Chapter 6

Baily–Borel compactification

Let $(\mathbf{G}, X)$ be a Shimura datum. By abuse of notation use X to denote a connected component. Let $\Gamma < \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup.

Throughout the whole chapter, we will assume $\mathbf{G}$ to be quasi-simple, *i.e.* $\mathbf{G}^{\mathrm{der}}$ is a simple group. For the purpose of compactifying $\Gamma\backslash X$ we can easily reduce to this case. Notice that $\mathbf{G}^{\mathrm{der}}_{\mathbb{R}}$ may not be simple as an $\mathbb{R}$-group, so that X is not necessarily irreducible.

6.1 Baily–Borel compactification as a topological space

Consider the Harish–Chandra embedding

$$X \simeq \mathcal{D} \subseteq \mathfrak{m}^+ \simeq \mathbb{C}^N$$

with $\overline{\mathcal{D}}$ the closure of $\mathcal{D}$ in $\mathfrak{m}^+$. Let F be an analytic boundary component of X, with normalizer $N(F) = \{g \in \mathbf{G}^{\mathrm{der}}(\mathbb{R})^+ : gF = F\}$.

Recall (5.1.3) the fundamental set Σ constructed from Siegel sets associated with a minimal rational proper parabolic subgroup $\mathbf{P}_0$ of $\mathbf{G}^{\mathrm{der}}$.

Let $\overline{\Sigma}$ be the closure of $\Sigma \subseteq X \simeq \mathcal{D} \subseteq \mathfrak{m}^+$. Then $\overline{\Sigma} \subseteq \overline{\mathcal{D}}$, with an induced topology.

Theorem 6.1.1. *The followings are equivalent:*

(1) $\Gamma F \cap \overline{\Sigma} \neq \emptyset$,

(2) $\mathbf{P}_{0,\mathbb{R}} < N(F)$, *and F is a rational analytic boundary component (i.e. $N(F)$ equals $\mathbf{P}_F(\mathbb{R})$ for a parabolic subgroup $\mathbf{P}_F$ of $\mathbf{G}^{\mathrm{der}}$).*

Moreover in these cases, $\mathbf{P}_F$ is a maximal proper parabolic subgroup of $\mathbf{G}^{\mathrm{der}}$.

Theorem 6.1.1 indicates that we can do the following compactification of $\Gamma\backslash X$:

(i) Define $\overline{X}^{\mathrm{BB}} := \mathcal{D} \cup \bigsqcup_{\mathbf{P}} F_{\mathbf{P}} \subseteq \overline{\mathcal{D}}$, where $\mathbf{P}$ runs over all *maximal* proper parabolic subgroup of $\mathbf{G}^{\mathrm{der}}$ and $F_{\mathbf{P}}$ is the rational analytic boundary component $F_{\mathbf{P}}$.

(ii) Endow $\overline{X}^{\mathrm{BB}}$ with the Satake topology.

(iii) The space $\Gamma\backslash\overline{X}^{\mathrm{BB}}$ is then a compact Hausdorff space containing $\Gamma\backslash X$ as an open dense subset.

Then $(\Gamma\backslash\overline{X})^{\mathrm{BB}} := \Gamma\backslash\overline{X}^{\mathrm{BB}}$ is called the *Baily–Borel compactification* of $\Gamma\backslash X$, and

$$(\Gamma\backslash\overline{X})^{\mathrm{BB}} = \Gamma\backslash X \cup \bigsqcup_{j=1}^{m} \Gamma_{F_j}\backslash F_j, \tag{6.1.1}$$

where $F_1, \ldots, F_m$ are rational analytic boundary components such that $\{\mathbf{P}_{F_1}, \ldots, \mathbf{P}_{F_m}\}$ is a set of representatives of Γ-conjugacy classes of maximal proper parabolic subgroups of $\mathbf{G}^{\mathrm{der}}$, with $\Gamma_{F_j} := \Gamma \cap \mathbf{P}_{F_j}(\mathbb{Q})$.

6.1.1 Satake topology on $\overline{X}^{\mathrm{BB}}$

The *Satake topology* on $\overline{X}^{\mathrm{BB}}$ is defined as follows. For each $x \in \overline{X}^{\mathrm{BB}} \subseteq \overline{\mathcal{D}}$, the neighborhoods of any point $x \in X^*$ is the saturations of the neighborhoods of the corresponding points in $\overline{\Sigma}$ under the action of $\Gamma_x := \{\gamma \in \Gamma : \gamma \cdot x = x\}$. More precisely, a fundamental system of neighborhoods of x is given by all subsets $U \subseteq \overline{\mathcal{D}}$ such that

$$\Gamma_x \cdot U = U,$$

and such that $\gamma U \cap \overline{\Sigma}$ is a neighborhood of $\gamma \cdot x$ in $\overline{\Sigma}$ whenever $\gamma \cdot x \in \overline{\Sigma}$.

