Chapter 4

Boundary components

Starting from this chapter, we will discuss compactifications of Shimura varieties $\mathrm{Sh}_K(\mathbf{G}, X)$, or locally Hermitian symmetric spaces $\Gamma\backslash X^+$. This chapter introduces boundary components of X^+.

4.1 Example: modular curves

Consider the modular curves $\mathrm{Sh}_K(\mathbf{GL}_2, \mathfrak{H}^\pm)$, *i.e.* the Siegel modular variety from §3.3 with $d = 1$. In the particular case where $K = \mathbf{GL}_2(\widehat{\mathbb{Z}})$, we are working with

$$Y(1) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}.$$

It is a well-known result that $Y(1) \simeq \mathbb{C}$ via the j-function $j\colon \mathfrak{H} \to \mathbb{C}$. Hence a compactification of $Y(1)$ is $\mathbb{P}^1(\mathbb{C})$. This is the *Baily–Borel compactification* or the *toroidal compactification* of $Y(1)$ (but not the *Borel–Serre compactification*). In this section, we explain how to view this compactification as the *Baily–Borel compactification* of $Y(1)$. A large part is to study the *boundary components*, which is important for other compactifications we will discuss (*toroidal compactification* and *Borel–Serre compactification*).

4.1.1 Boundary components of $\mathfrak{H}$

The *boundary* of $\mathfrak{H}$ in $\mathbb{C} \cup \{\infty\}$ is the union of the real axis and $\{\infty\}$; in other words, the boundary of $\mathfrak{H}$ in $\mathbb{P}^1(\mathbb{C})$ is $\mathbb{P}^1(\mathbb{R})$. This is better seen via the Cayley transformation (2.3.4)

$$\mathfrak{H} \xrightarrow{\sim} \mathcal{D} := \{z \in \mathbb{C} : |z| < 1\}, \qquad \tau \mapsto (\tau - \sqrt{-1})(\tau + \sqrt{-1})^{-1},$$

and the boundary of $\mathcal{D}$ is the unit circle. Denote by $\overline{\mathcal{D}}$ the closure of $\mathcal{D}$ in $\mathbb{C}$, *i.e.* $\overline{\mathcal{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$, and $\partial\mathcal{D} := \overline{\mathcal{D}} \setminus \mathcal{D}$. Then ∞ corresponds to $1 \in \overline{\mathcal{D}}$.

Call each point in $\partial\mathcal{D}$ a *boundary component* of $\mathcal{D}$. It is justified by the following fact: Any holomorphic map $\mathcal{D} \to \overline{\mathcal{D}}$ either has image in $\mathcal{D}$ or is constant.[1]

4.1.2 Extension of the group action to $\overline{\mathcal{D}}$

The group $\mathrm{GL}_2(\mathbb{R})^+$ acts on $\mathcal{D}$, via its action on $\mathfrak{H}$ and the Cayley transformation above, by the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{(a - \sqrt{-1}c)(z+1) + (b - \sqrt{-1}d)\sqrt{-1}(z-1)}{(a + \sqrt{-1}c)(z+1) + (b + \sqrt{-1}d)\sqrt{-1}(z-1)}, \qquad \forall z \in \mathcal{D}.$$

[1]This is a consequence of the Open Mapping Theorem in complex analysis, which asserts that any holomorphic function on a connected set in the complex plane is open.

Lemma 4.1.1. *The action of* $\mathrm{GL}_2(\mathbb{R})^+$ *on* $\mathcal{D}$ *extends to* $\overline{\mathcal{D}}$. *Moreover, the action of* $\mathrm{GL}_2(\mathbb{R})^+$ *on* $\partial\mathcal{D}$ *is transitive.*

Proof. Take $z \in \overline{\mathcal{D}}$, and set

$$u_\pm := (a \pm \sqrt{-1}c)(z+1) + (b \pm \sqrt{-1}d)\sqrt{-1}(z-1).$$

For the first part of the lemma, we need to show that $u_+ \neq 0$ and $u_- u_+^{-1} \in \overline{\mathcal{D}}$.

Then

$$\begin{bmatrix} u_- \\ u_+ \end{bmatrix} = \begin{bmatrix} 1 & -\sqrt{-1} \\ \sqrt{-1} & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \sqrt{-1} & -\sqrt{-1} \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix},$$

and one can compute that

$$u_+\overline{u}_+ - u_-\overline{u}_- = 4(1 - z\overline{z}).$$

So $u_+\overline{u}_+ \geq u_-\overline{u}_-$ because $z \in \overline{\mathcal{D}}$. If $u_+ = 0$, then $u_+ = u_- = 0$, contradiction to rank $\begin{bmatrix} u_- \\ u_+ \end{bmatrix} =$ rank $\begin{bmatrix} z \\ 1 \end{bmatrix} = 1$. So $u_+ \neq 0$, and $(u_- u_+^{-1})\overline{u_- u_+^{-1}} = \frac{u_-\overline{u}_-}{u_+\overline{u}_+} \leq 1$. Hence $u_- u_+^{-1} \in \overline{\mathcal{D}}$. We are done.

Let us prove the "Moreover" part. We have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot 1 = \frac{a^2 - c^2}{a^2 + c^2} + \frac{-2ac}{a^2 + c^2}\sqrt{-1}.$$

The right hand side is easily checked to be in $\partial\mathcal{D} = \{z \in \mathbb{C} : |z| = 1\}$. Conversely any $z \in \partial\mathcal{D}$ can be written as the right hand side for some 2×2-matrix in $\mathrm{GL}_2(\mathbb{R})^+$. Hence we are done. □

4.1.3 Compactifying at each boundary component

To see how to compactify $\mathcal{D} \simeq \mathfrak{H}$ at each boundary component, we need to study the stabilizer of each $z \in \overline{\mathcal{D}}$. Since $Z(\mathrm{GL}_2)(\mathbb{R})$ acts trivially on $\overline{\mathcal{D}}$, it suffices to consider the stabilizer in $\mathrm{SL}_2(\mathbb{R})$. By Lemma 4.1.1, it suffices to study this for $1 \in \overline{\mathcal{D}}$. For this purpose, it is easier to use the upper half plan. Define

$$P := \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} : b \in \mathbb{R},\ a \neq 0 \right\} \tag{4.1.1}$$

Then it is easy to check that $\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(g \cdot \infty) = gP(\mathbb{R})^+ g^{-1}$ for any $g \in \mathrm{SL}_2(\mathbb{R})$. Indeed, it suffices to check this with $g = I_2$, and then it suffices to notice that elements on the right hand side of (4.1.1) correspond to translations along the real axis.

Lemma 4.1.2. *The followings hold true:*

(i) $\mathrm{SL}_2(\mathbb{C})/P(\mathbb{C})$ *is a projective space.*

(ii) For any $g \in \mathrm{SL}_2(\mathbb{R})$, *the group* gPg^{-1} *is defined over* $\mathbb{Q}$ *if and only if* $g \in \mathrm{SL}_2(\mathbb{Q})$.

(iii) Let $\tau \in \mathbb{P}^1(\mathbb{R})$ *(the boundary of* $\mathfrak{H}$ *in* $\mathbb{P}^1(\mathbb{C})$*). Then* $\tau \in \mathbb{P}^1(\mathbb{Q}) \Leftrightarrow \tau = g \cdot \infty$ *for some* $g \in \mathrm{SL}_2(\mathbb{Q}) \Leftrightarrow \tau = g \cdot \infty$ *for some* $g \in \mathrm{SL}_2(\mathbb{Z})$.

Proof. (ii) and (iii) are simple computations. For (i), it suffices to notice that the homogeneous space $\mathrm{SL}_2(\mathbb{C})/P(\mathbb{C}) \simeq \mathrm{GL}_2(\mathbb{C})/\left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b, d \in \mathbb{C},\ ad \neq 0 \right\}$ is the Grassmannian parametrizing 1-dimensional $\mathbb{C}$-subspaces in $\mathbb{C}^2$. □

Let us go further. We have:

Lemma 4.1.3. *For each $g \in \mathrm{SL}_2(\mathbb{R})$, the group $gP(\mathbb{R})^+g^{-1}$ acts transitively on $\mathfrak{H}$.*

The proof itself is important. As a preparation, the group P has the following subgroups:

- The *unipotent radical* $N_P := \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : b \in \mathbb{R} \right\}$, where elements act on $\mathfrak{H}$ as $\tau \mapsto \tau + b$.
- the *split torus* $A_P := \left\{ \begin{bmatrix} a^{-1} & 0 \\ 0 & a \end{bmatrix} : a > 0 \right\}$,[2] where elements act on $\mathfrak{H}$ as $\tau \mapsto a^{-2}\tau$.
- $M_P := \{\pm I_2\}$, which acts trivially on $\mathfrak{H}$.

such that

$$P = N_P A_P M_P \tag{4.1.2}$$

and the map $N_P \times A_P \times M_P \to P$, $(n, a, m) \mapsto nam$, is a diffeomorphism.

Proof. We only need to prove this lemma for P. For any $\tau = x + \sqrt{-1}y \in \mathfrak{H}$, we have

$$\tau = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{bmatrix} \sqrt{-1}.$$

Hence we are done. ☐

Now we are ready to explain how the point ∞ is added to $\mathfrak{H}$ via the group P (in other words, how compactify $\mathfrak{H}$ at ∞). The decomposition 4.1.2 induces, by Lemma 4.1.3,

$$\mathfrak{H} \simeq P/(P \cap \mathrm{SO}(2)) = P/M_P \simeq N_P \times A_P \simeq \mathbb{R} \times \mathbb{R}_{>0}, \qquad \tau = x + \sqrt{-1}y \mapsto (x, \sqrt{y}^{-1}). \tag{4.1.3}$$

The A_P-factor is isomorphic to $\mathbb{R}_{>0}$, and a natural way to add a boundary to $\mathbb{R}_{>0}$ is to add 0 and make it into $\mathbb{R}_{\geq 0}$. In doing this, we are adding the point $x + \sqrt{-1}0^{-2} = \infty$ to $\mathfrak{H}$.

This process can be carried out for $g \cdot \infty \in \mathbb{P}^1(\mathbb{R})$ for any $g \in \mathrm{SL}_2(\mathbb{R})$, by replacing N_P and A_P by gN_Pg^{-1} and gA_Pg^{-1}. In this way, the point $g \cdot \infty \in \mathbb{P}^1(\mathbb{R})$ is added to $\mathfrak{H}$ by "compactifying" $gA_Pg^{-1} \simeq \mathbb{R}_{>0}$ into $\mathbb{R}_{\geq 0}$.

4.1.4 Rational vs real boundaries, and Siegel sets

We wish to compactify the quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H} \simeq \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{D}$. The idea is to do the quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\overline{\mathcal{D}}$, for the extended action of $\mathrm{SL}_2(\mathbb{R})$ on $\overline{\mathcal{D}}$ defined in Lemma 4.1.1. However, $\partial\mathcal{D} = \overline{\mathcal{D}} \setminus \mathcal{D} \simeq \mathbb{P}^1(\mathbb{R})$ contains infinitely many $\mathrm{SL}_2(\mathbb{Z})$-orbits.

A solution to this is to consider the *rational boundary components*, which are precisely the points in $\mathbb{P}^1(\mathbb{Q}) \subseteq \mathbb{P}^1(\mathbb{R})$. Equivalently by (ii) and (iii) of Lemma 4.1.2, a boundary component $z \in \partial\mathcal{D}$ is called a *rational boundary component* if its stabilizer in $\mathrm{SL}_2(\mathbb{R})$ is defined over $\mathbb{Q}$. Now part (iii) of Lemma 4.1.2 asserts that there is only one $\mathrm{SL}_2(\mathbb{Z})$-class of rational boundary components.

Another important notion is the *Siegel sets* associated with $P = \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(\infty)$ defined as follows; one needs this for example to pass from (partial) compactification of $\mathfrak{H}$ to compactification of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$. For each $t > 0$ and any compact bounded set $U \subseteq N_P \simeq \mathbb{R}$, define

$$\Sigma_{P,U,t} := U \times \{a \in \mathbb{R}_{>0} : a \leq t\} \simeq \{\tau = x + \sqrt{-1}y : x \in U,\ y \geq t^{-2}\} \subseteq \mathfrak{H}.$$

[2]Notice that A_P is not an algebraic subgroup of P, but only a Lie subgroup. This is a minor issue: Indeed, if we replace GL_2 by $\mathrm{PGL}_2 = \mathrm{SL}_2/\{\pm I_2\}$, then the quotient of A_P becomes an algebraic subgroup.

Then we have the following classical result on the j-function:[3] for a suitable U and suitable $t \gg 1$, $\Sigma_{P,U,t}$ is a fundamental set for the uniformization $j\colon \mathfrak{H} \to \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H} \simeq \mathbb{C}$ (*i.e.* $j|_{\Sigma_{P,U,t}}$ is surjective and has finite fibers). Then one can define the Siegel sets associated with $gPg^{-1} = \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(g\cdot\infty)$ (for any $g \in \mathrm{SL}_2(\mathbb{R})$) to be $g\cdot\Sigma_{P,U,t}$.

We can also compactify $\Gamma\backslash\mathfrak{H}$ to be, as a set, $\Gamma\backslash(\mathfrak{H}\cup\mathbb{P}^1(\mathbb{Q}))$ for any finite-indexed subgroup $\Gamma < \mathrm{SL}_2(\mathbb{Z})$, by the following lemma which is a direct consequence of the discussion above.

Lemma 4.1.4. *(i) There are finitely many rational boundary components $\alpha_1,\ldots,\alpha_n$ of $\mathfrak{H}$ such that $\mathbb{P}^1(\mathbb{Q}) = \bigcup_j \Gamma\cdot\alpha_j$.*

(ii) Let $P_j := \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(\alpha_j)$. Then there are suitable Siegel sets Σ_j associated with P_j for $j \in \{1,\ldots,n\}$ such that $\bigcup_j \Sigma_j$ is a fundamental set for the uniformization $u\colon \mathfrak{H} \to \Gamma\backslash\mathfrak{H}$.

4.1.5 Satake topology on $\overline{\mathcal{D}}$

This subsection is for the Baily–Borel compactification of $\Gamma\backslash\mathfrak{H}$. We will revisit the materials later in more generality.

Our desired compactification is $\Gamma\backslash(\mathfrak{H}\cup\mathbb{P}^1(\mathbb{Q}))$. We yet to explain the topology on this set, so that it is Hausdorff and compact. Notice that we cannot take the one induced by the usual topology on $\mathbb{C}$ because $x \in \mathbb{P}^1(\mathbb{Q})$ there are infinitely many $\gamma \in \Gamma$ which fixed x, and hence the quotient $\Gamma\backslash(\mathfrak{H}\cup\mathbb{P}^1(\mathbb{Q}))$ is not Hausdorff under this topology.

