Chapter 1

Hermitian line bundles on $\mathrm{Spec}\mathcal{O}_K$ and positivity

In the whole chapter, let K be a number field and $\mathcal{O}_K$ be its ring of integers. It is known that $\mathrm{Spec}\mathcal{O}_K$ is not a projective scheme. A key idea in Arakelov Geometry is to identify $\mathrm{Spec}\mathcal{O}_K$ with the set of finite places of K and then compactify $\mathrm{Spec}\mathcal{O}_K$ by adding the archimedean places.

1.1 Hermitian line bundles and arithmetic divisors on $\mathrm{Spec}\mathcal{O}_K$

Definition 1.1.1. *A* **Hermitian line bundle** *on* $\mathrm{Spec}\mathcal{O}_K$ *is a pair* $\overline{\mathcal{L}} := (\mathcal{L}, \|\cdot\|)$*, where* $\mathcal{L}$ *is a line bundle on* $\mathrm{Spec}\mathcal{O}_K$ *and* $\|\cdot\| = \{\|\cdot\|_\sigma\}_{\sigma\colon K\hookrightarrow\mathbb{C}}$ *is a collection of Hermitian metrics* $\|\cdot\|_\sigma$ *on each* $\mathcal{L}_\sigma = H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L}) \otimes_\sigma \mathbb{C}$ *satisfying* $\|s\|_\sigma = \|s\|_{\overline{\sigma}}$ *for all* $s \in H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})$*.*

We say that such collections of metrics are *invariant under complex conjugation*. Notice that $H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})$ is a projective $\mathcal{O}_K$-module of rank 1, and each $\mathcal{L}_\sigma$ is a $\mathbb{C}$-vector space of dimension 1. Thus $\|\cdot\|_\sigma$ is determined by $\|s\|_\sigma$ for any non-zero rational section s of $\mathcal{L}$.

Next we introduce the *group of isometric classes of Hermitian line bundles* on $\mathrm{Spec}\mathcal{O}_K$, denoted by $\widehat{\mathrm{Pic}}(\mathrm{Spec}\mathcal{O}_K)$. The identity element is the trivial Hermitian line bundle, the multiplication is the tensor product, and the inverse is the dual.

Definition 1.1.2. *(i) An* **isometry** *between two Hermitian line bundles* $\overline{\mathcal{L}}$ *and* $\overline{\mathcal{L}}'$ *on* $\mathrm{Spec}\mathcal{O}_K$ *is an isomorphism*

$$i\colon \mathcal{L} \to \mathcal{L}'$$

of line bundles on $\mathrm{Spec}\mathcal{O}_K$ *satisfying*

$$\|s\|_\sigma = \|i(s)\|_\sigma, \qquad \forall s \in H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L}),\ \forall \sigma\colon K \hookrightarrow \mathbb{C}.$$

(ii) The **trivial Hermitian line bundle** *on* $\mathrm{Spec}\mathcal{O}_K$ *is* $(\mathcal{O}_{\mathrm{Spec}\mathcal{O}_K}, |\cdot|)$ *where* $|\cdot|_\sigma$ *is the absolute value at each archimedean place* σ*.*

(iii) The **tensor product** *of two Hermitian line bundles* $\overline{\mathcal{L}}$ *and* $\overline{\mathcal{L}}'$ *on* $\mathrm{Spec}\mathcal{O}_K$ *is defined to be*

$$\overline{\mathcal{L}} \otimes \overline{\mathcal{L}}' := (\mathcal{L} \otimes \mathcal{L}', \|\cdot\|\|\cdot\|').$$

(iv) The **dual** *of a Hermitian line bundle* $\overline{\mathcal{L}}$ *on* $\mathrm{Spec}\mathcal{O}_K$ *is defined to be*

$$\overline{\mathcal{L}}^\vee := (\mathcal{L}^\vee, \|\cdot\|^\vee)$$

where $\mathcal{L}^\vee := \mathrm{Hom}(\mathcal{L}, \mathcal{O}_{\mathrm{Spec}\mathcal{O}_K})$ *and, for each* $t \in H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L}^\vee)$,

$$\|t\|_\sigma^\vee := \frac{|t(s)|}{\|s\|_\sigma} \quad \textit{for any non-zero } s \in \mathcal{L}_\sigma.$$

Definition-Lemma 1.1.3. *Let* $\overline{\mathcal{L}}$ *be a Hermitian line bundle on* $\mathrm{Spec}\mathcal{O}_K$. *For any non-zero* $s \in H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})$, *the number*

$$\widehat{\deg}(\overline{\mathcal{L}}) := \log \#\left(H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})/s\mathcal{O}_K\right) - \sum_{\sigma\colon K \hookrightarrow \mathbb{C}} \log \|s\|_\sigma \tag{1.1.1}$$

does not depend on the choice of s.

This number $\widehat{\deg}(\overline{\mathcal{L}})$ *is called the* **arithmetic degree** *of* $\overline{\mathcal{L}}$.

The proof is an application of the product formula. We will postpone it to Proposition 1.1.7, using the relation between Hermitian line bundles and arithmetic divisors introduced below.

In Algebraic Geometry, line bundles and (Cartier) divisors are closely related. In Arakelov Geometry, we also have the notion of arithmetic divisors.

Definition 1.1.4. *An* **arithmetic divisor** *is a formal finite sum*

$$\overline{D} = \sum_{\mathfrak{p} \in M_{K,f}} n_\mathfrak{p}[\mathfrak{p}] + \sum_{\sigma\colon K \hookrightarrow \mathbb{C}} n_\sigma[\sigma] \tag{1.1.2}$$

with $n_\mathfrak{p} \in \mathbb{Z}$, $n_\sigma \in \mathbb{R}$ *and* $n_\sigma = n_{\overline{\sigma}}$.

A **principal arithmetic divisor** *is of the form*

$$\widehat{\mathrm{div}}(\alpha) := \sum_{\mathfrak{p} \in M_{K,f}} \mathrm{ord}_\mathfrak{p}(\alpha)[\mathfrak{p}] - \sum_{\sigma\colon K \hookrightarrow \mathbb{C}} \log|\sigma(\alpha)|[\sigma]$$

for some $\alpha \in K^*$.

