Chapter 0

Quick Summary on the Height Machine

0.1 Weil height on projective spaces

Let us start with the simplest case. Let $x \in \mathbb{P}^1(\mathbb{Q})$. There is a unique way to write x as $[a : b]$ with $a, b \in \mathbb{Z}$ and $\gcd(a, b) = 1$. Set

$$h(x) := \log \max\{|a|, |b|\}.$$

For a general number field K, we use the following *normalized valuations* at places of K:

(i) For $v \in M_{K,f}$ a non-archimedean place, v is above a prime number $p \in \mathbb{Z}$. We take the absolute value $\|\cdot\|_v \colon K \to \mathbb{R}$ such that $\|p\|_v = p^{-1}$;

(ii) For $v \in M_{K,\infty} = \mathrm{Hom}(K, \mathbb{C})$ an archimedean place, v corresponds to an embedding $\sigma \colon K \to \mathbb{C}$. We take $\|\cdot\|_v \colon K \to \mathbb{R}$ to be $\|x\|_v := |\sigma(x)|^{[K_v:\mathbb{R}]}$.

Notice that $\|\cdot\|_v$ is an absolute value unless v is a complex place, *i.e.* $K_v = \mathbb{C}$.

Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$.

Definition 0.1.1. *Let $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}^n(\overline{\mathbb{Q}})$. The **(absolute logarithmic Weil) height** of x is defined to be*

$$h(\mathbf{x}) := \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} \log \max\{\|x_0\|_v, \ldots, \|x_n\|_v\},$$

where $K \subseteq \overline{\mathbb{Q}}$ is a number field such that $x_j \in K$ for all j.

*We also set $H(\mathbf{x}) := e^{h(\mathbf{x})}$ to be the **multiplicative height**.*

The Weil height is a well-defined function on $\mathbb{P}^n(\overline{\mathbb{Q}})$, *i.e.* it is independent of the choice of K and independent of the choice of the homogeneous coordinates. This can be proved using the product formula. Also one can check that this definition coincides with the one for $\mathbb{P}^1(\mathbb{Q})$ above.

The following properties are of fundamental importance for the Height Machine.

Theorem 0.1.2. *We have:*

- *(Positivity/Lower Bound) $h(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}})$;*

- *(Northcott Property) For each $B \geq 0$ and $D \geq 1$, the following set is a finite set*

$$\{\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}}) : h(\mathbf{x}) \leq B,\ [\mathbb{Q}(\mathbf{x}) : \mathbb{Q}] \leq D\}.$$

Lemma 0.1.3. *The action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathbb{P}^n(\overline{\mathbb{Q}})$ leaves the height invariant. More precisely, for any $\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}})$ and any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we have $h(\sigma(\mathbf{x})) = h(\mathbf{x})$.*

0.2 Height Machine

Let X be an irreducible *projective* variety defined over $\overline{\mathbb{Q}}$. Denote by $\mathbb{R}^{X(\overline{\mathbb{Q}})}$ the set of functions $X(\overline{\mathbb{Q}}) \to \mathbb{R}$, and by $O(1)$ the subset of bounded functions.

The ***Height Machine*** associates to each line bundle $L \in \mathrm{Pic}(X)$ a unique class of functions $\mathbb{R}^{X(\overline{\mathbb{Q}})}/O(1)$, *i.e.* a map

$$\mathbf{h}_X \colon \mathrm{Pic}(X) \to \mathbb{R}^{X(\overline{\mathbb{Q}})}/O(1), \quad L \mapsto \mathbf{h}_{X,L}. \tag{0.2.1}$$

Let $h_{X,L} \colon X(\overline{\mathbb{Q}}) \to \mathbb{R}$ a representative of the class $\mathbf{h}_{X,L}$; it is called a *height function associated with* (X, L).

