Generic positivity of the Beilinson–Bloch height (joint with Shouwu Zhang)

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Motivation: Weil height

A. Weil (1928) defined height to measure the “size” of algebraic points.

- ✎ On $\mathbb{Q}$: $h(a/b) = \log\max\{|a|, |b|\}$, for $a, b \in \mathbb{Z}$ and $\gcd(a, b) = 1$.
- ✎ On $\mathbb{P}^n(\mathbb{Q})$: $h([x_0 : \cdots : x_n]) = \log\max\{|x_0|, \ldots, |x_n|\}$, for $x_i \in \mathbb{Z}$ and $\gcd(x_0, \ldots, x_n) = 1$.
- ✎ Arbitrary number field K: For $[x_0 : \cdots : x_n] \in \mathbb{P}^n(K)$,

$$h([x_0 : \cdots : x_n]) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} \log\max\{\|x_0\|_v, \ldots, \|x_n\|_v\}.$$

$\rightsquigarrow$ (logarithmic) Weil height on $\mathbb{P}^n(\overline{\mathbb{Q}})$.

Motivation: Weil height

Two important properties $\rightarrow$

$\downarrow$

Positivity

$h(\mathbf{x}) \geq 0$ *for all* $\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}})$.

Northcott Property (1949)

For all B and $d \geq 1$,

$$\{\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}}) : h(\mathbf{x}) \leq B, [\mathbb{Q}(\mathbf{x}) : \mathbb{Q}] \leq d\}$$

is finite.

Motivation: (naive) Height Machine

X projective variety defined over a number field K_0.

- X can be embedded into $\mathbb{P}^N$ $\rightsquigarrow$ naive height h_{Weil} on $X(\overline{\mathbb{Q}})$
- Different embeddings $\rightsquigarrow$ well-defined up to a bounded function.

Two important properties $\rightarrow$ $\downarrow$

Northcott Property

For all B and $d \geq 1$,

$$\{x \in X(\overline{\mathbb{Q}}) : h(x) \leq B, [K_0(x) : K_0] \leq d\}$$

is finite.

Bounded from below

There exists C such that $h_{\mathrm{Weil}}(x) \geq C$ for all $x \in X(\overline{\mathbb{Q}})$.

Motivation: Dominant height function

- X quasi-projective variety defined over $\overline{\mathbb{Q}}$;
- $h\colon X(\overline{\mathbb{Q}}) \to \mathbb{R}$.

Definition

h is called a dominant height if it has a lower bound and satisfies the Northcott property.

Two famous examples:

Example

Néron–Tate height on abelian variety A, with lower bound 0. $\rightsquigarrow$ Mordell–Weil theorem, formulation of Birch and Swinnerton-Dyer Conjecture, etc.

Example (On the moduli space $\mathbb{M}_g$ of smooth projective curves of genus g)

$h_{\mathrm{Fal}}\colon \mathbb{M}_g(\overline{\mathbb{Q}}) \to \mathbb{R}$, sending each curve C to the Faltings height of its Jacobian. $\rightsquigarrow$ Mordell Conjecture.

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Motivation: Beilinson–Bloch height and conjecture

Aim (from 1980s):

- Extend height from points to higher cycles which are homologically trivial (Beilinson–Bloch height).
- Positivity of BB height.
- Finiteness of the rank of Chow group.
- Generalization of BSD.

Known results

- Conjecturally defined. Unconditional in some cases (Gross–Schoen, Künnemann, S. Zhang).
- Some sporadic families.
- ???
- ???

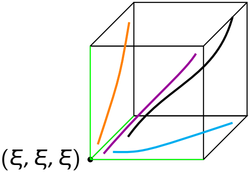
Motivation: Gross–Schoen and Ceresa cycles

Example (BB height is known to be unconditionally defined)

- *C smooth projective curve of genus $g \geq 2$;*
- *$\xi \in \mathrm{Pic}^1(C)$ such that $(2g-2)\xi = \omega_C$.*

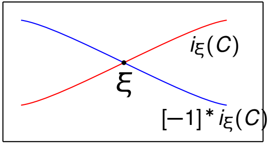
From these data, we obtain homologically trivial 1-cycles:

- *(Gross–Schoen) $\Delta_{\mathrm{GS}}(C) \in \mathrm{Ch}_1(C^3)$ the modified diagonal;*
- *(Ceresa) $\mathrm{Ce}(C) := i_\xi(C) - [-1]^* i_\xi(C) \in \mathrm{Ch}_1(J)$, with $J = \mathrm{Jac}(C)$.*



modified diagonal

$\Delta_{123} - \Delta_{12} - \Delta_{23} - \Delta_{13} + \Delta_1 + \Delta_2 + \Delta_3$.



