

# Chapter 6

## Baily–Borel compactification

Let  $(\mathbf{G}, X)$  be a Shimura datum. By abuse of notation use  $X$  to denote a connected component. Let  $\Gamma < \mathbf{G}(\mathbb{Q})$  be an arithmetic subgroup.

Throughout the whole chapter, we will assume  $\mathbf{G}$  to be quasi-simple, *i.e.*  $\mathbf{G}^{\text{der}}$  is a simple group. For the purpose of compactifying  $\Gamma \backslash X$  we can easily reduce to this case. Notice that  $\mathbf{G}_{\mathbb{R}}^{\text{der}}$  may not be simple as an  $\mathbb{R}$ -group, so that  $X$  is not necessarily irreducible.

We also fix a maximal  $\mathbb{Q}$ -split torus  $\mathbf{S}$  of  $\mathbf{G}^{\text{der}}$ , and a minimal parabolic subgroup  $\mathbf{P}_0$  of  $\mathbf{G}^{\text{der}}$  which contains  $\mathbf{S}$ .

### 6.1 Baily–Borel compactification as a topological space

Consider the Harish–Chandra embedding

$$X \simeq \mathcal{D} \subseteq \mathfrak{m}^+ \simeq \mathbb{C}^N$$

with  $\overline{\mathcal{D}}$  the closure of  $\mathcal{D}$  in  $\mathfrak{m}^+$ . Let  $F \neq X$  be an analytic boundary component of  $X$ , with normalizer  $N(F) = \{g \in \mathbf{G}^{\text{der}}(\mathbb{R})^+ : gF = F\}$ .

Recall (5.1.3) the fundamental set  $\Sigma$  constructed from Siegel sets associated with the minimal rational parabolic subgroup  $\mathbf{P}_0$  of  $\mathbf{G}^{\text{der}}$ .

Let  $\overline{\Sigma}$  be the closure of  $\Sigma \subseteq X \simeq \mathcal{D} \subseteq \mathfrak{m}^+$ . Then  $\overline{\Sigma} \subseteq \overline{\mathcal{D}}$ , with an induced topology.

**Theorem 6.1.1.** *The followings are equivalent:*

- (1)  $\Gamma F \cap \overline{\Sigma} \neq \emptyset$ ,
- (2)  $F$  is a rational analytic boundary component (*i.e.*  $N(F)$  equals  $\mathbf{P}_F(\mathbb{R})$  for a parabolic subgroup  $\mathbf{P}_F$  of  $\mathbf{G}^{\text{der}}$ ), and  $\mathbf{P}_F$  is a maximal proper parabolic subgroup of  $\mathbf{G}^{\text{der}}$ .

Theorem 6.1.1 indicates that we can do the following compactification of  $\Gamma \backslash X$ :

- (i) Define  $\overline{X}^{\text{BB}} := \mathcal{D} \cup \bigsqcup_{\mathbf{P}} F_{\mathbf{P}} \subseteq \overline{\mathcal{D}}$ , where  $\mathbf{P}$  runs over all *maximal* proper parabolic subgroup of  $\mathbf{G}^{\text{der}}$  and  $F_{\mathbf{P}}$  is the rational analytic boundary component  $F_{\mathbf{P}}$ .
- (ii) Endow  $\overline{X}^{\text{BB}}$  with the Satake topology.
- (iii) The space  $\Gamma \backslash \overline{X}^{\text{BB}}$  is then a compact Hausdorff space containing  $\Gamma \backslash X$  as an open dense subset.

Then  $(\Gamma \backslash \overline{X})^{\text{BB}} := \Gamma \backslash \overline{X}^{\text{BB}}$  is called the *Baily–Borel compactification* of  $\Gamma \backslash X$ , and

$$(\Gamma \backslash \overline{X})^{\text{BB}} = \Gamma \backslash X \cup \bigsqcup_{j=1}^m \Gamma_{F_j} \backslash F_j, \quad (6.1.1)$$

where  $F_1, \dots, F_m$  are rational analytic boundary components such that  $\{\mathbf{P}_{F_1}, \dots, \mathbf{P}_{F_m}\}$  is a set of representatives of  $\Gamma$ -conjugacy classes of maximal proper parabolic subgroups of  $\mathbf{G}^{\text{der}}$ , with  $\Gamma_{F_j} := \Gamma \cap \mathbf{P}_{F_j}(\mathbb{Q})$ .

### 6.1.1 Satake topology on $\overline{X}^{\text{BB}}$

The *Satake topology* on  $\overline{X}^{\text{BB}}$  is defined as follows. For each  $x \in \overline{X}^{\text{BB}} \subseteq \overline{\mathcal{D}}$ , the neighborhoods of any point  $x \in X^*$  is the saturations of the neighborhoods of the corresponding points in  $\overline{\Sigma}$  under the action of  $\Gamma_x := \{\gamma \in \Gamma : \gamma \cdot x = x\}$ . More precisely, a fundamental system of neighborhoods of  $x$  is given by all subsets  $U \subseteq \overline{\mathcal{D}}$  such that

$$\Gamma_x \cdot U = U,$$

and such that  $\gamma U \cap \overline{\Sigma}$  is a neighborhood of  $\gamma \cdot x$  in  $\overline{\Sigma}$  whenever  $\gamma \cdot x \in \overline{\Sigma}$ .

**Proposition 6.1.2.** *The Satake topology is the unique topology on  $\overline{X}^{\text{BB}}$  such that the followings hold:*

(i) *it induces the original topologies on  $\overline{\Sigma}$  and on  $X$ ,*

(ii) *the  $\mathbf{G}^{\text{der}}(\mathbb{Q})$ -action on  $\overline{X}^{\text{BB}}$  is continuous,*

(iii) *for any  $x \in \overline{X}^{\text{BB}}$ , there exists a fundamental system of neighborhoods  $\{U\}$  of  $x$  such that*

$$\gamma U = U \text{ for all } \gamma \in \Gamma_x \quad \text{and} \quad \gamma U \cap U = \emptyset \text{ for all } \gamma \notin \Gamma_x,$$

(iv) *if  $x, x' \in \overline{X}^{\text{BB}}$  are not in one  $\Gamma$ -orbit, then there exist neighborhoods  $U$  of  $x$  and  $U'$  of  $x'$  such that*

$$\Gamma U \cap U' = \emptyset.$$

**Corollary 6.1.3.**  $\Gamma \backslash \overline{X}^{\text{BB}}$  is compact and Hausdorff.

### 6.1.2 $\mathbb{Q}$ -roots vs $\mathbb{R}$ -roots, and $\mathbb{Q}$ -polydisc

Let  $K_\infty$  be a maximal compact subgroup of  $\mathbf{G}_\mathbb{R}^{\text{der}}$  such that  $\text{Lie}K_\infty \cap \text{Lie}\mathbf{S}_\mathbb{R} = 0$ , for the maximal  $\mathbb{Q}$ -split torus  $\mathbf{S} < \mathbf{P}_0$ .<sup>[1]</sup> Then there exists  $x_0 \in X$  such that  $\text{Stab}_{\mathbf{G}^{\text{der}}(\mathbb{R})}(x_0) = K_\infty$ .