In (1.1.2), we usually denote by $D_\mathrm{f} := \sum_{\mathfrak{p} \in M_{K,f}} n_\mathfrak{p}[\mathfrak{p}]$ the *finite part of* $\overline{D}$ and by $D_\infty := \sum_{\sigma\colon K \hookrightarrow \mathbb{C}} n_\sigma[\sigma]$ the *infinite part of* $\overline{D}$.

We will also introduce the following groups, where the group law is clear:

$$\begin{aligned}
\widehat{\mathrm{Div}}(\mathrm{Spec}\mathcal{O}_K) &:= \{\text{arithmetic divisors on } \mathrm{Spec}\mathcal{O}_K\},\\
\widehat{\mathrm{Prin}}(\mathrm{Spec}\mathcal{O}_K) &:= \{\text{principal arithmetic divisors on } \mathrm{Spec}\mathcal{O}_K\},\\
\widehat{\mathrm{Cl}}(\mathrm{Spec}\mathcal{O}_K) &:= \widehat{\mathrm{Div}}(\mathrm{Spec}\mathcal{O}_K)/\widehat{\mathrm{Prin}}(\mathrm{Spec}\mathcal{O}_K).
\end{aligned}$$

Definition 1.1.5. *The* **arithmetic degree** *of an arithmetic divisor* $\overline{D}$ *of the form* (1.1.2) *is defined to be*

$$\widehat{\deg}(\overline{D}) := \sum_{\mathfrak{p} \in M_{K,f}} n_\mathfrak{p} \log \#(\mathcal{O}_K/\mathfrak{p}) + \sum_{\sigma\colon K \hookrightarrow \mathbb{C}} n_\sigma.$$

The product formula immediately implies that any principal arithmetic divisor has arithmetic degree 0. Thus we get a group homomorphism

$$\widehat{\deg}\colon \widehat{\mathrm{Cl}}(\mathrm{Spec}\mathcal{O}_K) \to \mathbb{R}. \tag{1.1.3}$$

Proposition 1.1.6. *We have a group homomorphism*

$$\widehat{\mathrm{Div}}(\mathrm{Spec}\mathcal{O}_K) \to \widehat{\mathrm{Pic}}(\mathrm{Spec}\mathcal{O}_K), \quad \overline{D} \mapsto \mathcal{O}(\overline{D}), \tag{1.1.4}$$

where for $\overline{D} = D_{\mathrm{f}} + \sum_\sigma n_\sigma[\sigma]$*, the Hermitian line bundle* $\mathcal{O}(\overline{D})$ *is defined to be* $(\mathcal{O}(D_{\mathrm{f}}), \|\cdot\|_\sigma)$ *with* $\|1\|_\sigma := \exp(-n_\sigma)$ *for the canonical* 1 *of* $\mathcal{O}(D_{\mathrm{f}})$ *(i.e. the divisor of* 1 *is* D_{f}*).*

And this group homomorphism induces a group isomorphism

$$\widehat{\mathrm{Cl}}(\mathrm{Spec}\mathcal{O}_K) \xrightarrow{\sim} \widehat{\mathrm{Pic}}(\mathrm{Spec}\mathcal{O}_K). \tag{1.1.5}$$

The inverse of (1.1.5) is called the *arithmetic first Chern class* and is denoted by $\widehat{c}_1$.

By constructions, the group homomorphism (1.1.4) is compatible with the forgetful maps $\widehat{\mathrm{Div}}(\mathrm{Spec}\mathcal{O}_K) \to \mathrm{Div}(\mathrm{Spec}\mathcal{O}_K)$, $\overline{D} \mapsto D_{\mathrm{f}}$, and $\widehat{\mathrm{Pic}}(\mathrm{Spec}\mathcal{O}_K) \to \mathrm{Pic}(\mathrm{Spec}\mathcal{O}_K)$, $\overline{\mathcal{L}} \mapsto \mathcal{L}$. Thus the isomorphism (1.1.5) is an extension of the natural isomorphism $\mathrm{Cl}(\mathcal{O}_K) \simeq \mathrm{Pic}(\mathrm{Spec}\mathcal{O}_K)$.

Proof. It is easy to check that (1.1.4) is a group homomorphism.

For any $\widehat{\mathrm{div}}(\alpha) \in \widehat{\mathrm{Prin}}(\mathrm{Spec}\mathcal{O}_K)$, it is not hard to check that the isomorphism $\alpha\colon \mathcal{O}_{\mathrm{Spec}\mathcal{O}_K} \to \mathcal{O}(\mathrm{div}(\alpha))$ induces an isometry between the trivial Hermitian line bundle on $\mathrm{Spec}\mathcal{O}_K$ and $\widehat{\mathrm{div}}(\alpha)$. Thus we have a group homomorphism $\widehat{\mathrm{Cl}}(\mathrm{Spec}\mathcal{O}_K) \to \widehat{\mathrm{Pic}}(\mathrm{Spec}\mathcal{O}_K)$.

The inverse is defined as follows. For any $\overline{\mathcal{L}} \in \widehat{\mathrm{Pic}}(\mathrm{Spec}\mathcal{O}_K)$, let s be a non-zero rational section of $\mathcal{L}_K$ and set

$$\widehat{\mathrm{div}}(s) := \mathrm{div}(s) + \sum_\sigma (-\log\|s\|_\sigma)[\sigma]. \tag{1.1.6}$$

If we have two non-zero rational sections s and s', then $s = \alpha s'$ for some $\alpha \in K^*$. Then $\widehat{\mathrm{div}}(s) - \widehat{\mathrm{div}}(s')$ is a principal arithmetic divisor. Thus we obtain a group homomorphism

$$\widehat{\mathrm{Pic}}(\mathrm{Spec}\mathcal{O}_K) \to \widehat{\mathrm{Cl}}(\mathrm{Spec}\mathcal{O}_K), \quad \overline{L} \mapsto \widehat{\mathrm{div}}(s).$$

It is not hard to check that this is the desired inverse. $\square$

Proposition 1.1.7. *The following diagram of group homomorphisms commutes:*

$$\begin{tikzcd}
\widehat{\mathrm{Cl}}(\mathrm{Spec}\mathcal{O}_K) \arrow[r, "\sim"] \arrow[d, "\widehat{\deg}"'] & \widehat{\mathrm{Pic}}(\mathrm{Spec}\mathcal{O}_K) \arrow[d, "\widehat{\deg}"] \\
\mathbb{R} \arrow[r, "="] & \mathbb{R},
\end{tikzcd} \tag{1.1.7}$$

where the top arrow is the one induced by (1.1.4).

