Chapter 6

Height theory via adelic line bundles

In the whole chapter, we take K to be a number field, and X to be a quasi-projective variety defined over K. Let $n = \dim X$.

6.1 Height via adelic line bundles

In §2.3.2, we defined height functions on projective varieties via Hermitian line bundles, using the arithmetic degree of Hermitian line bundles over $\mathrm{Spec}\mathcal{O}_K$. This degree map was generalized to the intersection pairing (Definition 2.4.8).

In this section, we explain how the definitions extend when we use adelic line bundles on X.

6.1.1 Adelic line bundles on $\mathrm{Spec}K$ and arithmetic degree

Let us start by computing $\widehat{\mathrm{Pic}}(\mathrm{Spec}K/\mathbb{Z})$. It is easier to do the computation with adelic divisors.

Denote by $\mathcal{X} = \mathrm{Spec}\mathcal{O}_K$. For any open subscheme $\mathcal{U}$ of $\mathcal{X}$, we have

$$\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}} = \widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U}), \quad \widehat{\mathrm{Prin}}(\mathcal{U})_{\mathrm{mod}} = \widehat{\mathrm{Prin}}(\mathcal{X})$$

since $\mathcal{X}$ is the only normal projective model of $\mathcal{U}$.

Set $\mathcal{E} := \mathcal{X} \setminus \mathcal{U}$ endowed with the reduced scheme structure. Then we have

$$\begin{aligned}\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U}) &= \{\sum_{v\in|\mathcal{U}|} n_v[v] + \sum_{v'\in|\mathcal{E}|} n_{v'}[v'] + \sum_{\sigma:\,K\hookrightarrow\mathbb{C}} n_\sigma[\sigma] : n_v \in \mathbb{Z},\ n_{v'} \in \mathbb{Q},\ n_\sigma = n_{\overline{\sigma}} \in \mathbb{R} \text{ for all } \sigma\} \\ &\simeq (\bigoplus_{v\in|\mathcal{U}|} \mathbb{Z}) \oplus (\bigoplus_{v'\in|\mathcal{E}|} \mathbb{Q}) \oplus (\bigoplus_{\sigma\in M_{K,\infty}} \mathbb{R}).\end{aligned}$$

Taking the boundary divisor $\overline{\mathcal{E}} := (\mathcal{E}, 1) = \sum_{v\in|\mathcal{E}|\cup M_{K,\infty}}[v]$, we can compute the completion and get

$$\begin{aligned}\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z}) &= \{\sum_{v\in|\mathcal{U}|} n_v[v] + \sum_{v'\in|\mathcal{E}|} n_{v'}[v'] + \sum_{\sigma:\,K\hookrightarrow\mathbb{C}} n_\sigma[\sigma] : n_v \in \mathbb{Z},\ n_{v'} \in \mathbb{R},\ n_\sigma = n_{\overline{\sigma}} \in \mathbb{R} \text{ for all } \sigma\} \\ &\simeq (\bigoplus_{v\in|\mathcal{U}|} \mathbb{Z}) \oplus (\bigoplus_{v\in|\mathcal{E}|\cup M_{K,\infty}} \mathbb{R}).\end{aligned} \tag{6.1.1}$$

Hence we have

$$\widehat{\mathrm{Div}}(\mathrm{Spec}K/\mathbb{Z}) = \varinjlim_{\mathcal{U}} (\bigoplus_{v\in|\mathcal{U}|} \mathbb{Z}) \oplus (\bigoplus_{v\in|\mathcal{E}|\cup M_{K,\infty}} \mathbb{R}), \tag{6.1.2}$$

and an arithmetic degree map

$$\widehat{\deg}\colon \widehat{\mathrm{Div}}(\mathrm{Spec}K/\mathbb{Z}) \to \mathbb{R} \tag{6.1.3}$$

induced by the group homomorphism $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z}) \to \mathbb{R}$, $\sum_{v\in|\mathcal{U}|} n_v[v]+\sum_{v'\in|\mathcal{E}|} n_{v'}[v']+\sum_{\sigma\colon K\hookrightarrow\mathbb{C}} n_\sigma[\sigma] \mapsto \sum_v n_v + \sum_{v'} n_{v'} + \sum_\sigma n_\sigma$.

It is clear that (6.1.3) factors through $\widehat{\mathrm{Cl}}(\mathrm{Spec}K/\mathbb{Z})$, and hence we have the arithmetic degree map via Proposition 5.2.10

$$\widehat{\deg}\colon \widehat{\mathrm{Pic}}(\mathrm{Spec}K/\mathbb{Z}) \to \mathbb{R}. \tag{6.1.4}$$

This arithmetic degree is compatible with the arithmetic degree of Hermitian line bundles on $\mathrm{Spec}\mathcal{O}_K$ (1.1.1) in the following sense. By definition of $\widehat{\mathrm{Div}}(\mathrm{Spec}\mathcal{O}_K)$ and (6.1.2), we have a natural group homomorphism

$$\widehat{\mathrm{Div}}(\mathrm{Spec}\mathcal{O}_K) = (\bigoplus_{v\in|\mathcal{X}|}\mathbb{Z}) \oplus (\bigoplus_{v\in M_{K,\infty}}\mathbb{R}) \longrightarrow \widehat{\mathrm{Div}}(\mathrm{Spec}K/\mathbb{Z}),$$

which induces

$$\iota\colon \widehat{\mathrm{Pic}}(\mathrm{Spec}\mathcal{O}_K) \longrightarrow \widehat{\mathrm{Pic}}(\mathrm{Spec}K/\mathbb{Z}).$$

Then $\widehat{\deg}\circ\iota$ is precisely the arithmetic degree map defined by (1.1.1).

