Chapter 5

Adelic line bundles

In §0.3, we have seen that polarized dynamical systems can sometimes give normalized height functions, which are genuine functions in contrast to the abstract height machine. The Weil height on $\mathbb{P}^N$ can be obtained in this way. Another important case is the Néron–Tate height on abelian varieties.

In §2.3.2, we explained how to use arithmetic models (with Hermitian line bundles) to find representatives of each class of height functions constructed by the height machine.

It is desirable to express each normalized height in §0.3 in the framework of §2.3.2. This is the case for the Weil height, as shown in Example 2.3.8. When an abelian variety has good reduction everywhere, it is also possible to do so using the Néron model and the *cubist metric*. However, if the abelian variety does not have good reduction everywhere, it is not possible to define the Néron–Tate height using arithmetic models as in §2.3.2.

To solve this problem, S. Zhang defined and studied *adelically metrized line bundles* (*adelic line bundles* for short) over projective varieties, by putting suitable metrics at the places of bad reduction. All the normalized heights from §0.3 can be defined in this framework. This tool is fundamental in the solution of the famous Bogomolov Conjecture by Ullmo and S. Zhang.

More recently, Yuan and S. Zhang generalized this framework to *adelic line bundles over quasi-projective varieties*. On the one hand, this allows to study the normalized height functions *in family*. On the other hand, it turns out that many other height functions can be defined in this framework, for example the Faltings height as a function on the moduli space of principally polarized abelian varieties. This powerful theory opens another chapter of Arakelov Geometry.

In the whole chapter, we take K to be a number field, and X to be an irreducible quasi-projective variety defined over K.

5.1 Limit construction for the geometric setting

Via $\mathbb{Q} \subseteq K$, we can see X as a quasi-projective variety over $\mathrm{Spec}\mathbb{Q}$.

In this section, we construct the category of *geometric* adelic line bundles on X, denoted by $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})$. Roughly speaking, they are line bundles on X which can be extended to a line bundle on "some compatification" of X.

If X itself is projective, then the construction is void. Nevertheless, in practice we often need to work with quasi-projective varieties which are not projective, for example moduli spaces.

5.1.1 $\mathbb{Q}$-line bundles

We define the category of $\mathbb{Q}$-line bundles on X, denoted by $\mathcal{P}\mathrm{ic}(X)_{\mathbb{Q}}$, as follows:

Definition 5.1.1. *A* **$\mathbb{Q}$-line bundle** *on X is a pair (a, L) (often written as aL) with $a \in \mathbb{Q}$ and L a line bundle on X. A* **morphism** *of two $\mathbb{Q}$-line bundles aL and $a'L'$ is defined to be*

$$\mathrm{Hom}(aL, a'L') := \varinjlim_{m\to\infty} \mathrm{Hom}(amL, a'mL')$$

where m runs over all positive integers such that $am, a'm \in \mathbb{Z}$.

Denote by $\mathrm{Pic}(X)_{\mathbb{Q}}$ the group of isomorphism classes of $\mathbb{Q}$-line bundles on X. We can define *nef, ample, big* $\mathbb{Q}$-line bundles on projective varieties.

Definition 5.1.2. *A $\mathbb{Q}$-line bundle aL on X is said to be* **nef (ample, big)** *if amL is for some positive integer m such that amL is a usual line bundle on X.*

Next we define sections of $\mathbb{Q}$-line bundles.

Definition 5.1.3. *Let $aL \in \mathcal{P}\mathrm{ic}(X)_{\mathbb{Q}}$.*

(i) A **(global) section** *of aL on X is an element of $H^0(X, aL) := \mathrm{Hom}(\mathcal{O}_X, aL) = \varinjlim_m H^0(X, amL)$ where m runs over all positive integers with $am \in \mathbb{Z}$.*

(ii) A **rational section** *of aL on X is an element of $\mathrm{Hom}(\mathcal{O}_\eta, aL_\eta) = \varinjlim_m H^0(\eta, amL)$, where η is the generic point of X and m runs over all positive integers with $am \in \mathbb{Z}$.*

(iii) For a (rational) section s of aL on X, represented by $(s_m)_m$, define

$$\mathrm{div}(s) := (1/m)\mathrm{div}(s_m) \in \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q} =: \mathrm{Div}(X)_{\mathbb{Q}}.$$

For two elements $D_1, D_2 \in \mathrm{Div}(X)_{\mathbb{Q}}$, we write $D_1 \le D_2$ if $m(D_2 - D_1)$ is a usual effective Cartier divisor for some positive integer m.

5.1.2 Model geometric adelic line bundles and boundary norm/topology

Now we are ready to define *model geometric adelic line bundles* on X as follows.

Definition 5.1.4. *The category of model geometric adelic line bundles on X, denoted by $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})_{\mathrm{mod}}$, is defined to be the category of pairs (X', L') with*

- *X' is a compactification of X, i.e. a projective variety defined over K which contains X as an open subset;*
- *L' is a $\mathbb{Q}$-line bundle on X', such that $L'|_X$ is isomorphic to a usual line bundle on X.*

Adelic line bundles are, roughly speaking, limits of sequences of model adelic line bundles. In order for the limit process to make sense, we need to introduce a suitable *Cauchy condition* for a sequence of model adelic line bundles. Let us explain it now.

For each compatification X' of X, denote by $\mathrm{Div}(X', X) := \mathrm{Div}(X')_{\mathbb{Q}} \otimes_{\mathrm{Div}(X)_{\mathbb{Q}}} \mathrm{Div}(X)$, *i.e.* the group of $\mathbb{Q}$-divisors on X' whose restriction to X is a usual Cartier divisor.

Definition 5.1.5. *The group of* **model geometric adelic divisors** *is defined to be*

$$\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}} := \varinjlim_{X'} \mathrm{Div}(X', X) \tag{5.1.1}$$

with X' running over all compactifications of X.

Notice that there is a partial order $\leq$ on $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$. For any $D_1, D_2 \in \widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$, there exists a compactification X' such that both D_1 and D_2 can be represented by elements (by abuse of notation still denoted by D_1, D_2) in $\mathrm{Div}(X', X) \subseteq \mathrm{Div}(X')_{\mathbb{Q}}$. We say that $D_1 \leq D_2$ as elements in $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$ if this is the case in $\mathrm{Div}(X')_{\mathbb{Q}}$. It is not hard to check that this partial ordering on $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$ is well-defined.

With this in hand, we can define the *boundary topology* on $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$ as follows. Fix a compactification X_0 of X such that $X_0 \backslash X$ is a divisor, which we call D_0. Then $D_0 \in \mathrm{Div}(X_0, X)$, which gives rise to an element in $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$ which is still denoted by D_0. The following *boundary norm* on $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$ (we use the convention $\inf(\emptyset) = \infty$)

$$\| \cdot \|_{D_0} \colon \widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}} \to [0, \infty], \qquad D \mapsto \inf\{\epsilon \in \mathbb{Q}_{>0} : -\epsilon D_0 \leq D \leq \epsilon D_0\} \tag{5.1.2}$$

then induces a topology on $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$, by defining a neighborhood basis at 0. This is the boundary topology.

Here is an easy lemma on the properties of the boundary norm. The "Moreover" part implies that the boundary topology does not depend on the choice of X_0.

Lemma 5.1.6. *For any $D, D' \in \widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$, we have*

(i) $\|D\|_{D_0} = 0$ *if and only if* $D = 0$,

(ii) $\|D + D'\|_{D_0} \leq \|D\|_{D_0} + \|D'\|_{D_0}$,

(iii) $\|aD\|_{D_0} \leq |a| \cdot \|D\|_{D_0}$ *for any* $a \in \mathbb{Z} \setminus \{0\}$, *with* $<$ *if and only if* $D \neq 0$ *and* $aD = 0$ *both hold in* $\mathrm{Div}(X)$.

Moreover, if X_0' is another compactification of X such that $D_0' := X_0' \backslash X$ is a divisor, then there exists a real number $r > 1$ such that $r^{-1}\| \cdot \|_{D_0} \leq \| \cdot \|_{D_0'} \leq r \| \cdot \|_{D_0}$.

5.1.3 Geometric adelic line bundles and adelic divisors

Definition 5.1.7. *A* **geometric adelic divisor** *on X is an equivalence class of Cauchy sequences in $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$, Cauchy for the boundary topology on $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$.*

The group of geometric adelic divisors on X is denoted by $\widehat{\mathrm{Div}}(X/\mathbb{Q})$, with the obvious binary operation.

Definition 5.1.8. *A* **geometric adelic line bundle** *on X is a pair $(L, (X_i, L_i, \ell_i)_{i \geq 1})$ with*

- *L is a line bundle on X;*

- *$(X_i, L_i) \in \mathcal{P}\mathrm{ic}(X/\mathbb{Q})_{\mathrm{mod}}$;*

- *$\ell_i \colon L \to L_i|_X$ is an isomorphism in $\mathcal{P}\mathrm{ic}(X)_{\mathbb{Q}}$;*

such that the sequence $\{\mathrm{div}(\ell_i \ell_1^{-1})\}_{i \geq 1}$ satisfies the Cauchy condition defined using the boundary topology on $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$.

The category of geometric adelic line bundles is denoted by $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})$. The group of isomorphism classes of geometric adelic line bundles, with $\otimes$ being the binary operation, is denoted by $\widehat{\mathrm{Pic}}(X/\mathbb{Q})$.

We need to explain that the sequence $\{\mathrm{div}(\ell_i\ell_1^{-1})\}_{i\geq 1}$ is indeed a sequence in $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$. For each $i \geq 1$, we have an isomorphism $\ell_i\ell_1^{-1}\colon L_1|_X \to L_i|_X$ of $\mathbb{Q}$-line bundles on X, and hence a rational map $\ell_i\ell_1^{-1}\colon L_1 \to L_i$. Hence $\mathrm{div}(\ell_i\ell_1^{-1})$ is a model adelic divisor for each $i \geq 1$, *i.e.* $\mathrm{div}(\ell_i\ell_1^{-1}) \in \widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$.

