

## Chapter 3

# Shimura data and Shimura varieties

### 3.1 Basic definitions

#### 3.1.1 Shimura data

**Definition 3.1.1.** A Shimura datum is a pair  $(\mathbf{G}, X)$  where

- $\mathbf{G}$  is a reductive group defined over  $\mathbb{Q}$ ,
- $X$  is a  $\mathbf{G}(\mathbb{R})$ -orbit in  $\mathrm{Hom}(\mathbb{S}, \mathbf{G}_{\mathbb{R}})$

such that for one (and hence all)  $h \in X$ , we have

(SV1) the Hodge structure  $\mathrm{Ad} \circ h$  on  $\mathrm{Lie} \mathbf{G}$  has type  $(-1, 1) + (0, 0) + (1, -1)$ ,

(SV2)  $\mathrm{Int}(h(\sqrt{-1}))$  is a Cartan involution of  $\mathbf{G}_{\mathbb{R}}^{\mathrm{ad}}$ ,

(SV3) for every  $\mathbb{Q}$ -simple factor  $\mathbf{H}$  of  $\mathbf{G}^{\mathrm{ad}}$ , the morphism  $\mathbb{S} \xrightarrow{h} \mathbf{G}_{\mathbb{R}} \rightarrow \mathbf{H}_{\mathbb{R}}$  is non-trivial.

A (Shimura) morphism between two Shimura data  $\rho: (\mathbf{G}', X') \rightarrow (\mathbf{G}, X)$  is a morphism  $\rho$  on the underlying groups such that  $\rho \circ h \in X$  for all  $h \in X'$ . In particular, we call the image of such a Shimura morphism to be a **sub-Shimura datum** of  $(\mathbf{G}, X)$ .

The main difference of a Shimura datum and the pair  $(G, X^+)$  from §2.3 is the definition field of the group (over  $\mathbb{Q}$  or over  $\mathbb{R}$ ). A similar assumption to (SV3) for  $(G, X^+)$  has been discussed in Remark 2.3.3. By Theorem 2.3.1, each connected component  $X^+$  of  $X$  is a Hermitian symmetric domain (and the complex structure on  $X$  is  $\mathbf{G}(\mathbb{R})$ -invariant). By Proposition 1.3.5, each representation  $V$  of  $\mathbf{G}$  gives rise to a  $\mathbb{Q}$ -VHS on  $X^+$  by (SV1), which furthermore carries  $\mathbb{R}$ -polarization by Proposition 2.2.6 and (SV2).<sup>[1]</sup>

The following two further assumptions guarantee that this  $\mathbb{Q}$ -VHS carries a  $\mathbb{Q}$ -polarization. Notice that they may not be satisfied by an arbitrary Shimura datum.

(SV4) the morphism  $w_h: \mathbb{G}_{\mathrm{m}, \mathbb{R}} \rightarrow Z(\mathbf{G})_{\mathbb{R}}$  is defined over  $\mathbb{Q}$ ,

(SV2')  $\mathrm{Int}(h(\sqrt{-1}))$  is a Cartan involution of  $\mathbf{G}_{\mathbb{R}}/w_h(\mathbb{G}_{\mathrm{m}, \mathbb{R}})$ .

**Example 3.1.2** (0-dimensional Shimura datum). *The set  $X$  is a finite set if and only if  $\mathbf{G}$  is abelian (and hence an algebraic torus). This case shows up when we study CM abelian varieties.*

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<sup>[1]</sup>(SV1) implies that  $w_h: \mathbb{G}_{\mathrm{m}} \xrightarrow{w} \mathbb{S} \xrightarrow{h} \mathbf{G}_{\mathbb{R}}$  factors through  $Z(\mathbf{G})_{\mathbb{R}}$ , so we can apply Proposition 2.2.6

**Example 3.1.3** (Siegel Shimura datum). *Let us take the example of the Siegel case from Example 2.3.1, except that the vector space and the groups are defined over  $\mathbb{Q}$ . More precisely,  $V = \mathbb{Q}^{2d}$  and  $\psi: V \times V \rightarrow \mathbb{Q}$  is  $(x, y) \mapsto x^t J y$  with  $J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}$ . The  $\mathbb{Q}$ -group is*

$$\begin{aligned} \mathbf{GSp}(\psi) &= \mathbf{GSp}_{2d} := \{g \in \mathrm{GL}(V) : \psi(gx, gy) = c\psi(x, y) \text{ for some } c \in \mathbb{Q}^\times\} \\ &= \{g \in \mathrm{GL}_{2d, \mathbb{Q}} : gJg^t = cJ \text{ for some } c \in \mathbb{Q}^\times\}, \end{aligned}$$

and  $h_0: \mathbb{S} \rightarrow \mathbf{GSp}_{2d, \mathbb{R}}$  maps  $a + b\sqrt{-1} \mapsto aI_{2d} + bJ$ . The derived subgroup is  $\mathbf{Sp}(\psi) = \mathbf{Sp}_{2d}$  by requesting the  $c \in \mathbb{Q}^\times$  in the definition to be 1.

The  $\mathbf{G}(\mathbb{R})$ -orbit is  $\mathbf{GSp}_{2d}(\mathbb{R})h_0 \subseteq \mathrm{Hom}(\mathbb{S}, \mathbf{GSp}_{2d, \mathbb{R}})$ . Under the identification similar to 2.3.1, we have

$$\mathbf{GSp}_{2d}(\mathbb{R})h_0 \xrightarrow{\sim} \mathfrak{H}_d^\pm := \{\tau \in \mathrm{Mat}_{d \times d}(\mathbb{C}) : \tau = \tau^t, \text{ either } \mathrm{Im} \tau > 0 \text{ or } \mathrm{Im} \tau < 0\}. \quad (3.1.1)$$

Then  $(\mathbf{GSp}_{2d}, \mathfrak{H}_d^\pm)$  is a Shimura datum by Example 2.3.1 (see Remark 2.3.3 for (SV3)). It is called the **Siegel Shimura datum**. Moreover, (SV4) and (SV2') are easily seen to be also satisfied. In fact,  $V$  is a representation of  $\mathbf{GSp}_{2d}$ , and  $\psi$  is the desired  $\mathbb{Q}$ -polarization on the induced  $\mathbb{Q}$ -VHS.

Next we present an example where (SV4) and (SV2') are not satisfied. We also see in this example that two Shimura data may have the same underlying  $\mathbb{R}$ -group and the same underlying space, but the  $\mathbb{Q}$ -groups are different.