Proposition 6.1.2. *The Satake topology is the unique topology on $\overline{X}^{\mathrm{BB}}$ such that the followings hold:*

(i) it induces the original topologies on $\overline{\Sigma}$ and on X,

(ii) the $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$-action on $\overline{X}^{\mathrm{BB}}$ is continuous,

(iii) for any $x \in \overline{X}^{\mathrm{BB}}$, there exists a fundamental system of neighborhoods $\{U\}$ of x such that

$$\gamma U = U \text{ for all } \gamma \in \Gamma_x \quad \text{and} \quad \gamma U \cap U = \emptyset \text{ for all } \gamma \notin \Gamma_x,$$

(iv) if $x, x' \in \overline{X}^{\mathrm{BB}}$ are not in one Γ-orbit, then there exist neighborhoods U of x and U' of x' such that

$$\Gamma U \cap U' = \emptyset.$$

Corollary 6.1.3. *$\Gamma\backslash\overline{X}^{\mathrm{BB}}$ is compact and Hausdorff.*

6.1.2 $\mathbb{Q}$-roots vs $\mathbb{R}$-roots, and $\mathbb{Q}$-polydisc

Let $\mathbf{S} < \mathbf{P}_0$ be a maximal $\mathbb{Q}$-split tours, and let K_∞ be a maximal compact subgroup of $\mathbf{G}^{\mathrm{der}}_{\mathbb{R}}$ such that $\mathrm{Lie}K_\infty \cap \mathrm{Lie}\mathbf{S}_{\mathbb{R}} = 0$.[1] Then there exists $x_0 \in X$ such that $\mathrm{Stab}_{\mathbf{G}^{\mathrm{der}}(\mathbb{R})}(x_0) = K_\infty$.

In §4.6.2 and §4.6.3, we discussed the complex roots and real roots respectively. Now we need to analyze the $\mathbb{Q}$-roots of $\mathbf{G}^{\mathrm{der}}$. First, we can make an appropriate choice of K_∞ such that $\mathbf{S}_{\mathbb{R}} < A$ with A from §4.6.3.

Let ${}_{\mathbb{R}}\Psi = \{\gamma_1, \ldots, \gamma_r\}$ be as in (4.6.13); it is obtained from a set of strongly orthogonal complex roots $\Psi = \{\alpha_1, \ldots, \alpha_r\}$ from (4.6.4). If $\mathbf{G}^{\mathrm{der}}_{\mathbb{R}}$ is simple, we described the real roots ${}_{\mathbb{R}}\Phi = \Phi(A, \mathbf{G}^{\mathrm{der}}_{\mathbb{R}})$ in terms of $\gamma_1, \ldots, \gamma_r$ in Proposition 4.6.12. It turns out that one can also do this for the rational roots ${}_{\mathbb{Q}}\Phi := \Phi(\mathbf{S}, \mathbf{G}^{\mathrm{der}})$ when $\mathbf{G}^{\mathrm{der}}$ is simple.

[1] Even strongly, $\mathrm{Lie}K_\infty$ is orthogonal to $\mathrm{Lie}\mathbf{S}_{\mathbb{R}}$ for the Killing form on $\mathrm{Lie}\mathbf{G}^{\mathrm{der}}_{\mathbb{R}}$.

Proposition 6.1.4. *Let $s = \dim \mathbf{S}$. There is a partition*

$$\{1, \ldots, r\} = I_0 \cup I_1 \cup \cdots \cup I_s \tag{6.1.2}$$

such that

$$\begin{aligned} X(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{Q} &\simeq \frac{X(A) \otimes_{\mathbb{Z}} \mathbb{Q}}{\{\text{subspace spanned by } \gamma_i \text{ for } i \in I_0 \text{ and } \gamma_i - \gamma_j \text{ for } i, j \in I_\ell,\ \ell > 0\}} \\ &\simeq \sum_{\ell=1}^{s} \mathbb{Q}\beta_\ell, \qquad \text{where } \beta_\ell = \text{image of any } \gamma_j \text{ with } j \in I_\ell. \end{aligned}$$

In particular, $\mathbf{S} = \{\gamma_i = 1 \text{ for } i \in I_0;\ \gamma_i = \gamma_j \text{ for } i, j \in I_\ell, \text{ where } \ell > 0\}$.

