Chapter 1

Preparation on Hodge theory

1.1 Hodge structure and polarizations

Take $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.
Let $n \in \mathbb{Z}$.

1.1.1 Hodge decomposition and Hodge filtration

Definition 1.1.1. *An R-**Hodge structure of weight** n is a torsion-free R-module of finite type V endowed with a bigrading (called the **Hodge decomposition**)*

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}, \qquad \text{with} \quad \overline{V^{q,p}} = V^{p,q}.$$

*An R-linear map $\varphi\colon V \to W$ between two Hodge structures of weight n is said to be a **morphism of Hodge structures** if $\varphi(V^{p,q}) \subseteq W^{p,q}$ for all p, q.*

We thus have the *category of R-Hodge structures of weight n*, denoted by HS_R^n. One can define direct sums in HS_R^n, and hence makes it into an *abelian category.*

We can also consider the *category of R-Hodge structures*, denoted by HS_R. The objects are R-Hodge structures of any weight. Then we can define tensor products, duals, and internal homs in HS_R as follows. Let $V \in \mathrm{HS}_R^n$ and $W \in \mathrm{HS}_R^m$,

(i) the bigrading on $V \otimes W \in \mathrm{HS}_R^{n+m}$ is given by $(V \otimes W)^{p,q} = \bigoplus_{r+r'=p,\ s+s'=q} V^{r,s} \otimes W^{r',s'}$;

(ii) the bigrading on $V^\vee \in \mathrm{HS}_R^{-n}$ is given by $(V^\vee)^{p,q} = (V^{-p,-q})^\vee$;

(iii) $\mathrm{Hom}(V, W) := V^\vee \otimes W$.

Here are some examples.

Example 1.1.2 (Tate twist)**.** *For each $m \in \mathbb{Z}$, set $R(m) \in \mathrm{HS}_R^{-2m}$ to be*

$$R(m) = (2\pi i)^m R, \qquad R(m)_{\mathbb{C}} = R^{-m,-m}.$$

Then $R(0) = R$, $R(m) = R(1)^{\otimes m}$ with $R(-1) = R(1)^\vee$.

Example 1.1.3 (cohomology from geometry)**.** *Let X be a connected smooth projective variety defined over $\mathbb{C}$. For each $n \geq 0$, the Betti cohomology $H^n(X, \mathbb{Z})/\mathrm{tor}$ admits a $\mathbb{Z}$-Hodge structure of weight n via the Betti–de Rham comparison $H^n(X, \mathbb{C}) \simeq H^n_{\mathrm{dR}}(X)$ and the decomposition of $H^n_{\mathrm{dR}}(X)$ into the direct sum of subspaces arising from (p, q)-forms.*

Example 1.1.4 (Complex tori)**.** *We explain in this example the following equivalence of categories:*

$$\{complex\ tori\} \xrightarrow{\sim} \{\mathbb{Z} - Hodge\ structures\ of\ type\ (-1,0)+(0,-1)\}\,.$$

The direction $\to$ is by sending $T \mapsto H_1(T,\mathbb{Z})$. Let T be a complex torus of dimension $g \geq 1$. Set

$$V_{\mathbb{Z}} := H_1(T,\mathbb{Z}).$$

As a real manifold, we then have $T \simeq V_{\mathbb{R}}/V_{\mathbb{Z}}$. Moreover, as a real space $V_{\mathbb{R}}$ is isomorphic to $\mathrm{Lie}(T_{\mathbb{R}})$*, the Lie algebra with $T_{\mathbb{R}}$ seen as a real Lie group. The complex structure on T gives an action of J on $V_{\mathbb{R}}$, with*

$$J := \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix},$$

and hence the desired Hodge decomposition

$$V_{\mathbb{C}} = V^{-1,0} \bigoplus V^{0,-1}$$

with $V^{-1,0}$ the eigenspace of $\sqrt{-1}$ and $V^{0,-1}$ the eigenspace of $-\sqrt{-1}$.

The direction $\leftarrow$ is given as follows. Let $V_{\mathbb{Z}}$ be a $\mathbb{Z}$-Hodge structure of type $(-1,0)+(0,-1)$. Then $V_{\mathbb{C}}/V^{0,-1}$ is a complex space of dimension $\frac{1}{2}\mathrm{rank} V_{\mathbb{Z}}$. Thus we obtain the desired complex torus

$$V_{\mathbb{Z}}\backslash V_{\mathbb{C}}/V^{0,-1} \simeq V_{\mathbb{Z}}\backslash V^{-1,0}.$$

Notice that we have implicitly an isomorphism of real vector spaces $V_{\mathbb{R}} \simeq V_{\mathbb{C}}/V^{0,-1} = V^{-1,0}$ given as the composite $V_{\mathbb{R}} \subseteq V_{\mathbb{C}} \to V_{\mathbb{C}}/V^{0,-1} = V^{-1,0}$.

An alternative way to see the Hodge decomposition is the following Hodge filtration. It is of particular importance when we consider families of Hodge structures.