**Example 3.1.4** (Shimura curves). *Let  $B$  be a simple quaternion algebra over a totally real number field  $F$ . Assume that  $B$  is split at exactly one real place of  $F$ , i.e. there exists a real embedding  $\sigma: K \rightarrow \mathbb{R}$  such that*

$$B_\sigma \simeq \begin{cases} M_2(\mathbb{R}) & \text{if } \sigma = \sigma_0 \\ \mathbb{H} & \text{otherwise} \end{cases}$$

for all real embeddings  $\sigma: K \rightarrow \mathbb{R}$ , where  $\mathbb{H}$  is the Hamilton quaternion algebra over  $\mathbb{R}$ .

Define the  $\mathbb{Q}$ -group  $\mathbf{G}$

$$\mathbf{G}(R) := (B \otimes_{\mathbb{Q}} R)^\times \quad \text{for all } \mathbb{Q}\text{-algebra } R,$$

and let

$$h_0: \mathbb{S} \rightarrow \mathbf{G}_{\mathbb{R}} \simeq \mathrm{GL}_{2, \mathbb{R}} \times \mathbb{H}^\times \times \cdots \times \mathbb{H}^\times, \quad a + b\sqrt{-1} \mapsto \left( \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, 1, \dots, 1 \right).$$

Thus all but the first components of  $\mathbf{G}(\mathbb{R})h_0$  are the identity map, and so  $\mathbf{G}(\mathbb{R})h_0 \subseteq \mathrm{Hom}(\mathbb{S}, \mathbf{G}_{\mathbb{R}})$  is isomorphic to  $\mathfrak{H}_1^\pm$ , via an isomorphism similar to (3.1.1) (with  $d = 1$ ). Both (SV1) and (SV2) hold true for the pair  $(\mathbf{G}, \mathfrak{H}_1^\pm)$  similarly to the Siegel case. To see (SV3), it suffices to observe that  $\mathbf{G}^{\mathrm{ad}}$  is a simple group because  $B$  is a simple quaternion algebra over  $F$ .

So  $(\mathbf{G}, \mathfrak{H}_1^\pm)$  is a Shimura datum. However, if  $[F : \mathbb{Q}] > 1$ , then (SV4) and (SV2') do not hold true, by looking at the action of  $\mathrm{Aut}(\mathbb{R}/\mathbb{Q})$ .

And even in the case  $F = \mathbb{Q}$ , the group  $\mathbf{G}$  is not necessarily  $\mathbf{GL}_2$ . So  $(\mathbf{G}, \mathfrak{H}_1^\pm)$  needs not be the Siegel Shimura datum in this case.

### 3.1.2 Shimura varieties

Denote by  $\mathbb{A}_f \subseteq \prod_{p \in M_{\mathbb{Q},f}} \mathbb{Q}_p$  the ring of finite adèles over  $\mathbb{Q}$ , and by  $\widehat{\mathbb{Z}} := \prod \mathbb{Z}_p$ . Then  $\widehat{\mathbb{Z}}$  is a (maximal) compact open subgroup of  $\mathbb{A}_f$ , and  $\mathbb{Q}$  is dense in  $\mathbb{A}_f$ .

Let  $(\mathbf{G}, X)$  be a Shimura datum. Then  $\mathbf{G}(\mathbb{Q})$  acts on  $X$  by definition of Shimura data, and consider the action of  $\mathbf{G}(\mathbb{Q})$  on  $\mathbf{G}(\mathbb{A}_f)$  by multiplication on the left.

**Definition 3.1.5.** Let  $(\mathbf{G}, X)$  be a Shimura datum. A **Shimura variety** is a double coset

$$\mathrm{Sh}_K(\mathbf{G}, X) := \mathbf{G}(\mathbb{Q}) \backslash X \times \mathbf{G}(\mathbb{A}_f) / K$$

where  $K \subseteq \mathbf{G}(\mathbb{A}_f)$  is a compact open subset. Here  $\mathbf{G}(\mathbb{Q})$  acts on both  $X$  and  $\mathbf{G}(\mathbb{A}_f)$  on the left as above, and  $K$  acts on  $\mathbf{G}(\mathbb{A}_f)$  by the multiplication on the right; i.e.  $q(x, g)k = (q \cdot x, qgk)$  for all  $q \in \mathbf{G}(\mathbb{Q})$ ,  $(x, g) \in X \times \mathbf{G}(\mathbb{A}_f)$  and  $k \in K$ .

We will prove in this course that the double coset  $\mathrm{Sh}_K(\mathbf{G}, X)$  is the set of  $\mathbb{C}$ -points of an algebraic variety. This justifies the name of Shimura variety.

**Example 3.1.6.** In the Siegel example above, the group  $\mathbf{GSp}_{2d}$  is defined over  $\mathbb{Z}$ ; indeed we can take  $V$  to be  $\mathbb{Z}^{2d}$  and  $\psi$  maps  $V \times V$  to  $\mathbb{Z}$ . Then  $\mathbf{GSp}_{2d}(\widehat{\mathbb{Z}})$  is a compact open subgroup of  $\mathbf{GSp}_{2d}(\mathbb{A}_f)$ . Other compact open subgroups include  $gKg^{-1}$  for any  $g \in \mathbf{GSp}_{2d}(\mathbb{A}_f)$  and any finite-indexed subgroup  $K$  of  $\mathbf{GSp}_{2d}(\widehat{\mathbb{Z}})$ . We will come back to this example in §3.3 and prove that the Siegel Shimura varieties are moduli spaces of abelian varieties.

**Definition 3.1.7.** A **(Shimura) morphism**  $[\rho]: \mathrm{Sh}_{K'}(\mathbf{G}', X') \rightarrow \mathrm{Sh}_K(\mathbf{G}, X)$  between two Shimura varieties is a morphism of Shimura data  $\rho: (\mathbf{G}', X') \rightarrow (\mathbf{G}, X)$  such that  $\rho(K') \subseteq K$ .

**Example 3.1.8.** Let  $\mathrm{Sh}_K(\mathbf{G}, X)$  be a Shimura variety.

Let  $K' \subseteq K$  be another compact open subgroup of  $\mathbf{G}(\mathbb{A}_f)$ . Then the identity map on  $(\mathbf{G}, X)$  induces a Shimura morphism  $\mathrm{Sh}_{K'}(\mathbf{G}, X) \rightarrow \mathrm{Sh}_K(\mathbf{G}, X)$ , with finite fibers since  $K'$  must have finite index in  $K$ . In fact, this is finite morphism in the category of algebraic varieties.