Construction 0.2.1. *One can construct $h_{X,L}$ as follows. In each case below, $h_{X,L}$ depends on some extra data and hence is not unique. However, it can be shown that any two choices differ by a bounded functions on $X(\overline{\mathbb{Q}})$, and thus the class of $h_{X,L}$ is well-defined.*

(i) If L is very ample, then the global sections of L give rise to a closed immersion $\iota \colon X \to \mathbb{P}^n$ for some n, such that $\iota^*\mathcal{O}(1) \simeq L$. Set $h_{X,L} = h \circ \iota$, with h the Weil height on $\mathbb{P}^n$ from Definition 0.1.1.

(ii) If L is ample, then $L^{\otimes m}$ is very ample for some $m \gg 1$. Set $h_{X,L} = (1/m)h_{X,L^{\otimes m}}$.

(iii) For an arbitrary L, there exist ample line bundles L_1 and L_2 on X such that $L \simeq L_1 \otimes L_2^{\otimes -1}$ by general theory of Algebraic Geometry. Set $h_{X,L} = h_{X,L_1} - h_{X,L_2}$.

Here are some basic properties of the Height Machine. Moreover, the construction (0.2.1) is also uniquely determined by the normalization, additivity, and functoriality.

Proposition 0.2.2. *We have*

- *(Normalization) Let h be the Weil height from Definition 0.1.1. Then for all $\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}})$, we have*

$$h_{\mathbb{P}^n,\mathcal{O}(1)}(\mathbf{x}) = h(\mathbf{x}) + O(1).$$

- *(Additivity) Let L and M be two line bundles on X. Then for all $x \in X(\overline{\mathbb{Q}})$, we have*

$$h_{X,L\otimes M}(x) = h_{X,L}(x) + h_{X,M}(x) + O(1).$$

- *(Functoriality) Let $\phi \colon X \to Y$ be a morphism of irreducible projective varieties and let L be a line bundle on Y. Then for all $x \in X(\overline{\mathbb{Q}})$, we have*

$$h_{X,\phi^*L}(x) = h_{Y,L}(\phi(x)) + O(1).$$

- *(Lower Bound) If $s \in H^0(X, L)$ is a global section, then for all $x \in (X \setminus \mathrm{div}(s))(\overline{\mathbb{Q}})$ we have*

$$h_{X,L}(x) \geq O(1).$$

- *(Northcott property) Assume L is ample. Let K_0 be a number field on which X is defined. Then for any $d \geq 1$ and any constant B, the set*

$$\{x \in X(K) : [K : K_0] \leq d,\ h_{X,L}(x) \leq B\}$$

is a finite set.

The $O(1)$'s that appear in the proposition depend on the varieties, line bundles, morphisms, and the choices of the representatives in the classes of height functions. But they are independent of the points on the varieties.

A natural question arises at this point:

Question: What should one do to get a genuine function $X(\overline{\mathbb{Q}}) \to \mathbb{R}$ from a line bundle L? Or, in other words, to choose a nice representative $h_{X,L}$?

Here is a naive way: one can always fix a representative by fixing every operation needed to define h_L (for example, the basis of $H^0(X, L)$ giving the embedding of X into some $\mathbb{P}^N$ if L is very ample).

In the next section, we will see that a canonical choice of $h_{X,L}$ exists when (X, L) defines a polarized dynamical system, after Néron and Tate.

In general, we use *Arakelov Geometry* for this purpose. This is the main content of this course.

0.3 Normalized height function, after Néron and Tate

Let X be an irreducible projective variety defined over $\overline{\mathbb{Q}}$. Let $L \in \mathrm{Pic}(X)$.

Assume there exists $\phi\colon X \to X$ is a morphism satisfying $\phi^* L \simeq L^{\otimes \alpha}$ for some integer $\alpha > 1$. The following theorem gives a canonical representative of $\mathbf{h}_{X,L}$.

Theorem 0.3.1. *There exists a unique height function*

$$\widehat{h}_{X,\phi,L}\colon X(\overline{\mathbb{Q}}) \to \mathbb{R}$$

with the following properties.

