Chapter 4

Proof of Arithmetic Hilbert–Samuel

With the preparation in the previous chapter, we prove the arithmetic Hilbert–Samuel theorem for vol_χ and $\overline{\mathcal{L}}$ in the following setup in this chapter. We follow the approach of Abbès–Bouche.

Let K be a number field and let $\mathcal{O}_K$ be its ring of integers.

Let $\mathcal{X}$ be a projective arithmetic variety of dimension $n+1$ and let $\overline{\mathcal{L}}$ be a smooth Hermitian line bundle. We furthermore consider the case where $\mathcal{X} \to \mathrm{Spec}\mathbb{Z}$ factors through $\mathrm{Spec}\mathcal{O}_K$ and that the generic fiber $\mathcal{X}_K$ is smooth and irreducible.

Theorem 4.0.1. *Assume $\mathcal{L}$ is very ample on $\mathcal{X}$ and $c_1(\overline{\mathcal{L}}) > 0$. Then*

$$\lim_{k\to\infty} \frac{\chi(k\overline{\mathcal{L}})}{k^{n+1}/(n+1)!} \longrightarrow \overline{\mathcal{L}}^{n+1} \tag{4.0.1}$$

when $k \to \infty$.

In other words, the sup-limit in the definition of $\mathrm{vol}_\chi(\overline{\mathcal{L}})$ (Definition 2.5.10) is an actually limit under the assumption of the theorem, and $\mathrm{vol}_\chi(\overline{\mathcal{L}}) = \overline{\mathcal{L}}^{n+1}$.

In the proof, we will use the Hilbert–Samuel theorem in algebraic geometry. Let P be the Hilbert polynomial of $\mathcal{L}_K$ on $\mathcal{X}_K$, *i.e.* $P(k) = \dim H^0(\mathcal{X}_K, k\mathcal{L}_K)$ for $k \gg 1$. It is known that $\deg P = n$ with leading coefficient $\mathcal{L}_K^n/n!$.

4.1 Framework of the proof

4.1.1 Revision on the statement

We start by recalling the objects appearing in the statement of Theorem 4.0.1.

First we have the embedding

$$0 \to H^0(\mathcal{X}, \mathcal{L}) \to H^0(\mathcal{X}, \mathcal{L})_\mathbb{R}, \tag{4.1.1}$$

with $H^0(\mathcal{X}, \mathcal{L})_\mathbb{R}$ a real vector space of finite dimension defined in (2.5.2) and $H^0(\mathcal{X}, \mathcal{L})$ a lattice. In fact, the structural morphism $\mathcal{X} \to \mathrm{Spec}\mathbb{Z}$ factors through $\mathrm{Spec}R$ for an order R in a number field K, such that the generic fiber $\mathcal{X}_K$ is irreducible, and

$$H^0(\mathcal{X}, \mathcal{L})_\mathbb{R} := \{s = (s_\sigma)_\sigma \in H^0(\mathcal{X}, \mathcal{L})_\mathbb{C} = \bigoplus\nolimits_{\sigma\colon K\hookrightarrow\mathbb{C}} H^0(\mathcal{X}_\sigma, \mathcal{L}_\sigma) : s_\sigma = \overline{s_{\overline{\sigma}}} \text{ for all } \sigma\}.$$

We shall use the *sup-norm* on $H^0(\mathcal{X}, \mathcal{L})_\mathbb{R}$ defined as follows:

- For any $s = (s_\sigma)_\sigma \in H^0(\mathcal{X}, \mathcal{L})_\mathbb{R}$, define $\|s\|_{\sup} := \sup_{\sigma, x\in\mathcal{X}_\sigma}(\|s_\sigma(x)\|)$.

Set $B(\overline{\mathcal{L}})$ to be the unit ball in $H^0(\mathcal{X},\mathcal{L})_{\mathbb{R}}$ defined by the sup-norm $\|\cdot\|_{\sup}$. Then

$$\chi(\overline{\mathcal{L}}) := \log \frac{\mathrm{vol}(B(\overline{\mathcal{L}}))}{\mathrm{covol}(H^0(\mathcal{X},\mathcal{L})_{\mathbb{R}}/H^0(\mathcal{X},\mathcal{L}))} \tag{4.1.2}$$

is independent of the choice of the Haar measure on $H^0(\mathcal{X},\mathcal{L})_{\mathbb{R}}$. This finishes the explanation of the limit in (4.0.1).

We also need to define an L^2-norm on $H^0(\mathcal{X},\mathcal{L})_{\mathbb{R}}$ for the proof of Theorem 4.0.1. For this, consider the positive $(1,1)$-form $c_1(\overline{\mathcal{L}})$ on $\mathcal{X}(\mathbb{C}) = \coprod_\sigma \mathcal{X}_\sigma$. For each $\sigma\colon K \hookrightarrow \mathbb{C}$, the positive $(1,1)$-form $c_1(\overline{\mathcal{L}}_\sigma)$ is a Kähler form on $\mathcal{X}_\sigma$. We normalize it to

$$\omega_\sigma := \frac{c_1(\overline{\mathcal{L}}_\sigma)}{(\int_{\mathcal{X}_\sigma} c_1(\overline{\mathcal{L}}_\sigma)^{\wedge n}/n!)^{1/n}},$$

which is still a Kähler form on $\mathcal{X}_\sigma$, and the volume of $\mathcal{X}_\sigma$ for the associated volume form $\mathrm{d}V_\sigma$ is 1.

Now we define the L^2*-norm* on $H^0(\mathcal{X},\mathcal{L})_{\mathbb{R}}$ to be:

- For any $s = (s_\sigma)_\sigma \in H^0(\mathcal{X},\mathcal{L})_{\mathbb{R}}$, define $\|s\|_{L^2} := \sup_\sigma(\int_{\mathcal{X}_\sigma} \|s(x)\|^2 \mathrm{d}V_\sigma)^{1/2}$.

It is a fundamental question in Arakelov Geometry to compare the sup-norm and the L^2-norm. We shall prove later on, using the distortion function discussed in §3.1, the following result.

Proposition 4.1.1. *There exists a constant $c > 0$ such that for all $N \geq 1$ and all $s \in H^0(\mathcal{X}, k\mathcal{L})$, we have*

$$\|s\|_{L^2} \leq \|s\|_{\sup} \leq cP(k)^{1/2}\|s\|_{L^2}.$$

In fact, this c can be chosen to be $\sqrt{\sup_{x\in\mathcal{X}(\mathbb{C})} b_k(x)/P(k)}$, where b_k is the distortion function. Theorem 3.1.6 guarantees $c > 0$.