We close this subsection by the following formula for $\widehat{\deg}$. Let $\overline{L} \in \widehat{\mathrm{Pic}}(\mathrm{Spec}K/\mathbb{Z})$. Write $(L, \|\cdot\|_{\mathbb{A}})$ for the image of $\overline{L}$ under the canonical map $\widehat{\mathrm{Pic}}(\mathrm{Spec}K/\mathbb{Z}) \simeq \widehat{\mathrm{Pic}}((\mathrm{Spec}K/\mathbb{Z})^{\mathrm{an}})_{\mathrm{eqv}}$ from (5.4.1). Then $\|\cdot\|_{\mathbb{A}}$ is uniquely determined by the collection of K_v-metrics $\{\|\cdot\|_v$ on $L\otimes_K K_v\}_{v\in M_K}$ by norm-equivariance. Moreover, for any $\ell \in L\setminus\{0\}$, we have $\|\ell\|_v = 1$ for all but finitely many $v \in M_K$. The following lemma is not hard to check and we leave it as an exercise.

Lemma 6.1.1. *Under the notation above, we have*

$$\widehat{\deg}(\overline{L}) = -\sum_{v\in M_K} \log\|\ell\|_v^{\epsilon_v} \qquad \text{for any } \ell\in L\setminus\{0\},$$

where $\epsilon_v = 2$ if v is a complex place and $\epsilon_v = 1$ otherwise. The RHS is well-defined by the Product Formula.

In this terminology, ι sends $(\mathcal{L}, \|\cdot\|)$ to $(\mathcal{L}_K, \|\cdot\|_{\mathbb{A}})$, with $\|\ell\|_v := \inf\{|a| : a\in\mathbb{Q},\ \ell \in a\mathcal{L}\otimes_{\mathcal{O}_K}\mathcal{O}_{K_v}\}$.

6.1.2 Height function defined by adelic line bundles

Let $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$.

Definition 6.1.2. *The* **height function** *defined by $\overline{L}$ is*

$$h_{\overline{L}}\colon X(\overline{\mathbb{Q}}) \to \mathbb{R}, \qquad x \mapsto \frac{\widehat{\deg}(\overline{L}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)x})}{[K(x):K]}.$$

Here, $\overline{L}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)}$ is the image of $\overline{L}$ under $\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{int}} \to \widehat{\mathrm{Pic}}(\mathrm{Gal}(\overline{\mathbb{Q}}/K)x/\mathbb{Z})_{\mathrm{int}}$, and $\widehat{\deg}$ is defined below (6.1.8), *i.e.* this intersection pairing for $n = 0$, with X replaced by $\mathrm{Gal}(\overline{\mathbb{Q}}/K)x$.

Example 6.1.3. *Assume X is irreducible projective. Assume $L \in \mathcal{P}\mathrm{ic}(X)$ ample such that $f^*L \simeq qL$ for some $f\colon X\to X$ and $q\in\mathbb{Z}_{>1}$. Then by Theorem 5.5.1, there exists $\overline{L}_f \in \widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z})_{\mathrm{nef}}$ extending L such that $f^*\overline{L}_f \simeq q\overline{L}_f$. Then $h_{\overline{L}_f}$ is in the class of the height function* *of $h_{X,L}$ and satisfies $h_{\overline{L}_f}(f(x)) = qh_{\overline{L}_f}(x)$ for all $x \in X(\overline{\mathbb{Q}})$. Hence $h_{\overline{L}_f}$ equals the normalized height function $\widehat{h}_{X,f,L}$ from Theorem 0.3.1.*

This in particular applies to any abelian variety A and any symmetric ample line bundle L on A, both defined over $\overline{\mathbb{Q}}$. So the Néron–Tate height on A is a height function defined by an adelic line bundle $\overline{L}$ on A.

Better, if we have an abelian scheme $\mathcal{A} \to S$ with S an irreducible quasi-projective variety, and $\mathcal{L}$ a relatively ample symmetric line bundle on $\mathcal{A}$; all defined over $\overline{\mathbb{Q}}$. Then Theorem 5.5.1 gives an $\overline{\mathcal{L}} \in \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{A}/\mathbb{Z})_{\mathrm{nef}}$ such that $h_{\overline{\mathcal{L}}}$ is precisely the fiberwise Néron–Tate height defined by $\mathcal{L}$.