Definition 1.1.5. *Let V be an R-Hodge structure of weight n. The* **Hodge filtration** *is the decreasing chain $\cdots \supseteq F^pV_{\mathbb{C}} \supseteq F^{p-1}V_{\mathbb{C}} \supseteq \cdots$ with*

$$F^pV_{\mathbb{C}} := \bigoplus\nolimits_{r\geq p} V^{r,s}. \tag{1.1.1}$$

Conversely, the Hodge decomposition can be recovered by the Hodge filtration via

$$V^{p,q} = F^pV_{\mathbb{C}} \cap \overline{F^qV_{\mathbb{C}}}. \tag{1.1.2}$$

1.1.2 Polarizations

Let V be an R-Hodge structure of weight n.

The *Weil operator* $C \in \mathrm{End}(V_{\mathbb{C}})$ is defined as follows: It acts on $V^{p,q}$ by multiplication by $\sqrt{-1}^{q-p}$. We have $Cx = \overline{Cx}$ for all $x \in V_{\mathbb{R}}$.[1] So $C \in \mathrm{End}(V_{\mathbb{R}})$.

Definition 1.1.6. *A* **polarization** *on V is a morphism of Hodge structures*

$$\psi\colon V \otimes V \to R(-n)$$

such that the bi-linear map

$$V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}, \qquad (x,y) \mapsto \psi_C(x,y) := (2\pi\sqrt{-1})^n\psi(x,Cy) \tag{1.1.3}$$

is symmetric and positive definite.

[1]Indeed, for $x = \sum_{p,q} x_{p,q} \in V_{\mathbb{R}}$, we have $\overline{x_{p,q}} = x_{q,p}$ because $\overline{V^{p,q}} = V^{q,p}$. So $\overline{Cx} = \sum_{p,q} \overline{\sqrt{-1}}^{q-p}\,\overline{x_{p,q}} = \sum_{p,q} \sqrt{-1}^{p-q}x_{q,p} = \sum_{p,q} \sqrt{-1}^{p-q}x_{q,p} = Cx$, and hence $Cx \in V_{\mathbb{R}}$.

The Hermitian pairing associated with the bi-linear map (1.1.3) is $(x, y) \mapsto \psi_C(x, \overline{y})$.

Lemma 1.1.7. *Let $V \in \mathrm{HS}^n_R$, and let ψ be a polarization. Then*

(i) ψ is $(-1)^n$-symmetric, i.e. is alternating if n is odd and is symmetric if n is even.

(ii) the decomposition $V_{\mathbb{C}} = \bigoplus V^{p,q}$ is orthogonal with respect to the Hermitian pairing associated with (1.1.3).

Proof. We start by proving (ii). Take $x \in V^{p,q}$ and $y \in V^{r,s}$. Then

$$(2\pi\sqrt{-1})^{-n}\psi_C(x, \overline{y}) = \psi(x, C\overline{y}) = \psi(x, \sqrt{-1}^{r-s}\overline{y}) = \sqrt{-1}^{r-s}\psi(x, \overline{y})$$

Now $(x, \overline{y}) \in V^{p,q} \times V^{s,r} \subseteq (V \times V)^{p+s,q+r}$. So $\psi(x, \overline{y}) \in R(-n)^{p+s,q+r}$ since ψ is a morphism of Hodge structures. Assume $\psi(x, \overline{y}) \neq 0$. Then $p + s = q + r = n$. But $p + q = r + s = n$. So $p = r$ and $q = s$. Thus $\psi_C(V^{p,q}, \overline{V^{r,s}}) = 0$ unless $p = r$ and $q = s$. This establishes (ii).

Now we turn to (i). The proof will be much easier if we apply Proposition 1.3.5. Here we give a direct computation without using this proposition.

For each $x, y \in V_{\mathbb{R}}$, write $x = \sum_{p,q} x_{p,q}$ and $y = \sum_{p,q} y_{p,q}$ under $V_{\mathbb{C}} = \bigoplus V^{p,q}$. Then $(y_{p,q}, x_{r,s}) \subseteq (V \times V)^{p+r,q+s}$, and hence $\psi(y_{p,q}, x_{r,s}) \in R(-n)^{p+r,q+s}$ is 0 unless $p + r = q + s = n$. So

$$\psi(y, x) = \sum_{p,q} \psi(y_{p,q}, x_{p,q}).$$

On the other hand, $x_{p,q} = \overline{x_{q,p}}$ and $y_{p,q} = \overline{y_{q,p}}$ since $\overline{V^{p,q}} = V^{q,p}$. So

$$\begin{aligned}
\psi_C(Cy, x) &= \psi_C\Big(\sum_{p,q} \sqrt{-1}^{q-p} y_{p,q}, \sum_{p,q} x_{p,q}\Big) \\
&= \psi_C\Big(\sum_{p,q} \sqrt{-1}^{q-p} y_{p,q}, \sum_{p,q} \overline{x_{p,q}}\Big) \\
&= \sum_{p,q} \psi_C(\sqrt{-1}^{q-p} y_{p,q}, \overline{x_{p,q}}) \\
&= \sum_{p,q} \psi_C(\sqrt{-1}^{q-p} y_{p,q}, x_{q,p}) \\
&= (2\pi\sqrt{-1})^n \sum_{p,q} \psi(\sqrt{-1}^{q-p} y_{p,q}, Cx_{q,p}) \\
&= (2\pi\sqrt{-1})^n \sum_{p,q} \psi(\sqrt{-1}^{q-p} y_{p,q}, \sqrt{-1}^{p-q} x_{q,p}) \\
&= (2\pi\sqrt{-1})^n \sum_{p,q} \psi(y_{p,q}, x_{q,p}).
\end{aligned}$$

