Chapter 4

Proof of Arithmetic Hilbert–Samuel

With the preparation in the previous chapter, we prove the arithmetic Hilbert–Samuel theorem for vol_χ and $\overline{\mathcal{L}}$ in the following setup in this chapter. We follow the approach of Abbès–Bouche.

Let K be a number field and let $\mathcal{O}_K$ be its ring of integers.

Let $\mathcal{X}$ be a projective arithmetic variety of dimension $n+1$ and let $\overline{\mathcal{L}}$ be a smooth Hermitian line bundle. We furthermore consider the case where $\mathcal{X} \to \mathrm{Spec}\mathbb{Z}$ factors through $\mathrm{Spec}\mathcal{O}_K$ and that the generic fiber $\mathcal{X}_K$ is smooth and irreducible.

Theorem 4.0.1. *Assume $\mathcal{L}$ is very ample on $\mathcal{X}$ and $c_1(\overline{\mathcal{L}}) > 0$. Then*

$$\lim_{k \to \infty} \frac{\chi(k\overline{\mathcal{L}})}{k^{n+1}/(n+1)!} \longrightarrow \overline{\mathcal{L}}^{n+1} \tag{4.0.1}$$

when $k \to \infty$.

In other words, the sup-limit in the definition of $\mathrm{vol}_\chi(\overline{\mathcal{L}})$ (Definition 2.5.10) is an actually limit under the assumption of the theorem, and $\mathrm{vol}_\chi(\overline{\mathcal{L}}) = \overline{\mathcal{L}}^{n+1}$.

In the proof, we will use the Hilbert–Samuel theorem in algebraic geometry. Let P be the Hilbert polynomial of $\mathcal{L}_K$ on $\mathcal{X}_K$, *i.e.* $P(k) = \dim H^0(\mathcal{X}_K, k\mathcal{L}_K)$ for $k \gg 1$. It is known that $\deg P = n$ with leading coefficient $\mathcal{L}_K^n/n!$.

4.1 Framework of the proof

4.1.1 Revision on the statement

We start by recalling the objects appearing in the statement of Theorem 4.0.1.

First we have the embedding

$$0 \to H^0(\mathcal{X}, \mathcal{L}) \to H^0(\mathcal{X}, \mathcal{L})_{\mathbb{R}}, \tag{4.1.1}$$

with $H^0(\mathcal{X}, \mathcal{L})_{\mathbb{R}}$ a real vector space of finite dimension defined in (2.5.2) and $H^0(\mathcal{X}, \mathcal{L})$ a lattice. In fact, the structural morphism $\mathcal{X} \to \mathrm{Spec}\mathbb{Z}$ factors through $\mathrm{Spec}R$ for an order R in a number field K, such that the generic fiber $\mathcal{X}_K$ is irreducible, and

$$H^0(\mathcal{X}, \mathcal{L})_{\mathbb{R}} := \{s = (s_\sigma)_\sigma \in H^0(\mathcal{X}, \mathcal{L})_{\mathbb{C}} = \bigoplus\nolimits_{\sigma\colon K \hookrightarrow \mathbb{C}} H^0(\mathcal{X}_\sigma, \mathcal{L}_\sigma) : s_\sigma = \overline{s_{\overline{\sigma}}} \text{ for all } \sigma\}.$$

We shall use the *sup-norm* on $H^0(\mathcal{X}, \mathcal{L})_{\mathbb{R}}$ defined as follows:

- For any $s = (s_\sigma)_\sigma \in H^0(\mathcal{X}, \mathcal{L})_{\mathbb{R}}$, define $\|s\|_{\sup} := \sup_{\sigma, x \in \mathcal{X}_\sigma}(\|s_\sigma(x)\|)$.

Set $B(\overline{\mathcal{L}})$ to be the unit ball in $H^0(\mathcal{X},\mathcal{L})_{\mathbb{R}}$ defined by the sup-norm $\|\cdot\|_{\sup}$. Then

$$\chi(\overline{\mathcal{L}}) := \log \frac{\mathrm{vol}(B(\overline{\mathcal{L}}))}{\mathrm{covol}(H^0(\mathcal{X},\mathcal{L})_{\mathbb{R}}/H^0(\mathcal{X},\mathcal{L}))} \tag{4.1.2}$$

is independent of the choice of the Haar measure on $H^0(\mathcal{X},\mathcal{L})_{\mathbb{R}}$. This finishes the explanation of the limit in (4.0.1).