As an application of Lemma 6.1.1, we have the following:

Lemma 6.1.4. *Denote by $(L, \|\cdot\|) \in \widehat{\mathrm{Pic}}(X^{\mathrm{an}})_{\mathrm{eqv}}$ the image of $\overline{L}$ under the canonical map $\widehat{\mathrm{Pic}}(X/\mathbb{Z}) \xrightarrow{\sim} \widehat{\mathrm{Pic}}(X^{\mathrm{an}})_{\mathrm{eqv}}$ from (5.4.1). Then for any $x \in X(\overline{\mathbb{Q}})$, we have*

$$h_{\overline{L}}(x) = -\frac{1}{[K(x):K]} \sum_{v \in M_K} \sum_{z \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)x \times_K K_v} \log \|s(z)\|_v^{\deg_{K_v} z} \tag{6.1.5}$$

for any non-zero rational section s of L on X with $x \notin |\mathrm{div}(s)|$.

6.1.3 Top intersection number of adelic line bundles

For any projective arithmetic variety $\mathcal{X}$ of dimension $n+1$, Definition 2.4.8 defines an intersection pairing

$$\widehat{\mathrm{Pic}}(\mathcal{X})_{\mathrm{int}}^{n+1} \to \mathbb{R}. \tag{6.1.6}$$

Here, the index int refers to the integrability condition on the Hermitian metrics we consider; see below Definition 2.3.2.

Let us extend this intersection pairing to adelic line bundles over quasi-projective arithmetic varieties.

Proposition 6.1.5. *Let $\mathcal{U}$ be a quasi-projective arithmetic variety of dimension $n+1$. Then there exists a canonical multi-linear homomorphism, which is symmetric in the $n+1$ variables,*

$$\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{int}}^{n+1} \to \mathbb{R} \tag{6.1.7}$$

extending the intersection pairing (6.1.6) above. Moreover, if $\overline{\mathcal{L}}_1, \ldots, \overline{\mathcal{L}}_{n+1}$ are nef adelic line bundles on $\mathcal{U}$, then the intersection number $\overline{\mathcal{L}}_1 \cdot \overline{\mathcal{L}}_2 \cdot \ldots \cdot \overline{\mathcal{L}}_{n+1} \geq 0$.

Before moving on to the proof, let us explain how (6.1.7) induces an intersection pairing on $\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{int}}$. Indeed, $\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{int}} = \varinjlim_{\mathcal{U}} \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{int}}$, and hence (6.1.7) gives rise to a canonical multi-linear homomorphism (still called the *intersection pairing*)

$$\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{int}}^{n+1} \to \mathbb{R} \tag{6.1.8}$$

such that $\overline{L}_1 \cdot \ldots \cdot \overline{L}_{n+1} \geq 0$ if all $\overline{L}_i$'s are nef. When $n = 0$, this map is exactly $\widehat{\deg}$.

Similarly we have a canonical multi-linear homomorphism

$$\widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathrm{int}}^{n} \to \mathbb{R}, \tag{6.1.9}$$

such that $\widetilde{L}_1 \cdot \ldots \cdot \widetilde{L}_n \geq 0$ if all $\widetilde{L}_i$'s are nef.

Proof of Proposition 6.1.5. By linearity, it suffices to define (6.1.7) for strongly nef adelic line bundles. The proof is easier to write down in terms of adelic divisors. So we take $\overline{D}_1, \dots, \overline{D}_{n+1} \in \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$ with each $\mathcal{O}(\overline{D}_j)$ being a strongly nef adelic line bundle. We will define $\overline{D}_1 \cdot \dots \cdot \overline{D}_{n+1}$.

Fix a boundary divisor $(\mathcal{X}_0, \overline{D}_0)$ of $\mathcal{U}$, which define the boundary topology of $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$. We may furthermore assume $\mathcal{O}(\overline{D}_0)$ to be a nef Hermitian line bundle.

For $j \in \{1, \dots, n+1\}$, the adelic divisor $\overline{D}_j$ is represented by a Cauchy sequence $\{(\mathcal{X}_i, \overline{D}_{j,i})\}_{i \geq 1}$, where each $\mathcal{O}(\overline{D}_{j,i})$ is a nef Hermitian line bundle on the projective model $\mathcal{X}_i$ dominating $\mathcal{X}_0$. Here, we assume that the model $\mathcal{X}_i$ is independent of j which is always possible. There exists a sequence $\{\epsilon_i \in \mathbb{Q}_{>0}\}_{i \geq 1}$ with $\epsilon_i \to 0$ such that

$$-\epsilon_i \overline{D}_0 \leq \overline{D}_{j,i'} - \overline{D}_{j,i} \leq \epsilon_i \overline{D}_0, \quad \forall i' > i$$

for any $j \in \{1, \dots, n+1\}$.

For any subset $J \subseteq \{1, \dots, n+1\}$, consider the intersection number

$$\alpha_{J,i} := \overline{D}_0^{d-|J|} \prod\nolimits_{j \in J} \overline{D}_{j,i}.$$

We shall prove, by induction on $|J|$, that $\{\alpha_{J,i}\}_{i \geq 1}$ is a Cauchy sequence, and hence converges in $\mathbb{R}$. Then the limit of the Cauchy sequence gives our desired definition of $\overline{D}_1 \cdot \dots \cdot \overline{D}_{n+1}$ when $J = \{1, \dots, n+1\}$.

