Chapter 4

Boundary components

Starting from this chapter, we will discuss compactifications of Shimura varieties $\mathrm{Sh}_K(\mathbf{G}, X)$, or locally Hermitian symmetric spaces $\Gamma\backslash X^+$. This chapter introduces boundary components of X^+.

4.1 Example: modular curves

Consider the modular curves $\mathrm{Sh}_K(\mathbf{GL}_2, \mathfrak{H}^{\pm})$, *i.e.* the Siegel modular variety from §3.3 with $d = 1$. In the particular case where $K = \mathbf{GL}_2(\widehat{\mathbb{Z}})$, we are working with

$$Y(1) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}.$$

It is a well-known result that $Y(1) \simeq \mathbb{C}$ via the j-function $j\colon \mathfrak{H} \to \mathbb{C}$. Hence a compactification of $Y(1)$ is $\mathbb{P}^1(\mathbb{C})$. This is the *Baily–Borel compactification* or the *toroidal compactification* of $Y(1)$ (but not the *Borel–Serre compactification*). In this section, we explain how to view this compactification as the *Baily–Borel compactification* of $Y(1)$. A large part is to study the *boundary components*, which is important for other compactifications we will discuss (*toroidal compactification* and *Borel–Serre compactification*).

4.1.1 Boundary components of $\mathfrak{H}$

The *boundary* of $\mathfrak{H}$ in $\mathbb{C} \cup \{\infty\}$ is the union of the real axis and $\{\infty\}$; in other words, the boundary of $\mathfrak{H}$ in $\mathbb{P}^1(\mathbb{C})$ is $\mathbb{P}^1(\mathbb{R})$. This is better seen via the Cayley transformation (2.3.4)

$$\mathfrak{H} \xrightarrow{\sim} \mathcal{D} := \{z \in \mathbb{C} : |z| < 1\}, \qquad \tau \mapsto (\tau - \sqrt{-1})(\tau + \sqrt{-1})^{-1},$$

and the boundary of $\mathcal{D}$ is the unit circle. Denote by $\overline{\mathcal{D}}$ the closure of $\mathcal{D}$ in $\mathbb{C}$, *i.e.* $\overline{\mathcal{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$, and $\partial\mathcal{D} := \overline{\mathcal{D}} \setminus \mathcal{D}$. Then ∞ corresponds to $1 \in \overline{\mathcal{D}}$.

Call each point in $\partial\mathcal{D}$ a *boundary component* of $\mathcal{D}$. It is justified by the following fact: Any holomorphic map $\mathcal{D} \to \overline{\mathcal{D}}$ either has image in $\mathcal{D}$ or is constant.[1]

4.1.2 Extension of the group action to $\overline{\mathcal{D}}$

The group $\mathrm{GL}_2(\mathbb{R})^+$ acts on $\mathcal{D}$, via its action on $\mathfrak{H}$ and the Cayley transformation above, by the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{(a - \sqrt{-1}c)(z+1) + (b - \sqrt{-1}d)\sqrt{-1}(z-1)}{(a + \sqrt{-1}c)(z+1) + (b + \sqrt{-1}d)\sqrt{-1}(z-1)}, \qquad \forall z \in \mathcal{D}.$$

[1] This is a consequence of the Open Mapping Theorem in complex analysis, which asserts that any holomorphic function on a connected set in the complex plane is open.

Lemma 4.1.1. *The action of* $\mathrm{GL}_2(\mathbb{R})^+$ *on* $\mathcal{D}$ *extends to* $\overline{\mathcal{D}}$. *Moreover, the action of* $\mathrm{GL}_2(\mathbb{R})^+$ *on* $\partial\mathcal{D}$ *is transitive.*

Proof. Take $z \in \overline{\mathcal{D}}$, and set

$$u_\pm := (a \pm \sqrt{-1}c)(z+1) + (b \pm \sqrt{-1}d)\sqrt{-1}(z-1).$$

For the first part of the lemma, we need to show that $u_+ \neq 0$ and $u_- u_+^{-1} \in \overline{\mathcal{D}}$.

Then

$$\begin{bmatrix} u_- \\ u_+ \end{bmatrix} = \begin{bmatrix} 1 & -\sqrt{-1} \\ \sqrt{-1} & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \sqrt{-1} & -\sqrt{-1} \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix},$$

and one can compute that

$$u_+\overline{u}_+ - u_-\overline{u}_- = 4(1 - z\overline{z}).$$

So $u_+\overline{u}_+ \geq u_-\overline{u}_-$ because $z \in \overline{\mathcal{D}}$. If $u_+ = 0$, then $u_+ = u_- = 0$, contradiction to $\operatorname{rank} \begin{bmatrix} u_- \\ u_+ \end{bmatrix} = \operatorname{rank} \begin{bmatrix} z \\ 1 \end{bmatrix} = 1$. So $u_+ \neq 0$, and $(u_- u_+^{-1})\overline{u_- u_+^{-1}} = \frac{u_-\overline{u}_-}{u_+\overline{u}_+} \leq 1$. Hence $u_- u_+^{-1} \in \overline{\mathcal{D}}$. We are done.

