

TOWARDS THE ANDRE-OORT CONJECTURE FOR MIXED SHIMURA VARIETIES: THE AX-LINDEMANN THEOREM AND LOWER BOUNDS FOR GALOIS ORBITS OF SPECIAL POINTS

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ABSTRACT. We prove in this paper the Ax-Lindemann-Weierstraß theorem for all mixed Shimura varieties and discuss the lower bounds for Galois orbits of special points of mixed Shimura varieties. In particular we reprove a result of Silverberg [57] in a different approach. Then combining these results we prove the André-Oort conjecture unconditionally for any mixed Shimura variety whose pure part is a subvariety of \mathcal{A}_g^n and under GRH for all mixed Shimura varieties of abelian type.

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1. INTRODUCTION

1.1. Background. Every connected Shimura variety, being the quotient of a Hermitian symmetric domain by an arithmetic group, can be realized as a moduli space for pure Hodge structures plus tensors. Unlike the Hermitian symmetric domains themselves, connected Shimura varieties are algebraic varieties. This was proved by Baily-Borel [7]. The prototype for all

Shimura varieties is the Siegel moduli space of principally polarized abelian varieties of dimension g with a level structure. The points in this moduli space corresponding to CM abelian varieties, which are called *special points*, play a particularly important role in the theory of Shimura varieties. A major reason is that the Galois actions on special points are fairly completely determined by the Shimura-Taniyama theorem [20, Theorem 4.19] and its generalization by Milne-Shih [22] to Galois conjugates of CM abelian varieties. The concept of special points and the results concerning the Galois action on them have been generalized to arbitrary Shimura varieties. Every Shimura variety has a Zariski dense subset of special points ([20, Proposition 5.2]), and hence the results above have led to the concept of the *canonical model* of a Shimura variety over a number field: see Deligne [21] and Milne [38].

It is of course a natural problem to look for suitable compactifications of a given Shimura variety. The first compactification is the Baily-Borel (or minimal) compactification [7], which is canonical. However this compactification has bad singularities along the boundary. Next we have the toroidal compactifications [4], which are no longer canonical but provide smooth compactifications of Shimura varieties. To construct these compactifications one needs to study the boundary of a Shimura variety. As one approaches the boundary of a Hermitian symmetric domain, pure Hodge structures degenerate into mixed Hodge structures, and as one approaches the boundary of a Shimura variety, abelian varieties degenerate into 1-motives. This will lead to a new object, generalizing the notion of Shimura varieties and parametrizing variations of mixed Hodge structures (all of whose pure constituents are polarizable), which we shall call a mixed Shimura variety. In order to distinguish, we will use the term “pure Shimura variety” to denote the Shimura varieties in the first paragraph. Here we copy a list of some objects attached to a Shimura variety and the corresponding object attached to a mixed Shimura variety from Milne [38, Introduction],

(pure) Shimura variety	mixed Shimura variety
Hermitian (or bounded) symmetric domain	Siegel domain (of the third kind)
pure Hodge structure	mixed Hodge structure
reductive group over \mathbb{Q}	algebraic group over \mathbb{Q} with 3-step filtration
abelian variety	1-motive
motive	mixed motive

Here are several important examples of mixed Shimura varieties:

- the universal family of abelian varieties of dimension g with a level structure;
- the \mathbb{G}_m -torsor over such a universal family which corresponds to an ample line bundle over this family;
- Poincaré bi-extension;
- products of the above.

There is also the concept of special points for mixed Shimura varieties, e.g. special points of the universal family of abelian varieties are those which correspond to torsion points on CM abelian varieties. Similar results about the Galois action on special points and the canonical models of pure Shimura varieties hold for mixed Shimura varieties: see Pink [53].

An irreducible component of a mixed Shimura subvariety of a mixed Shimura variety, or its image under a Hecke operator, is called a special subvariety. In particular, special points are precisely special subvarieties of dimension 0. As for pure Shimura varieties, every special

subvariety contains a Zariski dense subset of special points ([53, 11.7]). The aim of this article is to study the following conjecture, which is the converse of this fact.

Conjecture 1.1 (Generalized André-Oort). *Let Y be a closed irreducible subvariety of a mixed Shimura variety. If Y contains a Zariski-dense set of special points, then it is special.*

The original André-Oort conjecture, in which one replaces “mixed Shimura variety” by “pure Shimura variety”, has been established in many cases (unconditionally or under GRH without using \mathfrak{o} -minimality) by Moonen [40], André [2], Edixhoven [24, 25], Edixhoven-Yafaev [27] and Yafaev [68, 69]. It was proved under GRH by Klingler, Ullmo and Yafaev [63, 36], where equidistribution results of Clozel-Ullmo [16] were used. Later Daw [18] removed the ergodic theory from Klingler-Yafaev’s result. Our generalized version of the André-Oort conjecture was suggested by Y. André in [3, Lecture 3], where he also gave a proof of the case of the universal family of elliptic curves. Results for mixed Shimura varieties have been obtained by Habegger [30] for fibered powers of elliptic surfaces. Remark that Conjecture 1.1 is not only a direct generalization of the original the André-Oort conjecture, but also contains the Manin-Mumford conjecture for any complex semi-abelian variety whose abelian variety part is CM.

1.2. Ax-Lindemann-Weierstraß. A recent development of this conjecture was made by Pila-Zannier. Its origin was the proof of the Manin-Mumford conjecture [52]. Afterwards using tools of \mathfrak{o} -minimality and Pila-Wilkie’s counting theorem [51], Pila proved the André-Oort conjecture for \mathcal{A}_1^n (product of modular curves) unconditionally [47]. Daw-Yafaev later proved the André-Oort conjecture unconditionally for Hilbert modular surfaces [19]. In this strategy of proving the André-Oort conjecture, a key step is to establish the following generalization of the functional Lindemann-Weierstrass theorem [5], which is the main result of this paper:

Theorem 1.2 (Ax-Lindemann-Weierstraß Theorem). *Let S be a connected mixed Shimura variety, let $\text{unif}: \mathcal{X}^+ \rightarrow S$ be its uniformization and let Y be a closed irreducible subvariety of S . Let \tilde{Z} be an irreducible algebraic subset of \mathcal{X}^+ contained in $\text{unif}^{-1}(Y)$, maximal for these properties. Then $\text{unif}(\tilde{Z})$ is weakly special.*

We explain briefly the notions in this theorem. A connected mixed Shimura variety is defined to be a connected component of a mixed Shimura variety. As for the pure case, the uniformizing space \mathcal{X}^+ can be realized as an open subset (w.r.t. the archimedean topology) of a complex algebraic variety \mathcal{X}^\vee (§4), and an *irreducible algebraic subset* of \mathcal{X}^+ is defined to be an a complex analytic irreducible component of the intersection of a closed algebraic subvariety of \mathcal{X}^\vee and \mathcal{X}^+ (Definition 6.1). Consider Shimura morphisms of connected mixed Shimura varieties $T' \xleftarrow{[\varphi]} T \xrightarrow{[i]} S$ and a point $t' \in T'$. Following Pink [54], an irreducible component of $[i]([\varphi]^{-1}(t'))$ is called a *weakly special subvariety* of S (Definition 5.1). In the case of pure Shimura varieties, Moonen [40] proved that the weakly special subvarieties are precisely the totally geodesic subvarieties.

It is worth remarking that if we prove Conjecture 1.1 via Theorem 1.2, then we also prove the Manin-Mumford conjecture for all semi-abelian varieties (not only those whose abelian variety parts are CM). Theorem 1.2 was proved for (semi-)abelian varieties by Ax [6], and then refound and reproved by Pila-Zannier [52] and Peterzil-Starchenko [46], where proofs of Manin-Mumford via Ax-Lindemann-Weierstraß can be found. For the hyperbolic case (pure Shimura varieties), this theorem has also been established in several different cases by Pila (for \mathcal{A}_1^n [47] and later a

product of universal families of elliptic curves [48]¹), Ullmo-Yafaev (for projective pure Shimura varieties [64]) and Pila-Tsimerman (for \mathcal{A}_2 [50] and then \mathcal{A}_g [49]). Klingler-Ullmo-Yafaev have recently proved this theorem for all pure Shimura varieties [35] using the idea of calculating the volumes of algebraic curves near the boundary, which was firstly executed in [64] and then in [49]. Our proof is based on the result of [35]. A main ingredient to prove all the results above (including the whole Theorem 1.2) is Pila-Wilkie’s counting theorem; however unlike the pure case, the “family version” of this counting theorem is crucially used in this paper. Some other difficulties to prove Theorem 1.2 for mixed Shimura varieties which we do not encounter in the pure case are listed in §9 before Lemma 9.5. We hope that this may make the strategy of our proof more clear.

We close this subsection with the following comment about weakly special subvarieties. To study them, it is useful to describe the smallest weakly special subvariety containing a given subvariety Y of a connected mixed Shimura variety S . We shall prove in Theorem 8.1 (sometimes called ***Ax’s theorem of log type***²), generalizing Moonen’s result [40, 3.6, 3.7] for pure Shimura varieties, that this smallest weakly special subvariety is precisely the one defined by the connected algebraic monodromy group of Y^{sm} . The proof uses André’s [1] and Wildeshaus’ [67] earlier work about variations of mixed Hodge structure (over mixed Shimura varieties). As a consequence of this description, we shall prove a characterization of weakly special subvarieties in terms of “bi-algebraicity” (Corollary 8.3), which is a direct generalization of the main result of Ullmo-Yafaev [62].

1.3. From Ax-Lindemann-Weierstraß to André-Oort. Ullmo and Pila-Tsimerman explained separately in [61] [49] how to deduce the André-Oort conjecture from the Ax-Lindemann-Weierstraß theorem for pure Shimura varieties. The proof of Ullmo is generalized to mixed Shimura varieties in this paper (§12). They showed that in order to prove the André-Oort conjecture for pure Shimura varieties of abelian type, the only ingredient (and obstacle) left is a suitable lower bound for the Galois orbit of a special point of a pure Shimura variety conjectured by Edixhoven [26]. We prove that what we need to prove Conjecture 1.1 (for any mixed Shimura variety whose pure part is of abelian type) is the same lower bound. More explicitly, we prove that the naturally expected good lower bound for the Galois orbit of a special point, i.e. the product of the lower bounds of the base and the fiber, is fulfilled (Proposition 13.3). As a special case, this provides a new proof for the result of Silverberg [57] (Corollary 13.4).

Theorem 1.3. *Let A be a complex abelian variety of CM type of dimension g . Its field of definition k is then a number field by CM theory. Let t be a torsion point of A of order $N(t)$. If we denote by $k(t)$ the field of definition of t over k , then $\forall \varepsilon \in (0, 1)$,*

$$[k(t) : k] \gg_{g, \varepsilon} N(t)^{1-\varepsilon}.$$

In Silverberg’s work, the constant on the right hand side also depends on the field k . The lower bound for pure Shimura varieties is known under GRH ([59], [65]). The best unconditional result is given by Tsimerman [59]. He established the lower bound unconditionally for $g \leq 6$ (for $g \leq 3$ this was also proved by Ullmo-Yafaev by a similar method [65]). Therefore as a consequence we prove (Theorem 13.6)

¹The definition of weakly special subvarieties in this paper looks quite different and a lot more complicated than the one we use here. They probably coincide but I did not check it.

²This is pointed to me by Daniel Bertrand.

Theorem 1.4. *Under GRH, the generalized André-Oort conjecture (Conjecture (1.1)) holds for any mixed Shimura variety whose pure part is a closed subvariety of \mathcal{A}_g^n . This result is unconditional (i.e. we do not need GRH) if $g \leq 6$.*

1.4. Zilber-Pink. Finally it is worth remarking that Conjecture 1.1 is part of the more general Zilber-Pink Conjecture [54, 70, 56]. Some unlikely intersections results of type André-Pink [54, Conjecture 1.6] about pure Shimura varieties beyond the André-Oort conjecture have been obtained by Pink [54, Theorem 7.6] (Galois generic points in \mathcal{A}_g), Habegger-Pila [31] (curves in \mathcal{A}_1^n) and Orr [43] (curves in \mathcal{A}_g). I shall not talk about the case of algebraic groups (see [15] for a summary). As for mixed Shimura varieties, Bertrand, Bertrand-Edixhoven, Bertrand-Pillay and Bertrand-Masser-Pillay-Zannier have recently been working on Poincaré biextensions [8, 12, 9, 10, 11]. They have got several interesting results, some of which provide good examples for this paper.

Structure of this paper. In §2 we recall some basic facts about mixed Shimura varieties following Pink [53]. §3 is a summary of variations of mixed Hodge structure. In §4 we discuss the realization of the uniformizing space of any given mixed Shimura variety. In particular we give a realization of it which is at the same time semi-algebraic and complex analytic (Proposition 4.3). §5 is exploited to study (weakly) special subvarieties following Pink [54]. In §6 we define algebraic subsets of the uniformizing space and prove the functoriality of the algebraicity. In §7 we list and prove some results for the unipotent part, with the statement of the Ax-Lindemann-Weierstraß Theorem for the unipotent part which we will eventually prove in §11. In §8 we will have our first important results, i.e. the description of the smallest weakly special subvariety containing a given subvariety Y of a connected mixed Shimura variety S (Theorem 8.1) and a criterion of weakly special subvarieties in terms of “bi-algebraicity” (Corollary 8.3). The core of this paper is the proof of Theorem 1.2, and it is executed in §9, §10 and §11. The proof is quite technical, and for readers’ convenience we organize it as follows: the outline of the proof is presented in §9, a key proposition leading to the theorem is proved in §10 by using Pila-Wilkie’s counting theorem and we shall prove Ax-Lindemann-Weierstraß for the unipotent part (the fiber) separately in §11. In §12 we derive a corollary from Theorem 1.2, which will be used to prove Theorem 1.4 in §13.2 together with a suitable lower bound discussed in §13.1. In the Appendix we reprove the Ax-Lindemann-Weierstraß theorem for algebraic tori over \mathbb{C} and complex abelian varieties by this method of calculating volumes and counting points.

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and language. I would also like to thank them here. Finally I thank the referee for his/her careful reading and helpful suggestions thanks to which this article has been improved.

Conventions. For $x = (x_1, \dots, x_n) \in \mathbb{Q}^n$, we define the height of x as $H(x) = \text{Max}(H(x_1), \dots, H(x_n))$ where for $a, b \in \mathbb{Z} \setminus \{0\}$ coprime $H(a/b) = \text{Max}(|a|, |b|)$, and $H(0) = 0$.

For an algebraic group P over a field k , when we talk about a subgroup of P , we always mean a k -subgroup unless it is claimed not to be.

For the theory of o-minimality, we refer to [64, §3] (for a concise version) and [47, §2,3] (for a more detailed version). In this paper, “semi-algebraic” will always mean \mathbb{R} -semi-algebraic. The o-minimal structure we consider is $\mathbb{R}_{an, \exp}$, i.e. by saying a set A is definable we mean that A is definable in $\mathbb{R}_{an, \exp}$.

2. CONNECTED MIXED SHIMURA VARIETIES

2.1. Definition and basic properties. (cf. [53, Chapter 1,2,3]) Let $\mathbb{S} := \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$ be the Deligne-torus.

Definition 2.1. A *mixed Shimura datum* (P, \mathcal{X}, h) is a triple where

- P is a connected linear algebraic group over \mathbb{Q} with unipotent radical W and with another algebraic subgroup $U \subset W$ which is normal in P and uniquely determined by \mathcal{X} using condition (3) below;
- \mathcal{X} is a left homogeneous space under the subgroup $P(\mathbb{R})U(\mathbb{C}) \subset P(\mathbb{C})$, and $\mathcal{X} \xrightarrow{h} \text{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$ is a $P(\mathbb{R})U(\mathbb{C})$ -equivariant map s.t. every fibre of h consists of at most finitely many points,

such that for some (equivalently for all) $x \in \mathcal{X}$,

- (1) the composite homomorphism $\mathbb{S}_{\mathbb{C}} \xrightarrow{h_x} P_{\mathbb{C}} \rightarrow (P/U)_{\mathbb{C}}$ is defined over \mathbb{R} ,
- (2) the adjoint representation induces on $\text{Lie } P$ a rational mixed Hodge structure of type

$$\{(-1, 1), (0, 0), (1, -1)\} \cup \{(-1, 0), (0, -1)\} \cup \{(-1, -1)\},$$

- (3) the weight filtration on $\text{Lie } P$ is given by

$$W_n(\text{Lie } P) = \begin{cases} 0 & \text{if } n < -2 \\ \text{Lie } U & \text{if } n = -2 \\ \text{Lie } W & \text{if } n = -1 \\ \text{Lie } P & \text{if } n \geq 0 \end{cases},$$

- (4) the conjugation by $h_x(\sqrt{-1})$ induces a Cartan involution on $G_{\mathbb{R}}^{\text{ad}}$ where $G := P/W$, and G^{ad} possesses no \mathbb{Q} -factor H s.t. $H(\mathbb{R})$ is compact,
- (5) $P/P^{\text{der}} = Z(G)$ is an almost direct product of a \mathbb{Q} -split torus with a torus of compact type defined over \mathbb{Q} .

In practice, we often omit the map “ h ” and write a mixed Shimura datum as a pair (P, \mathcal{X}) . If in addition P is reductive, then (P, \mathcal{X}) is called a **pure Shimura datum**.

Remark 2.2. (1) Let $\omega : \mathbb{G}_{m, \mathbb{R}} \hookrightarrow \mathbb{S}$ be $t \in \mathbb{R}^* \mapsto t \in \mathbb{C}^*$. Conditions (2) and (3) together imply that the composite homomorphism $\mathbb{G}_{m, \mathbb{C}} \xrightarrow{\omega} \mathbb{S}_{\mathbb{C}} \xrightarrow{h_x} P_{\mathbb{C}} \rightarrow (P/U)_{\mathbb{C}}$ is a co-character

of the center of P/W defined over \mathbb{R} . This map is called the weight. Furthermore, condition (5) implies that the weight is defined over \mathbb{Q} .

- (2) Condition (5) also implies that every sufficiently small congruence subgroup Γ of $P(\mathbb{Q})$ is contained in $P^{\text{der}}(\mathbb{Q})$ (cf [53, the proof of 3.3(a)]). Fix a Levi decomposition $P = W \rtimes G$ ([55, Theorem 2.3]), then $P^{\text{der}} = W \rtimes G^{\text{der}}$, and hence for any congruence subgroup $\Gamma < P^{\text{der}}(\mathbb{Q})$, Γ is Zariski dense in P^{der} by condition (4) ([55, Theorem 4.10]).
- (3) Condition (5) in the definition does not make the situation less general because we are only interested in a connected component of \mathcal{X} ([53, 1.29]).

Definition 2.3. Let (P, \mathcal{X}) be a mixed Shimura datum and let K be an open compact subgroup of $P(\mathbb{A}_f)$ where \mathbb{A}_f is the ring of finite adèle of \mathbb{Q} . The corresponding **mixed Shimura variety** is defined as

$$M_K(P, \mathcal{X}) := P(\mathbb{Q}) \backslash \mathcal{X} \times P(\mathbb{A}_f) / K,$$

where $P(\mathbb{Q})$ acts diagonally on both factors on the left and K acts on $P(\mathbb{A}_f)$ on the right.

In this article, we only consider connected mixed Shimura data and connected mixed Shimura varieties except in §13.

- Definition 2.4.** (1) A **connected mixed Shimura datum** is a pair (P, \mathcal{X}^+) satisfying the conditions of Definition 2.1, where $\mathcal{X}^+ \xrightarrow{h} \text{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$ is an orbit under the subgroup $P(\mathbb{R})^+ U(\mathbb{C}) \subset P(\mathbb{C})$.
- (2) A **connected mixed Shimura variety** S associated with (P, \mathcal{X}^+) is of the form $\Gamma \backslash \mathcal{X}^+$ for some congruence subgroup $\Gamma \subset P(\mathbb{Q}) \cap P(\mathbb{R})_+$, where $P(\mathbb{R})_+$ is the stabilizer of $\mathcal{X}^+ \subset \text{Hom}_{\mathbb{C}}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$.

Every connected mixed Shimura variety is a connected component of a mixed Shimura variety, and vice versa ([53, 3.2]). A connected mixed Shimura variety is a complex analytic space with at most finite quotient singularities, and if Γ is sufficiently small (for example if Γ is neat), then $\Gamma \backslash \mathcal{X}^+$ is smooth. For details we refer to [54, Fact 2.3] or [53, 1.18, 3.3, 9.24].

Recall the following definition, which Pink calls “irreducible” in [53, 2.13].

Definition 2.5. A connected mixed Shimura datum (P, \mathcal{X}^+) is said to **have generic Mumford-Tate group** if P possesses no proper normal subgroup P' defined over \mathbb{Q} s.t. for one (equivalently all) $x \in \mathcal{X}^+$, h_x factors through $P'_{\mathbb{C}} \subset P_{\mathbb{C}}$. We shall denote this case by $P = \text{MT}(\mathcal{X}^+)$. (This terminology will be explained in Remark 3.8).

Proposition 2.6. Let (P, \mathcal{X}^+) be a connected mixed Shimura datum, then

- (1) there exists a connected mixed Shimura datum $(P', \mathcal{X}'^+) \hookrightarrow (P, \mathcal{X}^+)$ s.t. $P' = \text{MT}(\mathcal{X}'^+)$ and $\mathcal{X}'^+ \simeq \mathcal{X}^+$ under this embedding;
- (2) if (P, \mathcal{X}^+) has generic Mumford-Tate group, then P acts on U via a scalar. In particular, any subgroup of U is normal in P .

Proof. [53, 2.13, 2.14]. □

Definition 2.7. A **(Shimura) morphism** of connected mixed Shimura data $(Q, \mathcal{Y}^+) \rightarrow (P, \mathcal{X}^+)$ is a homomorphism $\varphi: Q \rightarrow P$ of algebraic groups over \mathbb{Q} which induces a map $\mathcal{Y}^+ \rightarrow \mathcal{X}^+$, $y \mapsto \varphi \circ y$. A Shimura morphism of connected mixed Shimura varieties is a morphism of varieties induced by a Shimura morphism of connected mixed Shimura data.

Fact 1. (cf [53, 2.9]) Let (P, \mathcal{X}^+) be a connected mixed Shimura datum and let P_0 be a normal subgroup of P . Then there exists a quotient connected mixed Shimura datum $(P, \mathcal{X}^+)/P_0$ and a morphism $(P, \mathcal{X}^+) \rightarrow (P, \mathcal{X}^+)/P_0$ unique up to isomorphism s.t. every morphism $(P, \mathcal{X}^+) \rightarrow (P', \mathcal{X}'^+)$, where the homomorphism $P \rightarrow P'$ factors through P/P_0 , factors in a unique way through $(P, \mathcal{X}^+)/P_0$. Such a Shimura morphism $(P, \mathcal{X}^+) \rightarrow (P, \mathcal{X}^+)/P_0$ is called a quotient Shimura morphism.

Notation 2.8. For convenience, we fix some notation here. Given a connected mixed Shimura datum (P, \mathcal{X}^+) , we always denote by $W = \mathcal{R}_u(P)$ the unipotent radical of P , $G := P/W$ the reductive part, $U \triangleleft P$ the weight -2 part, $V := W/U$ the weight -1 part and $(P/U, \mathcal{X}_{P/U}^+) := (P, \mathcal{X}^+)/U$ (resp. $(G, \mathcal{X}_G^+) := (P, \mathcal{X}^+)/W$) the corresponding connected mixed Shimura datum whose weight -2 part is trivial (resp. pure Shimura datum). If we have several connected mixed Shimura data, say (P, \mathcal{X}^+) and (Q, \mathcal{Y}^+) , then we distinguish the different parts associated with them by adding subscript W_P, W_Q, G_P, G_Q , etc. For a connected mixed Shimura variety S , we denote by $S_{P/U}$ (resp. S_G) its image under the Shimura morphism induced by $(P, \mathcal{X}^+) \rightarrow (P/U, \mathcal{X}_{P/U}^+)$ (resp. $(P, \mathcal{X}^+) \rightarrow (G, \mathcal{X}_G^+)$).

Proposition 2.9. Let $(Q, \mathcal{Y}) \xrightarrow{f} (P, \mathcal{X})$ be a Shimura morphism, then $f(W_Q) \subset W_P$ (resp. $f(U_Q) \subset f(U_P)$), and hence f induces

$$\bar{f} : (G_Q, \mathcal{Y}_{G_Q}) \rightarrow (G_P, \mathcal{X}_{G_P}) \quad (\text{resp. } \bar{f}' : (Q/Q_U, \mathcal{Y}_{Q/U_Q}) \rightarrow (P/U_P, \mathcal{X}_{P/U_P})).$$

Furthermore, if the underlying homomorphism of algebraic groups f is injective, then so are \bar{f} and \bar{f}' .

Proof. Since

$$\text{Lie } W_P = W_{-1}(\text{Lie } P) \quad \text{and} \quad \text{Lie } W_Q = W_{-1}(\text{Lie } Q),$$

by the following commutative diagram

$$\begin{array}{ccc} \text{Lie } W_Q & \longrightarrow & \text{Lie } W_P \\ \downarrow \wr \exp & & \downarrow \exp \\ W_Q & \xrightarrow{f} & P \end{array}$$

(here \exp is algebraic and is an isomorphism as a morphism between algebraic varieties because W_Q is unipotent), $f(W_Q) \subset W_P$.

Hence f induces a map $G_Q \rightarrow G_P$. Now the existence of \bar{f} follows from the universal property of the quotient Shimura datum ([53, 2.9]).

Furthermore, suppose now that f is injective. By Levi decomposition, the exact sequence

$$1 \rightarrow W_Q \rightarrow Q \xrightarrow{\pi_Q} G_Q \rightarrow 1$$

splits. Choose a splitting $s_Q : G_Q \rightarrow Q$, then we have the following diagram whose solid arrows commute:

$$\begin{array}{ccccccc} 1 & \longrightarrow & W_Q & \longrightarrow & Q & \xrightarrow{s_Q} & G_Q \longrightarrow 1 \\ & & \downarrow & & \downarrow f & \nearrow \pi_Q & \downarrow \bar{f} \\ 1 & \longrightarrow & W_P & \longrightarrow & P & \xrightarrow{\pi_P} & G_P \longrightarrow 1 \end{array},$$

where $\lambda := f \circ s_Q$. Then λ is injective since f, s_Q are. And $\pi_P \circ \lambda = \pi_P \circ f \circ s_Q = \bar{f} \circ \pi_Q \circ s_Q = \bar{f}$, so we have

$$\text{Ker}(\bar{f}) = G_Q \cap W_P$$

where the intersection is taken in P . $(G_Q \cap W_P)^\circ$ is smooth (since we are in the characteristic 0), connected unipotent (since it is in W_P) and normal in G_Q (since W_P is normal in P), so it is trivial since G_Q is reductive. So $G_Q \cap W_P$ is finite, hence trivial because W_P is unipotent over \mathbb{Q} . To sum it up, \bar{f} is injective.

The proof for the statements with U 's is similar. \square

2.2. Structure of the underlying group. (cf [53, 2.15]).

For a given connected mixed Shimura datum (P, \mathcal{X}^+) , we can associate to P a 4-tuple (G, V, U, Ψ) which is defined as follows:

- $G := P/\mathcal{R}_u(P)$ is the reductive part of P ;
- U is the normal subgroup of P as in Definition 2.1 and $V := \mathcal{R}_u(P)/U$. Both of them are vector groups with an action of G induced by conjugation in P (which factors through G for reason of weight);
- The commutator on $W := \mathcal{R}_u(P)$ induces a G -equivariant alternating form $\Psi: V \times V \rightarrow U$ by reason of weight as explained by Pink in [53, 2.15]. Moreover, Ψ is given by a polynomial with coefficients in \mathbb{Q} .

On the other hand, P is uniquely determined up to isomorphism by this 4-tuple in the following sense:

- let W be the central extension of V by U defined by Ψ . More concretely, $W = U \times V$ as a \mathbb{Q} -variety and the group law on W is (this can be proved using the Baker-Campbell-Hausdorff formula)

$$(u, v)(u', v') = (u + u' + \frac{1}{2}\Psi(v, v'), v + v');$$

- define the action of G on W by $g((u, v)) := (gu, gv)$;
- define $P := W \rtimes G$.

2.3. Siegel type. (cf [53, 2.7, 2.25] for mixed Shimura data of Siegel type and [53, 10.1-10.14] for mixed Shimura varieties of Siegel type)

Let $g \in \mathbb{N}_{>0}$. Let V_{2g} be a \mathbb{Q} -vector space of dimension $2g$ and let

$$\Psi: V_{2g} \times V_{2g} \rightarrow U_{2g} := \mathbb{G}_{a, \mathbb{Q}}$$

be a non-degenerate alternating form. Define

$$\text{GSp}_{2g} := \{g \in \text{GL}(V_{2g}) \mid \Psi(gv, gv') = \nu(g)\Psi(v, v') \text{ for some } \nu(g) \in \mathbb{G}_m\},$$

and \mathbb{H}_g the set of all homomorphisms

$$\mathbb{S} \rightarrow \text{GSp}_{2g, \mathbb{R}}$$

which induce a pure Hodge structure of type $\{(-1, 0), (0, -1)\}$ on V_{2g} and for which either Ψ or $-\Psi$ defines a polarization. Let \mathbb{H}_g^+ be the set of all such homomorphisms s.t. Ψ defines a polarization.

GSp_{2g} acts on U_{2g} by the scalar ν , which induces a pure Hodge structure of type $(-1, -1)$ on U_{2g} . Let W_{2g} be the central extension of V_{2g} by U_{2g} defined by Ψ , then the action of GSp_{2g} on W_{2g} induces a Hodge structure of type $\{(-1, 0), (0, -1), (-1, -1)\}$ on W_{2g} .

By [53, 2.16, 2.17], there exist connected mixed Shimura data $(P_{2g,a}, \mathcal{X}_{2g,a}^+)$ and $(P_{2g}, \mathcal{X}_{2g}^+)$, where $P_{2g,a} := V_{2g} \rtimes \mathrm{GSp}_{2g}$ and $P_{2g} := W_{2g} \rtimes \mathrm{GSp}_{2g}$.