Next we will establish a canonical isomorphism between $\widehat{\mathrm{Pic}}(X/\mathbb{Q})$ with $\widehat{\mathrm{Cl}}(X/\mathbb{Q})$, the group of geometric adelic divisor classes. We should first of all define $\widehat{\mathrm{Cl}}(X/\mathbb{Q})$. We start by defining $\widehat{\mathrm{Prin}}(X/\mathbb{Q})_{\mathrm{mod}} := \varinjlim_{X'} \mathrm{Prin}(X')$, where X' runs over all the compactifications of X and $\mathrm{Prin}(X')$ is the group of principal divisors on X'. Then we can define

$$\widehat{\mathrm{Cl}}(X/\mathbb{Q}) := \widehat{\mathrm{Div}}(X/\mathbb{Q})/\widehat{\mathrm{Prin}}(X/\mathbb{Q})_{\mathrm{mod}}. \tag{5.1.3}$$

Lemma 5.1.9. *The group $\widehat{\mathrm{Prin}}(X/\mathbb{Q})_{\mathrm{mod}}$ is discrete in $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$ under the boundary topology.*

Before moving on to the proof, let us see an immediate corollary. If we let

$$\widehat{\mathrm{Cl}}(X/\mathbb{Q})_{\mathrm{mod}} := \widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}/\widehat{\mathrm{Prin}}(X/\mathbb{Q})_{\mathrm{mod}} = \varinjlim_{X'}(\mathrm{Div}(X', X)/\mathrm{Prin}(X')), \tag{5.1.4}$$

then $\widehat{\mathrm{Cl}}(X/\mathbb{Q})_{\mathrm{mod}}$ is dense in $\widehat{\mathrm{Cl}}(X/\mathbb{Q})$ by Lemma 5.1.9. In other words, $\widehat{\mathrm{Cl}}(X/\mathbb{Q})$ is the completion of $\widehat{\mathrm{Cl}}(X/\mathbb{Q})_{\mathrm{mod}}$.

Proof of Lemma 5.1.9. Assume that there exists a sequence $\{D_i\}_{i\geq 1}$ in $\widehat{\mathrm{Prin}}(X/\mathbb{Q})_{\mathrm{mod}}$ converging to 0. Then there exists a sequence $\{\epsilon_i \in \mathbb{Q}_{>0}\}_{i\geq 1}$ such that $\epsilon_i \to 0$ and $\epsilon D_0 \pm D_i \geq 0$ in $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$ for all $i \geq 1$. Assume D_i is represented by $\mathrm{div}(f_i)$ for a compactification X_i of X and a rational function $f_i \in \mathbb{Q}(X_i)^* = \mathbb{Q}(X)^*$. Recall the compactification X_0 used to define the boundary topology. Then $\epsilon_i D_0 \pm \mathrm{div}(f_i) \geq 0$ in $\mathrm{Div}(X_0)_{\mathbb{Q}}$. Hence $\mathrm{Div}(f_i) = 0$ on X_0 by taking ϵ_i to be small enough. We are done. □

Proposition 5.1.10. *There is a canonical isomorphism*

$$\widehat{\mathrm{Cl}}(X/\mathbb{Q}) \xrightarrow{\sim} \widehat{\mathrm{Pic}}(X/\mathbb{Q}).$$

Proof. We write the two morphisms.

For any $\{D_i\}_{i\geq 1} \in \widehat{\mathrm{Div}}(X/\mathbb{Q})$, assume each D_i is defined on the compactification X_i. Then $L_i := \mathcal{O}(D_i)$ is a $\mathbb{Q}$-line bundle on X_i. Notice that $D_i|_X = D_1|_U$ for all $i \geq 1$. Hence we get a line bundle $L := \mathcal{O}(D_1|_X)$ on X and isomorphisms $\ell_i\colon L \to L_i|_X$ for each $i \geq 1$. It is not hard to check the Cauchy condition for the sequence $\mathrm{div}(\ell_i\ell_1^{-1}) = D_i - D_1$. This defines the desired homomorphism

$$\widehat{\mathrm{Div}}(X/\mathbb{Q}) \to \widehat{\mathrm{Pic}}(X/\mathbb{Q}).$$

It is not hard to check that $\widehat{\mathrm{Prin}}(X/\mathbb{Q})_{\mathrm{mod}}$ is in the kernel.

To see the surjectivity: given any $(L, (X_i, L_i, \ell_i)_{i\geq 1})$ in $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})$, take a nonzero rational section s of L on X, and set

$$\widehat{\mathrm{div}}(s) := \{\mathrm{div}_{(X_1,L_1)}(s) + \mathrm{div}(\ell_i\ell_1^{-1})\}_{i\geq 1}, \tag{5.1.5}$$

where $\mathrm{div}_{(X_1,L_1)}(s)$ means to see s as a rational section of L_1 on X_1, and take the corresponding divisor. This defines the desired element in $\widehat{\mathrm{Cl}}(X/\mathbb{Q})$. □

5.1.4 Positivity

Definition 5.1.11. *An adelic line bundle $\widetilde{L} \in \widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})$ is said to be:*

(i) **strongly nef** *if it is isomorphic to an object $(L, (X_i, L_i, \ell_i)_{i\geq 1})$ where each L_i is nef on X_i;*

(ii) **nef** *if there exists a strongly nef $\widetilde{M} \in \widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})$ such that $a\widetilde{L} + \widetilde{M}$ is strongly nef for all positive integers a;*

(iii) **integrable** *if it is isomorphic to the difference of two strongly nef adelic line bundles.*

We will use $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})_{\mathrm{snef}}$ (resp. $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})_{\mathrm{nef}}$, $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})_{\mathrm{int}}$) to denote the full subcategories of $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})$ of strongly nef (resp. nef, integrable) ones. We will use $\widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathrm{snef}}$ (resp. $\widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathrm{nef}}$, $\widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathrm{int}}$) to denote the corresponding subsets of $\widehat{\mathrm{Pic}}(X/\mathbb{Q})$. It is a semi-subgroup (resp, semi-subgroup, subgroup).

For any $\widetilde{L} = (L, \{X_i, L_i, \ell_i\}_{i\geq 1}) \in \widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})$, we define

$$\widehat{H}^0(X, \widetilde{L}) := \{s \in H^0(X, L) : \widehat{\mathrm{div}}(s) \geq 0\} \tag{5.1.6}$$

This is a vector space. We shall prove later on () that $\dim \widehat{H}^0(X, \widetilde{L}) < \infty$.

In height theory, elements in (5.1.6) play the same role as global sections on X when X is projective. Indeed, given a non-zero element $s \in \widehat{H}^0(X, \widetilde{L})$, then roughly speaking the height function defined by $\widetilde{L}$ has a lower bound outside $\widehat{\mathrm{div}}(s)$ which is proper Zariski closed.

Definition-Theorem 5.1.12. *The following limit exists and is defined to be the* **volume** *of $\widetilde{L} = (L, \{X_i, L_i, \ell_i\}_{i\geq 1})$:*

$$\mathrm{vol}(X, \widetilde{L}) := \lim_{m\to\infty} \frac{(\dim X)!}{m^{\dim X}} \dim \widehat{H}^0(X, m\widetilde{L}). \tag{5.1.7}$$

Moreover,

$$\mathrm{vol}(X, \widetilde{L}) = \lim_{i\to\infty} \mathrm{vol}(X_i, L_i).$$

Definition 5.1.13. *An adelic line bundle $\widetilde{L} \in \widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})$ is said to be* **big** *if $\mathrm{vol}(X, \widetilde{L}) > 0$.*

Let $d = \dim X$. We also have an intersection pairing in this situation

$$\widehat{\mathrm{Pic}}(X/\mathbb{Q})^d_{\mathrm{int}} \to \mathbb{R}, \qquad (\widetilde{L}_1, \ldots, \widetilde{L}_d) \mapsto \widetilde{L}_1 \cdot \cdots \cdot \widetilde{L}_d. \tag{5.1.8}$$

Theorem 5.1.14 (Hilbert–Samuel)**.** *Assume $\widetilde{L}$ is nef. Then $\mathrm{vol}(X, \widetilde{L}) = \widetilde{L}^d$.*

Theorem 5.1.15 (Siu)**.** *If $\widetilde{L}$ and $\widetilde{M}$ are nef adelic line bundles, then*

$$\mathrm{vol}(X, \widetilde{L} - \widetilde{M}) \geq \widetilde{L}^d - d\widetilde{L}^{d-1}\widetilde{M}.$$

5.2 Adelic line bundles as limits of the model ones

Next we turn to the arithmetic setting and try to find arithmetic objects which will define the height functions as desired. We will do the limit construction in the following steps.

(i) Consider all the quasi-projective models $\mathcal{U}$ of X, *i.e.* $\mathcal{U}$ is an integral scheme which is quasi-projective over $\mathrm{Spec}\mathcal{O}_K$ such that X is open in the generic fiber $\mathcal{U}_K$. These quasi-projective models form an inverse system.

(ii) Define for each quasi-projective model $\mathcal{U}$ the category of *adelic line bundles* $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$ and the group of *adelic divisors* $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$.

(iii) Define $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z})$ to be $\varinjlim_{\mathcal{U}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$ where the limit is taken on the inverse system of quasi-projective models of X. Similarly define $\widehat{\mathrm{Div}}(X/\mathbb{Z}) := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$.

We call $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z})$ the category of adelic line bundles on X, and $\widehat{\mathrm{Div}}(X/\mathbb{Z})$ the group of adelic divisors on X. The group of isomorphism classes of adelic line bundles on X will be denoted by $\widehat{\mathrm{Pic}}(X/\mathbb{Z})$, which $\otimes$ being the binary operation. Similarly, we use the notation $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$ and $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$ to denote the groups of isomorphism classes of (model) adelic line bundles on $\mathcal{U}$.

Steps (i) and (iii) are formal. Step (ii) is the crucial step. In this section, we will define $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$ as a suitable completion of *model adelic line bundles* $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$ and define $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$ as a suitable completion of *model adelic divisors* $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$.

5.2.1 Model adelic line bundles on $\mathcal{U}$ and boundary topology

Let $\mathcal{U}$ be an integral scheme which is quasi-projective over $\mathrm{Spec}\mathcal{O}_K$.

Definition 5.2.1. *A* **model adelic line bundle** *on $\mathcal{U}$ is a pair $(\mathcal{X}, \overline{\mathcal{L}})$ consisting of:*

- *a projective model $\mathcal{X}$ of $\mathcal{U}$, i.e. an integral scheme which is projective over* $\mathrm{Spec}\mathcal{O}_K$ *and which contains $\mathcal{U}$ as an open subscheme;*

- *a $\mathbb{Q}$-Hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ on $\mathcal{X}$ such that $\mathcal{L}|_{\mathcal{U}}$ is a isomorphic to a usual line bundle on $\mathcal{U}$.*

Here, $\mathbb{Q}$-Hermitian line bundles are defined in the same way to the geometric case (Definition 5.1.1) with L replaced by $\overline{\mathcal{L}}$, and we also have the corresponding nefness, ampleness, and bigness for $\mathbb{Q}$-Hermitian line bundles.

The category of model adelic line bundles on $\mathcal{U}$ is denoted by $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$, and the group of isomorphism classes of model adelic line bundles is denoted by $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$.

To define model adelic divisors, we need to first of all define arithmetic $(\mathbb{Q}, \mathbb{Z})$-divisors. Let $\mathcal{X}$ be a projective model of $\mathcal{U}$.