Let  $g \in \mathbf{G}(\mathbb{A}_f)$ . Then  $gKg^{-1}$  is a compact open subgroup of  $\mathbf{G}(\mathbb{A}_f)$ , and we have a Shimura morphism  $[g]: \mathrm{Sh}_{gKg^{-1}}(\mathbf{G}, X) \rightarrow \mathrm{Sh}_K(\mathbf{G}, X)$ , sending  $[x, g'] \mapsto [x, gg']$ . More generally, if  $K'$  is a compact open subgroup of  $\mathbf{G}(\mathbb{A}_f)$  such that  $K' \subseteq gKg^{-1}$ , then we have a Shimura morphism  $[g]: \mathrm{Sh}_{K'}(\mathbf{G}, X) \rightarrow \mathrm{Sh}_K(\mathbf{G}, X)$  which is a finite morphism.

**Example 3.1.9** (Hecke operator). Let  $\mathrm{Sh}_K(\mathbf{G}, X)$  be a Shimura variety.

Any  $g \in \mathbf{G}(\mathbb{A}_f)$  induces a correspondence on  $\mathrm{Sh}_K(\mathbf{G}, X)$  as follows. Write  $K' := K \cap gKg^{-1}$  for simplicity; it is a compact open subgroup of  $\mathbf{G}(\mathbb{A}_f)$  and  $K' \subseteq gKg^{-1}$ . We have Shimura morphisms

$$\begin{array}{ccc} & \mathrm{Sh}_{K'}(\mathbf{G}, X) & \\ \swarrow [g] & & \searrow [1] \\ \mathrm{Sh}_K(\mathbf{G}, X) & & \mathrm{Sh}_K(\mathbf{G}, X) \end{array}$$

where the right one is induced by identity on  $(\mathbf{G}, X)$ . Both are finite morphisms, so we have a correspondence on  $\mathrm{Sh}_K(\mathbf{G}, X)$ , which is called the **Hecke correspondence/operator** and denoted by  $T_g$ .

**Definition 3.1.10.** Let  $\mathrm{Sh}_K(\mathbf{G}, X)$  be a Shimura variety. We call any irreducible component of  $(T_g \circ [\rho])(\mathrm{Sh}_{K'}(\mathbf{G}', X'))$ , where  $[\rho]$  is a Shimura morphism and  $g \in \mathbf{G}(\mathbb{A}_f)$ , to be a **special subvariety** of  $\mathrm{Sh}_K(\mathbf{G}, X)$ . A special subvariety of dimension 0 is called a **special point**.

Of course in the definition of special subvarieties, it suffices to consider the Shimura morphisms arising from sub-Shimura data of  $(\mathbf{G}, X)$ . Thus special points arise from sub-Shimura data  $(\mathbf{T}, X_{\mathbf{T}})$  of  $(\mathbf{G}, X)$  where  $\mathbf{T}$  is an algebraic torus.

### 3.2 Decomposition of Shimura varieties into Hermitian locally symmetric domains

Let  $(\mathbf{G}, X)$  be a Shimura datum. Then any connected component  $X$  is a Hermitian symmetric domain. Fix one such  $X^+$ .

Let  $K$  be a compact open subgroup of  $\mathbf{G}(\mathbb{A}_f)$ . Then we have a Shimura variety  $\mathrm{Sh}_K(\mathbf{G}, X)$  defined as the double coset  $\mathbf{G}(\mathbb{Q}) \backslash X \times \mathbf{G}(\mathbb{A}_f) / K$ . We wish to prove that this double coset is the  $\mathbb{C}$ -points of an algebraic variety.

In this section, we start with the first step, by endowing  $\mathrm{Sh}_K(\mathbf{G}, X)$  with a structure of complex varieties.

**Theorem 3.2.1.** *There exists a finite-indexed subgroup  $K'$  of  $K$  such that*

$$\mathrm{Sh}_{K'}(\mathbf{G}, X) \simeq \bigsqcup_{g \in \mathcal{C}} \Gamma_g \backslash X^+, \quad (3.2.1)$$

for a finite set  $\mathcal{C} \subseteq \mathbf{G}(\mathbb{A}_f)$ , with each  $\Gamma_g$  a torsion-free discrete group acting on  $X^+$ .

The actual decomposition will be given later on (3.2.3), where the definitions of  $\mathcal{C}$  and  $\Gamma_g$  are given. At this stage, let us make the following observation: since  $\Gamma_g$  is torsion-free discrete, the quotient  $\Gamma_g \backslash X^+$  has a natural structure of complex manifolds and even more is a Hermitian locally symmetric domain. So  $\mathrm{Sh}_{K'}(\mathbf{G}, X)$  is a finite disjoint union of Hermitian locally symmetric domains. As for  $\mathrm{Sh}_K(\mathbf{G}, X)$ , the finite-to-1 map  $\mathrm{Sh}_{K'}(\mathbf{G}, X) \rightarrow \mathrm{Sh}_K(\mathbf{G}, X)$  then makes  $\mathrm{Sh}_K(\mathbf{G}, X)$  into a finite union of complex orbifolds.

#### 3.2.1 Two approximation theorems for algebraic groups

Let  $\mathbf{H}$  be an algebraic group defined over  $\mathbb{Q}$ . We will use the following approximation theorems.

- (*Real Approximation*)  $\mathbf{H}(\mathbb{Q})$  is dense in  $\mathbf{H}(\mathbb{R})$ .
- (*Strong Approximation*) If  $\mathbf{H}$  is semi-simple and simply-connected, then  $\mathbf{H}(\mathbb{Q})$  is dense in  $\mathbf{H}(\mathbb{A}_f)$ .

The definition of “simply-connected” will be recalled later in §3.2.5.

#### 3.2.2 Preparation and adjoint Shimura data

Now let us introduce the *adjoint Shimura datum*  $(\mathbf{G}^{\mathrm{ad}}, \overline{X})$  of  $(\mathbf{G}, X)$ . Take  $h \in X^+$ . Then  $h$  induces a morphism

$$\overline{h}: \mathbb{S} \xrightarrow{h} \mathbf{G}_{\mathbb{R}} \rightarrow \mathbf{G}_{\mathbb{R}}^{\mathrm{ad}}.$$

Hence we obtain a  $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})$ -orbit  $\overline{X} := \mathbf{G}^{\mathrm{ad}}(\mathbb{R})\overline{h}$  in  $\mathrm{Hom}(\mathbb{S}, \mathbf{G}_{\mathbb{R}}^{\mathrm{ad}})$ , with a natural map  $X \rightarrow \overline{X}$ . The image of  $X^+$  in  $\overline{X}$  is connected, and the following lemma (applied to  $G = \mathbf{G}(\mathbb{R})$ )<sup>[2]</sup> easily implies that this image is again a connected component of  $\overline{X}$ . So by abuse of notation, we will also use  $X^+$  to denote a connected component of  $\overline{X}$ .