The base step $|J| = 0$ is trivial since there is nothing to prove in this case. Assume the claim holds true for any $|J| < r$ for some $r > 0$. We need to prove the result for $|J| = r$. Without loss of generality assume that $J = \{1, 2, \dots, r\}$. Then

$$\begin{aligned} \alpha_{J,i'} - \alpha_{J,i} &= \overline{D}_0^{d-r} \overline{D}_{1,i'} \cdots \overline{D}_{r,i'} - \overline{D}_0^{d-r} \overline{D}_{1,i} \cdots \overline{D}_{r,i} \\ &\leq \overline{D}_0^{d-r} (\overline{D}_{1,i} + \epsilon_i \overline{D}_0) \cdots (\overline{D}_{r,i} + \epsilon_i \overline{D}_0) - \overline{D}_0^{d-r} \overline{D}_{1,i} \cdots \overline{D}_{r,i} \\ &= \sum_{J' \subsetneq J} \epsilon_i^{r-|J'|} \alpha_{J',i} \end{aligned}$$

and similarly

$$\alpha_{J,i} - \alpha_{J,i'} \leq \sum_{J' \subsetneq J} \epsilon_i^{r-|J'|} \alpha_{J',i'}.$$

So

$$|\alpha_{J,i'} - \alpha_{J,i}| \leq \sum_{J' \subsetneq J} \epsilon_i^{r-|J'|} |\alpha_{J',i'} - \alpha_{J',i}|.$$

This shows that $\{\alpha_{J,i}\}_i$ is a Cauchy sequence by induction hypothesis. Hence we are done for the definition of (6.1.7).

The intersection pairing (6.1.7) is symmetric in the $n+1$ variables because (6.1.6) is. Moreover, $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})^{n+1}_{\mathrm{snef}}$ is mapped to $\mathbb{R}_{\geq 0}$ since (6.1.6) maps $\widehat{\mathrm{Pic}}(\mathcal{X})^{n+1}_{\mathrm{nef}}$ to $\mathbb{R}_{\geq 0}$. Now if $\overline{\mathcal{L}}_1, \dots, \overline{\mathcal{L}}_{n+1}$ are nef adelic line bundles on $\mathcal{U}$, then there exist strongly nef adelic line bundle $\overline{\mathcal{M}}_j$ ($j \in \{1, \dots, n+1\}$) such that $a\overline{\mathcal{L}}_j + \overline{\mathcal{M}}_j$ is strongly nef for all $a \in \mathbb{Z}_{>0}$ for all j. Hence

$$(a\overline{\mathcal{L}}_1 + \overline{\mathcal{M}}_1) \cdots (a\overline{\mathcal{L}}_{n+1} + \overline{\mathcal{M}}_{n+1}) \geq 0$$

for all $a \in \mathbb{Z}_{>0}$. And hence the leading coefficient $\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n+1}$ is non-negative. □

We also have the following projection formula, by taking limits of Proposition 2.4.10.

Proposition 6.1.6. *Let $f \colon X' \to X$ be a morphism of varieties defined over K. Assume $\dim X' = \dim X = n$. Then for any $\overline{L}_1, \dots, \overline{L}_{n+1} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$, we have*

$$f^*\overline{L}_1 \cdot \dots \cdot f^*\overline{L}_{n+1} = \deg(f) \overline{L}_1 \cdot \dots \cdot \overline{L}_{n+1}$$

with $\deg(f) = [K(X') : K(X)]$ if f is dominant and $\deg(f) = 0$ otherwise.

We close this section by defining the height of an arbitrary dimensional closed subvariety of X.

Definition 6.1.7. *Assume $\overline{L}$ is integrable. Let Z be a closed subvariety of X defined over $\overline{\mathbb{Q}}$. Define the* **height of** Z **for** $\overline{L}$ *to be*

$$h_{\overline{L}}(Z) := \frac{(\overline{L}|_{Z'})^{\dim Z+1}}{(\dim Z+1)(\widetilde{L}|_{Z'})^{\dim Z}}. \tag{6.1.10}$$

Here $Z' = \mathrm{Gal}(\overline{\mathbb{Q}}/K)Z$, and $\overline{L} \mapsto \overline{L}|_{Z'} \mapsto \widetilde{L}_{Z'}$ is the image of $\overline{L}$ under $\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{int}} \to \widehat{\mathrm{Pic}}(Z'/\mathbb{Z})_{\mathrm{int}} \to \widehat{\mathrm{Pic}}(Z'/\mathbb{Q})_{\mathrm{int}}$.

On the RHS of (6.1.10), the numerator is the arithmetic intersection pairing (6.1.8), and the second term of the denominator is the geometric intersection pairing (6.1.9). Of course this height is well-defined only if $(\widetilde{L}|_{Z'})^{\dim Z} \neq 0$.

6.2 Volume and bigness of adelic line bundles

We explained in §5.1.4 the volume and bigness of geometric adelic line bundles on X. As for the geometric-arithmetic analogue in the classical situation, we can generalize the discussion to adelic line bundles on X.

