Chapter 1

Preparation on Hodge theory

1.1 Hodge structure and polarizations

Take $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

Let $n \in \mathbb{Z}$.

1.1.1 Hodge decomposition and Hodge filtration

Definition 1.1.1. *An R-**Hodge structure of weight** n is a torsion-free R-module of finite type V endowed with a bigrading (called the **Hodge decomposition**)*

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}, \qquad \text{with} \quad \overline{V^{q,p}} = V^{p,q}.$$

*For a subset $A \subseteq \mathbb{Z} \oplus \mathbb{Z}$, we say that V has **Hodge type** A if $V^{p,q} = 0$ for all $(p,q) \notin A$.*

*An R-linear map $\varphi \colon V \to W$ between two Hodge structures of weight n is said to be a **morphism of Hodge structures** if $\varphi(V^{p,q}) \subseteq W^{p,q}$ for all p, q.*

We thus have the *category of R-Hodge structures of weight n*, denoted by HS_R^n. One can define direct sums in HS_R^n, and hence makes it into an *abelian category.*

We can also consider the *category of R-Hodge structures*, denoted by HS_R. The objects are R-Hodge structures of any weight. Then we can define tensor products, duals, and internal homs in HS_R as follows. Let $V \in \mathrm{HS}_R^n$ and $W \in \mathrm{HS}_R^m$,

(i) the bigrading on $V \otimes W \in \mathrm{HS}_R^{n+m}$ is given by $(V \otimes W)^{p,q} = \bigoplus_{r+r'=p,\ s+s'=q} V^{r,s} \otimes W^{r',s'}$;

(ii) the bigrading on $V^\vee \in \mathrm{HS}_R^{-n}$ is given by $(V^\vee)^{p,q} = (V^{-p,-q})^\vee$;

(iii) $\mathrm{Hom}(V, W) := V^\vee \otimes W$.

Here are some examples.

Example 1.1.2 (Tate twist)**.** *For each $m \in \mathbb{Z}$, set $R(m) \in \mathrm{HS}_R^{-2m}$ to be*

$$R(m) = (2\pi i)^m R, \qquad R(m)_{\mathbb{C}} = R^{-m,-m}.$$

Then $R(0) = R$, $R(m) = R(1)^{\otimes m}$ with $R(-1) = R(1)^\vee$.

Example 1.1.3 (cohomology from geometry)**.** *Let X be a connected smooth projective variety defined over $\mathbb{C}$. For each $n \geq 0$, the Betti cohomology $H^n(X, \mathbb{Z})/\mathrm{tor}$ admits a $\mathbb{Z}$-Hodge structure of weight n via the Betti–de Rham comparison $H^n(X, \mathbb{C}) \simeq H^n_{\mathrm{dR}}(X)$ and the decomposition of $H^n_{\mathrm{dR}}(X)$ into the direct sum of subspaces arising from (p,q)-forms.*

Example 1.1.4 (Complex tori)**.** *We explain in this example the following equivalence of categories:*

$$\{complex\ tori\} \xrightarrow{\sim} \{\mathbb{Z} - Hodge\ structures\ of\ type\ (-1,0)+(0,-1)\}.$$

The direction $\to$ is by sending $T \mapsto H_1(T,\mathbb{Z})$. Let T be a complex torus of dimension $g \geq 1$. Set

$$V_{\mathbb{Z}} := H_1(T,\mathbb{Z}).$$

As a real manifold, we then have $T \simeq V_{\mathbb{R}}/V_{\mathbb{Z}}$. Moreover, as a real space $V_{\mathbb{R}}$ is isomorphic to $\mathrm{Lie}(T_{\mathbb{R}})$*, the Lie algebra with $T_{\mathbb{R}}$ seen as a real Lie group. The complex structure on T gives an action of J on $V_{\mathbb{R}}$, with*

$$J := \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix},$$

and hence the desired Hodge decomposition

$$V_{\mathbb{C}} = V^{-1,0} \bigoplus V^{0,-1}$$

with $V^{-1,0}$ the eigenspace of $\sqrt{-1}$ and $V^{0,-1}$ the eigenspace of $-\sqrt{-1}$.