Definition 2.10. *The connected mixed Shimura data $(\mathrm{GSp}_{2g}, \mathbb{H}_g^+)$, $(P_{2g,a}, \mathcal{X}_{2g,a}^+)$ and $(P_{2g}, \mathcal{X}_{2g}^+)$ are called **of Siegel type** (of genus g).*

Next we introduce connected mixed Shimura varieties of Siegel type. They have good modular interpretation ([53, 10.8-10.14]).

For $N \geq 4$ and even, define

$$(2.1) \quad \Gamma_{\mathrm{GSp}}(N) := \{g \in \mathrm{GSp}_{2g}(\mathbb{Z}) \mid g \equiv 1 \pmod{N}\}$$

and

$$\Gamma_W(N) := (N \cdot U_{2g}(\mathbb{Z})) \times (N \cdot V_{2g}(\mathbb{Z}))$$

under the identification $W \simeq U \times V$ in §2.2. $\Gamma_W(N)$ is indeed a subgroup of $W(\mathbb{Z})$ by the group operation (defined by Ψ). Let $\Gamma_V(N) := N \cdot V_{2g}(\mathbb{Z})$, and write

$$(2.2) \quad \mathcal{A}_g(N) := \Gamma_{\mathrm{GSp}}(N) \backslash \mathbb{H}_g^+$$

$$(2.3) \quad \mathfrak{A}_g(N) := (\Gamma_V(N) \rtimes \Gamma_{\mathrm{GSp}}(N)) \backslash \mathcal{X}_{2g,a}^+$$

$$(2.4) \quad \mathfrak{L}_g(N) := (\Gamma_W(N) \rtimes \Gamma_{\mathrm{GSp}}(N)) \backslash \mathcal{X}_{2g}^+,$$

Then $\mathcal{A}_g(N)$ is a moduli space of abelian varieties of dimension g with a level structure, $\mathfrak{A}_g(N) \rightarrow \mathcal{A}_g(N)$ is the universal family of abelian varieties (and hence a principally polarized abelian scheme of relative dimension g), and $\mathfrak{L}_g(N) \rightarrow \mathfrak{A}_g(N)$ is a \mathbb{G}_m -torsor which (up to replacing the \mathbb{G}_m -action by its inverse) corresponds to a relatively ample line bundle over $\mathfrak{A}_g(N) \rightarrow \mathcal{A}_g(N)$. For more details see [53, 10.5, 10.9, 10.10].

Definition 2.11. *The connected mixed Shimura varieties $\mathcal{A}_g(N)$, $\mathfrak{A}_g(N)$ and $\mathfrak{L}_g(N)$ are called **of Siegel type of level N** (and of genus g).*

Denote by $\mathrm{GSp}_0 := \mathbb{G}_m$ and $P_0 := \mathbb{G}_a \rtimes \mathbb{G}_m$ with the standard action of \mathbb{G}_m on \mathbb{G}_a . Pink proved the following lemma ([53, 2.26])

Lemma 2.12 (Reduction Lemma). *Let (P, \mathcal{X}^+) be a connected mixed Shimura datum with generic Mumford-Tate group.*

(1) *If V is trivial, then there exists an embedding*

$$(P, \mathcal{X}^+) \hookrightarrow (G_0, \mathcal{D}^+) \times \prod_{i=1}^r (P_0, \mathcal{X}_0^+)$$

where $r = \dim(U)$ (see [53, 2.8, 2.14] for definition of (P_0, \mathcal{X}_0^+));

(2) *If V is not trivial, then there exist a connected pure Shimura datum (G_0, \mathcal{D}^+) and Shimura morphisms*

$$(P', \mathcal{X}'^+) \twoheadrightarrow (P, \mathcal{X}^+)$$

$$\text{and } (P', \mathcal{X}'^+) \xhookrightarrow{\lambda} (G_0, \mathcal{D}^+) \times \prod_{i=1}^r (P_{2g}, \mathcal{X}_{2g}^+)$$

s.t. $\text{Ker}(P' \rightarrow P)$ is of dimension 1 and of weight -2. Moreover $\lambda|_V: V \simeq V_{2g} \rightarrow \bigoplus_{i=1}^r V_{2g}$ is the diagonal map, $\lambda|_{U'}: U' \simeq \bigoplus_{i=1}^r U_{2g}$ and $G \xrightarrow{\lambda|_G} G_0 \times \prod_{i=1}^r \text{GSp}_{2g} \rightarrow \text{GSp}_{2g}$ is non-trivial for each projection.

Proof. The statement except the last claim of the “Moreover” part is [53, 2.26 statement & pp 45]. For the last part, call $p_i: G \rightarrow \text{GSp}_{2g}$ the composite with the i -th projection. If p_i is trivial, then $p_i(P', \mathcal{X}'^+)$ is trivial since a connected mixed Shimura datum is trivial if its pure part is trivial. This contradicts the dimension of V . \square

2.4. Decomposition of V and U as G -modules. We start this subsection by the following group theoretical proposition.

Proposition 2.13. *Let $1 \rightarrow N \rightarrow Q \xrightarrow{\varphi} Q' \rightarrow 1$ be an exact sequence of algebraic groups over \mathbb{Q} . Then the following diagram with solid arrows is commutative and all the lines and columns are exact:*

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & W_N := \mathcal{R}_u(N) & \longrightarrow & N & \xrightarrow[\pi_N]{s_N} & G_N := N/W_N \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & W_Q := \mathcal{R}_u(Q) & \longrightarrow & Q & \xrightarrow[\pi_Q]{s_Q} & G_Q := Q/W_Q \longrightarrow 1 \\
 & & \downarrow & & \downarrow \varphi & & \downarrow \bar{\varphi} \\
 1 & \longrightarrow & W_{Q'} := \mathcal{R}_u(Q') & \longrightarrow & Q' & \xrightarrow[\pi_{Q'}]{s_{Q'}} & G_{Q'} := Q'/W_{Q'} \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 1 & & 1
 \end{array}$$

Moreover, if we fix a morphism s_Q which splits the middle line (such an s_Q exists by Levi decomposition), then we can deduce s_N and $s_{Q'}$ which split the other two lines. Note that in this case, the action of G_N on $W_{Q'}$ induced by s_Q is trivial.

Proof. The two bottom lines are already exact. By group theory, we know $\varphi(W_Q(\overline{\mathbb{Q}})) = W_{Q'}(\overline{\mathbb{Q}})$ ([14, Corollary 14.11]), and since the set of closed points of W_Q (resp. $W_{Q'}$) is dense on W_Q (resp. $W_{Q'}$), we have $\varphi(W_Q) = W_{Q'}$. In consequence, we have the map $\bar{\varphi}$, which is surjective since φ is. Now we get the solid diagram by snake-lemma. G_N is reductive ([14, 14.2 Corollary(b)]).

If we have an s_Q , then to get a desired $s_{Q'}$ (and s_N) is equivalent to prove that $\varphi \circ s_Q(G_N)$ is trivial, i.e. the intersection of this image with $W_{Q'}$ (in Q') is trivial and the projection of this image to $G_{Q'}$ (under $\pi_{Q'}$) is trivial. The projection is trivial by a simple diagram-chasing. The neutral component of the intersection is trivial since it is reductive and unipotent, and hence the intersection is trivial since $W_{Q'}$ is unipotent over \mathbb{Q} . Now the triviality of the action of G_N on $W_{Q'}$ induced by s_Q is automatic. \square

Corollary 2.14. *Let (P, \mathcal{X}^+) be a connected mixed Shimura datum. Suppose $N \triangleleft P$. Then there are decompositions*

$$V = V_N \oplus V_N^\perp \quad (\text{resp. } U = U_N \oplus U_N^\perp)$$

as G -modules, where $V_N := V \cap N$ (resp. $U_N := U \cap N$), s.t. the action of $G_N := N/V_N$ on V_N^\perp (resp. U_N^\perp) is trivial.

Proof. To prove the decomposition of V , apply Proposition 2.13 to the exact sequence

$$1 \rightarrow V_N \rtimes G_N \rightarrow V \rtimes G \rightarrow (V/V_N) \rtimes (G/G_N) \rightarrow 1,$$

then since G is reductive, the vertical line on the left (in the diagram of the proposition) splits. The conjugation by P on V factors through G by reason of weights, and hence equals to the action of G on V induced by any Levi decomposition s_P . So the action of G_N on V_N^\perp is trivial by the last assertion of Proposition 2.13.

To prove the decomposition of U , it suffices to apply Proposition 2.13 to the exact sequence

$$1 \rightarrow U_N \rtimes G_N \rightarrow U \rtimes G \rightarrow (U/U_N) \rtimes (G/G_N) \rightarrow 1.$$

□

In fact we have a better result if (P, \mathcal{X}^+) has generic Mumford-Tate group.

Proposition 2.15. *Let (P, \mathcal{X}^+) be a connected mixed Shimura datum s.t. $P = \text{MT}(\mathcal{X}^+)$. Suppose $N \triangleleft P$ s.t. N possesses no non-trivial torus quotient. Then G_N acts trivially on U .*

Proof. By Reduction Lemma (Lemma 2.12), we may assume that $(P, \mathcal{X}^+) \hookrightarrow (G_0, \mathcal{D}^+) \times \prod_{i=1}^r (P_{2g}, \mathcal{X}_{2g}^+)$ ($g \geq 0$). Since N possesses no non-trivial torus quotient, G_N is semi-simple (the last line of the proof of Proposition 5.4). So

$$G_N = G_N^{\text{der}} < G^{\text{der}} < (G_0 \times \prod_{i=1}^r \text{GSp}_{2g})^{\text{der}} = G_0^{\text{der}} \times \prod_{i=1}^r \text{Sp}_{2g}$$

where $\text{Sp}_0 := 1$. Hence G_N acts trivially on U since $G_0^{\text{der}} \times \prod_{i=1}^r \text{Sp}_{2g}$ acts trivially on $\oplus_{i=1}^r U_{2g}$. □

3. VARIATIONS OF MIXED (\mathbb{Z}) -HODGE STRUCTURE

3.1. Arbitrary variation of mixed Hodge structure.

Definition 3.1. ([44, Definition 14.44]) *Let S be a complex manifold. A **variation of mixed Hodge structure** on S is a triple $(\mathbb{V}, W_\bullet, \mathcal{F})$ with*

- (1) *a local system \mathbb{V} of free \mathbb{Z} -module of finite rank on S ;*
- (2) *a finite increasing filtration $\{W_m\}$ of the local system $\mathbb{V}_\mathbb{Q} := \mathbb{V} \otimes_\mathbb{Z} \mathbb{Q}$ by local sub-systems (this is called the **weight filtration**);*
- (3) *a finite decreasing filtration $\{\mathcal{F}^p\}$ of the holomorphic vector bundle $\mathcal{V} := \mathbb{V} \otimes_\mathbb{Z} \mathcal{O}_S$ by holomorphic sub-bundles (this is called the **Hodge filtration**).*

s.t.

- (1) *for each $s \in S$, the filtrations $\{\mathcal{F}^p(s)\}$ and $\{W_m\}$ of $\mathbb{V}(s) \simeq \mathbb{V}_s \otimes_\mathbb{Z} \mathbb{C}$ define a mixed Hodge structure on the \mathbb{Z} -module of finite rank \mathbb{V}_s ;*
- (2) *the connection $\nabla: \mathcal{V} \rightarrow \mathcal{V} \otimes_{\mathcal{O}_S} \Omega_S^1$ ($\mathcal{V} := \mathbb{V} \otimes_\mathbb{Z} \mathcal{O}_S$) whose sheaf of horizontal sections is $\mathbb{V}_\mathbb{C}$ satisfies the Griffiths' transversality condition*

$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega_S^1.$$

Definition 3.2. *A variation of mixed Hodge structure on S is called **(graded-)polarizable** if the induced variations of pure Hodge structure $\mathrm{Gr}_k^W \mathbb{V}$ are all polarizable, i.e. for each k , there exists a flat morphism of variations*

$$Q_k: \mathrm{Gr}_k^W \mathbb{V} \otimes \mathrm{Gr}_k^W \mathbb{V} \rightarrow \mathbb{Z}(-k)_S$$

which induces on each fibre a polarization of the corresponding Hodge structure of weight k .

Let $\pi: \tilde{S} \rightarrow S$ be a universal covering and choose a trivialization $\pi^* \mathbb{V} \simeq \tilde{S} \times V$. For $s \in S$, $\mathrm{MT}_s \subset \mathrm{GL}(\mathcal{V}_s)$ denote the Mumford-Tate group of its fibre. The choice of a point $\tilde{s} \in \tilde{S}$ with $\pi(\tilde{s}) = s$ gives an identification $\mathcal{V}_s \simeq V$, whence an injective homomorphism $i_{\tilde{s}}: \mathrm{MT}_s \hookrightarrow \mathrm{GL}(V)$. By [1, §4, Lemma 4], on $S^\circ := S \setminus \Sigma$ where Σ is a meager subset of S , $M := \mathrm{Im}(i_{\tilde{s}}) \subset \mathrm{GL}(V)$ does not depend on s , nor on the choice of \tilde{s} . We call S° the “Hodge-generic” locus and the group M the *generic Mumford-Tate group* of $(\mathbb{V}, W, \mathcal{F})$.

On the other hand, if we choose a base-point $s \in S$ and a point $\tilde{s} \in \tilde{S}$ with $\pi(\tilde{s}) = s$, then the local system \mathbb{V} corresponds to a representation $\rho: \pi_1(S, s) \rightarrow \mathrm{GL}(V)$, called the monodromy representation. The algebraic monodromy group is defined as the smallest algebraic subgroup of $\mathrm{GL}(V)$ over \mathbb{Q} which contains the image of ρ . We write H_s^{mon} for its connected component of the identity, called the *connected algebraic monodromy group*. Given the trivialization of $\pi^* \mathbb{V}$, the group $H_s^{\mathrm{mon}} \subset \mathrm{GL}(V)$ is independent of the choice of s and \tilde{s} .

Suppose now that $(\mathbb{V}, W, \mathcal{F})$ is (graded-)polarizable, then $H_s^{\mathrm{mon}} < M$ for any $s \in S^\circ$ by [1, §4, Lemma 4].

3.2. Admissible variations of mixed Hodge structure. We now recall the concept of “admissible” variations of mixed Hodge structure which was introduced by Steenbrink-Zucker [58] and studied by Kashiwara [34] and Hain-Zucker [32]. We give the definition here, but instead of the exact definition, we shall only use the notion of “admissibility” and the fact that it can be defined using “curve test”. We will use Δ (resp. Δ^*) to denote the unit disc (resp. punctured unit disc).

Definition 3.3. (see [44, Definition 14.49])

- (1) *A variation of mixed Hodge structure $(\mathbb{V}, W, \mathcal{F})$ over the punctured unit disc Δ^* is called **admissible** if*
 - *it is (graded-)polarizable;*
 - *the monodromy T is unipotent and the weight filtration $M(N, W)$ of $N := \log T$ relative to W exists;*
 - *the filtration \mathcal{F} extends to a filtration $\tilde{\mathcal{F}}$ of $\tilde{\mathbb{V}}$ which induced ${}^k \tilde{\mathcal{F}}$ on $\mathrm{Gr}_k^W \tilde{\mathbb{V}}$ for each k .*
- (2) *Let S be a smooth connected complex algebraic variety and let \bar{S} be a compactification of S s.t. $\bar{S} \setminus S$ is a normal crossing divisor. A (graded-)polarizable variation of mixed Hodge structure $(\mathbb{V}, W, \mathcal{F})$ on S is called **admissible** if for every holomorphic map $i: \Delta \rightarrow \bar{S}$ which maps Δ^* to S and s.t. $i^* \mathbb{V}$ has unipotent monodromy, the variation $i^*(\mathbb{V}, W, \mathcal{F})$ is admissible. (This definition is sometimes called the “curve test” version).*

Remark 3.4. *This definition is equivalent to the one given in [32, 1.5]. See [58, Properties 3.13 & Appendix], [34, §1 & Theorem 4.5.2] and [32, 1.5] for details.*

The following lemma is an easy property of admissibility and is surely known by many people, but I cannot find any reference, so I give a proof here.

Lemma 3.5. *Let S be a smooth connected complex algebraic variety and let $(\mathbb{V}, W, \mathcal{F})$ be an admissible variation of mixed Hodge structure on S . Then for any smooth connected (not necessarily closed) subvariety $j: Y \hookrightarrow S$, $j^*(\mathbb{V}, W, \mathcal{F})$ is also admissible on Y .*

Proof. Take smooth compactifications \bar{Y} of Y and \bar{S} of S s.t. $\bar{Y} \setminus Y$ and $\bar{S} \setminus S$ are normal crossing divisors and s.t. $j: Y \hookrightarrow S$ extends to a morphism $\bar{j}: \bar{Y} \rightarrow \bar{S}$. This can be done by first choosing any compactifications of Y^{cp} of Y and S^{cp} of S with normal crossing divisors and then taking a suitable resolution of singularities of the closure of the graph of j in $Y^{\text{cp}} \times S^{\text{cp}}$. Now the conclusion follows from our “curve test” version of the definition. \square

3.3. Consequences of admissibility. Y.André proved that:

Theorem 3.6. *Let $(\mathbb{V}, W, \mathcal{F})$ be an admissible variation of mixed Hodge structure over a smooth connected complex algebraic variety S , then for any $s \in S$, the connected monodromy group H_s^{mon} is a normal subgroup of the generic Mumford-Tate group M and also its derived group M^{der} .*

Proof. [1, §5, Theorem 1] states that $H_s^{\text{mon}} \triangleleft M^{\text{der}}$, and in the proof he first proved that $H_s^{\text{mon}} \triangleleft M$. \square

Now we state a theorem which roughly says that all the variations of mixed Hodge structure obtained from representations of the underlying group of a connected mixed Shimura datum are admissible. Explicitly, let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum (P, \mathcal{X}^+) and let $\text{unif}: \mathcal{X}^+ \rightarrow S = \Gamma \backslash \mathcal{X}^+$ be the uniformization. Suppose that Γ is neat. Consider any \mathbb{Q} -representation $\xi: P \rightarrow \text{GL}(V)$. By [55, Proposition 4.2], there exists a Γ -invariant lattice $V_{\mathbb{Z}}$ of V . ξ and $V_{\mathbb{Z}}$ together give rise to a VMHS on S whose underlying local system is $\Gamma \backslash (\mathcal{X}^+ \times V_{\mathbb{Z}})$. This variation is (graded-)polarizable by [53, 1.18(d)].

Theorem 3.7. *Suppose that S , (P, \mathcal{X}^+) , $\xi: P \rightarrow \text{GL}(V)$ and $V_{\mathbb{Z}}$ are as in the previous paragraph, then the variation of mixed Hodge structure obtained as above is admissible.*

Proof. [67, Theorem 2.2] says that the corresponding \mathbb{Q} -variation is admissible, and Γ gives a \mathbb{Z} -structure as in the discussion above. \square

Remark 3.8. *In this language, we can rephrase Definition 2.5 as: P is the generic Mumford-Tate group (of the variation in Theorem 3.7). It is clear that for any Hodge generic point $x \in \mathcal{X}^+$, the only \mathbb{Q} -subgroup N of P^{der} s.t. $N(\mathbb{R})^+ U_N(\mathbb{C})$, where $U_N := U \cap N$, stabilizes x is the trivial group.*

4. REALIZATION OF \mathcal{X}^+

Let (P, \mathcal{X}^+) be a connected mixed Shimura datum. We first define the dual \mathcal{X}^{\vee} of \mathcal{X}^+ (see [53, 1.7(a)] or [38, Chapter VI, Proposition 1.3]):

Let M be a faithful representation of P and take any $x_0 \in \mathcal{X}^+$. The weight filtration on M is constant, so the Hodge filtration $x \mapsto \text{Fil}_x(M_{\mathbb{C}})$ gives an injective map $\mathcal{X}^+ \hookrightarrow \text{Grass}(M)(\mathbb{C})$ to a certain flag variety. In fact, this injective map factors through

$$\mathcal{X}^+ = P(\mathbb{R})^+ U(\mathbb{C}) / C(x_0) \hookrightarrow P(\mathbb{C}) / F_{x_0}^0 P(\mathbb{C}) \hookrightarrow \text{Grass}(M)(\mathbb{C})$$

where $C(x_0)$ is the stabilizer of x_0 in $P(\mathbb{R})^+ U(\mathbb{C})$. The first injection is an open immersion ([53, 1.7(a)] or [38, Chapter VI, (1.2.1)]). We define the dual \mathcal{X}^{\vee} of \mathcal{X}^+ to be

$$\mathcal{X}^{\vee} := P(\mathbb{C}) / F_{x_0}^0 P(\mathbb{C}).$$

Then \mathcal{X}^\vee is clearly a connected smooth complex algebraic variety.

Proposition 4.1. *Under the open immersion $\mathcal{X}^+ \hookrightarrow \mathcal{X}^\vee$, \mathcal{X}^+ is realized as a semi-algebraic set which is also a complex manifold.*

Proof. \mathcal{X}^+ is smooth since it is a homogeneous space, and the open immersion endows it with a complex structure. For semi-algebraicity, consider the diagram

$$\begin{array}{ccc} \mathcal{X}^+ & \hookrightarrow & \mathcal{X}^\vee \\ \pi \downarrow & & \downarrow \pi^\vee \\ \mathcal{X}_G^+ & \hookrightarrow & \mathcal{X}_G^\vee \end{array}.$$

Now $\mathcal{X}^+ = \{x \in \mathcal{X}^\vee \mid \pi^\vee(x) \in \mathcal{X}_G^+\}$ and π^\vee is algebraic, so the conclusion follows from [61, Lemme 2.1]. \square

Remark 4.2. *It is not hard to see that \mathcal{X}^\vee is a projective variety if and only if (P, \mathcal{X}^+) is pure. The argument is as follows: \mathcal{X}^\vee is a holomorphic vector bundle over \mathcal{X}_G^\vee where the fibre is homeomorphic to $W(\mathbb{R})U(\mathbb{C})$. But \mathcal{X}_G^\vee is projective, so \mathcal{X}^\vee is projective if and only if it is a trivial vector bundle over \mathcal{X}_G^\vee , i.e. if and only if W is trivial.*

Let us take a closer look at the semi-algebraic structure of \mathcal{X}^+ . By [67, pp 6], there exists a Shimura morphism $i: (G, \mathcal{X}_G^+) \rightarrow (P, \mathcal{X}^+)$ s.t. $\pi \circ i = \text{id}$. Then i defines a Levi decomposition of $P = W \rtimes G$. By definition $\mathcal{X}^+ \subset \text{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$. Define a bijective map

$$\begin{aligned} W(\mathbb{R})U(\mathbb{C}) \times \mathcal{X}_G^+ &\longrightarrow \mathcal{X}^+ \\ (w, x) &\mapsto \text{int}(w) \circ i(x) \end{aligned}$$

Identify P with the 4-tuple (G, V, U, Ψ) as in §2.2. Since $W \simeq U \times V$ as \mathbb{Q} -varieties, we can define a bijection induced by the one above

$$(4.1) \quad \rho: U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}_G^+ \xrightarrow{\sim} \mathcal{X}^+$$

$P(\mathbb{R})^+U(\mathbb{C})$ acts on \mathcal{X}^+ by definition. There is also a natural action of $P(\mathbb{R})^+U(\mathbb{C})$ on $U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}_G^+$ which is defined as follows. Under the notation of §2.2, for any $(u, v, g) \in P(\mathbb{R})^+U(\mathbb{C})$ and $(u', v', x) \in U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}_G^+$,

$$(4.2) \quad (u, v, g) \cdot (u', v', x) := (u + gu' + \frac{1}{2}\Psi(v, v'), v + gv', gx).$$

This action is algebraic since Ψ is a polynomial over \mathbb{Q} (see §2.2). The morphism ρ is $P(\mathbb{R})^+U(\mathbb{C})$ -equivariant by an easy calculation.

Proposition 4.3. *The map ρ is semi-algebraic.*

Proof. It is enough to prove that the graph of ρ is semi-algebraic. This is true since ρ is $P(\mathbb{R})^+U(\mathbb{C})$ -equivariant and the actions of $P(\mathbb{R})^+U(\mathbb{C})$ on both sides are algebraic and transitive. Explicitly, fix a point $x_0 \in U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}_G^+$, the graph of ρ

$$\begin{aligned} \text{Gr}(\rho) &= \{(gx_0, \rho(gx_0)) \in (U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}_G^+) \times \mathcal{X}^+ \mid g \in P(\mathbb{R})^+U(\mathbb{C})\} && \text{(transitivity)} \\ &= \{(gx_0, g\rho(x_0)) \in (U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}_G^+) \times \mathcal{X}^+ \mid g \in P(\mathbb{R})^+U(\mathbb{C})\} && \text{(equivariance)} \\ &= P(\mathbb{R})^+U(\mathbb{C}) \cdot (x_0, \rho(x_0)) \end{aligned}$$

is semi-algebraic since the action of $P(\mathbb{R})^+U(\mathbb{C})$ on $(U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}_G^+) \times \mathcal{X}^+$ is algebraic. \square

Remark 4.4. If U is trivial, then the complex structure of \mathcal{X}^+ given by \mathcal{X}^\vee is the same as the one given by [54, Construction 2.9] since for the projection $\mathcal{X}^+ \xrightarrow{\pi} \mathcal{X}_G^+$, the complex structure of any fibre $\mathcal{X}_{x_G}^+$ ($x_G \in \mathcal{X}_G^+$) given by \mathcal{X}^\vee is the same as the one obtained from $\mathcal{X}_{x_G}^+ \simeq V(\mathbb{C})/F_{x_G}^0 V(\mathbb{C})$ (see [53, 3.13, 3.14]). In particular this holds for $\mathcal{X}_{2g,a}^+$ (see §2.3 for notation). Therefore for any $\mathfrak{A}_g(N)$, the fundamental set $[0, N)^{2g} \times \mathcal{F}_G \subset V_{2g}(\mathbb{R}) \times \mathbb{H}_g^+ \simeq \mathcal{X}_{2g,a}^+$ is the one considered in [45].

5. (WEAKLY) SPECIAL SUBVARIETIES

5.1. Weakly special subvarieties.

Definition 5.1. (Pink, [54, Definition 4.1(b)]) Let S be a connected mixed Shimura variety. Consider any Shimura morphisms $T' \xleftarrow{[\varphi]} T \xrightarrow{[i]} S$ and any point $t' \in T'$. Then any irreducible component of $[i]([\varphi]^{-1}(t'))$ is called a **weakly special subvariety** of S . We will prove later in Remark 5.5 that weakly special subvarieties of S are indeed closed subvarieties.

Since any Shimura morphism is related to a Shimura morphism between Shimura data, we will try to rephrase this definition in the context of Shimura data:

Definition 5.2. Given a connected mixed Shimura datum (P, \mathcal{X}^+) , a weakly special subset of \mathcal{X}^+ is a connected component of $i(\varphi^{-1}(y')) \subset \mathcal{X}^+$ for a point $y' \in \mathcal{Y}'^+$, where $i, \varphi, \mathcal{Y}'^+$ are in the following diagram of Shimura morphisms

$$\begin{array}{ccc} & (Q, \mathcal{Y}^+) & \\ \varphi \swarrow & & \searrow i \\ (Q', \mathcal{Y}'^+) & & (P, \mathcal{X}^+) \end{array}$$

Remark 5.3. (1) In the definition above, let $N := \text{Ker}(Q \rightarrow Q')$ and let $U_N := U_Q \cap N$, then $i(\varphi^{-1}(y'))$ is a connected component of $N(\mathbb{R})U_N(\mathbb{C})y$ where $\varphi(y) = y'$. So $i(\varphi^{-1}(y'))$ is smooth as an analytic variety. In particular, its connected components and complex analytic irreducible components coincide. As a result, we can replace “a connected component” by “a complex analytic irreducible component” in Definition 5.2.

(2) If furthermore N is connected, then $i(\varphi^{-1}(y'))$ itself is connected (hence also complex analytic irreducible). The proof is as follows: Consider the image of $\varphi^{-1}(y')$ under the projection $(Q, \mathcal{Y}^+) \xrightarrow{\pi} (G_Q, \mathcal{Y}_{G_Q}^+) := (Q, \mathcal{Y}^+)/W_Q$. By the decomposition ([40, 3.6])

$$(G_Q^{\text{ad}}, \mathcal{Y}_{G_Q}^+) = (G_N^{\text{ad}}, \mathcal{Y}_1^+) \times (G_2, \mathcal{Y}_2^+)$$

where $G_N := N/W \cap N$, $\pi(\varphi^{-1}(y')) = \mathcal{Y}_1^+ \times \{y_2\}$. So $\pi(\varphi^{-1}(y')) = G_N(\mathbb{R})^+ \pi(y)$. But $W_N(\mathbb{R})U_N(\mathbb{C})$ ($W_N := W \cap N$) is connected, hence $\varphi^{-1}(y') = N(\mathbb{R})^+U_N(\mathbb{C})y$, which is connected. In consequence, $i(\varphi^{-1}(y'))$ also is connected.

Proposition 5.4. For any weakly special subvariety of S (resp. weakly special subset of \mathcal{X}^+), the Shimura morphisms in Definition 5.1 (resp. Definition 5.2) can be chosen such that

- the underlying homomorphism of algebraic groups i is injective, and hence i is an embedding in the sense of [53, 2.3];

- the underlying homomorphism of algebraic groups φ is surjective, and its kernel N is connected. Moreover, N possesses no non-trivial torus quotient (or equivalently, $G_N := N/(W \cap N)$ is semi-simple);
- φ is a quotient Shimura morphism.

Proof. If $P = \text{MT}(\mathcal{X}^+)$, then the first two points except the statement in the bracket are proved by [54, Proposition 4.4]. The general case follows directly from Proposition 2.6(1). The third assertion can be proved by the universal property of quotient Shimura data ([53, 2.9]). Now we are left to prove the statement in the bracket.