We also need to define an L^2-norm on $H^0(\mathcal{X},\mathcal{L})_{\mathbb{R}}$ for the proof of Theorem 4.0.1. For this, consider the positive $(1,1)$-form $c_1(\overline{\mathcal{L}})$ on $\mathcal{X}(\mathbb{C}) = \coprod_\sigma \mathcal{X}_\sigma$. For each $\sigma\colon K \hookrightarrow \mathbb{C}$, the positive $(1,1)$-form $c_1(\overline{\mathcal{L}}_\sigma)$ is a Kähler form on $\mathcal{X}_\sigma$. We normalize it to

$$\omega_\sigma := \frac{c_1(\overline{\mathcal{L}}_\sigma)}{(\int_{\mathcal{X}_\sigma} c_1(\overline{\mathcal{L}}_\sigma)^{\wedge n}/n!)^{1/n}},$$

which is still a Kähler form on $\mathcal{X}_\sigma$, and the volume of $\mathcal{X}_\sigma$ for the associated volume form $\mathrm{d}V_\sigma$ is 1.

Now we define the L^2-*norm* on $H^0(\mathcal{X},\mathcal{L})_{\mathbb{R}}$ to be:

- For any $s = (s_\sigma)_\sigma \in H^0(\mathcal{X},\mathcal{L})_{\mathbb{R}}$, define $\|s\|_{L^2} := \sup_\sigma(\int_{\mathcal{X}_\sigma} \|s(x)\|^2 \mathrm{d}V_\sigma)^{1/2}$.

It is a fundamental question in Arakelov Geometry to compare the sup-norm and the L^2-norm. We shall prove later on, using the distortion function discussed in §3.1, the following result.

Proposition 4.1.1. *There exists a constant $c > 0$ such that for all $N \geq 1$ and all $s \in H^0(\mathcal{X}, k\mathcal{L})$, we have*

$$\|s\|_{L^2} \leq \|s\|_{\sup} \leq cP(k)^{1/2}\|s\|_{L^2}.$$

In fact, this c can be chosen to satisfy $b_k(x)/P(k) \leq c^2$ for all k and all $x \in \mathcal{X}(\mathbb{C})$, where b_k is the distortion function.

4.1.2 A tale of three volumes

Consider the embedding (4.1.1). We shall define three volume forms on $H^0(\mathcal{X}, k\mathcal{L})_{\mathbb{R}}$ for each $k \geq 1$:

(i) $V^k_{X,\sup}$ such that the volume of $B(k\overline{\mathcal{L}})$ has volume 1;

(ii) V^k_{X,L^2} such that the volume of the unit ball in $H^0(\mathcal{X}, k\mathcal{L})_{\mathbb{R}}$ defined by the L^2-norm $\|\cdot\|_{L^2}$ has volume 1;

(iii) $V^k_{X,\alpha}$ for each real number $\alpha \in \mathbb{R}$ defined below by Definition 4.1.2 (which we will call $\eta_{k,\alpha}$), with $M = \bigoplus_{k\geq 0} H^0(\mathcal{X}, k\mathcal{L})$.

A key point to prove arithmetic Hilbert–Samuel is to compare $V^k_{X,\sup}$ with $V^k_{X,\alpha}$, and the comparison is done via V^k_{X,L^2}. The statements of these comparisons and their consequence on arithmetic Hilbert–Samuel will be discussed in the next subsection.