Corollary 6.1.5. *Recall our assumption that $\mathbf{G}^{\mathrm{der}}$ is simple. One of the two cases occurs:*

- `(Type` C_s`)` ${}_{\mathbb{Q}}\Phi = \{\pm\frac{1}{2}(\beta_i + \beta_j) \text{ for } i \geq j,\ \pm\frac{1}{2}(\beta_i - \beta_j) \text{ for } i > j\}$.
- `(Type` BC_s`)` ${}_{\mathbb{Q}}\Phi = \{\pm\frac{1}{2}(\beta_i + \beta_j) \text{ for } i \geq j,\ \pm\frac{1}{2}(\beta_i - \beta_j) \text{ for } i > j,\ \pm\frac{1}{2}\beta_i\}$.

If we order the roots such that $\beta_1 > \ldots > \beta_s$, then the set of simple roots is:

- `(Type` C_s`)` ${}_{\mathbb{Q}}\Delta = \{\nu_1 := \frac{1}{2}(\beta_1 - \beta_2), \ldots, \nu_{s-1} := \frac{1}{2}(\beta_{s-1} - \beta_s), \nu_s := \beta_s\}$.
- `(Type` BC_s`)` ${}_{\mathbb{Q}}\Delta = \{\nu_1 := \frac{1}{2}(\beta_1 - \beta_2), \ldots, \nu_{s-1} := \frac{1}{2}(\beta_{s-1} - \beta_s), \nu_s := \frac{1}{2}\beta_s\}$.

The proof goes as follows: We have the group-theoretic result that $\mathbf{G}^{\mathrm{der}} = \mathrm{Res}_{k/\mathbb{Q}}\mathbf{G}'$ for some absolutely simple k-group $\mathbf{G}'$ with k a totally real number field. Then $\mathbf{G}^{\mathrm{der}}_{\mathbb{R}} = \prod_{\sigma\colon k \hookrightarrow \mathbb{R}} \mathbf{G}'_\sigma$ with each $\mathbf{G}'_\sigma$ a group defined over $\sigma(k) \subseteq \mathbb{R}$. Then one analyzes each factor and use the Galois action.

Next we turn to the $\mathbb{Q}$-polydisc. Recall from the Polydisc Theorem (Theorem 4.6.7) that we have a totally geodesic embedding $D^r \to X$ (with $D = \{z \in \mathbb{C} : |z| < 1\}$ the Poincaré unit disc) arising from a group morphism

$$\varphi\colon \mathrm{SL}_2(\mathbb{R})^r \to \mathbf{G}^{\mathrm{der}}(\mathbb{R}),$$

and $X = K_\infty \cdot D^r$. This embedding gives rise to the analytic boundary components as in the diagram (4.6.15). Let us rephrase it here. Recall $\mathfrak{H} \simeq D$ with the Cayley transformation sending $\sqrt{-1} \mapsto 0$ and $\infty \mapsto 1$. Then we have the diagram

$$\begin{tikzcd} \mathfrak{H}^r \arrow[r, "f_1"] \arrow[d, "\subseteq"'] & \mathcal{D} \arrow[d, "\subseteq"] \\ (\mathbb{P}^1)^r \arrow[r, "f_3"] & X^\vee \end{tikzcd} \tag{6.1.3}$$

where f_1 is the natural composite $\mathfrak{H}^r \simeq D^r \to X \simeq \mathcal{D}$, with $D^r \to X$ the geodesic embedding as above and $X \simeq \mathcal{D}$ the Harish–Chandra realization, and $\mathcal{D} \subseteq X^\vee$ from (4.6.3). Then for any subset $S \subseteq \{1, \ldots, r\}$, the unique standard analytic boundary component containing the point $f_3((\sqrt{-1})_{j \notin S}, (\infty)_{j \in S})$ is F_S. In general, an analytic boundary component of X is of the form $g \cdot F_S$ for some $g \in \mathbf{G}^{\mathrm{der}}(\mathbb{R})$.

We wish to do this discussion and obtain the relevant results over $\mathbb{Q}$. First of all, any rational analytic boundary component is easily seen to be of the form $g \cdot F_S$, with $g \in \mathbf{G}^{\mathrm{der}}(\mathbb{Q})$ and F_S rational. Next we prove the following lemma.