$G_N \triangleleft G$ since $G_N = N/(W \cap N) \hookrightarrow G = P/W$ and $N \triangleleft P$, and hence G_N is reductive ([14, 14.2, Corollary(b)]). By [14, 14.2 Proposition(2)], G_N is the almost-product of G_N^{der} and $Z(G_N)^\circ$, and $Z(G_N)^\circ$ equals the radical of G_N which is a torus. So N possesses no non-trivial torus quotient iff G_N possesses no non-trivial torus quotient iff G_N is semi-simple. \square

Remark 5.5. We can now prove that weakly special subvarieties of S are closed. By the proposition above, we can choose i to be injective. Then $[i]$ is finite by [53, 3.8]. Hence $[i](\varphi^{-1}(t'))$ is closed.

Lemma 5.6. Suppose that the Shimura morphisms $T' \xleftarrow{[\varphi]} T \xrightarrow{[i]} S$ are associated with the morphisms of mixed Shimura data

$$(Q', \mathcal{Y}^+) \xleftarrow{\varphi} (Q, \mathcal{Y}^+) \xrightarrow{i} (P, \mathcal{X}^+)$$

so that we have the following commutative diagram

$$\begin{array}{ccccc} \mathcal{Y}'^+ & \xleftarrow{\varphi} & \mathcal{Y}^+ & \xrightarrow{i} & \mathcal{X}^+ \\ \text{unif}_{\mathcal{Y}'^+} \downarrow & & \text{unif}_{\mathcal{Y}^+} \downarrow & & \text{unif}_{\mathcal{X}^+} \downarrow \\ T' = \Delta' \backslash \mathcal{Y}'^+ & \xleftarrow{[\varphi]} & T = \Delta \backslash \mathcal{Y}^+ & \xrightarrow{[i]} & S = \Gamma \backslash \mathcal{X}^+ \end{array},$$

then for any point $y' \in \mathcal{Y}'^+$, any irreducible component of $\text{unif}_{\mathcal{X}^+}(i(\varphi^{-1}(y')))$ is also an irreducible component of $[i](\varphi^{-1}(\text{unif}_{\mathcal{Y}^+}(y')))$.

Proof. Let $N := \text{Ker}(\varphi)$ and let U_Q be the weight -2 part of Q , then we have

$$\text{unif}_{\mathcal{X}^+}(i(\varphi^{-1}(y')))) \subset [i](\varphi^{-1}(\text{unif}_{\mathcal{Y}^+}(y'))),$$

and both of them are of constant dimension d , where d is the dimension of any orbit of $N(\mathbb{R})^+(U_Q \cap N)(\mathbb{C})$. This allows us to conclude. \square

The following Proposition tells us that the two definitions of weak specialness are compatible.

Proposition 5.7. Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum (P, \mathcal{X}^+) and let $\text{unif}: \mathcal{X}^+ \rightarrow S = \Gamma \backslash \mathcal{X}^+$ be the uniformization, then a subvariety Z of S is weakly special if and only if Z is the image of some weakly special subset of \mathcal{X}^+ .

Proof. The “if” part is immediate by Lemma 5.6. We prove the “only if” part. We assume that i, φ are as in Proposition 5.4. For any weakly special subvariety $Z \subset S$, suppose that we have

a diagram as in Lemma 5.6 and Z is an irreducible component of $[i](\varphi^{-1}(t'))$. Since

$$[i](\varphi^{-1}(t')) \subset \bigcup_{y' \in \text{unif}_{\mathcal{Y}^+}^{-1}(t')} \text{unif}_{\mathcal{X}^+}(i(\varphi^{-1}(y'))) = \text{unif}_{\mathcal{X}^+}(i(\varphi^{-1}(\text{unif}_{\mathcal{Y}^+}^{-1}(t')))),$$

there exists a $y' \in \mathcal{Y}^+$ lying over t' s.t. Z is an irreducible component of $\text{unif}_{\mathcal{X}^+}(i(\varphi^{-1}(y')))$ by Lemma 5.6. By Remark 5.3.2, $i(\varphi^{-1}(y'))$ is complex analytic irreducible, so $\text{unif}_{\mathcal{X}^+}(i(\varphi^{-1}(y')))$ is also complex analytic irreducible when S is regarded as an analytic variety. Hence $\text{unif}_{\mathcal{X}^+}(i(\varphi^{-1}(y')))$ is irreducible as an algebraic variety. So $Z = \text{unif}_{\mathcal{X}^+}(i(\varphi^{-1}(y')))$. \square

We close this subsection by proving that this definition of weakly special subvarieties is compatible with the one (which is already known) for pure Shimura varieties.

Proposition 5.8. *A weakly special subvariety of a pure Shimura variety S is a subvariety of the same form as in [62, Definition 2.1].*

Proof. This is pointed out in [54, Remark 4.5]. We give a (relatively) detailed proof here. We prove the result for weakly special subsets. Assume that S is associated with the connected pure Shimura datum (P, \mathcal{X}^+) . For a subset of the same form as in [62, Definition 2.1], take $(Q, \mathcal{Y}^+) = (H, X_H^+)$ and $(Q', \mathcal{Y}'^+) = (H_1, X_1^+)$ (same notation as [62, Definition 2.1]). Then by definition such a subset is weakly special (as in Definition 5.2).

On the other hand, suppose that we have a weakly special subset \tilde{F} defined by a diagram as in Definition 5.2 satisfying Proposition 5.4. Let $N := \text{Ker}(\varphi)$, then the homogeneous spaces of the connected pure Shimura data $(Q', \mathcal{Y}'^+) = (Q, \mathcal{Y}^+)/N$ and $(Q, \mathcal{Y}^+)/Z(Q)N = (Q^{\text{ad}}, \mathcal{Y}^{\text{ad}+})/N^{\text{ad}}$ are canonically isomorphic to each other ([39, Proposition 5.7]). Hence we may replace (Q', \mathcal{Y}'^+) by $(Q^{\text{ad}}, \mathcal{Y}^{\text{ad}+})/N^{\text{ad}}$. But by [40, 3.6, 3.7], $(Q^{\text{ad}}, \mathcal{Y}^{\text{ad}+}) = (N^{\text{ad}}, \mathcal{Y}_1^+) \times (Q_2, \mathcal{Y}_2^+)$. So \tilde{F} is of the same form as in [62, Definition 2.1]. \square

5.2. Special subvarieties.

Definition 5.9. *Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum (P, \mathcal{X}^+) .*

- (1) A **special subvariety** of S is the image of any Shimura morphism $T \rightarrow S$ of connected mixed Shimura varieties;
- (2) A point $x \in \mathcal{X}^+$ and its image in S are called **special** if the homomorphism $x : \mathbb{S}_{\mathbb{C}} \rightarrow P_{\mathbb{C}}$ factors through $T_{\mathbb{C}}$ for a torus $T \subset P$.

Remark 5.10. *By definition, $x \in \mathcal{X}^+$ is special if and only if it is the image of a Shimura morphism $(T, \mathcal{Y}^+) \hookrightarrow (P, \mathcal{X}^+)$. Hence a special point is just a special subvariety of dimension 0.*

The following result is easy to prove.

Lemma 5.11. *Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum (P, \mathcal{X}^+) and let $\text{unif} : \mathcal{X}^+ \rightarrow S$ be the uniformizing map, then a subvariety of S is special if and only if it is of the form $\text{unif}(\mathcal{Y}^+)$ for some $(Q, \mathcal{Y}^+) \hookrightarrow (P, \mathcal{X}^+)$.*

Proposition 5.12. *Every special subvariety of S contains a Zariski dense subset of special points.*

Proof. [54, Proposition 4.14]. \square

The relation between special and weakly special subvarieties is:

Proposition 5.13. *A subvariety of S is special if and only if it is weakly special and contains one special point.*

Proof. [54, Proposition 4.2, Proposition 4.14, Proposition 4.15]. \square

6. ALGEBRAICITY IN THE UNIFORMIZING SPACE

Definition 6.1. *Let \tilde{Y} be an analytic subset of \mathcal{X}^+ , then*

- (1) *\tilde{Y} is called an irreducible algebraic subset of \mathcal{X}^+ if it is a complex analytic irreducible component of the intersection of its Zariski closure in \mathcal{X}^\vee and \mathcal{X}^+ ;*
- (2) *\tilde{Y} is called algebraic if it is a finite union of irreducible algebraic subsets of \mathcal{X}^+ .*

Lemma 6.2. *Any weakly special subset of \mathcal{X}^+ is irreducible algebraic.*

Proof. Suppose that \tilde{Z} is a weakly special subset of \mathcal{X}^+ . Use the notation of Definition 5.2 and assume that i and φ satisfy the properties in Proposition 5.4. Let $N := \text{Ker}(Q \rightarrow Q')$ and let y be a point of the weakly special subset, then $\tilde{Z} = N(\mathbb{R})^+ U_N(\mathbb{C})y$ is complex analytic irreducible by Remark 5.3.2. But $N(\mathbb{R})^+ U_N(\mathbb{C})y = N(\mathbb{C})y \cap \mathcal{X}^+$ and $N(\mathbb{C})y$ is algebraic, so \tilde{Z} is irreducible algebraic by definition. \square

Lemma 6.3 (functoriality of algebraicity). *Let $f: (Q, \mathcal{Y}^+) \rightarrow (P, \mathcal{X}^+)$ be a Shimura morphism. Then there exists a unique morphism $f^\vee: \mathcal{Y}^\vee \rightarrow \mathcal{X}^\vee$ of algebraic varieties such that the diagram commutes:*

$$\begin{array}{ccc} \mathcal{Y}^+ & \hookrightarrow & \mathcal{Y}^\vee \\ f \downarrow & & f^\vee \downarrow \\ \mathcal{X}^+ & \hookrightarrow & \mathcal{X}^\vee \end{array}.$$

Furthermore, for any irreducible algebraic subset \tilde{Z} of \mathcal{Y}^+ , the closure w.r.t the archimedean topology of $f(\tilde{Z})$ is irreducible algebraic in \mathcal{X}^+ and $f(\tilde{Z})$ contains a dense open subset of this closure.

In particular, if f is an embedding, then an irreducible algebraic subset of \mathcal{Y}^+ is an irreducible component of the intersection of an irreducible algebraic subset of \mathcal{X}^+ with \mathcal{Y}^+ .

Proof. Fix a point $x_0 \in \mathcal{Y}^+$, then we have

$$\begin{array}{ccc} \mathcal{Y}^+ = Q(\mathbb{R})^+ U_Q(\mathbb{C})/C(x_0) & \hookrightarrow & \mathcal{Y}^\vee = Q(\mathbb{C})/F_{x_0}^0 Q(\mathbb{C}) \\ f \downarrow & & f^\vee \downarrow \\ \mathcal{X}^+ = P(\mathbb{R})^+ U_P(\mathbb{C})/C(f(x_0)) & \hookrightarrow & \mathcal{X}^\vee = P(\mathbb{C})/F_{f(x_0)}^0 P(\mathbb{C}) \end{array},$$

where $C(x_0)$ (resp. $C(f(x_0))$) denotes the stabilizer of x_0 (resp. $f(x_0)$) in $Q(\mathbb{R})U_Q(\mathbb{C})$ (resp. $P(\mathbb{R})U_P(\mathbb{C})$). f^\vee is unique since $Q(\mathbb{R})U_Q(\mathbb{C})/C(x_0)$ is dense in \mathcal{Y}^\vee .

To prove the second statement, it is enough to prove the result for $f^\vee(\overline{\tilde{Z}}^{\text{Zar}}) \subset \mathcal{X}^\vee$ where $\overline{\tilde{Z}}^{\text{Zar}}$ is the Zariski closure of \tilde{Z} in \mathcal{Y}^\vee . This is then an algebro-geometric result, which follows easily from Chevalley's Theorem ([29, Chapitre IV, 1.8.4]) and [41, I.10, Theorem 1]. \square

7. RESULTS FOR THE UNIPOTENT PART

Given a connected mixed Shimura variety S , let S_G be its pure part so that we have a projection $S \rightarrow S_G$. For any point $b \in S_G$, denote by E the fiber S_b . Suppose that S is associated with the mixed Shimura datum (P, \mathcal{X}^+) , which can be further assumed to be irreducible by Proposition 2.6. Let $\text{unif}: \mathcal{X}^+ \rightarrow S = \Gamma \backslash \mathcal{X}^+$ be the uniformization. Now $E = S_b \simeq \Gamma_W \backslash W(\mathbb{R})U(\mathbb{C})$ with the complex structure determined by $b \in S_G$ ($E = S_b = \Gamma_W \backslash W(\mathbb{C})/F_b^0 W(\mathbb{C})$), where $\Gamma_W := \Gamma \cap W(\mathbb{Q})$. Write $T := \Gamma_U \backslash U(\mathbb{C})$ and $A := \Gamma_A \backslash V(\mathbb{C})/F_b^0 V(\mathbb{C})$ where $\Gamma_U := \Gamma \cap U(\mathbb{Q})$ and $\Gamma_V := \Gamma_W/\Gamma_U$, then T is an algebraic torus over \mathbb{C} , A is a complex abelian variety and E is an algebraic torus over A whose fibers are isomorphic to T .

Lemma 7.1. *If E can be given the structure of an algebraic group whose group law is compatible with that of W , then W (hence E) is commutative. In this case E is a semi-abelian variety.*

Proof. If E is an algebraic group, then T is a normal subgroup of E . Hence E acts on T by conjugation, and this action factors via A , and then it is trivial by [14, 8.10 Proposition]. Therefore T is in the center of E . Now consider the commutator pairing $E \times E \rightarrow E$. This factors through a morphism $A \times A \xrightarrow{f} T$. But this morphism is then constant. So the commutator pairing $E \times E \rightarrow E$ is trivial, and hence E is commutative.

The commutator pairing $W \times W \rightarrow W$ induces an alternating form $\Psi: V \times V \rightarrow U$ (same as §2.2) which induces the morphism f above. We have proved in the last paragraph that $\Psi(V(\mathbb{R}), V(\mathbb{R})) \subset \Gamma_U$ with $\Gamma_U := \Gamma \cap U(\mathbb{Q})$. But $\Psi(V(\mathbb{R}), v)$ is continuous for any $v \in V(\mathbb{R})$ and $\Psi(0, V(\mathbb{R})) = 0$, so $\Psi(V(\mathbb{R}), V(\mathbb{R})) = 0$. Hence the commutator pairing $W \times W \rightarrow W$ is trivial, and therefore W is commutative. \square

7.1. Weakly special subvarieties of a complex semi-abelian variety.

Proposition 7.2. *Use the notation as at the beginning of the section. Weakly special subvarieties of E are precisely the subsets of E of the form*

$$\text{unif}(W_0(\mathbb{R})U_0(\mathbb{C})\tilde{z})$$

where W_0 is a $\text{MT}(b)$ -subgroup of W (i.e. a subgroup of W normalized by $\text{MT}(b)$), $U_0 := W_0 \cap U$, $\text{unif}(\tilde{z}_G) = b$ and $\tilde{z}_V \in (N_W(W_0)/U)(\mathbb{R})$ ($\tilde{z} = (\tilde{z}_U, \tilde{z}_V, \tilde{z}_G)$ under (4.1)).

In particular, if E can be given the structure of an algebraic group whose group law is compatible with that of W (i.e. W is commutative), then the weakly special subvarieties of E are precisely the translates of subgroups of E .

Proof. Let Z be a weakly special variety of E and let \tilde{Z} be a complex analytic irreducible component of $\text{unif}^{-1}(Z)$, then there exists a diagram as in Definition 5.2 s.t. $\tilde{z}: \mathbb{S}_{\mathbb{C}} \rightarrow P_{\mathbb{C}}$ factors through $Q_{\mathbb{C}}$, $N \triangleleft Q$ and $\tilde{Z} = N(\mathbb{R})^+ U_N(\mathbb{C}) \tilde{z}$ for some $\tilde{z} \in \tilde{Z}$. As is explained in [54, paragraph 2, pp 265], $G_N = 1$. We prove that $N = W_N$ satisfies the conditions which we require. Let $U_N := W_N \cap U$, then U_N is a $\text{MT}(b)$ -module by Proposition 2.6(2). Denote by $V_N := W_N/U_N$, $\pi_{P/U}: (P, \mathcal{X}^+) \rightarrow (P/U, \mathcal{X}_{P/U}^+)$ and $[\pi_{P/U}]: S \rightarrow S_{P/U}$. Then $[\pi_{P/U}](Z)$ is a subvariety of A since Z is a subvariety of E . So $\pi_{P/U}(\tilde{Z}) = V_N(\mathbb{R}) + \pi_{P/U}(\tilde{z})$ is translate of a complex subspace of $V(\mathbb{R}) = V(\mathbb{C})/F_b^0 V(\mathbb{C})$, and therefore V_N is a $\text{MT}(b)$ -module. So W_N is stable under the action of $\text{MT}(b)$. Now $\tilde{z}_V \in (N_W(N)/U)(\mathbb{R})$ since $\tilde{z}: \mathbb{S}_{\mathbb{C}} \rightarrow P_{\mathbb{C}}$ factors through $N_P(N)_{\mathbb{C}}$.

Conversely let $\tilde{Z} = W_0(\mathbb{R})U_0(\mathbb{C})\tilde{z}$ with W_0, \tilde{z} as stated. Fix a Levi decomposition $P = W \rtimes G$. Let $G' := \text{MT}(b)$, let $W' := N_W(W_0)$ and define $Q := W' \rtimes G'$. By definition of Q , $W_0 \triangleleft Q$ and there exists a connected mixed Shimura datum $(Q, \mathcal{Y}^+) \hookrightarrow (P, \mathcal{X}^+)$ with $b \in \text{unif}(\mathcal{Y}^+)$. Now consider the morphisms of connected mixed Shimura data

$$(Q, \mathcal{Y}^+)/W_0 \xleftarrow{\varphi} (Q, \mathcal{Y}^+) \xrightarrow{i} (P, \mathcal{X}^+).$$

In the fibres above the point $b \in S_G$ these maps are simply

$$S_{Q,b}/Z \leftarrow S_{Q,b} \hookrightarrow E = S_b.$$

Hence Z is a weakly special subvariety by definition. \square

Corollary 7.3. (1) *Weakly special subvarieties of a complex abelian variety are precisely the translates of abelian subvarieties;*
 (2) *Weakly special subvarieties of an algebraic torus over \mathbb{C} are precisely the translates of subtori.*

Proof. This is a direct consequence of Proposition 7.2. \square

7.2. Smallest weakly special subvariety containing a given subvariety of an abelian variety or an algebraic torus over \mathbb{C} .

Proposition 7.4. (1) *Let X be a complex abelian variety and let Z be a closed irreducible subvariety of X . Denote by*

$$\tilde{X} = \pi_1(X, z) \otimes_{\mathbb{Z}} \mathbb{R} = H_1(X, \mathbb{R}) \simeq \mathbb{C}^n \xrightarrow{u} X$$

the universal cover of X ($z \in Z^{\text{sm}}$), then the smallest weakly special subvariety of X containing Z is a translate of $u(\pi_1(Z^{\text{sm}}, z) \otimes \mathbb{R})$.

(2) *Let X be an algebraic torus over \mathbb{C} and let Z be a closed irreducible subvariety of X . Denote by*

$$\tilde{X} = \pi_1(X, z) \otimes_{\mathbb{Z}} \mathbb{C} = H_1(X, \mathbb{C}) \simeq \mathbb{C}^n \xrightarrow{u} X$$

the universal cover of X ($z \in Z^{\text{sm}}$), then the smallest weakly special subvariety of X containing Z is a translate of $u(\pi_1(Z^{\text{sm}}, z) \otimes \mathbb{C})$.

Proof. (1) If X is a complex abelian variety, then the result is due to Ullmo-Yafaev. Their proof of [62, Proposition 5.1] has in fact revealed this property. Here we restate the proof with more details.

Let $Z^{\text{de}} \xrightarrow{s} Z$ be a desingularization of Z^{de} s.t. there exists a Zariski open subset Z_0^{de} of Z^{de} s.t. $Z_0^{\text{de}} \xrightarrow[s]{} Z^{\text{sm}}$. By the commutative diagram

$$\begin{array}{ccccc} \pi_1(Z_0^{\text{de}}, z) & \xrightarrow{\sim} & \pi_1(Z^{\text{sm}}, z) & & \\ \downarrow & & \downarrow & \searrow & \\ \pi_1(Z^{\text{de}}, z) & \longrightarrow & \pi_1(Z, z) & \longrightarrow & \pi_1(X, z) \end{array},$$

where $z \in Z^{\text{sm}}$ (the surjectivity on the left is due to [37, 2.10.1]), we know that the image of $\pi_1(Z^{\text{de}}, z)$ and the image of $\pi_1(Z^{\text{sm}}, z)$ in $\pi_1(X, z)$ are the same.

Let $\text{Alb}(Z^{\text{de}})$ be the Albanese variety of Z^{de} normalized by z , then the map $\tau: Z^{\text{de}} \rightarrow Z \rightarrow X$ factors uniquely through the Albanese morphism ([66, Theorem 12.15]):

$$\begin{array}{ccccc} Z^{\text{de}} & \longrightarrow & Z & \hookrightarrow & X \\ & \searrow \text{alb} & & \nearrow \Gamma & \\ & & \text{Alb}(Z^{\text{de}}) & & \end{array}$$

Let $A := \Gamma(\text{Alb}(Z^{\text{de}}))$, then it is the smallest weakly special subvariety (i.e. the translate of an abelian subvariety) of X containing Z since $\text{alb}(Z^{\text{de}})$ generates $\text{Alb}(Z^{\text{de}})$ ([66, Lemma 12.11]).

It suffices to prove that the image of $\pi_1(Z^{\text{de}}, z)$ in $\pi_1(X, z) \simeq H_1(X, \mathbb{Z})$ is of finite index in $H_1(A, \mathbb{Z})$. This is true since the image of $\pi_1(Z^{\text{de}}, z)$ in $H_1(X, \mathbb{Z})$ contains

$$(\Gamma \circ \text{alb})_* H_1(Z^{\text{de}}, \mathbb{Z}) \simeq \Gamma_* H_1(\text{Alb}(Z^{\text{de}}), \mathbb{Z}) \simeq \Gamma_* \pi_1(\text{Alb}(Z^{\text{de}}))$$

(the first isomorphism is given by the definition of Albanese varieties via Hodge theory, see e.g. the proof of [66, Lemma 12.11]), which is of finite index in $\pi_1(A, z) \simeq H_1(A, \mathbb{Z})$ by [37, 2.10.2].

- (2) If X is an algebraic torus over \mathbb{C} , then we can first of all translate Z by a point s.t. the translate contains the origin of X . Now we are done if we can prove that the smallest subtorus containing this translate of Z is $u(\pi_1(Z^{\text{sm}}, z) \otimes_{\mathbb{Z}} \mathbb{C})$.

Suppose $T \simeq (\mathbb{C}^*)^m$ is the smallest sub-torus of X containing Z with $j: Z^{\text{sm}} \hookrightarrow T$ the inclusion. We are done if we can prove $[\pi_1(T, z) : j_* \pi_1(Z^{\text{sm}}, z)] < \infty$. If not, then

$$(7.1) \quad j_* \pi_1(Z^{\text{sm}}, z) \subset \text{Ker}(\mathbb{Z}^m \xrightarrow{\rho} \mathbb{Z})$$

for some map ρ . Since the covariant functor $T \mapsto X_*(T)$ ($X_*(T)$ is the co-character group of T) is an equivalence between the category {algebraic tori over \mathbb{C} and their morphisms as algebraic groups} and the category {free \mathbb{Z} -modules of finite rank}, the map ρ corresponds to a surjective map (with connected kernel) of tori $p: T \rightarrow T'$. The composition of the maps $Z^{\text{sm}} \xrightarrow{j} T \xrightarrow{p} T' = \mathbb{G}_{m, \mathbb{C}}$ should be dominant by the choice of T . But then we have

$$[\pi_1(T', p(z)) : (p \circ j)_* \pi_1(Z^{\text{sm}}, z)] < \infty$$

([37, 2.10.2]), which contradicts (7.1) by the following lemma.

Lemma 7.5. *For any \mathbb{C} -split torus $T \simeq (\mathbb{C}^*)^n$, we have a canonical isomorphism*

$$X_*(T) \xrightarrow[\sim]{\psi_T} \pi_1(T, 1).$$

Here “canonical” means that for any morphism (between algebraic groups) $f: T \rightarrow T'$ between two such \mathbb{C} -split tori, the following diagram commutes:

$$\begin{array}{ccc} X_*(T) & \xrightarrow[\sim]{\psi_T} & \pi_1(T, 1) \\ X_*(f) \downarrow & & \downarrow f_* \\ X_*(T') & \xrightarrow[\sim]{\psi_{T'}} & \pi_1(T', 1) \end{array}$$

Proof. Denote by $U_1 := \{z \in \mathbb{C} \mid |z| = 1\}$ and $i: U_1 \hookrightarrow \mathbb{C}^*$ the inclusion. Then the map ψ_T is defined by

$$\begin{aligned} X_*(T) &\xrightarrow{\psi_T} \pi_1(T, 1) \\ \nu &\mapsto [\nu \circ i] \end{aligned}$$

This is clearly a group homomorphism. It is surjective since a representative of the generators of $\pi_1(T, 1)$ is given by the n coordinate embeddings $U_1 \hookrightarrow \mathbb{C}^* \hookrightarrow T = (\mathbb{C}^*)^n$. ψ_T is injective since $X_*(T) \simeq \pi_1(T, 1) \simeq \mathbb{Z}^n$ is torsion-free. The rest of the lemma is immediate by the construction of ψ_T . \square

\square

7.3. Ax-Lindemann-Weierstraß for the unipotent part. By abuse of notation we denote the uniformization of E by $\text{unif}: W(\mathbb{R})U(\mathbb{C}) = W(\mathbb{C})/F_b^0 W(\mathbb{C}) \rightarrow E$. It is then the restriction of $\text{unif}: \mathcal{X}^+ \rightarrow S$.

Theorem 7.6. *Let Y be a closed irreducible subvariety of E and let \tilde{Z} be a maximal irreducible algebraic subvariety which is contained in $\text{unif}^{-1}(Y)$. Then \tilde{Z} is weakly special.*

Proof. If E is an algebraic torus over \mathbb{C} , this is a consequence of the Ax-Schanuel theorem [42, Corollary 3.6]. If E is an abelian variety, this is Pila-Zannier [52, pp9, Remark 1]. A proof using volume calculation and points counting method for these two cases can be found in the Appendix. The general case will be proved in §11. \square

8. THE SMALLEST WEAKLY SPECIAL SUBVARIETY CONTAINING A GIVEN SUBVARIETY

In this section, our goal is to prove a theorem (Theorem 8.1) which (in some sense) generalizes [40, 3.6, 3.7]. In particular, we get a criterion of weak specialness as a corollary (Corollary 8.3) which generalizes [62, Theorem 4.1]. Before the proof, let us do some technical preparation at first.

Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum (P, \mathcal{X}^+) and let $\text{unif}: \mathcal{X}^+ \rightarrow S = \Gamma \backslash \mathcal{X}^+$ be the uniformization. We may assume $P = \text{MT}(\mathcal{X}^+)$ by Proposition 2.6. There exists a $\Gamma' \leq \Gamma$ of finite index s.t. Γ' is neat. Let $S' := \Gamma' \backslash \mathcal{X}^+$ and let $\text{unif}': \mathcal{X}^+ \rightarrow S'$ be its uniformization. Choose any faithful \mathbb{Q} -representation $\xi: P \rightarrow \text{GL}(M)$ of P , then Theorem 3.7 claims that ξ (together with a choice of a Γ' -invariant lattice of M) gives rise to an admissible variation of mixed Hodge structure on S' . The generic Mumford-Tate group of this variation is P .

Suppose that Y is a closed irreducible subvariety of S . Let Y' be an irreducible component of $p^{-1}(Y)$ under $p: S' = \Gamma' \backslash \mathcal{X}^+ \rightarrow S = \Gamma \backslash \mathcal{X}^+$, then Y' is a closed irreducible subvariety of S' which maps surjectively to Y under p . The variation we constructed above can be restricted to Y'^{sm} , and this restriction is still admissible by Lemma 3.5. The *connected algebraic monodromy group associated with Y'^{sm}* is defined to be the connected algebraic monodromy group of the restriction of the VMHS defined in the last paragraph to Y'^{sm} , i.e. the neutral component of the Zariski closure of the image of $\pi_1(Y'^{\text{sm}}, y') \rightarrow \pi_1(S', y') \rightarrow P$.

Let us briefly prove that the connected algebraic monodromy group associated with Y'^{sm} is well-defined. Suppose that we have another covering $S'' \xrightarrow{p'} S'$ with S'' smooth. Let Y'' be an

irreducible component of $p'^{-1}(Y')$. Let $Y_0''^{\text{sm}} := Y''^{\text{sm}} \cap p'^{-1}(Y'^{\text{sm}})$, then by the commutative diagram

$$\begin{array}{ccccc} \pi_1(Y_0''^{\text{sm}}, y'') & = & \pi_1(Y''^{\text{sm}}, y'') & \longrightarrow & \pi_1(S'', y'') & \longrightarrow & P \\ \downarrow & & & & \downarrow & & \downarrow \\ \pi_1(Y'^{\text{sm}}, y') & \longrightarrow & \pi_1(S', y') & \longrightarrow & P & & \end{array}$$

where the equality in the top-left corner is given by [37, 2.10.1] and the morphism on the left is of finite index by [37, 2.10.2], the neutral components of the Zariski closures of the images of $\pi_1(Y''^{\text{sm}}, y'')$ and $\pi_1(Y'^{\text{sm}}, y')$ in P coincide.