Proof. By the definitions of the arithmetic degrees ((1.1.1) and Definition 1.1.5) and the inverse of the top arrow (1.1.6), it suffices to prove the following claim. For any non-zero $s \in H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})$, we have

$$\#\left(H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})/s\mathcal{O}_K\right) = \prod_{\mathfrak{p}} \#(\mathcal{O}_K/\mathfrak{p})^{\mathrm{ord}_{\mathfrak{p}}(s)}.$$

Write $M := H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})$. Then for each $\mathfrak{p}$, the localization $M_{\mathfrak{p}}$ is a free $\mathcal{O}_{K,\mathfrak{p}}$-module of rank 1 and $M/s\mathcal{O}_K \simeq \oplus_{\mathfrak{p}} M_{\mathfrak{p}}/s\mathcal{O}_{K,\mathfrak{p}}$. Thus the desired equality holds true. We are done. $\square$

We finish this section by stating a lemma which compares $\widehat{\mathrm{Pic}}(\mathrm{Spec}\mathcal{O}_K) \simeq \widehat{\mathrm{Cl}}(\mathrm{Spec}\mathcal{O}_K)$ and $\mathrm{Pic}(\mathrm{Spec}\mathcal{O}_K) \simeq \mathrm{Cl}(\mathcal{O}_K)$. The proof is easy.

Lemma 1.1.8. *Let $\rho_1, \ldots, \rho_{r_1}$ be the real embeddings of K and $\sigma_1, \overline{\sigma}_1, \ldots, \sigma_{r_2}, \overline{\sigma}_{r_2}$ be the complex embeddings. Then We have the following exact sequence:*

$$1 \to \mu_K \to \mathcal{O}_K^* \xrightarrow{\log_K} \mathbb{R}^{r_1+r_2} \xrightarrow{\ell} \widehat{\mathrm{Cl}}(\mathrm{Spec}\mathcal{O}_K) \to \mathrm{Cl}(\mathcal{O}_K) \to 1,$$

where μ_K is the group of roots of unities contained in K, $\log_K$ is given by $\alpha \mapsto (\log|\sigma(\alpha)|)_{\sigma\colon K\hookrightarrow\mathbb{C}}$,

$$\ell\colon (a_1, \ldots, a_{r_1}, b_1, \ldots, b_{r_2}) \mapsto \sum_{i=1}^{r_1} a_i[\rho_i] + \sum_{i=1}^{r_2} b_i([\sigma_i] + [\overline{\sigma}_i]),$$

and $\widehat{\mathrm{Cl}}(\mathrm{Spec}\mathcal{O}_K) \to \mathrm{Cl}(\mathcal{O}_K)$ is the forgetful map.

1.2 Hermitian vector bundles on $\mathrm{Spec}\mathcal{O}_K$

Hermitian vector bundles are higher rank generalizations of Hermitian line bundles, for which there is a rich theory. In this course, we focus on: Even to study Hermitian line bundles on $\mathrm{Spec}\mathcal{O}_K$, it turns out to be sometimes helpful to study the more general Hermitian vector bundles as will be shown in §1.3.

Definition 1.2.1. *A* **Hermitian coherent sheaf** *on $\mathrm{Spec}\mathcal{O}_K$ is a pair $\overline{\mathcal{E}} := (\mathcal{E}, \|\cdot\|)$, where $\mathcal{E}$ is an $\mathcal{O}_K$-module of finite type and $\|\cdot\| = \{\|\cdot\|_\sigma\}_{\sigma\colon K\hookrightarrow\mathbb{C}}$ is a collection of Hermitian metrics $\|\cdot\|_\sigma$ on each $\mathcal{E}_\sigma := \mathcal{E} \otimes_\sigma \mathbb{C}$ such that $\|e\|_\sigma = \|e\|_{\overline{\sigma}}$ for all $e \in \mathcal{E}$ and all $\sigma\colon K \hookrightarrow \mathbb{C}$.*

If moreover $\mathcal{E}$ is a projective $\mathcal{O}_K$-module, then $\overline{\mathcal{E}}$ is called a **Hermitian vector bundle**.

We define the rank of $\overline{\mathcal{E}}$, denoted by $\mathrm{rk}(\overline{\mathcal{E}})$, to be the rank of $\mathcal{E}$ as an $\mathcal{O}_K$-module. A Hermitian coherent sheaf $\overline{\mathcal{E}}$ on $\mathrm{Spec}\mathcal{O}_K$ is a Hermitian vector bundle if and only if $\mathcal{E}$ is torsion-free.

The category of vector bundles on $\mathrm{Spec}\mathcal{O}_K$ is equivalent to the category projective $\mathcal{O}_K$-modules of finite rank. Using this one sees that any Hermitian line bundle on $\mathrm{Spec}\mathcal{O}_K$ is a Hermitian vector bundle on $\mathrm{Spec}\mathcal{O}_K$.

Definition 1.2.2. *Let $\overline{\mathcal{E}}$ and $\overline{\mathcal{F}}$ be Hermitian coherent sheaves (or Hermitian vector bundles) on $\mathrm{Spec}\mathcal{O}_K$. A* **morphism**

$$\varphi\colon \overline{\mathcal{E}} \to \overline{\mathcal{F}}$$

is a morphism between the underlying projective $\mathcal{O}_K$-modules such that $\|\varphi(e)\|_\sigma \leq \|e\|_\sigma$ for all $\sigma\colon K \hookrightarrow \mathbb{C}$ and all $e \in \mathcal{E}_\sigma$.

Thus we can define the category of Hermitian coherent sheaves on $\mathrm{Spec}\mathcal{O}_K$, and the full sub-category of Hermitian vector bundles on $\mathrm{Spec}\mathcal{O}_K$.

1.2.1 Several constructions on $\mathrm{Spec}\mathcal{O}_K$

Short exact sequence Let $\overline{\mathcal{E}}$ be a Hermitian coherent sheaf on $\mathrm{Spec}\mathcal{O}_K$.