Therefore

$$\psi(y, x) = (2\pi\sqrt{-1})^{-n}\psi_C(Cy, x).$$

Since ψ_C is symmetric, we furthermore have

$$\psi(y, x) = (2\pi\sqrt{-1})^{-n}\psi_C(x, Cy) = \psi(x, C^2 y).$$

Notice that C^2 acts on $V^{p,q}$ by multiplication by $(-1)^{q-p} = (-1)^{q+p} = (-1)^n$ for all p, q. Thus C^2 acts on V as multiplication by $(-1)^n$. So we have

$$\psi(y, x) = (-1)^n \psi(x, y).$$

This establishes (i). $\square$

Example 1.1.8 (Complex abelian varieties)**.** *We continue with Example 1.1.4 and prove*

$$\{\textit{complex abelian varieties}\} \xrightarrow{\sim} \{\textit{polarizable } \mathbb{Z}\textit{-Hodge structures of type } (-1,0)+(0,-1)\}\,.$$

Let T be a complex torus which corresponds to $V_{\mathbb{Z}} = H_1(T,\mathbb{Z})$. Then $T \simeq V_{\mathbb{R}}/V_{\mathbb{Z}}$ as real manifolds. Thus $\bigwedge^2 V_{\mathbb{Z}}^{\vee} \simeq \bigwedge^2 H^1(T,\mathbb{Z}) = H^2(T,\mathbb{Z})$. Therefore the set of alternating pairings

$$\psi\colon V_{\mathbb{Z}} \otimes V_{\mathbb{Z}} \to \mathbb{Z}(1)$$

is in bijection with $H^2(T,\mathbb{Z}(1))$.

The short exact sequence of sheaves $0 \to \underline{\mathbb{Z}}(1) \to \mathcal{O}_T \xrightarrow{\exp} \mathcal{O}_T^* \to 0$ induces

$$\mathrm{Pic}(T) = H^1(T,\mathcal{O}_T^*) \xrightarrow{c_1} H^2(T,\mathbb{Z}(1)) \to H^2(T,\mathcal{O}_T).$$

Assume T is an abelian variety. Then there exists an ample line bundle L on T. The Ampell–Hubert data for L then gives an alternating pairing $\psi \in H^2(T,\mathbb{Z}(1))$ such that the Hermitian pairing $(x,y) \mapsto \psi(x,\sqrt{-1}\overline{y})$ is the c_1 of L for a suitable Hermitian metric on L. But $V_{\mathbb{Z}}$ has Hodge type $(-1,0)+(0,-1)$ and the complex structure on $V_{\mathbb{R}}/V_{\mathbb{Z}}$ is by identifying $V_{\mathbb{R}} \simeq V^{-1,0}$. So $c_1(L)$ is precisely ψ_C. The ampleness of L implies that ψ_C is positive-definite. Thus ψ is a polarization on $V_{\mathbb{Z}}$.

Conversely assume ψ is a polarization on $V_{\mathbb{Z}}$. Then ψ can be seen as an element in $H^2(T,\mathbb{Z}(1))$, and ψ_C equals $(x,y) \mapsto \psi(x,\sqrt{-1}\overline{y})$ as above. So the Ampell–Hubert Theorem gives a line bundle L on T such that $c_1(L) = \psi_C$. The positivity of ψ_C thus implies the ampleness of L by Kodaira embedding. So T is an abelian variety.

Example 1.1.9 (Primitive cohomology and Lefschetz)**.** *We continue with Example 1.1.3. Assume $d = \dim X$. Let ω be a Kähler form on X^{an}, which is a closed $(1,1)$-form. It induces a homomorphism $L\colon H^n(X,\mathbb{Q}) \to H^{n+2}(X,\mathbb{Q})$, $[\alpha] \mapsto [\omega \wedge \alpha]$; here we are using $H^n(X,\mathbb{Q}) \subseteq H^n(X,\mathbb{C}) \simeq H^n_{\mathrm{dR}}(X)$. The Hard Lefschetz Theorem says that $L^r\colon H^{d-r}(X,\mathbb{Q}) \xrightarrow{\sim} H^{d+r}(X,\mathbb{Q})$ for all $r \geq 0$. Now let $r = d-n$. Set $H^n_{\mathrm{prim}}(X,\mathbb{Q})$ to be the kernel of $L^{r+1}\colon H^n(X,\mathbb{Q}) \to H^{2d-n+2}(X,\mathbb{Q})$. We have a morphism of Hodge structures*

$$\psi\colon H^n(X,\mathbb{Q}) \otimes H^n(X,\mathbb{Q}) \xrightarrow[\sim]{1\otimes L^r} H^n(X,\mathbb{Q}) \otimes H^{2d-n}(X,\mathbb{Q})(\dim X - n) \xrightarrow{\cup} H^{2d}(X,\mathbb{Q})(d-n) = \mathbb{Q}(-n).$$

The restriction of ψ to $H^n_{\mathrm{prim}}(X,\mathbb{Q})$ is a polarization. Thus we obtain a polarization on $H^n(X,\mathbb{Q})$ by the Lefschetz decomposition $H^n(X,\mathbb{Q}) = \bigoplus_{0 \leq s \leq \lfloor n/2 \rfloor} L^s(H^{n-2s}_{\mathrm{prim}}(X,\mathbb{Q}))$.