Theorem 8.1. *Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum (P, \mathcal{X}^+) and let $\text{unif}: \mathcal{X}^+ \rightarrow S = \Gamma \backslash \mathcal{X}^+$ be the uniformization. Let Y be a closed irreducible subvariety of S and*

- *let \tilde{Y} be an irreducible component of $\text{unif}^{-1}(Y)$;*
- *take $\tilde{y}_0 \in \tilde{Y}$;*
- *let N be the connected algebraic monodromy group associated with Y^{sm} .*

Then

- (1) *The set $\tilde{F} := N(\mathbb{R})^+ U_N(\mathbb{C}) \tilde{y}_0$, where $U_N := U \cap N$, is a weakly special subset of \mathcal{X}^+ (or equivalently, $F := \text{unif}(\tilde{F})$ is a weakly special subvariety of S). Moreover N is the largest subgroup of Q s.t. $N(\mathbb{R})^+ U_N(\mathbb{C})$ stabilizes \tilde{F} , where (Q, \mathcal{Y}^+) is the smallest connected sub-mixed Shimura datum with $\tilde{F} \subset \mathcal{Y}^+$;*
- (2) *The Zariski closure of \tilde{Y} in \mathcal{X}^+ (which means the complex analytic irreducible component of the intersection of the Zariski closure of \tilde{Y} in \mathcal{X}^\vee and \mathcal{X}^+ which contains \tilde{Y}) is \tilde{F} ;*
- (3) *The smallest weakly special subset containing \tilde{Y} is \tilde{F} and F is the smallest weakly special subvariety of S containing Y .*

Proof. (1) Let S_Y be the smallest special subvariety containing Y . Such an S_Y exists since the irreducible components of intersections of special subvarieties are special (which can easily be shown by means of generic Mumford-Tate group). By definition of special subvarieties, there exists a connected mixed Shimura subdatum (Q, \mathcal{Y}^+) s.t. S_Y is the image of $\Gamma_Q \backslash \mathcal{Y}^+$ in S where $\Gamma_Q := \Gamma \cap Q(\mathbb{Q})$. We may furthermore assume (Q, \mathcal{Y}^+) to have generic Mumford-Tate group by Proposition 2.6.

Let N be the connected algebraic monodromy group associated with Y^{sm} , then $N \triangleleft Q$ (and also $N \triangleleft Q^{\text{der}}$) by the discussion at the beginning of this section (which claims that the variation we use to define N is admissible), Remark 3.8 (which claims that the generic Mumford-Tate group of this variation is Q) and Theorem 3.6.

Then \tilde{F} is a weakly special subset of \mathcal{Y}^+ since it is the inverse image of the point $\varphi(\tilde{y}_0)$ under the Shimura morphism $(Q, \mathcal{Y}^+) \xrightarrow{\varphi} (Q, \mathcal{Y}^+)/N$. Then \tilde{F} is also a weakly special subset of \mathcal{X}^+ by definition. By the choice of (Q, \mathcal{Y}^+) , \tilde{F} is Hodge generic in \mathcal{Y}^+ , and hence $\varphi(\tilde{F})$ is a Hodge generic point in \mathcal{Y}^+ . Now $\text{Stab}_{Q^{\text{der}}(\mathbb{Q})}(\tilde{F})^\circ = N(\mathbb{Q})$ by Remark 3.8.

- (2) We prove that \tilde{F} is the Zariski closure of \tilde{Y} in \mathcal{X}^+ . We first show that it suffices to prove $\tilde{Y} \subset \tilde{F}$. Let $\tilde{\tilde{Y}}$ be the Zariski closure of \tilde{Y} in \mathcal{X}^+ , then $\tilde{\tilde{Y}} \subset \tilde{F}$ since $\tilde{Y} \subset \tilde{F}$ and

\tilde{F} is algebraic (Lemma 6.2). On the other hand, $\Gamma_{Y^{\text{sm}}} := \text{Im}(\pi_1(Y^{\text{sm}}) \rightarrow \pi_1(S) \rightarrow P)$ stabilizes \tilde{Y} , so $\Gamma_{Y^{\text{sm}}}\tilde{y}_0 \subset \tilde{Y}$. The group $\Gamma_{Y^{\text{sm}}}$ is Zariski dense in N , and hence Zariski dense in $N_{\mathbb{C}}$. As $\tilde{F} = N(\mathbb{C})\tilde{y}_0 \cap \mathcal{X}^+$, $\Gamma_{Y^{\text{sm}}}\tilde{y}_0$ is Zariski dense in \tilde{F} . Hence we have $\tilde{F} \subset \tilde{Y}$. As a result, $\tilde{F} = \tilde{Y}$.

Next we prove that $\tilde{Y} \subset \tilde{F}$ (or equivalently, $Y \subset F$).

The fact that $\tilde{Y} \subset \tilde{F}$ has nothing to do with the level structure. Hence we may assume $\Gamma = \Gamma_W \rtimes \Gamma_G$ with $\Gamma_W \subset W(\mathbb{Z})$, $\Gamma_U := \Gamma_W \cap U \subset U(\mathbb{Z})$, $\Gamma_V := \Gamma_W/\Gamma_U \subset V(\mathbb{Z})$ and $\Gamma_G \subset G(\mathbb{Z})$ small enough s.t. they are all neat and s.t. $\Gamma \subset P^{\text{der}}(\mathbb{Q})$ (Remark 2.2(2)). We write $\Gamma_{P/U} := \Gamma/\Gamma_U$.

We may replace (P, \mathcal{X}^+) by (Q, \mathcal{Y}^+) and S by S_Y (same notation as in (1)) since \tilde{Y} , $\tilde{F} \subset \mathcal{Y}^+$ and $Y, F \subset S_Y$. In other words, we may assume that Y is Hodge generic in S and (P, \mathcal{X}^+) is irreducible.

Consider the following diagram:

$$\begin{array}{ccccc} \mathcal{X}^+ & \xrightarrow{\pi_{P/U}} & \mathcal{X}_{P/U}^+ & \xrightarrow{\pi_G} & \mathcal{X}_G^+ \\ \text{\scriptsize pr} \downarrow & & \text{\scriptsize $unif_{P/U}$} \downarrow & & \text{\scriptsize $unif_G$} \downarrow \\ S = \Gamma \backslash \mathcal{X}^+ & \xrightarrow{[\pi_{P/U}]} & S_{P/U} := \Gamma_{P/U} \backslash \mathcal{X}_{P/U}^+ & \xrightarrow{[\pi_G]} & S_G := \Gamma_G \backslash \mathcal{X}_G^+ \end{array}$$

Denote by π and $[\pi]$ the composites of the maps in the two lines respectively. Denote by $\tilde{Y}_G := \pi(\tilde{Y})$, $Y_G := [\pi](Y)$ and $\tilde{Y}_{P/U} := \pi_{P/U}(\tilde{Y})$, $Y_{P/U} := [\pi_{P/U}](Y)$; $\tilde{F}_G := \pi(\tilde{F})$, $F_G := [\pi](F)$ and $\tilde{F}_{P/U} := \pi_{P/U}(\tilde{F})$, $F_{P/U} := [\pi_{P/U}](F)$. Denote by $\tilde{y}_{0,P/U} := \pi_{P/U}(\tilde{y}_0)$ and $\tilde{y}_{0,G} := \pi(\tilde{y}_0)$.

Now to make the proof more clear, we divide it into several steps.

Step I. Prove that $\tilde{Y}_G \subset \tilde{F}_G$.

We begin the proof with the following lemma:

Lemma 8.2. *In the context above, the connected algebraic monodromy group associated with $\overline{Y_G^{\text{sm}}}$ (resp. $\overline{Y_{P/U}^{\text{sm}}}$) is G_N (resp. N/U_N where $U_N := U \cap N$).*

Proof. We only prove the statement for $\overline{Y_G^{\text{sm}}}$. The proof for $\overline{Y_{P/U}^{\text{sm}}}$ is similar. Take $Y_0^{\text{sm}} := Y^{\text{sm}} \cap \pi^{-1}(Y_G^{\text{sm}})$, then we have the commutative diagram below:

$$\begin{array}{ccccc} \pi_1(Y_0^{\text{sm}}, y) & \rightarrow & \pi_1(Y_G^{\text{sm}}, y_G) & \twoheadrightarrow & \pi_1(\overline{Y_G^{\text{sm}}}, \zeta_G) \\ \downarrow & & \searrow & & \downarrow \\ \pi_1(Y^{\text{sm}}, y) & \longrightarrow & \pi_1(S, y) & \longrightarrow & \pi_1(S_G, y_G) \\ & & \downarrow & & \downarrow \\ & & P & \longrightarrow & G \end{array}$$

Here, the morphism on the left and the right morphism on the top are surjective since $\text{codim}_{Y^{\text{sm}}}(Y^{\text{sm}} - Y_0^{\text{sm}}) \geq 1$ and $\text{codim}_{\overline{Y_G^{\text{sm}}}}(\overline{Y_G^{\text{sm}}} - Y_G^{\text{sm}}) \geq 1$ ([37, 2.10.1]). Now [37, 2.10.2] shows that the image of $\pi_1(Y_0^{\text{sm}}, y)$ is of finite index in $\pi_1(Y_G^{\text{sm}}, y_G)$, so the neutral

components of the Zariski closures of $\pi_1(Y^{\text{sm}}, y)$ and $\pi_1(\overline{Y_G^{\text{sm}}}, y_G)$ in G coincide. Hence we are done. \square

Let \tilde{Z} be the closure (w.r.t. archimedean topology) of \tilde{Y}_G in \mathcal{X}_G^+ , then \tilde{Z} is a complex analytic irreducible component of $\text{unif}_G^{-1}(\overline{Y_G})$. For the pure connected Shimura datum $(G^{\text{ad}}, \mathcal{X}_G^+)$, we have a decomposition ([40, 3.6])

$$(G^{\text{ad}}, \mathcal{X}_G^+) = (G_N^{\text{ad}}, \mathcal{X}_{G,1}^+) \times (G_2, \mathcal{X}_{G,2}^+).$$

By [40, 3.6, 3.7] and Lemma 8.2, $\tilde{Z} \subset \mathcal{X}_{G,1}^+ \times \{\widetilde{y_{G,2}}\}$, i.e. $\tilde{Z} \subset G_N(\mathbb{R})^+ \tilde{x}_G$ for some $\tilde{x}_G \in \mathcal{X}_G^+$. But $\tilde{y}_{0,G} \in \tilde{Y}_G \subset \tilde{Z}$, so $\tilde{F}_G = G_N(\mathbb{R})^+ \tilde{y}_{0,G} \subset G_N(\mathbb{R})^+ \tilde{x}_G$. This implies that $\tilde{F}_G = G_N(\mathbb{R})^+ \tilde{x}_G$. As a result, $\tilde{Y}_G \subset \tilde{Z} \subset \tilde{F}_G$.

Step II. Consider the Shimura morphism

$$(P, \mathcal{X}^+) \xrightarrow{\rho} (P', \mathcal{X}^{+'}) := (P, \mathcal{X}^+)/N.$$

Then $\tilde{F} = \rho^{-1}(\rho(\tilde{F}))$ by definition of ρ . So in order to prove $\tilde{Y} \subset \tilde{F}$, it is enough to show that $\rho(\tilde{Y}) \subset \rho(\tilde{F})$. Hence we may replace (P, \mathcal{X}^+) by $(P', \mathcal{X}^{+'})$. In other words, we may assume $N = \mathbf{1}$.

In this case \tilde{F} is just a point $\tilde{x} \in \mathcal{X}^+$. Call $\tilde{x}_{P/U} := \pi_{P/U}(\tilde{x})$, $\tilde{x}_G := \pi(\tilde{x})$ and $x := \text{unif}(\tilde{x})$, $x_{P/U} := \text{unif}_{P/U}(\tilde{x}_{P/U})$, $x_G := \text{unif}_G(\tilde{x}_G)$. Then since $Y_G \subset F_G$, we have $Y \subset E$ where E is the fibre of $S \xrightarrow{[\pi]} S_G$ over x_G . Denote by A the fibre of $S_{P/U} \xrightarrow{[\pi]_G} S_G$ over x_G and T the fibre of $S \xrightarrow{[\pi_{P/U}]} S_{P/U}$ over $x_{P/U}$, then by [53, 3.13, 3.14] A is an abelian variety and T is an algebraic torus.

Step III. Prove that $\tilde{Y}_{P/U} \subset \tilde{F}_{P/U}$, i.e. $\tilde{Y}_{P/U} = \{\tilde{x}_{P/U}\}$.

By Step I, $Y_{P/U} \subset A$. We have the following morphisms

$$\pi_1(Y_{P/U}^{\text{sm}}) \rightarrow \pi_1(A) \rightarrow \pi_1(S_{P/U}) = \Gamma_{P/U} \rightarrow P/U = V \rtimes G.$$

The neutral component of the Zariski closure of $\pi_1(Y_{P/U}^{\text{sm}})$ (resp. $\pi_1(A)$) in $P/U = V \rtimes G$ is $\mathbf{1}$ (resp. V), so the image of

$$\pi_1(Y_{P/U}^{\text{sm}}) \rightarrow \pi_1(A)$$

is a finite group.

Now $Y_{P/U}$ is irreducible since Y is irreducible. So by Proposition 7.4, $Y_{P/U} \subset A$ is a point. Equivalently, $\tilde{Y}_{P/U}$ is a point. So $\tilde{Y}_{P/U} \subset \tilde{F}_{P/U}$ since $\tilde{Y}_{P/U} \cap \tilde{F}_{P/U} \neq \emptyset$ (both of them contain $\tilde{y}_{0,P/U}$).

Step IV. Prove that $\tilde{Y} \subset \tilde{F}$, i.e. $\tilde{Y} = \{\tilde{x}\}$.

By Step I, $Y \subset E$. By Step III, $Y_{P/U} = \{x_{P/U}\}$. So $Y \subset T$. We have the following morphisms

$$\pi_1(Y^{\text{sm}}) \rightarrow \pi_1(T) \rightarrow \pi_1(S) = \Gamma \rightarrow P = W \rtimes G.$$

The neutral component of the Zariski closure of $\pi_1(Y^{\text{sm}})$ (resp. $\pi_1(T)$) in $P = W \rtimes G$ is $\mathbf{1}$ (resp. U), so the image of

$$\pi_1(Y^{\text{sm}}) \rightarrow \pi_1(T)$$

is a finite group.

Now since Y is irreducible, by Proposition 7.4, $Y \subset T$ is a point. Equivalently, \tilde{Y} is a point. So $\tilde{Y} \subset \tilde{F}$ since $\tilde{Y} \cap \tilde{F} \neq \emptyset$ (both of them contain \tilde{y}_0).

- (3) Since every weakly special subset of \mathcal{X}^+ is algebraic by Lemma 6.2, \tilde{F} is also the smallest weakly special subset which contains \tilde{Y} . Therefore F is the smallest weakly special subvariety of S which contains Y . □

Corollary 8.3. *Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum (P, \mathcal{X}^+) and let $\text{unif}: \mathcal{X}^+ \rightarrow S = \Gamma \backslash \mathcal{X}^+$ be the uniformization map. Let Y be a closed irreducible subvariety of S , then Y is weakly special if and only if one (equivalently any) irreducible component of $\text{unif}^{-1}(Y)$ is algebraic.*

If Y is weakly special, then $Y = \text{unif}(N(\mathbb{R})^+ U_N(\mathbb{C}) \tilde{y})$ where N is the connected algebraic monodromy group associated with Y^{sm} , $U_N := U \cap N$ and \tilde{y} is any point of $\text{unif}^{-1}(Y)$.

Proof. The “only if” part is immediate by Lemma 6.2. Now we prove the “if” part.

We first of all quickly show that if one irreducible component of $\text{unif}^{-1}(Y)$ is algebraic, so are the others. The proof is the same as [62, first paragraph of the proof of Theorem 4.1]. Suppose that \tilde{Y} is an irreducible component of $\text{unif}^{-1}(Y)$ which is algebraic, i.e. \tilde{Y} is an irreducible component of $\mathcal{X}^+ \cap Z$ for some algebraic subvariety Z of \mathcal{X}^+ . Then for any $\gamma \in \Gamma \subset P(\mathbb{R})U(\mathbb{C})$,

$$\gamma \tilde{Y} = \gamma(\mathcal{X}^+ \cap Z) \subset \mathcal{X}^+ \cap \gamma Z = \gamma \gamma^{-1}(\mathcal{X}^+ \cap \gamma Z) \subset \gamma \tilde{Y}.$$

Hence it follows that $\gamma \tilde{Y} = \mathcal{X}^+ \cap \gamma Z$ is algebraic.

Next under the notation of Theorem 8.1, $\tilde{Y} = \tilde{\tilde{Y}} = \tilde{F}$ since \tilde{Y} is algebraic. Hence \tilde{Y} is weakly special, and so is Y .

Finally if Y is weakly special, then for any $\tilde{y} \in \text{unif}^{-1}(Y)$ and \tilde{Y} the irreducible component of $\text{unif}^{-1}(Y)$ which contains \tilde{y} , $\tilde{Y} = \tilde{F} = N(\mathbb{R})^+ U_N(\mathbb{C}) \tilde{y}$ by Theorem 8.1, and hence $Y = \text{unif}(N(\mathbb{R})^+ U_N(\mathbb{C}) \tilde{y})$. □

9. AX-LINDEMANN-WEIERSTRASS PART 1: OUTLINE OF THE PROOF

In the following three sections, we are going to prove Theorem 1.2. The organization of the proof is as follows: the outline of the proof is given in this section. After some preparation, the key proposition (Proposition 9.6) leading to the theorem will be stated and exploited (together with Theorem 7.6) to finish the proof in Theorem 9.8. We prove this key proposition in the next section using Pila-Wilkie’s counting theorem and Theorem 7.6 will be proved in §11, where a simple proof of Ax-Lindemann-Weierstraß for complex abelian varieties can be found.

Now let us fix some notation which will be used through the whole proof:

Notation 9.1. *Consider the following diagram:*

$$\begin{array}{ccc} \mathcal{X}^+ & \xrightarrow{\pi} & \mathcal{X}_G^+ \\ \text{pr} \downarrow & & \downarrow \text{unif}_G \\ S = \Gamma \backslash \mathcal{X}^+ & \xrightarrow{[\pi]} & S_G := \Gamma_G \backslash \mathcal{X}_G^+ \end{array}$$

Denote by $\tilde{Y}_G := \pi(\tilde{Y})$, $Y_G := [\pi](Y)$ and $\tilde{Z}_G := \pi(\tilde{Z})$.

Now we begin the proof of Theorem 1.2. Let us first of all do some reduction:

- Since every point of \mathcal{X}^+ is weakly special, we may assume $\dim(\tilde{Z}) > 0$.
- Let (Q, \mathcal{Y}^+) be the smallest mixed Shimura subdatum of (P, \mathcal{X}^+) s.t. $\tilde{Z} \subset \mathcal{Y}^+$ and let S_Q be the corresponding special subvariety of S . Then $Q = \text{MT}(\mathcal{Y}^+)$ by Proposition 2.6(1). If we replace (P, \mathcal{X}^+) by (Q, \mathcal{Y}^+) , S by S_Q , $\text{unif}: \mathcal{X}^+ \rightarrow S$ by $\text{unif}_Q: \mathcal{Y}^+ \rightarrow S_Q$ and Y by an irreducible component Y_0 of $Y \cap S_Q$, then \tilde{Z} is again a maximal irreducible algebraic subset of $\text{unif}_Q^{-1}(Y_0)$. By definition, \tilde{Z} is weakly special in \mathcal{X}^+ iff it is weakly special in \mathcal{Y}^+ . So we may assume $P = \text{MT}(\mathcal{X}^+)$ and that \tilde{Z} is Hodge generic.
- Furthermore, let Y_0 be the minimal irreducible subvariety of S such that $\tilde{Z} \subset \text{unif}^{-1}(Y_0)$, then \tilde{Z} is still maximal irreducible algebraic in $\text{unif}^{-1}(Y_0)$. Hence we may assume that $Y = Y_0$. In fact it is not hard to see that after this reduction, $Y = \overline{\text{unif}(\tilde{Z})}$ and \tilde{Z} is weakly special iff Y is weakly special.
- By the previous reduction, there is a unique complex analytic irreducible component of $\text{unif}^{-1}(Y)$ which contains \tilde{Z} . Denote it by \tilde{Y} . Denote by $\tilde{Y}_G := \pi(\tilde{Y})$, $Y_G := [\pi](Y)$ and $\tilde{Z}_G := \pi(\tilde{Z})$. Remark that by Lemma 6.3, \tilde{Z}_G is an algebraic subset of \mathcal{X}_G^+ .
- Replacing Γ by a subgroup of finite index does not matter for this problem, so we may assume that Γ is neat and $\Gamma \subset P^{\text{der}}(\mathbb{Q})$ (Remark 2.2(2)).

Let \tilde{F} be the smallest weakly special subset containing \tilde{Y} . By Theorem 8.1, $\tilde{F} = N(\mathbb{R})^+ U_N(\mathbb{C}) \tilde{z}$ some $\tilde{z} \in \tilde{Z} \subset \tilde{Y}$, where N is the connected algebraic monodromy group associated with Y^{sm} and $U_N := U \cap N$. The set \tilde{F} is Hodge generic in (P, \mathcal{X}^+) since \tilde{Z} is, so $N \triangleleft P$ and $N \triangleleft P^{\text{der}}$ by Theorem 3.6.

Define

$$\Gamma_{\tilde{Z}} := \{\gamma \in \Gamma \mid \gamma \cdot \tilde{Z} = \tilde{Z}\} \quad (\text{resp. } \Gamma_{G, \tilde{Z}_G} := \{\gamma_G \in \Gamma_G \mid \gamma_G \cdot \tilde{Z}_G = \tilde{Z}_G\})$$

and

$$H_{\tilde{Z}} := (\overline{\Gamma_{\tilde{Z}}}^{\text{Zar}})^{\circ} \quad (\text{resp. } H_{\tilde{Z}_G} := (\overline{\Gamma_{G, \tilde{Z}_G}}^{\text{Zar}})^{\circ}).$$

Define $U_{H_{\tilde{Z}}} := U \cap H_{\tilde{Z}}$ and $W_{H_{\tilde{Z}}} := W \cap H_{\tilde{Z}}$. Both of them are normal in $H_{\tilde{Z}}$. Then $H_{\tilde{Z}}$ (resp. $H_{\tilde{Z}_G}$) is the largest connected subgroup of P^{der} (resp. G^{der}) such that $H_{\tilde{Z}}(\mathbb{R})^+ U_{H_{\tilde{Z}}}(\mathbb{C})$ (resp. $H_{\tilde{Z}_G}(\mathbb{R})^+$) stabilizes \tilde{Z} (resp. \tilde{Z}_G).

Define $V_{H_{\tilde{Z}}} := W_{H_{\tilde{Z}}}/U_{H_{\tilde{Z}}}$ and $G_{H_{\tilde{Z}}} := H_{\tilde{Z}}/W_{H_{\tilde{Z}}} \hookrightarrow P/W = G$.

The following two lemmas were proved for the pure case in [49] and [35].

Lemma 9.2. *The set \tilde{Y} is stable under $H_{\tilde{Z}}(\mathbb{R})^+ U_{H_{\tilde{Z}}}(\mathbb{C})$.*

Proof. Every fiber of $\mathcal{X}^+ \rightarrow \mathcal{X}_{P/U}^+$ can be canonically identified with $U(\mathbb{C})$. So it is enough to prove that \tilde{Y} is stable under $H_{\tilde{Z}}(\mathbb{R})^+$: If $U_{H_{\tilde{Z}}}(\mathbb{R}) \tilde{y} \subset \tilde{Y}$ for $\tilde{y} \in \tilde{Y}$, then $U_{H_{\tilde{Z}}}(\mathbb{C}) \tilde{y} \subset \tilde{Y}$ because \tilde{Y} is complex analytic and $U_{H_{\tilde{Z}}}(\mathbb{C}) \tilde{y}$ is the smallest complex analytic subset of \mathcal{X}^+ containing $U_{H_{\tilde{Z}}}(\mathbb{R}) \tilde{y}$.

If not, then since $H_{\tilde{Z}}(\mathbb{Q})$ is dense (w.r.t. the archimedean topology) in $H_{\tilde{Z}}(\mathbb{R})^+$, there exists $h \in H_{\tilde{Z}}(\mathbb{Q})$ such that $h\tilde{Y} \neq \tilde{Y}$. The set \tilde{Z} is contained in $\tilde{Y} \cap h\tilde{Y}$ by definition of $H_{\tilde{Z}}$, and hence contained in a complex analytic irreducible component \tilde{Y}' of it.

Consider the Hecke operator T_h . Then $T_h(Y) = \text{unif}(h \cdot \text{unif}^{-1}(Y))$. Hence

$$Y \cap T_h(Y) = \text{unif}(\text{unif}^{-1}(Y) \cap (h \cdot \text{unif}^{-1}(Y))).$$

On the other hand, $T_h(Y)$ is equidimensional of the same dimension as Y by definition, hence by reason of dimension, $h\tilde{Y}$ is an irreducible component of $\text{unif}^{-1}(T_h(Y)) = \Gamma h \Gamma \tilde{Y}$. So $\text{unif}(h\tilde{Y})$ is an irreducible component of $T_h(Y)$.

Since \tilde{Y}' is a complex analytic irreducible component of $\tilde{Y} \cap h\tilde{Y}$, it is also a complex analytic irreducible component of $\text{unif}^{-1}(Y) \cap (h\tilde{Y}) = \Gamma \tilde{Y} \cap h\tilde{Y}$. So $Y' := \text{unif}(\tilde{Y}')$ is a complex analytic irreducible component of $Y \cap \text{unif}(h\tilde{Y})$. So Y' is a complex analytic irreducible component of $Y \cap T_h(Y)$, and hence is algebraic since $Y \cap T_h(Y)$ is.

Since $h\tilde{Y} \neq \tilde{Y}$ and Y is irreducible, $\dim(Y') < \dim(Y)$. But $\tilde{Z} \subset \tilde{Y} \cap h\tilde{Y} \subset \text{unif}^{-1}(Y')$. This contradicts the minimality of Y . \square

Lemma 9.3. $H_{\tilde{Z}} \triangleleft N$.

Proof. We have $\tilde{Z} \subset \tilde{F} = N(\mathbb{R})^+ U_N(\mathbb{C}) \tilde{z}$ for some $\tilde{z} \in \tilde{Z}$, so the image of \tilde{Z} under the morphism

$$(P, \mathcal{X}^+) \rightarrow (P, \mathcal{X}^+)/N$$

is a point. But $H_{\tilde{Z}}/(H_{\tilde{Z}} \cap N)$ stabilizes this point which is Hodge generic (since \tilde{F} is Hodge generic in \mathcal{X}^+), and therefore is trivial by Remark 3.8. So $H_{\tilde{Z}} < N$.

Let H' be the algebraic group generated by $\gamma^{-1} H_{\tilde{Z}} \gamma$ for all $\gamma \in \Gamma_{Y^{\text{sm}}}$, where $\Gamma_{Y^{\text{sm}}}$ is the monodromy group of Y^{sm} . Since H' is invariant under conjugation by $\Gamma_{Y^{\text{sm}}}$, it is invariant under $\overline{\Gamma_{Y^{\text{sm}}}}^{\text{Zar}}$, therefore invariant under conjugation by N .

By Lemma 9.2, \tilde{Y} is invariant under $H_{\tilde{Z}}(\mathbb{R})^+ U_{H_{\tilde{Z}}}(\mathbb{C})$. On the other hand, \tilde{Y} is also invariant under $\Gamma_{Y^{\text{sm}}}$ by definition. So \tilde{Y} is invariant under the action of $H'(\mathbb{R})^+ U_{H'}(\mathbb{C})$ where $U_{H'} := U \cap H'$. Since $H'(\mathbb{R})^+ U_{H'}(\mathbb{C}) \tilde{Z}$ is semi-algebraic, there exists an irreducible algebraic subset of \mathcal{X}^+ , say \tilde{E} , which contains $H'(\mathbb{R})^+ U_{H'}(\mathbb{C}) \tilde{Z}$ and is contained in \tilde{Y} by [50, Lemma 4.1]. Now $\tilde{Z} \subset \tilde{E} \subset \tilde{Y}$, so $\tilde{Z} = \tilde{E} = H'(\mathbb{R})^+ U_{H'}(\mathbb{C}) \tilde{Z}$ by maximality of \tilde{Z} , and therefore $H' = H_{\tilde{Z}}$ by definition of $H_{\tilde{Z}}$. So $H_{\tilde{Z}}$ is invariant under conjugation by N . Since $H_{\tilde{Z}} < N$, $H_{\tilde{Z}}$ is normal in N . \square

Corollary 9.4.

$$G_{H_{\tilde{Z}}}, H_{\tilde{Z}_G} \triangleleft G^{\text{der}} \text{ and } G_{H_{\tilde{Z}}} \triangleleft H_{\tilde{Z}_G}.$$

Proof. We have $G_{H_{\tilde{Z}}} \triangleleft G_N \triangleleft G^{\text{der}}$, and so $G_{H_{\tilde{Z}}} \triangleleft G^{\text{der}}$ since all the three groups are reductive.

Working with $((G, \mathcal{X}_G^+), \overline{Y_G}, \overline{\tilde{Z}_G})$ instead of $((P, \mathcal{X}^+), Y, \tilde{Z})$, we can prove (similar to Lemma 9.3) that $H_{\tilde{Z}_G} \triangleleft G_N$. Hence $H_{\tilde{Z}_G} \triangleleft G^{\text{der}}$ by the same reason for $G_{H_{\tilde{Z}}}$.