Definition 5.2.2. *An* **arithmetic** $(\mathbb{Q}, \mathbb{Z})$**-divisors** *on $(\mathcal{X}, \mathcal{U})$ is a $\mathbb{Q}$-arithmetic divisor $\overline{D} = (D, g_D) \in \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$ such that $D|_{\mathcal{U}}$ is a usual divisor on $\mathcal{U}$. It is said to be* **nef** *if $\overline{D}$ is nef in $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$.*

The group of arithmetic $(\mathbb{Q}, \mathbb{Z})$-divisors on $(\mathcal{X}, \mathcal{U})$ is denoted by $\widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$. There is a partial ordering $\leq$ on $\widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$: $\overline{D} \leq \overline{D}'$ if $\overline{D}' - \overline{D}$ is effective as a $\mathbb{Q}$-arithmetic divisor on $\mathcal{X}$ and $D'|_{\mathcal{U}} - D|_{\mathcal{U}} \geq 0$ on $\mathcal{U}$.

Definition 5.2.3. *The group of* **model adelic divisors** *on $\mathcal{U}$ is defined to be*

$$\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}} := \varinjlim_{\mathcal{X}} \widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U}) \tag{5.2.1}$$

where $\mathcal{X}$ runs over all projective models of $\mathcal{U}$.

Notice that the partial ordering on $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$ defined above induces a partial ordering $\leq$ on $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$.

We also have a *boundary topology* on $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$ defined as follows. Fix a projective model $\mathcal{X}_0$ of $\mathcal{U}$ and a strictly effective divisor $\overline{D}_0 = (D_0, g_0)$ on $\mathcal{X}_0$ such that $|D_0| = \mathcal{X}_0 \setminus \mathcal{U}$. Such a pair $(\mathcal{X}_0, \overline{D}_0)$ is called a *boundary divisor.* Then $\overline{D}_0$ gives rise to an element in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$ which we still denote by $\overline{D}_0$. Then the boundary norm is defined to be ($\inf(\emptyset)$ is set to be ∞)

$$\|\cdot\|_{\overline{D}_0} \colon \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}} \to [0,\infty], \qquad \overline{D} \mapsto \inf\{\epsilon \in \mathbb{Q}_{>0} : -\epsilon\overline{D}_0 \leq \overline{D} \leq \epsilon\overline{D}_0\}. \tag{5.2.2}$$

This boundary norm induces a topology on $\widehat{\mathrm{Div}}(X/\mathbb{Z})_{\mathrm{mod}}$, by defining a neighborhood basis at 0. This is the boundary topology.

As in the geometric, we also have the following lemma, which asserts that the boundary topology does not depend on the choice of the pair $(\mathcal{X}_0, \overline{D}_0)$.

Lemma 5.2.4. *For any $\overline{D}, \overline{D}' \in \widehat{\mathrm{Div}}(X/\mathbb{Z})_{\mathrm{mod}}$, we have*

(i) $\|\overline{D}\|_{\overline{D}_0} = 0$ *if and only if* $\overline{D} = 0$*,*

(ii) $\|\overline{D} + \overline{D}'\|_{\overline{D}_0} \leq \|\overline{D}\|_{\overline{D}_0} + \|\overline{D}'\|_{\overline{D}_0}$*,*

(iii) $\|a\overline{D}\|_{\overline{D}_0} \leq |a| \cdot \|\overline{D}\|_{\overline{D}_0}$ *for any* $a \in \mathbb{Z} \setminus \{0\}$*, with* $<$ *if and only if* $D_{\mathcal{U}} \neq 0$ *and* $aD|_{\mathcal{U}} = 0$ *both hold in* $\mathrm{Div}(\mathcal{U})$*.*

Moreover, if $(\mathcal{X}_0', \overline{D}_0')$ is another boundary divisor, then there exists a real number $r > 1$ such that $r^{-1}\|\cdot\|_{\overline{D}_0} \leq \|\cdot\|_{\overline{D}_0'} \leq r\|\cdot\|_{\overline{D}_0}$.

5.2.2 Adelic line bundles and adelic divisors on $\mathcal{U}$

Let $\mathcal{U}$ be an integral scheme which is quasi-projective over $\mathrm{Spec}\mathcal{O}_K$.

Definition 5.2.5. *An* **adelic divisor** *on $\mathcal{U}$ is an equivalence class of Cauchy sequences in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$, Cauchy for the boundary topology.*

The group of adelic divisors on X is denoted by $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$, with the obvious binary operation.

Definition 5.2.6. *An* **adelic line bundle** *on $\mathcal{U}$ is a pair $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ with*

- *$\mathcal{L}$ is a line bundle on $\mathcal{U}$;*
- *$(\mathcal{X}_i, \overline{\mathcal{L}}_i) \in \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$;*
- *$\ell_i \colon \mathcal{L} \to \mathcal{L}_i|_{\mathcal{U}}$ is an isomorphism in $\mathcal{P}\mathrm{ic}(\mathcal{U})_{\mathbb{Q}}$;*

such that the sequence $\{\widehat{\mathrm{div}}(\ell_i\ell_1^{-1})\}_{i\geq 1}$ satisfies the Cauchy condition defined using the boundary topology on $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$.

The category of adelic line bundles is denoted by $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$. The group of isomorphism classes of geometric adelic line bundles, with $\otimes$ being the binary operation, is denoted by $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$.

We need to explain that the sequence $\{\widehat{\mathrm{div}}(\ell_i\ell_1^{-1})\}_{i\geq 1}$ is indeed a sequence in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$. For each $i \geq 1$, we have an isomorphism $\ell_i\ell_1^{-1} \colon \mathcal{L}_1|_{\mathcal{U}} \to \mathcal{L}_i|_{\mathcal{U}}$ of $\mathbb{Q}$-line bundles on $\mathcal{U}$, and hence a rational map $\ell_i\ell_1^{-1} \colon \mathcal{L}_1 \to \mathcal{L}_i$. Hence $\widehat{\mathrm{div}}(\ell_i\ell_1^{-1})$ is a model adelic divisor for each $i \geq 1$, *i.e.* $\widehat{\mathrm{div}}(\ell_i\ell_1^{-1}) \in \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$.

Next we will establish a canonical isomorphism between $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$ with $\widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z})$, the group of adelic divisor classes. We should first of all define $\widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z})$. For each projective model $\mathcal{X}$ of $\mathcal{U}$, there is a natural homomorphism $\widehat{\mathrm{Div}}(\mathcal{X}) \to \widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$, which makes $\widehat{\mathrm{Prin}}(\mathcal{X})$ a subgroup of $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$. Hence we can define

$$\widehat{\mathrm{Prin}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}} := \varinjlim_{\mathcal{X}} \widehat{\mathrm{Prin}}(\mathcal{X}),$$

where $\mathcal{X}$ runs over all the projective models of $\mathcal{U}$. Then we can define

$$\widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z}) := \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})/\widehat{\mathrm{Prin}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}. \tag{5.2.3}$$

Lemma 5.2.7. *The group $\widehat{\mathrm{Prin}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$ is discrete in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$ under the boundary topology.*

We omit the proof but state the following immediate corollary. If we let

$$\widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}} := \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}/\widehat{\mathrm{Prin}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}} = \varinjlim_{\mathcal{X}}(\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})/\widehat{\mathrm{Prin}}(\mathcal{X})), \tag{5.2.4}$$

then $\widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$ is dense in $\widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z})$ by Lemma 5.2.7. In other words, $\widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z})$ is the completion of $\widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$.

Proposition 5.2.8. *There is a canonical isomorphism*

$$\widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z}) \xrightarrow{\sim} \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z}).$$

Proof. We write the two morphisms.

For any $\{\overline{D}_i\}_{i\geq 1} \in \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$, assume each $\overline{D}_i$ is defined on the projective model $\mathcal{X}_i$. Then $\overline{\mathcal{L}}_i := \mathcal{O}(\overline{D}_i)$ is a $\mathbb{Q}$-Hermitian line bundle on $\mathcal{X}_i$. Notice that $D_i|_{\mathcal{U}} = D_1|_{\mathcal{U}}$ for all $i \geq 1$. Hence we get a line bundle $\mathcal{L} := \mathcal{O}(D_1|_{\mathcal{U}})$ on $\mathcal{U}$ and isomorphisms $\ell_i \colon \mathcal{L} \to \mathcal{L}_i|_{\mathcal{U}}$ for each $i \geq 1$. It is not hard to check the Cauchy condition for the sequence $\widehat{\mathrm{div}}(\ell_i\ell_1^{-1}) = \overline{D}_i - \overline{D}_1$. This defines a homomorphism

$$\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z}) \to \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z}).$$

Now assume that $\{\overline{D}_i\}_{i\geq 1}$ is in the kernel of this homomorphism. Then there exists an isomorphism from $(\mathcal{O}_{\mathcal{U}}, (\mathcal{X}_0, \overline{\mathcal{O}}_{\mathcal{X}_0}, 1))$ to $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i))$. Hence we have an isomorphism $\mathcal{O}_{\mathcal{U}} \to \mathcal{O}(D_1|_{\mathcal{U}})$, which is given by the multiplication by some $f \in H^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}})^*$ with $\mathrm{div}(f) = D_1|_{\mathcal{U}} = 0$ on $\mathcal{U}$. The further properties of the isomorphism are equivalent to that $\overline{D}_i$ converges to $-\widehat{\mathrm{div}}(f)$ in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$. Hence the kernel of the group homomorphism above is $\widehat{\mathrm{Prin}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$. So we have an injective group homomorphism

$$\widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z}) \to \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z}).$$

To see the surjectivity: given any $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$, take a nonzero rational section s of $\mathcal{L}$ on $\mathcal{U}$, and set

$$\widehat{\mathrm{div}}(s) := \{\widehat{\mathrm{div}}_{(\mathcal{X}_1,\overline{\mathcal{L}}_1)}(s) + \widehat{\mathrm{div}}(\ell_i\ell_1^{-1})\}_{i\geq 1}, \tag{5.2.5}$$

where $\widehat{\mathrm{div}}_{(\mathcal{X}_1,\overline{\mathcal{L}}_1)}(s)$ means to see s as a rational section of $\overline{L}_1$ on $\mathcal{X}_1$, and take the corresponding arithmetic divisor. This defines the desired element in $\widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z})$. $\square$

5.2.3 Nefness and integrability

Let $\mathcal{U}$ be an integral scheme which is quasi-projective over $\mathrm{Spec}\mathcal{O}_K$.