Lemma 6.1.6. *For $S \subseteq \{1, \ldots, r\}$, the standard analytic boundary component F_S is rational if and only if $S = I_{\ell_1} \cup \cdots \cup I_{\ell_t}$, where $1 \le \ell_1 < \cdots < \ell_t \le s$, for the partition (6.1.2).*

Proof. For the proof, it is more convenient to use the description of parabolic subgroups given by Theorem 4.2.5. In §4.6.6, we explained that the normalizer $P_{F_S} = P(\lambda_S)$, with $\lambda_S \colon \mathbb{G}_{\mathrm{m},\mathbb{R}} \to \mathbf{G}^{\mathrm{der}}_{\mathbb{R}}$ sending

$$t \mapsto \varphi(\underbrace{\cdots, 1, \cdots}_{j \notin S}, \underbrace{\cdots, \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}, \cdots}_{j \in S}).$$

By Proposition 6.1.4, λ is defined over $\mathbb{Q}$ if and only if $S = I_{\ell_1} \cup \cdots \cup I_{\ell_t}$ for some $1 \le \ell_1 < \cdots < \ell_t \le s$. We are done. $\square$

With this lemma in hand, we obtain the $\mathbb{Q}$-version of (6.1.3)

$$\begin{array}{ccc} \mathfrak{H}^s & \xrightarrow{f_1} & \mathcal{D} \\ \subseteq \downarrow & & \downarrow \subseteq \\ (\mathbb{P}^1)^s & \xrightarrow{f_3} & X^\vee \end{array} \tag{6.1.4}$$

arising from

$$\varphi \colon \mathrm{SL}_2(\mathbb{R})^s \to G$$

such that φ(diagonal matrices) is the maximal $\mathbb{Q}$-split torus $\mathbf{S}$ of $\mathbf{G}^{\mathrm{der}}$. We can renumber the factors of $\mathfrak{H}^s$ and $\mathrm{SL}_2(\mathbb{R})^s$ such that: For the $\beta_1, \ldots, \beta_s \in {}_{\mathbb{Q}}\Phi$ from Proposition 6.1.4, we have

$$\beta_\ell \colon \varphi\left(\begin{bmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{bmatrix}, \ldots, \begin{bmatrix} t_s & 0 \\ 0 & t_s^{-1} \end{bmatrix}\right) \mapsto t_\ell^2. \tag{6.1.5}$$

Now for each subset $S \subseteq \{1, \ldots, s\}$, there exists a unique standard rational analytic boundary component F_S which contains the point

$$f_3(\underbrace{\cdots, \sqrt{-1}, \cdots}_{j \notin S}, \underbrace{\cdots, \infty, \cdots}_{j \in S}).$$

Proof of Theorem 6.1.1. Assume F meets $\overline{\Sigma}$.

Order the roots such that $\beta_1 > \ldots > \beta_s$, then $\mathbf{S}(\mathbb{R})^+$ consists of

$$\varphi\left(\begin{bmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{bmatrix}, \ldots, \begin{bmatrix} t_s & 0 \\ 0 & t_s^{-1} \end{bmatrix}\right)$$

where $t_1 \ge \ldots \ge t_s \ge 1$. Hence

$$\overline{\mathbf{S}(\mathbb{R})^+ x_0} = f_3\left(\{(\sqrt{-1}x_1, \ldots, \sqrt{-1}x_s) : \infty \ge x_1 \ge \cdots \ge x_s \ge 1\}\right).$$

Hence $\overline{\mathbf{S}(\mathbb{R})^+ x_0}$ meets the boundary components $F_1, \ldots, F_s$ with

$$f_3(\infty, \sqrt{-1}, \ldots, \sqrt{-1}) \in F_1,\ f_3(\infty, \infty, \sqrt{-1}, \ldots, \sqrt{-1}) \in F_2,\ \ldots,\ f_3(\sqrt{-1}, \sqrt{-1}, \ldots, \sqrt{-1}) \in F_s.$$

So $F = F_\ell$ for some $\ell \in \{1, \ldots, s\}$. We can compute the normalizer of each F_ℓ as in Theorem 4.6.19, and get

$$N(F_\ell) = \mathbf{P}_{{}_{\mathbb{Q}}\Delta \setminus \{\nu_\ell\}}(\mathbb{R})$$

for each $\ell \in \{1, \ldots, s\}$. Hence we are done. $\square$

6.2 First step towards the complex structure

6.2.1 A general criterion for a topological space to be complex analytic

Assume V is a compact Hausdorff space which can be written as a disjoint union

$$V = V_0 \bigsqcup V_1 \bigsqcup \cdots \bigsqcup V_m$$

with each V_j an irreducible normal complex analytic space. Assume that $\dim V_0 > \dim V_j$ for all $j \geq 1$, and that V_0 is open dense in V.