The direction $\leftarrow$ is given as follows. Let $V_{\mathbb{Z}}$ be a $\mathbb{Z}$-Hodge structure of type $(-1,0)+(0,-1)$. Then $V_{\mathbb{C}}/V^{0,-1}$ is a complex space of dimension $\frac{1}{2}\mathrm{rank}V_{\mathbb{Z}}$. Thus we obtain the desired complex torus

$$V_{\mathbb{Z}}\backslash V_{\mathbb{C}}/V^{0,-1} \simeq V_{\mathbb{Z}}\backslash V^{-1,0}.$$

Notice that we have implicitly an isomorphism of real vector spaces $V_{\mathbb{R}} \simeq V_{\mathbb{C}}/V^{0,-1} = V^{-1,0}$ given as the composite $V_{\mathbb{R}} \subseteq V_{\mathbb{C}} \to V_{\mathbb{C}}/V^{0,-1} = V^{-1,0}$.

An alternative way to see the Hodge decomposition is the following Hodge filtration. It is of particular importance when we consider families of Hodge structures.

Definition 1.1.5. *Let V be an R-Hodge structure of weight n. The* **Hodge filtration** *is the decreasing chain $\cdots \supseteq F^pV_{\mathbb{C}} \supseteq F^{p+1}V_{\mathbb{C}} \supseteq \cdots$ with*

$$F^pV_{\mathbb{C}} := \bigoplus_{r\geq p} V^{r,s}. \tag{1.1.1}$$

Conversely, the Hodge decomposition can be recovered by the Hodge filtration via

$$V^{p,q} = F^pV_{\mathbb{C}} \cap \overline{F^qV_{\mathbb{C}}}. \tag{1.1.2}$$

1.1.2 Polarizations

Let V be an R-Hodge structure of weight n.

The *Weil operator* $C \in \mathrm{End}(V_{\mathbb{C}})$ is defined as follows: It acts on $V^{p,q}$ by multiplication by $\sqrt{-1}^{q-p}$. We have $Cx = \overline{Cx}$ for all $x \in V_{\mathbb{R}}$.[1] So $C \in \mathrm{End}(V_{\mathbb{R}})$. A more elegant way to define the Weil operator will be given above Proposition 1.2.5, in terms of Deligne torus.

Definition 1.1.6. *A* **polarization** *on V is a morphism of Hodge structures*

$$\psi \colon V \otimes V \to R(-n)$$

such that the bi-linear map

$$V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}, \qquad (x,y) \mapsto \psi_C(x,y) := (2\pi\sqrt{-1})^n\psi(x,Cy) \tag{1.1.3}$$

is symmetric and positive definite.

[1] Indeed, for $x = \sum_{p,q} x_{p,q} \in V_{\mathbb{R}}$, we have $\overline{x_{p,q}} = x_{q,p}$ because $\overline{V^{p,q}} = V^{q,p}$. So $\overline{Cx} = \sum_{p,q} \overline{\sqrt{-1}}^{q-p}\overline{x_{p,q}} = \sum_{p,q} \sqrt{-1}^{p-q}x_{q,p} = \sum_{p,q} \sqrt{-1}^{p-q}x_{q,p} = Cx$, and hence $Cx \in V_{\mathbb{R}}$.

The Hermitian pairing associated with the bi-linear map (1.1.3) is $(x, y) \mapsto \psi_C(x, \overline{y})$.

Lemma 1.1.7. *Let $V \in \mathrm{HS}^n_R$, and let ψ be a polarization. Then*

(i) ψ is $(-1)^n$-symmetric, i.e. is alternating if n is odd and is symmetric if n is even.

(ii) the decomposition $V_{\mathbb{C}} = \bigoplus V^{p,q}$ is orthogonal with respect to the Hermitian pairing associated with (1.1.3).

Proof. We start by proving (ii). Take $x \in V^{p,q}$ and $y \in V^{r,s}$. Then

$$(2\pi\sqrt{-1})^{-n}\psi_C(x, \overline{y}) = \psi(x, C\overline{y}) = \psi(x, \sqrt{-1}^{r-s}\overline{y}) = \sqrt{-1}^{r-s}\psi(x, \overline{y})$$

Now $(x, \overline{y}) \in V^{p,q} \times V^{s,r} \subseteq (V \times V)^{p+s,q+r}$. So $\psi(x, \overline{y}) \in R(-n)^{p+s,q+r}$ since ψ is a morphism of Hodge structures. Assume $\psi(x, \overline{y}) \neq 0$. Then $p + s = q + r = n$. But $p + q = r + s = n$. So $p = r$ and $q = s$. Thus $\psi_C(V^{p,q}, \overline{V^{r,s}}) = 0$ unless $p = r$ and $q = s$. This establishes (ii).