In this subsection, we give the definition of $V^k_{X,\alpha}$. We start by defining the following generalization of the *arithmetic Euler characteristic* (1.3.1) in the context of geometry of numbers. For any M a finitely generated $\mathbb{Z}$-module of rank $r \geq 1$, define for each volume form η on $M_{\mathbb{R}}$ (*i.e.* an element $\eta \in \det_{\mathbb{R}}(M_{\mathbb{R}})$) the following

$$\chi(M,\eta) := -\log(\mathrm{covol}_\eta(M_{\mathbb{R}}/\overline{M})) + \log(\#M_{\mathrm{tor}}) \tag{4.1.3}$$

where $\overline{M} := M/M_{\mathrm{tor}}$. In fact, (4.1.2) can be defined in this context for $M = H^0(\mathcal{X}, \mathcal{L})$ and η the volume form determined by the sup-norm. The function χ is additive: for $0 \to M_1 \to M_2 \to M_3 \to 0$ an exact sequence and $\eta_j \in \det_{\mathbb{R}}(M_{j,\mathbb{R}})$ $(j \in \{1, 2, 3, \})$ such that $\eta_2 = \eta_1 \otimes \eta_3$ in the induced isomorphism $\det_{\mathbb{R}}(M_{2,\mathbb{R}}) \simeq \det_{\mathbb{R}}(M_{1,\mathbb{R}}) \otimes \det_{\mathbb{R}}(M_{3,\mathbb{R}})$, we have

$$\chi(M_2, \eta_2) = \chi(M_1, \eta_1) + \chi(M_3, \eta_3). \tag{4.1.4}$$

Back to our case. Let $M = \bigoplus_k M_k$ be a graduated $\mathcal{O}_K$-module of finite type, and let P_M be the Hilbert polynomial. A typical case for us is when $M = \bigoplus_{k \geq 0} H^0(\mathcal{X}, k\mathcal{L})$ with Hilbert polynomial P.

Definition 4.1.2. *Let $\alpha \in \mathbb{R}$ be a real number. Define $\eta_{k,\alpha} \in \det_{\mathbb{R}}(M_{\mathbb{R}})$ to be the volume form determined by the equation*

$$\chi(M_k, \eta_{k,\alpha}) = \alpha \sum_{j=0}^{k-1} P_M(j) + \chi(\mathcal{O}_K) P_M(k)$$

with the canonical volume form on $\mathcal{O}_K$.

The following lemma is easy to prove. It is the reason that the $\eta_{k,\alpha}$ is of interest to us. *It does not hold for the L^2-volume forms.*

Lemma 4.1.3. *Let $0 \to M^{(1)} \to M^{(2)} \to M^{(3)} \to 0$ be an exact sequence of graduated $\mathcal{O}_K$-modules of finite type, with $\Phi_k \colon \det(M^{(2)}_{k,\mathbb{R}}) \simeq \det(M^{(1)}_{k,\mathbb{R}}) \otimes \det(M^{(3)}_{k,\mathbb{R}})$ the induced isomorphism. Then for each $\alpha \in \mathbb{R}$, we have $\Phi_k(\eta^{(2)}_{k,\alpha}) = \eta^{(1)}_{k,\alpha} \otimes \eta^{(3)}_{k,\alpha}$ for the volume forms defined in Definition 4.1.2.*

4.1.3 Comparison of the three volumes and consequence on arithmetic Hilbert–Samuel

We need to compare the three volume forms on $H^0(\mathcal{X}, k\mathcal{L})_{\mathbb{R}}$. Define the positive functions

$$f_X(k, \alpha) := \frac{V^k_{X,L^2}}{V^k_{X,\alpha}}, \qquad h_X(k) := \frac{V^k_{X,\mathrm{sup}}}{V^k_{X,L^2}} \tag{4.1.5}$$

with $k \geq 1$ and $\alpha \in \mathbb{R}$.

Proposition 4.1.4. $\log h_X(k) = o(k^{n+1})$.

Proof. This follows immediately from Proposition 4.1.4. □

The following proposition will be proved in the next section.

Proposition 4.1.5. *There exists an affine function $\eta \colon \mathbb{R} \to \mathbb{R}$ such that*

$$\log f_X(k, \alpha) = \eta(\alpha) k^{n+1} + o(k^{n+1}). \tag{4.1.6}$$

In particular, there exists a unique real number α_0 such that $\log f_X(k, \alpha_0) = o(k^{n+1})$.