**Lemma 3.2.2.** *For any algebraic group  $G$  over  $\mathbb{R}$ , the adjoint quotient  $G^+ \rightarrow (G^{\mathrm{ad}})^+$  is surjective when restricted to the identity component.*

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<sup>[2]</sup>Here is a background for this lemma. Let  $\varphi: H \rightarrow H'$  be a morphism of algebraic groups defined over  $k$ . Assume  $\mathrm{char}(k) = 0$ . Then  $\varphi$  is called *surjective* if  $\varphi(H(\overline{k})) = H'(\overline{k})$ . If  $\varphi$  is surjective, it may happen that  $\varphi(H(k)) \neq H'(k)$ !

We omit the proof of this lemma. Define

$$\begin{aligned}\mathbf{G}(\mathbb{R})_+ &:= \text{inverse image of } \mathbf{G}^{\text{ad}}(\mathbb{R})^+ \text{ in } \mathbf{G}(\mathbb{R}) \\ \mathbf{G}(\mathbb{Q})_+ &:= \mathbf{G}(\mathbb{Q}) \cap \mathbf{G}(\mathbb{R})_+.\end{aligned}\tag{3.2.2}$$

**Lemma 3.2.3.**  $\mathbf{G}(\mathbb{R})_+$  is the stabilizer of  $X^+$ , i.e.  $\mathbf{G}(\mathbb{R})_+ = \{g \in \mathbf{G}(\mathbb{R}) : gX^+ = X^+\}$ .

With Lemma 3.2.3, we can complete our more precise version of 3.2.1:

$$\text{Sh}_K(\mathbf{G}, X) \simeq \bigsqcup_{[g] \in \mathbf{G}(\mathbb{Q})_+ \backslash \mathbf{G}(\mathbb{A}_f)/K} \Gamma_g \backslash X^+, \tag{3.2.3}$$

with  $\Gamma_g := gKg^{-1} \cap \mathbf{G}(\mathbb{Q})_+$ ; replacing  $K$  by a suitable finite-indexed subgroup  $K'$  guarantees that  $\Gamma_g$  is torsion-free, see 3.2.4. The finiteness of the double coset  $\mathbf{G}(\mathbb{Q})_+ \backslash \mathbf{G}(\mathbb{A}_f)/K$  will be proved in 3.2.5; the proof uses the *Strong Approximation Theorem*.

*Proof of Lemma 3.2.3.* Consider the action of  $\mathbf{G}^{\text{ad}}(\mathbb{R})$  on  $\overline{X}$ , and recall that  $X^+$  is a connected component of  $\overline{X}$ . It suffices to prove that  $\mathbf{G}^{\text{ad}}(\mathbb{R})^+ = \{g \in \mathbf{G}^{\text{ad}}(\mathbb{R}) : gX^+ = X^+\}$ . This follows from general theory of Hermitian symmetric domains (and some knowledge on  $\mathbb{R}$ -algebraic groups *v.s.* real Lie groups) which we will not cover in this course.  $\square$

### 3.2.3 Proof of 3.2.3

We start by showing that there is a bijection

$$\mathbf{G}(\mathbb{Q})_+ \backslash X^+ \times \mathbf{G}(\mathbb{A}_f) \xrightarrow{\sim} \mathbf{G}(\mathbb{Q}) \backslash X \times \mathbf{G}(\mathbb{A}_f), \quad [x, g] \mapsto [x, g]. \tag{3.2.4}$$

**Injectivity:** Assume  $(x, g), (x', g') \in X^+ \times \mathbf{G}(\mathbb{A}_f)$  are mapped to the same point on the right hand side. Then there exists  $q \in \mathbf{G}(\mathbb{Q})$  such that  $(x', g') = q(x, g) = (qx, qg)$ . Hence  $qX^+ \cap X^+$  is non-empty as it contains  $qx = x'$ . So  $qX^+ = X^+$ . So  $q \in \mathbf{G}(\mathbb{R})_+ \cap \mathbf{G}(\mathbb{Q}) = \mathbf{G}(\mathbb{Q})_+$ . This proves the injectivity of the map above.

**Surjectivity:** Assume  $(x, g) \in X \times \mathbf{G}(\mathbb{A}_f)$ . By the *Real Approximation* in 3.2.1,  $\mathbf{G}(\mathbb{Q})x$  is dense in  $\mathbf{G}(\mathbb{R})x = X$ . So  $\mathbf{G}(\mathbb{Q})x \cap X^+ \neq \emptyset$ , and hence there exists  $q \in \mathbf{G}(\mathbb{Q})$  such that  $qx \in X^+$ . Then  $(qx, qg) \in X^+ \times \mathbf{G}(\mathbb{A}_f)$ , and its image under 3.2.4 is  $[x, g]$ . We are done for the surjectivity of 3.2.3.

Now let us prove the bijectivity of the map

$$\bigsqcup_{[g] \in \mathbf{G}(\mathbb{Q})_+ \backslash \mathbf{G}(\mathbb{A}_f)/K} \Gamma_g \backslash X^+ \rightarrow \mathbf{G}(\mathbb{Q})_+ \backslash X^+ \times \mathbf{G}(\mathbb{A}_f)/K, \quad \Gamma_g x \mapsto [x, g]. \tag{3.2.5}$$

**Injectivity:** If  $[x', g'] = [x, g]$ , then  $(qx, qgk) = (x', g')$  for some  $q \in \mathbf{G}(\mathbb{Q})_+$  and  $k \in K$ . So  $[g] = [g']$  in  $\mathbf{G}(\mathbb{Q})_+ \backslash \mathbf{G}(\mathbb{A}_f)/K$ . Hence it suffices to prove the injectivity of  $\Gamma_g \backslash X^+ \rightarrow \mathbf{G}(\mathbb{Q})_+ \backslash X^+ \times \mathbf{G}(\mathbb{A}_f)/K$ . Now if  $[x', g] = [x, g]$ , then  $(qx, qgk) = (x', g)$  for some  $q \in \mathbf{G}(\mathbb{Q})_+$  and  $k \in K$ . So  $q = gk^{-1}g^{-1} \in gKg^{-1}$ . So  $q \in \Gamma_g = gKg^{-1} \cap \mathbf{G}(\mathbb{Q})_+$ . Thus we have proved the injectivity of 3.2.5.

