} \dot{q},
\end{aligned}
$$

Mass Matrix ← A constant

No nonlinear Coriolis-type forces

With V the total potential energy the free ABD is then simply the equations of motion:

$$\boxed{\mathrm{M}}\ddot{q} = -\nabla V(q) + f,$$

where external forces $f_k \in \mathbb{R}^3$, applied at material points k, are included as $f = \sum_k \mathrm{J}(\bar{x}_k)^T f_k$.

ORTHOGONALITY POTENTIAL

Rigidify each affine body with a stiff orthogonality potential in place of SE(3) coordinates:

$$V_\perp(q) = \kappa v \|\mathrm{A}\mathrm{A}^T - \mathrm{I}_3\|_F^2,$$

scaled by the stiffness κ and the body's volume v.

It can be computed as a polynomial

$$V_\perp = \kappa v \left(\sum (a_i \cdot a_i - 1)^2 + \sum_{i \neq j} (a_i \cdot a_j)^2 \right),$$

Gradient and Hessian:

$$\frac{\partial V_\perp}{\partial a_i} = 2\kappa v \left(2(a_i \cdot a_i - 1)a_i + 2 \sum (a_j \otimes a_j) a_i \right),$$

$$\frac{\partial^2 V_\perp}{\partial a_i^2} = 2\kappa v \left(4a_i \otimes a_i + 2(\|a_i\|^2 - 1)\mathrm{I}_3 + 2 \sum a_j \otimes a_j \right).$$

AFFINE IPC

Unconstrained time step update

The incremental potential (IP), E_b, for each affine body $b \in \mathcal{B}$:

$$q_b^{t+1} = \arg\min_{q_b} E_b(q_b), \quad E_b = \frac{1}{2}\|q_b - \widetilde{q}_b\|_M^2 + \Delta t^2 V_\perp(q_b).$$

Here, Δt is the time step size and $\widetilde{q}_b = q_b^t + \Delta t \dot{q}_b^t + \Delta t^2 M^{-1} f_b^{t+1}$.

Adding Constraints

IPC potentials for contact

$$V_C(q) = \kappa \sum_{\boxed{i \in \mathcal{C}}} B\left(d_i(q)\right), \quad B(d, \hat{d}) = \begin{cases} -(d-\hat{d})^2 \ln\left(\frac{d}{\hat{d}}\right), & 0 < d < \hat{d} \\ 0 & d \geq \hat{d} \end{cases},$$

All possible pairings for contact

Friction potential

$$V_F(q) = \sum_{\boxed{j \in \mathcal{F}}} \mu \lambda_j m_j(q),$$

Active subset of contact pair

Solve for the minimizer of the global IP for the full contact coupled system:

$$E = \sum E_b + V_C + V_F$$

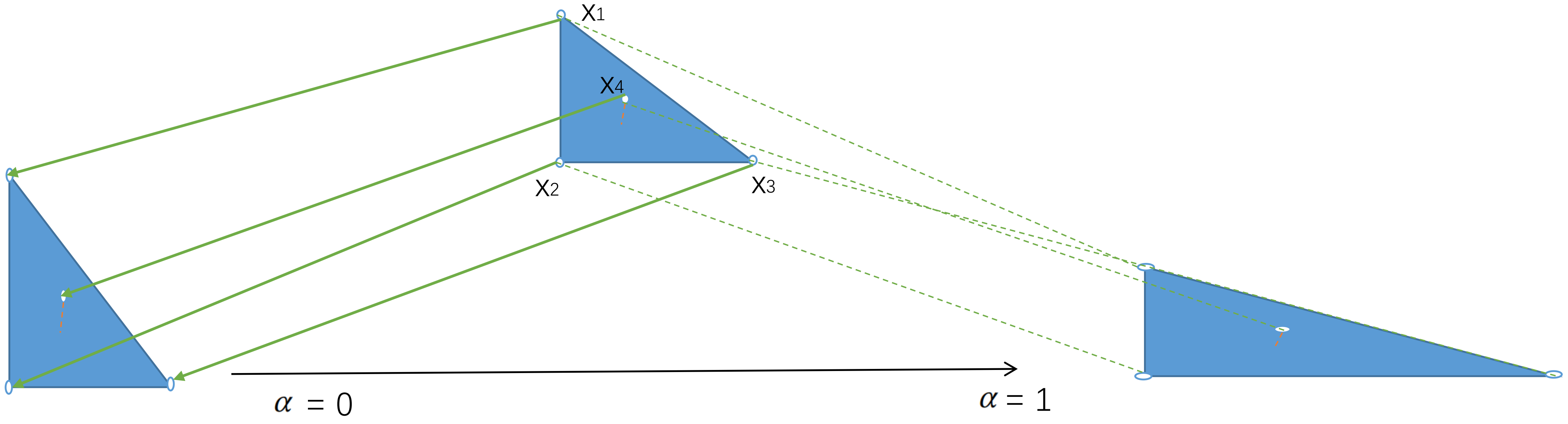
AFFINE CCD

At any iteration ℓ of a Newton solve, new positions of the vertices x_j in each possible contacting pair, per participating body b, is $\mathrm{A}_b^\ell \bar{x}_j + p_b^\ell$.

Search direction

$$\Delta \mathrm{A}_b^\ell = \mathrm{A}_b^\ell - \mathrm{A}_b^{\ell-1}$$
$$\Delta p_b^\ell = p_b^\ell - p_b^{\ell-1}$$

$$\Rightarrow \quad (\mathrm{A}_b^{\ell-1} + \alpha \Delta \mathrm{A}_b^\ell)\bar{x}_j + p_b^\ell + \alpha \Delta p_b^{\ell-1}, \quad \alpha \in [0,1]$$



ABD PSEUDO-CODE

$\hat{C} \leftarrow$ ComputeConstraintSet$(x, \hat{d})$, $E_{prev} \leftarrow B_t(x, \hat{d}, \hat{C})$

Newton loop (IPC): $B_t(x, \hat{d}, \hat{C}) = \boxed{E(x) + \kappa \sum_{k \in \hat{C}} b(D_k(x), \hat{d})}$