By definition $G_{H_{\tilde{Z}}} < H_{\tilde{Z}_G}$. So $G_{H_{\tilde{Z}}} \triangleleft H_{\tilde{Z}_G}$ since $G_{H_{\tilde{Z}}} \triangleleft G^{\text{der}}$. \square

So far the proof looks similar to the pure case. From now on it will be quite different. For the readers' convenience, we list here some differences between the proof of Ax-Lindemann-Weierstraß for mixed Shimura varieties and for the pure case:

- We shall prove that \tilde{Z} is an $H_{\tilde{Z}}(\mathbb{R})^+ U_{H_{\tilde{Z}}}(\mathbb{C})$ -orbit. To prove this, it suffices to prove $\dim H_{\tilde{Z}} > 0$ when S is a pure Shimura variety. However this is far from enough for the

mixed case, since this does not exclude the naive counterexample when $\dim \tilde{Z}_G > 0$ but $H_{\tilde{Z}}$ is unipotent. To overcome it, we should at least prove $\dim G_{H_{\tilde{Z}}} > 0$. In fact we shall directly prove $G_{H_{\tilde{Z}}} = H_{\overline{\tilde{Z}_G}}$ (Proposition 9.6). This equality is not obvious because, as appears in the proof of Lemma 9.5, there is no reason a priori why $\overline{\tilde{Z}_G}$, which is obviously algebraic in $\text{unif}^{-1}(Y_G)$, should be maximal for this property. If one could prove directly this is the case, then Klingler-Ullmo-Yafaev [35, Theorem 1.3] would give directly the result.

- As mentioned in the Introduction, we shall make essential use of the “family” version of Pila-Wilkie’s theorem (Remark 10.4);
- If $P = G$ is reductive, then $H_{\tilde{Z}} \triangleleft N \triangleleft P$ implies directly $H_{\tilde{Z}} \triangleleft P$. This is obviously false when P is not reductive.
- For a general mixed Shimura variety S , the fiber of $S \xrightarrow{[\pi]} S_G$ is not necessarily an algebraic group (Lemma 7.1), hence not a semi-abelian variety. We do not have Ax-Lindemann-Weierstraß for the fiber for this case. Thus we should execute a proof of Ax-Lindemann-Weierstraß for the fiber. As the readers will see in §11, the proof of this case calls for much more careful study of \tilde{Z} . First of all, when doing the estimate and using the family version of Pila-Wilkie for the fiber (*Step I*), we should introduce a seemingly strange subgroup which serves as G_N in the section. The reason for this will be explained in Remark 11.1. Secondly, to prove that $W_{H_{\tilde{Z}}}$ is normal in W is not trivial, and the key to the solution (*Step IV*) is a well-known fact: any holomorphic morphism from a complex abelian variety to an algebraic torus over \mathbb{C} is trivial.

Before proceeding, we prove the following lemma:

Lemma 9.5. (1) $\overline{Y_G}$ is weakly special. Hence $\overline{Y_G} = G_N(\mathbb{R})^+ \tilde{z}_G$ for any point $\tilde{z}_G \in \tilde{Z}_G$;
 (2) $\text{unif}_G(\overline{\tilde{Z}_G}) = \overline{Y_G}$.

Proof. (1) Let \tilde{Z}' be an irreducible algebraic subset of \mathcal{X}_G^+ which contains $\overline{\tilde{Z}_G}$ and is contained in $\text{unif}^{-1}(\overline{Y_G})$, maximal for these properties. By [35, Theorem 1.3], $Z' := \text{unif}_G(\tilde{Z}')$ is weakly special, and therefore Zariski closed by definition. Now $\tilde{Z} \subset \pi^{-1}(\tilde{Z}') \cap \text{unif}^{-1}(Y)$. However,

$$\text{unif}(\pi^{-1}(\tilde{Z}') \cap \text{unif}^{-1}(Y)) = \text{unif}(\pi^{-1}(\tilde{Z}')) \cap Y = [\pi]^{-1}(Z') \cap Y.$$

Then we must have $Y \subset [\pi]^{-1}(Z')$ since Y is the minimal irreducible closed subvariety of S such that $\tilde{Z} \subset \text{unif}^{-1}(Y)$. Therefore $\overline{Y_G} \subset Z'$. But $Z' \subset \overline{Y_G}$ by definition of Z' , so $Z' = \overline{Y_G}$. This means that $\overline{Y_G}$ is weakly special.

- (2) Let $Y' := \text{unif}_G(\tilde{Z}_G)$, then $\tilde{Z}_G \subset \text{unif}_G^{-1}(Y')$. So $\tilde{Z} \subset \pi^{-1}(\text{unif}_G^{-1}(Y')) = \text{unif}^{-1}([\pi]^{-1}(Y'))$, and so

$$\tilde{Z} \subset \text{unif}^{-1}([\pi]^{-1}(Y')) \cap \text{unif}^{-1}(Y) = \text{unif}^{-1}([\pi]^{-1}(Y') \cap Y).$$

Hence there exists an irreducible component Y'' of $[\pi]^{-1}(Y') \cap Y$ such that $\tilde{Z} \subset \text{unif}^{-1}(Y'')$. But

$$[\pi](Y'') \subset [\pi]([\pi]^{-1}(Y') \cap Y) = Y' \cap Y_G,$$

so $\dim([\pi](Y'')) \leq \dim(Y' \cap Y_G)$. If $Y' \neq \overline{Y_G}$, then $\dim(Y' \cap Y_G) < \dim(Y_G)$ and therefore $\dim(Y'') < \dim(Y)$, which contradicts the minimality of Y . So $Y' = \overline{Y_G}$. \square

Proposition 9.6 (key proposition). *The set $\overline{\tilde{Z}_G}$ is weakly special and $G_{H_{\tilde{Z}}} = H_{\overline{\tilde{Z}_G}}$. In other words,*

$$\overline{\tilde{Z}_G} = G_{H_{\tilde{Z}}}(\mathbb{R})^+ \tilde{z}_G$$

for any point $\tilde{z}_G \in \tilde{Z}_G$.

Now let us show how this proposition together with Theorem 7.6 implies Theorem 1.2. Before proceeding to the final argument, we shall prove the following group theoretical lemma:

Lemma 9.7. *Fixing a Levi decomposition $H_{\tilde{Z}} = W_{H_{\tilde{Z}}} \rtimes G_{H_{\tilde{Z}}}$, there exists a compatible Levi decomposition $P = W \rtimes G$.*

Proof. Suppose that the fixed Levi decomposition of $H_{\tilde{Z}}$ is given by $s_1: G_{H_{\tilde{Z}}} \rightarrow H_{\tilde{Z}}$. Define $P_* := \pi^{-1}(G_{H_{\tilde{Z}}})$, then $H_{\tilde{Z}} < P_*$. Now choose any Levi decomposition $P = W \rtimes G$ defined by $s_2: G \rightarrow P$. Then $G_{H_{\tilde{Z}}}$, being a subgroup of G , is realized as a subgroup of P via s_2 . Hence s_2 induces a Levi-decomposition $P_* = W \rtimes^{s_2} G_{H_{\tilde{Z}}}$. We have thus a diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & W_{H_{\tilde{Z}}} & \longrightarrow & H_{\tilde{Z}} & \xrightarrow{s_1} & G_{H_{\tilde{Z}}} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & W & \longrightarrow & P_* & \xrightarrow{s_1} & G_{H_{\tilde{Z}}} \longrightarrow 1 \end{array},$$

where the morphism s_1 in the second line is induced by the one in the first line. Now s_1, s_2 define two Levi decompositions of P_* . They differ by the conjugation by an element w_0 of $W(\mathbb{Q})$ by [55, Theorem 2.3]. So replacing s_2 by its conjugation by w_0 we can find a Levi decomposition of P which is compatible with the fixed $H_{\tilde{Z}} = W_{H_{\tilde{Z}}} \rtimes G_{H_{\tilde{Z}}}$. \square

Theorem 9.8. (1) $\tilde{Z} = H_{\tilde{Z}}(\mathbb{R})^+ U_{H_{\tilde{Z}}}(\mathbb{C}) \tilde{z}$ for any $\tilde{z} \in \tilde{Z}$;
(2) $H_{\tilde{Z}} \triangleleft P$.

Hence \tilde{Z} is weakly special by definition.

Proof. (1) Consider a fibre of \tilde{Z} over a Hodge-generic point $\tilde{z}_G \in \tilde{Z}_G$ such that $\pi|_{\tilde{Z}}$ is flat at \tilde{z}_G (such a point exists by [1, §4, Lemma 1.4] and generic flatness). Suppose that \tilde{W} is an irreducible algebraic component of $\tilde{Z}_{\tilde{z}_G}$ such that $\dim(\tilde{Z}_{\tilde{z}_G}) = \dim(\tilde{W})$, then since $\pi|_{\tilde{Z}}$ is flat at \tilde{z}_G ,

$$\dim(\tilde{Z}) = \dim(\tilde{Z}_G) + \dim(\tilde{Z}_{\tilde{z}_G}) = \dim(\tilde{Z}_G) + \dim(\tilde{W}).$$

Consider the set $\tilde{E} := H_{\tilde{Z}}(\mathbb{R})^+ U_{H_{\tilde{Z}}}(\mathbb{C}) \tilde{W}$. It is semi-algebraic (since \tilde{W} is algebraic and the action of $P(\mathbb{R})^+ U(\mathbb{C})$ on \mathcal{X}^+ is algebraic). The fact $\tilde{W} \subset \tilde{Z}$ implies that $\tilde{E} \subset \tilde{Z}$. By [50, Lemma 4.1], there exists an irreducible algebraic subset of \mathcal{X}^+ , say \tilde{E}_{alg} , which contains \tilde{E} and is contained in \tilde{Z} . Now we have by Proposition 9.6

$$\pi(\tilde{E}) = G_{H_{\tilde{Z}}}(\mathbb{R})^+ \tilde{z}_G = H_{\overline{\tilde{Z}_G}}(\mathbb{R})^+ \tilde{z}_G = \tilde{Z}_G$$

and that the \mathbb{R} -dimension of every fiber of $\pi|_{\widetilde{E}}$ is at least $\dim_{\mathbb{R}}(\widetilde{W})$. So

$$\dim(\widetilde{E}_{\text{alg}}) \geq \dim(\pi(\widetilde{E})) + \dim(\widetilde{W}) = \dim(\widetilde{Z}_G) + \dim(\widetilde{W}) = \dim(\widetilde{Z}).$$

So $\widetilde{E} = \widetilde{Z}$ since \widetilde{Z} is irreducible.

Next let \widetilde{W}' be an irreducible algebraic subset which contains $\widetilde{Z}_{\widetilde{z}_G}$ and is contained in $\text{unif}^{-1}(Y)_{\widetilde{z}_G}$, maximal for these properties. Then \widetilde{W}' is weakly special by Theorem 7.6. We have $\widetilde{W}' \subset \widetilde{Y}$ since \widetilde{Y} is an irreducible component of $\pi^{-1}(Y)$. Consider $\widetilde{E}' := H_{\widetilde{Z}}(\mathbb{R})^+ U_{H_{\widetilde{Z}}}(\mathbb{C}) \widetilde{W}'$. Then $\widetilde{E}' \subset \widetilde{Y}$ by Lemma 9.2. But \widetilde{E}' is semi-algebraic, so by [50, Lemma 4.1], there exists an irreducible algebraic subset of \mathcal{X}^+ , say $\widetilde{E}'_{\text{alg}}$ which contains \widetilde{E}' and is contained in \widetilde{Y} . So $\widetilde{Z} = \widetilde{E} \subset \widetilde{E}'_{\text{alg}} \subset \widetilde{Y}$, and hence $\widetilde{Z} = \widetilde{E}'_{\text{alg}} = \widetilde{E}'$ by the maximality of \widetilde{Z} . So $\widetilde{Z}_{\widetilde{z}_G} = \widetilde{W}'$ is weakly special.

Write $\widetilde{Z}_{\widetilde{z}_G} = W'(\mathbb{R})U'(\mathbb{C})\widetilde{z}$ with $W' < W$, $U' = W' \cap U$ and $\widetilde{z} \in \widetilde{Z}_{\widetilde{z}_G}$. Then $W_{H_{\widetilde{Z}}} < W'$. The complex structure of $\pi^{-1}(\widetilde{z}_G)$ comes from $W(\mathbb{R})U(\mathbb{C}) \simeq W(\mathbb{C})/F_{\widetilde{z}_G}^0 W(\mathbb{C})$, where $F_{\widetilde{z}_G}^0 W(\mathbb{C}) = \exp(F_{\widetilde{z}_G}^0 \text{Lie } W_{\mathbb{C}})$. So the fact that $\widetilde{Z}_{\widetilde{z}_G}$ is a complex subspace of $\pi^{-1}(\widetilde{z}_G)$ implies that W'/U' is a $\text{MT}(\widetilde{z}_G) = G$ -module. Hence W' is a G -group.

Define $P' := W'H_{\widetilde{Z}}$, then P' is a subgroup of P since $W' > W_{H_{\widetilde{Z}}}$ and $G_{H_{\widetilde{Z}}}W' = W'$. Now we have

$$\widetilde{Z} = H_{\widetilde{Z}}(\mathbb{R})^+ U_{H_{\widetilde{Z}}}(\mathbb{C}) \widetilde{Z}_{\widetilde{z}_G} = H_{\widetilde{Z}}(\mathbb{R})^+ U_{H_{\widetilde{Z}}}(\mathbb{C}) W'(\mathbb{R})U'(\mathbb{C})\widetilde{z} = P'(\mathbb{R})^+ U'(\mathbb{C})\widetilde{z}.$$

So $H_{\widetilde{Z}} = P'$ because $H_{\widetilde{Z}}$ is the largest subgroup of P^{der} such that $H_{\widetilde{Z}}(\mathbb{R})^+ U_{H_{\widetilde{Z}}}(\mathbb{C})$ stabilizes \widetilde{Z} . So we have $\widetilde{Z} = H_{\widetilde{Z}}(\mathbb{R})^+ U_{H_{\widetilde{Z}}}(\mathbb{C})\widetilde{z}$.

(2) First of all, $U_{H_{\widetilde{Z}}} \triangleleft P$ by Proposition 2.6(2).

Next consider the complex structure of $\pi^{-1}(\widetilde{z}_G)$ which comes from $W(\mathbb{R})U(\mathbb{C}) \simeq W(\mathbb{C})/F_{\widetilde{z}_G}^0 W(\mathbb{C})$. So the fact that $\widetilde{Z}_{\widetilde{z}_G}$ is a complex subspace of $\pi^{-1}(\widetilde{z}_G)$ implies that $V_{H_{\widetilde{Z}}}$ is a $\text{MT}(\widetilde{z}_G) = G$ -module. Hence $W_{H_{\widetilde{Z}}}$ is a G -group. Besides, $G_{H_{\widetilde{Z}}} \triangleleft G$ by Proposition 9.6. In particular, $G_{H_{\widetilde{Z}}}$ is reductive.

Then let us prove $W_{H_{\widetilde{Z}}} \triangleleft P$. It suffices to prove $W_{H_{\widetilde{Z}}} \triangleleft W$. For any $\widetilde{z} \in \widetilde{Z}$, we have proved in (1) that $\widetilde{Z}_{\widetilde{z}_G} = W_{H_{\widetilde{Z}}}(\mathbb{R})U_{H_{\widetilde{Z}}}(\mathbb{C})\widetilde{z}$ is weakly special. Hence by Proposition 5.4, there is a connected mixed Shimura subdatum $(Q, \mathcal{Y}^+) \hookrightarrow (P, \mathcal{X}^+)$ such that $\widetilde{z} \in \mathcal{Y}^+$ and $W_{H_{\widetilde{Z}}} \triangleleft Q$. Define W^* to be the G -subgroup (of W) generated by $W_Q := \mathcal{R}_u(Q)$, then $W_{H_{\widetilde{Z}}} \triangleleft W^*$ since $W_{H_{\widetilde{Z}}}$ is a G -group.

Fix a Levi decomposition $H_{\widetilde{Z}} = W_{H_{\widetilde{Z}}} \rtimes G_{H_{\widetilde{Z}}}$ and choose a compatible Levi decomposition $P = W \rtimes G$ (as is shown in Lemma 9.7). Let P^* be the group generated by GQ , then $\mathcal{R}_u(P^*) = W^*$ and $P^*/W^* = G$. The group P^* defines a connected mixed Shimura datum (P^*, \mathcal{X}^{*+}) with $\mathcal{X}^{*+} = P^*(\mathbb{R})^+ U^*(\mathbb{C})\widetilde{z}$. Now $\widetilde{Z} = H_{\widetilde{Z}}(\mathbb{R})^+ U_{H_{\widetilde{Z}}}(\mathbb{C})\widetilde{z} \subset \mathcal{X}^{*+}$. But \widetilde{Z} is Hodge generic in \mathcal{X}^+ by assumption, hence $P = P^*$ and $W = W^*$. So $W_{H_{\widetilde{Z}}} \triangleleft W$ and hence $W_{H_{\widetilde{Z}}} \triangleleft P$.

Use the notation in §2.2. We are done if we can prove:

$$\forall u \in U, \forall v \in V, \text{ and } \forall g \in G_{H_{\widetilde{Z}}}, (u, v, 1)(0, 0, g)(-u, -v, 1) \in H_{\widetilde{Z}}.$$

By Corollary 2.14, there exist decompositions

$$U = U_N \oplus U_N^\perp \quad V = V_N \oplus V_N^\perp$$

as G -modules such that G_N acts trivially on U_N^\perp and V_N^\perp . Now

$$\begin{aligned} (u, v, 1)(0, 0, g)(-u, -v, 1) &= (u, v, g)(-u, -v, 1) \\ &= (u - g \cdot u, v - g \cdot v, g) \\ &= ((u_N + u_N^\perp) - g \cdot (u_N + u_N^\perp), (v_N + v_N^\perp) - g \cdot (v_N + v_N^\perp), g) \\ &= (u_N - g \cdot u_N, v_N - g \cdot v_N, g) \\ &= (u_N, v_N, 1)(0, 0, g)(-u_N, -v_N, 1) \in H_{\bar{Z}}, \end{aligned}$$

where the last inclusion follows from Lemma 9.3. \square

10. AX-LINDEMANN-WEIERSTRASS PART 2: ESTIMATE

This section is devoted to prove Proposition 9.6. The proof uses essentially the “block family” version of Pila-Wilkie’s counting theorem [47, Theorem 3.6].

Keep notation and assumptions as in the last section and denote by $\pi: (P, \mathcal{X}^+) \rightarrow (G, \mathcal{X}_G^+)$. The group $G = Z(G)^\circ H_1 \dots H_r$ is an almost direct product, where H_i ’s are non-trivial simple groups and are normal in G . We have a decomposition

$$(G^{\text{ad}}, \mathcal{X}_G^+) \simeq \prod_{i=1}^r (H_i^{\text{ad}}, \mathcal{X}_{H_i}^+)$$

by [40, 3.6]. Let $S_G^{\text{ad}} := \Gamma_G^{\text{ad}} \backslash \mathcal{X}_G^+$. Shrinking Γ_G^{ad} if necessary, we may assume $S_G^{\text{ad}} \simeq \prod_{i=1}^r S_{H_i}$, where S_{H_i} is a connected pure Shimura variety associated with $(H_i^{\text{ad}}, \mathcal{X}_{H_i}^+)$.

Without loss of generality we may assume $G_N = H_1 \dots H_l$. It suffices to prove $H_i < G_{H_{\bar{Z}}}$ for each $i = 1, \dots, l$. The case $l = 0$ is trivial, so we assume that $l \geq 1$. Define $Q_i := \pi^{-1}(H_i)$.

10.1. Fundamental set and definability. The goal of this subsection is to prove that there exists $\mathcal{F} \subset \mathcal{X}^+$ a fundamental set for the action of Γ on \mathcal{X}^+ such that $\text{unif}|_{\mathcal{F}}$ is definable.

First of all, by the Reduction Lemma (Lemma 2.12), it suffices to prove the existence of such a fundamental set for (P, \mathcal{X}^+) pure and $(P, \mathcal{X}^+) = (P_{2g}, \mathcal{X}_{2g}^+)$. The case where (P, \mathcal{X}^+) is pure is guaranteed by Klingler-Ullmo-Yafaev [35, Theorem 4.1]. Now we prove the case $(P, \mathcal{X}^+) = (P_{2g}, \mathcal{X}_{2g}^+)$.

We draw the following diagram to make the notation more clear:

$$\begin{array}{ccc} \mathcal{X}_{2g}^+ & \xrightarrow{\pi_{P/U}} & \mathcal{X}_{2g,a}^+ \\ \text{unif} \downarrow & & \downarrow \text{unif}_{P/U} \\ S & \xrightarrow{[\pi_{P/U}]} & S_{P/U} \end{array} .$$

In this case, $[\pi_{P/U}]: S \rightarrow S_{P/U}$ is an algebraic \mathbb{G}_m -torsor. By Peterzil-Starchenko [45, Theorem 1.3], there exists a fundamental set $\mathcal{F}_{P/U}$ for the action of Γ/Γ_U on $\mathcal{X}_{2g,a}^+$ such that $\text{unif}_{P/U}|_{\mathcal{F}_{P/U}}$ is definable (recall that if $g = 0$, then $\mathcal{X}_{2g}^+ = \mathbb{C}$, $S = \mathbb{C}^*$, $\text{unif} = \exp$ and $S_{P/U}$ is a

point). Let us now construct a fundamental set for the action of Γ on \mathcal{X}_{2g}^+ such that $\text{unif}|_{\mathcal{F}}$ is definable and $\pi_{P/U}(\mathcal{F}) = \mathcal{F}_{P/U}$.

Since any variety over a field is quasi-compact in the Zariski topology, there exists a finite Zariski open covering $\{V_\alpha\}_{\alpha \in \Lambda}$ of $S_{P/U}$ such that $S|_{V_\alpha} \simeq \mathbb{C}^* \times V_\alpha$ and these isomorphisms are algebraic. Define $U_\alpha := S|_{V_\alpha} = [\pi_{P/U}]^{-1}(V_\alpha)$ for every $\alpha \in \Lambda$. Then we have

$$\text{unif}|_{\text{unif}^{-1}(U_\alpha)} : \text{unif}^{-1}(U_\alpha) \xrightarrow[\varphi]{} U_{2g}(\mathbb{C}) \times \text{unif}_{P/U}^{-1}(V_\alpha) \rightarrow (\mathbb{C}^*) \times V_\alpha \simeq U_\alpha,$$

where φ is semi-algebraic (Proposition 4.3), the last isomorphism is algebraic and the middle morphism is $(\exp, \text{unif}_{P/U}|_{\text{unif}_{P/U}^{-1}(V_\alpha)})$. Let $\mathcal{F}_U := \{s \in \mathbb{C} \mid -1 < \Re \Im(s) < 1\}$ and let $\mathcal{F}_\alpha := \varphi^{-1}(\mathcal{F}_U \times \mathcal{F}_{P/U, \alpha})$. Then $\text{unif}|_{\mathcal{F}_\alpha}$ is definable. Now $\mathcal{F} := \cup \mathcal{F}_\alpha$ (remember that this is a finite union) satisfies the conditions we want.

Now we return to arbitrary (P, \mathcal{X}^+) . We have proved the existence of an \mathcal{F} as stated at the beginning of this subsection. Let us choose such an \mathcal{F} more carefully. First of all replace \mathcal{F} by $\gamma \mathcal{F}$ if necessary to make sure $\mathcal{F} \cap \tilde{Z} \neq \emptyset$. Next define $\mathcal{F}_G := \pi(\mathcal{F}) \subset \mathcal{X}_G^+ \simeq \prod_{i=1}^r \mathcal{X}_{H,i}^+$. Denote by q_i the i -th projection and $\mathcal{F}_{H,i} := q_i(\mathcal{F}_G)$. There exist some $\gamma_1 = 1, \dots, \gamma_s \in \Gamma_G < \Gamma$ such that $\prod_{i=1}^r \mathcal{F}_{H,i} \subset \cup_{j=1}^s \gamma_j \mathcal{F}_G$. Consider

$$\mathcal{F}' := \left(\bigcup_{j=1}^s \gamma_j \mathcal{F} \right) \cap \pi^{-1} \left(\prod_{i=1}^r \mathcal{F}_{H,i} \right),$$

then \mathcal{F}' is a fundamental set for the action of Γ on \mathcal{X}^+ and $\text{unif}|_{\mathcal{F}'}$ is definable. Furthermore, $\pi(\mathcal{F}') = \prod_{i=1}^r \mathcal{F}_{H,i}$ and $\mathcal{F}_{H,i} = q_i \pi(\mathcal{F}')$. We still have $\mathcal{F}' \cap \tilde{Z} \neq \emptyset$ since $\mathcal{F} \subset \mathcal{F}'$. Now replace \mathcal{F} by \mathcal{F}' .

10.2. Counting points and conclusion. We shall work from now on with an \mathcal{F} satisfying the conditions in the last paragraph of the previous subsection. By Lemma 9.5, $\overline{Y_G} = \prod_{i=1}^l H_i(\mathbb{R})^+ \tilde{Z}_G$. Fix a point $\tilde{z} \in \mathcal{F} \cap \tilde{Z}$. Define the following Shimura morphisms for each $i = 1, \dots, l$

$$\begin{array}{ccc} (G, \mathcal{X}_G^+) & \xrightarrow{p_i} & (G_i, \mathcal{X}_{G,i}^+) := (G^{\text{ad}}, \mathcal{X}_G^+) / \prod_{j \neq i} H_j^{\text{ad}} \\ \text{unif}_G \downarrow & & \text{unif}_{G,i} \downarrow \\ S_G & \xrightarrow{[p_i]} & S_{G,i} \end{array}.$$

Fix $i \in \{1, \dots, l\}$. Define $\tilde{Y}_{G,i} := p_i(\tilde{Y}_G) = H_i^{\text{ad}}(\mathbb{R})^+ \pi_i(\tilde{z}_G)$, $\tilde{Z}_{G,i} := p_i(\tilde{Z}_G)$ and $Y_{G,i} := [p_i](Y_G)$, then $\text{unif}_{G,i}(\tilde{Z}_{G,i})$ is Zariski dense in $\overline{Y_{G,i}}$ by Lemma 9.5. If $\dim(\tilde{Z}_{G,i}) = 0$, then $\tilde{Z}_{G,i}$ is a finite set of points since it is algebraic. But then $\text{unif}_{G,i}(\tilde{Z}_{G,i})$, and hence $\overline{Y_{G,i}} = \text{unif}_{G,i}(\tilde{Z}_{G,i})$ is also a finite set of points. So $\dim(Y_{G,i}) = 0$, which contradicts $\tilde{Y}_{G,i} = H_i^{\text{ad}}(\mathbb{R})^+ \pi_i(\tilde{z}_G)$. To sum it up, $\dim(\tilde{Z}_{G,i}) > 0$. For further convenience, we will denote by $\pi_i := p_i \circ \pi$.

Take an algebraic curve $C_{G,i} \subset \tilde{Z}_{G,i}$ passing through $\pi_i(\tilde{z})$. Now $\pi_i(\tilde{Z} \cap \pi_i^{-1}(C_{G,i})) = \tilde{Z}_{G,i} \cap C_{G,i} = C_{G,i}$, and hence there exists an algebraic curve $C \subset \tilde{Z} \cap \pi_i^{-1}(C_{G,i})$ passing through \tilde{z} such that $\dim(\pi_i(C)) = 1$.

Let $\mathcal{F}_{G,i} := p_i(\mathcal{F}_G)$, then it is a fundamental set of $\text{unif}_{G,i}$ and $\text{unif}_{G,i}|_{\mathcal{F}_{G,i}}$ is definable. We define for any irreducible semi-algebraic subvariety A (resp. $A_{G,i}$) of $\text{unif}^{-1}(Y)$ (resp. $\text{unif}_{G,i}^{-1}(\overline{Y_{G,i}})$)

the following sets: define

$$\begin{aligned} \Sigma^{(i)}(A) &:= \{g \in Q_i(\mathbb{R}) \mid \dim(gA \cap \text{unif}^{-1}(Y) \cap \mathcal{F}) = \dim(A)\} \\ (\text{resp. } \Sigma_G^{(i)}(A_{G,i}) &:= \{g \in H_i^{\text{ad}}(\mathbb{R}) \mid \dim(gA_{G,i} \cap \text{unif}_{G,i}^{-1}(\overline{Y_{G,i}}) \cap \mathcal{F}_{G,i}) = \dim(A_{G,i})\}) \end{aligned}$$

and

$$\begin{aligned} \Sigma'^{(i)}(A) &:= \{g \in Q_i(\mathbb{R}) \mid g^{-1}\mathcal{F} \cap A \neq \emptyset\} \\ (\text{resp. } \Sigma'_G{}^{(i)}(A_{G,i}) &:= \{g \in H_i^{\text{ad}}(\mathbb{R}) \mid g^{-1}\mathcal{F}_{G,i} \cap A_{G,i} \neq \emptyset\}). \end{aligned}$$

Then $\Sigma^{(i)}(A)$ and $\Sigma_G^{(i)}(A_{G,i})$ are by definition definable. Let $\Gamma_{G,i}^{\text{ad}} := p_i(\Gamma_G^{\text{ad}})$.

Lemma 10.1. $\Sigma'^{(i)}(A) \cap \Gamma = \Sigma^{(i)}(A) \cap \Gamma$ (resp. $\Sigma'_G{}^{(i)}(A_{G,i}) \cap \Gamma_{G,i}^{\text{ad}} = \Sigma_G^{(i)}(A_{G,i}) \cap \Gamma_{G,i}^{\text{ad}}$).

Proof. The proof, which we include for completeness, is the same as [64, Lemma 5.2]. First of all $\Sigma^{(i)}(A) \cap \Gamma \subset \Sigma'^{(i)}(A) \cap \Gamma$ by definition. Conversely for any $\gamma \in \Sigma'^{(i)}(A) \cap \Gamma$, $\gamma^{-1}\mathcal{F} \cap A$ contains an open subspace of A since \mathcal{F} is by choice open in \mathcal{X}^+ . Hence $\gamma A \cap \text{unif}^{-1}(Y) \cap \mathcal{F} = \gamma A \cap \mathcal{F}$ contains an open subspace of γA which must be of dimension $\dim(A)$. Hence $\gamma \in \Sigma^{(i)}(A) \cap \Gamma$. The proof for $A_{G,i}$ is the same. \square

This lemma implies

$$(10.1) \quad \begin{aligned} \Sigma^{(i)}(C) \cap \Gamma &= \Sigma'^{(i)}(C) \cap \Gamma \subset \Sigma'^{(i)}(\tilde{Z}) \cap \Gamma = \Sigma^{(i)}(\tilde{Z}) \cap \Gamma \\ (\text{resp. } \Sigma_G^{(i)}(C_{G,i}) \cap \Gamma_{G,i}^{\text{ad}} &= \Sigma'_G{}^{(i)}(C_{G,i}) \cap \Gamma_{G,i}^{\text{ad}} \subset \Sigma'_G{}^{(i)}(\tilde{Z}_{G,i}) \cap \Gamma_{G,i}^{\text{ad}} = \Sigma_G^{(i)}(\tilde{Z}_{G,i}) \cap \Gamma_{G,i}^{\text{ad}}) \end{aligned}$$

Lemma 10.2. $\pi_i(\Gamma \cap \Sigma^{(i)}(C)) = \Gamma_{G,i}^{\text{ad}} \cap \Sigma_G^{(i)}(C_{G,i})$.