Goal of the project

Propose a systematic way to study the positivity of the Beilinson–Bloch height $\langle \bullet, \bullet \rangle_{\mathrm{BB}}$.

- Starting point: Use $\langle \bullet, \bullet \rangle_{\mathrm{BB}}$ to define a function on a suitable parametrizing space.

Setup for our main result

Two functions on $\mathbb{M}_g$:

$$h_{\mathrm{GS}}\colon \mathbb{M}_g(\overline{\mathbb{Q}}) \to \mathbb{R}, \qquad [C] \mapsto \langle \Delta_{\mathrm{GS}}(C), \Delta_{\mathrm{GS}}(C)\rangle_{\mathrm{BB}}$$

$$h_{\mathrm{Ce}}\colon \mathbb{M}_g(\overline{\mathbb{Q}}) \to \mathbb{R}, \qquad [C] \mapsto \langle \mathrm{Ce}(C), \mathrm{Ce}(C)\rangle_{\mathrm{BB}}$$

Facts:

- Both vanish on the hyperelliptic locus;
- $h_{\mathrm{GS}} = 6h_{\mathrm{Ce}}$

Question (in different grades)

Assume $g \geq 3$.

- ✎ *(i) Is h_{GS} a dominant height (lower bound + Northcott property) on a Zariski open dense subset U of $\mathbb{M}_g$ defined over $\mathbb{Q}$? $\rightsquigarrow$ generic positivity*
- ✎ *(ii) Can we determine U?*
- ✎ *(iii) Is the lower bound ≥ 0?*

Our main result

Theorem (G'–S.Zhang, 2024)

Assume $g \geq 3$. Let $\mathbb{M}_g^{\mathrm{amp}}$ be the maximal $\overline{\mathbb{Q}}$-Zariski open subset of $\mathbb{M}_g$ on which h_{GS} is a dominant height.
Then $\mathbb{M}_g^{\mathrm{amp}}$ is non-empty and is defined over $\mathbb{Q}$. ✓ *for (i)*
Moreover, $\mathbb{M}_g^{\mathrm{amp}}$ can be "constructed". ✓ *partially for (ii)*

Still, we need to express $\mathbb{M}_g^{\mathrm{amp}}$ more explicitly and need to show that the lower bound is ≥ 0. But already, we have

Corollary (Generic positivity)

For any number field K, there are at most finitely many C/K lying in $\mathbb{M}_g^{\mathrm{amp}}(\overline{\mathbb{Q}})$ such that $h_{\mathrm{GS}}([C]) \leq 0$.

Key steps of our proof

Steps:

- h_{GS} defined by an a.l.b. $\overline{\mathscr{L}}$
- volume identity for $\mathrm{vol}(\widetilde{\mathscr{L}})$

Tools:

- Adelic line bundle (Yuan–Zhang 2021).
- Morse Inequality (Demailly 1991).

Bridged via:

- ✎ Abel–Jacobi periods (Griffiths 1960s)
- ✎ archimedean local heights (Hain 1990s)

- Algebraicity of Betti strata
- Non-vanishing of Betti form

- Mixed Ax–Schanuel (Chiu/Gao–Klingler 2021).
- O-minimality for period map (Bakker, Brunebarbe, Klingler, Tsimerman 2018–2020...).

Adelic line bundle

Theorem

There exists an adelic line bundle $\overline{\mathscr{L}}$ on $\mathbb{M}_g$ such that $h_{\mathrm{GS}} = h_{\overline{\mathscr{L}}}$.

A construction was given by Yuan. We give a new construction using:

- Polarized dynamical system on the universal Jacobian $\mathrm{Jac}(\mathscr{C}_g/\mathbb{M}_g) \to \mathbb{M}_g$.
- Deligne pairing to "push-forward" adelic line bundles on $\mathscr{C}_g \times_{\mathbb{M}_g} \mathscr{C}_g$ to $\mathbb{M}_g$.
- Explicit computation.
- We use our construction to prove the volume identity.

Adelic line bundle

✎ What is an adelic line bundle, and what is the motivation/idea behind?

Let (X, L) projective variety with a line bundle, defined over a number field K.