Proof of Theorem 4.0.1 assuming Proposition 4.1.5. By (4.1.2) and the definition of $V^k_{X,\sup}$, we have

$$\chi(k\overline{\mathcal{L}}) = -\log \mathrm{covol}_{V^k_{X,\sup}}(H^0(\mathcal{X},\mathcal{L})_{\mathbb{R}}/H^0(\mathcal{X},\mathcal{L})).$$

Thus Definition 4.1.2, Proposition 4.1.4, and Proposition 4.1.5 together yield

$$\chi(H^0(\mathcal{X},k\mathcal{L}),V^k_{X,\alpha_0}) + \log h_X(k) + \log f_X(k,\alpha_0) = \alpha_0 \sum_{j=0}^{k-1} P(j) + o(k^{n+1})$$

Since $\deg P = n$ and P has leading coefficient $\mathcal{L}^n_K/n!$, we then have

$$\chi(k\overline{\mathcal{L}}) = \frac{\alpha_0 \mathcal{L}^n_K}{(n+1)!} k^{n+1} + o(k^{n+1}).$$

Thus the LHS converges to $\alpha_0 \mathcal{L}^n_K$ when $k \to \infty$. The real number α_0 can be read off in the proof of Proposition 4.1.5, where we will see that $\alpha_0 \mathcal{L}^n_K = \overline{\mathcal{L}}^{n+1}$. We are done. $\square$

4.2 Algebraic part of the proof of Proposition 4.1.5

The goal of this section is to prove Proposition 4.1.5, assuming an analytic result which will be proved in the next section.

4.2.1 Fundamental short exact sequence

Recall our assumption that $\mathcal{L}$ is very ample on $\mathcal{X}$. Hence there exists a closed immersion $\iota\colon \mathcal{X} \to \mathbb{P}^N_{\mathcal{O}_K}$ with $\iota^*\mathcal{O}(1) \simeq \mathcal{L}$. By Bertini's theorem, up to taking a finite extension of K there exists a non-zero global section ℓ of $\mathcal{O}(1)$ on $\mathbb{P}^N_{\mathcal{O}_K}$ such that $\mathrm{div}(\ell) \cap \mathcal{X}_K$ is a subvariety of dimension $n-1$ of $\mathcal{X}_K$, which is smooth if $n \geq 2$ and is reduced if $n = 1$.

Set $s := \iota^*(\ell)$. Then $s \in H^0(\mathcal{X},\mathcal{L})$. The ideal sheaf of $\mathrm{div}(s)$, which is $\mathcal{L}^{\otimes -1}$, admits a primary decomposition $\mathcal{L}^{\otimes -1} = \mathcal{I} \cap \mathcal{J}$ where $\mathcal{J}$ has vertical support and $\mathcal{I}$ defines a flat closed subscheme $\mathcal{Y}$ over $\mathrm{Spec}\mathcal{O}_K$ with smooth generic fiber. Moreover $\dim \mathcal{Y} = n = \dim \mathcal{X} - 1$.

Thus for $k \gg 1$, we have the following exact sequence:

$$0 \to H^0(\mathcal{X}, k\mathcal{L} + \mathcal{I}) \to H^0(\mathcal{X}, k\mathcal{L}) \to H^0(\mathcal{Y}, k\mathcal{L}|_{\mathcal{Y}}) \to 0. \tag{4.2.1}$$

Tensoring $\mathbb{R}$ yields, by definition of $\mathcal{I}$,

$$0 \to H^0(\mathcal{X},(k-1)\mathcal{L})_{\mathbb{R}} \xrightarrow{\cdot s} H^0(\mathcal{X},k\mathcal{L})_{\mathbb{R}} \to H^0(\mathcal{Y},k\mathcal{L}|_{\mathcal{Y}})_{\mathbb{R}} \to 0. \tag{4.2.2}$$

4.2.2 Volume forms on the spaces

Our goal is to compare the volume forms V^k_{X,L^2} and $V^k_{X,\alpha}$ on $H^0(\mathcal{X},k\mathcal{L})_{\mathbb{R}}$ for each real number $\alpha \in \mathbb{R}$, *by induction on* $n = \dim \mathcal{X}_K$. Hence it is desirable to study the respective volume forms on $H^0(\mathcal{Y},k\mathcal{L}|_{\mathcal{Y}})_{\mathbb{R}}$ and on $H^0(\mathcal{X},(k-1)\mathcal{L})_{\mathbb{R}}$.