Now we turn to (i). The proof will be much easier and more elegant if we apply Proposition 1.2.5. Here we give a direct computation without using this proposition.

For each $x, y \in V_{\mathbb{R}}$, write $x = \sum_{p,q} x_{p,q}$ and $y = \sum_{p,q} y_{p,q}$ under $V_{\mathbb{C}} = \bigoplus V^{p,q}$. Then $(y_{p,q}, x_{r,s}) \subseteq (V \times V)^{p+r,q+s}$, and hence $\psi(y_{p,q}, x_{r,s}) \in R(-n)^{p+r,q+s}$ is 0 unless $p + r = q + s = n$. So

$$\psi(y, x) = \sum_{p,q} \psi(y_{p,q}, x_{p,q}).$$

On the other hand, $x_{p,q} = \overline{x_{q,p}}$ and $y_{p,q} = \overline{y_{q,p}}$ since $\overline{V^{p,q}} = V^{q,p}$. So

$$\begin{aligned}
\psi_C(Cy, x) &= \psi_C(\sum_{p,q} \sqrt{-1}^{q-p} y_{p,q}, \sum_{p,q} x_{p,q}) \\
&= \psi_C(\sum_{p,q} \sqrt{-1}^{q-p} y_{p,q}, \sum_{p,q} \overline{x_{p,q}}) \\
&= \sum_{p,q} \psi_C(\sqrt{-1}^{q-p} y_{p,q}, \overline{x_{p,q}}) \\
&= \sum_{p,q} \psi_C(\sqrt{-1}^{q-p} y_{p,q}, x_{q,p}) \\
&= (2\pi\sqrt{-1})^n \sum_{p,q} \psi(\sqrt{-1}^{q-p} y_{p,q}, Cx_{q,p}) \\
&= (2\pi\sqrt{-1})^n \sum_{p,q} \psi(\sqrt{-1}^{q-p} y_{p,q}, \sqrt{-1}^{p-q} x_{q,p}) \\
&= (2\pi\sqrt{-1})^n \sum_{p,q} \psi(y_{p,q}, x_{q,p}).
\end{aligned}$$

Therefore

$$\psi(y, x) = (2\pi\sqrt{-1})^{-n}\psi_C(Cy, x).$$

Since ψ_C is symmetric, we furthermore have

$$\psi(y, x) = (2\pi\sqrt{-1})^{-n}\psi_C(x, Cy) = \psi(x, C^2 y).$$

Notice that C^2 acts on $V^{p,q}$ by multiplication by $(-1)^{q-p} = (-1)^{q+p} = (-1)^n$ for all p, q. Thus C^2 acts on V as multiplication by $(-1)^n$. So we have

$$\psi(y, x) = (-1)^n \psi(x, y).$$

This establishes (i). $\square$

Example 1.1.8 (Complex abelian varieties)**.** *We continue with Example 1.1.4 and prove*

$$\{\text{complex abelian varieties}\} \xrightarrow{\sim} \{\text{polarizable } \mathbb{Z}\text{-Hodge structures of type } (-1,0)+(0,-1)\}.$$

Let T be a complex torus which corresponds to $V_{\mathbb{Z}} = H_1(T,\mathbb{Z})$. Then $T \simeq V_{\mathbb{R}}/V_{\mathbb{Z}}$ as real manifolds. Thus $\bigwedge^2 V_{\mathbb{Z}}^{\vee} \simeq \bigwedge^2 H^1(T,\mathbb{Z}) = H^2(T,\mathbb{Z})$. Therefore the set of alternating pairings

$$\psi\colon V_{\mathbb{Z}} \otimes V_{\mathbb{Z}} \to \mathbb{Z}(1)$$

is in bijection with $H^2(T,\mathbb{Z}(1))$.