In §4.6.2 and §4.6.3, we discussed the complex roots and real roots respectively. Now we need to analyze the  $\mathbb{Q}$ -roots of  $\mathbf{G}^{\text{der}}$ . First, we can make an appropriate choice of  $K_\infty$  such that  $\mathbf{S}_\mathbb{R} < A$  with  $A$  from §4.6.3.

Let  ${}_{\mathbb{R}}\Psi = \{\gamma_1, \dots, \gamma_r\}$  be as in (4.6.13); it is obtained from a set of strongly orthogonal complex roots  $\Psi = \{\alpha_1, \dots, \alpha_r\}$  from (4.6.4). If  $\mathbf{G}_\mathbb{R}^{\text{der}}$  is simple, we described the real roots  ${}_{\mathbb{R}}\Phi = \Phi(A, \mathbf{G}_\mathbb{R}^{\text{der}})$  in terms of  $\gamma_1, \dots, \gamma_r$  in Proposition 4.6.12. It turns out that one can also do this for the rational roots  ${}_{\mathbb{Q}}\Phi := \Phi(\mathbf{S}, \mathbf{G}^{\text{der}})$  when  $\mathbf{G}^{\text{der}}$  is simple.

<sup>[1]</sup>Even strongly,  $\text{Lie}K_\infty$  is orthogonal to  $\text{Lie}\mathbf{S}_\mathbb{R}$  for the Killing form on  $\text{Lie}\mathbf{G}_\mathbb{R}^{\text{der}}$ .

**Proposition 6.1.4.** *Let  $s = \dim \mathbf{S}$ . There is a partition*

$$\{1, \dots, r\} = I_0 \cup I_1 \cup \dots \cup I_s \quad (6.1.2)$$

such that

$$\begin{aligned} X(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{Q} &\simeq \frac{X(A) \otimes_{\mathbb{Z}} \mathbb{Q}}{\{subspace \text{ spanned by } \gamma_i \text{ for } i \in I_0 \text{ and } \gamma_i - \gamma_j \text{ for } i, j \in I_\ell, \ell > 0\}} \\ &\simeq \sum_{\ell=1}^s \mathbb{Q}\beta_\ell, \quad \text{where } \beta_\ell = \text{image of any } \gamma_j \text{ with } j \in I_\ell. \end{aligned}$$

In particular,  $\mathbf{S} = \{\gamma_i = 1 \text{ for } i \in I_0; \gamma_i = \gamma_j \text{ for } i, j \in I_\ell, \text{ where } \ell > 0\}$ .

**Corollary 6.1.5.** *Recall our assumption that  $\mathbf{G}^{\text{der}}$  is simple. One of the two cases occurs:*

- (Type  $C_s$ )  ${}_{\mathbb{Q}}\Phi = \{\pm \frac{1}{2}(\beta_i + \beta_j) \text{ for } i \geq j, \pm \frac{1}{2}(\beta_i - \beta_j) \text{ for } i > j\}$ .
- (Type  $BC_s$ )  ${}_{\mathbb{Q}}\Phi = \{\pm \frac{1}{2}(\beta_i + \beta_j) \text{ for } i \geq j, \pm \frac{1}{2}(\beta_i - \beta_j) \text{ for } i > j, \pm \frac{1}{2}\beta_i\}$ .

If we order the roots such that  $\beta_1 > \dots > \beta_s$ , then the set of simple roots is:

- (Type  $C_s$ )  ${}_{\mathbb{Q}}\Delta = \{\nu_1 := \frac{1}{2}(\beta_1 - \beta_2), \dots, \nu_{s-1} := \frac{1}{2}(\beta_{s-1} - \beta_s), \nu_s := \beta_s\}$ .
- (Type  $BC_s$ )  ${}_{\mathbb{Q}}\Delta = \{\nu_1 := \frac{1}{2}(\beta_1 - \beta_2), \dots, \nu_{s-1} := \frac{1}{2}(\beta_{s-1} - \beta_s), \nu_s := \frac{1}{2}\beta_s\}$ .

The proof goes as follows: We have the group-theoretic result that  $\mathbf{G}^{\text{der}} = \text{Res}_{k/\mathbb{Q}} \mathbf{G}'$  for some absolutely simple  $k$ -group  $\mathbf{G}'$  with  $k$  a totally real number field. Then  $\mathbf{G}_{\mathbb{R}}^{\text{der}} = \prod_{\sigma: k \hookrightarrow \mathbb{R}} \mathbf{G}'_\sigma$  with each  $\mathbf{G}'_\sigma$  a group defined over  $\sigma(k) \subseteq \mathbb{R}$ . Then one analyzes each factor and use the Galois action.

Next we turn to the  $\mathbb{Q}$ -polydisc. Recall from the Polydisc Theorem (Theorem 4.6.7) that we have a totally geodesic embedding  $D^r \rightarrow X$  (with  $D = \{z \in \mathbb{C} : |z| < 1\}$  the Poincaré unit disc) arising from a group morphism

$$\varphi: \text{SL}_2(\mathbb{R})^r \rightarrow \mathbf{G}^{\text{der}}(\mathbb{R}), \quad (6.1.3)$$

and  $X = K_\infty \cdot D^r$ . This embedding gives rise to the analytic boundary components as in the diagram (4.6.15). Let us rephrase it here. Recall  $\mathfrak{H} \simeq D$  with the Cayley transformation sending  $\sqrt{-1} \mapsto 0$  and  $\infty \mapsto 1$ . Then we have the diagram

$$\begin{array}{ccc} \mathfrak{H}^r & \xrightarrow{f_1} & \mathcal{D} \\ \subseteq \downarrow & & \downarrow \subseteq \\ (\mathbb{P}^1)^r & \xrightarrow{f_3} & X^\vee \end{array} \quad (6.1.4)$$

where  $f_1$  is the natural composite  $\mathfrak{H}^r \simeq D^r \rightarrow X \simeq \mathcal{D}$ , with  $D^r \rightarrow X$  the geodesic embedding as above and  $X \simeq \mathcal{D}$  the Harish–Chandra realization, and  $\mathcal{D} \subseteq X^\vee$  from (4.6.3). Then for any subset  $S \subseteq \{1, \dots, r\}$ , the unique standard analytic boundary component containing the point  $f_3((\sqrt{-1})_{j \notin S}, (\infty)_{j \in S})$  is  $F_S$ . In general, an analytic boundary component of  $X$  is of the form  $g \cdot F_S$  for some  $g \in \mathbf{G}^{\text{der}}(\mathbb{R})$ .

We wish to do this discussion and obtain the relevant results over  $\mathbb{Q}$ . First of all, any rational analytic boundary component is easily seen to be of the form  $g \cdot F_S$ , with  $g \in \mathbf{G}^{\text{der}}(\mathbb{Q})$  and  $F_S$  rational. Next we prove the following lemma.