Let us prove the "Moreover" part. We have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot 1 = \frac{a^2 - c^2}{a^2 + c^2} + \frac{-2ac}{a^2 + c^2}\sqrt{-1}.$$

The right hand side is easily checked to be in $\partial\mathcal{D} = \{z \in \mathbb{C} : |z| = 1\}$. Conversely any $z \in \partial\mathcal{D}$ can be written as the right hand side for some 2×2-matrix in $\mathrm{GL}_2(\mathbb{R})^+$. Hence we are done. $\square$

4.1.3 Compactifying at each boundary component

To see how to compactify $\mathcal{D} \simeq \mathfrak{H}$ at each boundary component, we need to study the stabilizer of each $z \in \overline{\mathcal{D}}$. Since $Z(\mathrm{GL}_2)(\mathbb{R})$ acts trivially on $\overline{\mathcal{D}}$, it suffices to consider the stabilizer in $\mathrm{SL}_2(\mathbb{R})$. By Lemma 4.1.1, it suffices to study this for $1 \in \overline{\mathcal{D}}$. For this purpose, it is easier to use the upper half plan. Define

$$P := \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} : b \in \mathbb{R},\ a \neq 0 \right\} \tag{4.1.1}$$

Then it is easy to check that $\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(g \cdot \infty) = gP(\mathbb{R})^+ g^{-1}$ for any $g \in \mathrm{SL}_2(\mathbb{R})$. Indeed, it suffices to check this with $g = I_2$, and then it suffices to notice that elements on the right hand side of (4.1.1) correspond to translations along the real axis.

Lemma 4.1.2. *The followings hold true:*

(i) $\mathrm{SL}_2(\mathbb{C})/P(\mathbb{C})$ *is a projective space.*

(ii) For any $g \in \mathrm{SL}_2(\mathbb{R})$, *the group* gPg^{-1} *is defined over* $\mathbb{Q}$ *if and only if* $g \in \mathrm{SL}_2(\mathbb{Q})$.

(iii) Let $\tau \in \mathbb{P}^1(\mathbb{R})$ *(the boundary of* $\mathfrak{H}$ *in* $\mathbb{P}^1(\mathbb{C})$*). Then* $\tau \in \mathbb{P}^1(\mathbb{Q}) \Leftrightarrow \tau = g \cdot \infty$ *for some* $g \in \mathrm{SL}_2(\mathbb{Q}) \Leftrightarrow \tau = g \cdot \infty$ *for some* $g \in \mathrm{SL}_2(\mathbb{Z})$.

Proof. (ii) and (iii) are simple computations. For (i), it suffices to notice that the homogeneous space $\mathrm{SL}_2(\mathbb{C})/P(\mathbb{C}) \simeq \mathrm{GL}_2(\mathbb{C})/\left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b, d \in \mathbb{C},\ ad \neq 0 \right\}$ is the Grassmannian parametrizing 1-dimensional $\mathbb{C}$-subspaces in $\mathbb{C}^2$. $\square$

Let us go further. We have:

Lemma 4.1.3. *For each $g \in \mathrm{SL}_2(\mathbb{R})$, the group $gP(\mathbb{R})^+g^{-1}$ acts transitively on $\mathfrak{H}$.*

The proof itself is important. As a preparation, the group P has the following subgroups:

- The *unipotent radical* $N_P := \left\{\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix} : b \in \mathbb{R}\right\}$, where elements act on $\mathfrak{H}$ as $\tau \mapsto \tau + b$.
- the *split torus* $A_P := \left\{\begin{bmatrix}a^{-1} & 0\\ 0 & a\end{bmatrix} : a > 0\right\}$,[2] where elements act on $\mathfrak{H}$ as $\tau \mapsto a^{-2}\tau$.
- $M_P := \{\pm I_2\}$, which acts trivially on $\mathfrak{H}$.

such that

$$P = N_P A_P M_P \tag{4.1.2}$$

and the map $N_P \times A_P \times M_P \to P$, $(n, a, m) \mapsto nam$, is a diffeomorphism.

Proof. We only need to prove this lemma for P. For any $\tau = x + \sqrt{-1}y \in \mathfrak{H}$, we have

$$\tau = \begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}\begin{bmatrix}\sqrt{y} & 0\\ 0 & \sqrt{y}^{-1}\end{bmatrix}\sqrt{-1}.$$

Hence we are done. □

Now we are ready to explain how the point ∞ is added to $\mathfrak{H}$ via the group P (in other words, how compactify $\mathfrak{H}$ at ∞). The decomposition 4.1.2 induces, by Lemma 4.1.3,

$$\mathfrak{H} \simeq P/(P \cap \mathrm{SO}(2)) = P/M_P \simeq N_P \times A_P \simeq \mathbb{R} \times \mathbb{R}_{>0}, \qquad \tau = x + \sqrt{-1}y \mapsto (x, \sqrt{y}^{-1}). \tag{4.1.3}$$

The A_P-factor is isomorphic to $\mathbb{R}_{>0}$, and a natural way to add a boundary to $\mathbb{R}_{>0}$ is to add 0 and make it into $\mathbb{R}_{\geq 0}$. In doing this, we are adding the point $x + \sqrt{-1}0^{-2} = \infty$ to $\mathfrak{H}$.

This process can be carried out for $g \cdot \infty \in \mathbb{P}^1(\mathbb{R})$ for any $g \in \mathrm{SL}_2(\mathbb{R})$, by replacing N_P and A_P by gN_Pg^{-1} and gA_Pg^{-1}. In this way, the point $g \cdot \infty \in \mathbb{P}^1(\mathbb{R})$ is added to $\mathfrak{H}$ by "compactifying" $gA_Pg^{-1} \simeq \mathbb{R}_{>0}$ into $\mathbb{R}_{\geq 0}$.

4.1.4 Rational vs real boundaries, and Siegel sets

We wish to compactify the quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H} \simeq \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{D}$. The idea is to do the quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\overline{\mathcal{D}}$, for the extended action of $\mathrm{SL}_2(\mathbb{R})$ on $\overline{\mathcal{D}}$ defined in Lemma 4.1.1. However, $\partial\mathcal{D} = \overline{\mathcal{D}} \setminus \mathcal{D} \simeq \mathbb{P}^1(\mathbb{R})$ contains infinitely many $\mathrm{SL}_2(\mathbb{Z})$-orbits.