4.1.2 A tale of three volumes

Consider the embedding (4.1.1). We shall define three volume forms on $H^0(\mathcal{X}, k\mathcal{L})_{\mathbb{R}}$ for each $k \geq 1$:

(i) $V^k_{X,\sup}$ such that the volume of $B(k\overline{\mathcal{L}})$ has volume 1;

(ii) V^k_{X,L^2} such that the volume of the unit ball in $H^0(\mathcal{X}, k\mathcal{L})_{\mathbb{R}}$ defined by the L^2-norm $\|\cdot\|_{L^2}$ has volume 1;

(iii) $V^k_{X,\alpha}$ for each real number $\alpha \in \mathbb{R}$ defined below by Definition 4.1.2 (which we will call $\eta_{k,\alpha}$), with $M = \bigoplus_{k\geq 0} H^0(\mathcal{X}, k\mathcal{L})$.

A key point to prove arithmetic Hilbert–Samuel is to compare $V^k_{X,\sup}$ with $V^k_{X,\alpha}$, and the comparison is done via V^k_{X,L^2}. The statements of these comparisons and their consequence on arithmetic Hilbert–Samuel will be discussed in the next subsection.

In this subsection, we give the definition of $V^k_{X,\alpha}$. We start by defining the following generalization of the *arithmetic Euler characteristic* (1.3.1) in the context of geometry of numbers. For any M a finitely generated $\mathbb{Z}$-module of rank $r \geq 1$, define for each volume form η on $M_{\mathbb{R}}$ (*i.e.* an element $\eta \in \det_{\mathbb{R}}(M_{\mathbb{R}})$) the following

$$\chi(M,\eta) := -\log(\mathrm{covol}_\eta(M_{\mathbb{R}}/\overline{M})) + \log(\# M_{\mathrm{tor}}) \tag{4.1.3}$$

where $\overline{M} := M/M_{\mathrm{tor}}$. In fact, (4.1.2) can be defined in this context for $M = H^0(\mathcal{X}, \mathcal{L})$ and η the volume form determined by the sup-norm. The function χ is additive: for $0 \to M_1 \to M_2 \to M_3 \to 0$ an exact sequence and $\eta_j \in \det_{\mathbb{R}}(M_{j,\mathbb{R}})$ $(j \in \{1, 2, 3, \})$ such that $\eta_2 = \eta_1 \otimes \eta_3$ in the induced isomorphism $\det_{\mathbb{R}}(M_{2,\mathbb{R}}) \simeq \det_{\mathbb{R}}(M_{1,\mathbb{R}}) \otimes \det_{\mathbb{R}}(M_{3,\mathbb{R}})$, we have

$$\chi(M_2, \eta_2) = \chi(M_1, \eta_1) + \chi(M_3, \eta_3). \tag{4.1.4}$$

Back to our case. Let $M = \bigoplus_k M_k$ be a graduated $\mathcal{O}_K$-module of finite type, and let P_M be the Hilbert polynomial. A typical case for us is when $M = \bigoplus_{k\geq 0} H^0(\mathcal{X}, k\mathcal{L})$ with Hilbert polynomial P.

Definition 4.1.2. *Let $\alpha \in \mathbb{R}$ be a real number. Define $\eta_{k,\alpha} \in \det_{\mathbb{R}}(M_{\mathbb{R}})$ to be the volume form determined by the equation*

$$\chi(M_k, \eta_{k,\alpha}) = \alpha \sum_{j=0}^{k-1} P_M(j) + \chi(\mathcal{O}_K) P_M(k)$$

with the canonical volume form on $\mathcal{O}_K$.

The following lemma is easy to prove. It is the reason that the $\eta_{k,\alpha}$ is of interest to us. *It does not hold for the L^2-volume forms.*

Lemma 4.1.3. *Let $0 \to M^{(1)} \to M^{(2)} \to M^{(3)} \to 0$ be an exact sequence of graduated $\mathcal{O}_K$-modules of finite type, with $\Phi_k \colon \det(M^{(2)}_{k,\mathbb{R}}) \simeq \det(M^{(1)}_{k,\mathbb{R}}) \otimes \det(M^{(3)}_{k,\mathbb{R}})$ the induced isomorphism. Then for each $\alpha \in \mathbb{R}$, we have $\Phi_k(\eta^{(2)}_{k,\alpha}) = \eta^{(1)}_{k,\alpha} \otimes \eta^{(3)}_{k,\alpha}$ for the volume forms defined in Definition 4.1.2.*

4.1.3 Comparison of the three volumes and consequence on arithmetic Hilbert–Samuel

We need to compare the three volume forms on $H^0(\mathcal{X}, k\mathcal{L})_{\mathbb{R}}$. Define the positive functions

$$f_X(k, \alpha) := \frac{V^k_{X,L^2}}{V^k_{X,\alpha}}, \qquad h_X(k) := \frac{V^k_{X,\sup}}{V^k_{X,L^2}} \tag{4.1.5}$$

with $k \geq 1$ and $\alpha \in \mathbb{R}$.

Proposition 4.1.4. $\log h_X(k) = o(k^{n+1})$.

Proof. This follows immediately from Proposition 4.1.4. □

The following proposition will be proved in the next section.

Proposition 4.1.5. *There exists an affine function $\eta \colon \mathbb{R} \to \mathbb{R}$ such that*

$$\log f_X(k, \alpha) = \eta(\alpha) k^{n+1} + o(k^{n+1}). \tag{4.1.6}$$

In particular, there exists a unique real number α_0 such that $\log f_X(k, \alpha_0) = o(k^{n+1})$.

Proof of Theorem 4.0.1 assuming Proposition 4.1.5. By (4.1.2) and the definition of $V^k_{X,\sup}$, we have

$$\chi(k\overline{\mathcal{L}}) = -\log \mathrm{covol}_{V^k_{X,\sup}}(H^0(\mathcal{X}, k\mathcal{L})_{\mathbb{R}}/H^0(\mathcal{X}, k\mathcal{L})).$$

Thus Definition 4.1.2, Proposition 4.1.4, and Proposition 4.1.5 together yield

$$\chi(k\overline{\mathcal{L}}) = \chi(H^0(\mathcal{X}, k\mathcal{L}), V^k_{X,\alpha_0}) + \log h_X(k) + \log f_X(k, \alpha_0) = \alpha_0 \sum_{j=0}^{k-1} P(j) + o(k^{n+1})$$

Since $\deg P = n$ and P has leading coefficient $\mathcal{L}^n_K/n!$, we then have

$$\chi(k\overline{\mathcal{L}}) = \frac{\alpha_0 \mathcal{L}^n_K}{(n+1)!} k^{n+1} + o(k^{n+1}).$$

Thus the LHS converges to $\alpha_0 \mathcal{L}^n_K$ when $k \to \infty$. The real number α_0 can be read off in the proof of Proposition 4.1.5, where we will see that $\alpha_0 \mathcal{L}^n_K = \overline{\mathcal{L}}^{n+1}$. We are done. □

4.2 Algebraic part of the proof of Proposition 4.1.5

The goal of this section is to prove Proposition 4.1.5, assuming an analytic result which will be proved in the next section.