The topology which we consider is the Satake topology, induced from the Satake topology on $\mathfrak{H}\cup\mathbb{P}^1(\mathbb{Q})$ defined as follows. On $\mathfrak{H}$, the Satake topology is the usual topology, induced from $\mathbb{C}$. Next, an open neighborhood basis of ∞ consists of the open sets $U_t := \{z \in \mathfrak{H} : \mathrm{Im}(z) > t\}$ for all $t \geq 2$; equivalently a sequence $\tau_j = x_j + \sqrt{-1}y_j \in \mathfrak{H}$ converges to ∞ if and only if $y_j \to \infty$. Finally, an open neighborhood basis of $g\cdot\infty \in \mathbb{P}^1(\mathbb{Q})$ (with $g \in \mathrm{SL}_2(\mathbb{Q})$) consists of $g\cdot U_t$ for all $t \geq 2$. We state without proof the following assertions (whose proof needs to use Siegel sets):

(i) For any $x \in \mathfrak{H}\cup\mathbb{P}^1(\mathbb{Q})$, there exists a fundamental system of neighborhoods $\{U\}$ of x such that

$$\gamma U = U,\ \forall\gamma\in\Gamma_x; \quad \gamma U\cap U = \emptyset,\ \forall\gamma\notin\Gamma_x$$

where $\Gamma_x = \{\gamma\in\Gamma : \gamma x = x\}$.

(ii) If $x, x' \in \mathfrak{H}\cup\mathbb{P}^1(\mathbb{Q})$ are not in one Γ-orbit, then there exist neighborhoods U of x and U' of x' such that

$$\Gamma U \cap U' = \emptyset.$$

These properties guarantee that $\Gamma\backslash(\mathfrak{H}\cup\mathbb{P}^1(\mathbb{Q}))$ is Hausdorff under the Satake topology. The compactness follows easily from part (ii) of Lemma 4.1.4.

4.2 Parabolic subgroups and Levi subgroups: definitions and statements

For the simplest Siegel Shimura datum $(\mathbf{GL}_2, \mathfrak{H}^{\pm})$, Lemma 4.1.2.(i) suggests that parabolic subgroups of SL_2 (*i.e.* subgroups of SL_2 such that the homogeneous space $\mathrm{SL}_2(\mathbb{C})/P(\mathbb{C})$ is a projective variety) are closely related to the boundary components of $\mathfrak{H}$. This is true for an arbitrary Shimura datum $(\mathbf{G}, X)$.

In this section, we review background knowledge on parabolic subgroups of reductive groups over algebraically closed fields. In the next section, we do it over an arbitrary field.

Let k be a field, and let G be a reductive group defined over k. Let $\overline{k}$ be an algebraic closed field containing k. For our purpose, we will take $k = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ and $\overline{k} = \mathbb{C}$.

[3] A well-known fundamental domain of the j-function is $\{z \in \mathbb{C} : |z| \geq 1,\ -1 \leq \mathrm{Re}(z) < 1\}$.

Definition 4.2.1. *A subgroup P of G is called a* **parabolic subgroup** *if the homogeneous space $G(\overline{k})/P(\overline{k})$ is a projective variety.*

It is a theorem of Chevalley that *parabolic subgroups are always connected.* We are more interested in the *proper* parabolic subgroups.

Example 4.2.2. *For $G = \mathrm{GL}_N$. Let P be the subgroup of upper triangular matrices in blocks (with the length of the ℓ-th diagonal block being n_ℓ). Then if we write $G = \mathrm{GL}(V)$ with $V \simeq k^N$, then P is the stabilizer of a flag $F^\bullet = (0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{m-1} \subsetneq V_m = V)$ of subspaces of V, with $\dim V_\ell - \dim V_{\ell-1} = n_\ell$ for each ℓ. Hence G/P is a flag variety and hence is projective. So P is a parabolic subgroup of GL_N.*

Let P be a parabolic subgroup of G. The unipotent radical $\mathcal{R}_u(P)$ is a closed normal subgroup of P, and hence P acts on $\mathcal{R}_u(P)$ via conjugation. This induces an action of any subgroup of H on $\mathcal{R}_u(H)$.

Definition 4.2.3. *A* **Levi subgroup** *of P is a closed subgroup L of P such that $H = \mathcal{R}_u(P) \rtimes L$.*

A Levi subgroup, if exists, is then isomorphic to $P/\mathcal{R}_u(P)$ and hence is a reductive group (in particular is connected).

Theorem 4.2.4. *P has Levi subgroups, and any two Levi subgroups of P are conjugate by a unique element in $\mathcal{R}_u(P)$.*

We are more interested in more concrete constructions of Levi subgroups of P. This will be given in combinatorial data in the next two sections.

The following construction of parabolic subgroups of G is useful, although we will not use it in our course. Let λ be a cocharacter of G, *i.e.* a morphism of algebraic groups $\mathbb{G}_m \to G$.

Theorem 4.2.5. *(i) The set*

$$P(\lambda) := \{x \in G : \lim_{t \to 0} \lambda(t) x \lambda(t)^{-1} \text{ exists}\}$$

is a parabolic subgroup of G, and the centralizer of $\lambda(\mathbb{G}_m)$ is a Levi subgroup of $P(\lambda)$. Moreover $\mathcal{R}_u(P(\lambda)) = \{x \in G : \lim_{t \to 0} \lambda(t) x \lambda(t)^{-1} = 1\}$.

(ii) Any parabolic subgroup of G is $P(\lambda)$ for some λ.

If $\lambda(\mathbb{G}_m) < Z(G)$, then $P(\lambda) = G$. In fact, this theorem will serve as a bridge from the theory over algebraically closed fields to the theory over an arbitrary field.

4.3 Parabolic subgroups via root systems: over algebraically closed fields

In this section, we take $k = \overline{k}$ to be an algebraically closed field, and G a reductive group defined over k. For our purpose, it is harmless to take $k = \mathbb{C}$. We will explain the combinatorial construction of parabolic subgroups of G, and Example 4.2.2 will be revisited in this language as Example 4.3.15.

Let $\mathfrak{g} := \mathrm{Lie} G$. Then we have the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ whose kernel is $Z(G)$. Notice that $Z(G)^\circ$ is an algebraic torus since G is reductive.

4.3.1 Root system for G

Let T be a maximal torus of G, *i.e.* an algebraic torus contained in G and maximal under the inclusion. For example if $G = \mathrm{GL}_N$, we can take $T = D_N$ to be the subgroup of diagonal matrices with non-zero diagonal entries. We have the standard properties:

Lemma 4.3.1. *(i) Any maximal torus of G equals gTg^{-1} for some $g \in G(\overline{k})$.*

(ii) $T = Z_G(T) = \{g \in G(\overline{k}) : gtg^{-1} = t \text{ for all } t \in T(\overline{k})\}$.

(iii) $W(T, G) := N_G(T)/T$ is finite and is called the **Weyl group**.

Thus $T \supseteq Z(G)^\circ$.

Now consider the action of T on $\mathfrak{g}$ via $T < G$ and the adjoint action. Let $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_\mathrm{m})$ be the group of characters of T. For each $\alpha \in X^*(T)$, define $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} : t \cdot x = \alpha(t)x \text{ for all } t \in T\}$ to be the eigenspace for α. Then we have a decomposition as in (1.2.2)

$$\mathfrak{g} = \mathfrak{g}^T \oplus \bigoplus_{\alpha \in \Phi(T,G)} \mathfrak{g}_\alpha \tag{4.3.1}$$

where $\mathfrak{g}^T := \{x \in \mathfrak{g} : T \cdot x = x\}$ is the eigenspace for the trivial character, and $\Phi(T, G) \subseteq X^*(T) \setminus \{\text{trivial character}\}$ is the subset of non-trivial characters α of T such that $\mathfrak{g}_\alpha \neq 0$. By Lemma 4.3.1.(ii), we have $\mathfrak{g}^T = \mathfrak{t} := \mathrm{Lie}T$.

Denote for simplicity by $\Phi = \Phi(T, G)$. Elements in Φ are called *roots of T*. The following theorem, which gives combinatorial data associated with G and T, is extremely important in the theory of reductive groups.

Theorem 4.3.2. *(1) Φ generates a subgroup of finite index in $X^*(T/Z(G)^\circ) \subseteq X^*(T)$.*

(2) Let $\alpha \in \Phi$ and $\beta \in X^*(T)$ which is a multiple of α. Then $\beta \in \Phi \Leftrightarrow \beta = \pm\alpha$.

(3) Let $\alpha \in \Phi$, and set $G_\alpha := Z_G((\mathrm{Ker}\alpha)^\circ)$. Then

- *(a) $\dim \mathfrak{g}_\alpha = 1$, and there is a unique connected T-stable (unipotent) subgroup U_α of G such that $\mathrm{Lie}U_\alpha = \mathfrak{g}_\alpha$;*[4]
- *(b) G_α is a reductive group and $\mathrm{Lie}G_\alpha = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$;*[5] *and $G_\alpha^{\mathrm{ad}} \simeq \mathrm{PGL}_2$;*[6]
- *(c) the subgroup $W(T, G_\alpha)$ is $W(T, G)$ is generated by a reflection r_α such that $r_\alpha(\alpha) = -\alpha$.*

(4) Let $\alpha \in \Phi$ and $r_\alpha \in W(T, G)$ be as in (3.c). Then for any $\beta \in \Phi$, we have

$$r_\alpha(\beta) = \beta - n_{\beta,\alpha}\alpha$$

with $n_{\beta,\alpha} \in \mathbb{Z}$. Moreover, $n_{\alpha,\alpha} = 2$.

Thus Φ is a reduced root system in the vector space $E := X^*(T/Z(G)^\circ)_\mathbb{R}$ with Weyl group $W(T, G)$ in the sense below.

Definition 4.3.3. *Let E be a finite-dimensional real vector space with a Euclidean inner product $\langle, \rangle$. A* **root system** *Φ in E is a finite set of non-zero vectors (called* **roots***) such that:*

[4]Thus U_α is isomorphic to $\mathbb{G}_\mathrm{a}$ since it is a unipotent group of dimension 1.

[5]In other words, G_α is generated by T, U_α and $U_{-\alpha}$.

[6]Indeed, we can choose a generator X_α of $\mathfrak{g}_\alpha$ for each $\alpha \in \Phi$ such that X_α, $X_{-\alpha}$, $[X_\alpha, X_{-\alpha}]$ is an $\mathfrak{sl}_2$-triple for all $\alpha \in \Phi$.

(1) Φ spans E,

(2) If $\alpha, c\alpha \in \Phi$ for some $c \neq 0$, then $c \in \{1, -1, 1/2, -1/2\}$,

(3) For any $\alpha \in \Phi$, the set Φ is closed under the reflection through the hyperplane perpendicular to α (which we denote by r_α),

(4) For any $\alpha, \beta \in \Phi$, we have $r_\alpha(\beta) = \beta - n_{\beta,\alpha}\alpha$ with $n_{\beta,\alpha} \in \mathbb{Z}$.

A root system is called **reduced** *if furthermore it satisfies:*

(2') The only scalar multiples of a root $\alpha \in \Phi$ that belong to Φ are $\pm\alpha$.

We call $\dim E$ the **rank** *of Φ.*

The **Weyl group** *of Φ, denoted by $W(\Phi)$, is the group of $\mathrm{Aut}(\Phi)$ generated by r_α for all $\alpha \in \Phi$.*

Conversely, given a *root datum* (root system and "coroot system") one can associate a unique reductive group. We shall not go into details for this, but restrict our discussion to root systems. In practice, we often take G to be semi-simple, so that $\Phi(T, G)$ is a reduced root system in $X^*(T)_{\mathbb{R}}$.

Example 4.3.4. *Let $G = \mathrm{GL}_N$ and $T = D_N$. The Weyl group is isomorphic to the permutation group $\mathfrak{S}_N$. For each $j \in \{1, \ldots, N\}$, define $e_j \in X^*(D_N)$ to be $\mathrm{diag}(t_1, \ldots, t_N) \mapsto t_j$. Then we have an isomorphism $X^*(D_N) \simeq \bigoplus_{j=1}^N \mathbb{Z}e_j$. One can check that $\Phi(D_N, \mathrm{GL}_N) = \{e_i - e_j : i \neq j\}$.*

Highly related to this example is $G = \mathrm{SL}_N$ and $T = D_N \cap \mathrm{SL}_N$. Then $X^*(T) \simeq \bigoplus_{j=1}^N \mathbb{Z}e_j/\mathbb{Z}(e_1 + \ldots + e_N)$. And $\Phi(T, G)$ in this case is precisely the image of $\Phi(D_N, \mathrm{GL}_N)$ under the natural projection $X^*(D_N) \to X^*(T)$.

Example 4.3.5. *Let $G = \mathrm{Sp}_{2d}$ and $T = \mathrm{Sp}_{2d} \cap D_{2d} = \{\mathrm{diag}(t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}) : t_1 \cdots t_d \neq 0\}$. The Weyl group is isomorphic to $\{\pm 1\}^d \rtimes \mathfrak{S}_d$. For each $j \in \{1, \ldots, d\}$, define $e_j \in X^*(T)$ to be $\mathrm{diag}(t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}) \mapsto t_j$. Then $X^*(T) \simeq \bigoplus_{j=1}^d \mathbb{Z}e_j$. One can check that $\Phi(T, \mathrm{Sp}_{2d}) = \{\pm 2e_i, \pm e_i \pm e_j : 1 \leq i, j \leq d,\ i \neq j\}$.*

Root systems in Example 4.3.4 are called *of type A_{N-1}*, and root systems in Example 4.3.5 are called *of type C_d*. We also have root systems of type B_n (dual to C_n; coming from SO_{2n+1}) and D_n (coming from SO_{2n}), and exceptional types E_6, E_7, E_8, F_4, G_2). We will not go into details for this, but only point out that the last 3 types do not show up in the theory of Shimura varieties and that a Shimura variety is of abelian type unless the underlying group has $\mathbb{Q}$-factors of mixed type D or of exceptional types.

4.3.2 Positive roots and Borel subgroups

We start with the abstract theory of root systems $\Phi \subseteq E$.

Definition 4.3.6. *A* **basis** *of Φ is a subset Δ of Φ which is a basis of E such that each root $\beta \in \Phi$ is a linear combination $\beta = \sum_{\alpha \in \Delta} m_\alpha \alpha$ with $m_\alpha \in \mathbb{Z}$ of the same sign.*

Given a basis Δ of Φ, a root $\beta \in \Phi$ is said to be **positive (with respect to Δ)** *if $m_\alpha \geq 0$ for the decomposition above. Denote by Φ^+ the set of positive roots, and $\Phi^- := -\Phi^+$. Then $\Phi = \Phi^+ \sqcup \Phi^-$.*

A root $\alpha \in \Phi^+$ is said to be **simple** *if it is not the sum of two other positive roots.*

Lemma 4.3.7. *Δ is precisely the set of simple roots in Φ^+.*

In practice, one can start from a subset Φ^+ of Φ such that $\Phi = \Phi^+ \sqcup (-\Phi^+)$ and that $\alpha \in \Phi^+ \Rightarrow -2\alpha \notin \Phi^+$, and call these roots *positive*. Then we get a basis Δ consisting of simple roots in Φ^+, with respect to which Φ^+ is the set of positive roots. See Lemma 4.3.7.