A solution to this is to consider the *rational boundary components*, which are precisely the points in $\mathbb{P}^1(\mathbb{Q}) \subseteq \mathbb{P}^1(\mathbb{R})$. Equivalently by (ii) and (iii) of Lemma 4.1.2, a boundary component $z \in \partial\mathcal{D}$ is called a *rational boundary component* if its stabilizer in $\mathrm{SL}_2(\mathbb{R})$ is defined over $\mathbb{Q}$. Now part (iii) of Lemma 4.1.2 asserts that there is only one $\mathrm{SL}_2(\mathbb{Z})$-class of rational boundary components.

Another important notion is the *Siegel sets* associated with $P = \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(\infty)$ defined as follows; one needs this for example to pass from (partial) compactification of $\mathfrak{H}$ to compactification of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$. For each $t > 0$ and any compact bounded set $U \subseteq N_P \simeq \mathbb{R}$, define

$$\Sigma_{P,U,t} := U \times \{a \in \mathbb{R}_{>0} : a \leq t\} \simeq \{\tau = x + \sqrt{-1}y : x \in U,\ y \geq t^{-2}\} \subseteq \mathfrak{H}.$$

[2]Notice that A_P is not an algebraic subgroup of P, but only a Lie subgroup. This is a minor issue: Indeed, if we replace GL_2 by $\mathrm{PGL}_2 = \mathrm{SL}_2/\{\pm I_2\}$, then the quotient of A_P becomes an algebraic subgroup.

Then we have the following classical result on the j-function:[3] for a suitable U and suitable $t \gg 1$, $\Sigma_{P,U,t}$ is a fundamental set for the uniformization $j\colon \mathfrak{H} \to \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H} \simeq \mathbb{C}$ (*i.e.* $j|_{\Sigma_{P,U,t}}$ is surjective and has finite fibers). Then one can define the Siegel sets associated with $gPg^{-1} = \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(g\cdot\infty)$ (for any $g \in \mathrm{SL}_2(\mathbb{R})$) to be $g\cdot\Sigma_{P,U,t}$.

We can also compactify $\Gamma\backslash\mathfrak{H}$ to be, as a set, $\Gamma\backslash(\mathfrak{H}\cup\mathbb{P}^1(\mathbb{Q}))$ for any finite-indexed subgroup $\Gamma < \mathrm{SL}_2(\mathbb{Z})$, by the following lemma which is a direct consequence of the discussion above.

Lemma 4.1.4. *(i) There are finitely many rational boundary components $\alpha_1,\dots,\alpha_n$ of $\mathfrak{H}$ such that $\mathbb{P}^1(\mathbb{Q}) = \bigcup_j \Gamma\cdot\alpha_j$.*

(ii) Let $P_j := \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(\alpha_j)$. Then there are suitable Siegel sets Σ_j associated with P_j for $j \in \{1,\dots,n\}$ such that $\bigcup_j \Sigma_j$ is a fundamental set for the uniformization $u\colon \mathfrak{H} \to \Gamma\backslash\mathfrak{H}$.

4.1.5 Satake topology on $\overline{\mathcal{D}}$

This subsection is for the Baily–Borel compactification of $\Gamma\backslash\mathfrak{H}$. We will revisit the materials later in more generality.

Our desired compactification is $\Gamma\backslash(\mathfrak{H}\cup\mathbb{P}^1(\mathbb{Q}))$. We yet to explain the topology on this set, so that it is Hausdorff and compact. Notice that we cannot take the one induced by the usual topology on $\mathbb{C}$ because $x \in \mathbb{P}^1(\mathbb{Q})$ there are infinitely many $\gamma \in \Gamma$ which fixed x, and hence the quotient $\Gamma\backslash(\mathfrak{H}\cup\mathbb{P}^1(\mathbb{Q}))$ is not Hausdorff under this topology.

The topology which we consider is the Satake topology, induced from the Satake topology on $\mathfrak{H}\cup\mathbb{P}^1(\mathbb{Q})$ defined as follows. On $\mathfrak{H}$, the Satake topology is the usual topology, induced from $\mathbb{C}$. Next, an open neighborhood basis of ∞ consists of the open sets $U_t := \{z \in \mathfrak{H} : \mathrm{Im}(z) > t\}$ for all $t \geq 2$; equivalently a sequence $\tau_j = x_j + \sqrt{-1}y_j \in \mathfrak{H}$ converges to ∞ if and only if $y_j \to \infty$. Finally, an open neighborhood basis of $g\cdot\infty \in \mathbb{P}^1(\mathbb{Q})$ (with $g \in \mathrm{SL}_2(\mathbb{Q})$) consists of $g\cdot U_t$ for all $t \geq 2$. We state without proof the following assertions (whose proof needs to use Siegel sets):

(i) For any $x \in \mathfrak{H}\cup\mathbb{P}^1(\mathbb{Q})$, there exists a fundamental system of neighborhoods $\{U\}$ of x such that

$$\gamma U = U,\ \forall\gamma\in\Gamma_x; \quad \gamma U \cap U = \emptyset,\ \forall\gamma\notin\Gamma_x$$

where $\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}$.

(ii) If $x, x' \in \mathfrak{H}\cup\mathbb{P}^1(\mathbb{Q})$ are not in one Γ-orbit, then there exist neighborhoods U of x and U' of x' such that

$$\Gamma U \cap U' = \emptyset.$$

These properties guarantee that $\Gamma\backslash(\mathfrak{H}\cup\mathbb{P}^1(\mathbb{Q}))$ is Hausdorff under the Satake topology. The compactness follows easily from part (ii) of Lemma 4.1.4.

4.2 Parabolic subgroups and Levi subgroups: definitions and statements

For the simplest Siegel Shimura datum $(\mathbf{GL}_2, \mathfrak{H}^{\pm})$, Lemma 4.1.2.(i) suggests that parabolic subgroups of SL_2 (*i.e.* subgroups of SL_2 such that the homogeneous space $\mathrm{SL}_2(\mathbb{C})/P(\mathbb{C})$ is a projective variety) are closely related to the boundary components of $\mathfrak{H}$. This is true for an arbitrary Shimura datum $(\mathbf{G}, X)$.