4.2.1 Fundamental short exact sequence

Recall our assumption that $\mathcal{L}$ is very ample on $\mathcal{X}$. Hence there exists a closed immersion $\iota\colon \mathcal{X} \to \mathbb{P}^N_{\mathcal{O}_K}$ with $\iota^*\mathcal{O}(1) \simeq \mathcal{L}$. By Bertini's theorem, up to taking a finite extension of K there exists a non-zero global section ℓ of $\mathcal{O}(1)$ on $\mathbb{P}^N_{\mathcal{O}_K}$ such that $\mathrm{div}(\ell) \cap \mathcal{X}_K$ is a subvariety of dimension $n-1$ of $\mathcal{X}_K$, which is furthermore irreducible smooth if $n \geq 2$ (if $n = 1$ we can only guarantee the reducedness).

Set $s := \iota^*(\ell)$. Then $s \in H^0(\mathcal{X}, \mathcal{L})$. The ideal sheaf of $\mathrm{div}(s)$, which is $\mathcal{L}^{\otimes -1}$, admits a primary decomposition $\mathcal{L}^{\otimes -1} = \mathcal{I} \cap \mathcal{J}$ where $\mathcal{J}$ has vertical support and $\mathcal{I}$ defines a flat closed subscheme $\mathcal{Y}$ over $\mathrm{Spec}\mathcal{O}_K$ whose generic fiber is irreducible smooth if $n \geq 2$ and is reduced if $n = 1$. Moreover $\dim \mathcal{Y} = n = \dim \mathcal{X} - 1$.

Thus for $k \gg 1$, we have the following exact sequence:

$$0 \to H^0(\mathcal{X}, k\mathcal{L} + \mathcal{I}) \to H^0(\mathcal{X}, k\mathcal{L}) \to H^0(\mathcal{Y}, k\mathcal{L}|_{\mathcal{Y}}) \to 0. \tag{4.2.1}$$

Tensoring $\mathbb{R}$ yields, by definition of $\mathcal{I}$,

$$0 \to H^0(\mathcal{X}, (k-1)\mathcal{L})_{\mathbb{R}} \xrightarrow{\cdot s} H^0(\mathcal{X}, k\mathcal{L})_{\mathbb{R}} \to H^0(\mathcal{Y}, k\mathcal{L}|_{\mathcal{Y}})_{\mathbb{R}} \to 0. \tag{4.2.2}$$

4.2.2 Volume forms on the spaces

Our goal is to compare the volume forms V^k_{X,L^2} and $V^k_{X,\alpha}$ on $H^0(\mathcal{X}, k\mathcal{L})_{\mathbb{R}}$ for each real number $\alpha \in \mathbb{R}$, *by induction on* $n = \dim \mathcal{X}_K$. Hence it is desirable to study the respective volume forms on $H^0(\mathcal{Y}, k\mathcal{L}|_{\mathcal{Y}})_{\mathbb{R}}$ and on $H^0(\mathcal{X}, (k-1)\mathcal{L})_{\mathbb{R}}$.

On $H^0(\mathcal{Y}, k\mathcal{L}|_{\mathcal{Y}})_{\mathbb{R}}$, we have the volume forms

- V^k_{Y,L^2}, where the L^2-norm is defined using the same construction above Proposition 4.1.4 but with $\overline{\mathcal{L}}|_{\mathcal{Y}}$;

- $V^k_{Y,\alpha}$ defined by Defintion 4.1.2.

They are related by $V^k_{Y,L^2} = f_Y(k,\alpha)V^k_{Y,\alpha}$.

On $H^0(\mathcal{X}, (k-1)\mathcal{L})_{\mathbb{R}}$, we have the volume forms

- V^{k-1}_{X,L^2};

- Z^{k-1}_α, by applying Definition 4.1.2 to $M = \bigoplus_{k\geq 0} H^0(\mathcal{X}, (k+1)\mathcal{L} + \mathcal{I})$.

Set $t_X(k-1,\alpha) := V^{k-1}_{X,L^2}/Z^{k-1}_\alpha$.

Apply Lemma 4.1.3 to the exact sequence (4.2.1). Then we get $V^k_{X,\alpha} = V^k_{Y,\alpha} \otimes Z^{k-1}_\alpha$. Thus

$$\frac{V^k_{X,L^2}}{f_X(k,\alpha)} = \frac{V^k_{Y,L^2} \otimes V^{k-1}_{X,L^2}}{f_Y(k,\alpha)t_X(k-1,\alpha)}.$$

Denoting by

$$g(k) := \frac{V^k_{X,L^2}}{V^k_{Y,L^2} \otimes V^{k-1}_{X,L^2}}. \tag{4.2.3}$$

Then we have

$$\log f_X(k,\alpha) = \log t_X(k-1,\alpha) + \log f_Y(k,\alpha) - \log g(k). \tag{4.2.4}$$

The second term on the RHS will be handled by induction hypothesis.

The following proposition will be proved in the next section using analytic method. It handles the third term of the RHS of (4.2.4).

Proposition 4.2.1. *When $k \to \infty$, we have*

$$\frac{1}{P(k-1)} \log g(k) \to -\sum_{\sigma\colon K\hookrightarrow\mathbb{C}} \int_{\mathcal{X}_\sigma} \log \|s(x)\|^2 \mathrm{d}V_\sigma$$

with the volume form $\mathrm{d}V_\sigma$ on $\mathcal{X}_\sigma$ defined above Proposition 4.1.4 (via $c_1(\overline{\mathcal{L}}_\sigma)$).