Back to the theory of reductive groups, choosing Φ^+ is equivalently to taking a *Borel group*.

Definition 4.3.8. *A* **Borel group** *B is G is a closed connected solvable subgroup G, which is maximal for these properties.*

Example 4.3.9. *If $G = \mathrm{GL}_N$, then the subgroup T_N of upper triangular matrices is a Borel subgroup. Notice that T_N is a parabolic subgroup; see Example 4.2.2.*

Here are some basic properties of Borel subgroups. Part (iv) asserts that Borel subgroups are precisely the minimal parabolic subgroups (as we are working over $\overline{k}$).

Theorem 4.3.10. *(i) Any two Borel subgroups of G are conjugate.*

(ii) Every element of G lies in a Borel subgroup. And the intersection of all Borel subgroups of G is $Z(G)$.

(iii) (Lie–Kolchin) Assume $G < \mathrm{GL}_N$. Then there exists $x \in \mathrm{GL}_N(\overline{k})$ such that xGx^{-1} is contained in the subgroup of upper triangular matrices.

(iv) A closed subgroup of G is parabolic if and only if it contains a Borel subgroup.

Back to our root system $\Phi(T, G)$ constructed from a maximal torus T of G. Let B be a Borel subgroup containing T. For each $\alpha \in \Phi(T, G)$, Theorem 4.3.2.(3) constructs a reductive group G_α with $\mathrm{Lie}G_\alpha = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$.

Theorem 4.3.11. *For each $\alpha \in \Phi(T, G)$, the intersection $B \cap G_\alpha$ is a Borel subgroup of G_α, and $\mathrm{Lie}(B \cap G_\alpha)$ is either $\mathfrak{t} \oplus \mathfrak{g}_\alpha$ or $\mathfrak{t} \oplus \mathfrak{g}_{-\alpha}$.*

Now define

$$\Phi^+(B) := \{\alpha \in \Phi(T, G) : \mathrm{Lie}(B \cap G_\alpha) = \mathfrak{t} \oplus \mathfrak{g}_\alpha\}. \tag{4.3.2}$$

Then $\Phi(T, G) = \Phi^+(B) \sqcup (-\Phi^+(B))$ by Theorem 4.3.11. Thus we obtain the subset of positive roots determined by B, and the basis $\Delta(B)$ of $\Phi(T, G)$ consisting of simple (positive) roots in $\Phi^+(B)$ as below Lemma 4.3.7.

Conversely given any subset of positive roots Φ^+ of Φ, we can construct a subgroup B of G such that $\mathrm{Lie}B = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ (so that B is generated by T and U_α for all $\alpha \in \Phi^+$, with U_α from Theorem 4.3.2.(3a)).

Example 4.3.12. *In Example 4.3.4 with $(G, T) = (\mathrm{GL}_N, D_N)$, a set of positive roots is $\Phi^+ = \{e_i - e_j : 1 \le i < j \le N\}$, and the corresponding basis is $\Delta = \{e_i - e_{i+1} : 1 \le i \le N-1\}$. The corresponding Borel subgroup is the subgroup of upper triangular matrices T_N.*

Example 4.3.13. *In Example 4.3.5 with $G = \mathrm{Sp}_{2d}$, a set of positive roots is $\Phi^+ = \{2e_i, e_i \pm e_j : 1 \le i < j \le d\}$, and the corresponding basis is $\{e_i - e_{i+1} : 1 \le i \le d-1\} \cup \{2e_d\}$. The corresponding Borel subgroup consists of upper triangular matrices.*

4.3.3 Standard parabolic subgroups

Consider the root system $\Phi = \Phi(T, G) \subseteq X^*(T)$ constructed from a maximal torus T in G. Let B be a Borel subgroup of G which contains T. Then B defines the set of positive roots $\Phi^+ = \Phi^+(B)$ as in (4.3.2) and hence the basis $\Delta = \Delta(B)$ of Φ. Recall that $\mathrm{Lie}B = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$.

A parabolic subgroup of G is said to be *standard* (with respect to B) if it contains B. By parts (i) and (iv) of Theorem 4.3.10, every parabolic subgroup of G is conjugate to a standard one.

For any subset $I \subseteq \Delta$, denote by $\Phi_I \subseteq \Phi$ the set of roots which are linear combinations of roots in I. Let $\Phi_I^+ := \Phi^+ \cap I$. Then Φ_I is a root system in which Φ_I^+ is the set of positive roots and I is the corresponding basis. The Weyl group of Φ_I is the subgroup W_I of $W = W(T, G) = N_G(T)/T$ generated by the reflections r_α for all $\alpha \in I$.

We will use w to denote either an element in W or its representative in $N_G(T)$, whenever it is clear from the context. Then $C(w) := BwB$ is a subset of G, which by *Bruhat decomposition* satisfies: (a) $C(w)$ is a locally closed subvariety of G, (b) $G = \bigsqcup_{w\in W} C(w)$, (c) the closure $\overline{C(w)}$ is a union of certain $C(w')$.

Theorem 4.3.14. *(i) $P_I := \bigcup_{w\in W_I} BwB$ is a parabolic subgroup of G which contains B, with $\mathrm{Lie}P_I = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi^+\cup\Phi_I} \mathfrak{g}_\alpha$. In other words, P_I is generated by T and U_α for all $\alpha \in \Phi^+ \cup \Phi_I$, with U_α from Theorem 4.3.2.(3a).*

(ii) If P is a parabolic subgroup of G which contains B, then $P = P_I$ for a unique subset $I \subseteq \Delta$.

(iii) $\mathrm{Lie}\mathcal{R}_u(P_I) = \bigoplus_{\alpha\in\Phi^+\setminus\Phi_I} \mathfrak{g}_\alpha$.

(iv) Let L_I be the subgroup of G such that $\mathrm{Lie}L_I = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi_I} \mathfrak{g}_\alpha$. Then L_I is a Levi subgroup of P_I, i.e. is a reductive group contained in P_I such that $P_I = \mathcal{R}_u(P_I) \rtimes L_I$.

This theorem gives a combinatorial construction of all the standard parabolic subgroups of G: we add to Φ^+ the roots in Φ_I, and there is an inclusion-preserving bijection $I \mapsto P_I$ between subsets of Δ and standard parabolic subgroups. We have $P_\emptyset = B$, $P_\Delta = G$, and the maximal proper standard parabolic subgroups $P_{\Delta\setminus\{\alpha\}}$ for all $\alpha \in \Delta$. Moreover, if we define $T_I =: \left(\bigcap_{\alpha\in I} \mathrm{Ker}\alpha\right)^\circ$, then $L_I = Z_G(T_I)$. This is a more precise version of Theorem 4.2.4 for parabolic subgroups of reductive groups, when $k = \bar{k}$.

We can say more about the pieces $C(w) := BwB$ in Theorem 4.3.14. To ease notation, for any root $\alpha \in \Phi$ we shall write $\alpha > 0$ if $\alpha \in \Phi^+$ and $\alpha < 0$ if $\alpha \notin \Phi^+$.

For any $w \in W$, we can define a subset of Φ

$$\Phi(w)' := \{\alpha > 0 : w\alpha < 0\} = \{\alpha \in \Phi^+ : -w\alpha \in \Phi^+\}.$$

and define U'_w to be the subgroup of $U := \mathcal{R}_u(B)$ such that $\mathrm{Lie}U'_w = \bigoplus_{\alpha\in\Phi(w)'} \mathfrak{g}_\alpha$. Then the map $U'_w \times B \to G$, $(u, b) \mapsto uwb$ is an isomorphism of varieties.

Example 4.3.15. *In the Example 4.3.12 with $(G, T) = (\mathrm{GL}_N, D_N)$ and the Borel group being the subgroup of upper triangular matrices, the basis is $\Delta = \{e_i - e_{i+1} : 1 \le i \le N-1\}$ which identify with $\{1, \ldots, N-1\}$ (with $e_i - e_{i+1} \leftrightarrow i$). Take a subset $I \subseteq \Delta$ and write its complement*

$$\Delta \setminus I = \{a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_{s-1}\}$$

with $a_j > 0$. Then P_I consists of upper triangular block matrices, with diagonal blocks of lengths $a_1, \ldots, a_{s-1}, a_s := N - \sum_{j=1}^{s-1} a_j$. And $L_I \simeq \mathrm{GL}_{a_1} \times \cdots \times \mathrm{GL}_{a_s}$ consists of diagonal block matrices, and $\mathcal{R}_u(P_I)$ consists of those matrices in P_I where the diagonal blocks are identity.

This is the combinatorial construction of Example 4.2.2.

The result for the Siegel case $G = \mathrm{Sp}_{2d}$ (corresponding to Example 4.3.13) will be given in later sections.

Remark 4.3.16. *Now, Theorem 4.2.5 in the case $k = \overline{k}$ follows easily from Theorem 4.3.14.*

4.4 Parabolic subgroups via root systems: over arbitrary fields

In this section, we take k to be a field, and G a reductive group defined over k. Then $Z(G)^\circ$ is an algebraic torus defined over k. Let $\mathfrak{g} := \mathrm{Lie}G$.

Let $\overline{k}$ be an algebraically closed field which contains k. For our purpose, it is harmless to take $k = \mathbb{Q}, \mathbb{R}$ and $\overline{k} = \mathbb{C}$.

By a *subgroup of* G, we mean a closed algebraic subgroup of G defined over k. In this section, we will discuss the combinatorial construction of parabolic subgroups of G, similar to the case $k = \overline{k}$.

4.4.1 Relative root systems

The first thing to do is to take a maximal torus T of $G_{\overline{k}}$ which is defined over k. It is known that such maximal tori always exist. But this is not enough, since characters of T may not be defined over k. We need:

Definition 4.4.1. *Let k'/k be an extension of fields. An algebraic torus A defined over k is said to be* **k'-split** *if $A_{k'} \simeq \mathbb{G}^r_{\mathrm{m},k'}$. Equivalently, A is k'-split if all characters of A are defined over k.*

Theorem 4.4.2. *(i) G contains a proper parabolic subgroup if and only if G contains a k-split torus which is not contained in $Z(G)$.*

(ii) Two maximal k-split tori contained in G are conjugate by an element of $G(k)$.

Here is a brief explanation to (i). Indeed, all parabolic subgroups of $G_{\overline{k}}$ are described by Theorem 4.2.5 using cocharacters, and having a parabolic subgroup of G (which by our convention means a parabolic subgroup defined over k) amounts to having a cocharacter of G which is defined over k.

Now take A to be a *maximal k-split torus* contained in G. Then $A_{\overline{k}}$ is contained in some maximal torus T of $G_{\overline{k}}$ defined over k. For each $\alpha' \in X^*(A)$, define $\mathfrak{g}_{\alpha'} := \{x \in \mathfrak{g} : a \cdot x = \alpha'(a)x \text{ for all } s \in A\}$ to be the eigenspace for α'. Then the adjoint action of $A < G$ on $\mathfrak{g}$ induces a decomposition of $\mathfrak{g}$ similar to (4.3.1)

$$\mathfrak{g} = \mathfrak{g}^A \oplus \sum_{\alpha' \in \Phi(A,G)} \mathfrak{g}_{\alpha'} \tag{4.4.1}$$

where $\Phi(A, G) \subseteq X^*(A) \setminus \{\text{trivial character}\}$ is the subset of non-trivial characters α' of A such that $\mathfrak{g}_{\alpha'} \neq 0$. The decomposition (4.4.1) is defined over k since all characters of A are defined over k.

Denote by ${}_k\Phi := \Phi(A, G)$.

Theorem 4.4.3. *${}_k\Phi$ is a root system, whose Weyl group is isomorphic to*

$${}_kW = W(A, G) := N_G(A)/Z_G(A).$$

Unlike the case $k = \overline{k}$, this root system may not be reduced. We call ${}_k\Phi$ the *relative root system* and ${}_kW$ the *relative Weyl group.*

Let us explain the analogue of G_α from Theorem 4.3.2.(3) in this relative setting. For any $\alpha' \in \Phi(A, G)$, the torus $S_{\alpha'} := (\mathrm{Ker}\alpha')^\circ$ is defined over k, and denote by $(\alpha') \subseteq \Phi(A, G)$ the subset consisting of rational multiples of α'. Then

Proposition 4.4.4. *There exists a unique closed connected unipotent k-subgroup $U_{(\alpha')}$ normalized by $Z_G(A)$ such that $\mathrm{Lie}U_{(\alpha')} = \mathfrak{g}_{(\alpha')} := \sum_{\beta\in(\alpha')} \mathfrak{g}_\beta$.*

The subgroup $G_{\alpha'} := Z_G(A_{\alpha'})$ is a reductive group defined over k, has S as a maximal k-split torus, and is generated by $Z_G(A)$ and $U_{(\alpha')}$.

4.4.2 Standard parabolic subgroups

Over $\overline{k}$, we have seen in §4.3.2 that choosing a basis of a root system (equivalently assigning the positive roots) amounts to fixing a Borel subgroup, and that Borel subgroups are precisely the minimal parabolic subgroups (Theorem 4.3.10.(iv)). Now over arbitrary k, we shall work with *minimal parabolic subgroups.*

Assign a subset ${}_k\Phi^+ = \Phi^+(A, G)$ of positive roots in ${}_k\Phi = \Phi(A, G)$, as below Lemma 4.3.7. Define

$$\mathfrak{n} := \sum_{\alpha'\in{}_k\Phi^+} \mathfrak{g}_{(\alpha')}. \tag{4.4.2}$$

It is a Lie subalgebra of $\mathfrak{g}$, and the corresponding subgroup N is unipotent and normalized by $Z_G(A)$. It is known that $P_0 := NZ_G(A)$ is a minimal parabolic subgroup of G, and every minimal parabolic subgroup of G which contains A is obtained in this way.

Now fix a minimal parabolic subgroup P_0 which contains A. A parabolic subgroup of G is said to be *standard* (with respect to P_0) if it contains P_0. As in the case $k = \overline{k}$, we have:

Theorem 4.4.5. *Every parabolic subgroup of G is conjugate, by an element in $G(k)$, to a unique standard parabolic subgroup.*

Let us construct the standard parabolic subgroups in combinatorial terms. Let ${}_k\Phi^+$ be the set of positive roots determined by P_0. Then we obtain a basis ${}_k\Delta$ of ${}_k\Phi$ as below Lemma 4.3.7.