Let $\mathcal{F}$ be a submodule of $\mathcal{E}$ and consider the quotient $\mathcal{E} \to \mathcal{G} := \mathcal{E}/\mathcal{F}$. The restriction of the Hermitian metrics $\|\cdot\|_\sigma$ to $\mathcal{F}_\sigma$ for each $\sigma\colon K \hookrightarrow \mathbb{C}$ gives rise to a *Hermitian sub-coherent sheaf* $\overline{\mathcal{F}}$ of $\overline{\mathcal{E}}$. The quotient metrics, *i.e.* for each σ and each $g \in \mathcal{G}_\sigma$,

$$\|g\|_{\mathcal{G},\sigma} := \inf_{e\in\mathcal{E}_\sigma,\ e\mapsto g} \|e\|_\sigma,$$

define a *quotient Hermitian coherent sheaf* $\overline{\mathcal{G}}$ of $\overline{\mathcal{E}}$. We have a short exact sequence in the category of Hermitian coherent sheaves on $\mathrm{Spec}\mathcal{O}_K$

$$0 \to \overline{\mathcal{F}} \to \overline{\mathcal{E}} \to \overline{\mathcal{G}} \to 0.$$

If $\mathcal{F} = \mathcal{E}_{\mathrm{tor}}$, then $\overline{\mathcal{G}}$ is a Hermitian vector bundle.

Direct sum Let $\overline{\mathcal{E}}$ and $\overline{\mathcal{F}}$ be Hermitian coherent sheaves on $\mathrm{Spec}\mathcal{O}_K$. The *direct sum* $\overline{\mathcal{E}} \oplus \overline{\mathcal{F}}$ is defined to be $(\mathcal{E} \oplus \mathcal{F}, \|\cdot\|_{\mathcal{E}} + \|\cdot\|_{\mathcal{F}})$. It is a Hermitian vector bundle if both $\overline{\mathcal{E}}$ and $\overline{\mathcal{F}}$ are Hermitian vector bundles. The rank is $\mathrm{rk}(\overline{\mathcal{E}}) + \mathrm{rk}(\overline{\mathcal{F}})$.

Tensor product Let $\overline{\mathcal{E}}$ and $\overline{\mathcal{F}}$ be Hermitian coherent sheaves on $\mathrm{Spec}\mathcal{O}_K$. The *tensor product* $\overline{\mathcal{E}} \otimes \overline{\mathcal{F}}$ is defined to be $(\mathcal{E} \otimes \mathcal{F}, \|\cdot\|_{\mathcal{E}}\|\cdot\|_{\mathcal{F}})$. It is a Hermitian vector bundle if both $\overline{\mathcal{E}}$ and $\overline{\mathcal{F}}$ are Hermitian vector bundles. The rank is $\mathrm{rk}(\overline{\mathcal{E}})\mathrm{rk}(\overline{\mathcal{F}})$.

Dual and homomorphism Let $\overline{\mathcal{E}}$ be a Hermitian coherent sheaf on $\mathrm{Spec}\mathcal{O}_K$. Its *dual* $\overline{\mathcal{E}}^{\vee}$ is defined to be $(\mathcal{E}^{\vee}, \|\cdot\|^{\vee})$, where $\mathcal{E}^{\vee} := \mathrm{Hom}(\mathcal{E}, \mathcal{O}_K)$ and

$$\|v\|_{\sigma} := \sup_{e \in \mathcal{E}_{\sigma}} \frac{|v(e)|_{\sigma}}{\|e\|_{\sigma}} \quad \text{for all } \sigma \in M_{K,\infty} \text{ and all } v \in \mathcal{E}_{\sigma}^{\vee}.$$

It is a Hermitian vector bundle if $\overline{\mathcal{E}}$ is a Hermitian vector bundle. The rank is $\mathrm{rk}(\overline{\mathcal{E}})$.

More generally, let $\overline{\mathcal{E}}$ and $\overline{\mathcal{F}}$ be Hermitian coherent sheaves on $\mathrm{Spec}\mathcal{O}_K$. Then the *homomorphism* $\mathrm{Hom}(\overline{\mathcal{E}}, \overline{\mathcal{F}})$ is defined to be $\overline{\mathcal{E}}^{\vee} \otimes \overline{\mathcal{F}}$. It is a Hermitian vector bundle if both $\overline{\mathcal{E}}$ and $\overline{\mathcal{F}}$ are Hermitian vector bundles. The rank is $\mathrm{rk}(\overline{\mathcal{E}})\mathrm{rk}(\overline{\mathcal{F}})$.

Determinant Let $\overline{\mathcal{E}}$ be a Hermitian vector bundle on $\mathrm{Spec}\mathcal{O}_K$ of rank n. The *determinant* of $\overline{\mathcal{E}}$ is defined to be $\det \overline{\mathcal{E}} := (\bigwedge^n \mathcal{E}, \|\cdot\|_{\det})$, where for each $\sigma\colon K \hookrightarrow \mathbb{C}$, the metric $\|\cdot\|_{\det,\sigma}$ is the unique metric on $(\bigwedge^n \mathcal{E})_{\sigma}$ such that

$$\|e_1 \wedge \cdots \wedge e_n\|_{\det,\sigma} = 1$$

for any orthonormal basis $\{e_1, \ldots, e_n\}$ of the normed Euclidean space $(\mathcal{E}_{\sigma}, \|\cdot\|_{\sigma})$.

Notice that the determinant is always a Hermitian line bundle on $\mathrm{Spec}\mathcal{O}_K$. Now we can define:

Definition 1.2.3. *Let* $\overline{\mathcal{E}}$ *be a Hermitian vector bundle on* $\mathrm{Spec}\mathcal{O}_K$. *The* **arithmetic degree** *of* $\overline{\mathcal{E}}$ *is defined to be*

$$\widehat{\deg}(\overline{\mathcal{E}}) := \widehat{\deg}(\det \overline{\mathcal{E}}).$$

Let us look at the example and particularly important case where $K = \mathbb{Q}$. Since the class number of $\mathbb{Q}$ is 1, any projective module of finite rank is a free module. Consider a Hermitian vector bundle $\overline{\mathcal{E}} = (\mathcal{E}, \|\cdot\|)$. Let $\{v_1, \ldots, v_n\}$ be a $\mathbb{Z}$-basis of $\mathcal{E}$. Then $v := v_1 \wedge \cdots \wedge v_n$ is a $\mathbb{Z}$-basis of $\det \mathcal{E} := \bigwedge^n \mathcal{E}$. Thus

$$\widehat{\deg}(\overline{\mathcal{E}}) = \log \#(\det \mathcal{E}/\mathbb{Z}v) - \log \|v\| = -\log \|v\| = -\frac{1}{2} \log \det \left(h(v_i, v_j)\right),$$

where $h(\cdot,\cdot)$ is the Hermitian form on $\mathcal{E}_{\mathbb{C}}$, *i.e.* $h(v', v') = \|v'\|^2$ for all $v' \in \mathcal{E}_{\mathbb{C}}$.