Definition 5.2.9. *An adelic line bundle $\overline{\mathcal{L}} \in \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$ is said to be:*

(i) **strongly nef** *if it is isomorphic to an object $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ where each $\overline{\mathcal{L}}_i$ is nef on $\mathcal{X}_i$;*

(ii) **nef** *if there exists a strongly nef $\overline{\mathcal{M}} \in \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$ such that $a\overline{\mathcal{L}} + \overline{\mathcal{M}}$ is strongly nef for all positive integers a;*

(iii) **integrable** *if it is isomorphic to the difference of two strongly nef adelic line bundles.*

We will use $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{snef}}$ (resp. $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{nef}}$, $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{int}}$) to denote the full subcategories of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$ of strongly nef (resp. nef, integrable) ones. We will use $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{snef}}$ (resp. $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{nef}}$, $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{int}}$) to denote the corresponding subsets of $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$. It is a semi-subgroup (resp, semi-subgroup, subgroup).

5.2.4 Generic fiber of adelic line bundles

Now we go back to our original situation, where X is an irreducible quasi-projective variety defined over K.

Recall the definition at the beginning of this section that

$$\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z}) := \varinjlim_{\mathcal{U}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z}), \quad \widehat{\mathrm{Pic}}(X/\mathbb{Z}) := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z}), \quad \widehat{\mathrm{Div}}(X/\mathbb{Z}) := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$$

with $\mathcal{U}$ running over all quasi-projective models of X.

Proposition 5.2.8 implies immediately

Proposition 5.2.10. *There is a canonical isomorphism*

$$\widehat{\mathrm{Cl}}(X/\mathbb{Z}) := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z}) \xrightarrow{\sim} \widehat{\mathrm{Pic}}(X/\mathbb{Z}).$$

For any projective model $\mathcal{X}$ of X, *i.e.* an integral scheme which is projective over $\mathrm{Spec}\mathcal{O}_K$ such that X is open in $\mathcal{X}_K$, the generic fiber $\mathcal{X}_K$ is by definition a projective model of X. Hence the natural map $\widehat{\mathrm{Pic}}(\mathcal{X}) \to \widehat{\mathrm{Pic}}(\mathcal{X}_K)$ induces a group homomorphism

$$\widehat{\mathrm{Pic}}(X/\mathbb{Z}) \to \widehat{\mathrm{Pic}}(X/\mathbb{Q}). \tag{5.2.6}$$

Definition 5.2.11. *For any adelic line bundle $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$, the image under (5.2.6) is called the* **generic fiber** *of $\overline{L}$. It is often denoted by $\widetilde{L}$.*

Let $\mathbf{P}$ be one of the symbols {snef, nef, int}. Then we define

$$\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z})_{\mathbf{P}} := \varinjlim_{\mathcal{U}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})_{\mathbf{P}}, \quad \widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathbf{P}} := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathbf{P}}.$$

It is not hard to check that (5.2.6) restricts to

$$\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathbf{P}} \to \widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathbf{P}}.$$

5.3 Metrized line bundles on Berkovich analytification

A second way to understand adelic line bundles is to see them as *metrized line bundles* on the Berkovich analytification of X. In this section, we explain Berkovich analytification and metrized line bundles.

Let k be a Banach ring, *i.e.* a ring with a norm $|\cdot|_{\mathrm{Ban}}$ which is complete for the induced topology. For example, $\mathbb{Z}$ with the archimedean absolute value $|\cdot|_\infty$, $\mathbb{Z}_p$ with the p-adic absolute value $|\cdot|_p$, or *any field* endowed with the trivial absolute value $|\cdot|_0$ ($|a|_0 = 1$ for all $a \neq 0$).

Let Y be a scheme over $\mathrm{Spec}k$. In our discussion, we assume Y to be separated and of finite type.

5.3.1 Berkovich analytification

In this subsection, we explain and recollect some results on Berkovich analytifications.

Definition 5.3.1. *The* **(Berkovich) analytification** *of Y, denoted by $(Y/k)^{\mathrm{an}}$ or Y^{an} for short, is defined as follows.*

If $Y = \mathrm{Spec}A$, then

- *as a set, Y^{an} is defined to be the space $\mathcal{M}(A) = \mathcal{M}(A/k)$ of multiplicative semi-norms on A whose restriction to k is bounded by $|\cdot|_{\mathrm{Ban}}$. For each $y \in \mathcal{M}(A)$, denote the corresponding semi-norm on A by $|\cdot|_y \colon A \to \mathbb{R}$. For any $f \in A$, write $|f|_y$ as $|f(y)|$, which give a real-valued function $|f|$ on $\mathcal{M}(A)$.*
- *the topology on X^{an} is the weakest one such that the function $|f| \colon \mathcal{M}(A) \to \mathbb{R}$ is continuous for all $f \in A$.*

In general, take an affine open cover $\{\mathrm{Spec}A_i\}$ of Y, and define Y^{an} to be the union of $\mathcal{M}(A_i)$, glued canonically. The topology on Y^{an} is the weakest one such that each $\mathcal{M}(A_i)$ is open.

It is known that Y^{an} is locally compact and Hausdorff. If $k = \mathbb{C}$ with the standard absolute value, then Y^{an} is homeomorphic to $Y(\mathbb{C})$ (and so coincides with the usual analytification). If $k = \mathbb{R}$ with the standard absolute value, then Y^{an} is homeomorphic to $Y(\mathbb{C})$ quotient by the complex conjugation.

In general, we have a decomposition

$$Y^{\mathrm{an}} = Y^{\mathrm{an}}[\infty] \cup Y^{\mathrm{an}}[\mathrm{f}] \tag{5.3.1}$$

into the subsets of archimedean and non-archimedean semi-norms. The trivial norm is by definition non-archimedean.

In what follows, when $k = \mathbb{Z}$, we always take $|\cdot|_{\mathrm{Ban}}$ on $\mathbb{Z}$ to be the absolute value $|\cdot|_\infty$

Example 5.3.2. *Let us look at $(\mathrm{Spec}\mathbb{Z})^{\mathrm{an}}$. It is the union of the closed-line segments*

$$[0,1]_\infty := \{|\cdot|_\infty^t : 0 \leq t \leq 1\}$$

and the closed-line segments

$$[0,\infty]_p := \{|\cdot|_p^t : 0 \leq t \leq \infty\}$$

for all finite prime numbers $p > 0$, by identifying the endpoints $|\cdot|_\infty^0$ and $|\cdot|_p^0$ for all p with the trivial norm $|\cdot|_0$ on $\mathbb{Z}$. In particular, $(\mathrm{Spec}\mathbb{Z})^{\mathrm{an}}$ is compact and path-connected.

For convenience, denote by

$$v_0 = |\cdot|_0,\ v_\infty = |\cdot|_\infty,\ v_\infty^t = |\cdot|_\infty^t,\ v_p = |\cdot|_p,\ v_p^t = |\cdot|_p^t,$$

and by

$$(0,1]_\infty,\ (0,1)_\infty,\ (0,\infty]_p,\ [0,\infty)_p,\ (0,\infty)_p$$

the sub-intervals of the line segments obtained by removing one or two endpoints; for example $(0,\infty)_p = \{|\cdot|_p^t : 0 < t < \infty\}$.

And $(\mathrm{Spec}\mathbb{Q}/\mathbb{Z})^{\mathrm{an}}$ *is* $(\mathrm{Spec}\mathbb{Z})^{\mathrm{an}}$ *with* v_p^∞ *removed for all* $p > 0$*, when we see* $\mathrm{Spec}\mathbb{Q}$ *as a scheme over* $\mathrm{Spec}\mathbb{Z}$ *via* $\mathbb{Z} \subseteq \mathbb{Q}$*. If we consider the trivial norm on* $\mathbb{Q}$*, then* $(\mathrm{Spec}\mathbb{Q}/\mathbb{Q})^{\mathrm{an}}$ *is* $\{v_0\}$.

Lemma 5.3.3. *If* Y *is projective over* k*, then* Y^{an} *is compact.*

Here are several basic notions concerning Y^{an}.

Definition 5.3.4. *(i) (Residue field) For each* $y \in Y^{\mathrm{an}}$*, define the* **residue field** H_y *as follows. Take an affine open* $\mathcal{M}(A)$ *such that* $y \in \mathcal{M}(A)$*. The semi-norm* $|\cdot|_y$ *induces a norm on the integral domain* $A/\mathrm{Ker}(|\cdot|_y)$*. Then* H_y *is defined to be the completion of the fraction field of* $A/\mathrm{Ker}(|\cdot|_y)$*. Notice that* $|\cdot|_y\colon A \to \mathbb{R}$ *can be decomposed into*

$$A \to H_y \xrightarrow{|\cdot|} \mathbb{R} \tag{5.3.2}$$

where $|\cdot|$ *is the multiplicative norm on* H_y *induced by* $|\cdot|_y$*. We thus write* $A \to H_y$*,* $f \mapsto f(y)$*. This notion generalizes to an arbitrary* Y^{an}*. By (5.3.2), each* $y \in Y^{\mathrm{an}}$ *gives rise to a* k*-morphism*

$$\phi_y\colon \mathrm{Spec}H_y \to Y. \tag{5.3.3}$$

(ii) (Contraction) The **contraction map** $\kappa\colon Y^{\mathrm{an}} \to Y$ *is defined as follows. It suffices to define for* $\mathcal{M}(A)$*. For each* $y \in \mathcal{M}(A)$*, define* $\kappa(y) := \mathrm{Ker}(|\cdot|_y) \in \mathrm{Spec}A$.

(iii) (Injection) For each $x \in \mathrm{Spec}k$*, the trivial norm on the integral domain* k/x *induces a semi-norm* $|\cdot|_{x,0}$ *on* k*. Assume that each such* $|\cdot|_{x,0}$ *is bounded by* $|\cdot|_{\mathrm{Ban}}$*. This assumption holds true in the three cases considered at the beginning of this subsection (*$\mathbb{Z}$*,* $\mathbb{Z}_p$*, any field with the trivial norm).*

The **injection map** $\iota\colon Y \to Y^{\mathrm{an}}$ *is defined as follows. It suffices to define for* $Y = \mathrm{Spec}A$*. For* $\mathfrak{p} \in \mathrm{Spec}A$*, denote by* $|\cdot|_{\mathfrak{p},0}$ *the semi-norm on* A *induced by the trivial norm on* $A/\mathfrak{p}$*. Then set* $\iota(y) := |\cdot|_{\mathfrak{p},0}$.

(iv) (Reduction) If Y *is proper over* k*, then we can also define a* **reduction map** $r\colon Y^{\mathrm{an}} \to Y$ *as follows.*

Each $y \in Y^{\mathrm{an}}[\mathrm{f}]$ *gives rise to a* k*-morphism* $\mathrm{Spec}H_y \to Y$ *by (5.3.3), and the valuative criterion of properness gives a uniquely extends it to a* k*-morphism* $\mathrm{Spec}R_y \to Y$ *(where* R_x *is the valuation ring of* H_y*). Then* $r(y)$ *is the image of the unique closed point of* $\mathrm{Spec}R_y$.