For any subset $I \subseteq {}_k\Delta$, denote by ${}_k\Phi_I \subseteq {}_k\Phi$ the set of roots which are linear combinations of roots in I.

Let $A_I := \left(\bigcap_{\alpha'\in I} \mathrm{Ker}\alpha'\right)^\circ < A$. Then the group $L_I := Z_G(A_I)$ satisfies

$$\mathrm{Lie}L_I = \mathfrak{g}^A + \sum_{\alpha'\in{}_k\Phi_I} \mathfrak{g}_{(\alpha')}.$$

The Lie subalgebra of $\mathfrak{g}$

$$\mathfrak{n}_I := \sum_{\alpha'\in{}_k\Phi^+\setminus{}_k\Phi_I} \mathfrak{g}_{(\alpha')}$$

defines a unipotent subgroup N_I of G which is normalized by L_I, and we have:

Theorem 4.4.6. *The product $P_I := N_I \cdot L_I$ is a standard parabolic subgroup, with $N_I = \mathcal{R}_u(P_I)$ and L_I a Levi subgroup of P_I.*

Any standard parabolic subgroup of G equals P_I for some $I \subseteq {}_k\Delta$.

Moreover, observe that A_I a k-split torus, which is not contained in $Z(P_I)$. But A_I is the maximal k-split torus in $Z(L_I)$.

We close this subsection by the following immediate consequence of the construction above.

Lemma 4.4.7. *Assume $I \subseteq I' \subseteq {}_k\Delta$. Then $A_I > A_{I'}$ and $P_I < P_{I'}$.*

4.5 Horospherical decompositions and Siegel sets

Let $(\mathbf{G}, X)$ be a Shimura datum, and X^+ a connected component of X. We will use the following notation:

$$G = \mathbf{G}^{\mathrm{der}}_{\mathbb{R}}, \quad \mathfrak{g} := \mathrm{Lie}\mathbf{G}^{\mathrm{der}}, \quad \mathfrak{g}_{\mathbb{R}} = \mathrm{Lie}G. \tag{4.5.1}$$

To ease notation, we will also *use X to denote X^+*.

We need to use maximal $\mathbb{Q}$-split (resp. $\mathbb{R}$-split) torus contained in $\mathbf{G}^{\mathrm{der}}$, for which we make the following definition.

Definition 4.5.1. *The* **$\mathbb{Q}$-rank** *(resp.* **$\mathbb{R}$-rank***) of an algebraic group H defined over $\mathbb{Q}$ is the dimension of the maximal $\mathbb{Q}$-split torus in H (resp. of the maximal $\mathbb{R}$-split torus in H), and is denoted by $\mathrm{rk}_{\mathbb{Q}}H$ (resp. by $\mathrm{rk}_{\mathbb{R}}H$).*

Theorem 4.5.2. *The followings are equivalent:*

(i) $\Gamma\backslash X$ is compact for any arithmetic subgroup Γ of $\mathbf{G}^{\mathrm{der}}$;

(ii) $\mathrm{rk}_{\mathbb{Q}}\mathbf{G}^{\mathrm{der}} = 0$;

(iii) $\mathbf{G}^{\mathrm{der}}$ does not contain proper parabolic subgroups.

The equivalence of (ii) and (iii) follows immediately from Theorem 4.2.5, and can be read off from the relative root system construction of parabolic subgroups.

Thus to discuss on compactifications of $\Gamma\backslash X$, we may assume $\mathrm{rk}_{\mathbb{Q}}\mathbf{G}^{\mathrm{der}} \geq 1$ and that $\mathbf{G}^{\mathrm{der}}$ contains proper parabolic subgroups. In this section, we discuss about the horospherical decomposition and Siegel sets associated with each proper parabolic subgroup $\mathbf{P}$.

4.5.1 Horospherical decompositions over $\mathbb{R}$

Let P be a parabolic subgroup of G. We start with the discussion for standard parabolic subgroups, for which we need to fix a maximal $\mathbb{R}$-split torus and a minimal parabolic subgroup of G. The general case will be reduced to the standard case by Theorem 4.4.5.

Fix $x_0 \in X$. Then (SV3) gives a Cartan involution θ of G which induces the Cartan decomposition (4.6.1)

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{m}.$$

Let $K_\infty := \exp(\mathfrak{k})$ which is a maximal compact subgroup of $G(\mathbb{R})^+$; see Lemma 2.3.2. Let $\mathfrak{a}$ to be a maximal Lie subalgebra contained in $\mathfrak{m}$.

Theorem 4.5.3. *There exists a maximal $\mathbb{R}$-split torus A in G such that $\mathrm{Lie}A = \mathfrak{a}$.*

Proof. First $\mathfrak{a}$ is abelian since $[\mathfrak{a}, \mathfrak{a}] \subseteq \mathfrak{a} \cap [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{m} \cap \mathfrak{k} = 0$. Hence $\exp\colon \mathfrak{a} \to \exp(\mathfrak{a})$ is an isomorphism as Lie groups, and thus $\exp(\mathfrak{a}) \simeq (\mathbb{R}_{>0})^r \times \mathbb{R}^s$ (as Lie groups) for some $r, s \geq 0$. This gives rise to an $\mathbb{R}$-algebraic subgroup A_0 of G with $A_0(\mathbb{R})^+ = \exp(\mathfrak{a})$; indeed, $A_0(\mathbb{R}) \simeq (\mathbb{R}^\times)^r \times \mathbb{R}^s$.

We claim that $s = 0$. Indeed, for $\mathfrak{g}_{\mathbb{R},c} := \mathfrak{k} \oplus \sqrt{-1}\mathfrak{m}$, we know that $\exp(\mathfrak{g}_{\mathbb{R},c})$ is a compact Lie group, and hence $\exp(\sqrt{-1}\mathfrak{a}) \simeq \mathbb{T}^r \times \mathbb{R}^s$ (with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$) is compact, and hence $s = 0$.

Thus A_0 is an $\mathbb{R}$-split torus in G. It is contained in a maximal torus T of G defined over $\mathbb{R}$, and hence $T = A \cdot A'$ for some algebraic torus A' defined over $\mathbb{R}$. Then $\mathrm{Lie}A' \cap \mathfrak{m} = 0$ by the maximality of $\mathfrak{a}$ in $\mathfrak{m}$. One can choose A' such that $\mathrm{Lie}A' \subseteq \mathfrak{k}$, and then $A'(\mathbb{R}) < K_\infty$ which is compact. Hence A' has no $\mathbb{R}$-split factor; otherwise $\mathbb{R}^\times$ is a closed subset in $A'(\mathbb{R})$, contradiction to $A'(\mathbb{R})$ being compact. This finishes the proof. $\square$

Thus we have the relative root system ${}_{\mathbb{R}}\Phi := \Phi(A, G)$ as below (4.4.1). Assign a subset ${}_{\mathbb{R}}\Phi^+$ of positive roots in ${}_{\mathbb{R}}\Phi$ as below Lemma 4.3.7. It defines a basis ${}_{\mathbb{R}}\Delta$ of ${}_{\mathbb{R}}\Phi$ (as below Lemma 4.3.7) and a minimal parabolic subgroup P_0 of G (as below (4.4.2)).

Remark 4.5.4. *An alternative approach to study the theory over $\mathbb{R}$ is to use Cartan's theory of symmetric spaces and the restricted root system (to $\mathbb{R}$). We shall not take this point of view in our course to have a uniform treatment over $\mathbb{R}$ and over $\mathbb{Q}$, but only point out that these two points of view are equivalent for our study by the the following easy observation: $\mathfrak{g}^A \cap \mathfrak{m} = \mathfrak{a}$.*

Standard parabolic subgroups. Any parabolic subgroup P of G which contains P_0 is of the form P_I for some subset $I \subseteq {}_{\mathbb{R}}\Delta$, where P_I is defined in Theorem 4.4.6. Now P_I has unipotent radical N_I and a Levi subgroup $L_I = Z_G(A_I)$. Moreover, $A < Z(L_I)$ since $A_I < A$. It is not hard to construct a θ-stable subgroup M_I of L_I such that $L_I = A \times M_I$ (inner direct product).[7] Then we have the following *real Langlands decomposition based at $x_0 \in X^+$*

$$P_I(\mathbb{R})^+ = N_I(\mathbb{R})A_I(\mathbb{R})^+M_I(\mathbb{R}) \simeq N_I(\mathbb{R}) \times A_I(\mathbb{R})^+ \times M_I(\mathbb{R}) \tag{4.5.2}$$

where the first equality is as groups, and the second isomorphism is in the category of real algebraic manifolds (the inverse map is $(n, a, m) \mapsto nam$).

We have more. The reductive subgroup M_I is θ-stable, and thus $K_{I,\infty} := M_I \cap K_\infty$ is maximal compact in $M_I(\mathbb{R})^+$. So

$$X_I := M_I(\mathbb{R})^+/K_{I,\infty} = P_I(\mathbb{R})^+/K_{I,\infty}A_I(\mathbb{R})^+N_I(\mathbb{R}) \tag{4.5.3}$$

is a symmetric space, called the *boundary symmetric space* associated with P_I. Notice however X_I may not admit an $M_I(\mathbb{R})^+$-invariant complex structure.

Lemma 4.5.5. *$P_I(\mathbb{R})^+$ acts transitively on X.*

Proof. It is not hard to check that $\mathfrak{g}_{\mathbb{R}} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$ with $\mathfrak{n}$ from (4.4.2). Thus $G = NAK_\infty$, which is called the *Iwasawa decomposition* of G. On the other hand, $\mathfrak{n} \oplus \mathfrak{a} \subseteq \mathrm{Lie}P_I$ by construction of P_I. Hence $NA \subseteq P_I$ and we are done. □

Thus $X = P_I(\mathbb{R})^+x_0$, and by (4.5.2) and (4.5.3) (and $\mathfrak{n}_I \cap \mathfrak{k} = 0$) we then have the following *real horospherical decomposition based at $x_0 \in X$*

$$X \simeq N_I(\mathbb{R}) \times A_I(\mathbb{R})^+ \times X_I \tag{4.5.4}$$

where the isomorphism is in the category of real algebraic manifolds.

General parabolic subgroups. Now let P be an arbitrary parabolic subgroup of G. By Theorem 4.4.5, P is conjugate to a unique standard parabolic subgroup P_I for some $I \subseteq {}_{\mathbb{R}}\Delta$. But $G = NAK_\infty$ and $NA \subseteq P_I$. So there exists $k \in K_\infty$ such that $P = kP_Ik^{-1}$. Define

$$N_P := kN_Ik^{-1} = \mathcal{R}_u(P), \quad A_P := kA_Ik^{-1}, \quad M_P := kM_Ik^{-1}.$$

Then both A_P and M_P are θ-stable, and $L_P := A_PM_P$ is a Levi subgroup of P, and A_P is an $\mathbb{R}$-split torus in P. We have the *real Langlands decomposition (based at $x_0 \in X$)*

$$P(\mathbb{R})^+ = N_P(\mathbb{R})A_P(\mathbb{R})^+M_P(\mathbb{R}) \simeq N_P(\mathbb{R}) \times A_P(\mathbb{R})^+ \times M_P(\mathbb{R}) \tag{4.5.5}$$

which induces *real horospherical decomposition (based at $x_0 \in X$)*

$$X \simeq N_P(\mathbb{R}) \times A_P(\mathbb{R})^+ \times X_P \tag{4.5.6}$$

with $X_P := M_P(\mathbb{R})/(M_P \cap K_\infty)$ called the *boundary symmetric space associated with P*.

[7]One can construct using Lie algebras: $\mathrm{Lie}M_I$ is the direct sum of $\mathrm{Lie}Z_G(A) \cap \mathfrak{k}$, $\sum_{\alpha' \in {}_k\Phi_I} \mathfrak{g}_{(\alpha')}$ and the (orthogonal) complement of $\mathrm{Lie}A_I$ in $\mathfrak{a}$.

4.5.2 Horospherical decompositions over $\mathbb{Q}$

Before the discussion over $\mathbb{Q}$, let us state the following result.

Let $\mathbf{A}$ be a maximal $\mathbb{Q}$-split torus in $\mathbf{G}^{\mathrm{der}}$. Then we obtain a relative root system ${}_{\mathbb{Q}}\Phi := \Phi(\mathbf{A}, \mathbf{G}^{\mathrm{der}})$ as below (4.4.1). Assign a subset ${}_{\mathbb{Q}}\Phi^+$ of positive roots and thus get a basis ${}_{\mathbb{Q}}\Delta$ of ${}_{\mathbb{Q}}\Phi$ as below Lemma 4.3.7. Then we get a minimal parabolic subgroup $\mathbf{P}_0$ of $\mathbf{G}^{\mathrm{der}}$ as below (4.4.2). All standard parabolic subgroups (*i.e.* those containing $\mathbf{P}_0$) are of the form $\mathbf{P}_I$ for some $I \subseteq {}_{\mathbb{Q}}\Delta$, and every parabolic subgroup $\mathbf{P}$ of $\mathbf{G}^{\mathrm{der}}$ is conjugate to a unique $\mathbf{P}_I$ by some element in $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$; see Theorem 4.4.6. We have the unipotent radical $\mathbf{N}_I$ of $\mathbf{P}_I$ and a Levi subgroup $\mathbf{L}_I = Z_{\mathbf{G}^{\mathrm{der}}}(\mathbf{A}_I)$ with $\mathbf{A}_I := \left(\bigcap_{\alpha' \in {}_{\mathbb{Q}}\Phi_I} \mathrm{Ker}\alpha'\right)^{\circ} < \mathbf{A}$. Moreover, $\mathbf{A}_I$ is the maximal $\mathbb{Q}$-split torus in $Z(\mathbf{L}_I)$. Notice that for $P_I = \mathbf{P}_{I,\mathbb{R}}$, our $\mathbf{A}_{I,\mathbb{R}}$ is a subgroup of the A_I constructed in the real case (which is the maximal $\mathbb{R}$-split torus in $Z(\mathbf{L}_{I,\mathbb{R}})$) and is *proper* if $\mathrm{rk}_{\mathbb{Q}}\mathbf{P}_I < \mathrm{rk}_{\mathbb{R}}\mathbf{P}_I$. So we need to further decompose A_I into the product of $\mathbf{A}_{I,\mathbb{R}}$ and an $\mathbb{R}$-algebraic torus $A_I^{\perp}$ whose $\mathbb{Q}$-rank is 0.[8] For this purpose, define $\mathbf{M}_I := \bigcap_{\chi} \mathrm{Ker}\chi^2$ where χ runs over all non-trivial $\mathbf{L}_I \to \mathbb{G}_{\mathrm{m}}$. Then $\mathbf{M}_I$ is a reductive group with $\mathrm{rk}_{\mathbb{Q}} Z(\mathbf{M}_I) = 0$. Then we have $\mathbf{L}_I = \mathbf{A}_I\mathbf{M}_I$ and $A_I = \mathbf{A}_{I,\mathbb{R}} A_I^{\perp}$. Denote by $\Delta(\mathbf{A}_I, \mathbf{P}_I) := {}_{\mathbb{Q}}\Delta \setminus I$.