**Lemma 6.1.6.** *For  $S \subseteq \{1, \dots, r\}$ , the standard analytic boundary component  $F_S$  is rational if and only if  $S = I_{\ell_1} \cup \dots \cup I_{\ell_t}$ , where  $1 \leq \ell_1 < \dots < \ell_t \leq s$ , for the partition (6.1.2).*

*Proof.* For the proof, it is more convenient to use the description of parabolic subgroups given by Theorem 4.2.5. In §4.6.6, we explained that the normalizer  $P_{F_S} = P(\lambda_S)$ , with  $\lambda_S: \mathbb{G}_{m,\mathbb{R}} \rightarrow \mathbf{G}_{\mathbb{R}}^{\text{der}}$  sending

$$t \mapsto \varphi(\underbrace{\dots, 1, \dots, \dots}_{j \notin S}, \underbrace{\left[ \begin{array}{cc} t & 0 \\ 0 & t^{-1} \end{array} \right], \dots}_{j \in S}),$$

with  $\varphi$  from (6.1.3). By Proposition 6.1.4,  $\lambda$  is defined over  $\mathbb{Q}$  if and only if  $S = I_{\ell_1} \cup \dots \cup I_{\ell_t}$  for some  $1 \leq \ell_1 < \dots < \ell_t \leq s$ . We are done.  $\square$

With this lemma in hand, we obtain the  $\mathbb{Q}$ -version of (6.1.4)

$$\begin{array}{ccc} \mathfrak{H}^s & \xrightarrow{f_{1,\mathbb{Q}}} & \mathcal{D} \\ \subseteq \downarrow & & \downarrow \subseteq \\ (\mathbb{P}^1)^s & \xrightarrow{f_{3,\mathbb{Q}}} & X^\vee \end{array} \quad (6.1.5)$$

arising from

$$\varphi_{\mathbb{Q}}: \text{SL}_2(\mathbb{R})^s \rightarrow \mathbf{G}_{\mathbb{R}}^{\text{der}} \quad (6.1.6)$$

such that  $\varphi_{\mathbb{Q}}$ (diagonal matrices) is the maximal  $\mathbb{Q}$ -split torus  $\mathbf{S}$  of  $\mathbf{G}_{\mathbb{R}}^{\text{der}}$ . We can renumber the factors of  $\mathfrak{H}^s$  and  $\text{SL}_2(\mathbb{R})^s$  such that: For the  $\beta_1, \dots, \beta_s \in \Phi$  from Proposition 6.1.4, we have

$$\beta_\ell: \varphi_{\mathbb{Q}} \left( \left[ \begin{array}{cc} t_1 & 0 \\ 0 & t_1^{-1} \end{array} \right], \dots, \left[ \begin{array}{cc} t_s & 0 \\ 0 & t_s^{-1} \end{array} \right] \right) \mapsto t_\ell^2. \quad (6.1.7)$$

Now for each subset  $S_{\mathbb{Q}} \subseteq \{1, \dots, s\}$ , the unique standard analytic boundary component which contains the point

$$f_{3,\mathbb{Q}}(\underbrace{\dots, \sqrt{-1}, \dots, \dots, \infty, \dots}_{\ell \notin S_{\mathbb{Q}}}, \underbrace{\dots, \infty, \dots}_{\ell \in S_{\mathbb{Q}}})$$

is  $F_S$  with  $S = \bigcup_{\ell \in S_{\mathbb{Q}}} I_\ell$ . In particular,  $F_S$  is rational.

*Proof of Theorem 6.1.1.* Assume  $F$  meets  $\bar{\Sigma}$ .

Order the roots such that  $\beta_1 > \dots > \beta_s$ , then  $\mathbf{S}(\mathbb{R})^+$  consists of

$$\varphi_{\mathbb{Q}} \left( \left[ \begin{array}{cc} t_1 & 0 \\ 0 & t_1^{-1} \end{array} \right], \dots, \left[ \begin{array}{cc} t_s & 0 \\ 0 & t_s^{-1} \end{array} \right] \right)$$

where  $t_1 \geq \dots \geq t_s \geq 1$ . Hence

$$\overline{\mathbf{S}(\mathbb{R})^+ x_0} = f_{3,\mathbb{Q}} \left( \{(\sqrt{-1}x_1, \dots, \sqrt{-1}x_s) : \infty \geq x_1 \geq \dots \geq x_s \geq 1\} \right).$$

Hence  $\overline{\mathbf{S}(\mathbb{R})^+ x_0}$  meets exactly the standard boundary components  $F_1, \dots, F_s$  with

$$f_{3,\mathbb{Q}}(\infty, \sqrt{-1}, \dots, \sqrt{-1}) \in F_1, f_3(\infty, \infty, \sqrt{-1}, \dots, \sqrt{-1}) \in F_2, \dots, f_3(\sqrt{-1}, \sqrt{-1}, \dots, \sqrt{-1}) \in F_s.$$

So  $F = F_\ell$  for some  $\ell \in \{1, \dots, s\}$ . We can compute the normalizer of each  $F_\ell$  as in Theorem 4.6.19, and get

$$N(F_\ell) = \mathbf{P}_{\mathbb{Q}\Delta \setminus \{\nu_\ell\}}(\mathbb{R})$$

for each  $\ell \in \{1, \dots, s\}$ . Hence we are done.  $\square$

## 6.2 First step towards the complex structure

### 6.2.1 A general criterion for a topological space to be complex analytic

Assume  $V$  is a compact Hausdorff space which can be written as a disjoint union

$$V = V_0 \sqcup V_1 \sqcup \cdots \sqcup V_m$$

with each  $V_j$  an irreducible normal complex analytic space. Assume that  $\dim V_0 > \dim V_j$  for all  $j \geq 1$ , and that  $V_0$  is open dense in  $V$ .

Define a sheaf  $\mathcal{F}$  of  $\mathcal{A}$ -functions on  $V$  as follows. For any open subset  $U \subseteq V$ , a complex-valued continuous function on  $U$  is an  $\mathcal{A}$ -function if its restriction to each  $U \cap V_j$  ( $0 \leq j \leq m$ ) is complex analytic.

**Proposition 6.2.1.** *Assume:*

- (i) *For each integer  $d \geq 1$ , the union  $V_{(d)} := \bigcup_{\dim V_j \leq d} V_j$  is closed.*
- (ii) *Any  $v \in V$  has a countable fundamental set of open neighborhoods  $\{U_\ell\}$  such that  $U_\ell \cap V_0$  is connected for all  $\ell$ .*
- (iii) *The restriction to  $V_j$  of local  $\mathcal{A}$ -functions define the structure sheaf of  $V_j$ , for all  $j \geq 0$ .*
- (iv) *Any  $v \in V$  has a neighborhood  $U_v$  whose points are separated by the  $\mathcal{A}$ -functions defined on  $U$ .*

*Then  $V$  is an irreducible normal complex analytic space with structure sheaf  $\mathcal{F}$ . For each  $d \leq \dim V_0$ , the union  $V_{(d)}$  is an analytic subspace of  $V$  with dimension  $\max\{\dim V_j : V_j \subseteq V_{(d)}\}$ .*

### 6.2.2 Application to the Baily–Borel compactification

We shall apply Proposition 6.2.1 to the Baily–Borel compactification (6.1.1) (which is compact Hausdorff space by Corollary 6.1.3), with  $V_0 = \Gamma \backslash X$  and  $V_j = \Gamma_{F_j} \backslash F_j$  for  $1 \leq j \leq m$ .