For $y \in Y^{\mathrm{an}}[\infty]$*, we still have a morphism* $\mathrm{Spec}H_y \to Y$*. Here* H_y *is isomorphic to* $\mathbb{R}$ *or* $\mathbb{C}$*. Define* $r(y)$ *to be the image of* $\mathrm{Spec}H_y$.

Example 5.3.2′. *In* $(\mathrm{Spec}\mathbb{Z})^{\mathrm{an}}$*. For each finite prime* p*, the residue field of* $v_p^t = |\cdot|_p^t$ *is* $\mathbb{Q}_p$ *when* $t \in (0,\infty)$ *and is* $\mathbb{F}_p$ *when* $t = \infty$*. The residue field of* $v_\infty^t = |\cdot|_\infty^t$ *is* $\mathbb{R}$ *when* $t \in (0,1]$*. The residue field of* v_0 *is* $\mathbb{Q}$.

The contraction map leaves $v_p^\infty = |\cdot|_p^\infty$ *stable and sends all other points to* $v_0 = |\cdot|_0$.

The injection map sends the prime (p) *to* $v_p^\infty = |\cdot|_p^\infty$*, and sends* (0) *to* $v_0 = |\cdot|_0$.

Lemma 5.3.5. *Any morphism $f\colon Y \to Y'$ induces a continuous map $f^{\mathrm{an}}\colon Y^{\mathrm{an}} \to Y'^{\mathrm{an}}$. For any $v \in Y'^{\mathrm{an}}$, the* **fiber** $Y_v^{\mathrm{an}} := (f^{\mathrm{an}})^{-1}(v)$*, defined as a subspace of Y^{an}, is canonically homeomorphic to the Berkovich space $(Y_{H_v}/H_v)^{\mathrm{an}}$.*

With this lemma in hand, we have study the structure of the analytification of Y^{an} for $k = \mathbb{Z}$. This applies to any arithmetic variety $\mathcal{U} \to \mathrm{Spec}\mathbb{Z}$ and also to our quasi-projective variety X (defined over a number field K) via $\mathbb{Z} \subseteq \mathbb{Q} \subseteq K$.

We have a structure map $Y^{\mathrm{an}} = (Y/\mathbb{Z})^{\mathrm{an}} \to \mathcal{M}(\mathbb{Z})$, which gives a disjoint union

$$Y^{\mathrm{an}} = \bigcup_{v \in \mathcal{M}(\mathbb{Z})} Y_v^{\mathrm{an}}. \tag{5.3.4}$$

The most distinguished fibers are

$$Y_{v_\infty}^{\mathrm{an}} = Y_{\mathbb{R}}^{\mathrm{an}}, \quad Y_{v_p}^{\mathrm{an}} = Y_{\mathbb{Q}_p}^{\mathrm{an}}.$$

We can furthermore decompose, according to the structure of $\mathcal{M}(\mathbb{Z})$, into

(i) $Y_{\mathrm{triv}}^{\mathrm{an}} := Y_{v_0}^{\mathrm{an}} = (Y_{\mathbb{Q}}/\mathbb{Q})^{\mathrm{an}}$ under the trivial norm of $\mathbb{Q}$;

(ii) $Y_{v_p^\infty}^{\mathrm{an}} = (Y_{\mathbb{F}_p}/\mathbb{F}_p)^{\mathrm{an}}$ under the trivial norm of $\mathbb{F}_p$ for finite primes p;

(iii) $Y_{(0,\infty)_p}^{\mathrm{an}}$, homeomorphic to $Y_{\mathbb{Q}_p}^{\mathrm{an}} \times (0,\infty)$ for finite primes p;

(iv) $Y_{(0,1]_\infty}^{\mathrm{an}}$, homeomorphic to $Y_{\mathbb{R}}^{\mathrm{an}} \times (0,1]$.

Lemma 5.3.6. *The subset $Y^{\mathrm{an}} \setminus Y_{\iota(\mathrm{Spec}\mathbb{Z})}^{\mathrm{an}}$ is dense in Y^{an}.*

Let us go back to our situation where X is a quasi-projective variety defined over a number field K. We close this subsection with the following lemma.

Lemma 5.3.7. *Let $X \to \mathcal{U}$ be a quasi-projective model of X. Then the induced map $X^{\mathrm{an}} \to \mathcal{U}^{\mathrm{an}}$ is continuous, injective, and with a dense image. Better, the set of $v \in X^{\mathrm{an}}$ corresponding to discrete or archimedean valuations of H_v is dense in $\mathcal{U}^{\mathrm{an}}$.*

5.3.2 Metrized line bundle and arithmetic divisors on Y^{an}

Let L be a line bundle on Y. At each point $y \in Y^{\mathrm{an}}$, denote by $\overline{y} := \kappa(y)$ which is a point of Y. The *fiber* $L^{\mathrm{an}}(y)$ of L at y is defined to be the H_y-line $L(\overline{y}) \otimes_{k(\overline{y})} H_y$, or equivalently the completion of the fiber $L(\overline{y})$ of L on $\overline{y}$ for the semi-norm $|\cdot|_y$. In terms of (5.3.3), $L^{\mathrm{an}}(y) = \phi_y^* L$.

Definition 5.3.8. *A* **metrized line bundle** $\overline{L} = (L, \|\cdot\|)$ *on Y^{an} is a pair where L is a line bundle on Y and $\|\cdot\|$ is a continuous metric on Y^{an}. Here a continuous metric of L on Y^{an} is defined to be a continuous metric on $\coprod_{y \in Y^{\mathrm{an}}} L^{\mathrm{an}}(y)$ which is compatible with the semi-norms on $\mathcal{O}_Y$, i.e. for each $y \in Y^{\mathrm{an}}$, assign a norm $\|\cdot\|_y$ on $L^{\mathrm{an}}(y)$ such that $\|f\ell\|_y = |f|_y \|\ell\|_y$ for all $f \in H_y$ and all $\ell \in L^{\mathrm{an}}(y)$, and that for any local section ℓ of L on Y the function $\|\ell(y)\| := \|\ell(y)\|_y$ is continuous in $y \in Y^{\mathrm{an}}$.*

The category of metrized line bundle on Y^{an} is denoted by $\widehat{\mathcal{P}\mathrm{ic}}(Y^{\mathrm{an}})$, and the group of isomorphism classes of metrized line bundles on Y^{an} is denoted by $\widehat{\mathrm{Pic}}(Y^{\mathrm{an}})$.

Definition 5.3.9. *An* **arithmetic divisor** *on Y^{an} is a pair $\overline{D} = (D, g)$ where D is a Cartier divisor on Y and g is a continuous Green's function of $|D|^{\mathrm{an}}$ on Y^{an}, i.e. a continuous function $g\colon Y^{\mathrm{an}} \setminus |D|^{\mathrm{an}} \to \mathbb{R}$ such that for any rational function f on an open subset V of Y with $\mathrm{div}(f) = D|_V$, the function $g + \log|f|$ extends to a continuous function on V^{an}.*

An arithmetic divisor on Y^{an} is said to be **principal** *if it is of the form $\widehat{\mathrm{div}}(f) := (\mathrm{div}(f), -\log|f|)$ for some non-zero rational function of f on Y.*

The group of arithmetic divisors on Y^{an} is denoted by $\widehat{\mathrm{Div}}(Y^{\mathrm{an}})$, and the subgroup of principal arithmetic divisors is denoted by $\widehat{\mathrm{Prin}}(Y^{\mathrm{an}})$. We also have the following definition of effectiveness.

Definition 5.3.10. *An arithmetic divisor $\overline{D} = (D, g)$ on Y^{an} is called* **effective** *(resp.* **strictly effective***) if D is effective and $g \geq 0$ (resp. $g > 0$) on $Y^{\mathrm{an}} \setminus |D|^{\mathrm{an}}$.*

The Green's function g in this setting contains information not only on $Y^{\mathrm{an}}[\infty]$, but also $Y^{\mathrm{an}}[\mathrm{f}]$. Later on we shall see an example (Lemma 5.4.4) that the effectiveness of D is guaranteed by $g \geq 0$. This is not the case if we do not consider the Berkovich analytification.

In both $\widehat{\mathcal{P}\mathrm{ic}}(Y^{\mathrm{an}})$ and $\widehat{\mathrm{Div}}(Y^{\mathrm{an}})$, there is a distinguished class which is of particular interest.

Definition 5.3.11. *A metrized line bundle $\overline{L} = (L, \|\cdot\|)$ on Y^{an}, or its metric $\|\cdot\|$, is called* **norm-equivariance** *if any points $y, y' \in Y^{\mathrm{an}}$ satisfying $|\cdot|_y = |\cdot|_{y'}^t$ for some $0 \leq t < \infty$ locally on $\mathcal{O}_Y$, we have $\|\cdot\|_y = \|\cdot\|_{y'}^t$ (more precisely, for any rational section s of L on Y^{an} such that these two points y, y' are in $Y^{\mathrm{an}} \setminus |\mathrm{div}(s)|^{\mathrm{an}}$, we have $\|s(y)\| = \|s(y')\|^t$).*

An arithmetic divisor $\overline{D} = (D, g)$ on Y^{an}, or its Green's function g, is called **norm-equivariance** *if for any $y, y' \in Y^{\mathrm{an}} \setminus |D|^{\mathrm{an}}$ satisfying $|\cdot|_y = |\cdot|_{y'}^t$ for some $0 \leq t < \infty$ locally on $\mathcal{O}_Y$, we have $g(y) = tg(y')$.*

By definition, every principal arithmetic divisor is norm-equivariance. Denote by $\widehat{\mathcal{P}\mathrm{ic}}(Y^{\mathrm{an}})_{\mathrm{eqv}}$ the full sub-category of norm-equivariance metrized line bundles on Y^{an}, and $\widehat{\mathrm{Pic}}(Y^{\mathrm{an}})_{\mathrm{eqv}}$ and $\widehat{\mathrm{Div}}(Y^{\mathrm{an}})_{\mathrm{eqv}}$ similarly. We have the following proposition.

Proposition 5.3.12. *There is a natural group isomorphism*

$$\widehat{\mathrm{Cl}}(Y^{\mathrm{an}}) := \widehat{\mathrm{Div}}(Y^{\mathrm{an}})/\widehat{\mathrm{Prin}}(Y^{\mathrm{an}}) \xrightarrow{\sim} \widehat{\mathrm{Pic}}(Y^{\mathrm{an}}).$$

Moreover, it sends restricts to

$$\widehat{\mathrm{Cl}}(Y^{\mathrm{an}})_{\mathrm{eqv}} := \widehat{\mathrm{Div}}(Y^{\mathrm{an}})_{\mathrm{eqv}}/\widehat{\mathrm{Prin}}(Y^{\mathrm{an}}) \xrightarrow{\sim} \widehat{\mathrm{Pic}}(Y^{\mathrm{an}})_{\mathrm{eqv}}.$$

Proof. We write the two group homomorphisms.