**Surjectivity:** Let  $[x, g]$  be an element of the right hand side. Then it is the image of  $\Gamma_g x$ .

We have thus proved 3.2.3.  $\square$

### 3.2.4 Torsion-free subgroup

Here is a choice of  $K'$  so that  $gK'g^{-1} \cap \mathbf{G}(\mathbb{Q})_+$  is torsion-free for all  $g \in \mathbf{G}(\mathbb{A}_f)$ . Take a faithful representation  $V$  of  $\mathbf{G}$ . Then there exists a lattice  $L$  in  $V$  such that  $\widehat{L} := L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$  is fixed by  $K$ . Equivalently, we are embedding  $\mathbf{G}$  as a closed subgroup of  $\mathbf{GL}_N$  over  $\mathbb{Q}$  such that  $K$  is a subgroup of  $\mathbf{GL}_N(\widehat{\mathbb{Z}})$ . Let  $\ell \geq 3$  be an integer. Take  $K'$  to be the subgroup of  $K$  which acts trivially on  $\widehat{L}/\ell\widehat{L}$ , or equivalently

$$K' := \{g \in K < \mathbf{GL}_N(\widehat{\mathbb{Z}}) : g \equiv I_N \pmod{\ell}\}.$$

Then any element  $\gamma \in \Gamma_g := gK'g^{-1} \cap \mathbf{G}(\mathbb{Q})_+ < \mathbf{GL}(V)$  acts trivially on  $\widehat{g\widehat{L}}/\ell\widehat{g\widehat{L}}$ , so all the eigenvalues of  $\gamma$  are 1 (as they are 1 modulo  $\ell \geq 3$ ). So  $\gamma = 1$  if  $\gamma$  is torsion. So  $\Gamma_g$  is torsion-free.

### 3.2.5 The group of connected components of a Shimura variety

In this subsection, we prove the finiteness of the double coset  $\mathbf{G}(\mathbb{Q})_+ \backslash \mathbf{G}(\mathbb{A}_f)/K$ . This finishes the proof of Theorem [3.2.1](#), and shows that  $\pi_0(\text{Sh}_K(\mathbf{G}, X)) \simeq \mathbf{G}(\mathbb{Q})_+ \backslash \mathbf{G}(\mathbb{A}_f)/K$ .

#### Case: simply-connected derived subgroup

The result in this case is better, with a clear understanding of the group  $\pi_0(\text{Sh}_K(\mathbf{G}, X))$ . Consider the short exact sequence of  $\mathbb{Q}$ -groups

$$1 \rightarrow \mathbf{G}^{\text{der}} \rightarrow \mathbf{G} \rightarrow \mathbf{T} := \mathbf{G}/\mathbf{G}^{\text{der}} \rightarrow 1$$

with  $\mathbf{T}$  an algebraic torus defined over  $\mathbb{Q}$ .

**Definition 3.2.4.** *An algebraic group  $H$  defined over a field  $k$  of characteristic 0 is said to be **simply-connected** if any central isogeny  $H' \rightarrow H$  (i.e. a surjective morphism whose kernel is finite and contained in the center of  $H'$ ) is an isomorphism.*

**Theorem 3.2.5.** *Assume  $\mathbf{G}^{\text{der}}$  is simply-connected. Then  $\nu(\mathbf{G}(\mathbb{Q})_+)$  has finite index in  $\mathbf{G}(\mathbb{Q})$ ,  $\nu(K)$  is a compact open subgroup of  $\mathbf{T}(\mathbb{A}_f)$ , and  $\nu(\mathbf{G}(\mathbb{Q})_+) \backslash \mathbf{T}(\mathbb{A}_f)/\nu(K)$  is a finite abelian group. Moreover,  $\nu$  induces a natural isomorphism of groups*

$$\pi_0(\text{Sh}_K(\mathbf{G}, X)) \simeq \nu(\mathbf{G}(\mathbb{Q})_+) \backslash \mathbf{T}(\mathbb{A}_f)/\nu(K).$$

Before proving this theorem, we point out without proof that

$$\nu(\mathbf{G}(\mathbb{Q})_+) = \mathbf{T}(\mathbb{Q}) \cap \nu(Z(\mathbf{G})(\mathbb{R})) =: \mathbf{T}(\mathbb{Q})^\dagger. \quad (3.2.6)$$

*Proof.* General theory of semi-simple simply-connected  $\mathbb{Q}$ -groups asserts that  $\mathbf{G}^{\text{der}}(\mathbb{R})$  is connected. Therefore  $\mathbf{G}^{\text{der}}(\mathbb{R})$  stabilizes  $X^+$  and hence is contained in  $\mathbf{G}(\mathbb{R})_+$  by Lemma [3.2.3](#). So  $\mathbf{G}^{\text{der}}(\mathbb{Q}) \subseteq \mathbf{G}(\mathbb{Q})_+$ . By the *Strong Approximation Theorem* from [§3.2.1](#),  $\mathbf{G}^{\text{der}}(\mathbb{Q})$  is dense in  $\mathbf{G}^{\text{der}}(\mathbb{A}_f)$ . Hence

$$\mathbf{G}^{\text{der}}(\mathbb{A}_f) = \mathbf{G}^{\text{der}}(\mathbb{Q}) \cdot (K \cap \mathbf{G}^{\text{der}}(\mathbb{A}_f)) \subseteq \mathbf{G}(\mathbb{Q})_+ \cdot (K \cap \mathbf{G}^{\text{der}}(\mathbb{A}_f)). \quad (3.2.7)$$

Because  $\mathbf{G}^{\text{der}}$  is simply-connected, the short exact sequence of groups above Theorem [3.2.5](#) induces a short exact sequence

$$1 \rightarrow \mathbf{G}^{\text{der}}(\mathbb{A}_f) \rightarrow \mathbf{G}(\mathbb{A}_f) \xrightarrow{\nu} \mathbf{T}(\mathbb{A}_f) \rightarrow 1.$$

Here we use the knowledge on semi-simple simply-connected  $\mathbb{Q}$ -groups that  $H^1(\mathbb{Q}_p, \mathbf{G}^{\text{der}}) = 0$  for any prime  $p$ .

Now  $\nu$  induces a map

$$\mathbf{G}(\mathbb{Q})_+ \backslash \mathbf{G}(\mathbb{A}_f) / K \rightarrow \nu(\mathbf{G}(\mathbb{Q})_+) \backslash \mathbf{T}(\mathbb{A}_f) / \nu(K), \quad (3.2.8)$$

which, by (3.2.7), is a bijection. The right hand side is an abelian group because  $\mathbf{T}$  is an algebraic torus (hence abelian).

Now to prove the theorem, it remains to prove:

- (i)  $\nu(\mathbf{G}(\mathbb{Q}))$  has finite index in  $\mathbf{T}(\mathbb{Q})$ .
- (ii)  $\nu(K)$  is a compact open subgroup of  $\mathbf{T}(\mathbb{A}_f)$ .
- (iii) The right hand side of (3.2.8) is finite.