Conditions (i) and (ii) can be shown to hold by checking with the Satake topology from §6.1.1.

To check condition (iii), we define the projection

$$\pi_F: X \rightarrow F \tag{6.2.1}$$

for each analytic boundary component  $F$ . We focus on the rational ones. The example of the Siegel case will be presented in Example 6.3.6.

Recall our choice of a maximal  $\mathbb{Q}$ -split torus  $\mathbf{S}$  (from §6.1.2) in our minimal parabolic subgroup  $\mathbf{P}_0$  of  $\mathbf{G}^{\text{der}}$  (see above Theorem 6.1.1), and the basis  ${}_{\mathbb{Q}}\Delta = \{\nu_1, \dots, \nu_s\}$  (see Corollary 6.1.5) of the relative root system  ${}_{\mathbb{Q}}\Phi := \Phi(\mathbf{S}, \mathbf{G}^{\text{der}})$ . The root  $\nu_s$  is called the *distinguished root* because it has different length.

Over  $\mathbb{R}$ , we explained the relation between  $F$  and the boundary symmetric domain associated with  $P_F = N(F)$ ; see below Theorem 4.6.19. The discussion can be carried over  $\mathbb{Q}$ .

Let  $F$  be a rational analytic boundary component which meets  $\bar{\Sigma}$ . We have shown in the proof of Theorem 6.1.1 that  $\mathbf{P}_F = \mathbf{P}_{\mathbb{Q}\Delta \setminus \{\nu_\ell\}}$  for some  $\ell \in \{1, \dots, s\}$ . Let  $I_h := \{\nu_{\ell+1}, \dots, \nu_s\}$  and  $I_l := \{\nu_1, \dots, \nu_{\ell-1}\}$ . We thus have the refined rational horospherical decomposition

$$X \simeq N_{P_F}(\mathbb{R}) \times A_{\mathbf{P}_F}(\mathbb{R})^+ \times X_{\mathbf{P}_{I_h}} \times X_{\mathbf{P}_{I_l}}.$$

Moreover, the proof of Theorem 6.1.1 exhausts the possibilities of all  $F$ 's, and hence implies that  $F$  can be identified with the boundary symmetric domain associated with  $\mathbf{P}_{I_h}$ . Thus the refined rational horospherical decomposition above becomes

$$X \simeq N_{P_F}(\mathbb{R}) \times A_{\mathbf{P}_F}(\mathbb{R})^+ \times F \times X_{\mathbf{P}_{I_l}}, \quad (6.2.2)$$

and it induces a natural projection  $X \rightarrow F$ , which is our desired  $\pi_F$ . Although the decomposition is only real semi-algebraic, the projection  $\pi_F$  is also holomorphic.

If  $F$  is contained in  $\overline{F'}$  for another rational boundary component  $F'$ , then  $F$  is a rational boundary component of  $F'$ , and one gets a projection  $\pi_{F',F}: F' \rightarrow F$ . It is not hard to check that  $\pi_F$  is the composite of  $\pi_{F',F} \circ \pi_{F'}$ .

Now to check condition (iii) of Proposition 6.2.1, we only need to work locally and hence on the universal covering. But now for any rational boundary component  $F$  of  $X$ , any complex analytic function near  $F$  can be extended to an  $\mathcal{A}$ -function on a neighborhood of  $F$  in  $\overline{X}^{\text{BB}}$  by the discussion above. This establishes (iii).

Proving condition (iv) is the hardest part. We need to realize  $X$  as a Siegel domain of the third kind<sup>[2]</sup> and define the Poincaré–Eisenstein series.

### 6.3 $X$ as a Siegel domain of the third kind

Continue to use the notation from §6.1.2. In particular, we have the relative root system  ${}_{\mathbb{Q}}\Phi = \Phi(\mathbf{S}, \mathbf{G}^{\text{der}})$ , the roots  $\beta_1 > \dots > \beta_s$  which arise from the set of strongly orthogonal roots, and the basis  ${}_{\mathbb{Q}}\Delta = \{\nu_1 = \frac{1}{2}(\beta_1 - \beta_2), \dots, \nu_{s-1} = \frac{1}{2}(\beta_{s-1} - \beta_s)\} \cup \{\nu_s\}$ ;  $\nu_s$  is the distinguish roots which is either  $\beta_s$  or  $\frac{1}{2}\beta_s$ .

Let  $\mathbf{P}$  be a standard maximal proper parabolic subgroup of  $\mathbf{G}^{\text{der}}$ . Then  $\mathbf{P} = \mathbf{P}_{\mathbb{Q}\Delta \setminus \{\nu_\ell\}}$  for some  $\ell \in \{1, \dots, s\}$ . We have seen in the proof of Theorem 6.1.1 that  $\mathbf{P}(\mathbb{R})$  is the normalizer of the standard rational analytic boundary component  $F = F_S$  with  $S = I_1 \cup \dots \cup I_\ell$ .

The non-standard maximal proper parabolic subgroup of  $\mathbf{G}^{\text{der}}$  are  $\mathbf{G}^{\text{der}}(\mathbb{Q})$ -conjugates of the standard ones, and the rational analytic boundary components are all of the form  $g \cdot F_S$  with  $g \in \mathbf{G}^{\text{der}}(\mathbb{Q})$  and  $F_S$  as above. So all the discussion in this section applies to an arbitrary rational analytic boundary component by applying suitable  $\mathbf{G}^{\text{der}}(\mathbb{Q})$ -conjugation.

#### 6.3.1 Rational 5-group decomposition and refined horospherical decomposition

Let  $I_h := \{\nu_{\ell+1}, \dots, \nu_s\}$  and  $I_l := \{\nu_1, \dots, \nu_{\ell-1}\}$ .

Above Example 4.6.20, we defined several subgroups of  $\mathbf{P}_{\mathbb{R}}$ . We can also define the following  $\mathbb{Q}$ -subgroups of  $\mathbf{P}$  similarly:

- $\mathbf{W}(F) := \mathcal{R}_u(\mathbf{P})$ ,
- $\mathbf{L}(F)$  is a suitable Levi subgroup of  $\mathbf{P}$  which we shall define later using Lie algebra,
- $\mathbf{G}_h(F)$  is the semi-simple group whose  $\mathbb{Q}$ -root system is spanned by  $I_h$ . It is normal subgroup of  $\mathbf{L}(F)$ , has no compact  $\mathbb{Q}$ -factors, and  $F$  is a  $\mathbf{G}_h(F)(\mathbb{R})^+$ -orbit,

<sup>[2]</sup>In the original paper of Baily–Borel, this was done using partial Cayley transformation and the map  $\eta$  from Theorem 2.3.5. We will directly introduce the more explicit version using the 5-decomposition of the parabolic subgroup. This more explicit version will be crucial for the toroidal compactifications.