1.2 Variation of Hodge structures

In practice it is important for us to work with families. Let S be a complex manifold.

Definition 1.2.1. *A* **$\mathbb{Z}$-variation of Hodge structures** *($\mathbb{Z}$-VHS) of weight n on S is $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet)$ where*

- *$\mathbb{V}_{\mathbb{Z}}$ is a local system of free $\mathbb{Z}$-modules on S of finite rank,*
- *$\mathcal{F}^\bullet$ is a finite decreasing filtration (called the* **Hodge filtration***) of the holomorphic vector bundle $\mathcal{V} := \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}_S} \mathcal{O}_S$ by holomorphic subbundles,*

such that

(i) $(\mathbb{V}_{\mathbb{Z},s}, \mathcal{F}_s^\bullet)$ *is a* $\mathbb{Z}$*-Hodge structure of weight* n *for each* $s \in S$*,*

(ii) the connection $\nabla\colon \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}_S} \Omega_S^1$ *whose sheaf of horizontal sections is* $\mathbb{V}_{\mathbb{C}} := \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ *satisfies the* **Griffiths' transversality** *condition*

$$\nabla(\mathcal{F}^p) \subseteq \mathcal{F}^{p-1} \otimes \Omega_S^1 \qquad \textit{for all } p. \tag{1.2.1}$$

A **polarization** *on* $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet)$ *is a morphism of local systems*

$$\mathbb{V}_{\mathbb{Q}} \otimes \mathbb{V}_{\mathbb{Q}} \to \mathbb{Q}_S$$

inducing on each fiber a polarization of the corresponding $\mathbb{Q}$*-Hodge structure.*

Example 1.2.2. *Let* $f\colon X \to S$ *be a smooth projective morphism. Then* $\mathbb{V} := R^n f_* \mathbb{Z}_X$ *is a local system of* $\mathbb{Z}$*-modules on* S *with fiber* $\mathbb{V}_s = H^n(X_s, \mathbb{Z})$*. Replace* $\mathbb{V}$ *by its quotient by torsion. Under the isomorphism* $\mathcal{V} \simeq R^n f_* \Omega_{X/S}^\bullet$*, the Hodge filtration is* $\mathcal{F}^p\mathcal{V} = R^n f_* \Omega_{X/S}^{\geq p}$*. Notice that the subbundle of* (p,q)*-forms is not holomorphic if* $q \neq 0$*, but* $\mathcal{F}^p\mathcal{V}$ *is holomorphic. The fiberwise polarization from Example 1.1.9 gives a polarization on* $\mathbb{V}$*.*

1.3 Mumford–Tate group

1.3.1 Revision on algebraic tori

Let k be a field. A *linear algebraic group* defined over k is an affine group scheme G/k of finite type; it can be embedded as a closed subgroup scheme of GL_N for some N. If $\mathrm{char} k = 0$, then G is reduced and smooth. As an example, we have $\mathbb{G}_{\mathrm{m},k} := \mathrm{GL}_{1,k}$ which is defined by: for any k-algebra R, we have $\mathbb{G}_{\mathrm{m},k}(R) = R^\times$. When k is clear in the context, we simply write $\mathbb{G}_{\mathrm{m}}$.

Let k^{s} be a separable closure of k. If $\mathrm{char} k = 0$, then k^{s} is an algebraic closure of k.

Definition 1.3.1. *An* **algebraic torus** *defined over* k *is a linear algebraic group* T *defined over* k *such that its base change to* k^{s} *is isomorphic to* $\mathbb{G}_{\mathrm{m},k^{\mathrm{s}}}^r$ *for some* $r \geq 1$*.*

The *group of characters* (resp. *group of cocharacters*) of T is

$$X^*(T) := \mathrm{Hom}(T_{k^{\mathrm{s}}}, \mathbb{G}_{\mathrm{m},k^{\mathrm{s}}}), \quad (\text{resp. } X_*(T) := \mathrm{Hom}(\mathbb{G}_{\mathrm{m},k^{\mathrm{s}}}, T_{k^{\mathrm{s}}})).$$

Both $X^*(T)$ and $X_*(T)$ are isomorphic (as groups) to $\mathbb{Z}^{\dim T}$ and are naturally endowed with a $\mathrm{Gal}(k^{\mathrm{s}}/k)$-action. We also have a *perfect pairing* as $\mathrm{Gal}(k^{\mathrm{s}}/k)$-modules

$$X^*(T) \times X_*(T) \to \mathbb{Z} = \mathrm{End}(\mathbb{G}_{\mathrm{m},k^{\mathrm{s}}}), \qquad (\chi, \mu) \mapsto \langle \chi, \mu \rangle := \chi \circ \mu. \tag{1.3.1}$$

By definition, $T_{k'} \simeq \mathbb{G}_{\mathrm{m},k'}$ for some finite separable extension k'/k. So the Galois action of $\mathrm{Gal}(k^{\mathrm{s}}/k)$ on $X^*(T)$ factors through $\mathrm{Gal}(k'/k)$ which is a finite group. Therefore the $\mathrm{Gal}(k^{\mathrm{s}}/k)$-action on $X^*(T)$ is continuous. Same for the $\mathrm{Gal}(k^{\mathrm{s}}/k)$-action on $X_*(T)$. Thus the functor $T \mapsto X_*(T)$ gives an equivalence from the category of *algebraic tori defined over* k to the category of *free abelian groups of finite rank endowed with a continuous* $\mathrm{Gal}(k^{\mathrm{s}}/k)$*-action*.