In this section, we review background knowledge on parabolic subgroups of reductive groups over algebraically closed fields. In the next section, we do it over an arbitrary field.

Let k be a field, and let G be a reductive group defined over k. Let $\overline{k}$ be an algebraic closed field containing k. For our purpose, we will take $k = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ and $\overline{k} = \mathbb{C}$.

[3] A well-known fundamental domain of the j-function is $\{z \in \mathbb{C} : |z| \geq 1,\ -1 \leq \mathrm{Re}(z) < 1\}$.

Definition 4.2.1. *A subgroup P of G is called a* **parabolic subgroup** *if the homogeneous space $G(\overline{k})/P(\overline{k})$ is a projective variety.*

It is a theorem of Chevalley that *parabolic subgroups are always connected.* We are more interested in the *proper* parabolic subgroups.

Example 4.2.2. *For $G = \mathrm{GL}_N$. Let P be the subgroup of upper triangular matrices in blocks (with the length of the ℓ-th diagonal block being n_ℓ). Then if we write $G = \mathrm{GL}(V)$ with $V \simeq k^N$, then P is the stabilizer of a flag $F^\bullet = (0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{m-1} \subsetneq V_m = V)$ of subspaces of V, with $\dim V_\ell - \dim V_{\ell-1} = n_\ell$ for each ℓ. Hence G/P is a flag variety and hence is projective. So P is a parabolic subgroup of GL_N.*

Let P be a parabolic subgroup of G. The unipotent radical $\mathcal{R}_u(P)$ is a closed normal subgroup of P, and hence P acts on $\mathcal{R}_u(P)$ via conjugation. This induces an action of any subgroup of H on $\mathcal{R}_u(H)$.

Definition 4.2.3. *A* **Levi subgroup** *of P is a closed subgroup L of P such that $H = \mathcal{R}_u(P) \rtimes L$.*

A Levi subgroup, if exists, is then isomorphic to $P/\mathcal{R}_u(P)$ and hence is a reductive group (in particular is connected).

Theorem 4.2.4. *P has Levi subgroups, and any two Levi subgroups of P are conjugate by a unique element in $\mathcal{R}_u(P)$.*

We are more interested in more concrete constructions of Levi subgroups of P. This will be given in combinatorial data in the next two sections.

The following construction of parabolic subgroups of G is useful, although we will not use it in our course. Let λ be a cocharacter of G, *i.e.* a morphism of algebraic groups $\mathbb{G}_{\mathrm{m}} \to G$.

Theorem 4.2.5. *(i) The set*

$$P(\lambda) := \{x \in G : \lim_{t \to 0} \lambda(t) x \lambda(t)^{-1} \text{ exists}\}$$

is a parabolic subgroup of G, and the centralizer of $\lambda(\mathbb{G}_{\mathrm{m}})$ is a Levi subgroup of $P(\lambda)$. Moreover $\mathcal{R}_u(P(\lambda)) = \{x \in G : \lim_{t\to 0} \lambda(t)x\lambda(t)^{-1} = 1\}$.

(ii) Any parabolic subgroup of G is $P(\lambda)$ for some λ.

If $\lambda(\mathbb{G}_{\mathrm{m}}) < Z(G)$, then $P(\lambda) = G$. In fact, this theorem will serve as a bridge from the theory over algebraically closed fields to the theory over an arbitrary field.

4.3 Parabolic subgroups via root systems: over algebraically closed fields

In this section, we take $k = \overline{k}$ to be an algebraically closed field, and G a reductive group defined over k. For our purpose, it is harmless to take $k = \mathbb{C}$. We will explain the combinatorial construction of parabolic subgroups of G, and Example 4.2.2 will be revisited in this language as Example 4.3.15.

Let $\mathfrak{g} := \mathrm{Lie} G$. Then we have the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ whose kernel is $Z(G)$. Notice that $Z(G)^\circ$ is an algebraic torus since G is reductive.

4.3.1 Root system for G

Let T be a maximal torus of G, *i.e.* an algebraic torus contained in G and maximal under the inclusion. For example if $G = \mathrm{GL}_N$, we can take $T = D_N$ to be the subgroup of diagonal matrices with non-zero diagonal entries. We have the standard properties:

Lemma 4.3.1. *(i) Any maximal torus of G equals gTg^{-1} for some $g \in G(\overline{k})$.*

(ii) $T = Z_G(T) = \{g \in G(\overline{k}) : gtg^{-1} = t$ for all $t \in T(\overline{k})\}$.

(iii) $W(T,G) := N_G(T)/T$ is finite and is called the **Weyl group**.

Thus $T \supseteq Z(G)^\circ$.

Now consider the action of T on $\mathfrak{g}$ via $T < G$ and the adjoint action. Let $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_\mathrm{m})$ be the group of characters of T. For each $\alpha \in X^*(T)$, define $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} : t \cdot x = \alpha(t)x$ for all $t \in T\}$ to be the eigenspace for α. Then we have a decomposition as in (1.2.2)

$$\mathfrak{g} = \mathfrak{g}^T \oplus \bigoplus_{\alpha \in \Phi(T,G)} \mathfrak{g}_\alpha \tag{4.3.1}$$

where $\mathfrak{g}^T := \{x \in \mathfrak{g} : T \cdot x = x\}$ is the eigenspace for the trivial character, and $\Phi(T,G) \subseteq X^*(T) \setminus \{\text{trivial character}\}$ is the subset of non-trivial characters α of T such that $\mathfrak{g}_\alpha \neq 0$. By Lemma 4.3.1.(ii), we have $\mathfrak{g}^T = \mathfrak{t} := \mathrm{Lie}T$.

Denote for simplicity by $\Phi = \Phi(T,G)$. Elements in Φ are called *roots of* T. The following theorem, which gives combinatorial data associated with G and T, is extremely important in the theory of reductive groups.