For an arbitrary parabolic subgroup $\mathbf{P}$ of $\mathbf{G}^{\mathrm{der}}$, we can conjugate $\mathbf{P}$ to be a unique standard parabolic subgroup $\mathbf{P}_I$. Then we obtain the unipotent radical $\mathbf{N}_{\mathbf{P}}$ of $\mathbf{P}$, the Levi subgroup $\mathbf{L}_{\mathbf{P}}$ of $\mathbf{P}$, the maximal $\mathbb{Q}$-split torus $\mathbf{A}_{\mathbf{P}}$ in $Z(\mathbf{L}_{\mathbf{P}})$, and the subgroup $\mathbf{M}_{\mathbf{P}} = \bigcap_{\chi} \mathrm{Ker}\chi^2$ of $\mathbf{L}_{\mathbf{P}}$. Denote by $P := \mathbf{P}_{\mathbb{R}}$, $N_P := \mathbf{N}_{\mathbf{P},\mathbb{R}}$, $L_P := \mathbf{L}_{\mathbf{P},\mathbb{R}}$,

$$A_{\mathbf{P}} := \mathbf{A}_{\mathbf{P},\mathbb{R}}, \qquad M_{\mathbf{P}} := \mathbf{M}_{\mathbf{P},\mathbb{R}}. \tag{4.5.7}$$

Then we are in conformity with the notation in the real case, while $A_{\mathbf{P}}$ is a subgroup of A_P which is proper if $\mathrm{rk}_{\mathbb{Q}}\mathbf{P} < \mathrm{rk}_{\mathbb{R}}\mathbf{P}$. Denote by

$$\Delta(A_{\mathbf{P}}, P) \subseteq X^*(\mathbf{A}_{\mathbf{P}}) \tag{4.5.8}$$

to be the conjugate of ${}_{\mathbb{Q}}\Delta \setminus I$.

Now we have the *rational Langlands decomposition* of $\mathbf{P}$

$$P(\mathbb{R})^+ = N_P(\mathbb{R}) A_{\mathbf{P}}(\mathbb{R})^+ M_{\mathbf{P}}(\mathbb{R}) \simeq N_P(\mathbb{R}) \times A_{\mathbf{P}}(\mathbb{R})^+ \times M_{\mathbf{P}}(\mathbb{R}) \tag{4.5.9}$$

where the second isomorphism is in the category of real algebraic manifolds.

To get the rational horospherical decomposition, we need to fix a point $x_0 \in X$ and the associated Cartan involution θ on G, and require $A_{\mathbf{P}}$ and $M_{\mathbf{P}}$ to be θ-stable. To achieve this, we can work with the Levi quotient $\mathbf{P}/\mathbf{N}_{\mathbf{P}}$ instead of working with the Levi subgroup $\mathbf{L}_{\mathbf{P}}$ of $\mathbf{P}$, and then lift the resulting $\mathbf{A}_{\mathbf{P}}$ and $\mathbf{M}_{\mathbf{P}}$ to the $\mathbb{R}$-Levi subgroup L_P of P (the one constructed in the real case) which is θ-stable. The resulting groups may not be $\mathbb{Q}$-subgroups of $\mathbf{P}$, but this is enough for our purpose.

Remark 4.5.6. *In fact, it is known that for any $\mathbf{P}$, there exists a base point $x_1 \in X$ such that they are still defined over $\mathbb{Q}$.*

Let $K_\infty := \mathrm{Stab}_{G(\mathbb{R})^+}(x_0)$. Then $M_{\mathbf{P}} \cap K_\infty$ is maximal compact in $M_{\mathbf{P}}(\mathbb{R})^+$ by the θ-stability of $M_{\mathbf{P}}$. Now (4.5.5) induces the *rational horospherical decomposition* of $X = P(\mathbb{R})^+ x_0$

$$X \simeq N_P(\mathbb{R}) \times A_{\mathbf{P}}(\mathbb{R})^+ \times X_{\mathbf{P}} \tag{4.5.10}$$

[8] That is, there is no non-trivial subgroup of $A_I^{\perp} < G = \mathbf{G}_{\mathbb{R}}^{\mathrm{der}}$ which is defined over $\mathbb{Q}$.

with $X_{\mathbf{P}} = M_{\mathbf{P}}(\mathbb{R})^+/(M_{\mathbf{P}} \cap K_\infty)$ called the *boundary symmetric space associated with* $\mathbf{P}$. Moreover, let $A_{\mathbf{P}}^{\perp}$ be the orthogonal complement of $A_{\mathbf{P},\mathbb{R}}$ in A_P, *i.e.* $A_{\mathbf{P}}^{\perp}$ is θ-stable with $A_P(\mathbb{R})^+ = \mathbf{A}_{\mathbf{P}}(\mathbb{R})^+ \times A_{\mathbf{P}}^{\perp}(\mathbb{R})^+$. Then

$$X_{\mathbf{P}} \simeq X_P \times A_{\mathbf{P}}^{\perp}(\mathbb{R})^+, \qquad A_P(\mathbb{R})^+ = A_{\mathbf{P}}(\mathbb{R})^+ \times A_{\mathbf{P}}^{\perp}(\mathbb{R})^+. \tag{4.5.11}$$

While $A_{\mathbf{P}}^{\perp}$ has $\mathbb{Q}$-rank 0, taking the quotient by Γ will roll up the fact $A_{\mathbf{P}}^{\perp}(\mathbb{R})^+$ into circles and hence does not interfere with the compactification of $\Gamma\backslash X$.

4.5.3 Siegel sets

Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}^{\mathrm{der}}$. Continue to use the notation in the previous subsections. For $t > 0$, define

$$A_{\mathbf{P},t} := \{a \in A_{\mathbf{P}}(\mathbb{R})^+ : \alpha'(a) \geq t^{-1} \text{ for all } \alpha' \in \Delta(A_{\mathbf{P}}, P)\} \tag{4.5.12}$$

with $\Delta(A_{\mathbf{P}}, P)$ defined by (4.5.8).

Definition 4.5.7. *For any bounded sets $U \subseteq N_P(\mathbb{R})$ and $V \subseteq X_{\mathbf{P}}$, the subset*

$$S_{\mathbf{P},U,V,t} := U \times A_{\mathbf{P},t} \times V \subseteq N_P(\mathbb{R}) \times A_{\mathbf{P}}(\mathbb{R})^+ \times X_{\mathbf{P}} \simeq X$$

is called a **Siegel set** *in X associated with* $\mathbf{P}$.

4.6 Analytic boundary components

In this section focus on our discussion over $\mathbb{R}$ instead of $\mathbb{Q}$. The discussion over $\mathbb{Q}$ will be executed in Chapter 6.

Let (G, X^+) be a pair as in §2.3, *i.e.* G is a reductive group defined over $\mathbb{R}$ and X^+ is a $G(\mathbb{R})^+$-orbit contained in $\mathrm{Hom}(\mathbb{S}, G)$ satisfying conditions (i) and (ii) at the beginning of §2.3.[9]

To ease notation, we will replace G by G^{der}, so that G is from now on a semi-simple algebraic group defined over $\mathbb{R}$. We will also use X to denote X^+. We have shown that X is a Hermitian symmetric domain; see Theorem 2.3.1.

Denote by $\mathfrak{g} = \mathrm{Lie}G$.

It is known that under holomorphic isometry, X is isomorphic to an open bounded subset $\mathcal{D}$ in the affine space $\mathbb{C}^N$ where $N = \dim_{\mathbb{C}} X$; we shall review this *Harish–Chandra realization* later on at the end of §4.6.1. Let $\overline{\mathcal{D}}$ be the closure of $\mathcal{D}$ in $\mathbb{C}^N$ under the usual topology (we sometimes denote it by $\overline{X}^{\mathrm{BB}}$).

Definition 4.6.1. *An* **analytic boundary component** *of X is an equivalence class in $\overline{\mathcal{D}}$ under the equivalence relation generated by $x \sim y$ if there exists a holomorphic map $\rho\colon \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{C}^N$ such that $x, y \in \mathrm{Im}(\rho) \subseteq \overline{\mathcal{D}}$.*

Notice that $\mathcal{D}$ is a boundary component of X by definition. This definition generalizes the case where X is the upper half plane, in view of the last sentence of §4.1.1.

It is clear that $\overline{\mathcal{D}}$ is the disjoint union of its analytic boundary components. We shall prove:

[9]For our purpose, (G, X^+) are obtained as follows. Let $(\mathbf{G}, X)$ be a Shimura datum. Decompose $\mathbf{G}_{\mathbb{R}} = Z(\mathbf{G})_{\mathbb{R}} G_1 \cdots G_k$ for simple $\mathbb{R}$-groups $G_1, \ldots, G_k$, which induces a decomposition $X^+ \simeq X_1^+ \times \cdots \times X_k^+$. Then we can take our G to be $Z(\mathbf{G})_{\mathbb{R}} \prod_{j \in J} G_j$ and our new X^+ to be $\prod_{j \in J} X_j^+$, for any subset $J \subseteq \{1, \ldots, k\}$.

Theorem 4.6.2. *The action of $G(\mathbb{R})^+$ on $X \simeq \mathcal{D}$ extends to $\overline{\mathcal{D}}$. For any analytic boundary component F of X, its normalizer*

$$N(F) := \{g \in G(\mathbb{R})^+ : gF = F\}$$

is a parabolic subgroup of the Lie group $G(\mathbb{R})^+$, which means that it equals $P_F(\mathbb{R}) \cap G(\mathbb{R})^+$ for a parabolic subgroup P_F of G. If $F \neq X$, then P_F is proper.

Morever if X is irreducible as a Hermitian symmetric domain, then P_F is maximal proper parabolic in G. And the association $F \mapsto P_F$ defines a bijection between the set of analytic boundary components of X and the set of proper maximal parabolic subgroups of G.

In the statement, X is irreducible as a Hermitian symmetric domain if and only if X cannot be written as the product of two non-trivial Hermitian symmetric domains, or equivalently G is a simple group. [10]

In fact, we will prove a more precise version of Theorem 4.6.2, describing how P_F is constructed in terms of the root systems. Moreover, we will prove that the analytic boundary component F can be identified with the boundary symmetric space (defined below the real horospherical decomposition (4.5.6)) associated with some parabolic subgroup P'_F, and explain the relation of P_F and P'_F.

Before moving on, let us make the following very brief discussion over $\mathbb{Q}$; more details will be given in Chapter 6. Let us temporarily go back to our pair (G, X^+) at the beginning of this section (*i.e.* the pair from §2.3).

Definition 4.6.3. *Assume there exists a Shimura datum $(\mathbf{G}, X)$ such that $G = \mathbf{G}_{\mathbb{R}}$ and that X^+ is a connected component of X. Then an analytic boundary component F of X^+ is called* **rational** *if the parabolic subgroup P_F is defined over $\mathbb{Q}$.*

As hinted by Theorem 4.5.2, only the rational analytic boundary components should account for the compactification of $\Gamma\backslash X$. We will focus on the discussion of any analytic boundary component in this section, while in the end give a characterization of which ones are rational.

4.6.1 Complex structure on X and the Harish–Chandra realization

Take $x_0 \in X$ which corresponds to $h_0 \colon \mathbb{S} \to G$, and let $\theta = h_0(\sqrt{-1})$ be the Cartan involution on G given by condition (ii) at the beginning of §2.3. We thus have the Cartan decomposition (2.3.2)

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \tag{4.6.1}$$

with $\mathfrak{k}$ (resp. $\mathfrak{m}$) be the eigenspace for 1 (resp. for -1). Notice that $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$, and $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}$ by looking at the eigenvalues.

Then $K_\infty := \exp(\mathfrak{k})$ is a maximal compact subgroup of $G(\mathbb{R})^+$ by Lemma 2.3.2, and the real tangent space of X at x_0, denoted by $T_{\mathbb{R}}X$, can be identified as $\mathfrak{m}$.

The element $J := h_0(e^{\pi\sqrt{-1}/4})$ satisfies $J^2 = 1$. The action of J on X induces a decomposition of $\mathfrak{m}_{\mathbb{C}} = T_{\mathbb{R}}X \otimes_{\mathbb{R}} \mathbb{C}$

$$\mathfrak{m}_{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{m}^- \tag{4.6.2}$$

where J acts by multiplication by $\sqrt{-1}$ on $\mathfrak{m}^+$ and by $-\sqrt{-1}$ on $\mathfrak{m}^-$. Thus the holomorphic tangent space of X at x_0 can be identified with $\mathfrak{m}^+$. Moreover, as J acts on $T_{\mathbb{R}}X = \mathfrak{m}$, we have $J \in \exp(\mathfrak{k}) = K_\infty$, and thus $J \in Z(K_\infty)$.

[10]In general, $X \simeq X_1 \times \cdots \times X_n$ decomposes into the product of irreducible Hermitian symmetric domains, and analytic boundary components of X are precisely the products of the analytic boundary components of the X_j's. Then one can also obtain a description of the bijective association $F \mapsto P_F$.

Let us recall the *Harish–Chandra realization/embedding* $\mathcal{D}$ of X in Theorem 2.3.5. We only need a brief version: We can identify X with an open bounded subset $\mathcal{D}$ of $\mathfrak{m}^+$. The identification $X \simeq \mathcal{D}$ is called the *Harish–Chandra realization* and the inclusion $\mathcal{D} \subseteq \mathfrak{m}^+$ is called the *Harish–Chandra embedding*. Moreover, it is known that there exists an open holomorphic map $\mathfrak{m}^+ \to X^\vee$ which embeds $\mathfrak{m}^+$ as an open subset (in the usual topology) of the complex algebraic variety $X^\vee$. So we can summarize into:

$$X \simeq \mathcal{D} \subseteq \mathfrak{m}^+ \subseteq X^\vee. \tag{4.6.3}$$

Example 4.6.4. *In the Siegel case $(\mathbf{GSp}_{2d}, \mathfrak{H}_d^\pm)$ and the base point $x_0 = \sqrt{-1}I_{2d}$, we have $K_\infty = U(d) = O(2d) \cap \mathrm{Sp}_{2d}$ (and $G = \mathrm{Sp}_{2d}$). In this case, $\mathfrak{m}^+ \simeq \{\tau \in \mathrm{Mat}_{d\times d}(\mathbb{C}) : \tau = \tau^{\mathrm{t}}\}$, and the Harish–Chandra realization is $\mathcal{D}_d := \{Z \in \mathrm{Mat}_{d\times d}(\mathbb{C}) : Z = Z^{\mathrm{t}}$ and $I_d - Z\overline{Z} > 0\}$ as in Example 2.3.6.*

4.6.2 Complex roots and the Polydisc Theorem

Let T be a maximal torus of G contained in K_∞. Consider the root system $\Phi := \Phi(T, G_{\mathbb{C}})$. We have the root decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$$

with each $\mathfrak{g}_\alpha$ having dimension 1.