Proof. By Lemma 10.1, it suffices to prove $\pi_i(\Gamma \cap \Sigma'^{(i)}(C)) = \Gamma_{G,i}^{\text{ad}} \cap \Sigma'_G{}^{(i)}(C_{G,i})$. The inclusion \subset is clear by definition. For the other inclusion, $\forall \gamma_{G,i} \in \Gamma_{G,i}^{\text{ad}} \cap \Sigma'_G{}^{(i)}(C_{G,i})$, $\exists c_{G,i} \in C_{G,i}$ such that $\gamma_{G,i} \cdot c_{G,i} \in \mathcal{F}_{G,i}$.

Take a point $c \in C$ such that $\pi_i(c) = c_{G,i}$ and define $c_G := \pi(c) \in \mathcal{X}_G^+$. Suppose that under the decomposition

$$(G^{\text{ad}}, \mathcal{X}_G^+) \simeq \prod_{i=1}^r (H_i^{\text{ad}}, \mathcal{X}_{H,i}^+)$$

of [40, 3.6], $c_G = (c_{G,1}, \dots, c_{G,r})$. Then by choice of \mathcal{F}_G , there exists $\gamma'_G \in \Gamma_G^{\text{ad}}$ whose i -th coordinate is precisely the $\gamma_{G,i}$ in the last paragraph such that $\gamma'_G \cdot c_G \in \mathcal{F}_G$.

Let $\gamma_G \in \Gamma_G$ be such that its image under $\Gamma_G \rightarrow \Gamma_G^{\text{ad}}$ is γ'_G , then $\gamma_G \cdot c \in \pi^{-1}(\mathcal{F}_G)$. Therefore there exist $\gamma_V \in \Gamma_V$, $\gamma_U \in \Gamma_U$ such that $(\gamma_U, \gamma_V, \gamma_G)c \in \mathcal{F}$. Denote by $\gamma = (\gamma_U, \gamma_V, \gamma_G)$, then $\gamma \in \Gamma \cap \Sigma'^{(i)}(C)$ and $\pi_i(\gamma) = \gamma_{G,i}$. \square

For $T > 0$, define

$$\Theta_G^{(i)}(C_{G,i}, T) := \{\gamma_G \in \Gamma_{G,i}^{\text{ad}} \cap \Sigma_G^{(i)}(C_{G,i}) \mid H(\gamma_G) \leq T\}.$$

Proposition 10.3. *There exists a constant $\delta > 0$ s.t. for all $T \gg 0$, $|\Theta_G^{(i)}(C_{G,i}, T)| \geq T^\delta$.*

Proof. This follows directly from [35, Theorem 1.3] applied to $((G_i, \mathcal{X}_{G,i}^+, S_{G,i}, \overline{\tilde{Z}_{G,i}})$. \square

Let us prove how these facts imply $H_i < G_{H_{\bar{Z}}}$.

Take a faithful representation $G^{\text{ad}} \hookrightarrow \text{GL}_n$ which sends Γ_G^{ad} to $\text{GL}_n(\mathbb{Z})$. Consider the definable set $\Sigma_G^{(i)}(C_{G,i})$. By the theorem of Pila-Wilkie ([47, Theorem 3.6]), there exist $J = J(\delta)$ definable block families

$$B^j \subset \Sigma_G^{(i)}(C_{G,i}) \times \mathbb{R}^l, \quad j = 1, \dots, J$$

and $c = c(\delta) > 0$ such that for all $T \gg 0$, $\Theta_G^{(i)}(C_{G,i}, T^{1/2n})$ is contained in the union of at most $cT^{\delta/4n}$ definable blocks of the form B_y^j ($y \in \mathbb{R}^l$). By Proposition 10.3, there exist a $j \in \{1, \dots, J\}$ and a block $B_{G,i} := B_{y_0}^j$ of $\Sigma_G^{(i)}(C_{G,i})$ containing at least $T^{\delta/4n}$ elements of $\Theta_G^{(i)}(C_{G,i}, T^{1/2n})$.

Let $\Sigma^{(i)} := \Sigma^{(i)}(C) \cap \Sigma^{(i)}(\tilde{Z})$, which is by definition a definable set. Consider $X^j := (\pi_i \times 1_{\mathbb{R}^l})^{-1}(B^j) \cap (\Sigma^{(i)} \times \mathbb{R}^l)$, which is a definable family since π_i is algebraic.

By [23, Ch. 3, 3.6], there exists a number $n_0 > 0$ such that each fibre X_y^j has at most n_0 connected components. So the definable set $\pi_i^{-1}(B_{G,i}) \cap \Sigma^{(i)}$ has at most n_0 connected components. Now

$$\pi_i(\pi_i^{-1}(B_{G,i}) \cap \Sigma^{(i)} \cap \Gamma) = B_{G,i} \cap \pi_i(\Sigma^{(i)}(C) \cap \Gamma) = B_{G,i} \cap \Sigma_G^{(i)}(C_{G,i}) \cap \Gamma_{G,i}^{\text{ad}} = B_{G,i} \cap \Gamma_{G,i}^{\text{ad}}$$

by (10.1) and Lemma 10.2. So there exists a connected component B of $\pi_i^{-1}(B_{G,i}) \cap \Sigma^{(i)}$ such that $\pi_i(B \cap \Gamma)$ contains at least $T^{\delta/4n}/n_0$ elements of $\Theta_G^{(i)}(C_{G,i}, T^{1/2n})$.

We have $B\tilde{Z} \subset \text{unif}^{-1}(Y)$ since $\Sigma^{(i)}(\tilde{Z})\tilde{Z} \subset \text{unif}^{-1}(Y)$ by analytic continuation, and $\tilde{Z} \subset \sigma^{-1}B\tilde{Z}$ for any $\sigma \in B \cap \Gamma$. But B is connected, and therefore $\sigma^{-1}B\tilde{Z} = \tilde{Z}$ by maximality of \tilde{Z} and [50, Lemma 4.1]. So $\forall \sigma \in B \cap \Gamma$,

$$B \subset \sigma \text{Stab}_{Q_i(\mathbb{R})}(\tilde{Z}).$$

Fix a $\gamma_0 \in B \cap \Gamma$ such that $\pi_i(\gamma_0) \in \Theta_G^{(i)}(C_{G,i}, T^{1/2n})$. We have already shown that $\pi_i(B \cap \Gamma)$ contains at least $T^{\delta/4n}/n_0$ elements of $\Theta_G^{(i)}(C_{G,i}, T^{1/2n})$. For any $\gamma'_{G,i} \in \pi_i(B \cap \Gamma) \cap \Theta_G^{(i)}(C_{G,i}, T^{1/2n})$, let γ' be one of its pre-images in $B \cap \Gamma$. Then $\gamma := \gamma'^{-1}\gamma_0$ is an element of $\Gamma \cap \text{Stab}_{Q_i(\mathbb{R})}(\tilde{Z}) = \Gamma_{\tilde{Z}} \cap Q_i(\mathbb{R})$ such that $H(\pi_i(\gamma)) \ll T^{1/2}$. Therefore for $T \gg 0$, $\pi_i(\Gamma_{\tilde{Z}}) \cap H_i^{\text{ad}}(\mathbb{R})$ contains at least $T^{\delta/4n}/n_0$ elements $\gamma_{G,i}$ such that $H(\gamma_{G,i}) \leq T$. Hence $\dim(\pi_i(H_{\tilde{Z}}) \cap H_i^{\text{ad}}) > 0$ since $\pi_i(H_{\tilde{Z}}) \cap H_i^{\text{ad}}$ contains infinitely many rational points. But $\pi_i(H_{\tilde{Z}}) = p_i\pi(H_{\tilde{Z}}) = p_i(G_{H_{\tilde{Z}}})$ by definition. So $H_i^{\text{ad}} < p_i(G_{H_{\tilde{Z}}})$ since H_i^{ad} is simple and $p_i(G_{H_{\tilde{Z}}}) \cap H_i^{\text{ad}} \triangleleft H_i^{\text{ad}}$ by Corollary 9.4.

As a normal subgroup of G_N , $G_{H_{\tilde{Z}}}$ is the almost direct product of some H_j 's ($j = 1, \dots, l$). So $H_i^{\text{ad}} < p_i(G_{H_{\tilde{Z}}})$ implies $H_i < G_{H_{\tilde{Z}}}$. Now we are done.

Remark 10.4. In the proof of the pure case by Klingler-Ullmo-Yafaev [35], it suffices to use a non-family version of Pila-Wilkie ([35, Theorem 6.1]). However this is not enough for our proof, since otherwise the n_0 would depend on T . Hence it is important to use a family version of Pila-Wilkie ([47, Theorem 3.6]).

11. AX-LINDEMANN-WEIERSTRASS PART 3: THE UNIPOTENT PART

We prove in this section Theorem 7.6. We use the same notation as the first paragraph of §7 and §7.3. Assume $\dim_{\mathbb{C}} T = m$ and $\dim_{\mathbb{C}} A = n$.

Proof of Theorem 7.6. First of all we may assume that \tilde{Z} is of positive dimension since every point is a weakly special subvariety of dimension 0. For any fundamental set \mathcal{F} of the action of Γ_W on $W(\mathbb{R})U(\mathbb{C})$, define

$$\Sigma(\tilde{Z}) := \{g \in W(\mathbb{R}) \mid \dim(g\tilde{Z} \cap \text{unif}^{-1}(Y) \cap \mathcal{F}) = \dim(\tilde{Z})\}$$

and

$$\Sigma'(\tilde{Z}) := \{g \in W(\mathbb{R}) \mid g^{-1}\mathcal{F} \cap \tilde{Z} \neq \emptyset\},$$

then by Lemma 10.1,

$$(11.1) \quad \Sigma(\tilde{Z}) \cap \Gamma_W = \Sigma'(\tilde{Z}) \cap \Gamma_W$$

Let $\Gamma_U := \Gamma \cap U(\mathbb{Q})$ and let $\Gamma_V := \Gamma_W/\Gamma_U$.

Case i : $E=A$. This is [52, Theorem 2.1 and pp9 Remark 1]. A proof can be found in Appendix. In this case, $W = V$ and $\Gamma_V = \bigoplus_{i=1}^{2n} \mathbb{Z}e_i \subset \text{Lie}(A) = \mathbb{C}^n = \mathbb{R}^{2n}$ is a lattice. Denote by $\text{unif} : \text{Lie}(A) \rightarrow A$. Let $\mathcal{F}_V := \sum_{i=1}^{2n} (-1, 1)e_i$, then \mathcal{F}_V is a fundamental set for the action of Γ_V on $\text{Lie}(A)$ such that $\text{unif}|_{\mathcal{F}_V}$ is definable.

Case ii : $E=T$. This is a consequence of Ax's theorem [5] [42, Corollary 3.6]. A proof of this can be found in Appendix. In this case, $W = U$. Let $\mathcal{F}_U := \{s \in \mathbb{C} \mid -1 < \Re \mathfrak{F}(s) < 1\}^m$, then \mathcal{F}_U is a fundamental set for the action of Γ_U on $U(\mathbb{C})$ such that $\text{unif}|_{\mathcal{F}_U}$ is definable.

Case iii : general E . Unlike the rest of the paper, the symbol π in this section denotes the map

$$(11.2) \quad \begin{array}{ccc} W(\mathbb{R})U(\mathbb{C}) & \xrightarrow{\pi} & V(\mathbb{R}) \\ \downarrow \text{unif} & & \downarrow \text{unif}_V \\ E & \xrightarrow{[\pi]} & A \end{array}$$

Take $\mathcal{F}_V \subset V(\mathbb{R})$ any fundamental set for the action of Γ_V on $V(\mathbb{R})$ such that $\text{unif}_V|_{\mathcal{F}_V}$ is definable. We claim that:

$$(11.3) \quad \begin{array}{l} \text{There exists a fundamental set } \mathcal{F} \text{ for the action of } \Gamma_W \text{ on } W(\mathbb{R})U(\mathbb{C}) \\ \text{such that } \text{unif}|_{\mathcal{F}} \text{ is definable and } \pi(\mathcal{F}) = \mathcal{F}_V. \end{array}$$

By Reduction Lemma (Lemma 2.12), it suffices to prove this for $E = E_1 \times_A \dots \times_A E_m$ where E_i 's are \mathbb{G}_m -torsors over A . But then it suffices to prove for the case $m = 1$. For this case, the proof is similar to §10.1.

Let Y_0 be the minimal closed irreducible subvariety of E such that $\tilde{Z} \subset \text{unif}^{-1}(Y_0)$, then \tilde{Z} is maximal irreducible algebraic in $\text{unif}^{-1}(Y_0)$. Hence we may assume that $Y = Y_0$. Let N be the connected algebraic monodromy group of Y^{sm} and let $V_N := (N \cap W)/(N \cap U)$. Let \tilde{Y} be the complex analytic irreducible component of $\text{unif}^{-1}(Y)$ which contains \tilde{Z} . For further convenience, we will denote by $\tilde{Z}_V := \pi(\tilde{Z})$, $\tilde{Y}_V := \pi(\tilde{Y})$ and $Y_V := [\pi](Y)$.

Repeating the proof of Lemma 9.5 (but using the conclusion of *Case i* instead of [35, Theorem 1.1]), we get that $\overline{\tilde{Y}_V} = V_N(\mathbb{R}) + \tilde{z}_V$ for some $\tilde{z}_V \in \tilde{Z}_V$ is weakly special, and $\text{unif}_V(\tilde{Z}_V) = \overline{\tilde{Y}_V}$. Remark that by GAGA, these closures could be taken in the complex analytic topology (i.e. the topology whose closed sets are complex analytic sets) or the Zariski topology. If V_N is trivial, then we are actually in the situation of *Case ii*, and therefore \tilde{Z} is weakly special. From now

on, suppose that $\dim(V_N) > 0$. Replace S by its smallest special subvariety containing Y_0 , then $N \triangleleft P$ by Theorem 3.7. Hence V_N is a $G = \text{MT}(b)$ -submodule of V .

Define $W_0 := (\Gamma_W \cap \text{Stab}_{W(\mathbb{R})U(\mathbb{C})}(\tilde{Z})^{\text{Zar}})^\circ$, $U_0 := W_0 \cap U$ and $V_0 := \pi(W_0) = W_0/U_0$. The proof is somehow technical, so we will divide it into several steps.

Step I. Let V^\dagger be the smallest subgroup of V_N such that $\tilde{Z}_V \subset V^\dagger(\mathbb{R}) + \tilde{z}_V$. In Step I, we will prove $V^\dagger < V_0$.

Step I(i). We know that $A = \Gamma_V \backslash V(\mathbb{R})$ and $V(\mathbb{Q}) \simeq \Gamma_V \otimes_{\mathbb{Z}} \mathbb{Q}$. Consider any \mathbb{Q} -quotient group V' of V of dimension 1

$$p': V \rightarrow V'$$

such that $\dim(p'(V^\dagger)) = 1$. By abuse of notation, we shall denote its induced map $V(\mathbb{R}) \rightarrow V'(\mathbb{R})$ also by p' . Now let $\Gamma_{V'} := p'(\Gamma_V)$, then $\Gamma_{V'} \simeq \mathbb{Z}$ since p' is defined over \mathbb{Q} . Write $\Gamma_{V'} = \mathbb{Z}e'$, and let $\mathcal{F}_{V'} := (-1, 1)e'$. Then $\mathcal{F}_{V'}$ is a fundamental set for the action of $\Gamma_{V'}$ on $V'(\mathbb{R})$. Define $A' = \Gamma_{V'} \backslash V'(\mathbb{R}) \simeq \mathbb{Z} \backslash \mathbb{R}$, $\text{unif}_{V'}: V'(\mathbb{R}) \rightarrow A'$ the uniformization and $[p']: A \rightarrow A'$ the map induced by p' . Then $\text{unif}_{V'}|_{\mathcal{F}_{V'}}$ is definable (even in \mathbb{R}_{an}). Define $Y_{V'} := [p'](Y_V)$ and $\tilde{Y}_{V'} := p'(\tilde{Y}_V)$.

Let $V'' := \text{Ker}(p')$. The exact sequence of free \mathbb{Z} -modules

$$1 \rightarrow \Gamma_{V''} := \Gamma_V \cap V''(\mathbb{Q}) \simeq \mathbb{Z}^{2n-1} \rightarrow \Gamma_V \simeq \mathbb{Z}^{2n} \rightarrow \Gamma_{V'} \simeq \mathbb{Z} \rightarrow 1$$

splits, and hence $\Gamma_V \simeq \Gamma_{V''} \oplus \Gamma_{V'}$. This induces $V \simeq V'' \oplus V'$. Write $\Gamma_{V''} = \sum_{i=2}^{2n} \mathbb{Z}e''_i$ and take $\mathcal{F}_{V''} := \sum_{i=2}^n (-1, 1)e''_i$. Define $\mathcal{F}_V := \mathcal{F}_{V''} \oplus \mathcal{F}_{V'}$. Then \mathcal{F}_V is a fundamental set for the action of Γ_V on $V(\mathbb{R})$ such that $\text{unif}_V|_{\mathcal{F}_V}$ is definable (even in \mathbb{R}_{an}). Define \mathcal{F} as in (11.3).

Since $p(V^\dagger) = V'$ by choice of V' , $\dim_{\mathbb{R}} p'(\tilde{Z}_V) > 0$ by minimality of V^\dagger . Hence $p'(\tilde{Z}_V) = V'(\mathbb{R})$ since $p'(\tilde{Z}_V)$ is connected.

Remark 11.1. *If we only request (V', p') to satisfy $p'(V_N) = 1$, then we do not know whether $\dim_{\mathbb{R}}(p'(\tilde{Z}_V)) > 0$. This is because we are considering the real analytic topology (i.e. the topology whose closed sets are real analytic sets) on A' and the complex analytic topology (i.e. the topology whose closed sets are complex analytic sets) on A , and hence $\text{unif}_V(\tilde{Z}_V) = \overline{Y_V}$ does NOT imply $\text{unif}_{V'}(\tilde{Z}_{V'}) = \overline{Y_{V'}}$. To overcome this problem, we introduce the seemingly strange subgroup V^\dagger of V_N . We will prove (Step II) that V_0 is a $\text{MT}(b)$ -module with the help of V^\dagger . Then we prove the comparable result of Theorem 9.8(1) for the unipotent part in Step III.*

Let C be an \mathbb{R} -algebraic subvariety of \tilde{Z} of \mathbb{R} -dimension 1 such that $p'\pi(C) = V'(\mathbb{R})$. Define furthermore

$$\Sigma(C) := \{g \in W(\mathbb{R}) \mid \dim_{\mathbb{R}}(gC \cap \text{unif}^{-1}(Y) \cap \mathcal{F}) = 1\}$$

and

$$\Sigma'(C) := \{g \in W(\mathbb{R}) \mid g^{-1}\mathcal{F} \cap C \neq \emptyset\}.$$

The set $\Sigma(C)$ is by definition definable. By Lemma 10.1,

$$(11.4) \quad \Sigma'(C) \cap \Gamma_W = \Sigma(C) \cap \Gamma_W$$

For $M > 0$, define

$$\Theta_{V'}(V'(\mathbb{R}), M) = \{\gamma_{V'} \in \Gamma_{V'} \mid H(\gamma_{V'}) \leq M\}.$$

Then

$$(11.5) \quad |\Theta_{V'}(V'(\mathbb{R}), M)| \gg M.$$

Step I(ii) is quite similar to the end of §10. Consider the definable set $V'(\mathbb{R})$. By the theorem of Pila-Wilkie ([47, Theorem 3.6]), there exist J definable block families

$$B^j \subset V'(\mathbb{R}) \times \mathbb{R}^l, \quad j = 1, \dots, J$$

and $c > 0$ such that for all $M \gg 0$, $\Theta_{V'}(V'(\mathbb{R}), M^{1/4})$ is contained in the union of at most $cM^{\delta/8}$ definable blocks of the form B_y^j ($y \in \mathbb{R}^l$). By (11.5), there exist a $j \in \{1, \dots, J\}$ and a block $B_{V'} := B_{y_0}^j$ of $V'(\mathbb{R})$ containing at least $M^{\delta/8}$ elements of $\Theta_{V'}(V'(\mathbb{R}), M^{1/4})$.

Let $\Sigma := \Sigma(C) \cap \Sigma(\tilde{Z})$, which is by definition a definable set. Consider $X^j := ((p'\pi) \times 1_{\mathbb{R}^l})^{-1}(B^j) \cap (\Sigma \times \mathbb{R}^l)$, which is a definable family since $p'\pi$ is \mathbb{R} -algebraic.

By [23, Ch. 3, 3.6], there exists a number $n_0 > 0$ such that each fibre X_y^j has at most n_0 connected components. So the definable set $\pi^{-1}(B_{V'}) \cap \Sigma$ has at most n_0 connected components. Now

$$p'\pi((p'\pi)^{-1}(B_{V'}) \cap \Sigma \cap \Gamma_W) = B_{V'} \cap p'\pi(\Sigma(C) \cap \Gamma_W) = B_{V'} \cap (V'(\mathbb{R}) \cap \Gamma_{V'}) = B_{V'} \cap \Gamma_{V'}$$

by (11.1), (11.4) and the choice of \mathcal{F} (remember that $\Gamma_V = \Gamma_{V''} \oplus \Gamma_{V'}$ and $\mathcal{F}_V = \mathcal{F}_{V''} \oplus \mathcal{F}_{V'}$). So there exists a connected component B of $(p'\pi)^{-1}(B_{V'}) \cap \Sigma$ such that $p'\pi(B \cap \Gamma_W)$ contains at least $M^{\delta/8}/n_0$ elements of $\Theta_{V'}(V'(\mathbb{R}), M^{1/4})$.

We have $B\tilde{Z} \subset \text{unif}^{-1}(Y)$ since $B \subset \Sigma(\tilde{Z})$ by (complex) analytic continuation, and $\tilde{Z} \subset \sigma_W^{-1}B\tilde{Z}$ for any $\sigma_W \in B \cap \Gamma_W$. But B is connected, and therefore $\sigma_W^{-1}B\tilde{Z} = \tilde{Z}$ by maximality of \tilde{Z} and [50, Lemma 4.1]. So

$$B \subset \sigma_W \text{Stab}_{W(\mathbb{R})}(\tilde{Z}).$$

Fix a $\sigma_W \in B \cap \Gamma_W$ such that $p'\pi(\sigma_W) \in \Theta_{V'}(V'(\mathbb{R}), M^{1/4})$. We have shown that $p'\pi(B \cap \Gamma_W)$ contains at least $M^{\delta/8}/n_0$ elements of $\Theta_{V'}(V'(\mathbb{R}), M^{1/4})$. For any $\sigma_{V'} \in p'\pi(B \cap \Gamma) \cap \Theta_{V'}(V'(\mathbb{R}), M^{1/4})$, let σ'_W be one of its pre-images in $B \cap \Gamma_W$. Then $\gamma_W := \sigma_W^{-1}\sigma'_W$ is an element of $\Gamma_W \cap \text{Stab}_{W(\mathbb{R})}(\tilde{Z})$ and $H(p'\pi(\gamma_W)) \ll M^{1/2}$. Therefore for $M \gg 0$, $p'\pi(\Gamma_W \cap \text{Stab}_{W(\mathbb{R})}(\tilde{Z}))$ contains at least $M^{\delta/8}/n_0$ elements $\gamma_{V'}$ such that $H(\gamma_{V'}) \leq M$. Therefore $\dim(p'\pi(W_0)) > 0$ since it is an infinite set. So $p'\pi(W_0) = V'$ since $\dim(V') = 1$. But V' is an arbitrary 1-dimensional quotient of V such that $p'(V^\dagger) = V'$. Therefore $V^\dagger < \pi(W_0) = V_0$.

Step II. We prove in this step that V_0 is a $\text{MT}(b)$ -module. This implies that W_0 is a $\text{MT}(b)$ -subgroup of W by Proposition 2.6(2).

By definition of V^\dagger , $\tilde{Z}_V \subset V^\dagger(\mathbb{R}) + \tilde{z}_V$. By definition of V_0 , $V_0(\mathbb{R}) + \tilde{z}_V \subset \tilde{Z}_V$. Now the conclusion of *Step I* implies $V_0 = V^\dagger$ and $\tilde{Z}_V = V_0(\mathbb{R}) + \tilde{z}_V$. However \tilde{Z}_V is complex, so $V_0(\mathbb{R})$ is a complex subspace of $V(\mathbb{R})$. Therefore by considering the complex structure of $V(\mathbb{R})$, we get that $V_0(\mathbb{R})$ is a $\text{MT}(b)(\mathbb{R})$ -module. So V_0 is a $\text{MT}(b)$ -module.

Step III. can be seen as an analogue to the proof of Theorem 9.8(1). Consider a fibre of \tilde{Z} over a point $v \in \pi(\tilde{Z})$ such that $\pi: W(\mathbb{C})/F_b^0 W(\mathbb{C}) \rightarrow \text{Lie}(A)$ is flat at v (such a point exists by generic flatness). Let \tilde{W} be an irreducible algebraic component of \tilde{Z}_v such that $\dim(\tilde{Z}_v) = \dim(\tilde{W})$, then since π is flat at v ,

$$\dim(\tilde{Z}) = \dim(\pi(\tilde{Z})) + \dim(\tilde{Z}_v) = \dim(\pi(\tilde{Z})) + \dim(\tilde{W}).$$

Consider the set $\tilde{F} := W_0(\mathbb{R})U_0(\mathbb{C})\tilde{W}$. It is semi-algebraic. The fact $\tilde{W} \subset \tilde{Z}$ implies that $\tilde{F} \subset \tilde{Z}$. So by [50, Lemma 4.1], there exists an irreducible algebraic subvariety of $W(\mathbb{C})/F_b^0 W(\mathbb{C})$, say \tilde{F}_{alg} , which contains \tilde{F} and is contained in \tilde{Z} . Since

$$\pi(\tilde{F}) = \pi(W_0)(\mathbb{R}) + v = \overline{\pi(\tilde{Z})}$$

and every fiber of $\pi|_{\tilde{F}_{\text{alg}}}$ has \mathbb{R} -dimension at least $\dim_{\mathbb{R}}(\tilde{W})$, we have

$$\dim(\tilde{F}_{\text{alg}}) \geq \dim(\pi(\tilde{F})) + \dim(\tilde{W}) = \dim(\pi(\tilde{Z})) + \dim(\tilde{W}) = \dim(\tilde{Z}).$$

So $\tilde{F} = \tilde{Z}$ since \tilde{Z} is irreducible. In other words, $\tilde{Z} = W_0(\mathbb{R})U_0(\mathbb{C})\tilde{Z}_v$ and \tilde{Z}_v is irreducible for any $v \in \pi(\tilde{Z})$.

Next for any $v \in \pi(\tilde{Z})$, let \tilde{W}' be an irreducible algebraic subvariety which contains \tilde{Z}_v and is contained in $\text{unif}^{-1}(Y)_v$, maximal for these properties. Then \tilde{W}' is weakly special by *Case ii*. Consider $\tilde{F}' := W_0(\mathbb{R})U_0(\mathbb{C})\tilde{W}'$. Let \tilde{Y} be the irreducible component of $\text{unif}^{-1}(Y)$ which contains \tilde{Z} , then $\tilde{W}' \subset \tilde{Y}$ and so $\tilde{F}' \subset \tilde{Y}$ by Lemma 9.2. But \tilde{F}' is semi-algebraic, and hence by [50, Lemma 4.1] there exists an irreducible algebraic subvariety of $W(\mathbb{C})/F_b^0 W(\mathbb{C})$, say \tilde{F}'_{alg} , which contains \tilde{F}' and is contained in \tilde{Y} . So $\tilde{Z} = W_0(\mathbb{R})U_0(\mathbb{C})\tilde{Z}_v \subset \tilde{F}'_{\text{alg}} \subset \text{unif}^{-1}(Y)$, and hence $\tilde{Z} = \tilde{F}'_{\text{alg}} = \tilde{F}'$ by the maximality of \tilde{Z} . So $\tilde{Z}_v = \tilde{W}'$, i.e.

$$(11.6) \quad \text{For any } v \in \pi(\tilde{Z}), \tilde{Z}_v \text{ is a maximal irreducible algebraic subvariety of } W(\mathbb{C})/F^0 W(\mathbb{C}) \text{ contained in } \text{unif}^{-1}(Y)_v.$$

Now that $\tilde{Z}_v = \tilde{W}'$ is weakly special, we can write $\tilde{Z}_v = U'(\mathbb{C}) + \tilde{z}$ with $U' < U$ and $\tilde{z} \in \tilde{Z}_v$. Then $U_0 < U'$. The product $W' := W_0 U'$ is a subgroup of W , and hence

$$\tilde{Z} = W_0(\mathbb{R})U_0(\mathbb{C})\tilde{Z}_v = W_0(\mathbb{R})U'(\mathbb{C})\tilde{z} = W'(\mathbb{R})U'(\mathbb{C})\tilde{z}.$$

So $W_0 = W'$ and $U_0 = U'$. In other words,

$$(11.7) \quad \tilde{Z} = \tilde{E} = W_0(\mathbb{R})U_0(\mathbb{C})\tilde{z}$$

for some point $\tilde{z} \in \tilde{Z}_v$.

Step IV. Let us now conclude that \tilde{Z} is weakly special.