Theorem 4.3.2. *(1) Φ generates a subgroup of finite index in $X^*(T/Z(G)^\circ) \subseteq X^*(T)$.*

(2) Let $\alpha \in \Phi$ and $\beta \in X^*(T)$ which is a multiple of α. Then $\beta \in \Phi \Leftrightarrow \beta = \pm\alpha$.

(3) Let $\alpha \in \Phi$, and set $G_\alpha := Z_G((\mathrm{Ker}\alpha)^\circ)$. Then

(a) $\dim \mathfrak{g}_\alpha = 1$, and there is a unique connected T-stable (unipotent) subgroup U_α of G such that $\mathrm{Lie}U_\alpha = \mathfrak{g}_\alpha$;[4]

(b) G_α is a reductive group and $\mathrm{Lie}G_\alpha = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$,[5] *and $G_\alpha^{\mathrm{ad}} \simeq \mathrm{PGL}_2$;*[6]

(c) the subgroup $W(T,G_\alpha)$ is $W(T,G)$ is generated by a reflection r_α such that $r_\alpha(\alpha) = -\alpha$.

(4) Let $\alpha \in \Phi$ and $r_\alpha \in W(T,G)$ be as in (3.c). Then for any $\beta \in \Phi$, we have

$$r_\alpha(\beta) = \beta - n_{\beta,\alpha}\alpha$$

with $n_{\beta,\alpha} \in \mathbb{Z}$. Moreover, $n_{\alpha,\alpha} = 2$.

Thus Φ is a reduced root system in the vector space $E := X^*(T/Z(G)^\circ)_\mathbb{R}$ with Weyl group $W(T,G)$ in the sense below.

Definition 4.3.3. *Let E be a finite-dimensional real vector space with a Euclidean inner product $\langle , \rangle$. A* **root system** *Φ in E is a finite set of non-zero vectors (called* **roots***) such that:*

[4]Thus U_α is isomorphic to $\mathbb{G}_\mathrm{a}$ since it is a unipotent group of dimension 1.

[5]In other words, G_α is generated by T, U_α and $U_{-\alpha}$.

[6]Indeed, we can choose a generator X_α of $\mathfrak{g}_\alpha$ for each $\alpha \in \Phi$ such that X_α, $X_{-\alpha}$, $[X_\alpha, X_{-\alpha}]$ is an $\mathfrak{sl}_2$-triple for all $\alpha \in \Phi$.

(1) Φ spans E,

(2) If $\alpha, c\alpha \in \Phi$ for some $c \neq 0$, then $c \in \{1, -1, 1/2, -1/2\}$,

(3) For any $\alpha \in \Phi$, the set Φ is closed under the reflection through the hyperplane perpendicular to α (which we denote by r_α),

(4) For any $\alpha, \beta \in \Phi$, we have $r_\alpha(\beta) = \beta - n_{\beta,\alpha}\alpha$ with $n_{\beta,\alpha} \in \mathbb{Z}$.

A root system is called **reduced** *if furthermore it satisfies:*

(2') The only scalar multiples of a root $\alpha \in \Phi$ that belong to Φ are $\pm\alpha$.

We call $\dim E$ the **rank** *of Φ.*

The **Weyl group** *of Φ, denoted by $W(\Phi)$, is the group of $\mathrm{Aut}(\Phi)$ generated by r_α for all $\alpha \in \Phi$.*

Conversely, given a *root datum* (root system and "coroot system") one can associate a unique reductive group. We shall not go into details for this, but restrict our discussion to root systems. In practice, we often take G to be semi-simple, so that $\Phi(T, G)$ is a reduced root system in $X^*(T)_{\mathbb{R}}$.

Example 4.3.4. *Let $G = \mathrm{GL}_N$ and $T = D_N$. The Weyl group is isomorphic to the permutation group $\mathfrak{S}_N$. For each $j \in \{1, \ldots, N\}$, define $e_j \in X^*(D_N)$ to be $\mathrm{diag}(t_1, \ldots, t_N) \mapsto t_j$. Then we have an isomorphism $X^*(D_N) \simeq \bigoplus_{j=1}^N \mathbb{Z}e_j$. One can check that $\Phi(D_N, \mathrm{GL}_N) = \{e_i - e_j : i \neq j\}$.*

Highly related to this example is $G = \mathrm{SL}_N$ and $T = D_N \cap \mathrm{SL}_N$. Then $X^*(T) \simeq \bigoplus_{j=1}^N \mathbb{Z}e_j/\mathbb{Z}(e_1 + \ldots + e_N)$. And $\Phi(T, G)$ in this case is precisely the image of $\Phi(D_N, \mathrm{GL}_N)$ under the natural projection $X^*(D_N) \to X^*(T)$.

Example 4.3.5. *Let $G = \mathrm{Sp}_{2d}$ and $T = \mathrm{Sp}_{2d} \cap D_{2d} = \{\mathrm{diag}(t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}) : t_1 \cdots t_d \neq 0\}$. The Weyl group is isomorphic to $\{\pm 1\}^d \rtimes \mathfrak{S}_d$. For each $j \in \{1, \ldots, d\}$, define $e_j \in X^*(T)$ to be $\mathrm{diag}(t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}) \mapsto t_j$. Then $X^*(T) \simeq \bigoplus_{j=1}^d \mathbb{Z}e_j$. One can check that $\Phi(T, \mathrm{Sp}_{2d}) = \{\pm 2e_i, \pm e_i \pm e_j : 1 \leq i, j \leq d,\ i \neq j\}$.*

Root systems in Example 4.3.4 are called *of type* A_{N-1}, and root systems in Example 4.3.5 are called *of type* C_d. We also have root systems of type B_n (dual to C_n; coming from SO_{2n+1}) and D_n (coming from SO_{2n}), and exceptional types E_6, E_7, E_8, F_4, G_2). We will not go into details for this, but only point out that the last 3 types do not show up in the theory of Shimura varieties and that a Shimura variety is of abelian type unless the underlying group has $\mathbb{Q}$-factors of mixed type D or of exceptional types.