Next we turn to the representations of algebraic tori $\rho\colon T \to \mathrm{GL}(V)$. Passing to k', ρ becomes $T_{k'} \simeq \mathbb{G}_{\mathrm{m},k'}^r \to \mathrm{GL}(V_{k'})$. Then $V_{k'}$ can be decomposed into

$$V_{k'} = \bigoplus\nolimits_{\chi \in X^*(T)} V_\chi = \bigoplus_{(n_1,\ldots,n_r) \in \mathbb{Z}^r} V^{n_1,\ldots,n_r}$$

where $V_\chi = \{v \in V_{k'} : \rho(t)v = \chi(t)v\}$ and $V^{n_1,\ldots,n_r} = \{v \in V_{k'} : \rho(z_1,\ldots,z_r)v = z_1^{-n_1} \cdots z_r^{-n_r} v\}$. On the base field k, the decomposition is Galois compatible, *i.e.* $\sigma(V_\chi) = V_{\chi^\sigma}$ for all $\sigma \in \mathrm{Gal}(k'/k)$.

1.3.2 Deligne torus

View $\mathbb{C}$ as an $\mathbb{R}$-algebra using the inclusion $\mathbb{R} \subseteq \mathbb{C}$. Let $\mathbb{S}$ be the algebraic group $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{\mathrm{m}}$ defined over $\mathbb{R}$, *i.e.* for any $\mathbb{R}$-algebra R, we have

$$\mathbb{S}(R) = (R \otimes_{\mathbb{R}} \mathbb{C})^{\times}.$$

Then

$$\mathbb{S}(\mathbb{C}) = (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^{\times} = \big((\mathbb{R} \oplus \sqrt{-1}\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}\big)^{\times} = (\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C})^{\times} \times (\sqrt{-1}\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C})^{\times} = \mathbb{C}^{\times} \times \mathbb{C}^{\times}.$$

Hence $\mathbb{S}$ is an algebraic torus defined over $\mathbb{R}$, and $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$ acts on $\mathbb{S}(\mathbb{C})$ by $\sigma(z_1, z_2) = (\overline{z}_2, \overline{z}_1)$. Thus $\mathbb{S}(\mathbb{R}) = \{z \in \mathbb{S}(\mathbb{C}) : z = \sigma(z)\} = \{(z_1, z_2) \in \mathbb{C}^{\times} \times \mathbb{C}^{\times} : z_2 = \overline{z}_1\}$. In other words, the natural inclusion $\mathbb{S}(\mathbb{R}) \subseteq \mathbb{S}(\mathbb{C})$ is given by $z \mapsto (z, \overline{z})$.

Definition 1.3.2. *The algebraic torus $\mathbb{S}$ is called the* **Deligne torus***.*

The character group of the Deligne torus is

$$X^*(\mathbb{S}) = \mathrm{Hom}(\mathbb{S}(\mathbb{C}), \mathbb{C}^{\times}) = \mathrm{Hom}(\mathbb{C}^{\times} \times \mathbb{C}^{\times}, \mathbb{C}^{\times}) = \mathrm{Hom}(\mathbb{C}^{\times}, \mathbb{C}^{\times}) \oplus \mathrm{Hom}(\mathbb{C}^{\times}, \mathbb{C}^{\times}) \simeq \mathbb{Z} \oplus \mathbb{Z}, \tag{1.3.2}$$

where the last isomorphism is obtained from the inverse of

$$\mathbb{Z} \xrightarrow{\sim} \mathrm{Hom}(\mathbb{C}^{\times}, \mathbb{C}^{\times}), \qquad p \mapsto (z \mapsto z^{-p}). \tag{1.3.3}$$

The Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$ acts on $X^*(\mathbb{S})$ by $\sigma(p, q) = (q, p)$.

Among the cocharacters of $\mathbb{S}$, two are particularly important:

- the *weight cocharacter* $w\colon \mathbb{G}_{\mathrm{m},\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$, $z \mapsto (z, z)$, which descends to $\mathbb{R}$ (namely it is the base change to $\mathbb{C}$ of a morphism $\mathbb{G}_{\mathrm{m},\mathbb{R}} \to \mathbb{S}$).
- the *principal cocharacter* $\mu\colon \mathbb{G}_{\mathrm{m},\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$, $z \mapsto (z, 1)$.

An important character of $\mathbb{S}$ is the *norm character* $\mathrm{Nm}\colon \mathbb{S} \to \mathbb{G}_{\mathrm{m}}$, $z \mapsto z\sigma(z)$. It fits into the following short exact sequence:

$$0 \to U(1) \to \mathbb{S} \xrightarrow{\mathrm{Nm}} \mathbb{G}_{\mathrm{m}} \to 0. \tag{1.3.4}$$

Notice that $\mathrm{Nm} \circ w$ sends each $z \in \mathbb{G}_{\mathrm{m}}(\mathbb{R}) = \mathbb{R}^{\times}$ to z^2.