Do

$H \leftarrow$ SPDProject$(\nabla_x^2 B_t(x, \hat{d}, \hat{C}))$

$p \leftarrow -H^{-1} \nabla_x B_t(x, \hat{d}, \hat{C})$

$\alpha \leftarrow \min(1, CCD(x, x+p))$

Do

$x \leftarrow x_{prev} + \alpha p$

$\hat{C} \leftarrow$ ComputeConstraintSet$(x, \hat{d})$

$\alpha \leftarrow \alpha / 2$

While $B_t(x, \hat{d}, \hat{C}) > E_{prev}$

$E_{prev} \leftarrow B_t(x, \hat{d}, \hat{C})$, $x_{prev} \leftarrow x$

Update κ, boundary conditions, etc

While $||p||_\infty / h > \epsilon_d$

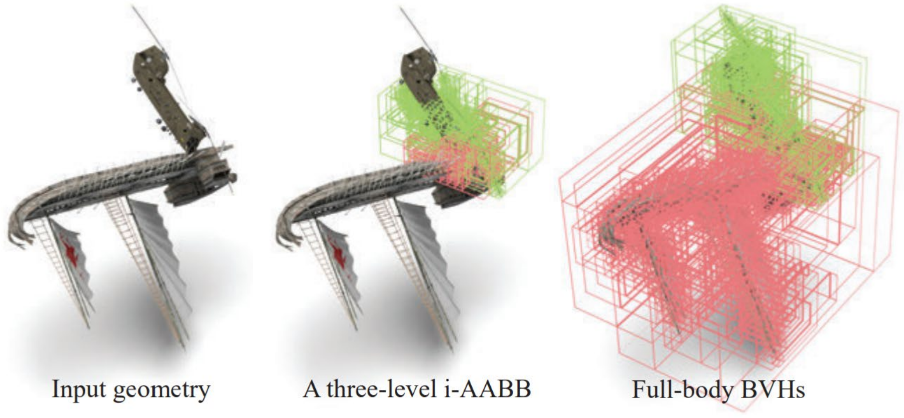
Replace with Affine IPC

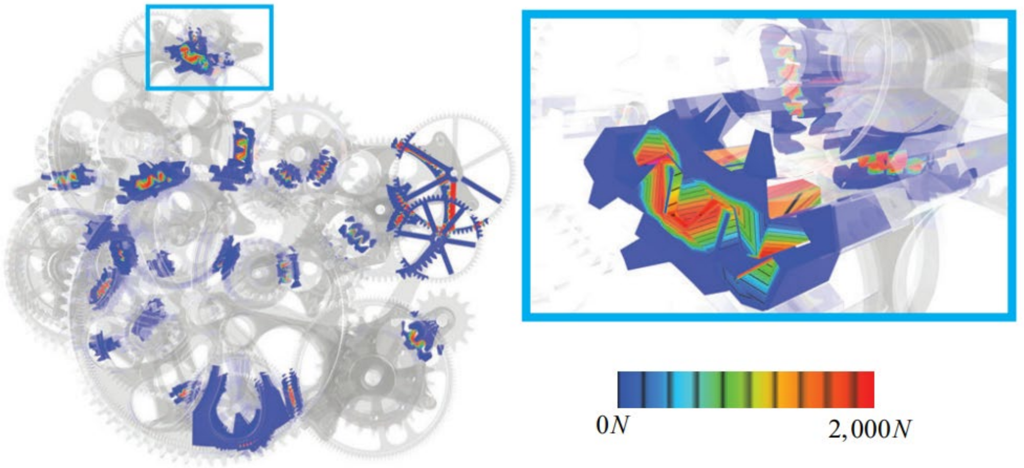
$E = \sum E_b + V_C + V_F$

Affine CCD

SPEED UP AND PARALLELISM

Contact Culling via i-AABB





Contact-Aware Hessian Construction

- Nonzero diagonal blocks are given by a constant mass term and the orthogonality potentials' Hessian
- Two-pass strategy to compute and assemble barrier and friction terms for the global Hessian

JOINT CONSTRAINTS

It is known that a non-degenerate tetrahedron uniquely defines an affine transform.

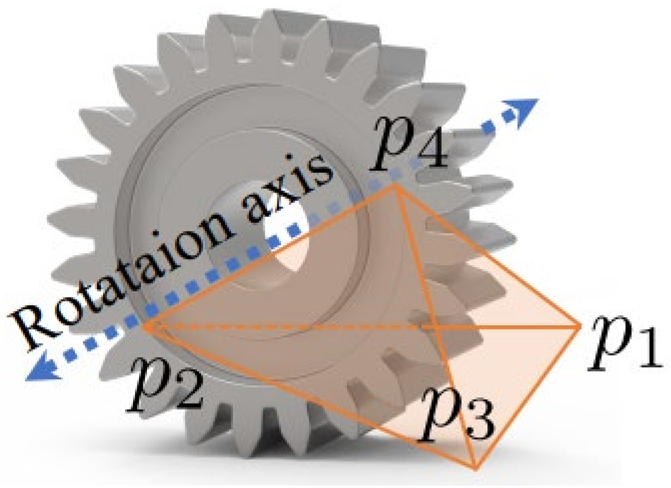
Let ϕ denote the map between q and this virtual tetrahedron such that $q = \phi(\mathrm{P})$.

$$\mathrm{P} = \begin{pmatrix} p_{1x} & p_{2x} & p_{3x} & p_{4x} \\ p_{1y} & p_{2y} & p_{3y} & p_{4y} \\ p_{1z} & p_{2z} & p_{3z} & p_{4z} \end{pmatrix}$$

$$\phi(\mathrm{P}) = \left[\frac{1}{4} \sum_i (p_i - \bar{p}_i)^T, \mathrm{vec}^T \left(\mathrm{P}\bar{\mathrm{P}}^T (\bar{\mathrm{P}}\bar{\mathrm{P}}^T)^{-1} \right) \right]^T$$

Clearly ϕ is a linear function of P. Therefore, the Jacobi of the system remains constant:

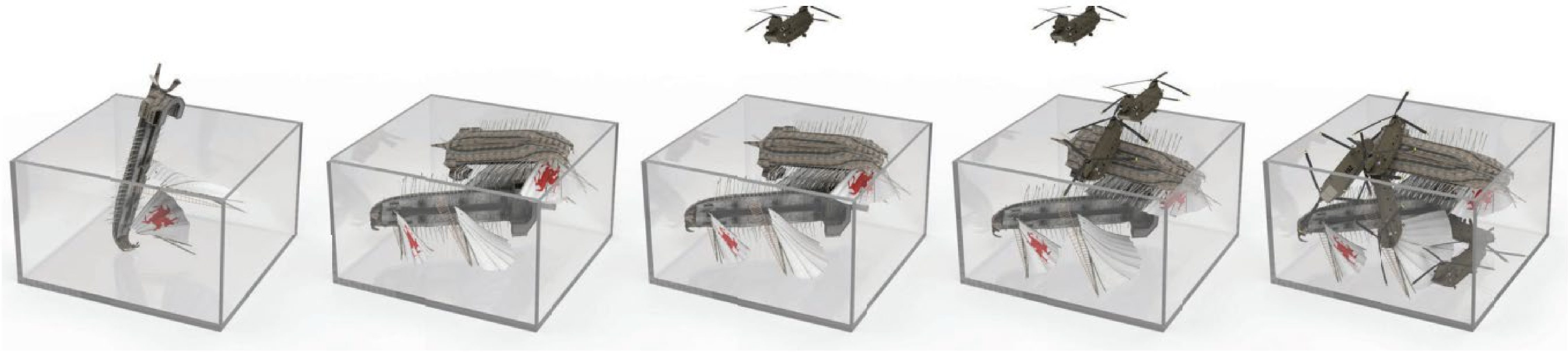
$$\mathrm{J}_i = \frac{\partial x_i}{\partial q} \cdot \frac{\partial \phi}{\partial \mathrm{vec}(\mathrm{P})} \in \mathbb{R}^{3\times 12}.$$



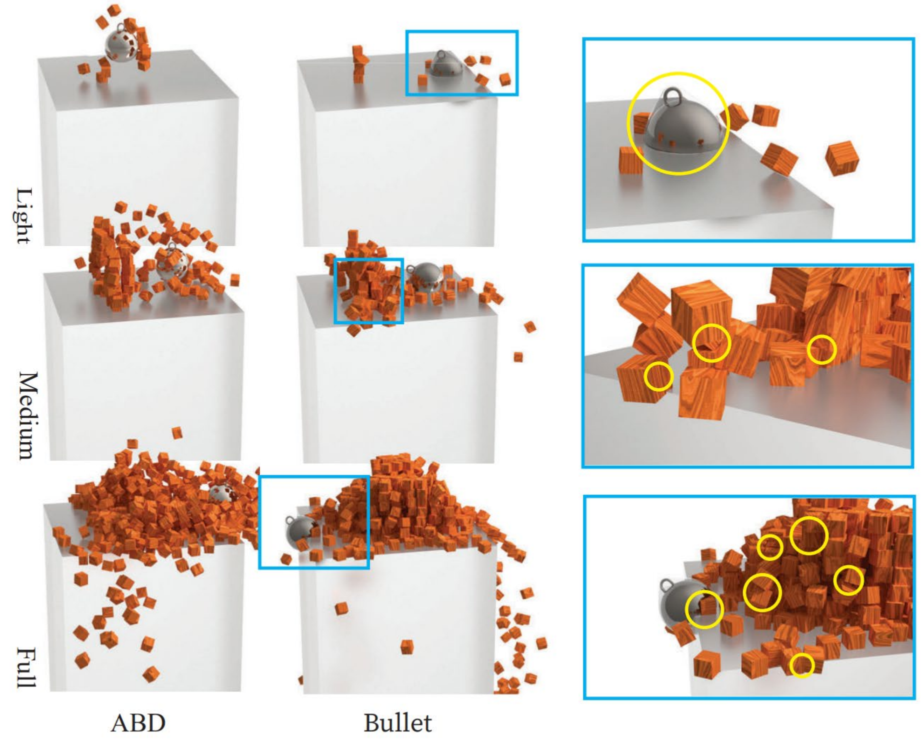
From this perspective, an affine body simulation can be viewed as a single-element FEM of which geometry can be setup flexibly

HYBRID SIMULATION

The collision between rigid and soft bodies can be handled uniformly using barrier-based penalties



COMPARE WITH BULLET



Test	# Bdy	# Tri./Edg.	Δt (sec)	# Iter.	Time (ms)
Light	16	1.2K/796	1/100	1.9	**3** | 2
			1/240	1.5	**2.2** | 1.5
			1/1000	1.1	**2** | 3
Medium	142	3.5K/2.3K	1/100	7.6	**92** | 68
			1/240	2.9	**41** | 58
			1/1000	1.3	**19** | 82
Full	562	11K/7.3K	1/100	11.0	**657** | 629
			1/240	4.4	**328** | 809
			1/1000	1.8	**102** | 804

COMPARE WITH BULLET