On $H^0(\mathcal{Y},k\mathcal{L}|_{\mathcal{Y}})_{\mathbb{R}}$, we have the volume forms

- V^k_{Y,L^2}, where the L^2-norm is defined using the same construction above Proposition 4.1.4 but with $\overline{\mathcal{L}}|_{\mathcal{Y}}$;
- $V^k_{Y,\alpha}$ defined by Defintion 4.1.2.

They are related by $V^k_{Y,L^2} = f_Y(k,\alpha)V^k_{Y,\alpha}$.

On $H^0(\mathcal{X},(k-1)\mathcal{L})_{\mathbb{R}}$, we have the volume forms

- V^{k-1}_{X,L^2};
- Z^{k-1}_{α}, by applying Definition 4.1.2 to $M = \bigoplus_{k\geq 0} H^0(\mathcal{X},(k+1)\mathcal{L}+\mathcal{I})$.

Set $t_X(k-1,\alpha) := V^{k-1}_{X,L^2}/Z^{k-1}_{\alpha}$.

Apply Lemma 4.1.3 to the exact sequence (4.2.1). Then we get $V^k_{X,\alpha} = V^k_{Y,\alpha}\otimes Z^{k-1}_{\alpha}$. Thus

$$\frac{V^k_{X,L^2}}{f_X(k,\alpha)} = \frac{V^k_{Y,L^2}\otimes V^{k-1}_{X,L^2}}{f_Y(k,\alpha)t_X(k-1,\alpha)}.$$

Denoting by

$$g(k) := \frac{V^k_{X,L^2}}{V^k_{Y,L^2}\otimes V^{k-1}_{X,L^2}}. \tag{4.2.3}$$

Then we have

$$\log f_X(k,\alpha) = \log t_X(k-1,\alpha) + \log f_Y(k,\alpha) - \log g(k). \tag{4.2.4}$$

The second term on the RHS will be handled by induction hypothesis.

The following proposition will be proved in the next section using analytic method. It handles the third term of the RHS of (4.2.4).

Proposition 4.2.1. *When $k\to\infty$, we have*

$$\frac{1}{P(k)}\log g(k+1) \to -\sum_{\sigma\colon K\hookrightarrow\mathbb{C}}\int_{\mathcal{X}_\sigma}\log\|s(x)\|^2 \mathrm{d}V_\sigma$$

with the volume form $\mathrm{d}V_\sigma$ on $\mathcal{X}_\sigma$ defined above Proposition 4.1.4 (via $c_1(\overline{\mathcal{L}}_\sigma)$).

4.2.3 Further treatment

Consider the following exact sequences of sheaves:

$$0\to\mathcal{I}\cdot\mathcal{J}\to\mathcal{I}\to\mathcal{I}\otimes\mathcal{O}_{\mathcal{X}}/\mathcal{J}\to 0,$$
$$0\to\mathcal{I}\cdot\mathcal{J}\to\mathcal{I}\cap\mathcal{J}=\mathcal{L}^{\otimes -1}\to\mathcal{T}:=\mathrm{Tor}^1(\mathcal{O}_{\mathcal{X}}/\mathcal{I},\mathcal{O}_{\mathcal{X}}/\mathcal{J})\to 0.$$

Then $\mathcal{T}$ has support in $\mathrm{Supp}(\mathcal{J})$, which is vertical over $\mathrm{Spec}\mathcal{O}_K$. For $k\gg 1$, we have exact sequences since $\mathcal{L}$ is ample

$$0\to H^0(\mathcal{X},k\mathcal{L}+\mathcal{I}\cdot\mathcal{J})\to H^0(\mathcal{X},k\mathcal{L}+\mathcal{I})\to H^0(\mathcal{X},k\mathcal{L}+\mathcal{I}+\mathcal{O}_{\mathcal{X}}/\mathcal{J})\to 0,$$
$$0\to H^0(\mathcal{X},k\mathcal{L}+\mathcal{I}\cdot\mathcal{J})\to H^0(\mathcal{X},(k-1)\mathcal{L})\to H^0(\mathcal{X},kL+\mathcal{T})\to 0$$

where we write $+$ for $\otimes$ as usual. The last terms in both short exact sequences are torsion. So applying the additivity of the arithmetic Euler characteristic (4.1.4) to both short exact sequences above and taking the difference, we obtain

$$\log t_X(k-1,\alpha)-\log f_X(k-1,\alpha) = \log\#H^0(\mathcal{X},k\mathcal{L}+\mathcal{I}+\mathcal{O}_X/\mathcal{J})-\log\#H^0(\mathcal{X},k\mathcal{L}+\mathcal{T}). \tag{4.2.5}$$