- $\mathbf{G}_l(F)\mathbf{M}(F)$  is the normal reductive subgroup of  $\mathbf{L}(F)$  which is the complement to  $\mathbf{G}_h(F)$ , with  $\mathbf{G}_l(F)$  with no compact  $\mathbb{Q}$ -factors and  $\mathbf{M}(F)(\mathbb{R})$  compact.

Then  $\mathbf{P} = \mathbf{W}(F) \rtimes (\mathbf{G}_h(F) \cdot \mathbf{G}_l(F) \cdot \mathbf{M}(F))$ .

**Remark 6.3.1.** Let us compare them with the subgroups of  $\mathbf{P}_{\mathbb{R}}$  defined above Example 4.6.20. We have  $W(F) = \mathbf{W}(F)_{\mathbb{R}}$  and  $L(F) = \mathbf{L}(F)_{\mathbb{R}}$  by definition. Later on, we will see that  $G_l(F) = \mathbf{G}_l(F)_{\mathbb{R}}$  in Corollary 6.3.4. However, in general  $\mathbf{G}_h(F)_{\mathbb{R}}$  is not  $G_h(F)$ . In other words,  $G_h(F)$  may not be defined over  $\mathbb{Q}$ , in which case we have  $G_h(F) = \mathbf{G}_h(F)_{\mathbb{R}} \cdot M'$  for some factor  $M'$  of  $\mathbf{M}(F)_{\mathbb{R}}$ .

Denote by  $\mathfrak{g} := \text{Lie}\mathbf{G}^{\text{der}}$ . Then one can compute that

$$\text{Lie}\mathbf{P} = \mathfrak{g}^S \oplus \sum_{\varphi = \frac{\pm\beta_i \pm \beta_j}{2} \text{ or } \frac{\pm\beta_i}{2}, \ell+1 \leq i, j \leq s} \mathfrak{g}_{\varphi} \oplus \sum_{\varphi = \frac{\gamma_i \pm \gamma_j}{2} \text{ or } \frac{\gamma_i}{2}, 1 \leq i \leq \ell} \mathfrak{g}_{\varphi}.$$

Now we define  $\mathbf{L}(F)$  to be the subgroup of  $\mathbf{P}$  whose Lie algebra is the direct sum of the first two factors. Then by construction,  $\mathbf{L}(F)$  is defined over  $\mathbb{Q}$  and  $\mathbf{L}(F)_{\mathbb{R}}$  is precisely the  $L(F)$  defined above Example 4.6.20.

The Lie algebra of  $\mathbf{W}(F) = \mathcal{R}_u(\mathbf{P})$  is, by computation, the direct sum of

$$\mathfrak{u} := \sum_{\substack{\varphi = \frac{\gamma_i + \gamma_j}{2} \\ 1 \leq i, j \leq \ell}} \mathfrak{g}_{\varphi}$$

and

$$\mathfrak{v} := \sum_{\substack{\varphi = \frac{\gamma_i \pm \gamma_j}{2} \text{ or } \frac{\gamma_i}{2} \\ 1 \leq i \leq \ell, \ell+1 \leq j \leq s}} \mathfrak{g}_{\varphi}.$$

Let  $\mathbf{U}(F)$  be the exponent of  $\mathfrak{u}$ . One can prove:

**Lemma 6.3.2.**  $\mathbf{U}(F)$  is the center of  $\mathbf{W}(F)$ , and  $\mathbf{V}(F) := \mathbf{W}(F)/\mathbf{U}(F)$  is a vector group (i.e. abelian and diffeomorphic to its Lie algebra).

Clearly,  $\text{Lie}\mathbf{V}(F)$  can be canonically identified with  $\mathfrak{v}$ .

Write  $U(F) = \mathbf{U}(F)_{\mathbb{R}}$  and  $V(F) = \mathbf{V}(F)_{\mathbb{R}}$ . Then the refined rational horospherical decomposition (6.2.2) can be furthermore refined to be

$$X \simeq U(F)(\mathbb{R}) \times V(F)(\mathbb{R}) \times A_{\mathbf{P}}(\mathbb{R})^+ \times F \times X_{\mathbf{P}, l}. \quad (6.3.1)$$

Notice that the  $\mathbb{R}$ -split torus  $A_{\mathbf{P}}$  is contained in  $\mathbf{G}_l(F)$  by definition of  $\mathbf{G}_l(F)$ , and  $X_{\mathbf{P}, l} \simeq (\mathbf{G}_l(F)/A_{\mathbf{P}})(\mathbb{R})^+$ . Denote by  $K_{l, \infty}$  this maximal compact subgroup of  $\mathbf{G}_l(F)(\mathbb{R})^+$ , then we have

$$X_{\mathbf{P}, l} \simeq \mathbf{G}_l(F)(\mathbb{R})^+ / K_{l, \infty} A_{\mathbf{P}}(\mathbb{R})^+. \quad (6.3.2)$$

### 6.3.2 Cone in $U(F)(\mathbb{R})$

We start with the following proposition. Let us go back to  $\mathbb{R}$ -groups, and recall the subgroups  $G_h(F)$ ,  $G_l(F)$ ,  $M(F)$  of  $\mathbf{P}_{\mathbb{R}}$  defined above Example 4.6.20. We have  $\mathbf{L}(F)_{\mathbb{R}} = G_h(F) \cdot G_l(F) \cdot M(F)$  as almost direct product.

The group  $\mathbf{L}(F)$  acts naturally on  $\mathbf{W}(F)$ , and hence on  $\mathbf{U}(F)$ . So  $G_h(F)$ ,  $G_l(F)$ , and  $M(F)$  act on  $U(F)$ .