Let us prove (i). The Hasse Principle for simply-connected  $\mathbb{Q}$ -groups says that the natural map  $H^1(\mathbb{Q}, \mathbf{G}^{\text{der}}) \rightarrow \prod_{p \leq \infty} H^1(\mathbb{Q}_p, \mathbf{G}^{\text{der}}) = H^1(\mathbb{R}, \mathbf{G}^{\text{der}})$  is injective; here we used again the fact that  $H^1(\mathbb{Q}_p, \mathbf{G}^{\text{der}}) = 0$  for any prime number  $p$  (as  $\mathbf{G}^{\text{der}}$  is furthermore semi-simple). So by the diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbf{G}^{\text{der}}(\mathbb{Q}) & \longrightarrow & \mathbf{G}(\mathbb{Q}) & \longrightarrow & \mathbf{T}(\mathbb{Q}) \longrightarrow H^1(\mathbb{Q}, \mathbf{G}^{\text{der}}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbf{G}^{\text{der}}(\mathbb{R}) & \longrightarrow & \mathbf{G}(\mathbb{R}) & \longrightarrow & \mathbf{T}(\mathbb{R}) \longrightarrow H^1(\mathbb{R}, \mathbf{G}^{\text{der}}) \end{array}$$

we get that  $\mathbf{T}(\mathbb{Q})/\nu(\mathbf{G}(\mathbb{Q})) \rightarrow \mathbf{T}(\mathbb{R})/\nu(\mathbf{G}(\mathbb{R}))$  is injective. But  $\nu(\mathbf{G}(\mathbb{R})^+) = \mathbf{T}(\mathbb{R})^+$ . So  $\mathbf{T}(\mathbb{R})/\nu(\mathbf{G}(\mathbb{R}))$  is finite, and hence  $\mathbf{T}(\mathbb{Q})/\nu(\mathbf{G}(\mathbb{Q}))$  is finite. This establishes the claim.

For (ii), we extend  $\mathbf{G} \rightarrow \mathbf{T}$  to a morphism of group schemes over  $\mathbb{Z}[1/N]$  for some integer  $N$ , and prove that  $\mathbf{G}(\mathbb{Z}_p) \rightarrow \mathbf{T}(\mathbb{Z}_p)$  is surjective for almost all prime  $p$ . We first work on  $\mathbb{F}_p$  and then list using an argument similar to Newton's Lemma. We omit this proof.

Now we prove (iii). It suffices to prove that  $\mathbf{T}(\mathbb{Q}) \backslash \mathbf{T}(\mathbb{A}_f) / \nu(K)$  is finite, and up to replacing  $\nu(K)$  by a smaller compact open subgroup we may assume  $\nu(K) \subseteq \mathbf{T}(\widehat{\mathbb{Z}})$ . As  $[\mathbf{T}(\widehat{\mathbb{Z}}) : \nu(K)]$  is finite (since  $\mathbf{T}(\widehat{\mathbb{Z}})$  is compact and  $\nu(K)$  is open), it suffices to prove that

$$\mathbf{T}(\mathbb{Q}) \backslash \mathbf{T}(\mathbb{A}_f) / \mathbf{T}(\widehat{\mathbb{Z}})$$

is finite. This is exactly the *class group* of the algebraic torus  $\mathbf{T}$  which is known to be finite by classical theory (and this number is called the *class number* of  $\mathbf{T}$ ). In the case where  $\mathbf{T} = \text{Res}_{K/\mathbb{Q}} \mathbb{G}_m$  for a number field  $K$ , this is exactly the class group of  $K$ .  $\square$

### General case

Let  $\tilde{\mathbf{G}}$  be the universal cover of  $\mathbf{G}^{\text{der}}$ , i.e.  $\tilde{\mathbf{G}}$  is simply-connected with a central isogeny (surjective with finite kernel contained in the center)  $u: \tilde{\mathbf{G}} \rightarrow \mathbf{G}^{\text{der}}$ . Then we have a surjective morphism of  $\mathbb{Q}$ -groups

$$\varphi: \mathbf{G}' := Z(\mathbf{G}) \times \tilde{\mathbf{G}} \rightarrow \mathbf{G}, \quad (z, g) \mapsto zu(g)$$

which is a central isogeny. Thus to prove the finiteness of  $\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}_f) / K$ , it suffices to prove the finiteness of

$$\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K'$$

for  $K'$  a compact open subgroup of  $\mathbf{G}'(\mathbb{A}_f)$ . But the derived subgroup of  $\mathbf{G}'$  is  $\tilde{\mathbf{G}}$  which is simply-connected. So we are back to the previous case, and hence  $\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K'$  is finite. So  $\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}_f) / K$  is finite.



### 3.2.6 An upshot of Theorem 3.2.1 on special subvarieties

Special subvarieties of  $\mathrm{Sh}_K(\mathbf{G}, X)$  can be better understood via Theorem 3.2.1 as follows. Let  $S$  be a special subvariety of  $\mathrm{Sh}_K(\mathbf{G}, X)$ , arising from the sub-Shimura datum  $(\mathbf{G}', X') \subseteq (\mathbf{G}, X)$  and the Hecke operator given by  $g \in \mathbf{G}(\mathbb{A}_f)$ . Then under the decomposition (3.2.1),  $S$  is the image of  $u((X')^+)$  under the uniformization  $u: X^+ \rightarrow \Gamma_g \backslash X^+$  for the suitable connected component  $(X')^+$  of  $X'$ .

## 3.3 Siegel modular variety

Take the example of Siegel case in Example 3.1.3 and Example 3.1.6. In particular  $V = \mathbb{Q}^{2d}$ ,  $\psi: V \times V \rightarrow \mathbb{Q}$  is  $(x, y) \mapsto x^t J y$  with  $J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}$ . Thus there is a lattice  $L$  in  $V$  such that  $\psi$  restricts to  $L \times L \rightarrow \mathbb{Z}$ . To simplify notation, denote by  $L = V(\mathbb{Z})$ .

The Siegel Shimura datum is  $(\mathbf{GSp}_{2d}, \mathfrak{H}_d^\pm)$ . For each  $N$ , set

$$\begin{aligned} K(N) &:= \left\{ g \in \mathbf{GSp}_{2d}(\mathbb{A}_f) : gV(\widehat{\mathbb{Z}}) \subseteq V(\widehat{\mathbb{Z}}) \text{ and acts trivially on } V(\widehat{\mathbb{Z}})/NV(\widehat{\mathbb{Z}}) \right\} \\ &= \left\{ g \in \mathbf{GSp}_{2d}(\widehat{\mathbb{Z}}) : g \equiv I_{2d} \pmod{N} \right\}. \end{aligned}$$

Then we have the Shimura variety  $\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d}, \mathfrak{H}_d^\pm)$ .