4.2.3 Further treatment

Consider the following exact sequences of sheaves:

$$0 \to \mathcal{I}\cdot\mathcal{J} \to \mathcal{I} \to \mathcal{I}\otimes\mathcal{O}_{\mathcal{X}}/\mathcal{J} \to 0,$$
$$0 \to \mathcal{I}\cdot\mathcal{J} \to \mathcal{I}\cap\mathcal{J} = \mathcal{L}^{\otimes -1} \to \mathcal{T} := \mathrm{Tor}^1(\mathcal{O}_{\mathcal{X}}/\mathcal{I}, \mathcal{O}_{\mathcal{X}}/\mathcal{J}) \to 0.$$

Then $\mathcal{T}$ has support in $\mathrm{Supp}(\mathcal{J})$, which is vertical over $\mathrm{Spec}\mathcal{O}_K$. For $k \gg 1$, we have exact sequences since $\mathcal{L}$ is ample

$$0 \to H^0(\mathcal{X}, k\mathcal{L} + \mathcal{I}\cdot\mathcal{J}) \to H^0(\mathcal{X}, k\mathcal{L}+\mathcal{I}) \to H^0(\mathcal{X}, k\mathcal{L}+\mathcal{I}+\mathcal{O}_{\mathcal{X}}/\mathcal{J}) \to 0,$$
$$0 \to H^0(\mathcal{X}, k\mathcal{L} + \mathcal{I}\cdot\mathcal{J}) \to H^0(\mathcal{X}, (k-1)\mathcal{L}) \to H^0(\mathcal{X}, kL+\mathcal{T}) \to 0$$

where we write $+$ for $\otimes$ as usual. The last terms in both short exact sequences are torsion. So applying the additivity of the arithmetic Euler characteristic (4.1.4) to both short exact sequences above and taking the difference, we obtain

$$\log t_X(k-1,\alpha) - \log f_X(k-1,\alpha) = \log \#H^0(\mathcal{X}, k\mathcal{L}+\mathcal{I}+\mathcal{O}_X/\mathcal{J}) - \log \#H^0(\mathcal{X}, k\mathcal{L}+\mathcal{T}). \tag{4.2.5}$$

Combining with (4.2.4), we thus obtain

$$\log f_X(k,\alpha) - \log f_X(k-1,\alpha) = \log f_Y(k,\alpha) - \log g(k) + \log \#H^0(\mathcal{X}, k\mathcal{L}+\mathcal{I}+\mathcal{O}_X/\mathcal{J}) - \log \#H^0(\mathcal{X}, k\mathcal{L}+\mathcal{T}). \tag{4.2.6}$$

4.2.4 Proof of Proposition 4.1.5 assuming Proposition 4.2.1

We do induction on $n = \dim \mathcal{X}_K \geq 0$.

$\boxed{\text{Base step}}$ When $n = 0$, we need to do a bit more, *i.e.* we assume $\mathcal{X}_K$ to be reduced but not necessarily irreducible. In this case $\mathcal{X} = \mathrm{Spec}R$ with R a finite $\mathcal{O}_K$-algebra which is reduced. By definition (4.1.5), we have

$$\log f_X(k, \alpha) = -\chi(k\mathcal{L}, V^k_{X,\alpha}) + \chi(k\mathcal{L}, V^k_{X,L^2}).$$

Notice that the Hilbert polynomial of $\mathcal{L}_K$ is constant. Hence Definition 4.1.2 implies that $\chi(k\mathcal{L}, V^k_{X,\alpha})$ is an affine function in k. The function $\chi(k\mathcal{L}, V^k_{Y,L^2})$ is also affine in k, by arithmetic Riemann–Roch applied to $\mathrm{Spec}R$ (we have seen this when $R = \mathcal{O}_K$ as Theorem 1.3.7 whose proof is a direct computation; in general we reduce to the case where R is an order of a number field and prove the similar result by computation). Hence we are done in this base step.

$\boxed{\text{Induction}}$ For general $n \geq 1$, we use (4.2.6) to analyze $\log f_X(k, \alpha) - \log f_X(k-1, \alpha)$.

When $n = 1$, recall our choice $s \in H^0(\mathcal{X}, \mathcal{L})$ satisfies that $\mathrm{div}(s_K)$ is reduced. When $n \geq 2$, the generic fiber $\mathcal{Y}_K$ is smooth by choice of the global section $s \in H^0(\mathcal{X}, \mathcal{L})$. In both cases, we can apply our induction hypothesis and get $\log f_Y(k, \alpha) = \eta'(\alpha)k^n + o(k^n)$. And $\log g(k) = \lambda k^n + o(k^n)$ by Proposition 4.2.1.

For $\log \#H^0(X, k\mathcal{L} + \mathcal{I} + \mathcal{O}_X/\mathcal{J})$ and $\log \#H^0(\mathcal{X}, k\mathcal{L} + \mathcal{T})$, decompose $\mathrm{Supp}(\mathcal{J})$ into disjoint union of connected subvarieties of dimension $\leq n$ (they are all contained in vertical fibers). The Hilbert–Samuel formula in algebraic geometry then implies that both terms are of the form $c'k^n + o(k^n)$.

Therefore, $\log f_X(k, \alpha) - \log f_X(k-1, \alpha) = c_0 k^n + o(k^n)$. So there exists a function $\eta \colon \mathbb{R} \to \mathbb{R}$ such that

$$\log f_X(k, \alpha) = \eta(\alpha)k^{n+1} + o(k^{n+1}).$$

It remains to show that η is affine. For this, notice that Definition 4.1.2 implies

$$\log f_X(k, \alpha) - \log f_X(k, \alpha') = c(\alpha - \alpha')k^{n+1} + o(k^{n+1}),$$

where c is the leading coefficient of $\sum_{j=0}^{k-1} P(j)$. Thus $\eta(\alpha) - \eta(\alpha') = c(\alpha - \alpha')$ for all $\alpha, \alpha' \in \mathbb{R}$. So η is affine. Better, we have $c = \mathcal{L}_K^n/(n+1)!$. We are done.

4.3 Analytic part of the proof

We will prove Proposition 4.1.4 and Proposition 4.2.1 in this section. This finishes the proof of Theorem 4.0.1.

Because both results are analytic, we rephrase our setting as follows to ease notation.

Let X be a projective manifold of dimension $n \geq 1$, and let $(L, \|\cdot\|)$ be a smooth Hermitian line bundle on X (so that $c_1(L, \|\cdot\|)$ is a Kähler form on X). Let ω be the scalar of $c_1(L, \|\cdot\|)$ such that $\int_X \mathrm{d}V = 1$ for the volume form $\mathrm{d}V = \omega^n/n!$ on X. Let P be the Hilbert polynomial, *i.e.* $P(k) = \dim H^0(X, kL)$ for $k \gg 1$. In the setting of Theorem 4.0.1 (see §4.1.1), these are $X = \mathcal{X}_\sigma$, $(L, \|\cdot\|) = (\overline{\sigma}, \|\cdot\|_\sigma)$, and $\omega = \omega_\sigma$ for each $\sigma \colon K \hookrightarrow \mathbb{C}$.

By abuse of notation, we still use $\|\cdot\|$ to denote the Hermitian metric on $kL = L^{\otimes k}$ for each $k \geq 1$ (in §3.1 it was denoted by $\|\cdot\|_k$). The norms $\|\cdot\|_{\sup}$ and $\|\cdot\|_{L^2}$ on $H^0(X, kL)$ are defined by $\|s\|_{\sup} := \sup_{x \in X} \|s(x)\|$ and $\|s\|^2_{L^2} := \int_X \|s(x)\|^2 \mathrm{d}V$ (for $s \in H^0(X, kL)$).