**Proposition 6.3.3.** *The centralizer of the action of  $L(F) = G_h(F) \cdot G_l(F) \cdot M(F)$  on  $U(F)$  is  $G_h(F) \cdot M(F)$ .*

*Proof.* The subgroups  $G_h(F)$ ,  $G_l(F)$ , and  $M(F)$  can all be defined using the (real) root decomposition (4.6.12), for example (4.6.14) for  $G_h(F)$ . Hence we can prove this proposition by checking roots and direct computation.  $\square$

This proposition immediately yields the following proposition:

**Corollary 6.3.4.** *The group  $G_l(F)$  is defined over  $\mathbb{Q}$ , and hence is precisely  $\mathbf{G}_l(F)_{\mathbb{R}}$ .*

*Proof.* Since  $U(F)$  is defined over  $\mathbb{Q}$  and  $L(F)$  is defined over  $\mathbb{Q}$ , the centralizer of the action of  $L(F)$  on  $U(F)$  is also defined over  $\mathbb{Q}$ . So  $G_h(F) \cdot M(F)$  is defined over  $\mathbb{Q}$  by Proposition 6.3.3. But  $G_l(F)$  is defined to be the complement of  $G_h(F) \cdot M(F)$  in  $L(F)$ . So we are done.  $\square$

For the morphism  $\varphi_{\mathbb{Q}}: \mathrm{SL}_2(\mathbb{R})^s \rightarrow \mathbf{G}_{\mathbb{R}}^{\mathrm{der}}$  from (6.1.6), take the point

$$\Omega_F := \varphi_{\mathbb{Q}} \left( \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \dots, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\ell\text{-components}}, 1, \dots, 1 \right) \in U(F)(\mathbb{R}).$$

Consider the action of  $G_l(F)(\mathbb{R})^+$  on  $U(F)(\mathbb{R})$ .

**Proposition 6.3.5.**  $\mathrm{Stab}_{G_l(F)(\mathbb{R})^+}(\Omega_F) = K_{l,\infty}$ .

*The orbit*

$$C(F) := \{g\Omega_F g^{-1} : g \in G_l(F)(\mathbb{R})^+\}$$

*is an open symmetric homogeneous cone in  $U(F)(\mathbb{R})$ .*

By (6.3.2), we have  $C(F) \simeq A_{\mathbf{P}}(\mathbb{R})^+ \times X_{\mathbf{P},l}$ . Hence (6.3.1) becomes

$$X \simeq U(F)(\mathbb{R}) \times C(F) \times V(F)(\mathbb{R}) \times F. \quad (6.3.3)$$

Denote by

$$\Phi_F: X \rightarrow C(F) \quad (6.3.4)$$

the natural projection.

**Example 6.3.6.** *In the Siegel case,  $s = r$  (i.e. the  $\mathbb{Q}$ -rank equals the  $\mathbb{R}$ -rank), and the partition (6.1.2) is simply  $I_0 = \emptyset$  and  $I_\ell = \{\ell\}$ .*

Take  $\mathbf{P} = \mathbf{P}_{\mathbb{Q}\Delta \setminus \{\nu_{d'}\}}$ . Then as in Example 4.6.20, we have

$$\begin{aligned}\mathbf{P} &= \left\{ \begin{bmatrix} A' & 0 & B' & * \\ * & u & * & * \\ C' & 0 & D' & * \\ 0 & 0 & 0 & (u^t)^{-1} \end{bmatrix} \in G : \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \in \mathrm{Sp}_{2d', \mathbb{Q}}, u \in \mathrm{GL}_{d-d', \mathbb{Q}} \right\}, \\ \mathbf{W}(F) &= \left\{ \begin{bmatrix} I_{d'} & 0 & 0 & n \\ m^t & I_{d-d'} & n^t & b \\ 0 & 0 & I_{d'} & -m \\ 0 & 0 & 0 & I_{d-d'} \end{bmatrix} : n^t m + b = m^t n + b^t \right\}, \\ \mathbf{G}_h(F) &= \left\{ \begin{bmatrix} A' & 0 & B' & 0 \\ 0 & I_{d-d'} & 0 & 0 \\ C' & 0 & D' & 0 \\ 0 & 0 & 0 & I_{d-d'} \end{bmatrix} : \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \in \mathrm{Sp}_{2d', \mathbb{Q}} \right\} \simeq \mathrm{Sp}_{2d', \mathbb{Q}}, \\ \mathbf{G}_l(F) &= \left\{ \begin{bmatrix} I_{d'} & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & I_{d'} & 0 \\ 0 & 0 & 0 & (u^t)^{-1} \end{bmatrix} : u \in \mathrm{GL}_{d-d', \mathbb{Q}} \right\} \simeq \mathrm{GL}_{d-d', \mathbb{Q}}, \\ \mathbf{M}(F) &= \{\pm I_{2d}\}.\end{aligned}$$

Moreover,

$$\mathbf{U}(F) = \left\{ \begin{bmatrix} I_{d'} & 0 & 0 & 0 \\ 0 & I_{d-d'} & 0 & b \\ 0 & 0 & I_{d'} & 0 \\ 0 & 0 & 0 & I_{d-d'} \end{bmatrix} : b = b^t \right\} \simeq \{b \in \mathrm{Mat}_{(d-d') \times (d-d')}(\mathbb{Q}) : b = b^t\} \ni \Omega_F = I_{d-d'},$$

and

$$C(F) = \{b \in \mathrm{Mat}_{(d-d') \times (d-d')}(\mathbb{R}) : b = b^t, b > 0\}.$$

Notice that  $F \simeq \mathfrak{H}_{d'}$  in this case. The projections (6.2.1) and (6.3.4) are

$$\begin{aligned}\pi_F: \mathfrak{H}_d &\rightarrow F \simeq \mathfrak{H}_{d'}, & \begin{bmatrix} \tau' & \tau_0 \\ \tau_0^t & \tau'' \end{bmatrix} &\mapsto \tau' \\ \Phi_F: \mathfrak{H}_d &\rightarrow C(F), & \begin{bmatrix} \tau' & \tau_0 \\ \tau_0^t & \tau'' \end{bmatrix} &\mapsto \mathrm{Im}\tau'' - (\mathrm{Im}\tau_0)^t (\mathrm{Im}\tau')^{-1} (\mathrm{Im}\tau_0).\end{aligned}\tag{6.3.5}$$

### 6.3.3 Fibered structure

Recall the Harish–Chandra embedding together with the Borel embedding  $X \simeq \mathcal{D} \subseteq \mathfrak{m}^+ \subseteq X^\vee$ , with  $X^\vee$  a  $\mathbf{G}^{\mathrm{der}}(\mathbb{C})$ -orbit.

Define

$$\mathcal{D}(F) := \mathbf{U}(F)(\mathbb{C}) \cdot \mathcal{D} = \bigcup_{g \in \mathbf{U}(F)(\mathbb{C})} g \cdot \mathcal{D} \subseteq X^\vee.\tag{6.3.6}$$

Then  $\mathcal{D}(F)$  has a natural complex structure, and  $U(F)(\mathbb{C})$  acts holomorphically on  $\mathcal{D}(F)$ . So the quotient

$$\mathcal{D}'(F) := \mathbf{U}(F)(\mathbb{C}) \backslash \mathcal{D}(F)\tag{6.3.7}$$

has a complex structure. Moreover since  $\mathcal{D}'(F) \simeq V(F)(\mathbb{R}) \times F$  real semi-algebraically, we have that  $\mathcal{D}'(F) \rightarrow F$  is a complex vector bundle, *i.e.* each  $x \in F$  determines a complex structure on  $V(F)(\mathbb{R})$ .