The short exact sequence of sheaves $0 \to \underline{\mathbb{Z}}(1) \to \mathcal{O}_T \xrightarrow{\exp} \mathcal{O}_T^* \to 0$ induces

$$\mathrm{Pic}(T) = H^1(T,\mathcal{O}_T^*) \xrightarrow{c_1} H^2(T,\mathbb{Z}(1)) \to H^2(T,\mathcal{O}_T).$$

Assume T is an abelian variety. Then there exists an ample line bundle L on T. The Ampell–Hubert data for L then gives an alternating pairing $\psi \in H^2(T,\mathbb{Z}(1))$ such that the Hermitian pairing $(x,y) \mapsto \psi(x,\sqrt{-1}\overline{y})$ is the c_1 of L for a suitable Hermitian metric on L. But $V_{\mathbb{Z}}$ has Hodge type $(-1,0)+(0,-1)$ and the complex structure on $V_{\mathbb{R}}/V_{\mathbb{Z}}$ is by identifying $V_{\mathbb{R}} \simeq V^{-1,0}$. So $c_1(L)$ is precisely ψ_C. The ampleness of L implies that ψ_C is positive-definite. Thus ψ is a polarization on $V_{\mathbb{Z}}$.

Conversely assume ψ is a polarization on $V_{\mathbb{Z}}$. Then ψ can be seen as an element in $H^2(T,\mathbb{Z}(1))$, and ψ_C equals $(x,y) \mapsto \psi(x,\sqrt{-1}\overline{y})$ as above. So the Ampell–Hubert Theorem gives a line bundle L on T such that $c_1(L) = \psi_C$. The positivity of ψ_C thus implies the ampleness of L by Kodaira embedding. So T is an abelian variety.

Example 1.1.9 (Primitive cohomology and Lefschetz)**.** *We continue with Example 1.1.3. Assume $d = \dim X$. Let ω be a Kähler form on X^{an}, which is a closed $(1,1)$-form. It induces a homomorphism $L\colon H^n(X,\mathbb{Q}) \to H^{n+2}(X,\mathbb{Q})$, $[\alpha] \mapsto [\omega \wedge \alpha]$; here we are using $H^n(X,\mathbb{Q}) \subseteq H^n(X,\mathbb{C}) \simeq H^n_{\mathrm{dR}}(X)$. The Hard Lefschetz Theorem says that $L^r\colon H^{d-r}(X,\mathbb{Q}) \xrightarrow{\sim} H^{d+r}(X,\mathbb{Q})$ for all $r \geq 0$. Now let $r = d-n$. Set $H^n_{\mathrm{prim}}(X,\mathbb{Q})$ to be the kernel of $L^{r+1}\colon H^n(X,\mathbb{Q}) \to H^{2d-n+2}(X,\mathbb{Q})$. We have a morphism of Hodge structures*

$$\psi\colon H^n(X,\mathbb{Q})\otimes H^n(X,\mathbb{Q}) \xrightarrow[\sim]{1\otimes L^r} H^n(X,\mathbb{Q})\otimes H^{2d-n}(X,\mathbb{Q})(\dim X-n) \xrightarrow{\cup} H^{2d}(X,\mathbb{Q})(d-n) = \mathbb{Q}(-n).$$

The restriction of ψ to $H^n_{\mathrm{prim}}(X,\mathbb{Q})$ is a polarization. Thus we obtain a polarization on $H^n(X,\mathbb{Q})$ by the Lefschetz decomposition $H^n(X,\mathbb{Q}) = \bigoplus_{0\leq s\leq \lfloor n/2\rfloor} L^s(H^{n-2s}_{\mathrm{prim}}(X,\mathbb{Q}))$.

1.2 Mumford–Tate group

1.2.1 Revision on algebraic tori

Let k be a field. A *linear algebraic group* defined over k is an affine group scheme G/k of finite type; it can be embedded as a closed subgroup scheme of GL_N for some N. If $\mathrm{char} k = 0$, then G is reduced and smooth. As an example, we have $\mathbb{G}_{\mathrm{m},k} := \mathrm{GL}_{1,k}$ which is defined by: for any k-algebra R, we have $\mathbb{G}_{\mathrm{m},k}(R) = R^{\times}$. When k is clear in the context, we simply write $\mathbb{G}_{\mathrm{m}}$.