Combining with (4.2.4), we thus obtain

$$\log f_X(k+1,\alpha)-\log f_X(k,\alpha) = \log f_Y(k,\alpha)-\log g(k)+\log\#H^0(\mathcal{X},k\mathcal{L}+\mathcal{I}+\mathcal{O}_X/\mathcal{J})-\log\#H^0(\mathcal{X},k\mathcal{L}+\mathcal{T}). \tag{4.2.6}$$

4.2.4 Proof of Proposition 4.1.5 assuming Proposition 4.2.1

We do induction on $n = \dim \mathcal{X}_K \geq 0$.

$\boxed{\text{Base step}}$ When $n = 0$, we need to do a bit more, *i.e.* we assume $\mathcal{X}_K$ to be reduced but not necessarily irreducible. In this case $\mathcal{X} = \mathrm{Spec}R$ with R a finite $\mathcal{O}_K$-algebra which is reduced. By definition (4.1.5), we have

$$\log f_X(k, \alpha) = -\chi(k\mathcal{L}, V^k_{X,\alpha}) + \chi(k\mathcal{L}, V^k_{X,L^2}).$$

Notice that the Hilbert polynomial of $\mathcal{L}_K$ is constant. Hence Definition 4.1.2 implies that $\chi(k\mathcal{L}, V^k_{X,\alpha})$ is an affine function in k. The function $\chi(k\mathcal{L}, V^k_{Y,L^2})$ is also affine in k, by arithmetic Riemann–Roch applied to $\mathrm{Spec}R$ (we have seen this when $R = \mathcal{O}_K$ as Theorem 1.3.7 whose proof is a direct computation; in general we reduce to the case where R is an order of a number field and prove the similar result by computation). Hence we are done in this base step.

$\boxed{\text{Induction}}$ For general $n \geq 1$, we use (4.2.6) to analyze $\log f_X(k+1, \alpha) - \log f_X(k, \alpha)$.

When $n = 1$, recall our choice $s \in H^0(\mathcal{X}, \mathcal{L})$ satisfies that $\mathrm{div}(s_K)$ is reduced. When $n \geq 2$, the generic fiber $\mathcal{Y}_K$ is smooth by choice of the global section $s \in H^0(\mathcal{X}, \mathcal{L})$. In both cases, we can apply our induction hypothesis and get $\log f_Y(k, \alpha) = \eta'(\alpha)k^n + o(k^n)$. And $\log g(k) = \lambda k^n + o(k^n)$ by Proposition 4.2.1.

For $\log \#H^0(X, k\mathcal{L} + \mathcal{I} + \mathcal{O}_X/\mathcal{J})$ and $\log \#H^0(\mathcal{X}, k\mathcal{L} + \mathcal{T})$, decompose $\mathrm{Supp}(\mathcal{J})$ into disjoint union of connected subvarieties of dimension $\leq n$ (they are all contained in vertical fibers). The Hilbert–Samuel formula in algebraic geometry then implies that both terms are of the form $c'k^n + o(k^n)$.

Therefore, $\log f_X(k+1, \alpha) - \log f_X(k, \alpha) = c_0 k^n + o(k^n)$. So there exists a function $\eta\colon \mathbb{R} \to \mathbb{R}$ such that

$$\log f_X(k, \alpha) = \eta(\alpha)k^{n+1} + o(k^{n+1}).$$

It remains to show that η is affine. For this, notice that Definition 4.1.2 implies

$$\log f_X(k, \alpha) - \log f_X(k, \alpha') = c(\alpha - \alpha')k^{n+1} + o(k^{n+1}),$$

where c is the leading coefficient of $\sum_{j=0}^{k-1} P(j)$. Thus $\eta(\alpha) - \eta(\alpha') = c(\alpha - \alpha')$ for all $\alpha, \alpha' \in \mathbb{R}$. So η is affine. Better, we have $c = \mathcal{L}^n_K/(n+1)!$. We are done.

4.3 Analytic part of the proof