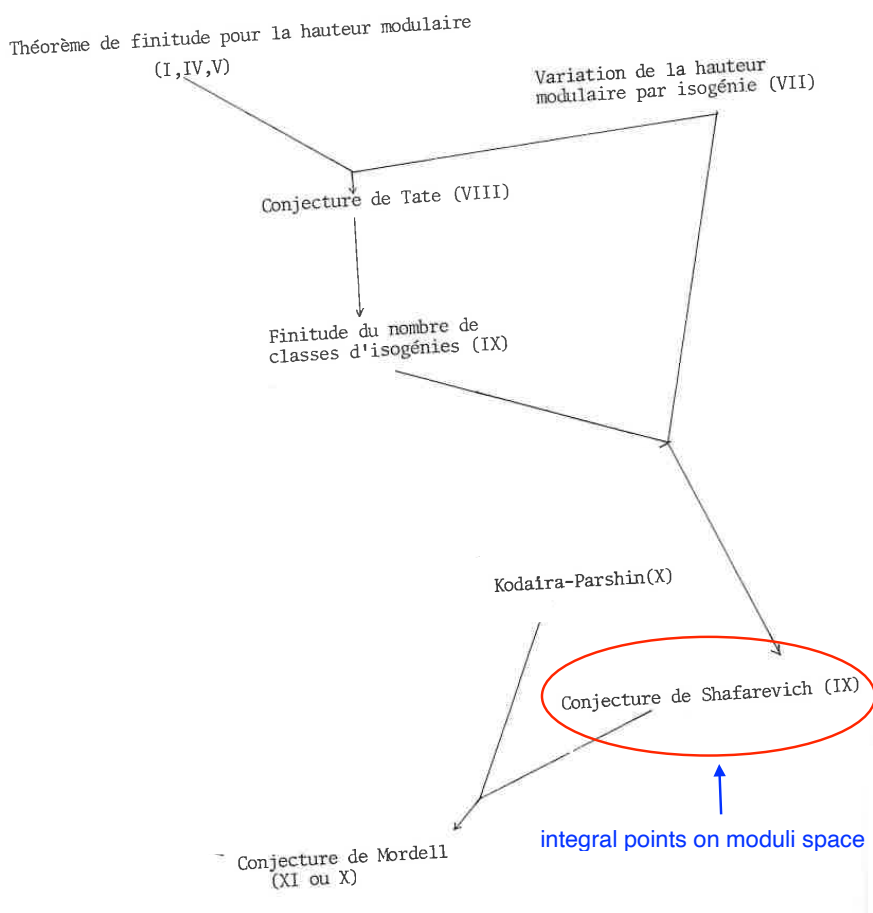


# Mordell Conjecture, after Faltings 1983

UCLA, Winter and Spring 2025

## Introduction

The aim of this participating seminar is to go through Faltings's proof of the Mordell Conjecture in 1983. Here is the sketch of the proof, extracted from [Ast127].



Inspired by this, we will divide the seminar into the following themes:

- Finiteness theorem for Faltings height;
- Isogeny and Tate Conjecture;
- Shafarevich Conjecture;
- Kodaira-Parshin construction and conclusion.

## Schedule

### 1. Finiteness theorem for Faltings height:

- 01/14 (Tom Han): Introduction to height theory. Define the logarithmic Weil height on projective spaces and state the Northcott property ([Gao22, §1.2.1], [HS00, §B.2]). Introduce the Height Machine ([Gao22, §5.1] or [HS00, §B.3]). Introduce Arakelov height defined by Hermitian line bundles on arithmetic varieties ([HS00, pp. 248 of §B.10]). Define the (stable) Faltings height of an abelian variety [Ast127, Exp. I, §3.3, Defn. 2] if time permits (assume the existence of the Néron model).
- 01/21 (Jacob Swenberg): Define the height associated with a Hermitian line bundle with logarithmic singularities, and prove the “Northcott property” in this case. See [FW84, Chapter I, §4] and [Ast127, Exp. 1, §3.2]. Recall the definition of the (stable) Faltings height of an abelian variety [Ast127, Exp. 1, §3.3, Defn. 2] (assume the existence of the Néron model). Prove [Ast127, Exp. I, Thm. 3.2].
- 01/28 (Jas Singh): Introduction to moduli spaces. Explain the general theory (without proof) of the moduli spaces of curves  $\mathcal{M}_g$  and principally polarized abelian varieties  $\mathcal{A}_g$  (over  $\mathbb{Z}$ ).
- 02/04 (Zach Baugher): Summary of the analytic theory of  $\mathcal{A}_g$  over  $\mathbb{C}$  and Baily–Borel compactification. Should cover [FW84, Chapter I, §5]. See also [Nam, §1 and 5].
- 02/11 (John Zhou): Summary of the toroidal compactification of  $\mathcal{A}_g$  over  $\mathbb{C}$ . Should cover [FW84, Chapter I, §6]. See also [AMRT10, I.1, I.3, I.4] and [Nam, §6, 7].
- 02/18 (Tom Han): Comparison of the Faltings height and the theta height. We take the analytic approach [FW84, Chapter 2, Thm. 3.1]. An alternative (more algebraic) approach is provided by [Ast127, Exp. IV].

### 2. Isogeny and Tate Conjecture:

- 02/25 (Matthew Hung), 03/04, 03/11: Introduction to finite group schemes and Raynaud’s result on group schemes of type  $(p, \dots, p)$ . Need to cover [FW84, Chapter III, §2 and §4].
- 04/01, 04/08: Introduction to  $p$ -divisible groups. See [FW84, Chapter III, §3 and §5].
- 04/15: Prove that the Faltings height is bounded within an isogeny class. See [FW84, Chapter III, §3] and/or [Ast127, Exp. VII].
- 04/22: Prove Tate Conjecture and the finiteness of isomorphism classes in an isogeny class. See [Ast127, Exp. VIII].

### 3. Shafarevich Conjecture:

- 04/29: Reformulation of Tate conjecture; see [FW84, Chapter IV, Thm. 1.1, Cor. 1.2, Cor. 1.3]. Proof of the finiteness of isogeny classes *using Tate conjecture*; see [FW84, Chapter V, §2] and/or [Ast127, Exp. IX].

### 4. Kodaira–Parshin construction and conclusion.

- 05/06: [Ast127, Exp. X].

## References

- [AMRT10] A. Ash, D. Mumford, M. Rapoport, Y. Tai: *Smooth Compactifications of Locally Symmetric Varieties (2nd ed)*. Cambridge University Press, 2010.

- [FW84] G. Faltings, G. Wüstholz et al.: *Rational Points: Seminar Bonn/Wuppertal 1983/1984*. Aspects of Mathematics, **E6**, 1984.
- [Gao22] Z. Gao: *An Introduction to Diophantine Geometry*. Lecture notes online <https://ziyangjeremygao.github.io/teaching/LectureNotes/DiophantineGeometry2022.pdf>.
- [HS00] M. Hindry, J. Silverman: *Diophantine Geometry, An Introduction*. GTM, **201**, 2000.
- [Nam] Y. Namikawa: *Toroidal Compactification of Siegel Spaces*. LNM, **812**, 1980.
- [Ast127] L. Szpiro: *Séminaire sur les pinceaux arithmétiques : la conjecture de Mordell*. Astérisque, **127**, 1985. Available at [http://www.numdam.org/item/AST\\_1985\\_\\_127\\_/](http://www.numdam.org/item/AST_1985__127_/).