We say that a root α is *compact* (resp. *non-compact*) if $\mathfrak{g}_\alpha \subseteq \mathfrak{k}_{\mathbb{C}}$ (resp. if $\mathfrak{g}_\alpha \subseteq \mathfrak{m}_{\mathbb{C}}$). Let Φ_K be the set of compact roots and Φ_M be the set of non-compact roots. One can check that $\Phi = \Phi_K \cup \Phi_M$.

Lemma 4.6.5. *There exists a choice of positive roots Φ^+ such that*

$$\mathfrak{m}^+ = \bigoplus\nolimits_{\alpha\in\Phi^+\cap\Phi_M} \mathfrak{g}_\alpha.$$

The proof uses the complex structure on X, or more precisely the action of J on $\mathfrak{m}_{\mathbb{C}}$. One can show that $J\mathfrak{g}_\alpha = \mathfrak{g}_\alpha$ for any non-compact root α.

Definition 4.6.6. *Two roots $\alpha, \beta \in \Phi$ are called* **strongly orthogonal** *if $\alpha \pm \beta$ are not roots.*

From now on, we fix a maximal subset of strongly orthogonal roots in $\Phi^+ \cap \Phi_M$, maximal under inclusions

$$\Psi = \{\alpha_1, \ldots, \alpha_r\}. \tag{4.6.4}$$

This can be done be choosing successively the lowest positive root.

For each $\alpha \in \Psi$, choose a non-zero $e_\alpha \in \mathfrak{g}_\alpha$ and set $e_{-\alpha}$ to be the complex conjugation on $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ of e_α. Then $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$, and $h_\alpha := \sqrt{-1}[e_\alpha, e_{-\alpha}] \in \mathfrak{t} \subseteq \mathfrak{k}$ and is non-zero. Set

$$\mathfrak{g}_{\mathbb{C}}[\alpha] := \mathbb{C}h_\alpha + \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} = \mathbb{C}h_\alpha + \mathbb{C}e_\alpha + \mathbb{C}e_{-\alpha}. \tag{4.6.5}$$

Then $\mathfrak{g}_{\mathbb{C}}[\alpha] \simeq \mathfrak{sl}_{2,\mathbb{C}}$, since $[h_\alpha, e_\alpha] = \alpha(\exp(h_\alpha))e_\alpha$ and $[h_\alpha, e_{-\alpha}] = -\alpha(\exp(h_\alpha))e_{-\alpha}$ are both non-zero. Hence

$$\mathfrak{g}[\alpha] := \mathfrak{g}_{\mathbb{C}}[\alpha] \cap \mathfrak{g} = \mathbb{R}h_\alpha + \mathbb{R}x_\alpha + \mathbb{R}y_\alpha \simeq \mathfrak{sl}_{2,\mathbb{R}}$$

where $x_\alpha := e_\alpha + e_{-\alpha}$ and $y_\alpha := \sqrt{-1}(e_\alpha - e_{-\alpha})$. Notice that $Jx_\alpha = y_\alpha$ and $Jy_\alpha = -x_\alpha$ by Lemma 4.6.5, and $\{x_\alpha, y_\alpha : \alpha \in \Phi^+ \cap \Phi_M\}$ is an $\mathbb{R}$-basis of $\mathfrak{m}$.

For each $\alpha \in \Psi$, let $G[\alpha]$ be the subgroup of G such that $\mathrm{Lie}G[\alpha] = \mathfrak{g}[\alpha]$. Let $G[\Psi]$ be the subgroup of G with $\mathrm{Lie}G[\Psi] = \sum_{\alpha\in\Psi} \mathfrak{g}[\alpha]$.

Theorem 4.6.7 (Polydisc Theorem). *The orbit $G[\Psi](\mathbb{R})^+x_0 \subseteq X$ is a totally geodesic submanifold which is isomorphic to a Poincaré polydisc D^r, and $X = \bigcup_{k\in K_\infty} k \cdot D^r$.*[11]

Recall that $\mathfrak{g}[\alpha] \simeq \mathfrak{sl}_{2,\mathbb{R}}$ for all $\alpha \in \Psi$. Hence the inclusion $G[\Psi](\mathbb{R})^+x_0 \subseteq X$ is induced by a morphism

$$\varphi\colon \mathrm{SL}_2(\mathbb{R})^r \to G(\mathbb{R}). \tag{4.6.6}$$

By general theory on holomorphic maps between bounded symmetric domains, φ is the second factor of a morphism $\widetilde{\varphi}\colon U(1)\times \mathrm{SL}_2(\mathbb{R})^r \to G(\mathbb{R})$ satisfying: $\left(e^{\sqrt{-1}\theta}, \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \ldots, \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\right) \mapsto h_0(e^{\sqrt{-1}\theta})$.

The Polydisc Theorem is a key step in the proof of the Harish–Chandra embedding. To study boundary components, we need to have a finer statement. Let $S \subseteq \{1,\ldots,r\}$ be a subset, and let $G[S]$ be the subgroup of G with $\mathrm{Lie}G[S] = \sum_{j\in S} \mathfrak{g}[\alpha_j]$; in particular $G[S] = G[\Psi]$ for $S = \{1,\ldots,r\}$. Then the orbit $G[S](\mathbb{R})^+x_0 \subseteq X$ is still totally geodesic in X and is isomorphic to $D^{|S|}$. We have the following compatibility:

Theorem 4.6.8. *For each $j \in \{1,\ldots,r\}$, the image of $G[j](\mathbb{R})^+x_0 \subseteq X$ under the Harish–Chandra embedding is the open unit disc D_j in $\mathbb{C}e_{\alpha_j} \subseteq \mathfrak{m}^+$ (with $1 \in D_j$ corresponding to e_{α_j}). The image of $G[S](\mathbb{R})^+x_0 \subseteq X$ under the Harish–Chandra embedding is the open unit polydisc $\prod_{j\in S} D_j$ in $\prod_{j\in S} \mathbb{C}e_{\alpha_j} \subseteq \mathfrak{m}^+$.*

We finish this subsection by the example of the Siegel case.

Example 4.6.9. *In the Siegel case $(\mathbf{GSp}_{2d}, \mathfrak{H}_d^\pm)$ and the base point $x_0 = \sqrt{-1}I_{2d}$, we have $K_\infty = U(d) = O(2d) \cap \mathrm{Sp}_{2d}$ (and $G = \mathrm{Sp}_{2d}$). Our maximal torus is not the usual one, but is*

$$T = \left\{ \mathrm{bdiag}(t_1,\ldots,t_d) := \begin{bmatrix} \cos t_1 & & & \sin t_1 & & \\ & \ddots & & & \ddots & \\ & & \cos t_d & & & \sin t_d \\ -\sin t_1 & & & \cos t_1 & & \\ & \ddots & & & \ddots & \\ & & -\sin t_d & & & \cos t_d \end{bmatrix} : t_1,\ldots,t_d \in \mathbb{R} \right\}.$$

Let $\lambda_j \in X^*(T)$ be $\mathrm{bdiag}(t_1,\ldots,t_d) \mapsto e^{\sqrt{-1}t_j}$. Then $\Phi = \{\pm\sqrt{-1}(\lambda_i+\lambda_j) : 1 \le i \le j \le d\} \cup \{\pm\sqrt{-1}(\lambda_i-\lambda_j) : 1 \le i < j \le d\}$ and $\Phi_M \cap \Phi^+ = \{\sqrt{-1}(\lambda_i+\lambda_j) : 1 \le i \le j \le d\}$. The basis for this choice of Φ^+ is $\{\sqrt{-1}(\lambda_i-\lambda_{i+1}) : 1 \le i \le d-1\} \cup \{2\sqrt{-1}\lambda_d\}$.

The set Ψ is $\{\alpha_j := 2\sqrt{-1}\lambda_j : 1 \le j \le d\}$ (so $r = d$). Then the corresponding normalized e_{α_j}, h_{α_j}, x_{α_j}, y_{α_j} are:

$$\frac{1}{2}\begin{bmatrix} 1_{j,j} & \sqrt{-1}_{d+j,j} \\ \sqrt{-1}_{j,d+j} & -1_{g+j,g+j} \end{bmatrix}, \begin{bmatrix} 0 & -\sqrt{-1}_{d+j,j} \\ \sqrt{-1}_{j,d+j} & 0 \end{bmatrix}, \begin{bmatrix} 1_{j,j} & 0 \\ 0 & -1_{g+j,g+j} \end{bmatrix}, \begin{bmatrix} 0 & -1_{d+j,j} \\ -1_{j,d+j} & 0 \end{bmatrix}.$$

Here for a number s, we use $s_{i,j}$ to denote the matrix with the (i,j)-entry being s and all the rest being 0.

[11]The Poincaré unit disc D is $\{z \in \mathbb{C} : |z| < 1\}$ endowed with the hyperbolic metric, and D^r is the r-copy of D.

The extension $U(1) \times \mathrm{SL}_2(\mathbb{R})^d \to G(\mathbb{R})$ of the morphism φ from (4.6.6) is:

$$\left(u, \begin{bmatrix} a_1 & b_1 \\ c_1 & s_1 \end{bmatrix}, \dots, \begin{bmatrix} a_d & b_d \\ c_d & s_d \end{bmatrix}\right) \mapsto \begin{bmatrix} a_1 & & & b_1 & & \\ & \ddots & & & \ddots & \\ & & a_d & & & b_d \\ c_1 & & & s_1 & & \\ & \ddots & & & \ddots & \\ & & c_d & & & s_d \end{bmatrix}.$$

4.6.3 Real roots and Cayley transformation

Next we need to study a relative root system over $\mathbb{R}$, for which we need to take a maximal $\mathbb{R}$-split torus A in G. Our construction is as follows. Recall $\Psi = \{\alpha_1, \dots, \alpha_r\}$ from (4.6.4) is the maximal subset of strongly orthogonal roots in $\Phi^+ \cap \Phi_M$. By definition of strong orthogonality, the sum

$$\mathfrak{a} := \sum_{\alpha \in \Psi} \mathbb{R} x_\alpha, \tag{4.6.7}$$

with x_α as below (4.6.5), is commutative, and hence is a Lie subalgebra. In fact we have more:

Proposition 4.6.10. *$\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{m}$.*

Thus by Theorem 4.5.3, there exists a maximal $\mathbb{R}$-split tours A in G with $\mathrm{Lie}A = \mathfrak{a}$, and hence we have the relative root system ${}_{\mathbb{R}}\Phi := \Phi(A, G)$.

Example 4.6.11. *In the Siegel case, A is the standard torus $\{\mathrm{diag}(t_1, \dots, t_d, t_1^{-1}, \dots, t_d^{-1}) : t_1, \dots, t_d \in \mathbb{R}^\times\}$.*

We wish to use the root system Φ constructed in §4.6.2 to study ${}_{\mathbb{R}}\Phi$. For this purpose, we need to conjugate the maximal torus T in §4.6.2, which is contained in K_∞, to a maximal torus which contains A. For this purpose, it suffices to find an abelian Lie subalgebra $\mathfrak{a}'$ in $\mathfrak{t} \subseteq \mathfrak{k}$ which is a conjugate of $\mathfrak{a}$. This is the *Cayley transformation* which we introduce now.

Let $h_\alpha \in \mathfrak{t} \subseteq \mathfrak{k}$ be as above (4.6.5). Define

$$\mathfrak{a}' := \sum_{\alpha \in \Psi} \mathbb{R} h_\alpha \subseteq \mathfrak{t}. \tag{4.6.8}$$

For each $\alpha \in \Psi$, set $C_\alpha := \exp(\pi\sqrt{-1} y_\alpha / 4) \in G(\mathbb{C})$.[12] Then $\mathrm{Ad}(C_\alpha) h_\alpha = [\pi\sqrt{-1} y_\alpha/4, h_\alpha] \in \mathbb{R} x_\alpha \subseteq \mathfrak{a}$. The *Cayley transformation* is defined to be:

$$\mathrm{Ad}(C_\Psi) \colon \mathfrak{a}' \xrightarrow{\sim} \mathfrak{a}, \qquad \text{with } C_\Psi = \textstyle\prod_{\alpha \in \Psi} C_\alpha. \tag{4.6.9}$$

In terms of the morphism $\varphi \colon \mathrm{SL}_2(\mathbb{R})^r \to G(\mathbb{R})$ from (4.6.6) based changed to $\mathbb{C}$,

$$C_\Psi = \varphi\left(\cdots, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & 1 \end{bmatrix}, \cdots\right). \tag{4.6.10}$$

Now, $\mathfrak{a}'$ gives rise to an $\mathbb{R}$-split torus A' with $\mathrm{Lie}A' = \mathfrak{a}'$, and the relative root system $\Phi(A', G)$ is exactly $\mathrm{Int}(C_\Psi)^* {}_{\mathbb{R}}\Phi$. Since $A' < T$ for the maximal torus T in §4.6.2, we can directly compare

[12]Notice that our y_α is well-defined up to scalar. We usually take a normalization in the definition of e_α and h_α, and hence x_α and y_α. Then the resulting C_Ψ will be as in (4.6.10).

$\Phi = \Phi(T,G)$ and $\mathrm{Int}(C_\Psi)^*{}_{\mathbb{R}}\Phi = \Phi'(A,G)$. More precisely, we can regroup the eigenspace decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ to be:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^{A'} \oplus \sum_{\alpha'\in\Phi'(A,G)} \mathfrak{g}_{\alpha'} \tag{4.6.11}$$

with $\mathfrak{g}_{\mathbb{C}}^{A'} = \mathfrak{a}'_{\mathbb{C}} \oplus \sum_{\beta\sim 0} \mathfrak{g}_\beta$ and $\mathfrak{g}_{\alpha'} = \sum_{\beta\in\Phi,\ \beta\sim\alpha'} \mathfrak{g}_\beta$. Here, the equivalence $\sim$ on Φ is defined by: $\beta_1 \sim \beta_2$ if and only if $\beta_1|_{A'} = \beta_2|_{A'}$. This decomposition is defined over $\mathbb{R}$ because A' is $\mathbb{R}$-split. Applying the Cayley transformation to (4.6.11), we get the eigenspace decomposition

$$\mathfrak{g} = \mathfrak{g}^A \oplus \sum_{\varphi\in_{\mathbb{R}}\Phi} \mathfrak{g}_\varphi \tag{4.6.12}$$

with each $\mathfrak{g}_\varphi$ being a suitable Ad-conjugate of a suitable $\mathfrak{g}_{\alpha'}$ above.