- Naive height $h_L\colon X(\overline{\mathbb{Q}}) \to \mathbb{R}$, *well-defined up to a bounded function*.
- Wish to get genuine functions. Sometimes okay, *e.g.* Néron–Tate height on abelian varieties.
- In general, integral model $(\mathscr{X}, \overline{\mathscr{L}})$, with $\overline{\mathscr{L}}$ a Hermitian line bundle.
  But cannot recover Néron–Tate height in this way!!
- Solution: Put a $\overline{K_v}$-metric of L on $X(\overline{K_v})$ for *all* $v \in M_K \rightsquigarrow$ metrized line bundle
  An adelic line bundle $\overline{\mathscr{L}}$ is a metrized line bundle which can be obtained as a "limit" of integral models.
- This construction can be generalized to quasi-projective varieties, "limit" of integral models of compactifications of $X \rightsquigarrow$ generic fiber $\widetilde{\mathscr{L}}$ of $\overline{\mathscr{L}}$.

Example ($X = \mathrm{Spec} K$)

An adelic line bundle on $\mathrm{Spec} K$ *is* $(L, \{\|\cdot\|_v\}_v)$ *with* $L =$*vector space of dim* 1 *and* $\|\cdot\|_v$ *a* K_v*-metric, satisfying:* $\forall \ell \in L \setminus \{0\}$, $\|\ell\|_v = 1$ *for all but finitely many* v.

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Volume identity

Recall our main theorem

Theorem (G'–S.Zhang, 2024)

Assume $g \geq 3$. Then h_{GS} is a dominant height on a Zariski open dense subset $\mathbb{M}_g^{\mathrm{amp}}$ of $\mathbb{M}_g$ defined over $\mathbb{Q}$. ✓ for (i)
Moreover, $\mathbb{M}_g^{\mathrm{amp}}$ can be "constructed". ✓ partially for (ii)

- Part (i) except "defined over $\mathbb{Q}$"$\Leftrightarrow$ $\widetilde{\mathscr{L}}$ is big, *i.e.* $\mathrm{vol}(\widetilde{\mathscr{L}}) > 0$.

A key property we prove is the following volume identity.

Theorem (GZ 2024)

$$\mathrm{vol}(\widetilde{\mathscr{L}}) = \int_{\mathbb{M}_g(\mathbb{C})} c_1(\overline{\mathscr{L}})^{\wedge \dim \mathbb{M}_g}.$$

Stronger: needed for "over $\mathbb{Q}$"

Theorem (GZ 2024)

For each subvariety S of $\mathbb{M}_{g,\mathbb{C}}$, we have

$$\mathrm{vol}(\widetilde{\mathscr{L}}_{\mathbb{C}}|_S) = \int_{S(\mathbb{C})} c_1(\overline{\mathscr{L}})^{\wedge \dim S}.$$

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- LHS defined using some kind of h^0, so invariant under $\mathrm{Aut}(\mathbb{C})$.
  $\rightsquigarrow$ Used for "over $\mathbb{Q}$" in the main theorem.
- In the flavor of (arithmetic) Hilbert–Samuel.
- Problem: $\widetilde{\mathscr{L}}$ is not known to be nef!!!
- Solution: Compute $\mathrm{vol}(\widetilde{\mathscr{L}}_{\mathbb{C}}|_S)$ directly, by our explicit construction of $\overline{\mathscr{L}} = \{(\mathscr{M}_i, \overline{\mathscr{L}}_i)\}_{i\geq 1}$ and the fact $\mathrm{vol}(\mathscr{L}_{i,\mathbb{Q}}|_{\overline{S}}) \longrightarrow \mathrm{vol}(\widetilde{\mathscr{L}}_{\mathbb{C}}|_S)$. Use Demailly's Morse Inequality to bound $h^0(m\mathscr{L}_{i,\mathbb{C}}|_{\overline{S}})$ and hence handle $\mathrm{vol}(\mathscr{L}_{i,\mathbb{C}}|_{\overline{S}})$. Need our explicit construction to get fast enough convergence.

A dévissage

Theorem (G'–S.Zhang, 2024)

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- "$\{x \in S(\mathbb{C}) : (c_1(\overline{\mathscr{L}})|_S^{\wedge \dim S})_x = 0\}$" is Zariski closed.

$\uparrow$

It remains to prove this.

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Some key ingredients of our proof

Key steps:

- ✓ h_{GS} defined by an a.l.b. $\overline{\mathscr{L}}$
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- ➢ Adelic line bundle.
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Bridged via:

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- ➢ Algebraicity of Betti strata
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Bridged by: (R. de Jong, GZ) $c_1(\overline{\mathscr{L}})$ equals the Betti form.

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General setup for studying Betti strata/form/foliation

- $f\colon X \to S$ projective morphism over quasi-projective variety, over $\mathbb{C}$,
- Z is a family of homologically trivial cycles, of codimension n.