4.3.1 Comparison of $\|\cdot\|_{\sup}$ and $\|\cdot\|_{L^2}$

Let us prove Proposition 4.1.4, *i.e.* there exists a real number $c > 0$ such that

$$\|s\|_{L^2} \leq \|s\|_{\sup} \leq cP(k)^{1/2}\|s\|_{L^2} \tag{4.3.1}$$

for all $k \geq 1$ and all $s \in H^0(X, kL)$.

The first inequality of (4.3.1) is clearly true by definition.

Let $s \in H^0(X, kL)$. Since X is compact, there exists $x \in X$ such that $\|s\|_{\sup} = \|s(x)\|$. Take an orthonormal basis $s_1, \ldots, s_N$ (with $N = P(k)$) of $H^0(X, kL)$ with respect to the L^2-norm. We may choose $s_2, \ldots, s_N$ such that $s_2(x) = \cdots = s_N(x) = 0$. Then we can write

$$s = \sum_{j=1}^{N} a_j s_j$$

with $a_j \in \mathbb{C}$. So $s(x) = a_1 s_1(x)$, $\|s\|_{L^2}^2 = \sum |a_j|^2$. Thus $\|s\|_{\sup}^2 = \|s(x)\|^2 = |a_1|^2\|s_1(x)\|^2$.

Now we use the distortion function $b_k \colon X \to \mathbb{R}$ defined by $b_k(x) = \sum_{j=1}^{N} \|s_j(x)\|^2$ from (3.1.3). Then $\|s\|_{\sup}^2 = |a_1|^2 b_k(x)$. Therefore

$$\|s\|_{\sup}^2 \leq \|s\|_{L^2} b_k(x).$$

Let $c := \sqrt{\sup_{x\in X} b_k(x)/P(k)}$. Notice that $\sup_{x\in X} b_k(x) < \infty$ since X is compact. So $c < \infty$. Moreover, $c > 0$ by the main theorem on the distortion function (Theorem 3.1.6). Hence we are done.

4.3.2 Setup and first estimates to prove Proposition 4.2.1

Let $s \in H^0(X, L)$ such that $Y := \mathrm{div}(s)$ is connected smooth if $n \geq 2$ (reduced if $n = 1$). For $k \gg 1$, we have the following exact sequence

$$0 \to H^0(X, (k-1)L) \xrightarrow{s} H^0(X, kL) \to H^0(Y, kL|_Y) \to 0. \tag{4.3.2}$$

For the L^2-volumes forms V_{X,L^2}^{k-1}, V_{X,L^2}^{k} and V_{Y,L^2}^{k} induced by the L^2-norms on the three spaces in the exact sequence, define the comparison function

$$g(k) := \frac{V_{X,L^2}^{k}}{V_{Y,L^2}^{k} \otimes V_{X,L^2}^{k-1}}.$$

We shall prove Proposition 4.2.1, *i.e.*

$$\frac{1}{P(k-1)} \log g(k) \to -\int_X \log\|s(x)\|^2 \mathrm{d}V \qquad \text{when } k \to \infty. \tag{4.3.3}$$

We shall make use of (4.3.2). The volume form V_{X,L^2}^{k} induces a quotient volume form V_{q,L^2}^{k} on $H^0(Y, kL|_Y)$, via the quotient L^2-norm $\|\cdot\|_{q,L^2}$ on $H^0(Y, kL|_Y)$. Define

$$\gamma(k) := \frac{V_{Y,L^2}^{k}}{V_{q,L^2}^{k}} > 0.$$

The volume form V^k_{X,L^2} also induces a subspace volume form V^{k-1}_{s,L^2} on $H^0(X,(k-1)L)$ via the multiplication by s, via the subspace L^2-norm $\|v\|_{s,L^2} := \|sv\|_{L^2}$ for all $v \in H^0(X,(k-1)L)$. Define

$$\delta(k) := \frac{V^k_{X,L^2}}{V^k_{s,L^2}} > 0.$$

Finally define

$$\varphi(k) := \frac{V^{k-1}_{s,L^2} \otimes V^k_{q,L^2}}{V^k_{X,L^2}} > 0.$$

Then we have

$$g(k) = \frac{\varphi(k)}{\delta(k-1)\gamma(k)}. \tag{4.3.4}$$

We prove the following estimates in this subsection.

Proposition 4.3.1. *$\log \varphi(k) = o(k^n)$.*

Proposition 4.3.2. *$\log \gamma(k) = o(k^n)$.*

The estimate of $\log \delta(k)$ will be proved in the next subsection (Proposition 4.3.3). These information will be put together to prove (4.3.3).

Let Q be the Hilbert polynomial of $L|_Y$ on Y, *i.e.* $Q(k) = \dim H^0(Y, kL|_Y)$ for $k \gg 1$. Then $\deg Q = \dim Y = n-1$.

Proof of Proposition 4.3.1. Let $a\colon \mathbb{R} \to \mathbb{R}$ which sends m to the volume of the unit ball of $\mathbb{R}^m$ for the usual volume form.

(4.3.2) is a short exact sequence of $\mathbb{C}$-vector spaces. Take a $\|\cdot\|_{L^2}$-orthonormal basis of $H^0(X,kL)$ over $\mathbb{C}$, $\{s_1,\dots,s_{P(k-1)},t_1,\dots,t_{Q(k)}\}$, such that $\{s^{-1}s_1,\dots,s^{-1}s_{P(k-1)}\}$ is a $\|\cdot\|_{s,L^2}$-orthonormal basis of $H^0(X,(k-1)L)$ and the quotients $\{[t_1],\dots,[t_{Q(k)}]\}$ is a $\|\cdot\|_{q,L^2}$-orthonormal basis of $H^0(Y,kL|_Y)$. Then

$$V^k_{X,L^2} = \frac{1}{a(P(k-1)+Q(k))} s_1\wedge\cdots\wedge s_{P(k-1)}\wedge t_1\wedge\cdots\wedge t_{Q(k)} \bigwedge \sqrt{-1}s_1\wedge\cdots\wedge\sqrt{-1}s_{P(k-1)}\wedge\sqrt{-1}t_1\wedge\cdots\wedge\sqrt{-1}t_{Q(k)}$$

and similarly for V^{k-1}_{s,L^2} and V^k_{q,L^2}. Thus we get

$$\varphi(k) = \frac{a(P(k-1))\cdot a(Q(k))}{a(P(k))}.$$

We are done. □

To prove Proposition 4.3.2, we need a comparison of $\|\cdot\|_{q,L^2}$ and $\|\cdot\|_{Y,L^2}$.