On the other hand, let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathcal{E}_{\mathbb{R}}$. Then we have an isomorphism $\mathcal{E}_{\mathbb{R}} \xrightarrow{\sim} \mathbb{R}^n$ with $\mathcal{E}$ identified with a lattice in $\mathbb{R}^n$. Let $\mathrm{covol}(\mathcal{E}_{\mathbb{R}}/\mathcal{E})$ denote the co-volume of this lattice, namely the volume of any fundamental domain of this lattice for the Lebesgue measure on $\mathcal{E}_{\mathbb{R}}$.

For each i, we have $v_i = \sum_j a_{ij} e_j$ for some $a_{ij} \in \mathbb{R}$. Then $h(v_i, v_j) = \sum_k a_{ik} a_{jk}$. Thus $\det\left(h(v_i, v_j)\right) = \det(A^{\mathrm{t}} A) = \det(A)^2$ for the matrix $A = (a_{ij})$. Therefore we have

$$\widehat{\deg}(\overline{\mathcal{E}}) = -\log \mathrm{covol}(\mathcal{E}_{\mathbb{R}}/\mathcal{E}). \tag{1.2.1}$$

1.2.2 Pullback, pushforward, norm

Let $K \subseteq K'$ be an inclusion of number fields, and $f\colon \mathrm{Spec}\mathcal{O}_{K'} \to \mathrm{Spec}\mathcal{O}_K$ the induced morphism; then f is finite of degree $d := [K' : K]$.

Pullback Let $\overline{\mathcal{E}}$ be a Hermitian vector bundle on $\mathrm{Spec}\mathcal{O}_K$ of rank n. Define its pullback $f^*\overline{\mathcal{E}}$ as follows. First, set $f^*\mathcal{E} := \mathcal{E} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$; then $f^*\mathcal{E}$ is a projective $\mathcal{O}_{K'}$-module of rank n. Next, for any embedding $\sigma'\colon K' \hookrightarrow \mathbb{C}$, its restriction to K (denoted by σ) is an embedding of K into $\mathbb{C}$, and the canonical isomorphism $(f^*\mathcal{E}) \otimes_{\sigma'} \mathbb{C} = \mathcal{E} \otimes_{\sigma} \mathbb{C}$ gives the desired metric $\|\cdot\|_{\sigma'}$ on $(f^*\mathcal{E})_{\sigma'} = (f^*\mathcal{E}) \otimes_{\sigma'} \mathbb{C}$.

Proposition 1.2.4. *The pullback f^* commutes with direct sums, tensor products, and taking determinants. Moreover,*

$$\widehat{\deg} f^*\overline{\mathcal{E}} = [K' : K]\widehat{\deg}\overline{\mathcal{E}}.$$

Proof. The first claim is easy to check and we leave it as an exercise. To prove the second claim, it then suffices to check for Hermitian line bundles.

Let $\ell \in \mathcal{E} \setminus \{0\}$. Then

$$\widehat{\deg}(\overline{\mathcal{E}}) = \log \#(\mathcal{E}/\ell\mathcal{O}_K) - \sum_{\sigma\colon K \hookrightarrow \mathbb{C}} \log \|\ell\|_\sigma = \sum_{\mathfrak{p} \in M_{K,f}} \mathrm{ord}_{\mathfrak{p}}(\ell) \log \#(\mathcal{O}_K/\mathfrak{p}) - \sum_{\sigma\colon K \hookrightarrow \mathbb{C}} \log \|\ell\|_\sigma.$$

Thus

$$\begin{aligned}
\widehat{\deg}(f^*\overline{\mathcal{E}}) &= \sum_{\mathfrak{p}' \in M_{K',f}} \mathrm{ord}_{\mathfrak{p}'}(\ell) \log \#(\mathcal{O}_{K'}/\mathfrak{p}') - \sum_{\sigma'\colon K' \hookrightarrow \mathbb{C}} \log \|\ell\|_{\sigma'} \\
&= \sum_{\mathfrak{p} \in M_{K,f}} \sum_{\mathfrak{p}'|\mathfrak{p}} \mathrm{ord}_{\mathfrak{p}'}(\ell) \log \#(\mathcal{O}_{K'}/\mathfrak{p}') - \sum_{\sigma\colon K \hookrightarrow \mathbb{C}} \sum_{\sigma'|\sigma} \log \|\ell\|_{\sigma'} \\
&= \sum_{\mathfrak{p} \in M_{K,f}} d \cdot \mathrm{ord}_{\mathfrak{p}}(\ell) \log \#(\mathcal{O}_K/\mathfrak{p}) - \sum_{\sigma\colon K \hookrightarrow \mathbb{C}} d \log \|\ell\|_\sigma \\
&= d \cdot \widehat{\deg}(\overline{\mathcal{E}}).
\end{aligned}$$

We are done. ∎

Pushforward Let $\overline{\mathcal{E}}'$ be a Hermitian vector bundle on $\mathrm{Spec}\mathcal{O}_{K'}$ of rank n. Define its pushforward $f_*\overline{\mathcal{E}}'$ as follows. First, the underlying projective module $f_*\mathcal{E}'$ is set to be $\mathcal{E}'$, viewed as an $\mathcal{O}_K$-module of rank dn which is again projective (locally free). Next, for any embedding $\sigma\colon K \hookrightarrow \mathbb{C}$, the tensor product $\mathcal{O}_{K'} \otimes_{\mathcal{O}_K,\sigma} \mathbb{C}$ is canonically isomorphic to

$$\bigoplus\nolimits_{\sigma'|\sigma} \mathbb{C} := \bigoplus\nolimits_{\sigma'\colon K' \hookrightarrow \mathbb{C},\ \sigma'|_K = \sigma} \mathbb{C}.$$

Thus we have a canonical isomorphism

$$(f_*\mathcal{E}')_\sigma = \mathcal{E}' \otimes_{\mathcal{O}_K,\sigma} \mathbb{C} = \mathcal{E}' \otimes_{\mathcal{O}_{K'}} (\mathcal{O}_{K'} \otimes_{\mathcal{O}_K,\sigma} \mathbb{C}) = \bigoplus\nolimits_{\sigma'|\sigma} \mathcal{E}'_{\sigma'}.$$

Thus the desired Hermitian metric is given by: for any $e = (e'_{\sigma'})_{\sigma'|\sigma} \in (f_*\mathcal{E}')_\sigma$, set

$$\|e\|_\sigma^2 := \sum_{\sigma'|\sigma} d_{\sigma'/\sigma} \|e'_{\sigma'}\|_{\sigma'}^2, \tag{1.2.2}$$

where $d_{\sigma'/\sigma} = 2$ if σ' is a complex place and σ is a real place, and $d_{\sigma'/\sigma} = 1$ otherwise.