4.3.2 Positive roots and Borel subgroups

We start with the abstract theory of root systems $\Phi \subseteq E$.

Definition 4.3.6. *A* **basis** *of Φ is a subset Δ of Φ which is a basis of E such that each root $\beta \in \Phi$ is a linear combination $\beta = \sum_{\alpha \in \Delta} m_\alpha \alpha$ with $m_\alpha \in \mathbb{Z}$ of the same sign.*

Given a basis Δ of Φ, a root $\beta \in \Phi$ is said to be **positive (with respect to Δ)** *if $m_\alpha \geq 0$ for the decomposition above. Denote by Φ^+ the set of positive roots, and $\Phi^- := -\Phi^+$. Then $\Phi = \Phi^+ \sqcup \Phi^-$.*

A root $\alpha \in \Phi^+$ is said to be **simple** *if it is not the sum of two other positive roots.*

Lemma 4.3.7. *Δ is precisely the set of simple roots in Φ^+.*

In practice, one can start from a subset Φ^+ of Φ such that $\Phi = \Phi^+ \sqcup (-\Phi^+)$ and that $\alpha \in \Phi^+ \Rightarrow -2\alpha \notin \Phi^+$, and call these roots *positive*. Then we get a basis Δ consisting of simple roots in Φ^+, with respect to which Φ^+ is the set of positive roots. See Lemma 4.3.7.

Back to the theory of reductive groups, choosing Φ^+ is equivalently to taking a *Borel group*.

Definition 4.3.8. *A* **Borel group** *B is G is a closed connected solvable subgroup G, which is maximal for these properties.*

Example 4.3.9. *If $G = \mathrm{GL}_N$, then the subgroup T_N of upper triangular matrices is a Borel subgroup. Notice that T_N is a parabolic subgroup; see Example 4.2.2.*

Here are some basic properties of Borel subgroups. Part (iv) asserts that Borel subgroups are precisely the minimal parabolic subgroups (as we are working over $\overline{k}$).

Theorem 4.3.10. *(i) Any two Borel subgroups of G are conjugate.*

(ii) Every element of G lies in a Borel subgroup. And the intersection of all Borel subgroups of G is $Z(G)$.

(iii) (Lie–Kolchin) Assume $G < \mathrm{GL}_N$. Then there exists $x \in \mathrm{GL}_N(\overline{k})$ such that xGx^{-1} is contained in the subgroup of upper triangular matrices.

(iv) A closed subgroup of G is parabolic if and only if it contains a Borel subgroup.

Back to our root system $\Phi(T, G)$ constructed from a maximal torus T of G. Let B be a Borel subgroup containing T. For each $\alpha \in \Phi(T, G)$, Theorem 4.3.2.(3) constructs a reductive group G_α with $\mathrm{Lie}G_\alpha = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$.

Theorem 4.3.11. *For each $\alpha \in \Phi(T, G)$, the intersection $B \cap G_\alpha$ is a Borel subgroup of G_α, and $\mathrm{Lie}(B \cap G_\alpha)$ is either $\mathfrak{t} \oplus \mathfrak{g}_\alpha$ or $\mathfrak{t} \oplus \mathfrak{g}_{-\alpha}$.*

Now define

$$\Phi^+(B) := \{\alpha \in \Phi(T, G) : \mathrm{Lie}(B \cap G_\alpha) = \mathfrak{t} \oplus \mathfrak{g}_\alpha\}. \tag{4.3.2}$$

Then $\Phi(T, G) = \Phi^+(B) \sqcup (-\Phi^+(B))$ by Theorem 4.3.11. Thus we obtain the subset of positive roots determined by B, and the basis $\Delta(B)$ of $\Phi(T, G)$ consisting of simple (positive) roots in $\Phi^+(B)$ as below Lemma 4.3.7.

Conversely given any subset of positive roots Φ^+ of Φ, we can construct a subgroup B of G such that $\mathrm{Lie}B = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ (so that B is generated by T and U_α for all $\alpha \in \Phi^+$, with U_α from Theorem 4.3.2.(3a)).

Example 4.3.12. *In Example 4.3.4 with $(G, T) = (\mathrm{GL}_N, D_N)$, a set of positive roots is $\Phi^+ = \{e_i - e_j : 1 \le i < j \le N\}$, and the corresponding basis is $\Delta = \{e_i - e_{i+1} : 1 \le i \le N - 1\}$. The corresponding Borel subgroup is the subgroup of upper triangular matrices T_N.*

Example 4.3.13. *In Example 4.3.5 with $G = \mathrm{Sp}_{2d}$, a set of positive roots is $\Phi^+ = \{2e_i, e_i \pm e_j : 1 \le i < j \le d\}$, and the corresponding basis is $\{e_i - e_{i+1} : 1 \le i \le d - 1\} \cup \{2e_d\}$. The corresponding Borel subgroup consists of upper triangular matrices.*

4.3.3 Standard parabolic subgroups

Consider the root system $\Phi = \Phi(T,G) \subseteq X^*(T)$ constructed from a maximal torus T in G. Let B be a Borel subgroup of G which contains T. Then B defines the set of positive roots $\Phi^+ = \Phi^+(B)$ as in (4.3.2) and hence the basis $\Delta = \Delta(B)$ of Φ. Recall that $\mathrm{Lie}B = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$.

A parabolic subgroup of G is said to be *standard* (with respect to B) if it contains B. By parts (i) and (iv) of Theorem 4.3.10, every parabolic subgroup of G is conjugate to a standard one.