Example (Gross–Schoen and Ceresa)

(GS) $f\colon \mathscr{C}_g \times_{\mathbb{M}_g} \mathscr{C}_g \times_{\mathbb{M}_g} \mathscr{C}_g \to \mathbb{M}_g$, *$Z$ is the family of Gross–Schoen cycles. $n = 2$.*

(Ce) $f\colon \mathrm{Jac}(\mathscr{C}_g/\mathbb{M}_g) \to \mathbb{M}_g$, *$Z$ is the family of Ceresa cycles. $n = g - 1$.*

- $\mathbb{V}_{\mathbb{Z}} := Rf_*^{2n-1}\mathbb{Z}_X$. Each fiber $\mathbb{V}_{\mathbb{Z},s} = H^{2n-1}(X_s, \mathbb{Z})$.
- de Rham–Betti comparison $\Rightarrow$ $\mathbb{V}_{\mathbb{Z}}$ is a VHS (variation of Hodge structures) of weight $2n-1$.
- Polarization (by Lefschetz)

$$Q\colon \mathbb{V} \otimes \mathbb{V} \to \mathbb{Q}_S(-n)$$

with $\mathbb{V} := \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}_S} \mathbb{Q}_S$.

Intermediate Jacobian and normal function

Definition

The n-th relative intermediate Jacobian is

$$J^n(X/S) := F^n\mathscr{V}\backslash\mathscr{V}/\mathbb{V}_{\mathbb{Z}},$$

with $\mathscr{V} := \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}_S} \mathscr{O}_S$ the holomorphic vector bundle.

The fibers are

$$(*) \qquad J^n(X_s) \begin{array}{ll} = F^n\backslash H^{2n-1}(X_s,\mathbb{C})/H^{2n-1}(X_s,\mathbb{Z}) & \text{compact complex torus} \\ \cong H^{2n-1}(X_s,\mathbb{R})/H^{2n-1}(X_s,\mathbb{Z}) & \text{real torus} \end{array}$$

- (Griffiths 1969) $\mathrm{AJ}\colon \mathrm{Ch}^n(X_s)_{\mathrm{hom}} \to J^n(X_s)$.

Definition (Normal function)

$$\nu = \nu_Z\colon S \to J^n(X/S), \qquad s \mapsto \mathrm{AJ}(Z_s).$$

Betti form, Betti foliation, Betti strata

Family version of (*) becomes

$$J^n(X/S) \xrightarrow{\sim} \mathbb{V}_{\mathbb{R}}/\mathbb{V}_{\mathbb{Z}}.$$

$$\begin{aligned}\nu_{\mathrm{Betti},s}\colon T_sS \xrightarrow{\mathrm{d}\nu} & T_{\nu(s)}J^n(X/S) \\ \cong\ & T_{\nu(s)}\mathbb{V}_{\mathbb{R}}/\mathbb{V}_{\mathbb{Z}} \\ =\ & T_sS \oplus \mathbb{V}_{\mathbb{R},s} \to \mathbb{V}_{\mathbb{R},s}\end{aligned}$$

$\mathbb{V}_{\mathbb{R}}/\mathbb{V}_{\mathbb{Z}} \to S$ local system of real tori

$\downarrow$

Betti foliation $\mathscr{F}_{\mathrm{Betti}}$ on $J^n(X/S)$

Definition (Betti form)

$\beta_\nu(u,v) := 2Q_s(\nu_{\mathrm{Betti},s}(u), \nu_{\mathrm{Betti},s}(v))$ *for all* $s \in S(\mathbb{C})$ *and* $u, v \in T_sS$.

Definition (Betti strata)

For any $t \geq 1$, $S^{\mathrm{Betti}}(t) := \{s \in S(\mathbb{C}) : \dim_s(\nu(S) \cap \mathscr{F}_{\mathrm{Betti}}) \geq t\}$.

- β_ν semi-positive $(1,1)$-form (Hain 1990s, using Griffiths' transversality)
- $\beta_\nu^{\wedge \dim S} \equiv 0$ " $\Leftrightarrow$ " $S^{\mathrm{Betti}}(1) = S$

Our result on Betti rank and Betti strata

Theorem (GZ 2024)

- *$S^{\mathrm{Betti}}(t)$ is Zariski closed in S.*
- *We have a checkable criterion for $S^{\mathrm{Betti}}(t) = S$ (equivalently a formula to compute the generic rank of $\nu_{\mathrm{Betti},s}$). In particular, a checkable criterion for $\beta_\nu^{\wedge \dim S} \equiv 0$.*

- O-minimality for period map to use definable Chow.
- Mixed Ax–Schanuel used twice, second time is through Geometric Zilber–Pink (an application of Ax–Schanuel; Ullmo, Daw–Ren, Gao, Baldi–Klingler–Ullmo, Baldi–Urbanik).