Let $\overline{D} = (D, g) \in \widehat{\mathrm{Div}}(Y^{\mathrm{an}})$. Define $\mathcal{O}(\overline{D}) := (\mathcal{O}(D), \|\cdot\|_g)$, with $\|s_D\|_g = e^{-g}$ where s_D is the canonical section of $\mathcal{O}(D)$ (*i.e.* $\mathrm{div}(s_D) = D$). If $\overline{D}$ is principal, then it is not hard to check that $\mathcal{O}(\overline{D})$ is isomorphic to the trivial metrized line bundle.

Conversely let $\overline{L} = (L, \|\cdot\|)$ be a metrized line bundle on Y^{an}. Let s be a rational section of L on Y, and define

$$\widehat{\mathrm{div}}_{Y^{\mathrm{an}}}(s) := (\mathrm{div}(s), -\log\|s\|).$$

This gives the desired inverse. $\square$

When $k = \mathbb{Z}$, a norm-equivariant Green's function or a norm-equivariant metric on a line bundle on Y^{an} is uniquely determined by its restriction to the disjoint union of the distinguished fibers $Y^{\mathrm{an}}_{v_\infty} = Y^{\mathrm{an}}_{\mathbb{R}}$ and $Y^{\mathrm{an}}_{v_p} = Y^{\mathrm{an}}_{\mathbb{Q}_p}$ for all finite primes p. This follows from Lemma 5.3.6.

5.4 Adelic line bundles as metrized line bundles

For our quasi-projective variety X defined over a number field K, we view X as a scheme over $\mathrm{Spec}\mathbb{Z}$ via $\mathbb{Z} \subseteq \mathbb{Q} \subseteq K$. Then we can apply the disjoint union decomposition (5.3.4) to X. In particular, we get a fiber $X^{\mathrm{an}}_{\mathrm{triv}} = X^{\mathrm{an}}_{v_0} = (X/\mathbb{Q})^{\mathrm{an}}$ (under the trivial norm of $\mathbb{Q}$) of $X^{\mathrm{an}} = (X/\mathbb{Z})^{\mathrm{an}}$ (with the Banach norm on $\mathbb{Z}$ being the archimedean absolute value).

The goal of this section is to prove the following theorem.

Theorem 5.4.1. *We have the following commutative diagram of homomorphisms of groups:*

$$\begin{array}{ccc} \widehat{\mathrm{Pic}}(X/\mathbb{Z}) & \xrightarrow{\sim} & \widehat{\mathrm{Pic}}(X^{\mathrm{an}})_{\mathrm{eqv}} \\ \downarrow & & \downarrow \\ \widehat{\mathrm{Pic}}(X/\mathbb{Q}) & \xrightarrow{\sim} & \widehat{\mathrm{Pic}}(X^{\mathrm{an}}_{\mathrm{triv}})_{\mathrm{eqv}} \end{array} \tag{5.4.1}$$

where the left vertical arrow is taking the generic fiber $\overline{L} \mapsto \widetilde{L}$, and the right vertical arrow is obtained by pulling back of $X^{\mathrm{an}}_{\mathrm{triv}} \subseteq X^{\mathrm{an}}$.

We will construct the horizontal homomorphisms and prove their injectivity. This suffices for many applications. In the proof we shall see that the top isomorphism in (5.4.1) holds true with X replaced by any quasi-projective arithmetic variety $\mathcal{U}$.

We also have the corresponding version for arithmetic divisors and arithmetic divisor classes, in view of Proposition 5.2.10 and Proposition 5.3.12.

Theorem 5.4.1′. *We have the following natural isomorphisms*

$$\widehat{\mathrm{Div}}(X/\mathbb{Z}) \xrightarrow{\sim} \widehat{\mathrm{Div}}(X^{\mathrm{an}})_{\mathrm{eqv}}, \qquad \widehat{\mathrm{Cl}}(X/\mathbb{Z}) \xrightarrow{\sim} \widehat{\mathrm{Cl}}(X^{\mathrm{an}})_{\mathrm{eqv}}. \tag{5.4.2}$$

Moreover the result holds true with X is replaced by any quasi-projective arithmetic variety.

5.4.1 Construction over projective arithmetic varieties

Let $\mathcal{X}$ be a projective arithmetic variety, *i.e.* a separated integral scheme of finite type over $\mathrm{Spec}\mathbb{Z}$ with projective structural morphism. Let us construct a functor

$$\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X}) \to \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X}^{\mathrm{an}})_{\mathrm{eqv}} \tag{5.4.3}$$

where $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})$ is the category of Hermitian line bundles on $\mathcal{X}$.

Let $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ be a Hermitian line bundle on $\mathcal{X}$. We define a metric of $\mathcal{L}$ on $\mathcal{X}^{\mathrm{an}}$ as follows. Recall the decomposition (5.3.1)

$$\mathcal{X}^{\mathrm{an}} = \mathcal{X}^{\mathrm{an}}[\infty] \cup \mathcal{X}^{\mathrm{an}}[\mathrm{f}]$$

and its refinement below (5.3.4). Now $\|\cdot\|$ gives a metric $\|\cdot\|^{\mathrm{an}}$ of $\mathcal{L}$ on $\mathcal{X}^{\mathrm{an}}_{v_\infty} = \mathcal{X}^{\mathrm{an}}_{\mathbb{R}}$, because $\mathcal{X}^{\mathrm{an}}_{\mathbb{R}} = \mathcal{X}(\mathbb{C})/\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and the metric $\|\cdot\|$ is invariant under the complex conjugation. This metric extends to $\mathcal{X}^{\mathrm{an}}[\infty]]$ by norm-equivariance (Definition 5.3.11) as follows: For any $x \in \mathcal{X}^{\mathrm{an}}[\infty]$, write $(x', t) \in \mathcal{X}^{\mathrm{an}}_{v_\infty} \times (0, 1]$ for the coordinate under the homeomorphism $\mathcal{X}^{\mathrm{an}}[\infty] = \mathcal{X}^{\mathrm{an}}_{(0,1]_\infty} \simeq \mathcal{X}^{\mathrm{an}}_{v_\infty} \times (0, 1]$ (with $(0, 1]_\infty = \{v^t_\infty : 0 < t \leq 1\}$), then set $\|\cdot\|^{\mathrm{an}}_x := (\|\cdot\|^{\mathrm{an}}_{x'})^t$. Notice that $\|\cdot\|^{\mathrm{an}}$ is continuous on $\mathcal{X}^{\mathrm{an}}[\infty]$ by construction.

To define the metric of $\mathcal{L}$ at a point $x \in \mathcal{X}^{\mathrm{an}}[\mathrm{f}]$, we use (the construction of the) specialization map $r \colon \mathcal{X}^{\mathrm{an}}[\mathrm{f}] \to \mathcal{X}$ by the properness of $\mathcal{X} \to \mathrm{Spec}\mathbb{Z}$. More precisely, the point x

gives a morphism $\phi_x\colon \mathrm{Spec} H_x \to \mathcal{X}$ which, by the valuative criterion of properness, extends to $\phi_x^\circ\colon \mathrm{Spec} R_x \to \mathcal{X}$ for the valuation ring R_x of H_x. Then $(\phi_x^\circ)^*\mathcal{L}$ is a free module over R_x of rank 1. Let s_x be the basis of this free module. Define the metric $\|\cdot\|_x^{\mathrm{an}}$ on $\mathcal{L}^{\mathrm{an}}(x) = \phi_x^*\mathcal{L}$ by letting $\|s_x\|_x^{\mathrm{an}} = 1$. Notice that this construction does not use the Hermitian metric on $\overline{\mathcal{L}}$.

Lemma 5.4.2. *Let $\mathcal{U}$ be a Zariski open subset of $\mathcal{X}$. Assume $x \in \mathcal{X}^{\mathrm{an}}$ satisfies $r(x) \in \mathcal{U}$. Then $x \in \mathcal{U}^{\mathrm{an}}$.*

Proof. This is clearly true if $x \in \mathcal{X}^{\mathrm{an}}[\infty]$, by definition of r. Now assume $x \in \mathcal{X}^{\mathrm{an}}[\mathrm{f}]$. Recall that $r(x) = \phi_x^\circ(\mathfrak{m}_x)$ where $\mathfrak{m}_x$ is the unique closed point of $\mathrm{Spec} R_x$.

It suffices to prove $\phi_x^\circ(R_x) \subseteq \mathcal{U}$. Assume not. Then $\phi_x\colon \mathrm{Spec} H_x \to \mathcal{X}$ factors through $\mathcal{X} \setminus \mathcal{U}$ which is itself proper, and hence its (unique) extension ϕ_x° also factors through $\mathcal{X} \setminus \mathcal{U}$, contradicting $r(x) \in \mathcal{U}$. $\square$

Lemma 5.4.3. *The metric $\|\cdot\|^{\mathrm{an}}$ is continuous.*

Proof. We first prove the continuity of $\|\cdot\|^{\mathrm{an}}$ on $\mathcal{X}^{\mathrm{an}}[\mathrm{f}]$. Let $r\colon \mathcal{X}^{\mathrm{an}} \to \mathcal{X}$ be the specialization map, *i.e.* $r(x) = \phi_x^\circ(\mathfrak{m}_x)$ where $\mathfrak{m}_x$ is the unique closed point of $\mathrm{Spec} R_x$. Let $\{x_m\}_{m\geq 1}$ be a sequence in $\mathcal{X}^{\mathrm{an}}[\mathrm{f}]$ converging to $x \in \mathcal{X}^{\mathrm{an}}[\mathrm{f}]$, and let ℓ be a local section of $\mathcal{L}$ on $\mathcal{X}$. We need to prove that $\|\ell(x_m)\|^{\mathrm{an}}$ converges to $\|\ell(x)\|^{\mathrm{an}}$.

Take an open cover $\mathcal{U}_1, \ldots, \mathcal{U}_n$ of $\mathcal{X}$ such that each $\mathcal{U}_i$ contains $r(x)$ and trivializes $\mathcal{L}$ (such an open cover exists). Then $x \in \mathcal{U}_i^{\mathrm{an}}$ by Lemma 5.4.2. On the other hand, $\ell|_{\mathcal{U}_i}$ can be seen as a regular function on $\mathcal{U}_i$, which we denote by f_i. Then $\|\ell(x)\|^{\mathrm{an}} = |f_i|_x$.