For any subset $I \subseteq \Delta$, denote by $\Phi_I \subseteq \Phi$ the set of roots which are linear combinations of roots in I. Let $\Phi_I^+ := \Phi^+ \cap I$. Then Φ_I is a root system in which Φ_I^+ is the set of positive roots and I is the corresponding basis. The Weyl group of Φ_I is the subgroup W_I of $W = W(T,G) = N_G(T)/T$ generated by the reflections r_α for all $\alpha \in I$.

We will use w to denote either an element in W or its representative in $N_G(T)$, whenever it is clear from the context. Then $C(w) := BwB$ is a subset of G, which by *Bruhat decomposition* satisfies: (a) $C(w)$ is a locally closed subvariety of G, (b) $G = \bigsqcup_{w\in W} C(w)$, (c) the closure $\overline{C(w)}$ is a union of certain $C(w')$.

Theorem 4.3.14. *(i) $P_I := \bigcup_{w\in W_I} BwB$ is a parabolic subgroup of G which contains B, with $\mathrm{Lie}P_I = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi^+\cup\Phi_I} \mathfrak{g}_\alpha$. In other words, P_I is generated by T and U_α for all $\alpha \in \Phi^+ \cup \Phi_I$, with U_α from Theorem 4.3.2.(3a).*

(ii) If P is a parabolic subgroup of G which contains B, then $P = P_I$ for a unique subset $I \subseteq \Delta$.

(iii) $\mathrm{Lie}\mathcal{R}_u(P_I) = \bigoplus_{\alpha\in\Phi^+\setminus\Phi_I} \mathfrak{g}_\alpha$.

(iv) Let L_I be the subgroup of G such that $\mathrm{Lie}L_I = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi_I} \mathfrak{g}_\alpha$. Then L_I is a Levi subgroup of P_I, i.e. is a reductive group contained in P_I such that $P_I = \mathcal{R}_u(P_I) \rtimes L_I$.

This theorem gives a combinatorial construction of all the standard parabolic subgroups of G: we add to Φ^+ the roots in Φ_I, and there is an inclusion-preserving bijection $I \mapsto P_I$ between subsets of Δ and standard parabolic subgroups. We have $P_\emptyset = B$, $P_\Delta = G$, and the maximal proper standard parabolic subgroups $P_{\Delta\setminus\{\alpha\}}$ for all $\alpha \in \Delta$. Moreover, if we define $T_I =: \left(\bigcap_{\alpha\in\Phi_I} \mathrm{Ker}\alpha\right)^\circ$, then $L_I = Z_G(T_I)$. This is a more precise version of Theorem 4.2.4 for parabolic subgroups of reductive groups, when $k = \overline{k}$.

We can say more about the pieces $C(w) := BwB$ in Theorem 4.3.14. To ease notation, for any root $\alpha \in \Phi$ we shall write $\alpha > 0$ if $\alpha \in \Phi^+$ and $\alpha < 0$ if $\alpha \notin \Phi^+$.

For any $w \in W$, we can define a subset of Φ

$$\Phi(w)' := \{\alpha > 0 : w\alpha < 0\} = \{\alpha \in \Phi^+ : -w\alpha \in \Phi^+\}.$$

and define U'_w to be the subgroup of $U := \mathcal{R}_u(B)$ such that $\mathrm{Lie}U'_w = \bigoplus_{\alpha\in\Phi(w)'} \mathfrak{g}_\alpha$. Then the map $U'_w \times B \to G$, $(u,b) \mapsto uwb$ is an isomorphism of varieties.

Example 4.3.15. *In the Example 4.3.12 with $(G,T) = (\mathrm{GL}_N, D_N)$ and the Borel group being the subgroup of upper triangular matrices, the basis is $\Delta = \{e_i - e_{i+1} : 1 \le i \le N-1\}$ which identify with $\{1,\dots,N-1\}$ (with $e_i - e_{i+1} \leftrightarrow i$). Take a subset $I \subseteq \Delta$ and write its complement*

$$\Delta \setminus I = \{a_1, a_1 + a_2, \dots, a_1 + \dots + a_{s-1}\}$$

with $a_j > 0$. Then P_I consists of upper triangular block matrices, with diagonal blocks of lengths $a_1, \dots, a_{s-1}, a_s := N - \sum_{j=1}^{s-1} a_j$. And $L_I \simeq \mathrm{GL}_{a_1} \times \dots \times \mathrm{GL}_{a_s}$ consists of diagonal block matrices, and $\mathcal{R}_u(P_I)$ consists of those matrices in P_I where the diagonal blocks are identity.

This is the combinatorial construction of Example 4.2.2.

The result for the Siegel case $G = \mathrm{Sp}_{2d}$ (corresponding to Example 4.3.13) will be given in later sections.

Remark 4.3.16. *Now, Theorem 4.2.5 in the case $k = \overline{k}$ follows easily from Theorem 4.3.14.*

4.4 Parabolic subgroups via root systems: over arbitrary fields

In this section, we take k to be a field, and G a reductive group defined over k. Then $Z(G)^\circ$ is an algebraic torus defined over k. Let $\mathfrak{g} := \mathrm{Lie}G$.

Let $\overline{k}$ be an algebraically closed field which contains k. For our purpose, it is harmless to take $k = \mathbb{Q}, \mathbb{R}$ and $\overline{k} = \mathbb{C}$.

By a *subgroup of* G, we mean a closed algebraic subgroup of G defined over k. In this section, we will discuss the combinatorial construction of parabolic subgroups of G, similar to the case $k = \overline{k}$.

4.4.1 Relative root systems

The first thing to do is to take a maximal torus T of $G_{\overline{k}}$ which is defined over k. It is known that such maximal tori always exist. But this is not enough, since characters of T may not be defined over k. We need:

Definition 4.4.1. *Let k'/k be an extension of fields. An algebraic torus A defined over k is said to be* **k'-split** *if $A_{k'} \simeq \mathbb{G}^r_{\mathrm{m},k'}$. Equivalently, A is k'-split if all characters of A are defined over k.*

Theorem 4.4.2. *(i) G contains a proper parabolic subgroup if and only if G contains a k-split torus which is not contained in $Z(G)$.*

(ii) Two maximal k-split tori contained in G are conjugate by an element of $G(k)$.