Let k^{s} be a separable closure of k. If $\mathrm{char} k = 0$, then k^{s} is an algebraic closure of k.

Definition 1.2.1. *An* **algebraic torus** *defined over k is a linear algebraic group T defined over k such that its base change to k^{s} is isomorphic to $\mathbb{G}^r_{\mathrm{m},k^{\mathrm{s}}}$ for some $r \geq 1$.*

The *group of characters* (resp. *group of cocharacters*) of T is

$$X^*(T) := \mathrm{Hom}(T_{k^{\mathrm{s}}}, \mathbb{G}_{\mathrm{m},k^{\mathrm{s}}}), \quad (\text{resp. } X_*(T) := \mathrm{Hom}(\mathbb{G}_{\mathrm{m},k^{\mathrm{s}}}, T_{k^{\mathrm{s}}})).$$

Both $X^*(T)$ and $X_*(T)$ are isomorphic (as groups) to $\mathbb{Z}^{\dim T}$ and are naturally endowed with a $\mathrm{Gal}(k^{\mathrm{s}}/k)$-action. We also have a *perfect pairing* as $\mathrm{Gal}(k^{\mathrm{s}}/k)$-modules

$$X^*(T) \times X_*(T) \to \mathbb{Z} = \mathrm{End}(\mathbb{G}_{\mathrm{m},k^{\mathrm{s}}}), \qquad (\chi, \mu) \mapsto \langle \chi, \mu \rangle := \chi \circ \mu. \tag{1.2.1}$$

By definition, $T_{k'} \simeq \mathbb{G}_{\mathrm{m},k'}$ for some finite separable extension k'/k. So the Galois action of $\mathrm{Gal}(k^{\mathrm{s}}/k)$ on $X^*(T)$ factors through $\mathrm{Gal}(k'/k)$ which is a finite group. Therefore the $\mathrm{Gal}(k^{\mathrm{s}}/k)$-action on $X^*(T)$ is continuous. Same for the $\mathrm{Gal}(k^{\mathrm{s}}/k)$-action on $X_*(T)$. Thus the functor $T \mapsto X_*(T)$ gives an equivalence from the category of *algebraic tori defined over* k to the category of *free abelian groups of finite rank endowed with a continuous* $\mathrm{Gal}(k^{\mathrm{s}}/k)$*-action.*

Next we turn to the representations of algebraic tori $\rho\colon T \to \mathrm{GL}(V)$. Passing to k', ρ becomes $T_{k'} \simeq \mathbb{G}^r_{\mathrm{m},k'} \to \mathrm{GL}(V_{k'})$. Then $V_{k'}$ can be decomposed into

$$V_{k'} = \bigoplus\nolimits_{\chi \in X^*(T)} V_\chi = \bigoplus_{(n_1,\ldots,n_r) \in \mathbb{Z}^r} V^{n_1,\ldots,n_r}$$

where $V_\chi = \{v \in V_{k'} : \rho(t)v = \chi(t)v\}$ and $V^{n_1,\ldots,n_r} = \{v \in V_{k'} : \rho(z_1,\ldots,z_r)v = z_1^{-n_1}\cdots z_r^{-n_r}v\}$. On the base field k, the decomposition is Galois compatible, *i.e.* $\sigma(V_\chi) = V_{\chi^\sigma}$ for all $\sigma \in \mathrm{Gal}(k'/k)$.

1.2.2 Deligne torus

View $\mathbb{C}$ as an $\mathbb{R}$-algebra using the inclusion $\mathbb{R} \subseteq \mathbb{C}$. Let $\mathbb{S}$ be the algebraic group $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{\mathrm{m}}$ defined over $\mathbb{R}$, *i.e.* for any $\mathbb{R}$-algebra R, we have

$$\mathbb{S}(R) = (R \otimes_{\mathbb{R}} \mathbb{C})^\times.$$

Then

$$\mathbb{S}(\mathbb{C}) = (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^\times = \big((\mathbb{R} \oplus \sqrt{-1}\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}\big)^\times = (\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C})^\times \times (\sqrt{-1}\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C})^\times = \mathbb{C}^\times \times \mathbb{C}^\times.$$