1.3.3 Hodge structures as representations of the Deligne torus

Now let V be an R-Hodge structure of weight n. Recall the Hodge decomposition $V_{\mathbb{C}} = \bigoplus V^{p,q}$. It gives rise to an action of $\mathbb{S}_{\mathbb{C}}$ on $V_{\mathbb{C}}$ by setting $V^{p,q}$ to be the eigenspace of the character $(p, q) \in X^*(\mathbb{S})$. More precisely, for each $(z_1, z_2) \in \mathbb{S}(\mathbb{C}) = \mathbb{C}^{\times} \times \mathbb{C}^{\times}$ and each $v = (v_{p,q})_{p,q} \in V_{\mathbb{C}} = \bigoplus V^{p,q}$, we have

$$(z_1, z_2) \cdot v = (z_1^{-p} z_2^{-q} v_{p,q})_{p,q}. \tag{1.3.5}$$

This action of $\mathbb{S}_{\mathbb{C}}$ on $V_{\mathbb{C}}$ induces a morphism

$$h\colon \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(V_{\mathbb{C}}). \tag{1.3.6}$$

Lemma 1.3.3. *The morphism h descends to $\mathbb{R}$, i.e. it is the base change to $\mathbb{C}$ of a morphism $\mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$.*

Proof. For $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$, we can do the following computation. Let $(z_1, z_2) \in \mathbb{S}(\mathbb{C})$ and $v = (v_{p,q})_{p,q} \in V_{\mathbb{C}}$.

Recall that the Hodge decomposition satisfies $\overline{V^{p,q}} = V^{q,p}$. So $\overline{v_{p,q}} \in \overline{V^{p,q}} = V^{q,p}$. Hence the decomposition of $\overline{v} = \sigma(v)$ under $V_{\mathbb{C}} = \bigoplus V^{p,q}$ is $\overline{v} = (\overline{v_{q,p}})_{p,q}$. In particular, $\overline{v}_{p,q} = \overline{v_{q,p}}$.

Now we have

$$h(\sigma(z_1, z_2))v = (\overline{z}_2, \overline{z}_1) \cdot v = (\overline{z}_2^{-p} \overline{z}_1^{-q} v_{p,q})_{p,q}$$

and

$$\sigma\left(h(z_1, z_2)\right) v = \sigma\left(h(z_1, z_2)\overline{v}\right) = \sigma\left((z_1, z_2) \cdot \overline{v}\right) = \sigma((z_1^{-p} z_2^{-q} \overline{v}_{p,q})_{p,q}) = \sigma((z_1^{-p} z_2^{-q} \overline{v_{q,p}})_{p,q}) = (\overline{z}_1^{-q} \overline{z}_2^{-p} v_{p,q})_{p,q}.$$

Hence h is $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant, and therefore descends to $\mathbb{R}$. $\square$

Thus from any R-Hodge structure V of weight n, we have constructed a morphism $\mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$. Conversely given any $h\colon \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$, we can define $V^{p,q}$ to be the eigenspace of the character $(p, q) \in X^*(\mathbb{S})$ of $\mathbb{S}_{\mathbb{C}}$. Then $V = \bigoplus V^{p,q}$, and $\overline{V^{q,p}} = V^{p,q}$ because h is defined over $\mathbb{R}$. Hence we have the following proposition.

Proposition 1.3.4. *Let $R = \mathbb{Z}, \mathbb{Q}$ and let V be a torsion-free R-module of finite type.*

Then there are bijections between the following sets of:

(i) Hodge structures of weight n on V;

(ii) morphisms $h\colon \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ such that the eigenspace of $(p, q) \in X^*(\mathbb{S})$ is 0 unless $p+q = n$.

(iii) morphisms $h\colon \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ such that the composite $h \circ w\colon \mathbb{G}_{\mathrm{m},\mathbb{R}} \to \mathrm{GL}(V_{\mathbb{R}})$ sends each $z \in \mathbb{R}^{\times}$ to the multiplication by z^{-n}.

If a Hodge structure on V corresponds to $h\colon \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$, by abuse of notation we use (V, h) to denote this Hodge structure. In this terminology, the Weil operator C of the Hodge structure (V, h) in the definition of polarizations (1.1.3) is simply $h(\sqrt{-1})$.

Proposition 1.3.5. *Let (V, h) and (W, h') be two R-Hodge structures of weight n, and let $\varphi\colon V \to W$ be an R-linear map.*

Then φ is a morphism of Hodge structures if and only if $\varphi(h(z)v) = h'(z)\varphi(v)$ for all $z \in \mathbb{S}(\mathbb{R})$ and $v \in V_{\mathbb{R}}$.

The proof of Lemma 1.1.7.(i) can be much simplified by this proposition: $\psi(y, x) = \psi(Cy, Cx) = (2\pi\sqrt{-1})^{-2n}\psi_C(Cy, x) = (2\pi\sqrt{-1})^{-2n}\psi_C(x, Cy) = \psi(x, C^2 y) = (-1)^n \psi(x, y)$, and hence ψ is $(-1)^n$-symmetric.