Here is a brief explanation to (i). Indeed, all parabolic subgroups of $G_{\overline{k}}$ are described by Theorem 4.2.5 using cocharacters, and having a parabolic subgroup of G (which by our convention means a parabolic subgroup defined over k) amounts to having a cocharacter of G which is defined over k.

Now take A to be a *maximal k-split torus* contained in G. Then $A_{\overline{k}}$ is contained in some maximal torus T of $G_{\overline{k}}$ defined over k. For each $\alpha' \in X^*(A)$, define $\mathfrak{g}_{\alpha'} := \{x \in \mathfrak{g} : a \cdot x = \alpha'(a)x \text{ for all } s \in A\}$ to be the eigenspace for α'. Then the adjoint action of $A < G$ on $\mathfrak{g}$ induces a decomposition of $\mathfrak{g}$ similar to (4.3.1)

$$\mathfrak{g} = \mathfrak{g}^A \oplus \sum_{\alpha' \in \Phi(A,G)} \mathfrak{g}_{\alpha'} \tag{4.4.1}$$

where $\Phi(A, G) \subseteq X^*(A) \setminus \{\text{trivial character}\}$ is the subset of non-trivial characters α' of A such that $\mathfrak{g}_{\alpha'} \neq 0$. The decomposition (4.4.1) is defined over k since all characters of A are defined over k.

Denote by ${}_k\Phi := \Phi(A, G)$.

Theorem 4.4.3. *${}_k\Phi$ is a root system, whose Weyl group is isomorphic to*

$${}_kW = W(A, G) := N_G(A)/Z_G(A).$$

Unlike the case $k = \overline{k}$, this root system may not be reduced. We call ${}_k\Phi$ the *relative root system* and ${}_kW$ the *relative Weyl group*.

Let us explain the analogue of G_α from Theorem 4.3.2.(3) in this relative setting. For any $\alpha' \in \Phi(A, G)$, the torus $S_{\alpha'} := (\mathrm{Ker}\alpha')^\circ$ is defined over k, and denote by $(\alpha') \subseteq \Phi(A, G)$ the subset consisting of rational multiples of α'. Then

Proposition 4.4.4. *There exists a unique closed connected unipotent k-subgroup $U_{(\alpha')}$ normalized by $Z_G(A)$ such that $\mathrm{Lie}U_{(\alpha')} = \mathfrak{g}_{(\alpha')} := \sum_{\beta \in (\alpha')} \mathfrak{g}_\beta$.*

The subgroup $G_{\alpha'} := Z_G(A_{\alpha'})$ is a reductive group defined over k, has S as a maximal k-split torus, and is generated by $Z_G(A)$ and $U_{(\alpha')}$.

4.4.2 Standard parabolic subgroups

Over $\overline{k}$, we have seen in §4.3.2 that choosing a basis of a root system (equivalently assigning the positive roots) amounts to fixing a Borel subgroup, and that Borel subgroups are precisely the minimal parabolic subgroups (Theorem 4.3.10.(iv)). Now over arbitrary k, we shall work with *minimal parabolic subgroups.*

Assign a subset ${}_k\Phi^+ = \Phi^+(A, G)$ of positive roots in ${}_k\Phi = \Phi(A, G)$, as below Lemma 4.3.7. Define

$$\mathfrak{n} := \sum_{\alpha' \in {}_k\Phi^+} \mathfrak{g}_{(\alpha')}. \tag{4.4.2}$$

It is a Lie subalgebra of $\mathfrak{g}$, and the corresponding subgroup N is unipotent and normalized by $Z_G(A)$. Now $P_0 := NZ_G(A)$ is a minimal parabolic subgroup of G, and every minimal parabolic subgroup of G which contains S is obtained in this way.

Now fix a minimal parabolic subgroup P_0 which contains S. A parabolic subgroup of G is said to be *standard* (with respect to P_0) if it contains P_0. As in the case $k = \overline{k}$, we have:

Theorem 4.4.5. *Every parabolic subgroup of G is conjugate, by an element in $G(k)$, to a unique standard parabolic subgroup.*

Let us construct the standard parabolic subgroups in combinatorial terms. Let ${}_k\Phi^+$ be the set of positive roots determined by P_0. Then we obtain a basis ${}_k\Delta$ of ${}_k\Phi$ as below Lemma 4.3.7.

For any subset $I \subseteq {}_k\Delta$, denote by ${}_k\Phi_I \subseteq {}_k\Phi$ the set of roots which are linear combinations of roots in I.

Let $A_I := \left(\bigcap_{\alpha' \in {}_k\Phi_I} \mathrm{Ker}\alpha'\right)^\circ < A$. Then the group $L_I := Z_G(A_I)$ satisfies

$$\mathrm{Lie}L_I = \mathfrak{g}^A + \sum_{\alpha' \in {}_k\Phi_I} \mathfrak{g}_{(\alpha')},$$

and

$$\mathfrak{n}_I := \sum_{\alpha' \in {}_k\Phi^+ \setminus {}_k\Phi_I} \mathfrak{g}_{(\alpha')}$$

is a Lie subalgebra of G. So $\mathfrak{n}_I$ defines a unipotent subgroup N_I of G which is normalized by L_I, and we have:

Theorem 4.4.6. *The product $P_I := N_I \cdot L_I$ is a standard parabolic subgroup, with $N_I = \mathcal{R}_u(P_I)$ and L_I a Levi subgroup of P_I.*

Any standard parabolic subgroup of G equals P_I for some $I \subseteq {}_k\Delta$.

Moreover, observe that A_I a k-split torus, which is not contained in $Z(P_I)$.