6.2.1 Effective/small sections

Let $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$. Denote by

$$\widehat{\mathrm{Pic}}(X/\mathbb{Z}) \to \widehat{\mathrm{Pic}}(X/\mathbb{Q}) \to \mathrm{Pic}(X), \quad \overline{L} \mapsto \widetilde{L} \mapsto L.$$

Define

$$H^0(X,\overline{L}) := \{s \in H^0(X,L) : \widehat{\mathrm{div}}(s) \geq 0\}, \qquad h^0(X,\overline{L}) := \log \# H^0(X,\overline{L}), \tag{6.2.1}$$

and recall

$$H^0(X,\widetilde{L}) = \{s \in H^0(X,L) : \widehat{\mathrm{div}}(s) \geq 0\}, \qquad h^0(X,\widetilde{L}) = \dim H^0(X,\widetilde{L}).$$

In the definition of $H^0(X,\overline{L})$, $\widehat{\mathrm{div}}(s)$ means the *(arithmetic)* adelic divisor (5.2.5), while in the definition of $H^0(X,\widetilde{L})$ it means the geometric adelic divisor (5.1.5).

We state the following lemma without giving the proof. The proof is not too complicated: one first proves the finiteness result for the model case and then passes to Cauchy sequences.

Lemma 6.2.1. *Both $h^0(X,\overline{L})$ and $h^0(X,\widetilde{L})$ are finite numbers.*

Next, recall the diagram (5.4.1). Let $(L, \|\cdot\|) \in \widehat{\mathrm{Pic}}(X^{\mathrm{an}})_{\mathrm{eqv}}$ be the metrized line bundle as the image of $\widehat{\mathrm{Pic}}(X/\mathbb{Z})$. Then for any $s \in H^0(X,L)$ and any $v \in (\mathrm{Spec}\mathbb{Z})^{\mathrm{an}}$, define

$$\|s\|_{\sup} := \sup_{x \in X^{\mathrm{an}}} \|s(x)\|, \qquad \|s\|_{v,\sup} := \sup_{x \in X^{\mathrm{an}}_v} \|s(x)\|. \tag{6.2.2}$$

Then we have by construction of (5.4.1) and Lemma 5.4.5, we have

$$H^0(X,\overline{L}) = \{s \in H^0(X,L) : \|s\|_{\sup} \leq 1\}, \qquad H^0(X,\widetilde{L}) = \{s \in H^0(X,L) : \|s\|_{v_0,\sup} \leq 1\}. \tag{6.2.3}$$

By Lemma 6.1.4, $h_{\overline{L}}$ is non-negative outside $|\mathrm{div}(s)|$ if we can find a non-zero small section $s \in H^0(X,\overline{L})$.

6.2.2 Volume and bigness

Definition-Theorem 6.2.2. *The following limit exists and is defined to be the* **volume** *of* $\overline{L}$:

$$\mathrm{vol}(X,\overline{L}) := \lim_{m\to\infty} \frac{(n+1)!}{m^n} h^0(X, m\overline{L}). \tag{6.2.4}$$

Moreover, assume that $\overline{L}$ *is represented by* $(\mathcal{L}, \{\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i\}_{i\geq 1})$ *on* $\mathcal{U}$ *for a quasi-projective model* $\mathcal{U}$ *of* X, *then*

$$\mathrm{vol}(X,\overline{L}) = \lim_{i\to\infty} \mathrm{vol}(\mathcal{X}_i, \overline{\mathcal{L}}_i).$$

Definition 6.2.3. *An adelic line bundle* $\overline{L} \in \widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z})$ *is said to be* **big** *if* $\mathrm{vol}(X,\overline{L}) > 0$.

Theorem 6.2.4 (Arithmetic Hilbert–Samuel)**.** *Assume* $\overline{L}$ *is nef. Then* $\mathrm{vol}(X,\overline{L}) = \overline{L}^{n+1}$.

Theorem 6.2.5 (Arithmetic Siu)**.** *If* $\overline{L}$ *and* $\overline{M}$ *are nef adelic line bundles on* X, *then*

$$\mathrm{vol}(X, \overline{L} - \overline{M}) \geq \overline{L}^{n+1} - (n+1)\overline{L}^n\overline{M}.$$

All the definitions and results extend to $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathbb{Q}}$, *i.e.* $\mathbb{Q}$-adelic line bundles on X.

Theorem 6.2.6 (continuity)**.** *Let* $\overline{L}, \overline{M}_1, \ldots, \overline{M}_r \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$. *Then*

$$\lim_{t_1,\ldots,t_r\to 0} \mathrm{vol}(\overline{L} + t_1\overline{M}_1 + \cdots + t_r\overline{M}_r) = \mathrm{vol}(\overline{L}),$$

with $t_1, \ldots, t_r$ *rational numbers.*

The following lemma states that the bigness of the generic fiber $\widetilde{L}$ of $\overline{L}$ is not far from the bigness of $\overline{L}$. In view of height theory, this is reasonable: having a non-zero small section yields the non-negativity of $h_{\overline{L}}$, whereas having a non-zero section yields a lower bound of $h_{\overline{L}}$ (both outside the support of the divisor of the section).

Write $f\colon X \to \mathrm{Spec}K$ for the structural morphism.