Proof. Write $v = (v_{p,q})_{p,q} \in V_{\mathbb{C}} = \bigoplus V^{p,q}$. Then $h(z)v = (z^{-p}\overline{z}^{-q} v_{p,q})_{p,q}$. So $\varphi(h(z)v) = (z^{-p}\overline{z}^{-q}\varphi(v_{p,q}))_{p,q}$ by linearity of φ.

Assume φ is a morphism of Hodge structures. Then $\varphi(V^{p,q}) \subseteq W^{p,q}$ for all p, q, and hence $\varphi(v_{p,q}) = \varphi(v)_{p,q}$ for all p, q. So $\varphi(h(z)v) = (z^{-p}\overline{z}^{-q}\varphi(v)_{p,q})_{p,q} = h'(z)\varphi(v)$.

Conversely assume $\varphi(h(z)v) = h'(z)\varphi(v)$ for all $z \in \mathbb{S}(\mathbb{R})$ and $v \in V_{\mathbb{R}}$. Let $v \in V^{p,q}$. By considering $v + \overline{v}$ and $(v - \overline{v})/\sqrt{-1}$, we have $\varphi(h(z)v) = h'(z)\varphi(v)$ for all $z \in \mathbb{S}(\mathbb{R})$. So $h'(z)\varphi(v) = \varphi(h(z)v) = \varphi(z^{-p}\overline{z}^{-q}v) = z^{-p}\overline{z}^{-q}\varphi(v)$ for all $z \in \mathbb{S}(\mathbb{R})$. Therefore $\varphi(v) \in W^{p,q}$. $\square$

This proposition has the following immediate corollary.

Corollary 1.3.6. *Let (V, h) be an R-Hodge structure of weight n, and let W be a torsion-free R-submodule of V.*

Then $h|_W$ is an R-Hodge structure if and only if $W_{\mathbb{R}}$ is an $h(\mathbb{S})$-submodule of V.

In this case, we call the Hodge structure $(W, h|_W)$ a *sub-R-Hodge structure* of (V, h).

1.3.4 Mumford–Tate group

In this subsection, assume $R = \mathbb{Z}$ or $\mathbb{Q}$. Let (V, h) be an R-Hodge structure.

Definition 1.3.7. *The* **Mumford–Tate group** *of* (V, h) *is the smallest* $\mathbb{Q}$*-subgroup* $\mathrm{MT}(h)$ *of* $\mathrm{GL}(V_{\mathbb{Q}})$ *such that* $h(\mathbb{S}) \subseteq \mathrm{MT}(h)(\mathbb{R})$.

It is easy to check that $\mathrm{MT}(h)$ is connected since $\mathbb{S}$ is, and $\mathrm{MT}(h)(\mathbb{C})$ is the subgroup of $\mathrm{GL}(V(\mathbb{C}))$ generated by $\sigma(h(\mathbb{S}(\mathbb{C})))$ for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. We also have the following characterization of $\mathrm{MT}(h)$ using the principal cocharacter μ defined above (1.3.4).

Lemma 1.3.8. $\mathrm{MT}(h)$ *is the smallest* $\mathbb{Q}$*-subgroup of* $\mathrm{GL}(V_{\mathbb{Q}})$ *such that* $\mu_h := h \circ \mu \colon \mathbb{G}_{\mathrm{m},\mathbb{C}} \to \mathrm{GL}(V_{\mathbb{C}})$ *factors through* $\mathrm{MT}(h)_{\mathbb{C}}$.

Proof. By definition $\mu_h(\mathbb{G}_{\mathrm{m},\mathbb{C}}) \subseteq \mathrm{MT}(h)_{\mathbb{C}}$. Conversely let M be a $\mathbb{Q}$-subgroup of $\mathrm{GL}(V_{\mathbb{Q}})$ which contains $\mu_h(\mathbb{G}_{\mathrm{m},\mathbb{C}}) = h(\mu(\mathbb{G}_{\mathrm{m},\mathbb{C}}))$. Then $M(\mathbb{C})$ contains $h(z, 1) \in \mathrm{GL}(V(\mathbb{C}))$ for all $z \in \mathbb{C}^\times$. Since M is defined over $\mathbb{Q}$ and h is defined over $\mathbb{R}$, we have that $M(\mathbb{C})$ contains $\sigma(h(z, 1)) = h(\sigma(z, 1)) = h(1, \overline{z})$ for all $z \in \mathbb{C}^\times$, where $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$. Hence $M(\mathbb{C})$, as a group, contains $h(z, 1)h(1, \overline{z}') = h(z, \overline{z}')$ for all $z, z' \in \mathbb{C}^\times$. Hence $h(\mathbb{S}_{\mathbb{C}}) \subseteq M_{\mathbb{C}}$, so $\mathrm{MT}(h) \subseteq M$. $\square$

It is not hard to check that the Mumford–Tate of the dual Hodge structure of (V, h) is still $\mathrm{MT}(h)$.

Now assume $R = \mathbb{Q}$. For $m, n \in \mathbb{Z}_{\geq 0}$, we have a Hodge structure $T^{m,n}V := V^{\otimes m} \otimes (V^\vee)^{\otimes n}$, and $\mathrm{MT}(h)$ acts on $T^{m,n}V$ componentwise. The following proposition is an immediate consequence of Corollary 1.3.6 (applied to $T^{m,n}V$).