Proof of Proposition 4.3.2. We claim: there exist $k_0 > 0$ and $B > 0$ such that

$$\|t\|_{q,L^2} \le B\|t\|_{Y,L^2} \tag{4.3.5}$$

for all $k \ge k_0$ and all $t \in H^0(Y,kL|_Y)$. To prove this, we use the L^2-extension Theorem 3.4.1. More precisely, $(L,\|\cdot\|)$ in Theorem 3.4.1 is taken to be $kL - K_X$ endowed with the natural smooth metric for $k \gg 1$ and $(L_0,\|\cdot\|)$ in Theorem 3.4.1 is taken to be $(L,\|\cdot\|)$. Then the assumptions of Theorem 3.4.1 are satisfied for $k \gg 1$. Hence for $t \in H^0(Y, kL|_Y)$, there exists $T \in H^0(X, kL)$ such that $T|_Y = t$ and

$$\int_X \frac{\|T\|_{L^2}^2}{\|s\|_{L^2}^2 (\log \|s\|_{L^2})^2} \mathrm{d}V \leq M \|t\|_{Y,L^2}^2$$

for a constant M depending only on Y. Thus (4.3.5) holds true because the LHS of the inequality above is $\geq c_0 \|T\|_{L^2}^2 \geq c_0 \|t\|_{q,L^2}^2$, with $c_0 := \|s\|_{\sup}^{-2} (\log \|s\|_{\sup})^{-2} > 0$ a positive real number.

On the other hand, $\|\cdot\|_{Y,L^2} \leq \|\cdot\|_{Y,\sup} \leq \|\cdot\|_{q,\sup}$, which by Proposition 4.1.4 is furthermore $\leq cQ(k)^{1/2} \|\cdot\|_{q,L^2}$. Hence

$$(cQ(k)^{1/2})^{-1} \|\cdot\|_{Y,L^2} \leq \|\cdot\|_{q,L^2} \leq B \|\cdot\|_{Y,L^2}.$$

Therefore $\log \gamma(k) = o(k^n)$ since $\deg Q = n - 1$. □

4.3.3 Last estimate for the proof of Proposition 4.2.1

We prove in this subsection the following proposition. Notice that this finishes the proof of (4.3.3) (*i.e.* Proposition 4.2.1) in view of (4.3.4), Proposition 4.3.1, and Proposition 4.3.2.

Proposition 4.3.3. $\log \delta(k) = P(k) \int_X \log \|s(x)\|^2 \mathrm{d}V + o(k^n)$.

We start the proof by giving another expression of $\delta(k)$. Write $m_s \colon H^0(X, kL) \xrightarrow{s} H^0(X, (k+1)L)$ for the first morphism in (4.3.2) with k replaced by $k+1$, and write m_s^* for its dual under the L^2-norms. Set $\phi_{k,s} := m_s^* \circ m_s$. Then

$$\langle u, \phi_{k,s} u \rangle_{L^2} = \int_X \|s\|^2 u^2 \mathrm{d}V \qquad \text{for all } u \in H^0(X, kL). \tag{4.3.6}$$

The eigenvalues of $\phi_{k,s}$ can be obtained as follows. There exists a $\|\cdot\|_{L^2}$-orthonormal basis $\{\widetilde{s}_1, \ldots, \widetilde{s}_N\}$ of $H^0(X, kL)$ (with $N = P(k)$) which is orthogonal for norm $\|\cdot\|_{s,L^2}$ on $H^0(X, kL)$. Then the eigenvalues of $\phi_{k,s}$ are

$$\lambda_j := \|s \otimes \widetilde{s}_j\|_{L^2}^2 = \int_X \|s\| \|\widetilde{s}_j\|^2 \mathrm{d}V$$

with $j \in \{1, \ldots, N\}$.

Lemma 4.3.4. $\delta(k) = \det \phi_{k,s} = \prod_{j=1}^N \lambda_j$.

Proof. By choice of the basis $\{\widetilde{s}_1, \ldots, \widetilde{s}_N\}$, the matrix of the Hermitian pairing $\langle \cdot, \cdot \rangle_{s,L^2}$ obtained from $\|\cdot\|_{s,L^2}$, under this basis of $H^0(X, kL)$, is $\mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. Hence

$$V_{X,L^2}^k = a(N) \widetilde{s}_1 \wedge \cdots \wedge \widetilde{s}_N \wedge \sqrt{-1} \widetilde{s}_1 \wedge \cdots \wedge \sqrt{-1} \widetilde{s}_N$$
$$V_{s,L^2}^k = a(N) \cdot \lambda_1^{-1/2} \widetilde{s}_1 \wedge \cdots \wedge \lambda_N^{-1/2} \widetilde{s}_N \wedge \sqrt{-1} \lambda_1^{-1/2} \widetilde{s}_1 \wedge \cdots \wedge \sqrt{-1} \lambda_N^{-1/2} \widetilde{s}_N.$$

So $\delta(k) = \prod_{j=1}^N \lambda_j = \det \phi_{k,s}$. □

This lemma tells us that $\frac{1}{P(k)} \log \delta(k)$ is precisely the logarithm of the geometric mean of the eigenvalues λ_j of $\phi_{k,s}$. Thus rescaling the metric $\|\cdot\|$ does not change the conclusion of Proposition 4.3.3. So from now on, we may and so assume

$$\|s\|_{\sup} < 1. \tag{4.3.7}$$

Thus $\lambda_j < 1$ for all $j \in \{1, \ldots, N\}$.

Now we are ready to proceed to the proof of Proposition 4.3.3.

$\boxed{\text{Proof of } \geq}$ Let us show

$$\liminf_{k\to\infty} P(k)^{-1} \log \delta(k) \geq \int_X \log \|s\|^2 \mathrm{d}V. \tag{4.3.8}$$

By assumption, $\int_X \|\widetilde{s}_j\|^2 \mathrm{d}V = 1$ for all $j \in \{1, \ldots, N\}$. So Jensen's inequality implies (where the measure is $\|\widetilde{s}_j\| \mathrm{d}V$)

$$\log \lambda_j = \log \int_X \|s\|^2 \|\widetilde{s}_j\|^2 \mathrm{d}V \geq \int_X (\log \|s\|^2) \|\widetilde{s}_j\|^2 \mathrm{d}V.$$

Taking sum over $j \in \{1, \ldots, N = P(k)\}$ and recalling the distortion function $b_k = \sum \|\widetilde{s}_j\|^2$ defined in (3.1.3), we get

$$P(k)^{-1} \log \delta(k) = P(k)^{-1} \sum_{j=1}^{N} \log \lambda_j \geq \int_X \log \|s\|^2 \frac{b_k}{P(k)} \mathrm{d}V. \tag{4.3.9}$$

Hence (4.3.8) follows from the main theorem on the distortion function (Theorem 3.1.6).