Lemma 6.2.7. *Let* $\overline{N} \in \widehat{\mathrm{Pic}}(K/\mathbb{Z})$ *be an adelic line bundle with* $\widehat{\deg}(\overline{N}) > 0$. *Let* $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$.

Assume that the generic fiber $\widetilde{L}$ *of* $\overline{L}$ *is big (see Definition 5.1.13). Then the adelic line bundle* $\overline{L} + cf^*\overline{N} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$ *is big for all rational numbers* $c \gg 1$.

6.2.3 The height inequality

In this subsection, we prove the following height inequality which plays a significant role in the solution of many problems recently.

Theorem 6.2.8. *Let* $\pi\colon X \to S$ *be a morphism of quasi-projective varieties defined over a number field* K. *Let* $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$ *and* $\overline{M} \in \widehat{\mathrm{Pic}}(S/\mathbb{Z})$. *Denote by* $\widetilde{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Q})$ *the generic fiber of* $\overline{L}$.

(i) If $\overline{L}$ *is big, then there exists* $\epsilon > 0$ *and a non-empty Zariski open subset* U *of* X *such that*

$$h_{\overline{L}}(x) \geq \epsilon h_{\overline{M}}(\pi(x)), \quad \forall x \in U(\overline{\mathbb{Q}}).$$

(ii) If $\widetilde{L}$ *is big, then there exist* $c > 0$ *and* $\epsilon > 0$ *and a non-empty Zariski open subset* U *of* X *such that*

$$h_{\overline{L}}(x) \geq \epsilon h_{\overline{M}}(\pi(x)) - c, \quad \forall x \in U(\overline{\mathbb{Q}}).$$

Proof. Let us prove (i). Assume $\overline{L}$ is big. Then $\mathrm{vol}(X, \overline{L}) > 0$.

We claim that there exists $\epsilon \in \mathbb{Q}_{>0}$ such that $\mathrm{vol}(X, \overline{L} - \epsilon\pi^*\overline{M}) > 0$. If $\overline{L}$ and $\overline{M}$, this follows from Arithmetic Siu (Theorem 6.2.5) and Arithmetic Hilbert–Samuel (Theorem 6.2.4). In general, we use the continuity (Theorem 6.2.6) to get that

$$\lim_{\epsilon \to 0} \mathrm{vol}(X, \overline{L} - \epsilon\pi^*\overline{M}) = \mathrm{vol}(X, \overline{L}) > 0.$$

Hence such an ϵ exists.

Therefore there exists $m \in \mathbb{Z}_{>0}$ and a non-zero $s \in H^0(X, m(\overline{L} - \epsilon\pi^*\overline{M}))$. Hence by (6.2.3), we have

$$h_{\overline{L} - \epsilon\pi^*\overline{M}}(x) \geq 0, \quad \forall x \in (X \setminus |\mathrm{div}(s)|)(\overline{\mathbb{Q}}).$$

Hence (i) holds true because $h_{\overline{L} - \epsilon\pi^*\overline{M}}(x) = h_{\overline{L}}(x) - \epsilon h_{\overline{M}}(\pi(x))$.

Now we prove (ii). Take $\overline{N} \in \widehat{\mathrm{Pic}}(\mathrm{Spec}K/\mathbb{Z})$ with $\widehat{\deg}(\overline{N}) = 1$. For the structural morphism $f\colon X \to \mathrm{Spec}K$, denote by $\overline{L}' = \overline{L} + cf^*\overline{N}$ for a rational number $c > 0$. By Lemma 6.2.7, $\overline{L}'$ is big for $c \gg 1$. Hence we can apply part (i) to $(\overline{L}', \overline{M})$ and conclude. ☐

6.2.4 A formula to compute the self-intersection of geometric adelic line bundles

Let $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$, and write $(L, \|\cdot\|)$ for its image under the injective homomorphism $\widehat{\mathrm{Pic}}(X/\mathbb{Z}) \to \widehat{\mathrm{Pic}}(X^{\mathrm{an}})_{\mathrm{eqv}}$. Each place v of K can be seen as a point in $(\mathrm{Spec}K/\mathbb{Z})^{\mathrm{an}}$, which is over the point $v_p \in (\mathrm{Spec}\mathbb{Q}/\mathbb{Z})^{\mathrm{an}}$ with $p \leq \infty$. Now $\|\cdot\|_v$ is a metric of $L|_{X_v^{\mathrm{an}}}$, and hence defines a curvature current $c_1(\overline{L})_v$; at archimedean places this is $-\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\|\cdot\|_v$, and we omit the discussion at non-archimedean places.