Define a sheaf $\mathcal{F}$ of $\mathcal{A}$-functions on V as follows. For any open subset $U \subseteq V$, a complex-valued continuous function on U is an $\mathcal{A}$-function if its restriction to each $U \cap V_j$ $(0 \leq j \leq m)$ is complex analytic.

Proposition 6.2.1. *Assume:*

(i) For each integer $d \geq 1$, the union $V_{(d)} := \bigcup_{\dim V_j \leq d} V_j$ is closed.

(ii) Any $v \in V$ has a countable fundamental set of open neighborhoods $\{U_\ell\}$ such that $U_\ell \cap V_0$ is connected for all ℓ.

(iii) The restriction to V_j of local $\mathcal{A}$-functions define the structure sheaf of V_j, for all $j \geq 0$.

(iv) Any $v \in V$ has a neighborhood U_v whose points are separated by the $\mathcal{A}$-functions defined on U.

Then V is an irreducible normal complex analytic space with structure sheaf $\mathcal{F}$. For each $d \leq \dim V_0$, the union $V_{(d)}$ is an analytic subspace of V with dimension $\max\{\dim V_j : V_j \subseteq V_{(d)}\}$.

6.2.2 Application to the Baily–Borel compactification

We shall apply Proposition 6.2.1 to the Baily–Borel compactification (6.1.1) (which is compact Hausdorff space by Corollary 6.1.3), with $V_0 = \Gamma\backslash X$ and $V_j = \Gamma_{F_j}\backslash F_j$ for $1 \leq j \leq m$.

Conditions (i) and (ii) can be shown to hold by checking with the Satake topology from §6.1.1.

To check condition (iii), we define the projection

$$\pi_F \colon X \to F \tag{6.2.1}$$

for each analytic boundary component F. We focus on the rational ones.

Recall our choice of a maximal $\mathbb{Q}$-split torus $\mathbf{S}$ (from §6.1.2) in our minimal parabolic subgroup $\mathbf{P}_0$ of $\mathbf{G}^{\mathrm{der}}$ (see above Theorem 6.1.1), and the basis ${}_{\mathbb{Q}}\Delta = \{\nu_1, \ldots, \nu_s\}$ (see Corollary 6.1.5) of the relative root system ${}_{\mathbb{Q}}\Phi := \Phi(\mathbf{S}, \mathbf{G}^{\mathrm{der}})$. The root ν_s is called the *distinguished root* because it has different length.

Over $\mathbb{R}$, we explained the relation between F and the boundary symmetric domain associated with $P_F = N(F)$; see below Theorem 4.6.19. The discussion can be carried over $\mathbb{Q}$.

Let F be a rational analytic boundary component which meets $\overline{\Sigma}$. We have shown in the proof of Theorem 6.1.1 that $\mathbf{P}_F = \mathbf{P}_{{}_{\mathbb{Q}}\Delta\backslash\{\nu_\ell\}}$ for some $\ell \in \{1, \ldots, s\}$. Let $I_h := \{\mu_{\ell+1}, \ldots, \mu_s\}$ and $I_l := \{1, \ldots, \ell - 1\}$. We thus have the refined rational horospherical decomposition

$$X \simeq N_{P_F}(\mathbb{R}) \times A_{\mathbf{P}_F}(\mathbb{R})^+ \times X_{\mathbf{P}_{I_h}} \times X_{\mathbf{P}_{I_l}}.$$

Moreover, the proof of Theorem 6.1.1 exhausts the possibilities of all F's, and hence implies that F can be identified with the boundary symmetric domain associated with $\mathbf{P}_{I_h}$. Thus the refined rational horospherical decomposition above induces a natural projection $X \to F$, which is our desired π_F. Although the decomposition is only real semi-algebraic, the projection π_F is also holomorphic.

If F is contained in $\overline{F'}$ for another rational boundary component F', then F is a rational boundary component of F', and one gets a projection $\pi_{F',F}\colon F' \to F$. It is not hard to check that π_F is the composite of $\pi_{F',F} \circ \pi_{F'}$.

Now to check condition (iii) of Proposition 6.2.1, we only need to work locally and hence on the universal covering. But now for any rational boundary component F of X, any complex analytic function near F can be extended to an $\mathcal{A}$-function on a neighborhood of F in $\overline{X}^{\mathrm{BB}}$ by the discussion above. This establishes (iii).

Proving condition (iv) is the hardest part. We need to realize X as a Siegel domain of the third kind and define the Poincaré–Eisenstein series.