$\boxed{\text{Proof of } \leq}$ It remains to prove the hard direction

$$\limsup_{k\to\infty} P(k)^{-1} \log \delta(k) \leq \int_X \log \|s\|^2 \mathrm{d}V. \tag{4.3.10}$$

The proof goes through *tilings* of X, *i.e.* a disjoint union of finitely many connected open subsets of X whose union is dense in X; we will furthermore assume each such connected open subset to have smooth boundary. We have assumed $\|s\|_{\sup} < 1$, so to control $\delta(k) = \det \phi_{k,s}$ it suffices to work on subspaces of $H^0(X, kL)$. Ideally, we would be able prove (4.3.10) if we could find a subspace of $H^0(X, kL)$ of dimension $\sim P(k)$ which has an orthonormal basis with supports in a suitable tiling, so that (4.3.9) eventually becomes an equality. This is not possible in the holomorphic category, and we need to extend our discussion to anti-holomorphic analysis discussed in §3.2.1. All is not lost: we can approximate holomorphic sections by *near holomorphic sections* subcoordinate to finer and finer tilings.

Fix a tiling Ω of X (which is an open subset of X). Then Ω is the disjoint union of finitely many connected open subsets $\Omega_1, \ldots, \Omega_l$ of X

Recall the anti-holomorphic Kodaira Laplacian Δ''_k from Definition 3.2.1 (with $q = 0$) and the heat operator $\overline{\square}_k := (2/k)\Delta''_k$. They acts on the Hilbert space $L^2(X, kL)$, and $H^0(X, kL)$ can be identified with the closed subspace $\mathrm{Ker}\overline{\square}_k \subseteq L^2(X, kL)$. The L^2-orthogonal decomposition $L^2(X, kL) = \mathrm{Ker}\overline{\square}_k \oplus \mathrm{Ker}\overline{\square}_k^{\perp}$ defines the *Bergman projector*

$$\Psi_k \colon L^2(X, kL) \to \mathrm{Ker}\overline{\square}_k = H^0(X, kL). \tag{4.3.11}$$

Let us consider the differential operator $\overline{\square}_{k,\Omega}$, which is the restriction of $\overline{\square}_k$ to Ω with the Dirichlet condition on the boundary $\partial\Omega$. Moreover, $\overline{\square}_{k,\Omega}$ also have discrete spectrum.

Now we can define near holomorphic sections. For any real number $\mu > 0$, denote by $\mathcal{H}_k(\Omega, \mu)$ the direct sum of eigenspaces of $\overline{\square}_{k,\Omega}$ associated with eigenvalues $\leq \mu$. We have a canonical way to obtain holomorphic sections from near holomorphic ones via $\Psi_k|_{\mathcal{H}_k(\Omega,\mu)}$.

Similarly we can define $\mathcal{H}_k(\Omega_j, \mu)$ for each $j \in \{1, \ldots, l\}$. Then $\mathcal{H}_k(\Omega, \mu) = \bigoplus_{j=1}^{l} \mathcal{H}_k(\Omega_j, \mu)$.

Recall our assumption that $c_1(L, \|\cdot\|) = 2\alpha_0\omega$ for some $\alpha_0 > 0$.

The following lemma says that the Bergman projector injects $\mathcal{H}_k(\Omega, \mu)$ quasi-isometrically into $H^0(X, kL)$, for μ small enough and $k \gg 1$.

Lemma 4.3.5. *Assume $\mu < \alpha_0$. Then for $k \gg 1$, $\Psi_k|_{\mathcal{H}_k(\Omega,\mu)}$ is injective and*

$$\|\Psi_k(u) - u\|_{L^2} \leq \frac{2\mu}{\alpha_0}\|u\|_{L^2} \qquad \text{for all } u \in \mathcal{H}_k(\Omega,\mu).$$

Proof. By Lemma 3.2.7, the smallest non-zero eigenvalue of $\overline{\Box}_k$ is $\geq \alpha_0$ for $k \gg 1$.

Let $u \in \mathrm{Ker}\Psi_k$. Then $u \in \mathrm{Ker}\overline{\Box}_k^{\perp}$, and hence $\|\overline{\Box}_k u\|_{L^2} \geq \alpha_0\|u\|_{L^2} \geq \mu\|u\|_{L^2}$ by the previous paragraph. Now if $u \in \mathcal{H}_k(\Omega,\mu) \cap \mathrm{Ker}\Psi_k$, then the definition of $\mathcal{H}_k(\Omega,\mu)$ furthermore implies $u = 0$. This establishes the injectivity.

For $u \in \mathcal{H}_k(\Omega,\mu)$, set $\widetilde{u} := u - \Psi_k(u)$. Then $\widetilde{u} \in \mathrm{Ker}\overline{\Box}_k^{\perp}$, and hence $\|\overline{\Box}_k\widetilde{u}\|_{L^2} \geq \alpha_0\|\widetilde{u}\|_{L^2}$ by the previous paragraph. So

$$\alpha_0\|\widetilde{u}\|_{L^2} \leq \|\overline{\Box}_k(\Psi_k u - u)\|_{L^2} \leq \|\Psi_k\overline{\Box}_k u\|_{L^2} + \|\overline{\Box}_k u\|_{L^2} \leq 2\|\overline{\Box}_k u\|_{L^2} \leq 2\mu\|u\|_{L^2},$$

where the second inequality is the triangular inequality, and the last inequality is by definition of $\mathcal{H}_k(\Omega,\mu)$. □

The next lemma says that the Bergman projector is also a quasi-isometry for the quadratic form q_k defined by $q_k(u) := \int_X \|s\|^2\|u\|^2 \mathrm{d}V$. Recall that $q_k(u) = \langle u, \phi_{k,s}(u)\rangle_{L^2}$ for $u \in H^0(X,kL)$ by (4.3.6).