Now we have a holomorphic isomorphism  $\mathcal{D}(F) \simeq U(F)(\mathbb{C}) \times \mathcal{D}'(F) = (U(F)(\mathbb{R}) \oplus \sqrt{-1}U(F)(\mathbb{R})) \times \mathcal{D}'(F)$ . The cone  $C(F)$  should be seen as a cone in  $\sqrt{-1}U(F)(\mathbb{R})$ .

**Theorem 6.3.7.** *The projection  $\Phi_F$  in (6.3.4) extends to  $\Phi_F: \mathcal{D}(F) \rightarrow U(F)(\mathbb{R})$  such that  $X \simeq \mathcal{D} = \Phi_F^{-1}(C(F))$ .*

We have the following  $\mathbf{P}(\mathbb{R})$ -equivariant commutative diagram of holomorphic maps

$$\begin{array}{ccc}
 C(F) & \subseteq & U(F)(\mathbb{R}) \\
 \Phi_F \uparrow & & \uparrow \Phi_F \\
 X \simeq \mathcal{D} & \subseteq & \mathcal{D}(F) \\
 & \searrow \pi_F & \swarrow \pi'_F \text{ mod } U(F)(\mathbb{C}) \\
 & \mathcal{D}'(F) & \\
 & \downarrow p_F & \\
 & F &
 \end{array}$$

The map  $p_F$  is a holomorphic vector bundle with each fiber  $\simeq V(F)(\mathbb{R})$  (real semi-algebraically).

**Example 6.3.8.** Continue with the Siegel case in Example 6.3.6. We have

$$\mathcal{D}(F) \simeq \left\{ \tau = \begin{bmatrix} \tau' & \tau_0 \\ \tau_0^t & \tau'' \end{bmatrix} \in \mathrm{Mat}_{d \times d}(\mathbb{C}) : \tau = \tau^t, \mathrm{Im}\tau' > 0 \right\}.$$

The map  $\Phi_F: \mathcal{D}(F) \rightarrow U(F)(\mathbb{R})$  is defined with the same formula as (6.3.5). The map  $\pi'_F$  is  $\mathrm{mod}\tau''$ , and the map  $p_F \circ \pi'_F$  is  $\begin{bmatrix} \tau' & \tau_0 \\ \tau_0^t & \tau'' \end{bmatrix} \mapsto \tau'$ .

For  $\tau' \in F \simeq \mathfrak{H}_{d'}$ , the complex structure on  $V(F)(\mathbb{R}) = W(F)(\mathbb{R})/U(F)(\mathbb{R})$  determined by  $\tau'$  is

$$\begin{bmatrix} I_{d'} & 0 & 0 & n \\ m^t & I_{d-d'} & n^t & b \\ 0 & 0 & I_{d'} & -m \\ 0 & 0 & 0 & I_{d-d'} \end{bmatrix} \mathrm{mod} b \mapsto \tau'm + n.$$

## 6.4 Poincaré–Eisenstein series and complex algebraic structure on $(\Gamma \backslash X)^{\mathrm{BB}}$

### 6.4.1 Bounded realization of the Poincaré series

Consider the Harish–Chandra realization  $X \simeq \mathcal{D} \subseteq \mathfrak{m}^+ \simeq \mathbb{C}^N$ , where  $\mathfrak{m}^+$  is identified with the holomorphic tangent space of  $o \in X$ .

For each  $g \in \mathbf{G}(\mathbb{R})$ , we have a map  $g \cdot: \mathcal{D} \rightarrow \mathcal{D}$ . Denote by  $J_g: \mathcal{D} \rightarrow \mathbb{C}^\times$ , sending each  $z \in \mathcal{D}$  to the determinant of the Jacobian of the action  $g \cdot$  on  $\mathcal{D}$  at  $z$ . In fact  $J_g(z)$  can be computed as follows; for simplicity we only write the formula for  $z = o$ . Denote by abuse of notation

$K_\infty := \text{Stab}_{\mathbf{G}^{\text{der}}(\mathbb{R})}(o)$ , and let  $M^+ \times K_{\infty, \mathbb{C}} \times M^- \rightarrow \mathbf{G}_\mathbb{C}$  be as in Theorem 2.3.5. The image of this map contains  $\mathbf{G}^{\text{der}}(\mathbb{R})$ . So each  $g \in \mathbf{G}^{\text{der}}(\mathbb{R})$  can be decomposed into  $m^+ k(g) m^-$  in a unique way. Then we have

$$J_g(o) = \text{Ad}_{m^+} k(g)^{-1}. \quad (6.4.1)$$

One can then prove that  $|J_g(o)|$  is bounded on  $\mathcal{D}$ .<sup>[3]</sup>

**Lemma 6.4.1.** *The function  $g \mapsto |J_g(o)|^m$  is in  $L^1(\mathbf{G}_\mathbb{R}^{\text{der}})$  for any  $m \geq 2$ , i.e.*

$$\int_{\mathbf{G}^{\text{der}}(\mathbb{R})} |J_g(o)|^m dg < \infty.$$

*Proof.* By (6.4.1), the function  $g \mapsto |J_g(o)|^m$  is left and right invariant under  $K_\infty$ , and in particular can be viewed as a function on  $\mathcal{D} \simeq \mathbf{G}^{\text{der}}(\mathbb{R})^+ / K_\infty$ . We have

$$\int_{\mathbf{G}^{\text{der}}(\mathbb{R})} |J_g(o)|^m dg = \int_{\mathcal{D}} |J_x(o)|^m dx$$

with  $dx$  a suitable invariant volume form, which up to a positive factor is  $|J_x(o)|^{-2}\omega$  for the Euclidean volume form  $\omega$  on  $\mathfrak{m}^+$ .

Now we can conclude because  $\mathcal{D}$  is bounded and  $|J_g(o)|$  is bounded.  $\square$

Now we are ready to define the *Poincaré series* associated with any polynomial  $f$  on  $\mathcal{D}$ .

**Definition 6.4.2.** *Let  $m \geq 2$ . The **Poincaré series** of weight  $m$  is  $P_{f,m}: \mathcal{D} \rightarrow \mathbb{C}$  defined by*

$$P_{f,m}(z) = \sum_{\gamma \in \Gamma} J_\gamma(z)^m f(\gamma z).$$

The series converges absolutely uniformly on compact sets by Lemma 6.4.1, and it satisfies the modularity condition by the chain rule. Indeed,  $P_{f,m}$  is a holomorphic automorphic form of weight  $m$ .

**Theorem 6.4.3.** *Suppose  $\Gamma$  is torsion-free. For any  $\Gamma$ -inequivalent points  $z_1, \dots, z_n \in \mathcal{D}$  and any complex numbers  $b_1, \dots, b_n$ , there exists a polynomial  $f$  on  $\mathcal{D}$  such that*

$$P_{f,m}(z_1) = b_1, \dots, P_{f,m}(z_n) = b_n$$

for all  $m \gg 1$ .