**Theorem 3.3.1.** *Assume  $N \geq 3$ . Then  $\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d}, \mathfrak{H}_d^\pm)$  is the fine moduli space of principally polarized abelian varieties of dimension  $d$  with a level- $N$ -structure, i.e. there is a canonical bijection between*

- the  $\mathbb{C}$ -points of  $\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d}, \mathfrak{H}_d^\pm)$ ,
- and the isomorphism classes of the triples  $(A, \lambda, \eta_N)$  where  $A$  is a complex abelian variety of dimension  $d$ ,  $\lambda$  is a principal polarization on  $A$ , and  $\eta_N$  is a level- $N$ -structure on  $A$ .

When  $N = 1, 2$ , the Shimura variety is a coarse moduli space.

Let us explain the meaning of this theorem. Let  $A$  be an abelian variety defined over  $\mathbb{C}$ .

- (i) A *principal polarization* on  $A$  is a polarization on the Hodge structure  $H_1(A, \mathbb{Z})$  with determinant 1, i.e. an alternating pairing  $\lambda: H_1(A, \mathbb{Z}) \times H_1(A, \mathbb{Z}) \rightarrow \mathbb{Z}$ , which under suitable  $\mathbb{Z}$ -basis of  $H_1(A, \mathbb{Z})$  is  $\begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}$ . In more geometric terms, it is an isomorphism  $\lambda: A \xrightarrow{\sim} A^\vee$ .
- (ii) A *(symplectic) level- $N$ -structure* on  $A$  is a basis of  $H_1(A, \mathbb{Z}/N\mathbb{Z})$  which is symplectic with respect to  $\lambda$ . In more geometric terms, it is a basis of the  $\mathbb{Z}/N\mathbb{Z}$ -module  $A[N]$  which is symplectic under  $e_N: A[N] \times A[N] \xrightarrow{(1, \lambda)} A[N] \times A^\vee[N] \rightarrow \mu_N$  where last map is the Weil pairing. Or more concretely, it is an isomorphism

$$\eta_N: A[N] \xrightarrow{\sim} H_1(A, \mathbb{Z}/N\mathbb{Z})$$

such that the two composites

$$\begin{aligned} A[N] \times A[N] &\xrightarrow{(\eta_N, \eta_N)} H_1(A, \mathbb{Z}/N\mathbb{Z}) \times H_1(A, \mathbb{Z}/N\mathbb{Z}) \xrightarrow{\bar{\lambda}} \mathbb{Z}/N\mathbb{Z} \\ \text{and } A[N] \times A[N] &\xrightarrow{e_N} \mu_N \xrightarrow{e^{2\pi\sqrt{-1}a/N} \mapsto [a]} \mathbb{Z}/N\mathbb{Z} \end{aligned}$$



differ from the multiplication by an element in  $[\ell] \in (\mathbb{Z}/N\mathbb{Z})^\times$ , and we say that this level- $N$ -structure *has twist*  $[\ell]$ .

*Proof.* Recall that each point in  $\mathfrak{H}_d^\pm$  parametrizes a  $\mathbb{Q}$ -Hodge structure on  $V$  of type  $(-1, 0) + (0, -1)$ ; see §2.3.1.

We shall use Theorem 3.2.1 and the more precise version (3.2.3), and better, Theorem 3.2.5 because  $\mathbf{Sp}_{2d}$  is simply-connected. One can compute that  $\mathbf{GSp}_{2d}(\mathbb{R})_+ = \mathbf{GSp}_{2d}(\mathbb{R})^+ = \{g \in \mathbf{GSp}_{2d}(\mathbb{R}) : \det(g) > 0\}$ . So  $\mathbf{GSp}_{2d}(\mathbb{Q})_+ = \{g \in \mathbf{GSp}_{2d}(\mathbb{Q}) : \det(g) > 0\}$ . Thus for the quotient

$$1 \rightarrow \mathbf{Sp}_{2d} \rightarrow \mathbf{GSp}_{2d} \xrightarrow{\nu} \mathbb{G}_m \rightarrow 1,$$

we have  $\nu(\mathbf{GSp}_{2d}(\mathbb{Q})_+) = \mathbb{Q}_{>0}$ .<sup>[3]</sup> It is not hard to compute that  $\nu(K(N)) = \{z \in \widehat{\mathbb{Z}} : z \equiv 1 \pmod{N}\} = 1 + N\widehat{\mathbb{Z}}$ . Thus

$$\pi_0(\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d}, \mathfrak{H}_d^\pm)) \simeq \mathbb{Q}_{>0} \backslash \mathbb{A}_f^\times / (1 + N\widehat{\mathbb{Z}}) \simeq (\mathbb{Z}/N\mathbb{Z})^\times.$$

Write  $\Gamma_{[\ell]} \backslash \mathfrak{H}_d^+$  for the connected component of  $\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d}, \mathfrak{H}_d^\pm)$  indexed by  $[\ell] \in (\mathbb{Z}/N\mathbb{Z})^\times$ . Below we only give the constructions of the two directions, without proving that they are inverse to each other.

Given a triple  $(A, \lambda, \eta_N)$ . Assume that the level- $N$ -structure has twist  $[\ell] \in (\mathbb{Z}/N\mathbb{Z})^\times$ . First  $H_1(A, \mathbb{Z})$  is a  $\mathbb{Z}$ -Hodge structure of type  $(-1, 0) + (0, -1)$ , and hence under suitable isomorphism  $(H_1(A, \mathbb{Z}), \lambda) \simeq (V(\mathbb{Z}), \psi)$  we obtain a point  $\tau \in \mathfrak{H}_d^+$ . Then we get a point in  $\Gamma_{[\ell]} \backslash \mathfrak{H}_d^+$  as the image of  $\tau$  under  $\mathfrak{H}_d^+ \rightarrow \Gamma_{[\ell]} \backslash \mathfrak{H}_d^+$ .