Finally each $\alpha_j \in \Psi$ defines a character $\alpha_j|_{A'} \in X^*(A')$, and hence a character $\gamma_j \in X^*(A)$ via the Cayley transformation. We thus have the following set

$${}_{\mathbb{R}}\Psi := \{\gamma_1, \ldots, \gamma_r\}. \tag{4.6.13}$$

Since $\Psi \subseteq \Phi$, we have ${}_{\mathbb{R}}\Psi \subseteq {}_{\mathbb{R}}\Phi$. In general, we have the following proposition, which is a consequence of the classification of (real) representations of $U(1) \times \mathrm{SL}_2(\mathbb{R})^r$ by analyzing the action of Weyl groups.

Proposition 4.6.12. *Assume X is irreducible as a Hermitian symmetric domain, i.e. X cannot be written as the product of two non-trivial Hermitian symmetric domains (equivalently, G is a simple group). Then one of the following possibilities occurs:*

- (Type C_r) ${}_{\mathbb{R}}\Phi = \{\pm\frac{1}{2}(\gamma_i + \gamma_j)$ *for* $i \geq j$, $\pm\frac{1}{2}(\gamma_i - \gamma_j)$ *for* $i > j\}$.
- (Type BC_r) ${}_{\mathbb{R}}\Phi = \{\pm\frac{1}{2}(\gamma_i + \gamma_j)$ *for* $i \geq j$, $\pm\frac{1}{2}(\gamma_i - \gamma_j)$ *for* $i > j$, $\pm\frac{1}{2}\gamma_i\}$.

If we order the roots such that $\gamma_1 > \ldots > \gamma_r$, then the set of simple roots is:

- (Type C_r) ${}_{\mathbb{R}}\Delta = \{\mu_1 := \frac{1}{2}(\gamma_1 - \gamma_2), \ldots, \mu_{r-1} := \frac{1}{2}(\gamma_{r-1} - \gamma_r), \mu_r := \gamma_r\}$.
- (Type BC_r) ${}_{\mathbb{R}}\Delta = \{\mu_1 := \frac{1}{2}(\gamma_1 - \gamma_2), \ldots, \mu_{r-1} := \frac{1}{2}(\gamma_{r-1} - \gamma_r), \mu_r := \frac{1}{2}\gamma_r\}$.

In each case, the simple root μ_r is called the *distinguished root*, and is the longest (resp. shortest) simple root in Type C_r (resp. Type BC_r).

Example 4.6.13. *In the Siegel case, $\gamma_j \colon A \to \mathbb{R}^\times$ is $\mathrm{diag}(t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}) \mapsto t_j^2$, and we are of Type C_d.*

4.6.4 Standard form of analytic boundary components

Recall the maximal subset of strongly orthogonal roots $\Psi = \{\alpha_1, \ldots, \alpha_r\}$ of Φ, and the induced subset ${}_{\mathbb{R}}\Psi = \{\gamma_1, \ldots, \gamma_r\}$ of ${}_{\mathbb{R}}\Phi$.

For any $S \subseteq \{1, \ldots, r\}$, define the Lie subalgebra

$$\mathfrak{l}_S := \sum_{\substack{\varphi\in_{\mathbb{R}}\Phi \\ \varphi \text{ is a linear combination of } \gamma_j \text{ with } j\notin S}} (\mathfrak{g}_\varphi + [\mathfrak{g}_\varphi, \mathfrak{g}_{-\varphi}]) \tag{4.6.14}$$

of $\mathfrak{g}$, with each $\mathfrak{g}_\varphi$ the eigenspace of φ for the adjoint action of A on $\mathfrak{g}$; see (4.6.12).

Let L_S be the subgroup of G with $\mathrm{Lie}L_S = \mathfrak{l}_S$. Denote by $\mathfrak{m}_S^+ := \mathfrak{m}^+ \cap \mathfrak{l}_S$.

Proposition 4.6.14. *L_S is a semi-simple subgroup of G without compact factors, and*

$$X_S := L_S(\mathbb{R})^+ x_0 \simeq L_S(\mathbb{R})^+ / (L_S(\mathbb{R}) \cap K_\infty)$$

is a sub-Hermitian symmetric domain in X.

Moreover, for the Harish–Chandra realizations $\mathcal{D}$ of X and $\mathcal{D}_S$ of X_S (see (4.6.3)), we have the following equivariant diagram of symmetric holomorphic maps

$$\begin{tikzcd} D^{|S|} \times \mathcal{D}_S \arrow[r, "f_1"] \arrow[d, "\subseteq"'] & \mathcal{D} \arrow[d, "\subseteq"] \\ \mathbb{C}^{|S|} \times \mathfrak{m}_S^+ \arrow[r, "f_2"] \arrow[d, "\subseteq"'] & \mathfrak{m}^+ \arrow[d, "\subseteq"] \\ (\mathbb{P}^1)^{|S|} \times X_S^\vee \arrow[r, "f_3"] & X^\vee \end{tikzcd} \tag{4.6.15}$$

where $D = \{z \in \mathbb{C} : |z| < 1\}$ is the Poincaré unit disc.

Proof. We have the following decomposition of $\mathfrak{l}_{S,\mathbb{C}}$ in terms of the complex roots in Φ by (4.6.11) and (4.6.12):

$$\mathfrak{l}_{S,\mathbb{C}} := \sum_{\substack{\alpha \in \Phi,\ \alpha \not\sim 0 \\ \alpha \sim \sum_{j \notin S} a_j \alpha_j}} (\mathfrak{g}_\alpha + [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]);$$

see below (4.6.11) for the definition of $\sim$. Hence $\mathfrak{l}_{S,\mathbb{C}}$ is stable under $\mathrm{Ad} h_0(e^{\sqrt{-1}\theta})$, and so

(a) $\mathfrak{l}_S = (\mathfrak{k} \cap \mathfrak{l}_S) \oplus (\mathfrak{m} \cap \mathfrak{l}_S)$,

(b) $\mathfrak{m}_\mathbb{C} \cap \mathfrak{l}_{S,\mathbb{C}} = \mathfrak{m}_S^+ \oplus \mathfrak{m}_S^-$ with $\mathfrak{m}_S^- := \mathfrak{m}^- \cap \mathfrak{l}_{S,\mathbb{C}}$.

Hence L_S is a reductive group and X_S is a sub-Hermitian symmetric domain of X. Better, L_S is semi-simple without compact factors because it is generated by unipotent elements; see Theorem 4.3.2.(3.a).

For the "Moreover" part, notice that L_S commutes with (modulo center) the subgroup $\varphi(\prod_{j \in S} \mathrm{SL}_2(\mathbb{R}))$ for the morphism φ from (4.6.6); this is an immediate consequence of the construction of $\mathfrak{l}_S$. Hence we are done. □

We state the following theorem without proof. The proof needs the Hermann convexity theorem.

Theorem 4.6.15. *The analytic boundary components of X defined in Definition 4.6.1 are precisely the sets the form $k \cdot F_S$, where $k \in K_\infty$, $S \subseteq \{1, \ldots, r\}$, and*

$$F_S := f_2\left((1, \ldots, 1) \times \mathcal{D}_S\right) \subseteq \mathfrak{m}^+$$

with f_2 from (4.6.15).

Two other ways to write F_S are given by (4.6.16) and, in terms of partial Cayley transformations, by (4.6.17) respectively.

Example 4.6.16. *In the Siegel case, $r = d$. Take the subset $S = \{d'+1, \ldots, d\} \subseteq \{1, \ldots, d\}$; then $|S| = d - d'$. In this case we have*

$$L_S = \left\{ \begin{bmatrix} A' & 0 & B' & 0 \\ 0 & I_{d-d'} & 0 & 0 \\ C' & 0 & D' & 0 \\ 0 & 0 & 0 & I_{d-d'} \end{bmatrix} : \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \in \mathrm{Sp}_{2d',\mathbb{R}} \right\} \simeq \mathrm{Sp}_{2d',\mathbb{R}},$$

and $X_S \simeq \mathfrak{H}_{d'}$ with Harish–Chandra realization being $\mathcal{D}_{d'}$. Under the natural identifications $\mathfrak{m}^+ \simeq \{Z \in \mathrm{Mat}_{d\times d}(\mathbb{C}) : Z = Z^{\mathrm{t}}\}$ and $\mathfrak{m}_S^+ \simeq \{Z' \in \mathrm{Mat}_{d'\times d'}(\mathbb{C}) : Z' = (Z')^{\mathrm{t}}\}$, the holomorphic map f_2 is

$$\big((a_{d'+1}, \ldots, a_d), Z'\big) \mapsto \mathrm{diag}(Z', a_{d'+1}, \ldots, a_d).$$

Hence in this case, we have

$$F_S = \left\{ \begin{bmatrix} Z' & 0 \\ 0 & I_{d-d'} \end{bmatrix} : Z' \in \mathcal{D}_{d'} \right\}.$$

Before moving on, let us see a corollary of Theorem 4.6.15. The proof presents an application of the construction of F_S in Theorem 4.6.15, and given another way (4.6.16) to write F_S.

Corollary 4.6.17. *An analytic component of an analytic component of X is an analytic component of X.*

Proof. Let $\mathcal{D}_1$ be an analytic component of X. Theorem 4.6.15 implies that $\mathcal{D}_1 = k \cdot F_S$ for some $k \in K_\infty$ and $S \subseteq \{1, \ldots, r\}$. By Theorem 4.6.8, we have

$$F_S = \sum_{j\in S} e_{\alpha_j} + \mathcal{D}_S. \tag{4.6.16}$$

Let $\mathcal{D}_2$ be an analytic component of $\mathcal{D}_1$. Then $\mathcal{D}_2 = k \cdot \left(\sum_{j\in S} e_{\alpha_j} + \mathcal{D}_2'\right)$ for some analytic component $\mathcal{D}_2'$ of $\mathcal{D}_S$. Theorem 4.6.15 implies that $\mathcal{D}_2' = k' \cdot \left(\sum_{i\in S'} e_{\alpha_j} + \mathcal{D}_{S\cup S'}\right)$ for some $k' \in L_S \cap K_\infty$ and $S' \subseteq \{1, \ldots, r\} \setminus S$. So

$$\mathcal{D}_2 = kk' \left(\sum_{j\in S\cup S'} e_{\alpha_j} + \mathcal{D}_{S\cup S'} \right) = kk' \cdot f_2\left((1, \ldots, 1) \times \mathcal{D}_{S\cup S'}\right) = kk' \cdot F_{S\cup S'}$$

which by Theorem 4.6.15 is an analytic component of X. □

4.6.5 Analytic boundary components and maximal parabolic subgroups

From now on, assume that X is irreducible as a Hermitian symmetric domain, *i.e.* X cannot be written as the product of two non-trivial Hermitian symmetric domains. Equivalently, G is a simple group.

Now let $S \subseteq \{1, \ldots, r\}$ be a subset. Recall the set of simple roots ${}_{\mathbb{R}}\Delta = \{\mu_1, \ldots, \mu_r\}$ from Proposition 4.6.12, with μ_r being the distinguished root. Then ${}_{\mathbb{R}}\Delta$ gives rise to a minimal parabolic subgroup P_0 of G as below (4.4.2). Denote by $I_S := \{\mu_j : j \notin S\}$. Then we have a standard parabolic subgroup P_{I_S} of G (*i.e.* a parabolic subgroup containing P_0) defined in Theorem 4.4.6. It has unipotent radical N_{I_S}, maximal $\mathbb{R}$-split torus $A_{I_S} := \left(\bigcap_{j\notin S} \mathrm{Ker}\mu_j\right)^\circ$, and Levi subgroup $L_{I_S} = Z_G(A_{I_S})$.

Proposition 4.6.18. *The analytic boundary component F_S can be identified with the boundary symmetric space $X_{P_{I_S}}$ associated with the parabolic subgroup P_{I_S}.*

See (4.5.3) and below (4.5.6) for the definition of $X_{P_{I_S}}$.

Proof. We shall use the Polydisc Theorem (Theorem 4.6.7) and its refinement Theorem 4.6.8, for which we need to go back to the complex roots.

Recall the maximal subset of strongly orthogonal roots $\Psi = \{\alpha_1, \ldots, \alpha_r\}$ from (4.6.4). Then the polydisc is $G[\Psi](\mathbb{R})^+ x_0 \subseteq X$, with $\mathrm{Lie}G[\Psi] = \sum_{j=1}^r \mathfrak{g}[\alpha_j] \simeq \mathfrak{sl}_{2,\mathbb{R}}^{\oplus r}$. For the maximal abelian Lie subalgebra $\mathfrak{a} = \sum_{j=1}^r \mathbb{R}x_{\alpha_j} \subseteq \mathfrak{m}$ from (4.6.7) and the corresponding maximal $\mathbb{R}$-split torus A, we have $A < G[\Psi]$ since $\mathfrak{a} \subseteq \mathrm{Lie}G[\Psi]$. Hence $A(\mathbb{R})^+ x_0 \subseteq G[\Psi](\mathbb{R})^+ x_0$. A refinement of the Cartan decomposition says that $G = K_\infty A K_\infty$ (more precisely, $\mathfrak{m} = \bigcup_{k \in K_\infty} \mathrm{Ad}(k)(\mathfrak{a})$). Hence $X = \bigcup_{k \in K_\infty} k \cdot A(\mathbb{R})^+ x_0$, which in fact improves the last sentence of Theorem 4.6.7. In view of $G[\Psi](\mathbb{R})^+ x_0 \simeq D^r$, the subset $A(\mathbb{R})^+ x_0$ can be identified with $(-1,1)^r$ and the multiplication by elements in $Z(K_\infty)$ correspond to rotations.

Denote by $S^c := \{1, \ldots, r\} \setminus S$. The discussion generalizes to $S^c \subsetneq \{1, \ldots, r\}$. We have the partial polydisc $G[S^c](\mathbb{R})^+ x_0 \subseteq X$, with $\mathrm{Lie}G[S^c] = \sum_{j \notin S} \mathfrak{g}[\alpha_j]$. For the sub-Hermitian symmetric domain X_S defined in Proposition 4.6.14, it is not hard to see that $G[S^c](\mathbb{R})^+ x_0$ is a maximal polydisc in X_S by Theorem 4.6.8. We also have an abelian Lie subalgebra $\mathfrak{a}_{S^c} := \sum_{j \notin S} \mathbb{R}x_{\alpha_j} \subseteq \mathfrak{m}$, which defines an $\mathbb{R}$-split torus A_{S^c} in $G[S]$. Moreover similar to Proposition 4.6.10, A_{S^c} is a maximal $\mathbb{R}$-split torus in L_S. So $X_S = \bigcup_{k \in L_S \cap K_\infty} k \cdot A_{S^c}(\mathbb{R})^+ x_0$ as in the last paragraph.