Sometimes, it is more convenient to put a singular metric, by changing (1.2.2) to

$$\|e\|_{\max,\sigma} := \max_{\sigma'|\sigma} \|e'_{\sigma'}\|_{\sigma'}. \tag{1.2.3}$$

We denote by $f_{\max,*}\overline{\mathcal{E}}' := (f_*\mathcal{E}', \|\cdot\|_{\max})$.

Example 1.2.5. *A particularly important case is when $f\colon \mathrm{Spec}\mathcal{O}_K \to \mathrm{Spec}\mathbb{Z}$ is induced by the inclusion $\mathbb{Q} \subseteq K$ (we changed our notation for this particular case). Let $\overline{\mathcal{L}}$ be a Hermitian line bundle on $\mathrm{Spec}\mathcal{O}_K$. Then $f_*\mathcal{L}$ is a vector bundle on $\mathrm{Spec}\mathbb{Z}$ which must be trivial since the class number of $\mathbb{Q}$ is 1. Under the identification of vector bundles and projective modules, this is equivalent to say that $H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})$ is a projective $\mathcal{O}_K$-module of rank 1, and is free if viewed as a $\mathbb{Z}$-module. Moreover, we have*

$$H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L}) \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus\nolimits_{\sigma\colon K \hookrightarrow \mathbb{C}} \mathcal{L}_\sigma.$$

For any $s = (s_\sigma)_\sigma \in H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L}) \otimes_{\mathbb{Z}} \mathbb{C}$, we then have

$$\|s\|_{\max} = \max_\sigma \{\|s_\sigma\|\}.$$

Set

$$H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})_{\mathbb{R}} := \{s = (s_\sigma)_\sigma \in H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L}) \otimes_{\mathbb{Z}} \mathbb{C} : s_\sigma = \overline{s}_{\overline{\sigma}} \text{ for all } \sigma\}. \tag{1.2.4}$$

Then $H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})$ is a lattice in $H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})_{\mathbb{R}}$, and $\|\cdot\|_{\max}$ induces a norm on $H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})_{\mathbb{R}}$.

We will come back to this example later.

Norm of Hermitian line bundles Let $\overline{\mathcal{L}}'$ be a Hermitian line bundle on $\mathrm{Spec}\mathcal{O}_{K'}$. We wish to define the *norm* $\mathrm{Norm}_{K'/K}(\overline{\mathcal{L}}') \in \widehat{\mathrm{Pic}}(\mathrm{Spec}\mathcal{O}_K)$, which corresponds to the pushforward of the arithmetic class group (even though we have not defined what it means), *i.e.*

$$f_*\widehat{c}_1(\overline{\mathcal{L}}') = \widehat{c}_1(\mathrm{Norm}_{K'/K}(\overline{\mathcal{L}}')).$$

for the arithmetic first Chern class $\widehat{c}_1$ (the inverse of (1.1.5)).

Let (U_i) be an open cover of $\mathrm{Spec}\mathcal{O}_K$ such that $\mathcal{L}'|_{f^{-1}(U_i)}$ is trivial for each i. Choose a section $\epsilon_i \in H^0(f^{-1}(U_i), \mathcal{L}')$ which generates $\mathcal{L}'$ everywhere on $f^{-1}(U_i)$. Then the line bundle $\mathcal{L}'$ is represented by the 1-cocycle (f_{ij}) defined as follows: for each pair (i,j) and $U_{ij} := U_i \cap U_j$, $f_{ij} \in H^0(f^{-1}(U_{ij}), \mathcal{O}^\times_{\mathrm{Spec}\mathcal{O}_{K'}})$ is the unique invertible function on $f^{-1}(U_{ij})$ such that $\epsilon_i = f_{ij}\epsilon_j$.

The underlying line bundle $\mathrm{Norm}_{K'/K}(\mathcal{L}')$ is then defined to be the line bundle on $\mathrm{Spec}\mathcal{O}_K$ determined by the 1-cocycle $\mathrm{Norm}_{K'/K}(f_{ij})$, relative to the open cover (U_i). It admits a canonical trivialization over U_i with generator $\mathrm{Norm}_{K'/K}(\epsilon_i)$.

The Hermitian metrics are defined as follows. Let $\sigma\colon K \hookrightarrow \mathbb{C}$. Then we have a canonical isomorphism

$$\mathrm{Norm}_{K'/K}(\mathcal{L}')_\sigma = \bigotimes\nolimits_{\sigma'|\sigma} \mathcal{L}'_{\sigma'}.$$

This defines a canonical Hermitian metric on $\mathrm{Norm}_{K'/K}(\mathcal{L}')_\sigma$.

1.3 Positivity of Hermitian line bundles on $\mathrm{Spec}\mathcal{O}_K$

Let $\overline{\mathcal{L}}$ be a Hermitian line bundle on $\mathrm{Spec}\mathcal{O}_K$.

Definition 1.3.1. *The Hermitian line bundle $\overline{\mathcal{L}}$ is said to be* **ample** *(resp.* **nef***) if $\widehat{\deg}(\overline{\mathcal{L}}) > 0$ (resp. $\widehat{\deg}(\overline{\mathcal{L}}) \geq 0$).*

We will prove a criterion for ampleness (Theorem 1.3.8) which is the arithmetic version of the criterion for ample line bundles over curves. For this we need to introduce the sets of effective sections and of strictly effective sections of $\overline{\mathcal{L}}$.