Lemma 4.3.6. *Assume $\mu < \alpha_0$. Then for $k \gg 1$, we have*

$$|q_k(u) - q_k(\Psi_k(u))| \leq \frac{4\mu}{\alpha_0}\left(\frac{\mu}{\alpha_0} + 1\right)\|u\|_{L^2}^2 < \frac{8\mu}{\alpha_0}\|u\|_{L^2}^2.$$

Proof. We have

$$|q_k(u) - q_k(\Psi_k(u))| = \left|\int_X \|s\|^2(\|u\|^2 - \|\Psi_k(u)\|^2)\mathrm{d}V\right| \leq \|s\|_{\sup}\int_X \left|\|u\|^2 - \|\Psi_k(u)\|^2\right| \mathrm{d}V.$$

We have assumed $\|s\|_{\sup} < 1$. So (by $\left||v_1|^2 - |v_2|^2\right| \leq |v_1 - v_2|^2 + 2|v_2||v_1 - v_2|$)

$$\begin{aligned}|q_k(u) - q_k(\Psi_k(u))| &\leq \|u - \Psi_k(u)\|_{L^2}^2 + 2\int_X \|\Psi_k(u)\| \cdot \left|\|u\| - \|\Psi_k(u)\|\right|\mathrm{d}V \\ &\leq \|u - \Psi_k(u)\|_{L^2}^2 + 2\|\Psi_k(u)\|_{L^2}\|u - \Psi_k(u)\|_{L^2} \\ &\leq \|u - \Psi_k(u)\|_{L^2}^2 + 2\|u\|_{L^2}\|u - \Psi_k(u)\|_{L^2}\end{aligned}$$

Now the conclusion follows from Lemma 4.3.5. □

The last estimate we need is the following:

Lemma 4.3.7. *For each $j \in \{1,\dots,l\}$, we have*

$$\dim \mathcal{H}_k(\Omega_j, k^{-1/6}) = P(k)\mathrm{vol}(\Omega_j) + o(k^n).$$

As a consequence, $\dim \mathcal{H}_k(\Omega, k^{-1/6}) = P(k) + o(k^n)$ *since* $\mathrm{vol}(\Omega) = \mathrm{vol}(X) = 1$.

Proof. The differential operator $\overline{\Box}_{k,\Omega_j}$ (restriction of $\overline{\Box}_k$ to Ω_j with Dirichlet condition on the boundary) also admits a heat kernel $e_{k,\Omega_j}(t,x,y)$, with condition (iii) of Proposition 3.2.4 replaced by $e_{k,\Omega_j}(t,\partial\Omega_j,y) = 0$ on the boundary $\partial\Omega_j$. Denote by $e_{k,\Omega_j}(t,x) := e_{k,\Omega_j}(t,x,x)$. Then the localization process of proving the heat kernel expansion implies

$$\lim_{k\to\infty}\|e_k(t,x) - e_{k,\Omega_j}(t,x)\| = 0$$

uniformly in $x \in X$ and $t \in [1, T_k]$, as long as $T_k = o(\sqrt{k})$. Consider an L^2-orthonormal eigenbasis $(\lambda, \psi_\lambda)_\lambda$ (resp. $(\widetilde{\lambda}, \psi_{\widetilde{\lambda}})$) for $\overline{\square}_k$ (resp. for $\overline{\square}_{k,\Omega_j}$). Integrating over X yields

$$(\sum_{\widetilde{\lambda}} e^{-\widetilde{\lambda} t}) \mathrm{vol}(\Omega_j) = \sum_\lambda e^{-\lambda t} + o(1)$$

for $k \gg 1$, uniformly in $t \in [1, T_k]$. The conclusion then follows by setting $t = k^{1/4}$, using an argument similar to the estimate of the second term of (3.2.5) (use $\overline{\square}_k^1$ and $\overline{\square}_{k,\Omega_j}^1$, and the injectivity of the positive spectrum of $\overline{\square}_{k(,\Omega_j)}^0$ into that of $\overline{\square}_{k(,\Omega_j)}^1$). □

Now we are ready to finish the proof of (4.3.10), which is what is left to prove Proposition 4.3.3.

Proof of (4.3.10). Since $\|s\|_{\sup} < 1$, we have

$$\delta(k) = \det \phi_{k,s} = \det(q_k|_{H^0(X,kL)}) \leq \det(q_k|_{\Psi_k(\mathcal{H}_k(\Omega,\mu))}).$$

Lemma 4.3.5 implies

$$\det(q_k|_{\Psi_k(\mathcal{H}_k(\Omega,\mu))}) \leq \left(\frac{1}{1 - 2\mu/\alpha_0}\right)^{2\dim \mathcal{H}_k(\Omega,\mu)} \det(q_k \circ \Psi_k|_{\mathcal{H}_k(\Omega,\mu)}).$$

We shall use the following elementary result in linear algebra: the determinant of a positive-definite Hermitian matrix is bounded above by the product of the diagonal entries. Recall $\Omega = \coprod_{j=1}^l \Omega_j$. Now for each k and each $j \in \{1, \dots, l\}$, take an orthonormal basis $\{h_m^{(j)}\}_m$ of $\mathcal{H}_k(\Omega_j, \mu)$ (hence $\int_{\Omega_j} \|h_m^{(j)}\|^2 \mathrm{d}V = 1$). Then $\{h_m^{(j)}\}_{m,j}$ is an orthonormal basis of $\mathcal{H}_k(\Omega, \mu) = \bigoplus_{j=1}^l \mathcal{H}_k(\Omega_j, \mu)$. Then

$$\det(q_k \circ \Psi_k|_{\mathcal{H}_k(\Omega,\mu)}) \leq \prod_{m,j} q_k \circ \Psi_k(h_m^{(j)}),$$

while Lemma 4.3.6 implies that

$$q_k \circ \Psi_k(h_m^{(j)}) \leq q_k(h_m^{(j)}) + \frac{8\mu}{\alpha_0} = \int_{\Omega_j} \|s\|^2 \|h_m^{(j)}\|^2 \mathrm{d}V + \frac{8\mu}{\alpha_0} \leq \sup_{\Omega_j} \|s\|^2 + \frac{8\mu}{\alpha_0}.$$

Combining the inequalities above, we get

$$\log \delta(k) \leq 2 \dim \mathcal{H}_k(\Omega, \mu) \log(1 - 2\mu/\alpha_0) + \sum_{j=1}^l \log\left(\sup_{\Omega_j} \|s\|^2 + 8\mu/\alpha_0\right) \dim \mathcal{H}_k(\Omega_j, \mu). \quad (4.3.12)$$

Fix $\epsilon \in (0, \frac{1 - \|s\|_{\sup}}{8})$. Take $\mu = \alpha_0 \epsilon$. Then $\|s\|_{\sup} + 8\mu/\alpha_0 < 1$. Now

$$\sum_j \log\left(\sup_{\Omega_j} \|s\|^2 + 8\mu/\alpha_0\right) \mathrm{vol}(\Omega_j) = \sum_j \sup_{\Omega_j} \log(\|s\|^2 + 8\epsilon) \mathrm{vol}(\Omega_j) \to \int_X \log(\|s\|^2 + 8\epsilon) \mathrm{d}V, \quad (4.3.13)$$

where the limit is on taking finer and finer tilings of X. More precisely, by letting the diameter of Ω tend to 0^+.

Thus the conclusion follows from Lemma 4.3.7, (4.3.12), and (4.3.13), by letting $\epsilon \to 0^+$. □