First of all, $U_0 \triangleleft P$ by Proposition 2.6(2). Consider $(P, \mathcal{X}^+) \xrightarrow{\rho} (P, \mathcal{X}^+)/U_0$, then by definition \tilde{Z} is weakly special iff $\rho(\tilde{Z})$ is. Replace (P, \mathcal{X}^+) (resp. $W, \tilde{Z}, W_0, \tilde{z}$) by $(P, \mathcal{X}^+)/U_0$ (resp. $W/U_0, \rho(\tilde{Z}), W_0/U_0 = V_0, \rho(\tilde{z})$), then V_0 is a subgroup of W and $\tilde{Z} = V_0(\mathbb{R})\tilde{z}$. Use the notation of §2.2 and §4 and suppose $\tilde{z} = (\tilde{z}_U, \tilde{z}_V)$. By Proposition 7.2, \tilde{Z} is weakly special iff $\tilde{z}_V \in (N_W(V_0)/U)(\mathbb{R})$ iff $\Psi(V_0(\mathbb{R}), \tilde{z}_V) = 0$. We shall prove the last claim.

Define $Z := \text{unif}(\tilde{Z})$, $z = \text{unif}(\tilde{z})$ and $z_V = [\pi](z) \in A$, then $\pi(\tilde{Z}) = V_0(\mathbb{R}) + \tilde{z}_V$ and $[\pi](Z) = A_0 + z_V$ where $A_0 = \Gamma_{V_0} \setminus V_0(\mathbb{R})$ is an abelian subvariety of A . We can compute the fiber

$$(11.8) \quad Z_{z_V} = \left(\text{unif}(\Gamma_W \tilde{Z}) \right)_{z_V} = \tilde{z}_U + \frac{1}{2} \Psi(\Gamma_V, \tilde{z}_V) + \Gamma_U \pmod{\Gamma_U}.$$

We have $\Psi(V(\mathbb{R}), V(\mathbb{R})) \subset U(\mathbb{R})$ since Ψ is defined over \mathbb{Q} . Let us prove $\Psi(\Gamma_V, \tilde{z}_V) \subset U(\mathbb{Q})$. Fix an isomorphism $\Gamma_U \simeq \mathbb{Z}^m$, which induces an isomorphism $U(\mathbb{Q}) \simeq \mathbb{Q}^m$. Suppose that there exists a $u \in \Psi(\Gamma_V, \tilde{z}_V) \setminus U(\mathbb{Q})$, then at least one of the coordinates of u is irrational. Without loss

of generality we may assume that its first coordinate $u_1 \in \mathbb{R} \setminus \mathbb{Q}$. Denote by U_1 the \mathbb{Q} -subgroup of U corresponding to the first factor of $U(\mathbb{Q}) \simeq \mathbb{Q}^m$, then

$$\text{unif}(\tilde{z}_U + U_1(\mathbb{R})) \subset \overline{Z_{z_V}}$$

since $\{lu_1 \bmod \mathbb{Z} \mid l \in \mathbb{Z}\}$ is dense in $[0, 1]$. So $\overline{Z_{z_V}}$ contains

$$\text{unif}(\tilde{z}_U + U_1(\mathbb{C})),$$

and so does Y_{z_V} since $\overline{Z} \subset Y$. Let $v := v_0 + \tilde{z}_V \in V(\mathbb{R})$, then $\tilde{z}_U + U_1(\mathbb{C}) \subset \text{unif}^{-1}(Y)_v$. However $\tilde{Z}_{\tilde{z}_v} = \tilde{z}_U$ by (11.7) (recall that we have reduced to $W_0 = V_0$ and $U_0 = 0$), which contradicts (11.6). Hence $\Psi(\Gamma_V, \tilde{z}_V) \subset U(\mathbb{Q})$, and therefore $(1/2)\Psi(N\Gamma_V, \tilde{z}_V) \subset \Gamma_U$ for some $N \gg 0$ (since $\text{rank } \Gamma_V < \infty$). Now we can construct a new lattice Γ'_W with $N\Gamma_V$ and Γ_U . Γ'_W is of finite index in Γ_W . Replacing Γ_W by Γ'_W does not change the assumption or the conclusion of Ax-Lindemann-Weierstraß, so we may assume $(1/2)\Psi(\Gamma_V, \tilde{z}_V) \subset \Gamma_U$. Now we can define C^∞ -morphisms

$$f: A_0 + z_V \longrightarrow T$$

$$a_0 + z_V \mapsto \tilde{z}_U + (1/2)\Psi(v_0, \tilde{z}_V) \bmod \Gamma_U$$

and

$$s: A_0 + z_V \longrightarrow E|_{A_0 + z_V}$$

$$a_0 + z_V \mapsto (\tilde{z}_U + (1/2)\Psi(v_0, \tilde{z}_V), a_0 + z_V) \bmod \Gamma_W$$

where v_0 is any point of $V_0(\mathbb{R})$ such that $\text{unif}_V(v_0) = a_0$. But Z_a is a single point for all $a \in A_0 + z_V$ by (11.8), so s is the inverse of $[\pi]|_Z$, and therefore s is a holomorphic section of $E|_{A_0 + z_V} \rightarrow A_0 + z_V$. Locally on $U_i \subset A_0 + z_V$, s is represented by a holomorphic morphism $U_i \rightarrow T$, which must equal to $f|_{U_i}$ by definition. Hence f is holomorphic since being holomorphic is a local condition. So f is constant.

But $\Psi(0, \tilde{z}_V) = 0$, and therefore $(1/2)\Psi(V_0(\mathbb{R}), \tilde{z}_V) \subset \Gamma_U$. But $\Psi(V_0(\mathbb{R}), \tilde{z}_V)$ is continuous and $\Psi(0, \tilde{z}_V) = 0$, so $\Psi(V_0(\mathbb{R}), \tilde{z}_V) = 0$. Hence we are done. \square

12. CONSEQUENCE OF AX-LINDEMANN-WEIERSTRASS

12.1. Weakly special subvarieties defined by a fixed \mathbb{Q} -subgroup. Let $S = \Gamma \backslash \mathcal{X}^+$ be a connected mixed Shimura variety associated with the connected mixed Shimura datum (P, \mathcal{X}^+) and let $\text{unif}: \mathcal{X}^+ \rightarrow S$ be the uniformization. Suppose that N is a connected subgroup of P s.t. $N/(W \cap N) \hookrightarrow G$ is semi-simple. A subvariety of S is said to be weakly special defined by N if it is of the form $\text{unif}(i(\varphi^{-1}(y')))$ under the notation of Definition 5.2 s.t. $N = \text{Ker}(\varphi)$. Let $\mathfrak{F}(N)$ be the set of all weakly special subvarieties of S defined by N . The goal of this subsection is to prove:

Proposition 12.1. *If $\mathfrak{F}(N) \neq \emptyset$ and $N \ntriangleleft P$, then $\cup_{Z \in \mathfrak{F}(N)} Z$ is a finite union of proper special subvarieties of S .*

Proof. Take any $F \in \mathfrak{F}(N)$. Let \mathcal{F} be a fundamental domain for the action Γ on \mathcal{X}^+ . Suppose that $x' \in \mathcal{F}$ is such that $F = \text{unif}(N(\mathbb{R})^+ U_N(\mathbb{C}) x')$. Consider $Q' := N_P(N)$, the normalizer of N in P . By definition of weakly special subvarieties, there exists $(R', \mathcal{Z}^+) \hookrightarrow (P, \mathcal{X}^+)$ such that $h_{x'}: \mathbb{S}_{\mathbb{C}} \rightarrow P_{\mathbb{C}}$ factors through $R'_{\mathbb{C}}$ and $N \triangleleft R'$. Hence $R' < Q'$. Define $G_{Q'} := Q'/(W \cap Q')$.

Then $G_{Q'}/(Z(G) \cap G_{Q'})$ is reductive by [17, Lemma 4.3] or [60, Proposition 3.28], and hence $G_{Q'}$ is reductive. Write

$$1 \rightarrow W \cap Q' \rightarrow Q' \xrightarrow{\pi_{Q'}} G_{Q'} \rightarrow 1.$$

The group $G_{Q'} = Z(G_{Q'})^\circ G_{Q'}^{\text{nc}} G_{Q'}^{\text{c}}$ is an almost-direct product, where $G_{Q'}^{\text{nc}}$ (resp. $G_{Q'}^{\text{c}}$) is the product of the \mathbb{Q} -simple factors whose set of \mathbb{R} -points is non-compact (resp. compact). Let $G_Q := Z(G_{Q'})^\circ G_{Q'}^{\text{nc}}$ and then define $Q := \pi_{Q'}^{-1}(G_Q)$, then $h_{x'}$ factors through $Q_{\mathbb{C}}$ and $R' < Q$ by Definition 2.1(4). So $N \triangleleft Q$ and (Q, \mathcal{Y}^+) , where $\mathcal{Y}^+ := Q(\mathbb{R})^+ U_Q(\mathbb{C}) x'$, is a connected mixed Shimura subdatum of (P, \mathcal{X}^+) . But then $F \subset \text{unif}(\mathcal{Y}^+) \subset \cup_{Z \in \mathfrak{F}(N)} Z$.

Define $\mathfrak{Y}_Q := \{x \in \mathcal{X}^+ | h_x \text{ factors through } Q_{\mathbb{C}}\}$, then $Q(\mathbb{R})^+ U_Q(\mathbb{C}) \mathfrak{Y}_Q = \mathfrak{Y}_Q$. The discussion of last paragraph tells us that $F \subset \text{unif}(\mathfrak{Y}_Q)$ for any $F \in \mathfrak{F}(N)$. On the other hand, for any $x \in \mathfrak{Y}_Q$, (Q, \mathcal{Y}^+) , where $\mathcal{Y}^+ := Q(\mathbb{R})^+ U_Q(\mathbb{C}) x$, is a connected mixed Shimura subdatum of (P, \mathcal{X}^+) and hence $\text{unif}(N(\mathbb{R})^+ U_N(\mathbb{C}) x) \in \mathfrak{F}(N)$. Therefore $\text{unif}(\mathfrak{Y}_Q) \subset \cup_{Z \in \mathfrak{F}(N)} Z$. To sum it up, $\cup_{Z \in \mathfrak{F}(N)} Z = \text{unif}(\mathfrak{Y}_Q)$.

Now we are done if we can prove

Claim. *The set \mathfrak{Y}_Q is a finite union of $Q(\mathbb{R})^+ U_Q(\mathbb{C})$ -conjugacy classes. In other words, \mathfrak{Y}_Q is a finite union of connected mixed Shimura subdata of (P, \mathcal{X}^+) .*

Fix a special point x of \mathcal{X}^+ contained in \mathfrak{Y}_Q . There exists by definition a torus $T_x \subset Q$ such that $h_x : \mathbb{S}_{\mathbb{C}} \rightarrow Q_{\mathbb{C}}$ factors through $T_{x, \mathbb{C}}$. Furthermore, we may and do assume that $T_{x, \mathbb{C}}$ is a maximal torus of $Q_{\mathbb{C}}$. Let T be a maximal torus of $P_{\mathbb{C}}$ defined over \mathbb{Q} such that $T > T_x$. Take a Levi decomposition $P = W \rtimes G$ such that $T < G < P$. Then the composite $\mathbb{S}_{\mathbb{C}} \xrightarrow{h_x} T_{x, \mathbb{C}} < P_{\mathbb{C}} \xrightarrow{\pi} G_{\mathbb{C}} < P_{\mathbb{C}}$ equals h_x and is defined over \mathbb{R} by Definition 2.1(1).

For any other special point y of \mathcal{X}^+ contained in \mathfrak{Y}_Q , there exists $g \in Q(\mathbb{C})$ such that $gT_{x, \mathbb{C}}g^{-1} = T_{y, \mathbb{C}}$. The number of the $Q(\mathbb{R})$ -conjugacy classes of maximal tori of $Q_{\mathbb{R}}$ defined over \mathbb{R} is at most

$$\#(\text{Ker}(H^1(\mathbb{R}, N_{Q(\mathbb{R})}(T_{x, \mathbb{R}})) \rightarrow H^1(\mathbb{R}, Q))) < \infty,$$

where $N_{Q(\mathbb{R})}(T_{x, \mathbb{R}})$ is the normalizer of $T_{x, \mathbb{R}}$ in $Q(\mathbb{R})$. So it is equivalent to prove the finiteness of the $Q(\mathbb{R})^+ U_Q(\mathbb{C})$ -conjugacy classes in \mathfrak{Y}_Q and to prove the finiteness of the $Q(\mathbb{R})^+$ -conjugacy classes of the morphisms $\mathbb{S} \rightarrow T_{x, \mathbb{R}}$. But $T_x < T < G$, so the $Q(\mathbb{R})^+$ -conjugacy classes of the morphisms $\mathbb{S} \rightarrow T_{x, \mathbb{R}}$ equals the $G_Q(\mathbb{R})^+$ -conjugacy classes of the morphisms $\mathbb{S} \rightarrow T_{x, \mathbb{R}}$. In other words, it suffices to prove the claim for (G, \mathcal{X}_G^+) . Now the result follows from [17, Lemma 4.4(ii)] (or [40, 2.4] or [63, Lemma 3.7]). \square

12.2. Consequence of Ax-Lindemann-Weierstraß. Now we use the result of the previous subsection to prove the following theorem, which will be used in the next section to prove the André-Oort Conjecture.

Theorem 12.2. *Let $S = \Gamma \backslash \mathcal{X}^+$ be a connected mixed Shimura variety associated with the connected mixed Shimura datum (P, \mathcal{X}^+) . Let Y be a Hodge generic irreducible subvariety of S . Then there exists an $N \triangleleft P$ (denote by $U_N := U \cap N$) s.t. for the diagram*

$$(12.1) \quad \begin{array}{ccc} (P, \mathcal{X}^+) & \xrightarrow{\rho} & (P, \mathcal{X}'^+) := (P, \mathcal{X}^+)/N \\ \downarrow \text{unif} & & \downarrow \text{unif}' \\ S & \xrightarrow{[\rho]} & S' \end{array},$$

- the union of positive-dimensional weakly special subvarieties which are contained in $Y' := \overline{[\rho](Y)}$ is NOT Zariski dense in Y' ;
- $Y = [\rho]^{-1}(Y')$.

Proof. Without any loss of generality, we assume that the union of positive-dimensional weakly special subvarieties which are contained in Y is Zariski dense in Y .

Take a fundamental domain \mathcal{F} for the action of Γ on \mathcal{X}^+ s.t. $\text{unif}|_{\mathcal{F}}$ is definable. Such an \mathcal{F} exists by e.g. §10.1.

By Reduction Lemma (Lemma 2.12), we may assume $(P, \mathcal{X}^+) \xrightarrow{\lambda} (G_0, \mathcal{D}^+) \times \prod_{i=1}^r (P_{2g}, \mathcal{X}_{2g}^+)$,

i.e. replace (P, \mathcal{X}^+) by (P', \mathcal{X}'^+) in the lemma if necessary. Identify (P, \mathcal{X}^+) with its image under λ .

Let \mathcal{T} be the set of the triples (U', V', G') consisting of an \mathbb{R} -subgroup of $U_{\mathbb{R}}$, an \mathbb{R} -sub-Hodge structure of $V_{\mathbb{R}}$ and a connected \mathbb{R} -subgroup of $G_{\mathbb{R}}$ which is semi-simple and has no compact factors. Let

$$\mathcal{G} := \mathbb{G}_m(\mathbb{R})^r \times \text{GSp}_{2g}(\mathbb{R}) \times G(\mathbb{R}),$$

then \mathcal{G} acts on \mathcal{T} by $(g_U, g_V, g) \cdot (U', V', G') := (g_U U', g_V V', g G' g^{-1})$. Also we define the action of a triple $(U'(\mathbb{R}), V'(\mathbb{R}), G'(\mathbb{R}))$ on $\mathcal{X}^+ \simeq U(\mathbb{C}) \times V(\mathbb{R}) \times \mathcal{X}_G^+$ as (4.2). This action is algebraic.

Lemma 12.3. *Up to the action of \mathcal{G} on \mathcal{T} , there exist only finitely many such triples.*

Proof. First of all by root system and Galois cohomology, there exist only finitely many semi-simple subgroups of $G_{\mathbb{R}}$ up to conjugation by $G(\mathbb{R})$.

Secondly, V' is by definition a symplectic subspace of $V_{\mathbb{R}}$. Hence a symplectic base of V' extends to a symplectic base of $V_{\mathbb{R}} = V_{2g, \mathbb{R}}$. But $\text{GSp}_{2g}(\mathbb{R})$ acts transitively on the set of symplectic bases of $V_{2g, \mathbb{R}}$, so there are only finitely many choices for V' up to the action of $\text{GSp}_{2g}(\mathbb{R})$ (in fact, there are only g choices).

Finally, observe that $\forall (\lambda_1, \dots, \lambda_r) \in \mathbb{G}_m(\mathbb{R})^r$ and $(u_1, \dots, u_r) \in U \simeq \oplus_{i=1}^r U_{2g}^{(i)}$,

$$(\lambda_1, \dots, \lambda_r) \cdot (u_1, \dots, u_r) = (\lambda_1 u_1, \dots, \lambda_r u_r)$$

Now it is clear that (u_1, \dots, u_r) and (u'_1, \dots, u'_r) are under the same orbit of the action of $\mathbb{G}_m(\mathbb{R})^r$ if and only if $u_i u'_i \geq 0$ with $u_i u'_i = 0 \Rightarrow u_i = u'_i = 0$ for all $i = 1, \dots, r$. Hence up to the action of $\mathbb{G}_m(\mathbb{R})^r$, there are only finitely many U' 's (in fact, there are $2 \binom{r}{s}$ U' 's of dimension s). \square

Let $\mathfrak{W}(Y)$ (resp. $\mathfrak{W}_l(Y)$) be the union of weakly special subvarieties of positive dimension (resp. of real dimension l) contained in Y .

For any l s.t. $\mathfrak{W}_l(Y) \neq \emptyset$, there exist by definition (and Proposition 5.4) a subgroup N_l of P^{der} and a point $x_0 \in \mathcal{F}$ s.t. $\text{unif}(N_l(\mathbb{R})^+ U_{N_l}(\mathbb{C}) x_0)$ is a weakly special subvariety of dimension l contained in Y . Note that the triple $(U_{N_l, \mathbb{R}}, V_{N_l, \mathbb{R}}, G_{N_l, \mathbb{R}}^{+\text{nc}}) \in \mathcal{T}$, where $G_{N_l, \mathbb{R}}^{+\text{nc}}$ is the product of the \mathbb{R} -simple factors of $G_{N_l, \mathbb{R}}^+$ which are non-compact. We say that two such subgroups N_l, N'_l of P are equivalent if $(U_{N_l, \mathbb{R}}, V_{N_l, \mathbb{R}}, G_{N_l, \mathbb{R}}^{+\text{nc}}) = (U_{N'_l, \mathbb{R}}, V_{N'_l, \mathbb{R}}, G_{N'_l, \mathbb{R}}^{+\text{nc}})$. By condition (4) of Definition 2.1, $\text{unif}(N_l(\mathbb{R})^+ U_{N_l}(\mathbb{C}) x_0) = \text{unif}(N'_l(\mathbb{R})^+ U_{N'_l}(\mathbb{C}) x_0)$ iff N_l and N'_l are equivalent.

Define

$$B(N_l, \mathbb{R}, Y) := \{(g_U, g_V, g, x) \in \mathcal{G} \times \mathcal{F} \mid \text{unif}((g_U U_{N_l}(\mathbb{C}), g_V V_{N_l}(\mathbb{R}), g G_{N_l}(\mathbb{R})^{+\text{nc}} g^{-1}) x) \subset Y \text{ and is not contained in } \cup_{l' > l} \mathfrak{W}_{l'}(Y)\}.$$

Then by analytic continuation,

$$(12.2) \quad B(N_{l,\mathbb{R}}, Y) = \{(g_U, g_V, g, x) \in \mathcal{G} \times \mathcal{F} \mid \text{pr}|_{\mathcal{F}}((g_U U_{N_l}(\mathbb{R}), g_V V_{N_l}(\mathbb{R}), g G_{N_l}(\mathbb{R})^{+\text{nc}} g^{-1})x) \subset Y \\ \text{and is not contained in } \cup_{l' > l} \mathfrak{M}_{l'}(Y)\}.$$

Lemma 12.4. *For any $(g_U, g_V, g, x) \in B(N_{l,\mathbb{R}}, Y)$, define*

$$\tilde{Z} := (g_U U_{N_l}(\mathbb{C}), g_V V_{N_l}(\mathbb{R}), g G_{N_l}(\mathbb{R})^{+\text{nc}} g^{-1})x.$$

Then $\text{unif}(\tilde{Z})$ is a weakly special subvariety of Y .

Proof. The set \tilde{Z} is a connected irreducible semi-algebraic subset of \mathcal{X}^+ which is contained in $\text{unif}^{-1}(Y)$. Let \tilde{Z}^\dagger be a connected irreducible semi-algebraic subset of \mathcal{X}^+ which is contained in $\text{unif}^{-1}(Y)$ and which contains \tilde{Z} , maximal for these properties. By [50, Lemma 4.1] (there is a typo in the proof: \mathbb{C}^{2n} should be \mathbb{C}^n) and Ax-Lindemann-Weierstraß (Theorem 1.2), \tilde{Z}^\dagger is complex analytic and each of its complex analytic irreducible component is weakly special. But \tilde{Z} is smooth, so \tilde{Z} is contained in one complex analytic irreducible component of \tilde{Z}^\dagger which we denote by \tilde{Z}' . Now we have

$$\begin{aligned} \dim(\tilde{Z}) - \dim(N_l(\mathbb{R})^+ U_{N_l}(\mathbb{C})x_0) &= \dim(g G_{N_l}(\mathbb{R})^+ g^{-1} \cdot x_G) - \dim(G_{N_l}(\mathbb{R})^+ x_{0,G}) \\ &= \dim(\text{Stab}_{G_{N_l}(\mathbb{R})^+}(x_{0,G})) - \dim(\text{Stab}_{g G_{N_l}(\mathbb{R})^+ g^{-1}}(x_G)) \\ &\geq 0 \end{aligned}$$

because $\text{Stab}_{g G_{N_l}(\mathbb{R})^+ g^{-1}}(x_G)$ is a compact subgroup of $g G_{N_l}(\mathbb{R})^+ g^{-1}$ and $\text{Stab}_{G_{N_l}(\mathbb{R})^+}(x_{0,G})$ is a maximal compact subgroup of $G_{N_l}(\mathbb{R})^+$. Hence

$$\dim(\tilde{Z}') \leq l = \dim(N_l(\mathbb{R})^+ U_{N_l}(\mathbb{C})x_0) \leq \dim(\tilde{Z}) \leq \dim(\tilde{Z}')$$

where the first inequality follows from the definition of $B(N_{l,\mathbb{R}}, Y)$. Therefore $\tilde{Z} = \tilde{Z}'$ is weakly special. So $\text{unif}(\tilde{Z})$ is weakly special. \square

Define

$$C(N_{l,\mathbb{R}}, Y) := \{\underline{t} := (g_U U_{N_l}(\mathbb{R}), g_V V_{N_l}(\mathbb{R}), g G_{N_l}(\mathbb{R})^{+\text{nc}} g^{-1}) \mid (g_U, g_V, g) \in \mathcal{G} \text{ s.t. } \exists x \in \mathcal{F} \\ \text{with } \text{unif}(\underline{t} \cdot x) \subset Y \text{ and is not contained in } \cup_{l' > l} \mathfrak{M}_{l'}(Y)\}.$$

Let

$$\begin{aligned} B(N_{l,\mathbb{R}}, Y) &\xrightarrow{\psi_l} (\mathbb{G}_m(\mathbb{R})^r / \text{Stab}_{\mathbb{G}_m(\mathbb{R})^r} U_{N_l}(\mathbb{R})) \times \text{GSp}_{2g}(\mathbb{R}) / \text{Stab}_{\text{GSp}_{2g}(\mathbb{R})} V_{N_l}(\mathbb{R}) \times G(\mathbb{R}) / N_G(\mathbb{R}) G_{N_l}(\mathbb{R})^{+\text{nc}} \\ (g_U, g_V, g, x) &\mapsto (g_U U_{N_l}(\mathbb{R}), g_V V_{N_l}(\mathbb{R}), g G_{N_l}(\mathbb{R})^{+\text{nc}} g^{-1}) \end{aligned}$$

then there is a bijection between $\psi_l(B(N_{l,\mathbb{R}}, Y))$ and $C(N_{l,\mathbb{R}}, Y)$.

Lemma 12.5. *The set $C(N_{l,\mathbb{R}}, Y)$ (hence $\psi_l(B(N_{l,\mathbb{R}}, Y))$) is countable.*

Proof. By Lemma 12.4, $\text{unif}((g_U U_{N_l}(\mathbb{C}), g_V V_{N_l}(\mathbb{R}), g G_{N_l}(\mathbb{R})^{+\text{nc}} g^{-1}) \cdot x)$ is weakly special. Hence by Proposition 5.4 there exists a \mathbb{Q} -subgroup N' of P^{der} s.t.

$$(12.3) \quad (g_U U_{N_l}(\mathbb{C}), g_V V_{N_l}(\mathbb{R}), g G_{N_l}(\mathbb{R})^{+\text{nc}} g^{-1}) = (U_{N'}(\mathbb{C}), V_{N'}(\mathbb{R}), G_{N'}(\mathbb{R})^{+\text{nc}}).$$

But $g_U U_{N_l}(\mathbb{R}) = g_U U_{N_l}(\mathbb{C}) \cap U(\mathbb{R})$ and $U_{N'}(\mathbb{R}) = U_{N'}(\mathbb{C}) \cap U(\mathbb{R})$, so

$$(g_U U_{N_l}(\mathbb{R}), g_V V_{N_l}(\mathbb{R}), g G_{N_l}(\mathbb{R})^{+\text{nc}} g^{-1}) = (U_{N'}(\mathbb{R}), V_{N'}(\mathbb{R}), G_{N'}(\mathbb{R})^{+\text{nc}}).$$

So $C(N_{l,\mathbb{R}}, Y)$, and therefore $\psi_l(B(N_{l,\mathbb{R}}, Y))$ is countable. \square

Proposition 12.6. *For any $l > 0$ and N_l ,*

- (1) *the set $C(N_{l,\mathbb{R}}, Y)$ (hence $\psi_l(B(N_{l,\mathbb{R}}, Y)))$ is finite;*
- (2) *the set $\cup_{l' \geq l} \mathfrak{W}_{l'}(Y)$ is definable;*

Proof. We prove the two statements together by induction on l .

Step I. Let d be the maximum of the dimensions of weakly special subvarieties of positive dimension contained in Y . For any N_d , $B(N_{d,\mathbb{R}}, Y)$ is definable by (12.2), and hence $\psi_d(B(N_{d,\mathbb{R}}, Y))$ is definable since ψ_d is algebraic. So $\psi_d(B(N_{d,\mathbb{R}}, Y))$, and therefore $C(N_{d,\mathbb{R}}, Y)$, is finite by Lemma 12.5.

Consider all the triples

$$\mathfrak{W}_d(Y, \mathcal{T}) := \{(U', V', G') \in \mathcal{T} \mid \begin{array}{l} \exists x \in \mathcal{F} \text{ with } \text{unif}((U'(\mathbb{C}), V'(\mathbb{R}), G'(\mathbb{R})^+)x) \\ \text{weakly special of dimension } d \text{ contained in } Y \end{array}\}.$$

By Lemma 12.3, any $\underline{t} \in \mathfrak{W}_d(Y, \mathcal{T})$ is of the form $\underline{g} \cdot (U'_i, V'_i, G'_i)$ for $\underline{g} \in \mathcal{G}$ and $i = 1, \dots, n$. Furthermore, by Proposition 5.4, we may assume

$$(U'_i, V'_i, G'_i) = (U_{N'_i, \mathbb{R}}, V_{N'_i, \mathbb{R}}, G_{N'_i, \mathbb{R}}^{+\text{nc}})$$

for some $N'_i < Q$ ($i = 1, \dots, n$). But we just proved that $C(N'_{i,\mathbb{R}}, Y)$ is finite ($\forall i = 1, \dots, n$). Hence $\mathfrak{W}_d(Y, \mathcal{T})$ is a finite set. Again by Proposition 5.4, each triple of $\mathfrak{W}_d(Y, \mathcal{T})$ equals $(U_{N', \mathbb{R}}, V_{N', \mathbb{R}}, G_{N', \mathbb{R}}^{+\text{nc}})$ for some $N' < P$. We shall denote this triple by N' for simplicity.

Hence

$$\mathfrak{W}_d(Y) = \bigcup_{N' \in \mathfrak{W}_d(Y, \mathcal{T})} \bigcup_{\substack{(0,0,1,x) \\ \in B(N'_{\mathbb{R}}, Y)}} \text{unif}((N'(\mathbb{R})^+ U_{N'}(\mathbb{C})x)$$

is definable.