For each i, denote by I_i the set of $m \geq 1$ such that $r(x_m) \in \mathcal{U}_i$. Then $I_1 \cup \cdots \cup I_n = \mathbb{Z}_{>0}$. Now take $i \in \{1, \ldots, n\}$ and $m \in I_i$. By Lemma 5.4.2, $x_m \in \mathcal{U}_i^{\mathrm{an}}$. And then $\|\ell(x_m)\|^{\mathrm{an}} = |f_i|_{x_m}$ by the discussion in the previous paragraph. So $\lim_{m\in I_i} \|\ell(x_m)\|^{\mathrm{an}} = \lim_{m\in I_i} |f_i|_{x_m} = |f_i|_x = \|\ell(x)\|^{\mathrm{an}}$ for each $i \in \{1, \ldots, n\}$.

Next, we check the continuity of $\|\cdot\|^{\mathrm{an}}$ when $\mathcal{X}^{\mathrm{an}}[\infty]$ approaches $\mathcal{X}_{v_0}^{\mathrm{an}}$. Let $\{x_m\}_{m\geq 1}$ be a sequence in $\mathcal{X}^{\mathrm{an}}[\infty]$ converging to a point $x \in \mathcal{X}_{v_0}^{\mathrm{an}}$ and let ℓ be a local section of $\mathcal{L}$ on $\mathcal{X}$. We need to prove that $\|\ell(x_m)\|^{\mathrm{an}}$ converges to $\|\ell(x)\|^{\mathrm{an}}$.

Write (z_m, t_m) for the point x_m under the homeomorphism $\mathcal{X}^{\mathrm{an}}[\infty] = \mathcal{X}^{\mathrm{an}}_{(0,1]_\infty} \simeq \mathcal{X}^{\mathrm{an}}_{v_\infty} \times (0,1]$. Then $t_m \to 0$ by assumption on $\{x_m\}_{m\geq 1}$. Assume I is a subsequence of $\mathbb{Z}_{>0}$ such that $\lim_{m\in I} z_m = z \in \mathcal{X}^{\mathrm{an}}_{v_\infty}$. Take an open subset $\mathcal{U}$ of $\mathcal{X}$ which contains $r(x)$ and $r(z)$ such that $\mathcal{L}|_{\mathcal{U}}$ is trivial (such an $\mathcal{U}$ exists). Then $x, z \in \mathcal{U}^{\mathrm{an}}$ by Lemma 5.4.2. Up to removing finitely many elements in I, we may and do assume that $x_m, z_m \in \mathcal{U}^{\mathrm{an}}$ for all $m \in I$. Notice that $\ell|_{\mathcal{U}}$ can be seen as a regular function on $\mathcal{U}$ which we denote by f, and $\|\ell(x)\|^{\mathrm{an}} = |f|_x$ since $x \in \mathcal{U}^{\mathrm{an}}_{v_0} \subseteq \mathcal{U}^{\mathrm{an}}[\mathrm{f}]$. Now f extends to a rational function on $\mathcal{X}$ which we still call f. Then $f^{-1}\ell$ is a rational section of $\mathcal{L}$ on $\mathcal{X}$ such that $\mathcal{U} \cap |\mathrm{div}(f^{-1}\ell)| = \emptyset$. In particular, we have $\|(f^{-1}\ell)(x_m)\|^{\mathrm{an}} = (\|(f^{-1}\ell)(z_m)\|^{\mathrm{an}})^{t_m} = \|(f^{-1}\ell)(z_m)\|^{t_m}$ by definition of $\|\cdot\|^{\mathrm{an}}$ (the first equality is the definition of norm-equivariance Definition 5.3.11), so $\|(f^{-1}\ell)(x_m)\|^{\mathrm{an}} \to 1$ when $m \to \infty$. So $\lim_{m\in I} \|\ell(x_m)\|^{\mathrm{an}} = \lim_{m\in I} |f|_{x_m} = |f|_x = \|\ell(x)\|^{\mathrm{an}}$.

Now the conclusion follows because $\|\cdot\|^{\mathrm{an}}$ is clearly continous on $\mathcal{X}^{\mathrm{an}}[\infty]$. $\square$

The construction (5.4.3) can be translated into a group homomorphism

$$\widehat{\mathrm{Div}}(\mathcal{X}) \to \widehat{\mathrm{Div}}(\mathcal{X}^{\mathrm{an}})_{\mathrm{eqv}}. \tag{5.4.4}$$

Let $\overline{D} = (D, g)$ be an arithmetic divisor on $\mathcal{X}$. The desired Green's function $\widetilde{g}$ of $|D|^{\mathrm{an}}$ on $\mathcal{X}^{\mathrm{an}}$ as follows. Now $g\colon \mathcal{X}(\mathbb{C}) \setminus |D(\mathbb{C})| \to \mathbb{R}$ naturally gives a Green's function on $\mathcal{X}^{\mathrm{an}}_{v_\infty} = \mathcal{X}^{\mathrm{an}}_{\mathbb{R}} = \mathcal{X}(\mathbb{C})/\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ since g is invariant under the complex conjugation. It extends to a

Green's function $\widetilde{g}$ on $\mathcal{X}^{\mathrm{an}}[\infty]$ by norm-equivariance (Definition 5.3.11): For any $x \in \mathcal{X}^{\mathrm{an}}[\infty]$, write $(x', t) \in \mathcal{X}^{\mathrm{an}}_{v_\infty} \times (0,1]$ for the coordinate under the homeomorphism $\mathcal{X}^{\mathrm{an}}[\infty] = \mathcal{X}^{\mathrm{an}}_{(0,1]_\infty} \simeq \mathcal{X}^{\mathrm{an}}_{v_\infty} \times (0,1]$ (with $(0,1]_\infty = \{v_\infty^t : 0 < t \leq 1\}$), and then set $\widetilde{g}(x) = tg(x')$.

For $x \in (\mathcal{X} \setminus |D|)^{\mathrm{an}}[\mathrm{f}]$, take a Zariski open $\mathcal{U}$ of $\mathcal{X}$ such that $r(x) \in \mathcal{U}$ and that $D|_{\mathcal{U}} = \mathrm{div}(f)$ for some $f \in \mathbb{Q}(\mathcal{U})^*$. Then $\widetilde{g}(x)$ is defined to be $-\log|f|_x$.

The continuity of $\widetilde{g}$ on $\mathcal{X}^{\mathrm{an}} \setminus |D|^{\mathrm{an}}$ follows from Lemma 5.4.3. It self-improves to that $\widetilde{g}$ is a Green's function of $|D|^{\mathrm{an}}$ on $\mathcal{X}^{\mathrm{an}}$, by applying the continuity to the arithmetic divisor $(D - \mathrm{div}_{\mathcal{X}}(f), g + \log|f|_\infty)$ for any rational function f on an open subset $\mathcal{V}$ of $\mathcal{X}$ with $\mathrm{div}(f) = D|_{\mathcal{V}}$.

The Green's function $\widetilde{g}$ contains much more information than g. As a particular instance, we have the following lemma.

Lemma 5.4.4. *Assume $\mathcal{X}$ is normal. Let $\overline{D} = (D, g)$ be an arithmetic divisor on $\mathcal{X}$ and let $\widetilde{g}$ be the associated Green's function on $\mathcal{X}^{\mathrm{an}}$. Then $\overline{D}$ is effective if and only if $\widetilde{g} \geq 0$ on $\mathcal{X}^{\mathrm{an}} \setminus |D|^{\mathrm{an}}$.*

Proof. Only the "if" part needs to be checked. Assume $\widetilde{g} \geq 0$. We only need to check the effectiveness of D. For any $v \in \mathcal{X}$ of codimension 1, we need to show that the valuation $\mathrm{ord}_v(D)$ in the local ring $\mathcal{O}_{\mathcal{X},v}$ is non-negative. Consider the point $\xi := \exp(-\mathrm{ord}_v)$ of $\mathcal{X}^{\mathrm{an}}$. Let f be a local equation of D in an open neighborhood of v in $\mathcal{X}$, then by definition we have

$$\widetilde{g}(\xi) = -\log|f|_\xi = -\log(\exp(-\mathrm{ord}_v f)) = \mathrm{ord}_v f = \mathrm{ord}_v(D).$$

Hence we are done. □

5.4.2 Construction over quasi-projective arithmetic varieties

Let $\mathcal{U}$ be a quasi-projective arithmetic variety, *i.e.* a separated integral scheme of finite type over $\mathrm{Spec}\mathbb{Z}$ with quasi-projective structural morphism. Now let us construct a functor

$$\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z}) \to \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}} \tag{5.4.5}$$

and prove that it is fully-faithful. Notice that this proves the existence of the top arrow in (5.4.1) and its injectivity, with X replaced by $\mathcal{U}$.

Construction of (5.4.5)

Let $\overline{\mathcal{L}} = (\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1}) \in \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$. Each $\overline{\mathcal{L}}_i$ induces a metric $\|\cdot\|_i^{\mathrm{an}}$ of $\mathcal{L}_i$ on $\mathcal{X}_i^{\mathrm{an}}$ as was done in the previous subsection. Then we get a metric $\|\cdot\|_i$ on $\mathcal{L}$ by pulling back via the isomorphism $\ell_i \colon \mathcal{L} \xrightarrow{\sim} \mathcal{L}_i|_{\mathcal{U}}$.

Let us show that $\|\cdot\|_i$ converges pointwise to a metric $\|\cdot\|$ of $\mathcal{L}$ on $\mathcal{U}^{\mathrm{an}}$; then the image of $\overline{\mathcal{L}}$ under (5.4.5) is set to be $\overline{\mathcal{L}}^{\mathrm{an}} := (\mathcal{L}, \|\cdot\|)$.

Let $(\mathcal{X}_0, \overline{D}_0)$ be a boundary divisor. Write $\widetilde{g}_0$ for the Green's function of D_0 on $\mathcal{X}_0^{\mathrm{an}}$ induced by $\overline{D}_0$ via (5.4.4). By the definition of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$, the sequence $\{\widehat{\mathrm{div}}(\ell_i\ell_1^{-1})\}_{i\geq 1}$ is Cauchy in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$, *i.e.* there exists a sequence $\{\epsilon_j\}_{j\geq 1}$ of positive rational numbers tending to 0 such that the following inequality holds true in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$:

$$-\epsilon_j\overline{D}_0 \leq \widehat{\mathrm{div}}(\ell_i\ell_1^{-1}) - \widehat{\mathrm{div}}(\ell_j\ell_1^{-1}) \leq \epsilon_j\overline{D}_0, \quad \forall i \geq j \geq 1.$$

Write $f_i := \log(\|\cdot\|_i/\|\cdot\|_1)$ as a continuous function on $\mathcal{U}^{\mathrm{an}}$. Then the condition above implies

$$-\epsilon_j\widetilde{g}_0 \leq f_i - f_j \leq \epsilon_j\widetilde{g}_0, \quad \forall i \geq j \geq 1. \tag{5.4.6}$$

The verification is by a detour of using the constructions of (5.4.3) and (5.4.4), and Proposition 5.2.8 relating $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$ with $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$ (resp. Proposition 5.3.12 ralating $\widehat{\mathrm{Pic}}(\mathcal{U}^{\mathrm{an}})$ with $\widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})$). Thus $\{f_i\}$ is uniformly convergent to a continuous function on any compact subset of $\mathcal{U}^{\mathrm{an}}$.