Lemma 6.2.9. *Let $\overline{L}_1, \ldots, \overline{L}_n \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{int}}$, and let $\widetilde{L}_1, \ldots, \widetilde{L}_n \in \widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathrm{int}}$ be their generic fibers. Then for any place v of K, we have*

$$\widetilde{L}_1 \cdot \ldots \cdot \widetilde{L}_n = \int_{X_v^{\mathrm{an}}} c_1(\overline{L}_1)_v \cdots c_1(\overline{L}_n)_v.$$

In practice, take $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{int}}$ with generic fiber $\widetilde{L}$, and let $\sigma\colon K \hookrightarrow \mathbb{C}$. Then $\widetilde{L}^n = \int_{X_\sigma(\mathbb{C})} c_1(\overline{L}_\sigma)^n$ by Lemma 6.2.9. If $\widetilde{L}$ is known to be nef, then we can use Hilbert–Samuel to get

$$\mathrm{vol}(X, \widetilde{L}) = \int_{X_\sigma(\mathbb{C})} c_1(\overline{L}_\sigma)^n.$$

6.3 A brief discussion on equidistribution

6.3.1 Essential minimum and fundamental inequality

Definition 6.3.1. *Let $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$. Define the* **essential minimum** *to be*

$$e_1(X, \overline{L}) := \sup_{U \subseteq X} \inf_{x \in U(\overline{\mathbb{Q}})} h_{\overline{L}}(x),$$

where U runs over all Zariski open subsets of X.

The *fundamental inequality* is:

Theorem 6.3.2. *Let $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$ be nef such that its generic fiber $\widetilde{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Q})$ is big. Then*

$$e_1(X, \overline{L}) \geq h_{\overline{L}}(X) \geq \frac{1}{n+1} e_1(X, \overline{L}).$$

The second inequality is a weak version of Zhang's *successive minima theorem*. The first inequality is a consequence of the arithmetic Hilbert–Samuel formula with the following lemma, which is an application of the Geometry of Numbers in §1.3.1 and the formula for $h_{\overline{L}}$ from Lemma 6.1.4.

Lemma 6.3.3. *For any positive integer $m > 0$ such that $h^0(X, m\overline{L}) > 0$, we have*

$$e_1(X, \overline{L}) \geq \frac{h^0(X, m\overline{L})}{m h^0(X, m\widetilde{L})} - \frac{2}{m}[K : \mathbb{Q}]$$

if the RHS is > 0.

6.3.2 Equidistribution

Let $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{nef}}$. Assume that $\deg_{\widetilde{L}}(X) = \widetilde{L}^n > 0$. Then the height

$$h_{\overline{L}}(X) = \frac{\overline{L}^{n+1}}{(n+1)\deg_{\widetilde{L}}(X)}$$

from (6.1.10) is well-defined. Define the *equibrilium measure*

$$\mathrm{d}\mu_{\overline{L},v} := \frac{1}{\deg_{\widetilde{L}}(X)} c_1(\overline{L})_v^n. \tag{6.3.1}$$

A sequence $\{x_m\}_{m\geq 1}$ in $X(\overline{\mathbb{Q}})$ is said to be *generic* if any proper closed subvariety of X contains only finitely many terms in the sequence. The sequence is said to be *small* if $h_{\overline{L}}(x_m) \to h_{\overline{L}}(X)$ when $m \to \infty$.

Let $x \in X(\overline{\mathbb{Q}})$. Define $O(x) := \mathrm{Gal}(\overline{\mathbb{Q}}/K)x \subseteq X(\overline{\mathbb{Q}})$, and set

$$\mu_{x,v} := \frac{1}{\#O(x)} \delta_{O(x)\times_K K_v} \tag{6.3.2}$$

where the RHS is the dirac measure.

Theorem 6.3.4. *Let $\{x_m\}_{m\geq 1}$ be a generic small sequence. Then the Galois orbit of $\{x_m\}_{m\geq 1}$ is equidistributed in X_v^{an} for $\mathrm{d}\mu_{\overline{L},v}$ for any place v of K. This means: the weak convergence $\mu_{x_m,v} \to \mathrm{d}\mu_{\overline{L},v}$ holds on X_v^{an}, i.e. for any compactly supported continuous function f on X_v^{an}, we have*

$$\frac{1}{\#O(x_m)} \sum_{y \in O(x_m)\times_K K_v} f(y) \longrightarrow \int_{X_v^{\mathrm{an}}} f \mathrm{d}\mu_{\overline{L},v}. \tag{6.3.3}$$

Proof. The key approach is the *variational principle* of Szpiro–Ullmo–Zhang. The conditions and the result do not change if we replace $\overline{L}$ by $\overline{L} + f^*\overline{N}$ for some $\overline{N} \in \widehat{\mathrm{Pic}}(\mathrm{Spec} K/\mathbb{Z})_{\mathrm{int}}$ with $\widehat{\deg}(\overline{N}) > 0$, where $f \colon X \to \mathrm{Spec} K$ is the structural morphism. So we may assume that $\overline{L}^{n+1} > 0$. Then $\overline{L}$ is big, and hence $\widetilde{L}$ is big.