Back to Gross–Schoen and Ceresa

Main theorem reduced to prove:

For our adelic line bundle $\overline{\mathscr{L}}$ on $\mathbb{M}_g$ with $h_{\overline{\mathscr{L}}} = h_{\mathrm{GS}}$:

- $c_1(\overline{\mathscr{L}}) \geq 0$,
- $c_1(\overline{\mathscr{L}})^{\wedge \dim \mathbb{M}_g} \not\equiv 0$ *if* $g \geq 3$,
- "$\left\{x \in S(\mathbb{C}) : (c_1(\overline{\mathscr{L}})|_S^{\wedge \dim S})_x = 0\right\}$" *is Zariski closed,* $\forall$ *subvariety* $S \subseteq \mathbb{M}_{g,\mathbb{C}}$.

$\Uparrow$ (R. de Jong, GZ) $c_1(\overline{\mathscr{L}})$ equals the Betti form β_{GS}.

Corollary (particular case of GZ 2024 on Betti rank and Betti strata)

- $\beta_{\mathrm{GS}} \geq 0$ *(Hain 1990s),*
- $\beta_{\mathrm{GS}}^{\wedge \dim \mathbb{M}_g} \not\equiv 0$ *if* $g \geq 3$ *(in this case independently by Hain 2024),*
- $S(1)$ *is Zariski closed,* $\forall$ *subvariety* $S \subseteq \mathbb{M}_{g,\mathbb{C}}$.

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Main theorem reduced to prove:

For our adelic line bundle $\overline{\mathscr{L}}$ on $\mathbb{M}_g$ with $h_{\overline{\mathscr{L}}} = h_{\mathrm{GS}}$:

- $c_1(\overline{\mathscr{L}}) \geq 0$,
- $c_1(\overline{\mathscr{L}})^{\wedge \dim \mathbb{M}_g} \not\equiv 0$ *if* $g \geq 3$,
- "$\left\{x \in S(\mathbb{C}) : (c_1(\overline{\mathscr{L}})|_S^{\wedge \dim S})_x = 0\right\}$" *is Zariski closed,* $\forall$ *subvariety* $S \subseteq \mathbb{M}_{g,\mathbb{C}}$.

$\Uparrow$ (R. de Jong, GZ) $c_1(\overline{\mathscr{L}})$ equals the Betti form β_{GS}.

Corollary (particular case of GZ 2024 on Betti rank and Betti strata)

- $\beta_{\mathrm{GS}} \geq 0$ *(Hain 1990s),*
- $\beta_{\mathrm{GS}}^{\wedge \dim \mathbb{M}_g} \not\equiv 0$ *if* $g \geq 3$ *(in this case independently by Hain 2024),*
- $S(1)$ *is Zariski closed,* $\forall$ *subvariety* $S \subseteq \mathbb{M}_{g,\mathbb{C}}$.

Our main result

Theorem (G'–S.Zhang, 2024)

Assume $g \geq 3$. Let $\mathbb{M}_g^{\mathrm{amp}}$ be the maximal $\overline{\mathbb{Q}}$-Zariski open subset of $\mathbb{M}_g$ on which h_{GS} is a dominant height.
Then $\mathbb{M}_g^{\mathrm{amp}}$ is non-empty and is defined over $\mathbb{Q}$. ✓ *for (i)*
Moreover, $\mathbb{M}_g^{\mathrm{amp}}$ can be "constructed". ✓ *partially for (ii)*

Corollary (Generic positivity)

For any number field K, there are at most finitely many C/K lying in $\mathbb{M}_g^{\mathrm{amp}}(\overline{\mathbb{Q}})$ such that $h_{\mathrm{GS}}([C]) \leq 0$.

- Extra result on torsion: For every non-$\overline{\mathbb{Q}}$ point $[C]$ in $\mathbb{M}_g^{\mathrm{amp}}$, the cycles $\Delta_{\mathrm{GS}}(C)$ and $\mathrm{Ce}(C)$ are not torsion in the Chow groups.
This is also independently proved by Hain (2024) for a non-empty real-analytic open subset and Kerr–Tayou (2024) for a $\mathbb{C}$-Zariski open dense subset.

Thanks!