Definition 1.3.2. *Define*

$$\widehat{H}^0(\overline{\mathcal{L}}) := \{s \in H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L}) : \|s\|_\sigma \leq 1,\ \forall \sigma\},$$
$$\widehat{H}^0_{\mathrm{s}}(\overline{\mathcal{L}}) := \{s \in H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L}) : \|s\|_\sigma < 1,\ \forall \sigma\}.$$

Lemma 1.3.3. *Both $\widehat{H}^0(\overline{\mathcal{L}})$ and $\widehat{H}^0_{\mathrm{s}}(\overline{\mathcal{L}})$ are finite sets.*

Proof. It suffices to prove the result for $\widehat{H}^0(\overline{\mathcal{L}})$. By Example 1.2.5, $\widehat{H}^0(\overline{\mathcal{L}})$ is the set of lattice points in $H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})_{\mathbb{R}}$ contained in the unit ball defined by the norm induced by $\|\cdot\|_{\max}$. Thus it is a finite set. □

Definition 1.3.4. *Define*

$$\widehat{h}^0(\overline{\mathcal{L}}) := \log \#\widehat{H}^0(\overline{\mathcal{L}}),$$
$$\widehat{h}^0_{\mathrm{s}}(\overline{\mathcal{L}}) := \log \#\widehat{H}^0_{\mathrm{s}}(\overline{\mathcal{L}}).$$

By definition of arithmetic degree (1.1.1), $\overline{\mathcal{L}}$ is ample if $\widehat{h}^0_{\mathrm{s}}(\overline{\mathcal{L}}) > 0$ and is nef if $\widehat{h}^0(\overline{\mathcal{L}}) > 0$.

As indicated by the proof of Lemma 1.3.3, we are interested in counting the number of lattice points in a unit ball, both contained in a Euclidean space. In general this is not an easy task. But there are tools in the theory of geometry of numbers which we can use.

1.3.1 Geometry of numbers

Consider the pairs $\overline{M} = (M, \|\cdot\|)$ where M is a free $\mathbb{Z}$-module of finite rank of $r \geq 1$ and $\|\cdot\|$ is a norm on $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. Thus the natural map $M \to M_{\mathbb{R}}$ makes M into a lattice in $M_{\mathbb{R}}$. An example is the one obtained from $H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})$ and $\|\cdot\|_{\max}$ from Example 1.2.5.

Set

$$\widehat{H}^0(\overline{M}) := \{m \in M : \|m\| \leq 1\}, \qquad \widehat{h}^0(\overline{M}) := \log \#\widehat{H}^0(\overline{M});$$
$$\widehat{H}^0_{\mathrm{s}}(\overline{M}) := \{m \in M : \|m\| < 1\}, \qquad \widehat{h}^0_{\mathrm{s}}(\overline{M}) := \log \#\widehat{H}^0_{\mathrm{s}}(\overline{M}).$$

Minkowski's First Theorem is a tool to prove the existence of a non-zero small lattice point, via the quantity $\chi(\overline{M})$ defined as below. Denote by $B(\overline{M}) := \{m \in M_{\mathbb{R}} : \|m\| \leq 1\}$ the unit ball in $M_{\mathbb{R}}$. Fix a Haar measure on $M_{\mathbb{R}}$ and let

$$\chi(\overline{M}) := \log \frac{\mathrm{vol}(B(\overline{M}))}{\mathrm{covol}(M_{\mathbb{R}}/M)}, \tag{1.3.1}$$

which is independent of the choice of the Haar measure. This is an arithmetic analogue of the Euler characteristic.

By Minkowski's First Theorem, $\widehat{h}^0(\overline{M}) > 0$ if $\chi(\overline{M}) > r \log 2$. The following is a quantitative version:

Theorem 1.3.5. *We have*

$$\widehat{h}^0(\overline{M}) \geq \chi(\overline{M}) - r \log 2. \tag{1.3.2}$$

Moreover, there exists a non-zero $m \in M$ such that

$$-\log \|m\| \geq \frac{\chi(\overline{M})}{r} - \log 2.$$

To prove Theorem 1.3.5, we use a common trick called *Variational Principle* in Arakelov Geometry. For any real number c, set

$$\overline{M}(c) := (M, e^{-c}\|\cdot\|).$$

It is not hard to check that

$$\chi(\overline{M}(c)) = \chi(\overline{M}) + cr.$$

Proof. Consider the universal covering

$$u\colon M_{\mathbb{R}} \to M_{\mathbb{R}}/M.$$

For $c \in \mathbb{R}$, there exists a point $y \in M_{\mathbb{R}}/M$ such that

$$\#\left(u^{-1}(y) \cap B(\overline{M}(c))\right) \geq \mathrm{vol}(B(\overline{M}(c)))/\mathrm{covol}(M_{\mathbb{R}}/M).$$

Otherwise we would have $\mathrm{vol}(B(\overline{M}(c))) < \mathrm{covol}(M_{\mathbb{R}}/M) \cdot \mathrm{vol}(B(\overline{M}(c)))/\mathrm{covol}(M_{\mathbb{R}}/M)$.

Take $m_0 \in u^{-1}(y) \cap B(\overline{M}(c))$. For any $m \in u^{-1}(y) \cap B(\overline{M}(c))$, we have $m - m_0 \in M$ and $\|m - m_0\| \leq 2e^c$, and therefore $m - m_0 \in \widehat{H}^0(\overline{M}(c + \log 2))$. Hence

$$\widehat{h}^0(\overline{M}(c + \log 2)) \geq \log \#\left(u^{-1}(y) \cap B(\overline{M}(c))\right).$$

The two inequalities above together with the definition of $\chi(\overline{M}(c))$ yield

$$\widehat{h}^0(\overline{M}(c + \log 2)) \geq \chi(\overline{M}(c)) = \chi(\overline{M}) + cr.$$

Thus we get (1.3.2) by letting $c = -\log 2$.