Take $\overline{M} \in \mathrm{Ker}(\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{int}} \to \widehat{\mathrm{Pic}}(X/\mathbb{Q}))$. Let $\epsilon \in \mathbb{Q}^*$. By the first part of Theorem 6.3.2, we have

$$e_1(X, \overline{L} + \epsilon\overline{M}) \geq \frac{(\overline{L} + \epsilon\overline{M})^{n+1}}{(n+1)\deg_{\widetilde{L}}(X)}$$

if $\overline{L} + \epsilon\overline{M}$ is nef. We have

$$(\overline{L} + \epsilon\overline{M})^{n+1} = \overline{L}^{n+1} + \epsilon(n+1)\overline{L}^n\overline{M} + O(\epsilon^2).$$

Hence the RHS is > 0 if $|\epsilon| \ll 1$ because $\overline{L}^{n+1} > 0$. So

$$e_1(X, \overline{L} + \epsilon\overline{M}) \geq \frac{\overline{L}^{n+1} + \epsilon(n+1)\overline{L}^n\overline{M}}{(n+1)\deg_{\widetilde{L}}(X)} + O(\epsilon^2).$$

By definitions of e_1 and of generic sequence, we have

$$\liminf_{m\to\infty} h_{\overline{L}+\epsilon\overline{M}}(x_m) \geq \frac{\overline{L}^{n+1} + \epsilon(n+1)\overline{L}^n\overline{M}}{(n+1)\deg_{\widetilde{L}}(X)} + O(\epsilon^2).$$

Since $\{x_m\}_{m\geq 1}$ is small, we have

$$\lim_{m\to\infty} h_{\overline{L}}(x_m) = h_{\overline{L}}(X) = \frac{\overline{L}^{n+1}}{(n+1)\deg_{\widetilde{L}}(X)}.$$

Therefore

$$\liminf_{m\to\infty} \epsilon h_{\overline{M}}(x_m) \geq \epsilon\frac{\overline{L}^n\overline{M}}{\deg_{\widetilde{L}}(X)} + O(\epsilon^2). \tag{6.3.4}$$

Now letting $\epsilon \to 0^+$ and $\epsilon \to 0^-$, we obtain

$$\lim_{m\to\infty} h_{\overline{M}}(x_m) = \frac{\overline{L}^n\overline{M}}{\deg_{\widetilde{L}}(X)}. \tag{6.3.5}$$

Now assume $\overline{L}$ is represented by a Cauchy sequence $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U})_{\mathrm{mod}}$ for a quasi-projective model $\mathcal{U}$ of X. Assume that $\psi\colon \mathcal{X}_i \to \mathcal{X}_1$ extends the identity morphism on $\mathcal{U}$, and denote by X_i to be the generic fiber of $\mathcal{X}_i$ which contains X as an open subvariety.

Let $\mathcal{X}_1'$ be another projective model of X_1. Let $\overline{\mathcal{M}} \in \widehat{\mathrm{Pic}}(\mathcal{X}_1')_{\mathbb{Q}}$ with a fixed isomorphism $\mathcal{M}_K \to \mathcal{O}_{X_1}$. Then it induces a metric $\|\cdot\|_w$ of $\mathcal{O}_{X_1}$ on $X_{1,w}^{\mathrm{an}}$ for any place w of K. Assume that the metric $\|1\|_w = 1$ for any $w \neq v$. Denote by $g = -\log\|1\|_v$; it is continuous on $X_{1,v}^{\mathrm{an}}$. Then by definition, we have

$$h_{\overline{\mathcal{M}}}(x_m) = \frac{1}{\#O(x_m)}\sum_{y\in O(x_m)\times_K K_v} g(y), \qquad \overline{L}^n\overline{M} = \lim_{i\to\infty}\overline{\mathcal{L}}_i^n\overline{\mathcal{M}} = \lim_{i\to\infty}\int_{X_v^{\mathrm{an}}} gc_1(\overline{\mathcal{L}}_i)_v^n.$$

So we get, by (6.3.5),

$$\lim_{m\to\infty}\frac{1}{\#O(x_m)}\sum_{y\in O(x_m)\times_K K_v} g(y) = \frac{1}{\deg_{\widetilde{L}}(X)}\lim_{i\to\infty}\int_{X_v^{\mathrm{an}}} gc_1(\overline{\mathcal{L}}_i)_v^n, \tag{6.3.6}$$

with g viewed as a function on $X_{i,v}^{\mathrm{an}}$ by the pullback via $\psi_{i,K}\colon X_i \to X_1$.

Now vary $g = -\log\|1\|_v$, which is a model function on $X_{1,v}^{\mathrm{an}}$ associated with $(\mathcal{X}_1', \overline{\mathcal{M}})$. Gubler's density theorem implies that the space of all such model functions is dense in $C(X_{1,v}^{\mathrm{an}})$ under the topology of uniform convergence. So (6.3.6) holds true for any function in $C(X_{1,v}^{\mathrm{an}})$.

Finally, assume $f \in C_c(X_v^{\mathrm{an}})$, viewed as an element of $C(X_{i,v}^{\mathrm{an}})$ by the open immersion $X \to X_i$. Then

$$\lim_{i\to\infty}\int_{X_v^{\mathrm{an}}} fc_1(\overline{\mathcal{L}}_i)_v^n = \lim_{i\to\infty}\int_{X_v^{\mathrm{an}}} fc_1(\overline{\mathcal{L}}_i)_v^n|_{X_v^{\mathrm{an}}} = \int_{X_v^{\mathrm{an}}} fc_1(\overline{L})_v^n.$$

And we can conclude by (6.3.6) applied to f. $\square$