(i) $\widehat{h}_{X,\phi,L}(x) = h_{X,L}(x) + O(1)$ *for all* $x \in X(\overline{\mathbb{Q}})$,

(ii) $\widehat{h}_{X,\phi,L}(\phi(x)) = \alpha \widehat{h}_{X,\phi,L}(x)$ *for all* $x \in X(\overline{\mathbb{Q}})$.

The height function $\widehat{h}_{X,\phi,L}$ depends only on the isomorphism class of L. Moreover, it can be computed as the limit

$$\widehat{h}_{X,\phi,L}(x) = \lim_{n\to\infty} \frac{1}{\alpha^n} h_{X,L}(\phi^n(x)) \tag{0.3.1}$$

with ϕ^n the n-fold iterate of ϕ.

Before moving on to the proof, let us have a digest. The morphism ϕ induces a $\mathbb{Z}$-linear map $\phi^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(X)$.[1] Tensoring with $\mathbb{R}$ gives a linear map $\phi^*\colon \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ of real vector spaces of finite dimension. Say L is non-trivial. Then the assumption $\phi^* L \simeq L^{\otimes \alpha}$ implies that L is an eigenvector for the eigenvalue α. The assumption $\alpha > 1$ guarantees that the *Tate Limit Process* (0.3.1) will work in the end.

We finish this section by two examples of normalized height.

[1] The "addition" on the group $\mathrm{Pic}(X)$ is $\otimes$.

Example 0.3.2. *Let $X = \mathbb{P}^n$ and $L = \mathcal{O}(1)$. Let $\phi\colon \mathbb{P}^n \to \mathbb{P}^n$ be given by homogeneous polynomials of degree $d > 1$, then $\phi^*\mathcal{O}(1) \simeq \mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$. If $\phi([x_0 : \cdots : x_n]) = [x_0^d : \cdots : x_n^d]$, then one can check that $\widehat{h}_{\mathbb{P}^n,\phi,\mathcal{O}(1)}$ is precisely the Weil height h.*

Notice that ϕ restricted to the algebraic torus $\mathbb{G}_{\mathrm{m}}^n \subseteq \mathbb{P}^n$ is precisely the multiplication-by-d morphism. Using this observation, one can prove the following Kronecker's Theorem: For any $\zeta := (\zeta_1, \ldots, \zeta_n) \in \mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}}) = (\overline{\mathbb{Q}}^*)^n$, we have $h(\zeta) = 0$ if and only if each component ζ_j is a root of unity.

A more important example for the Tate Limit Process (0.3.1) is the definition of the *Néron–Tate heights on abelian varieties.* Let $X = A$ be an abelian variety and L be a symmetric line bundle, *i.e.* $[-1]^*L \simeq L$. Then $[n]^*L \simeq L^{\otimes n^2}$ for the multiplication-by-n map. Taking $n = 2$ gives the *Néron–Tate height on A*, which we denote by $\widehat{h}_{A,L}$.

The following theorem summarizes some important properties of $\widehat{h}_{A,L}$. Notice that by (i), in the definition of the Néron–Tate height we can replace the morphism $[2]\colon A \to A$ by $[n]$ for any $n \geq 2$.

Theorem 0.3.3. *We have:*

(i) For each $N \in \mathbb{Z}$, we have $\widehat{h}_{A,L}([N]x) = N^2\widehat{h}_{A,L}(x)$ for all $x \in A(\overline{\mathbb{Q}})$.

(ii) $\widehat{h}_{A,L}(x) \geq 0$ for all $x \in A(\overline{\mathbb{Q}})$.

(iii) Assume L is ample. Then $\widehat{h}_{A,L}(x) = 0$ if and only if x is a torsion point.

(iv) For each finitely generated subgroup Γ of $A(\overline{\mathbb{Q}})$, the $\mathbb{R}$-linearly extension of $\widehat{h}_{A,L}$ is a quadratic form on $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ which is furthermore positive definite.