*Proof.* Fix  $0 < u < 1$ . The set  $\Gamma_u := \{\gamma \in \Gamma : |J_\gamma(z_i)| \geq u\}$  is finite by Lemma 6.4.1. Thus we can take a polynomial  $f$  such that  $f(z_j) = b_j$  and  $f(\gamma z_j) = 0$  for all  $j \in \{1, \dots, n\}$  and all  $\gamma \in \Gamma_u$ . It is not hard to check that  $|b_j - P_{f,m}(z_j)| = O(u^m)$ , and hence  $P_{f,m}(z_j) \rightarrow b_j$  with  $m \rightarrow \infty$  because  $0 < u < 1$ . Therefore the image of the linear map  $f \mapsto (P_{f,m}(z_1), \dots, P_{f,m}(z_n))$  contains a basis of  $\mathbb{C}^n$  and hence is surjective. Now we are done.  $\square$

---

<sup>[3]</sup>This follows from the *KAK*-decomposition of  $\mathbf{G}_\mathbb{R}^{\text{der}}$  and explicit computation for  $g \in A$ .

### 6.4.2 Poincaré–Eisenstein series with respect to an analytic boundary component

Now let  $F$  be a rational analytic boundary component and let  $\pi_F: X \rightarrow F$  be the holomorphic projection (6.2.1).

Through the whole subsection, we will identify  $F$  with its Harish–Chandra realization and identify  $X$  as the Siegel domain of the third kind, fibered over  $F$ , as in Theorem 6.3.7.

For each  $\gamma \in \Gamma$ , denote by  $J_\gamma^F: X \rightarrow \mathbb{C}^\times$  the map sending each  $x \in X$  to the determinant of the Jacobian of the action  $\gamma \cdot$  on  $X$  at  $x$ .

Let  $\Gamma_0 := \Gamma \cap (\mathbf{W}(F)\mathbf{G}_l(F))(\mathbb{Q})$  (see §6.3.1 for the notation). By (4.6.19) and Corollary 6.3.4,  $\gamma_0 \cdot z = z$  for all  $\gamma_0 \in \Gamma_0$  and  $z \in F$ .

**Definition 6.4.4.** *For any polynomial  $f$  on  $F$  and any  $m \geq 2$ , define the associated Poincaré–Eisenstein series of weight  $m$  to be*

$$E_{f,m}(x) := \sum_{\gamma \in \Gamma/\Gamma_0} f(\pi_F(\gamma x)) \cdot J_\gamma^F(x)^m.$$

For this definition to make sense, we need to check that every term in the sum of the right hand side is  $\Gamma_0$ -invariant. This is true for the first term by the discussion above, and is true for the second term for all  $m$  dividing a certain fixed integer  $m_0$ . From now on, we will take these  $m$ .

We also need to settle the convergence of the series defining  $E_{f,m}(x)$ . The key is the following proposition. Denote by  $j_g: F \rightarrow \mathbb{C}^\times$ , sending each  $z \in F$  to the determinant of the Jacobian of the action  $g \cdot$  on  $F$  at  $z$ .

**Proposition 6.4.5.** *For any  $g \in P_F(\mathbb{R})$ , there are rational numbers  $n$  and  $q > 0$  such that*

$$|J_g^F(x)| = |\chi(g)|^n |j_g(\pi_F(x))|^q$$

where  $\chi$  is a rational character of  $\mathbf{P}_F$ .

This settles the convergence issue: the series defining  $E_{f,m}(x)$  is absolutely uniformly convergent on compact sets. Hence  $E_{f,m}$  is a holomorphic function on  $X$ . Better, it is a holomorphic automorphic form.

### 6.4.3 Analytic structure on $\overline{\Gamma \backslash X}^{\text{BB}}$

We need to prove the separation property (Proposition 6.2.1(iv)) for  $\overline{\Gamma \backslash X}^{\text{BB}}$ . The key theorem to prove is:

**Theorem 6.4.6.** *Let  $F$  be a rational analytic boundary component of  $X$ . The Poincaré–Eisenstein series  $E_{f,m}$  associated with  $F$  extends to a holomorphic function on  $\overline{X}^{\text{BB}}$  (which by abuse of notation we still denote by  $E_{f,m}$ ) with the following properties:*

- (i) *the restriction of  $E_{f,m}$  to  $F$  is a Poincaré series on  $F$ ,*
- (ii)  *$E_{f,m}$  vanishes on any rational analytic boundary component  $F'$  if  $\dim F' \leq \dim F$  and  $F' \not\subseteq \Gamma F$ .*

Moreover, all Poincaré series on  $F$  can be obtained as restrictions of such extensions of  $E_{f,m}$ .

The ‘‘Moreover’’ part of the theorem immediately implies the separation property because Poincaré series separate points on each boundary component (Theorem 6.4.3). So

**Theorem 6.4.7.**  $\overline{\Gamma \backslash X}^{\text{BB}}$  carries a structure of complex analytic space, compatible with the complex structure on  $\Gamma \backslash X$ . In other words,  $\overline{\Gamma \backslash X}^{\text{BB}}$  is a compactification of  $\Gamma \backslash X$  in the category of complex analytic varieties.

#### 6.4.4 Algebraic structure on $\overline{\Gamma \backslash X}^{\text{BB}}$

Denote by  $\overline{S}^{\text{BB}} := \overline{\Gamma \backslash X}^{\text{BB}}$ . Consider the canonical line bundle (in the complex analytic category)

$$\omega_{\overline{S}^{\text{BB}}}.$$

Poincaré–Eisenstein series defined before are global sections of  $\omega_{\overline{S}^{\text{BB}}}^{\otimes m}$ . Since  $\overline{S}^{\text{BB}}$  is compact, it satisfies the descending chain condition for closed complex analytic subsets. So there exist finitely many global sections  $E_0, \dots, E_{N'}$  of  $\omega_{\overline{S}^{\text{BB}}}^{\otimes m}$  which separate points. Thus we get an injective analytic map

$$\varphi = [E_0 : \dots : E_{N'}] : \overline{S}^{\text{BB}} \longrightarrow \mathbb{P}^{N'}.$$

**Theorem 6.4.8.** This map  $\varphi$  endows  $\overline{S}^{\text{BB}}$  with the structure of a normal complex projective variety. In particular,  $\overline{\Gamma \backslash X}^{\text{BB}}$  carries a structure of normal projective complex varieties which induces the complex analytic structure in Theorem 6.4.7.

This theorem gives  $\Gamma \backslash X$  a complex algebraic structure. Moreover, the complex algebraic structure on  $\Gamma \backslash X$  is unique by the following theorem. Recall  $D = \{z \in \mathbb{C} : |z| < 1\}$  is the open unit disc and let  $D^\times := D \setminus \{0\}$  be the punctured disc.

**Theorem 6.4.9.** Assume  $\Gamma$  is torsion-free. Then any holomorphic map  $D^a \times (D^\times)^b \rightarrow \Gamma \backslash X$  extends to a holomorphic map  $D^a \times (D^\times)^b \rightarrow \overline{\Gamma \backslash X}^{\text{BB}}$ .