Proposition 1.3.9. *Let* W *be a* $\mathbb{Q}$*-subspace of* $T^{m,n}V$*. Then* W *is a sub-*$\mathbb{Q}$*-Hodge structure of* $T^{m,n}V$ *if and only if* W *is a* $\mathrm{MT}(h)$*-submodule of* $T^{m,n}V$.

This proposition gives rise to another useful characterization of $\mathrm{MT}(h)$, which is important in the study of (sub-)Shimura varieties. We make the following definition.

Definition 1.3.10. *The elements of* $(T^{m,n}V_{\mathbb{C}})^{0,0} \cap T^{m,n}V$*, with* m *and* n *running over all non-negative integers, are called the* **Hodge tensor** *for* (V, h).

Denote by Hdg_h the set of all Hodge tensors for (V, h).

Proposition 1.3.11. *We have* $\mathrm{MT}(h) = Z_{\mathrm{GL}(V)}(\mathrm{Hdg}_h)$.

In particular by dimension reasons, $\mathrm{MT}(h) = Z_{\mathrm{GL}(V)}(\mathfrak{I})$ *for some finite set* $\mathfrak{I} \subseteq \mathrm{Hdg}_h$.

Proof. Take $t \in \mathrm{Hdg}_h$. For any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, we have $\sigma(t) = t$ since t is a $\mathbb{Q}$-element. By (1.3.5) we have $h(z_1, z_2)t = z_1^0 \overline{z_2}^0 t = t$ for any $(z_1, z_2) \in \mathbb{S}(\mathbb{C})$. Applying the action of any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ and recalling that $\mathrm{MT}(h)(\mathbb{C})$ is generated by the $\sigma(h(\mathbb{S}(\mathbb{C})))$'s, we have that t is fixed by $\mathrm{MT}(h)(\mathbb{Q})$. This establishes "$\subseteq$".

To get $\mathrm{MT}(h) = Z_{\mathrm{GL}(V)}(\mathrm{Hdg}_h)$, notice that $\mathrm{MT}(h)$ is a closed subgroup of $\mathrm{GL}(V)$. By theory of algebraic groups, $\mathrm{MT}(h)$ is thus the stabilizer of some 1-dimensional $\mathbb{Q}$-subspace L in $\bigoplus_{(m,n)\in I} T^{m,n}V$ for some finite subset $I \subseteq \mathbb{Z}^2_{\geq 0}$. Now that L is a 1-dimensional $\mathrm{MT}(h)$-submodule of $\bigoplus_{(m,n)\in I} T^{m,n}V$, Proposition 1.3.9 implies that L is a 1-dimensional $\mathbb{Q}$-Hodge structure, and hence $L_{\mathbb{C}} = L^{p,q}$ for some p and q. But then $p = q$ since $L^{p,q} = \overline{L^{q,p}}$.[2] In other words, $L \simeq \mathbb{Q}(-p)$ has weight $2p$.

If $p = 0$, take a $\mathbb{Q}$-generator ℓ of L. Then $\mathrm{MT}(h)(\mathbb{Q})$ fixes ℓ by the same argument on proving "$\subseteq$". So $\mathrm{MT}(h)$, being the stabilizer of $\mathbb{Q}\ell$, equals $Z_{\mathrm{GL}(V)}(\ell)$. If $p \neq 0$, then the weight of (V, h) is not zero,

[2] To make the argument in this paragraph vigorous, we need to argue with *mixed* Hodge structures because $\bigoplus_{(m,n)\in I} T^{m,n}V$ may have more than one weight. However, since $\bigoplus_{(m,n)\in I} T^{m,n}V$ is a direct sum of (pure) Hodge structures and $\dim L = 1$, we are essentially working with a pure Hodge structure.

and hence the weight r of the Hodge structure $\det V := \bigwedge^{\dim V} V$ is non-zero (since $\det V$ can be realized as a $\mathrm{MT}(h)$-submodule of $V^{\otimes \dim V}$). We may assume $r > 0$ up to replacing V by $V^\vee$. The 1-dimensional $\mathbb{Q}$-space $L^{\otimes r} \otimes (\det V)^{\otimes -2p}$ is a Hodge structure of weight 0 and hence equals its $(0,0)$-piece. Let ℓ be a generator of $L^{\otimes r} \otimes (\det V)^{\otimes -2p}$. Then ℓ is fixed by $\mathrm{MT}(h)(\mathbb{Q})$ by the same argument on proving "$\subseteq$". Hence $\mathrm{MT}(h) = Z_{\mathrm{GL}(V)}(\ell)$ as in the case of $p = 0$.

To summarize, there exists a finite sum of Hodge tensors $t_1 + \cdots + t_N$ such that $\mathrm{MT}(h) = Z_{\mathrm{GL}(V)}(t_1 + \cdots + t_N)$. So $\mathrm{MT}(h) \subseteq \bigcap_{i=1}^N Z_{\mathrm{GL}(V)}(t_i) \subseteq Z_{\mathrm{GL}(V)}(t_1 + \cdots + t_N)$ becomes an equality. We are done. $\square$

Finally, we point out that the Mumford–Tate group of any *polarized* $\mathbb{Q}$-Hodge structure of weight n is a reductive group. A detailed discussion on this will be given in the next chapter.