Step II. For any l and N_l , $B(N_{l,\mathbb{R}}, Y)$ is definable by (12.2) and induction hypothesis (2). Arguing as in the previous case we get that $C(N_{l,\mathbb{R}}, Y)$ is finite. Define

$$\mathfrak{W}_l(Y, \mathcal{T}) := \{(U', V', G') \in \mathcal{T} \mid \begin{array}{l} \exists x \in \mathcal{F} \text{ with } \text{unif}((U'(\mathbb{C}), V'(\mathbb{R}), G'(\mathbb{R})^+)x) \text{ weakly special} \\ \text{of dimension } l \text{ contained in } Y \text{ but not contained in } \cup_{l' > l} \mathfrak{W}_{l'}(Y) \end{array}\}.$$

Arguing as in the previous case we can get that $\mathfrak{W}_l(Y, \mathcal{T})$ is a finite set and each element of it equals $(U_{N', \mathbb{R}}, V_{N', \mathbb{R}}, G_{N', \mathbb{R}}^{+\text{nc}})$ for some $N' < P$. Hence

$$\bigcup_{l' \geq l} \mathfrak{W}_{l'}(Y) = \bigcup_{l' > l} \mathfrak{W}_{l'}(Y) \cup \bigcup_{N' \in \mathfrak{W}_l(Y, \mathcal{T})} \bigcup_{\substack{(0,0,1,x) \\ \in B(N'_{\mathbb{R}}, Y)}} \text{unif}(N'(\mathbb{R})^+ U_{N'}(\mathbb{C})x)$$

is definable by induction hypothesis (2). \square

From now on, for any connected subgroup N^\dagger of P , we will denote by $\mathfrak{F}(N^\dagger)$ the set of all weakly special subvarieties of S defined by the group N^\dagger (see the beginning of this section) and $\mathfrak{F}(N^\dagger, Y) := \{Z \in \mathfrak{F}(N^\dagger) \text{ s.t. } Z \subset Y\}$. Remark that when proving Proposition 12.6, we have also given the following description of $\mathfrak{W}(Y) = \cup_{l=1}^d \mathfrak{W}_l(Y)$:

$$(12.4) \quad \mathfrak{W}(Y) = \bigcup_{N'} \text{unif}(N'(\mathbb{R})^+ U_{N'}(\mathbb{C})\text{-orbits contained in } \text{unif}^{-1}(Y)) = \bigcup_{N'} \bigcup_{Z \in \mathfrak{F}(N', Y)} Z$$

which is a finite union on N' 's and each N' is of positive dimension. We have assumed that $\mathfrak{W}(Y)$ is Zariski dense in Y (otherwise there is nothing to prove). Therefore by (12.4), there exists an N_1 of positive dimension s.t.

$$(12.5) \quad \bigcup_{Z \in \mathfrak{F}(N_1, Y)} Z$$

is Zariski dense in Y .

Prove now $N_1 \triangleleft P$. If not, then by Proposition 12.1, $\bigcup_{Z \in \mathfrak{F}(N_1)} Z$ equals a finite union of proper special subvarieties of S . The intersection of this union and Y is not Zariski dense in Y since Y is Hodge generic in S . This is a contradiction. Hence $N_1 \triangleleft P$.

Consider the diagram

$$(12.6) \quad \begin{array}{ccc} (P, \mathcal{X}^+) & \xrightarrow{\rho_1} & (P_1, \mathcal{X}_1^+) := (P, \mathcal{X}^+)/N_1 \\ \text{\scriptsize pr} \downarrow & & \downarrow \text{\scriptsize unif}_1 \\ S & \xrightarrow{[\rho_1]} & S_1 \end{array}$$

and let $Y_1 := \overline{[\rho_1](Y)}$, which is Hodge generic in S_1 . Since $\dim(N_1) > 0$, $\dim(S_1) < \dim(S)$. It is not hard to prove $[\rho]^{-1}(Y_1) = Y$ by the fact (12.5). If the union of positive-dimensional weakly special subvarieties contained in Y_1 is not Zariski dense in Y_1 , then take $N = N_1$. Otherwise by the same argument, there exists a normal subgroup $N_{1,2}$ of P_1 s.t. $\dim(N_{1,2}) > 0$ and $\bigcup_{Z \in \mathfrak{F}(N_{1,2}, Y_1)} Z$ is Zariski dense in Y_1 . Let $N_2 := \rho_1^{-1}(N_{1,2})$, then $N_2 \triangleleft P$. Draw the same diagram (12.6) with N_2 instead of N_1 , then we get a mixed Shimura variety S_2 with $\dim(S_2) < \dim(S_1)$ and a Hodge generic subvariety Y_2 of S_2 . Continue the process (if the union of positive-dimensional weakly special subvarieties contained in Y_2 is Zariski dense in Y_2).

Since $\dim(S) < \infty$, this process will end in a finite step. Hence there exists a number $k > 0$ s.t. the union of positive-dimensional weakly special subvarieties contained in Y_k is not Zariski dense in Y_k . Then $N := N_k$ is the desired subgroup of P . \square

13. FROM AX-LINDEMANN-WEIERSTRASS TO ANDRÉ-OORT

For pure Shimura varieties, Ullmo and Pila-Tsimerman have explained separately in [61, §5] [49, §7] how to deduce the André-Oort conjecture from the Ax-Lindemann-Weierstraß theorem with a suitable lower bound for Galois orbits of special points. In this section we first prove that in order to get a suitable lower bound for Galois orbits of special points for a mixed Shimura variety, it is enough to have one for its pure part. Then we show that the idea of Ullmo also works for mixed Shimura varieties.

13.1. Lower bounds for Galois orbits of special points. In this subsection, we will consider mixed Shimura data (resp. varieties) instead of only connected ones. See Definition 2.1.

Let (P, \mathcal{X}) be a mixed Shimura datum. Let $\pi: (P, \mathcal{X}) \rightarrow (G, \mathcal{X}_G)$ be the projection to its pure part. We use the notation of §2.2. In particular, we fix a Levi decomposition $P = W \rtimes G$ and an embedding $(G, \mathcal{X}_G) \hookrightarrow (P, \mathcal{X})$ as in [67, pp 6].

Let K be an open compact subgroup of $P(\mathbb{A}_f)$ defined as follows: for $M > 3$ even, $K_U := MU(\widehat{\mathbb{Z}})$, $K_V := MV(\widehat{\mathbb{Z}})$, $K_W := K_U \times K_V$ with the group law as in §2.2, $K_G := \{g \in G(\widehat{\mathbb{Z}}) | g \equiv 1 \pmod{M}\}$ and $K := K_W \rtimes K_G$.

Let s be a special point of $M_K(P, \mathcal{X})$ which corresponds to a special point $x \in \mathcal{X}$. The group $\text{MT}(x)$ is of the form wTw^{-1} for a torus $T \subset G$ and $w \in W(\mathbb{Q})$. Let $\text{ord}(w) \in \mathbb{Z}_{>0}$ be the smallest integer such that $\text{ord}(w)w \in W(\mathbb{Z})$. Define the order of s to be $N(s) := \text{ord}(w)$.

Remark 13.1. *It is not hard to show that if the fiber of $S \xrightarrow{[\pi]} S_G$ is a semi-abelian variety, then $N(s)$ coincides with the order of s as a torsion point on the fiber (up to a constant).*

Attached to (P, \mathcal{X}) there is a number field $E = E(P, \mathcal{X})$ called the **reflex field** and $M_K(P, \mathcal{X})$ is defined over E (cf. [53, 11.5]). We want a comparison of $|\text{Gal}(\overline{\mathbb{Q}}/E)s|$ and $|\text{Gal}(\overline{\mathbb{Q}}/E)[\pi](s)|$.

Define $(G^w, \mathcal{X}_{G^w}) := (wGw^{-1}, w^{-1} \cdot \mathcal{X}_G)$, $K_{G^w} := G^w(\mathbb{A}_f) \cap K$ and $K'_G := w^{-1}K_{G^w}w$, then we have the following commutative diagram:

$$\begin{array}{ccc} M_{K_{G^w}}(G^w, \mathcal{X}_{G^w}) & \hookrightarrow & M_K(P, \mathcal{X}) \\ \downarrow \wr [w^{-1} \cdot] & & \downarrow [\pi] \\ M_{K'_G}(G, \mathcal{X}_G) & \xrightarrow{\rho} & M_{K_G}(G, \mathcal{X}_G) \end{array}.$$

All the morphisms in this diagram are defined over E since the reflex field of (P, \mathcal{X}) , (G, \mathcal{X}_G) and (G^w, \mathcal{X}_{G^w}) are all E . Denote by $s' := [w^{-1}] \cdot s$. Let $T^w := wTw^{-1}$. Let $K'_T := K \cap T^w(\mathbb{A}_f)$ and let $K_T := K \cap T(\mathbb{A}_f)$. The following inequality follows essentially from [63, §2.2] (note that we do not need GRH for this inequality since [63, Lemma 2.13, 2.14] are not used!). We refer to [28, Theorem 1(1)] for a more precise version.

(13.1)

$$|\text{Gal}(\overline{\mathbb{Q}}/E)s| = |\text{Gal}(\overline{\mathbb{Q}}/E)s'| \geq B^{i(T)} |K_T/K'_T| |\text{Gal}(\overline{\mathbb{Q}}/E)\rho(s')| = B^{i(T)} |K_T/K'_T| |\text{Gal}(\overline{\mathbb{Q}}/E)[\pi](s)|$$

for some $B \in (0, 1)$ depending only on (P, \mathcal{X}) .

Write $w = (u, v)$ under the identification $W \simeq U \times V$ in §2.2. All elements of $w^{-1}Kw$ are of the form

$$(-u, -v, 1)(u', v', g')(u, v, 1) = (u' - (u - g'u) - \Psi(v, v'), v' - (v - g'v), g')$$

with $(u', v', g') \in K$. Since $K'_T = w^{-1}K_{T^w}w = w^{-1}Kw \cap T(\mathbb{A}_f)$, this element is in K'_T iff

- $u' = u - g'u + \Psi(v, v') \in K_U$
- $v' = v - g'v \in K_V$
- $g' \in T(\mathbb{A}_f) \cap K_G = K_T$.

So

$$\begin{aligned} (13.2) \quad & t \in K_T; \\ & t \in w^{-1}K_{T^w}w \iff v - tv \in K_V = MV(\widehat{\mathbb{Z}}); \\ & u - tu + \Psi(v, v - tv) \in K_U = MU(\widehat{\mathbb{Z}}). \end{aligned}$$

Lemma 13.2. $|K_T/K'_T| \geq \text{ord}(w) \prod_{p|\text{ord}(w)} (1 - \frac{1}{p})$.

Proof. Let T' be the image of $\mathbb{G}_{m, \mathbb{R}} \xrightarrow{\omega} \mathbb{S} \xrightarrow{w^{-1} \cdot x} G_{\mathbb{R}}$, then it is an algebraic torus defined over \mathbb{Q} by Remark 2.2(1). We always have $T' < T$. If T' is trivial, then $P = G$ is adjoint by reason of weight, and $\text{ord}(w) = 1$. If not, $T' \simeq \mathbb{G}_{m, \mathbb{Q}}$ and

$$T'(M) := \{t' \in T'(\widehat{\mathbb{Z}}) | t' \equiv 1 \pmod{(M)}\} \subset K_G \cap T(\mathbb{A}_f) = K_T.$$

So

$$T'(M)/(T'(M) \cap w^{-1}K_{T^w}w) \hookrightarrow K_T/w^{-1}K_{T^w}w.$$

Hence it is enough to prove that LHS is of cardinality $\geq \text{ord}(w)$.

Since T' acts on V and U via a scalar, $t' \in T'(M) \cap w^{-1}K_{T^w}w$ iff

- (1) $t' \in T'(M)$
- (2) $v - t'v \in MV(\widehat{\mathbb{Z}})$
- (3) $u - t'u \in MU(\widehat{\mathbb{Z}})$.

Let $t' \in T'(M) \subset T'(\widehat{\mathbb{Z}}) = \widehat{\mathbb{Z}}^*$. Suppose $\text{ord}(w) = \prod p^{n_p}$ and $M = \prod p^{m_p}$. If $n_p = 0$, then condition (2) and (3) are automatically satisfied. If $n_p > 0$, then condition (2) and (3) imply that $t'_p = 1 + a_{n_p+m_p}p^{n_p+m_p} + \dots \in \mathbb{Z}_p^*$, hence

$$(13.3) \quad |T'(\mathbb{Z}_p) \cap T'(M)/(T'(\mathbb{Z}_p) \cap T'(M) \cap w^{-1}K_{T^w,p}w)| = p^{n_p-1}(p-1).$$

To sum up,

$$(13.4) \quad |T'(M)/(T'(M) \cap w^{-1}K_{T^w}w)| = \text{ord}(w) \prod_{p|\text{ord}(w)} \left(1 - \frac{1}{p}\right).$$

□

Theorem 13.3. *For any $\varepsilon \in (0, 1)$, there exist a positive constant C_ε (depending only on (P, \mathcal{X}) and ε) such that*

$$|\text{Gal}(\overline{\mathbb{Q}}/E)s| \geq C_\varepsilon N(s)^{1-\varepsilon} |\text{Gal}(\overline{\mathbb{Q}}/E)[\pi](s)|.$$

Proof. We have proved in Lemma 13.2

$$(13.5) \quad p|\text{ord}(w) \iff K_{T,p} \neq K'_{T,p}.$$

Hence denoting by $\varsigma(M) := |\{p, p|M\}|$ for any $M \in \mathbb{Z}_{>0}$, we have by Lemma 13.2 and (13.1)

$$|\text{Gal}(\overline{\mathbb{Q}}/E)s| \geq B^{\varsigma(N(s))} N(s) \prod_{p|N(s)} \left(1 - \frac{1}{p}\right) |\text{Gal}(\overline{\mathbb{Q}}/E)\rho(s')|.$$

Now the theorem follows from the basic facts of elementary math:

$$(13.6) \quad \forall \varepsilon \in (0, 1), \text{ there exists } C_\varepsilon > 0 \text{ such that } B^{\varsigma(N(s))} N(s)^\varepsilon \geq C_\varepsilon.$$

$$(13.7) \quad \forall \varepsilon \in (0, 1), \text{ there exists } C'_\varepsilon > 0 \text{ such that } N(s)^\varepsilon \prod_{p|N(s)} \left(1 - \frac{1}{p}\right) \geq C'_\varepsilon.$$

□

Corollary 13.4. *For A an abelian variety over a number field $k \subset \mathbb{C}$ and t a torsion point of $A(\mathbb{C})$, denote by $N(t)$ its order and $k(t)$ the field of definition of t over k .*

Let $g \in \mathbb{N}_+$ and let $\varepsilon \in (0, 1)$. There exists $c > 0$ such that for all number fields $k \subset \mathbb{C}$, all g -dimensional CM abelian varieties A with definition field k and all torsion points t in $A(\mathbb{C})$,

$$[k(t) : k] \geq cN(t)^{1-\varepsilon}.$$

Proof. (compare with [57]) By Zarhin's trick, it suffices to give a proof for A principally polarized. Such an A can be realized as a fiber of $\mathfrak{A}_g(4) \rightarrow \mathcal{A}_g(4)$, and any torsion point t of A is a special point of $\mathfrak{A}_g(4)$. Now this result is a direct consequence of Proposition 13.3. □

Remark 13.5. *The lower bound of the Galois orbit of a special point for pure Shimura varieties is given by [61, Conjecture 2.7]. It has been proved under the Generalized Riemann Hypothesis by Ullmo-Yafaev [63]. For the case of \mathcal{A}_g , it is equivalent to the following conjectural lower bound (suggested and proved for $g = 2$ by Edixhoven [26, 25]): suppose that $x \in \mathcal{A}_g$ is a special point. Let A_x denote the CM abelian variety parametrised by x and let R_x be the center of $\text{End}(A_x)$, then there exists $\delta(g) > 0$ such that*

$$(13.8) \quad |\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})x| \gg_g |\text{disc}(R_x)|^{\delta(g)}.$$

For their equivalence see [59, Theorem 7.1]. The best unconditional result is given by Tsimerman [59, Theorem 1.1]: (13.8) is true when $g \leq 6$ (and for $g \leq 3$ by a similar method in [65]).

Hence for a mixed Shimura variety of Siegel type of genus g and any special point x , Theorem 13.3 tells us that if [61, Conjecture 2.7] is verified for the pure part, then for any $\varepsilon \in (0, 1)$, there exists $\delta(g) > 0$ such that

$$|\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})x| \gg_{g,\varepsilon} N(x)^{1-\varepsilon} |\text{disc}(R_{[\pi](x)})|^{\delta(g)}.$$

13.2. André-Oort.

Theorem 13.6. *Let S be a connected mixed Shimura variety of abelian type (i.e. its pure part is of abelian type). Let Y be a closed irreducible subvariety of S containing a Zariski-dense set of special points. If (13.8) holds for the pure part of S , then Y is special.*

In particular, by the main result of [59], the André-Oort conjecture holds unconditionally for any mixed Shimura variety whose pure part is a subvariety of \mathcal{A}_6^n .

Proof. Suppose S is associated with (P, \mathcal{X}^+) . Replacing Γ by a neat subgroup does not change the assumption or the conclusion, so we may assume that $\Gamma = \{\gamma \in P(\mathbb{Z}) | \gamma \equiv 1 \pmod{N}\}$ for some $N > 3$ even. Replacing S by the smallest connected mixed Shimura subvariety does not change the assumption or the conclusion, so we may assume that Y is Hodge generic in S . Since Y contains a Zariski-dense set of special points, we may assume that Y is defined over a number field k . Suppose that Y is not special.

If the set of positive-dimensional weakly special subvarieties of Y is Zariski dense in Y , then let N be the normal subgroup P as in Theorem 12.2. Consider the diagram (12.1), then Y is special iff $Y' := \overline{[\rho](Y)}$ is. It is clear that S' is again of abelian type. Replacing (S, Y) by (S', Y') , we may assume that the set of positive-dimensional special subvarieties of Y is not Zariski dense in Y .

Now we are left prove that the set of special points of Y which do not lie in any positive-dimensional special subvariety is finite.

By definition, there exists a Shimura morphism $(G, \mathcal{X}_G^+) \rightarrow \prod_{i=1}^r (\text{GSp}_{2g}^{(i)}, \mathbb{H}_g^{+(i)})$ (the upper-index (i) is to distinguish different factors) s.t. $G \rightarrow \prod_{i=1}^r \text{GSp}_{2g}^{(i)}$ has a finite kernel (contained in the center) and $\mathcal{X}_G^+ \hookrightarrow \prod_{i=1}^r \mathbb{H}_g^{+(i)}$. Therefore under Proposition 4.3, we can identify \mathcal{X}^+ as a subspace of $U(\mathbb{C}) \times V(\mathbb{R}) \times \mathbb{H}_g^{+r}$. Then any special point is contained in $U(\mathbb{Q}) \times V(\mathbb{Q}) \times (\mathbb{H}_g^{+r} \cap M_{2g}(\overline{\mathbb{Q}})^r)$ and hence we can define its height (for $\overline{\mathbb{Q}}$ -points, see [13, Definition 1.5.4 multiplicative height]).

Now take \mathcal{F} as in §10.1. For any special point $x \in S$, take a representative $\tilde{x} \in \text{unif}^{-1}(x)$ in \mathcal{F} , then by [50, Theorem 3.1], $H(\tilde{x}_{G,i}) \ll |\text{disc}(R_{[\pi](x)_i})|^{B_g}$ for a constant B_g ($\forall i = 1, \dots, r$). By

choice of \mathcal{F} , $H(\tilde{x}_V)$, $H(\tilde{x}_U) \ll N(x)$ (see Remark 4.4). If (13.8) holds, then by Proposition 13.3

$$\#(\text{Gal}(\overline{\mathbb{Q}}/k)x) \gg_g H(\tilde{x})^{\varepsilon(g)}$$

for some $\varepsilon(g) > 0$. Hence for $H(\tilde{x}) \gg 0$, Pila-Wilkie [47, 3.2] implies that $\exists \sigma \in \text{Gal}(\overline{\mathbb{Q}}/k)$ s.t. $\widetilde{\sigma(x)}$ is contained in a semi-algebraic subset of $\text{unif}^{-1}(Y) \cap \mathcal{F}$ of positive dimension. But by [61, pp6, last line],

$$\bigcup_{\substack{\tilde{Z} \subset \text{unif}^{-1}(Y) \text{ semi-algebraic} \\ \dim(\tilde{Z}) > 0}} \tilde{Z} = \bigcup_{\substack{\tilde{Z} \subset \text{unif}^{-1}(Y) \text{ irreducible algebraic} \\ \dim(\tilde{Z}) > 0}} \tilde{Z}$$

So $\widetilde{\sigma(x)}$ is contained in some maximal algebraic subset \tilde{Z} of $\text{unif}^{-1}(Y)$ of positive dimension. Theorem 1.2 tells us that \tilde{Z} is weakly special. Hence $\sigma^{-1}(Z)$ ($Z := \text{unif}(\tilde{Z})$) is weakly special containing a special point x . Hence $\sigma^{-1}(Z)$ is special of positive dimension. To sum up, the heights of the elements of

$\{\tilde{x} \in \text{unif}^{-1}(Y) \cap \mathcal{F} \text{ special and } \text{unif}(\tilde{x}) \text{ is not contained in a positive-dimensional special subvariety}\}$ is uniformly bounded, and hence this set is finite by Northcott's theorem [13, Theorem 1.6.8]. \square

14. APPENDIX

We prove here Theorem 7.6 when $E = T$ is an algebraic torus over \mathbb{C} (which corresponds to the case $W = U$) and when $E = A$ is a complex abelian variety (which corresponds to the case $W = V$). The proof is a rearrangement of existing proofs (combine the point counting of Pila-Zannier [52] and volume calculation of Ullmo-Yafaev [64]). Use notation in §11.

Case $i : E=A$. In this case, $W = V$ and $\Gamma_V = \oplus_{i=1}^{2n} \mathbb{Z}e_i \subset \text{Lie}(A) = \mathbb{C}^n = \mathbb{R}^{2n}$ is a lattice. Denote by $\text{univ} : \text{Lie}(A) \rightarrow A$. Let $\mathcal{F}_V := \sum_{i=1}^{2n} (-1, 1)e_i$, then \mathcal{F}_V is a fundamental set for the action of Γ_V on $\text{Lie}(A)$ s.t. $\text{univ}|_{\mathcal{F}_V}$ is definable. Define the norm of $z = (x_1, y_1, \dots, x_n, y_n) \in \text{Lie}(A) = \mathbb{R}^{2n}$ to be

$$\|z\| := \text{Max}(|x_1|, |y_1|, \dots, |x_n|, |y_n|).$$

It is clear that $\forall z \in \text{Lie}(A)$ and $\forall \gamma_V \in \Gamma_V$ s.t. $\gamma_V z \in \mathcal{F}_V$,

$$(14.1) \quad H(\gamma_V) \ll \|x_V\|.$$

Let $\omega_V := dz_1 \wedge d\bar{z}_1 + \dots + dz_n \wedge d\bar{z}_n$ be the canonical $(1, 1)$ -form of $\text{Lie}(A) = \mathbb{C}^n$. Let p_i ($i = 1, \dots, n$) be the n natural projections of $\text{Lie}(A) = \mathbb{C}^n$ to \mathbb{C} . Let C be an algebraic curve of \tilde{Z} and define $C_M := \{z \in C \mid \|z\| \leq M\}$. We have

$$(14.2) \quad \begin{aligned} \int_{C \cap \mathcal{F}_V} \omega_V &\leq d \sum_{i=1}^n \int_{p_i(C \cap \mathcal{F}_V)} dz_i \wedge d\bar{z}_i \\ &\leq d \sum_{i=1}^n \int_{p_i(\mathcal{F}_V)} dz_i \wedge d\bar{z}_i = d \cdot O(1) \end{aligned}$$

and

$$(14.3) \quad \int_{C_M} \omega_V \geq O(M^2)$$

with $d = \deg(C)$ by [33, Theorem 0.1].

By (14.1)

$$C_M \subset \bigcup_{\gamma_V \in \Theta(\tilde{Z}, M)} (C \cap \gamma^{-1}\mathcal{F}).$$

Integrating both side w.r.t. ω_V we have

$$M^2 \ll \#\Theta(\tilde{Z}, M)$$

by (14.2) and (14.3).

Let $\text{Stab}_V(\tilde{Z}) := \overline{\Gamma_V \cap \text{Stab}_{V(\mathbb{R})}(\tilde{Z})}^{\text{Zar}}$. Now by Pila-Wilkie [64, Theorem 3.4], there exists an semi-algebraic block $B \subset \Sigma(\tilde{Z})$ of positive dimension containing arbitrarily many points $\gamma_V \in \Gamma_V$. We have $B\tilde{Z} \subset \text{univ}^{-1}(Y)$ since $\Sigma(\tilde{Z})\tilde{Z} \subset \text{univ}^{-1}(Y)$ by definition. Hence for any $\gamma_V \in \Gamma_V \cap B$, $\tilde{Z} \subset \gamma_V^{-1}B\tilde{Z} \subset \text{univ}^{-1}(Y)$, and therefore $\tilde{Z} = \gamma_V^{-1}B\tilde{Z}$ by maximality of \tilde{Z} . So $\gamma_V^{-1}(B \cap \Gamma_V) \subset \text{Stab}_V(\tilde{Z})(\mathbb{Q})$, and hence $\dim(\text{Stab}_V(\tilde{Z})) > 0$. For any point $\tilde{z} \in \tilde{Z}$, $\text{Stab}_V(\tilde{Z})(\mathbb{R}) + \tilde{z} \subset \tilde{Z}$. By [52, Lemma 2.3], $\text{Stab}_V(\tilde{Z})(\mathbb{R})$ is full and complex. Define $V' := V/\text{Stab}_V(\tilde{Z})$ and $\Gamma_{V'} := \Gamma_V/(\Gamma_V \cap \text{Stab}_V(\tilde{Z})(\mathbb{Q}))$, and then $A' := V'(\mathbb{R})/\Gamma_{V'}$ is a quotient abelian variety of A . Let Y' (resp. \tilde{Z}') be the Zariski closure of the projection of Y (resp. \tilde{Z}) in A' (resp. $V'(\mathbb{R})$). We prove that the image of \tilde{Z}' is a point. If not, then proceeding as before for the triple (A', Y', \tilde{Z}') can we prove $\dim(\text{Stab}_{V'}(\tilde{Z}')) > 0$. This contradicts to the definition (maximality) of $\text{Stab}_V(\tilde{Z})$. Hence \tilde{Z} is a translate of $\text{Stab}_V(\tilde{Z})(\mathbb{R})$. So \tilde{Z} is weakly special.

Case ii : $E=T$. Define the norm of $x_U = (x_{U,1}, x_{U,2}, \dots, x_{U,m}) \in U(\mathbb{C})$ to be

$$\|x_U\| := \text{Max}(\|x_{U,1}\|, \|x_{U,2}\|, \dots, \|x_{U,m}\|).$$

It is clear that $\forall x_U \in U(\mathbb{C})$ and $\forall \gamma_U \in \Gamma_U$ s.t. $\gamma_U x_U \in \mathcal{F}_U$,

$$(14.4) \quad H(\gamma_U) \ll \|x_U\|.$$

Let $\omega|_T = dz_1 \wedge d\bar{z}_1 + \dots + dz_m \wedge d\bar{z}_m$ be the canonical $(1,1)$ -form of $U(\mathbb{C}) \simeq \mathbb{C}^m$. Let p_i ($i = 1, \dots, m$) be the m natural projections of $U(\mathbb{C}) \simeq \mathbb{C}^m$ to \mathbb{C} . Let C be an algebraic curve of \tilde{Z} and define $C_M := \{x \in C \mid \|x\| \leq M\}$. We have

$$(14.5) \quad \begin{aligned} \int_{C_M \cap \mathcal{F}_U} \omega|_T &\leq d \sum_{i=1}^m \int_{p_i(C_M \cap \mathcal{F}_U)} dz_i \wedge d\bar{z}_i \\ &\leq d \sum_{i=1}^m \int_{\{s \in \mathbb{C} \mid -1 < \Re(s) < 1, \|s\| \leq M\}} dz_i \wedge d\bar{z}_i = d \cdot O(M) \end{aligned}$$

where $d := \deg(C)$. On the other hand by [33, Theorem 0.1],

$$(14.6) \quad \int_{C_M} \omega|_T \geq O(M^2).$$

By (14.4)

$$C_M \subset \bigcup_{\gamma \in \Theta(\tilde{Z}, M)} (C_M \cap \gamma^{-1}\mathcal{F}).$$

Integrating both side w.r.t. $\omega|_T$ and taking into account that

$$\gamma \cdot C_M \subset (\gamma C)_{2M} \quad \text{if } H(\gamma) \leq M,$$

we have

$$M^2 \ll \#\Theta(\tilde{Z}, M) \cdot M$$

by (14.5) and (14.6). Hence $\#\Theta(\tilde{Z}, M) \gg M$.

Let $\text{Stab}_U(\tilde{Z}) := \overline{\Gamma_U \cap \text{Stab}_{U(\mathbb{C})}(\tilde{Z})}^{\text{Zar}}$. Now by Pila-Wilkie [47, Theorem 3.6], there exists an semi-algebraic subset $B \subset \Sigma(\tilde{Z})$ of positive dimension containing arbitrarily many points $\gamma_U \in \Gamma_U$. We have $B\tilde{Z} \subset \text{univ}^{-1}(Y)$ since $\Sigma(\tilde{Z})\tilde{Z} \subset \text{univ}^{-1}(Y)$ by definition. Hence for any $\gamma_U \in \Gamma_U \cap B$, $\tilde{Z} \subset \gamma_U^{-1}B\tilde{Z} \subset \text{univ}^{-1}(Y)$, and therefore $\tilde{Z} = \gamma_U^{-1}B\tilde{Z}$ by maximality of \tilde{Z} . So $\gamma_U^{-1}(B \cap \Gamma_U) \subset \text{Stab}_U(\tilde{Z})(\mathbb{Q})$, and hence $\dim(\text{Stab}_U(\tilde{Z})) > 0$. Let $U' := U/\text{Stab}_U(\tilde{Z})$, $\Gamma_{U'} := \Gamma_U/(\Gamma_U \cap \text{Stab}_U(\tilde{Z})(\mathbb{Q}))$ and $T' := U'(\mathbb{C})/\Gamma_{U'}$. T' is an algebraic torus over \mathbb{C} . Let Y' (resp. \tilde{Z}') be the Zariski closure of the projection of Y (resp. \tilde{Z}) in T' (resp. $U'(\mathbb{C})$). We prove that \tilde{Z}' is a point. If not, then proceeding as before for the triple (T', Y', \tilde{Z}') we can prove $\dim(\text{Stab}_{U'}(\tilde{Z}')) > 0$. This contradicts the definition (maximality) of $\text{Stab}_U(\tilde{Z})$. Hence \tilde{Z} is a translate of $\text{Stab}_U(\tilde{Z})(\mathbb{C})$. So \tilde{Z} is weakly special.

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