Now for any $c \in \mathbb{R}$, we have

$$\widehat{h}^0(\overline{M}(-c)) \geq \chi(\overline{M}(-c)) - r\log 2 = \chi(\overline{M}) - rc - r\log 2.$$

Thus for all $c < \chi(\overline{M})/r - \log 2$, there exists a non-zero $m \in M$ such that $e^c\|m\| \leq 1$. In other words, for any $\epsilon > 0$, there exists a non-zero $m_\epsilon \in M$ with

$$-\log\|m_\epsilon\| \geq \frac{\chi(\overline{M})}{r} - \log 2 - \epsilon.$$

Taking a sequence $\{\epsilon_n\}_{n\geq 1}$ decreasing to 0, the corresponding sequence $\{m_n\}_{n\geq 1}$ takes finitely many values since m_n are lattice points in a bounded ball. Thus we find an $m \in M$ with

$$-\log\|m\| \geq \frac{\chi(\overline{M})}{r} - \log 2 - \epsilon_n$$

with $\epsilon_n \to 0$. It suffices to take $n \to \infty$. $\square$

Proposition 1.3.6. *We have*

$$\widehat{h}^0_{\mathrm{s}}(\overline{M}) \leq \widehat{h}^0(\overline{M}) \leq \widehat{h}^0_{\mathrm{s}}(\overline{M}) + r\log 3.$$

Proof. We will prove the desired comparison by the following: For any $c > 0$, we have

$$\widehat{h}^0(\overline{M}(-c)) \leq \widehat{h}^0(\overline{M}) \leq \widehat{h}^0(\overline{M}(-c)) + rc + r\log 3. \tag{1.3.3}$$

In fact, the desired inequality follows directly from (1.3.3) by letting $c \to 0$.

Let us prove (1.3.3). We only need to prove the second inequality. For any $t > 0$, set $B(t) := \{m \in M_{\mathbb{R}} : \|m\| \le t\}$ the ball of radius t centered at 0. Then $\mathrm{vol}(B(t)) = t^r \mathrm{vol}(B(1))$. Notice that $B(t) = B(\overline{M}(\log t))$.

Now consider $B(1+2^{-1}e^{-c})$. Let us also consider all the balls of radius $2^{-1}e^{-c}$ centered at points in $M \cap B(1) = \widehat{H}^0(\overline{M})$; all these small balls are contained in $B(1+2^{-1}e^{-c})$. Thus there exists a point $m \in B(1+2^{-1}e^{-c})$ which is contained in N of these small balls, with

$$N \ge \frac{\#\widehat{H}^0(\overline{M}) \cdot \mathrm{vol}(B(2^{-1}e^{-c}))}{\mathrm{vol}(B(1+2^{-1}e^{-c}))} = \#\widehat{H}^0(\overline{M}) \frac{1}{(1+2e^c)^r}.$$

Thus

$$\log N \ge \widehat{h}^0(\overline{M}) - r(c + \log 3).$$

Let $x_1, \ldots, x_N$ be the centers of these small balls. Then $x_i - m \in B(2^{-1}e^{-c})$. Hence $x_i - x_1 \in B(e^{-c})$ for all i. In particular we find N points in $\widehat{H}^0(\overline{M}(-c))$. Therefore we can conclude. $\square$

1.3.2 Ampleness and nefness

Let $\overline{\mathcal{L}}$ be a Hermitian line bundle on $\mathrm{Spec}\mathcal{O}_K$.

We relate Hermitian line bundles with the theory of geometry of numbers as follows. Let $\overline{M} = (M, \|\cdot\|)$ be the pair as in §1.3.1 obtained from $H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})$ and $\|\cdot\|_{\max}$ from Example 1.2.5. Then by definition, we have

$$\widehat{H}^0(\overline{\mathcal{L}}) = \widehat{H}^0(\overline{M}), \qquad \widehat{h}^0(\overline{\mathcal{L}}) = \widehat{h}^0(\overline{M});$$
$$\widehat{H}^0_{\mathrm{s}}(\overline{\mathcal{L}}) = \widehat{H}^0_{\mathrm{s}}(\overline{M}), \qquad \widehat{h}^0_{\mathrm{s}}(\overline{\mathcal{L}}) = \widehat{h}^0_{\mathrm{s}}(\overline{M}).$$

We also set

$$\chi(\overline{\mathcal{L}}) := \chi(\overline{M}). \tag{1.3.4}$$

The following arithmetic Riemann–Roch theorem is not hard to prove.

Theorem 1.3.7 (Arithmetic Riemann–Roch over $\mathrm{Spec}\mathcal{O}_K$)**.** $\chi(\overline{\mathcal{L}}) = \widehat{\deg}\overline{\mathcal{L}} + \chi(\mathcal{O}_{\mathrm{Spec}\mathcal{O}_K}, |\cdot|)$. *Here $|\cdot|$ is the trivial norm on $\mathrm{Spec}\mathcal{O}_K$.*

Theorem 1.3.8. *The followings are equivalent:*

(i) $\overline{\mathcal{L}}$ is ample,

(ii) $\widehat{h}^0_{\mathrm{s}}(\overline{\mathcal{L}}^{\otimes m}) > 0$ for $m \gg 1$,

(iii) for any Hermitian line bundle $\overline{\mathcal{M}}$ on $\mathrm{Spec}\mathcal{O}_K$, we have $\widehat{h}^0_{\mathrm{s}}(\overline{\mathcal{L}}^{\otimes m} \otimes \overline{\mathcal{M}}) > 0$ for $m \gg 1$.

Proof. (iii) clearly implies (ii).

(ii) implies (i): Take a non-zero $s \in \widehat{H}^0_{\mathrm{s}}(\overline{\mathcal{L}}^{\otimes m})$. Then by definition of arithmetic degree (1.1.1), we have $\widehat{\deg}(\overline{\mathcal{L}}^{\otimes m}) > 0$. But $\widehat{\deg}(\overline{\mathcal{L}}^{\otimes m}) = m\widehat{\deg}(\overline{\mathcal{L}})$ by Proposition 1.1.7. Thus $\overline{\mathcal{L}}$ is ample.

(i) implies (iii): By Theorem 1.3.7, we have

$$\chi(\overline{\mathcal{L}}^{\otimes m} \otimes \overline{\mathcal{M}}) = m\widehat{\deg}(\overline{\mathcal{L}}) + \widehat{\deg}(\overline{\mathcal{M}}) + \chi(\mathcal{O}_{\mathrm{Spec}\mathcal{O}_K}, |\cdot|).$$

Since $\widehat{\deg}(\overline{\mathcal{L}}) > 0$, for $m \gg 1$ we have $\chi(\overline{\mathcal{L}}^{\otimes m} \otimes \overline{\mathcal{M}}) > [K : \mathbb{Q}] \log 6$. Thus $\widehat{h}^0_{\mathrm{s}}(\overline{\mathcal{L}}^{\otimes m} \otimes \overline{\mathcal{M}}) > 0$ by Theorem 1.3.5 and Proposition 1.3.6. $\square$