Next we turn to the Langlands decomposition $P_{I_S}(\mathbb{R})^+ \simeq N_{I_S}(\mathbb{R}) \times A_{I_S}(\mathbb{R})^+ \times M_{I_S}(\mathbb{R})$ (4.5.2). Both A_{S^c} and A_{I_S} are subgroups of A, so $A_{S^c} < Z_G(A_{I_S}) = L_{I_S} = A_{I_S} M_{I_S}$. But $A_{S^c} \cap A_{I_S} = \{1\}$ by definition, so $A_{S^c} < M_{I_S}$. Thus $A_{S^c}(\mathbb{R})^+ x_0$ is a subset of $X_{I_S} = M_{I_S}(\mathbb{R})^+ / K_{I_S,\infty}$ with $K_{I_S,\infty} := M_{I_S} \cap K_\infty$. Moreover, A_{S^c} is a maximal $\mathbb{R}$-split torus in M_{I_S} by reason of rank, and $A_{S^c}(\mathbb{R}) \cap K_{I_S,\infty} = \{1\}$. So the refined Cartan decomposition for M_{I_S} (recall that $M_{I_S} < L_{I_S}$ is chosen to be invariant under the Cartan involution) implies that $M_{I_S} = K_{I_S,\infty} A_{S^c} K_{I_S,\infty}$ as before. Hence $X_{I_S} = \bigcup_{k \in K_{I_S,\infty}} k \cdot A_{S^c}(\mathbb{R})^+ x_0$.

Now we are done by the conclusions of the previous two paragraphs, since F_S is a suitable translate of X_S. $\square$

In fact we can be more precise on this translate. More precisely we can write the analytic boundary component F_S in terms of X_S and the *partial Cayley transformation* defined as follows. Let $C_{S^c} := \prod_{j \notin S} C_{\alpha_j}$ with C_{α_j} defined above (4.6.9). Recall that C_{α_j} is the image of $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & 1 \end{bmatrix}$ under $\mathrm{SL}_2(\mathbb{C}) \xrightarrow{\iota_j} \mathrm{SL}_2(\mathbb{C})^r \xrightarrow{\varphi} G(\mathbb{C})$, where ι_j is the embedding as the j-th component; see (4.6.10) for the notation and explanation. Moreover on the closed unit disc $\overline{D} = \{z \in \mathbb{C} : |z| \le 1\}$, the matrix $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & 1 \end{bmatrix}$ sends 0 to 1 by the formula above Lemma 4.1.1. So

$$F_S = C_{S^c} \cdot X_S. \tag{4.6.17}$$

Theorem 4.6.19. *Assume $S = \{\ell, \ldots, r\}$. Then $N(F_S) = P_{{}_{\mathbb{R}}\Delta \setminus \{\mu_\ell\}}(\mathbb{R}) \cap G(\mathbb{R})^+$.*

Proof of Theorem 4.6.2 from Theorem 4.6.19. It is easy to reduce Theorem 4.6.2 to the case where X is irreducible. Next by Theorem 4.6.15, we may assume that the analytic boundary component is F_S for some subset $S \subseteq \{1, \ldots, r\}$.

Now S defines a subset $J_S := \{\gamma_j : j \in S\}$ of ${}_{\mathbb{R}}\Psi = \{\gamma_1, \ldots, \gamma_r\}$ defined in (4.6.13). By Proposition 4.6.12, the Weyl group $W({}_{\mathbb{R}}\Phi)$ contains all the signed permutations of the roots in ${}_{\mathbb{R}}\Psi$. So there exists an element $w \in W({}_{\mathbb{R}}\Phi)$ such that $w(J_S) = \{\gamma_\ell, \ldots, \gamma_r\}$ for some ℓ. Thus we may and do assume $S = \{\ell, \ldots, r\}$. Now the conclusion follows immediately from Theorem 4.6.19, because $P_{{}_{\mathbb{R}}\Delta \setminus \{\mu_\ell\}}$ is a maximal proper parabolic subgroup of G. $\square$

Proof of Theorem 4.6.19. We start by showing that

$$P_{\mathbb{R}\Delta\setminus\{\mu_\ell\}}(\mathbb{R}) \cap G(\mathbb{R})^+ \subseteq N(F_S). \tag{4.6.18}$$

Let $g \in P_{\mathbb{R}\Delta\setminus\{\mu_\ell\}}(\mathbb{R}) \cap G(\mathbb{R})^+$. Then g sends F_S to an analytic boundary component, which must be either F_S or disjoint from F_S. By (4.6.17), it suffices to show that $g \cdot (C_{S^c} \cdot x_0) \in C_{S^c}(X_S^\vee)$, which itself is equivalent to $(C_{S^c}^{-1} g C_{S^c}) \cdot x_0 \in X_S^\vee$. Thus it suffices to show

$$C_{S^c}^{-1} g C_{S^c} \in L_{S,\mathbb{C}} K_{\infty,\mathbb{C}} \exp(\mathfrak{m}^-).$$

Now consider $P' := C_{S^c}^{-1}(P_{\mathbb{R}\Delta\setminus\{\mu_\ell\}})_{\mathbb{C}} C_{S^c} = \mathrm{Int}(C_{S^c})^{-1}(P_{\mathbb{R}\Delta\setminus\{\mu_\ell\}})_{\mathbb{C}}$, which is a subgroup of $G_{\mathbb{C}}$. Our goal is to prove that

$$P' < L_{S,\mathbb{C}} K_{\infty,\mathbb{C}} \exp(\mathfrak{m}^-).$$

For this purpose, let us compute $\mathrm{Lie}P'$. We start with writing $\mathrm{Lie}P_{\mathbb{R}\Delta\setminus\{\mu_\ell\}}$ as above and in Theorem 4.4.6. Then using the Cayley transformation $\mathrm{Ad}(C_{S^c})$ and by (4.6.11) and (4.6.12), we have the following decomposition of $\mathrm{Lie}P'$ into weight spaces of roots in $\Phi = \Phi(T, G_{\mathbb{C}})$ (together with $\mathfrak{t}_{\mathbb{C}}$)

$$\mathrm{Lie}P' = \mathfrak{t}_{\mathbb{C}} \oplus \sum_{\substack{\alpha\sim\frac{\pm\alpha_i\pm\alpha_j}{2} \text{ or } \frac{\pm\alpha_i}{2} \\ i,j\notin S}} \mathfrak{g}_\alpha \oplus \sum_{\substack{\alpha\sim\frac{-\alpha_i-\alpha_j}{2} \text{ or } \frac{-\alpha_i}{2} \\ i\in S}} \mathfrak{g}_\alpha \oplus \sum_{\substack{\alpha\sim\frac{-\alpha_i+\alpha_j}{2} \text{ or } \frac{\pm\alpha_i}{2} \\ i,j\notin S}} \oplus \sum_{\alpha\sim 0} \mathfrak{g}_\alpha.$$

Again, the equivalence relation $\sim$ on Φ is defined by: $\beta_1 \sim \beta_2$ if and only if $\beta_1|_{A'} = \beta_2|_{A'}$.

Now $\mathfrak{t} \subseteq \mathfrak{k}$ by choice of T, the second term generates $\mathfrak{l}_{S,\mathbb{C}}$, the third term is in $\mathfrak{m}^-$, and the fourth and fifth terms are in $\mathfrak{k}_{\mathbb{C}}$. Now the first, third, and fourth terms together generate a normal subgroup which is contained in $K_{\infty,\mathbb{C}} \exp(\mathfrak{m}^-)$. Hence the first four terms generate a normal subgroup contained in $L_{S,\mathbb{C}} K_{\infty,\mathbb{C}} \exp(\mathfrak{m}^-)$. Hence $P' < L_{S,\mathbb{C}} K_{\infty,\mathbb{C}} \exp(\mathfrak{m}^-) K_{\infty,\mathbb{C}} = L_{S,\mathbb{C}} K_{\infty,\mathbb{C}} \exp(\mathfrak{m}^-)$. This establishes (4.6.18).

Now that

$$P_{\mathbb{R}\Delta\setminus\{\mu_\ell\}}(\mathbb{R}) \cap G(\mathbb{R})^+ \subseteq N(F_S) \subsetneq G(\mathbb{R})^+$$

and $P_{\mathbb{R}\Delta\setminus\{\mu_\ell\}}$ is a maximal proper parabolic subgroup of G by Theorem 4.4.6, we have $P_{\mathbb{R}\Delta\setminus\{\mu_\ell\}}(\mathbb{R})^+ = N(F_S)^+$. Hence $N(F_S)$ normalizes $\mathrm{Lie}N(F_S) = \mathrm{Lie}P_{\mathbb{R}\Delta\setminus\{\mu_\ell\}}$. Chevalley's Theorem (parabolic subgroups are always connected) says that the normalizer of $\mathrm{Lie}(P_{\mathbb{R}\Delta\setminus\{\mu_\ell\}})_{\mathbb{C}}$ in $G_{\mathbb{C}}$ is precisely $(P_{\mathbb{R}\Delta\setminus\{\mu_\ell\}})_{\mathbb{C}}$. So $N(F_S) \subseteq G(\mathbb{R})^+ \cap (P_{\mathbb{R}\Delta\setminus\{\mu_\ell\}})_{\mathbb{C}} = P_{\mathbb{R}\Delta\setminus\{\mu_\ell\}}(\mathbb{R}) \cap G(\mathbb{R})^+$. Now we are done. □

Now let us return to any X (not necessarily irreducible). Let $F = k \cdot F_S$ be an analytic boundary component of X, and let $N(F) = \{g \in G(\mathbb{R})^+ : gF = F\}$ be its normalizer, which by Theorem 4.6.2 equals $P_F(\mathbb{R}) \cap G(\mathbb{R})^+$ for a parabolic subgroup P_F of G. We have the following subgroups of $P_F = kP_{F_S}k^{-1}$:

- $W(F) := \mathcal{R}_u(P_F)$.
- $L(F)$ which is the Levi subgroup of P_F obtained as follows: Theorem 4.6.19 gives the construction of P_{F_S} in terms of the relative root system, and hence a Levi subgroup $L(F_S)$ of P_{F_S} as in Theorem 4.4.6; now $L(F) = kL(F_S)k^{-1}$.
- $G_h(F) := kL_Sk^{-1}$ which is a subgroup of a suitable Levi subgroup of P_F.
- $G_l(F)$ which is a reductive subgroup of $L(F)$ with no compact factors,
- $M(F)$ which is a compact reductive subgroup of $L(F)$,

such that $L(F) = G_h(F) \cdot G_l(F) \cdot M(F)$ and $L(F)^{\mathrm{ad}} = G_h(F)^{\mathrm{ad}} \times G_l(F)^{\mathrm{ad}} \times M(F)^{\mathrm{ad}}$. One can easily check that

$$\{g \in G(\mathbb{R})^+ : gx = x \text{ for all } x \in F\} = W(F) \rtimes (G_l(F) \cdot M(F)). \tag{4.6.19}$$

Example 4.6.20. *In the Siegel case, $r = d$. Take $S = \{d'+1, \ldots, d\} \subseteq \{1, \ldots, d\}$. Then*

$$P_F = \left\{ \begin{bmatrix} A' & 0 & B' & * \\ * & u & * & * \\ C' & 0 & D' & * \\ 0 & 0 & 0 & (u^{\mathrm{t}})^{-1} \end{bmatrix} \in G : \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \in \mathrm{Sp}_{2d',\mathbb{R}},\ u \in \mathrm{GL}_{d-d',\mathbb{R}} \right\},$$

$$W(F) = \left\{ \begin{bmatrix} I_{d'} & 0 & 0 & n \\ m^{\mathrm{t}} & I_{d-d'} & n^{\mathrm{t}} & b \\ 0 & 0 & I_{d'} & -m \\ 0 & 0 & 0 & I_{d-d'} \end{bmatrix} : n^{\mathrm{t}} m + b = m^{\mathrm{t}} n + b^{\mathrm{t}} \right\},$$

$$G_h(F) = \left\{ \begin{bmatrix} A' & 0 & B' & 0 \\ 0 & I_{d-d'} & 0 & 0 \\ C' & 0 & D' & 0 \\ 0 & 0 & 0 & I_{d-d'} \end{bmatrix} : \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \in \mathrm{Sp}_{2d',\mathbb{R}} \right\} \simeq \mathrm{Sp}_{2d',\mathbb{R}},$$

$$G_l(F) = \left\{ \begin{bmatrix} I_{d'} & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & I_{d'} & 0 \\ 0 & 0 & 0 & (u^{\mathrm{t}})^{-1} \end{bmatrix} : u \in \mathrm{GL}_{d-d',\mathbb{R}} \right\} \simeq \mathrm{GL}_{d-d',\mathbb{R}},$$

$$M(F) = \{\pm I_{2d}\}.$$

4.6.6 Some other remarks

The analytic boundary components are closely related to the Polydisc Theorem, since in (4.6.15) $(1, \ldots, 1) \in \mathbb{C}^{|S|} \subseteq (\mathbb{P}^1)^{|S|}$ is a point on the boundary of the polydisc $D^{|S|} \subseteq (\mathbb{P}^1)^{|S|}$. Thus one can also recover information of analytic boundary components in terms of the polydisc $\mathfrak{H}^r \simeq D^r \subseteq X$ and the corresponding homomorphism $\varphi\colon \mathrm{SL}_2(\mathbb{R})^r \to G$ in (4.6.6).

More precisely , for every analytic boundary component $F = kF_S$, there are holomorphic symmetric maps

$$\begin{tikzcd} \mathfrak{H} \arrow[r, "f_F"] \arrow[d, "\subseteq"'] & \mathcal{D} \arrow[d, "\subseteq"] \\ \mathbb{P}^1 \arrow[r, "f_F"] & X^{\vee} \end{tikzcd}$$

such that $f_F(\sqrt{-1}) = x_0$ and $f_F(\infty) \in F$, and equivariant with respect to a homomorphism

$$\varphi_F\colon U(1) \times \mathrm{SL}_2(\mathbb{R}) \to G$$

which sends $\left(e^{\sqrt{-1}\theta}, \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\right) \mapsto h_0(e^{\sqrt{-1}\theta})$. This homomorphism φ_F is defined using the Polydisc Theorem, or more precisely the map φ and its extension $\widetilde{\varphi}$ from (4.6.6) and below. Indeed, when $F = F_S$, then φ_{F_S} is precisely $(e^{\sqrt{-1}\theta}, x) \mapsto \widetilde{\varphi}(e^{\sqrt{-1}\theta}; (x_j)_{1 \le j \le r})$, where $x_j = e^{\sqrt{-1}\theta}$ if $j \notin S$ and $x_j = x$ if $j \in S$.

And in terms of Theorem 4.2.5, P_F is defined by the cocharacter $\lambda_F\colon \mathbb{G}_{\mathrm{m},\mathbb{R}} \to G$, $t \mapsto \varphi_F\left(1, \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}\right)$. This is easy to check with the computation in the proof of Theorem 4.6.19. Then F is a rational analytic boundary component if and only if λ_F is defined over $\mathbb{Q}$.