Recall that $\mathcal{U}^{\mathrm{an}}$ is locally compact. So $\{f_i\}$ converges pointwise to a continuous function f on $\mathcal{U}^{\mathrm{an}}$. Hence $\|\cdot\|_i$ converges pointwise to a continuous metric $\|\cdot\|$ such that

$$-\epsilon_j\widetilde{g}_0 \le \log(\|\cdot\|/\|\cdot\|_j) \le \epsilon_j\widetilde{g}_0, \quad \forall j \ge 1. \tag{5.4.7}$$

Fully-faithfulness of (5.4.5)

Let us show that there exists a canonical isomorphism

$$\Phi\colon \mathrm{Hom}(\overline{\mathcal{O}}_{\mathcal{X}_0}, \overline{\mathcal{L}}) \xrightarrow{\sim} \mathrm{Hom}(\overline{\mathcal{O}}_{\mathcal{U}}, \overline{\mathcal{L}}^{\mathrm{an}}) \tag{5.4.8}$$

where $\overline{\mathcal{O}}_{\mathcal{X}_0} = (\mathcal{O}_{\mathcal{U}}, (\mathcal{X}_0, \overline{\mathcal{O}}_{\mathcal{X}_0}, 1))$ and $\overline{\mathcal{O}}_{\mathcal{U}} = (\mathcal{O}_{\mathcal{U}}, \|\cdot\|_0)$ are the identity elements.

Elements of both sides of Φ are represented by regular sections s of $\mathcal{L}$ which are everywhere non-vanishing on $\mathcal{U}$. Such a section s gives an element of the RHS if and only if $\|s\| = 1$ on $\mathcal{U}^{\mathrm{an}}$, or equivalently if and only if $\widehat{\mathrm{div}}(s) = 0$ in $\widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})$. Such a section s gives an element of the LHS if and only if $\widehat{\mathrm{div}}(s) = 0$ in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$.

Recall that $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}} = \varinjlim_{\mathcal{X}} \widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$ with $\mathcal{X}$ running over all projective models of $\mathcal{U}$. We may assume $\mathcal{X}$ to be normal by taking normalization. Then by Lemma 5.4.4, an element in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$ is effective if and only if its image in $\widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}$ is effective. This gives the desired isomorphism (5.4.8).

Now let $\overline{\mathcal{L}}, \overline{\mathcal{L}}' \in \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$ with images $\overline{\mathcal{L}}^{\mathrm{an}}, \overline{\mathcal{L}}'^{\mathrm{an}} \in \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})$. Applying (5.4.8) to $(\overline{\mathcal{L}}')^\vee \otimes \overline{\mathcal{L}}$, we get a canonical isomorphism

$$\mathrm{Hom}(\overline{\mathcal{O}}_{\mathcal{X}_0}, (\overline{\mathcal{L}}')^\vee \otimes \overline{\mathcal{L}}) \xrightarrow{\sim} \mathrm{Hom}(\overline{\mathcal{O}}_{\mathcal{U}}, (\overline{\mathcal{L}}'^{\mathrm{an}})^\vee \otimes \overline{\mathcal{L}}^{\mathrm{an}}),$$

and hence a canonical isomorphism

$$\mathrm{Hom}(\overline{\mathcal{L}}', \overline{\mathcal{L}}) \xrightarrow{\sim} \mathrm{Hom}(\overline{\mathcal{L}}'^{\mathrm{an}}, \overline{\mathcal{L}}^{\mathrm{an}}).$$

This proves that the functor (5.4.5) is fully-faithful.

In terms of adelic divisors

The construction (5.4.5) can be converted to

$$\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z}) \to \widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}, \qquad \text{and} \quad \widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z}) \to \widehat{\mathrm{Cl}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}. \tag{5.4.9}$$

Here is a more concrete way for this construction of (5.4.9) for which we focus on the first homomorphism. For each projective model $\mathcal{X}$ of $\mathcal{U}$, the analytification map (5.4.4) induces a map

$$\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U}) \to \widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}, \quad \overline{D} = (D, g) \mapsto (D|_{\mathcal{U}}, \widetilde{g}).$$

By direct limit, this map gives $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}} \to \widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}$. Now we wish to extend this map to (5.4.9). Fix a boundary divisor $(\mathcal{X}_0, \overline{D}_0)$ of $\mathcal{U}$. Let $\{(D_i, g_i)\}_{i\ge 1} \in \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$, *i.e.* a Cauchy sequence in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$, with each D_i a divisor of a projective model $\mathcal{X}_i$ of $\mathcal{U}$. Then there exists a sequence $\{\epsilon_j \in \mathbb{Q}_{>0}\}_{j\ge 1}$ with $\epsilon_j \to 0$ and

$$-\epsilon_j(D_0, g_0) \le (D_i - D_j, g_i - g_j) \le \epsilon_j(D_0, g_0), \quad \forall i \ge j \ge 1. \tag{5.4.10}$$

Write $\widetilde{g}_i$ for the Green's function of D_i on $\mathcal{X}_i^{\mathrm{an}}$ induced by $\overline{D}_i = (D_i, g_i)$ via (5.4.4), for each $i \geq 0$ (this includes the boundary divisor). Notice that $D_1|_{\mathcal{U}} = D_2|_{\mathcal{U}} = \cdots$, and we denote by D this divisor on $\mathcal{U}$. Let us show that $\{\widetilde{g}_i\}_{i\geq 1}$ converges to a Green's function of D on $\mathcal{U}^{\mathrm{an}}$. Indeed, (5.4.10) implies that

$$-\epsilon_j \widetilde{g}_0 \leq \widetilde{g}_i - \widetilde{g}_j \leq \epsilon_j \widetilde{g}_0, \quad \forall i \geq j \geq 1. \tag{5.4.11}$$

Thus $\{\widetilde{g}_i\}$ is uniformly convergent to a continuous function on any compact subset of $\mathcal{U}^{\mathrm{an}}$. Recall that $\mathcal{U}^{\mathrm{an}}$ is locally compact. So $\{\widetilde{g}_i\}$ converges pointwise to a continuous function $\widetilde{g}$ on $\mathcal{U}^{\mathrm{an}}$, which is the desired Green's function.

Now (5.4.9) is defined by sending $\{(D_i, g_i)\}_{i\geq 1} \mapsto (D, \widetilde{g})$.

5.4.3 Proof of Theorem 5.4.1 (without surjectivity)

Consider the functor

$$\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z}) = \varinjlim_{\mathcal{U}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z}) \xrightarrow{\lim \text{ of } (5.4.5)} \varinjlim_{\mathcal{U}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}, \tag{5.4.12}$$

which is fully-faithful since (5.4.5) is. For any quasi-projective model $\mathcal{U}$ of X, the map $X^{\mathrm{an}} \to \mathcal{U}^{\mathrm{an}}$ induces a natural map $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}} \to \widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})_{\mathrm{eqv}}$. Thus we have a functor

$$\varinjlim_{\mathcal{U}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}} \to \widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})_{\mathrm{eqv}}. \tag{5.4.13}$$

Now composing the two functors above, we obtain

$$\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z}) \to \widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})_{\mathrm{eqv}} \tag{5.4.14}$$

which gives the top arrow of (5.4.1).

Now let us prove that (5.4.13) is fully-faithful. The upshot is that the top arrow of (5.4.1) is injective.

We start by showing that the natural functor

$$\varinjlim_{\mathcal{U}} \mathcal{P}\mathrm{ic}(\mathcal{U}) \to \mathcal{P}\mathrm{ic}(X) \tag{5.4.15}$$

is fully-faithful. Fix a quasi-projective model $\mathcal{U}_0$ of X. It is not hard to show that the system $\{\mathcal{U}\}$ can be taken to be the inverse system of open subscheme of $\mathcal{U}_0$ containing X. Now take $\mathcal{L}, \mathcal{L}'$ two line bundles on some open neighborhood of X in $\mathcal{U}_0$. Then the map

$$\varinjlim_{\mathcal{U}} H^0(\mathcal{U}, \mathcal{L}^\vee \otimes \mathcal{L}') \to H^0(X, \mathcal{L}^\vee \otimes \mathcal{L}')$$

is injective since both sides are subgroups of rational sections of $\mathcal{L}^\vee \otimes \mathcal{L}'$ on X, and is surjective because any rational section s of $\mathcal{L}^\vee \otimes \mathcal{L}'$ regular and nowhere vanishing on X must be regular and nowhere vanishing on a neighborhood of X in $\mathcal{U}_0$. In other words,

$$\varinjlim_{\mathcal{U}} \mathrm{Hom}(\mathcal{L}|_{\mathcal{U}}, \mathcal{L}'|_{\mathcal{U}}) \simeq \mathrm{Hom}(\mathcal{L}|_X, \mathcal{L}'|_X),$$

whereas the fully-faithfulness of (5.4.15). Hence (5.4.13) is fully-faithful by Lemma 5.3.7.

Next we turn to the bottom arrow of (5.4.1). In fact, we can simply repeat the construction in §5.4.1 if X is projective (replace $\mathcal{X}/\mathbb{Z}$ by $X/\mathbb{Q}$; notice that the construction is easier since

$X^{\mathrm{an}} = X^{\mathrm{an}}[\mathrm{f}]$ in this case), and then pass to quasi-projective X after a similar but easier construction as in §5.4.2. This establishes the bottom arrow of (5.4.1) and proves its injectivity, and at the same time proves the commutativity of the diagram (5.4.1). $\square$

As for Theorem 5.4.1′, the desired homomorphisms are

$$\widehat{\mathrm{Div}}(X/\mathbb{Z}) = \varinjlim_{\mathcal{U}} \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z}) \xrightarrow{\lim\ \text{of } (5.4.9)} \varinjlim_{\mathcal{U}} \widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}} \to \widehat{\mathrm{Div}}(X^{\mathrm{an}})_{\mathrm{eqv}},$$

$$\widehat{\mathrm{Cl}}(X/\mathbb{Z}) = \varinjlim_{\mathcal{U}} \widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z}) \xrightarrow{\lim\ \text{of } (5.4.9)} \varinjlim_{\mathcal{U}} \widehat{\mathrm{Cl}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}} \to \widehat{\mathrm{Cl}}(X^{\mathrm{an}})_{\mathrm{eqv}}.$$

Here the last maps in both compositions are induced by $X \subseteq \mathcal{U}$.