Conversely let  $x \in \Gamma_{[\ell]} \backslash \mathfrak{H}_d^+$ . Let  $\tau$  be a pre-image of  $x$  under the quotient  $\mathfrak{H}_d^+ \rightarrow \Gamma_{[\ell]} \backslash \mathfrak{H}_d^+$ . Recall that  $\tau$  parametrizes a  $\mathbb{Q}$ -Hodge structure on  $V$  of type  $(-1, 0) + (0, -1)$ , and thus we can endow  $V(\mathbb{R})$  with a complex structure by the bijection  $V(\mathbb{R}) \subseteq V(\mathbb{C}) \rightarrow V(\mathbb{C})/V_\tau^{0,-1}$ . This makes  $A_\tau := V(\mathbb{R})/V(\mathbb{Z})$  into a compact complex torus of dimension  $d$ , with  $H_1(A_\tau, \mathbb{Z}) = V(\mathbb{Z})$ . Thus  $\psi$  induces a principle polarization via  $H_1(A_\tau, \mathbb{Z})$ . Hence  $A_\tau$  is an abelian variety with a principal polarization which by abuse of notation we still use  $\psi$  to denote. The level- $N$ -structure on  $A_\tau$  is given as follows. We have  $A_\tau[N] = \frac{1}{N}V(\mathbb{Z})/V(\mathbb{Z}) = V(\mathbb{Z})/NV(\mathbb{Z}) = V(\mathbb{Z}/N\mathbb{Z})$ . Take  $g \in \mathbf{GSp}_{2d}(\widehat{\mathbb{Z}})$  such that  $\nu(g) \in \widehat{\mathbb{Z}}^\times$  is congruent to  $\ell$  modulo  $1 + N\widehat{\mathbb{Z}}$ . Then  $g$  induces an isomorphism  $g: V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}}) \xrightarrow{\sim} V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}})$ . But  $V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}}) = V(\mathbb{Z}/N\mathbb{Z}) = H_1(A_\tau, \mathbb{Z}/N\mathbb{Z})$ . Thus we have  $A_\tau[N] = V(\mathbb{Z}/N\mathbb{Z}) = V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}}) \xrightarrow{g} V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}}) = H_1(A_\tau, \mathbb{Z}/N\mathbb{Z})$ . This is the desired level- $N$ -structure because  $\psi(gx, gy) = \nu(g)\psi(x, y)$  by definition of  $\mathbf{GSp}_{2d}$ .  $\square$

### 3.4 CM abelian varieties and special points

Let  $\mathrm{Sh}_K(\mathbf{G}, X)$  be a Shimura variety. In Definition 3.1.10 we defined *special points* on  $\mathrm{Sh}_K(\mathbf{G}, X)$ . They are of particular importance. For example, there exists a natural number field  $E(\mathbf{G}, X)$ , called the *reflex field* of  $(\mathbf{G}, X)$ , on which  $\mathrm{Sh}_K(\mathbf{G}, X)$  is “naturally” defined (or in more vigorous terms, has a canonical model), characterized by the action of the Galois group of  $E(\mathbf{G}, X)$ . This action is explicitly defined for special points on  $\mathrm{Sh}_K(\mathbf{G}, X)$  via the class field theory, and is uniquely determined in this way by the following theorem whose proof we omit:

**Theorem 3.4.1.** *The set of special points is dense in  $\mathrm{Sh}_K(\mathbf{G}, X)$ .*

Here “dense” is true even for the usual topology. The hard part of this theorem is to prove the existence of one special point. Indeed, assume  $\mathrm{Sh}_K(\mathbf{G}, X) \simeq \bigsqcup \Gamma_g \backslash X^+$  has a special point

<sup>[3]</sup>In fact  $\nu(g) = (\det g)^{1/d}$ .

$[x]$ . Then its inverse image  $x$  in  $X^+$  gives rise to a morphism  $x: \mathbb{S} \rightarrow \mathbf{G}_{\mathbb{R}}$  which factors through  $\mathbf{T}_{\mathbb{R}}$  for an algebraic torus  $\mathbf{T} < \mathbf{G}$ . But then the morphism given by  $g \cdot x$  for any  $g \in \mathbf{G}(\mathbb{Q})$  factors through  $(g\mathbf{T}g^{-1})_{\mathbb{R}}$ , with  $g\mathbf{T}g^{-1}$  clearly an algebraic torus in  $\mathbf{G}$  (since it is abelian), and hence defines a Shimura datum  $(g\mathbf{T}g^{-1}, g \cdot \mathbf{T}(\mathbb{R})x)$ . But  $\mathbf{T}(\mathbb{R})x$  is a finite set of points since  $\mathbf{T}$  is abelian. So the image of  $\mathbf{G}(\mathbb{Q})x$  under the quotient  $X^+ \rightarrow \Gamma_g \backslash X^+$  consists of special points of  $\mathrm{Sh}_K(\mathbf{G}, X)$ . Notice that  $X^+ = \mathbf{G}(\mathbb{R})^+x$ . Now it suffice to use the Real Approximation that  $\mathbf{G}(\mathbb{Q})$  is dense in  $\mathbf{G}(\mathbb{R})$  to conclude.

For the existence of special points, we shall focus on the Siegel modular variety, for which we have:

**Theorem 3.4.2.** *Take  $[x] \in \mathrm{Sh}_K(\mathbf{GSp}_{2d}, \mathfrak{H}_d^{\pm})(\mathbb{C})$ . Then  $[x]$  is a special point if and only if the abelian variety  $A_x$  parametrized by  $[x]$  is CM, i.e.  $\mathrm{End}(A_x) \otimes_{\mathbb{Z}} \mathbb{Q}$  contains a commutative  $\mathbb{Q}$ -subalgebra of dimension  $2d$ . Equivalently, an abelian variety  $A$  defined over  $\mathbb{C}$  is CM if and only if the Mumford–Tate group of the  $\mathbb{Q}$ -Hodge structure  $H_1(A, \mathbb{Q})$  is an algebraic torus.*

We will not give a full proof of this theorem, but only recall the definition of CM abelian varieties and give a brief explanation why the associated Mumford–Tate group (which we call the Mumford–Tate group of  $A$ ) is an algebraic torus.

Assume  $A$  is a simple abelian variety. Then  $A$  is CM if and only if  $E := \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$  is a CM field, i.e. there exists a totally real field  $F$  such that  $E/F$  is a totally imaginary quadratic extension. Write  $\bar{(\cdot)}$  for the complex conjugation with respect to  $E/F$ . Then there exists an element  $\iota \in E$  such that  $\bar{\iota} = -\iota$  (totally imaginary element). Then  $E$  can be endowed with the  $\mathbb{Q}$ -symplectic form

$$\langle x, y \rangle := \mathrm{Tr}_{E/\mathbb{Q}}(\bar{x}\iota y).$$

This makes  $(E, \langle \cdot, \cdot \rangle) \simeq (\mathbb{Q}^{2d}, \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix})$  into a symplectic space. Set  $\mathbf{GU}_E$  to be the subgroup of  $\mathbf{GSp}_{2d}$  generated by  $\mathbb{G}_m = Z(\mathbf{GSp}_{2d})$  and

$$\mathbf{U}_E := \{x \in \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m : x\bar{x} = 1\}.$$

Then one can check that  $\mathbf{GU}_E$  is an algebraic torus which contains the Mumford–Tate group of  $A$ . Thus the Mumford–Tate group of  $A$  is abelian, and hence must be an algebraic torus. In fact, one can check that  $\mathbf{GU}_E$  is a maximal torus of  $\mathbf{GSp}_{2d}$ .