Hence $\mathbb{S}$ is an algebraic torus defined over $\mathbb{R}$, and $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$ acts on $\mathbb{S}(\mathbb{C})$ by $\sigma(z_1, z_2) = (\overline{z}_2, \overline{z}_1)$. Thus $\mathbb{S}(\mathbb{R}) = \{z \in \mathbb{S}(\mathbb{C}) : z = \sigma(z)\} = \{(z_1, z_2) \in \mathbb{C}^\times \times \mathbb{C}^\times : z_2 = \overline{z}_1\}$. In other words, the natural inclusion $\mathbb{S}(\mathbb{R}) \subseteq \mathbb{S}(\mathbb{C})$ is given by $z \mapsto (z, \overline{z})$.

Definition 1.2.2. *The algebraic torus* $\mathbb{S}$ *is called the* **Deligne torus**.

The character group of the Deligne torus is

$$X^*(\mathbb{S}) = \mathrm{Hom}(\mathbb{S}(\mathbb{C}), \mathbb{C}^\times) = \mathrm{Hom}(\mathbb{C}^\times \times \mathbb{C}^\times, \mathbb{C}^\times) = \mathrm{Hom}(\mathbb{C}^\times, \mathbb{C}^\times) \oplus \mathrm{Hom}(\mathbb{C}^\times, \mathbb{C}^\times) \simeq \mathbb{Z} \oplus \mathbb{Z}, \tag{1.2.2}$$

where the last isomorphism is obtained from the inverse of

$$\mathbb{Z} \xrightarrow{\sim} \mathrm{Hom}(\mathbb{C}^\times, \mathbb{C}^\times), \qquad p \mapsto (z \mapsto z^{-p}). \tag{1.2.3}$$

The Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$ acts on $X^*(\mathbb{S})$ by $\sigma(p, q) = (q, p)$.

Among the cocharacters of $\mathbb{S}$, two are particularly important:

- the *weight cocharacter* $w\colon \mathbb{G}_{\mathrm{m},\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$, $z \mapsto (z, z)$, which descends to $\mathbb{R}$ (namely it is the base change to $\mathbb{C}$ of a morphism $\mathbb{G}_{\mathrm{m},\mathbb{R}} \to \mathbb{S}$).
- the *principal cocharacter* $\mu\colon \mathbb{G}_{\mathrm{m},\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$, $z \mapsto (z, 1)$.

An important character of $\mathbb{S}$ is the *norm character* $\mathrm{Nm}\colon \mathbb{S} \to \mathbb{G}_{\mathrm{m}}$, $z \mapsto z\sigma(z)$. It fits into the following short exact sequence:

$$0 \to U(1) \to \mathbb{S} \xrightarrow{\mathrm{Nm}} \mathbb{G}_{\mathrm{m}} \to 0. \tag{1.2.4}$$

Notice that $\mathrm{Nm} \circ w$ sends each $z \in \mathbb{G}_{\mathrm{m}}(\mathbb{R}) = \mathbb{R}^{\times}$ to z^2.

1.2.3 Hodge structures as representations of the Deligne torus

Now let V be an R-Hodge structure of weight n. Recall the Hodge decomposition $V_{\mathbb{C}} = \bigoplus V^{p,q}$. It gives rise to an action of $\mathbb{S}_{\mathbb{C}}$ on $V_{\mathbb{C}}$ by setting $V^{p,q}$ to be the eigenspace of the character $(p, q) \in X^*(\mathbb{S})$. More precisely, for each $(z_1, z_2) \in \mathbb{S}(\mathbb{C}) = \mathbb{C}^{\times} \times \mathbb{C}^{\times}$ and each $v = (v_{p,q})_{p,q} \in V_{\mathbb{C}} = \bigoplus V^{p,q}$, we have

$$(z_1, z_2) \cdot v = (z_1^{-p} z_2^{-q} v_{p,q})_{p,q}. \tag{1.2.5}$$

This action of $\mathbb{S}_{\mathbb{C}}$ on $V_{\mathbb{C}}$ induces a morphism

$$h\colon \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(V_{\mathbb{C}}). \tag{1.2.6}$$

Lemma 1.2.3. *The morphism h descends to $\mathbb{R}$, i.e. it is the base change to $\mathbb{C}$ of a morphism $\mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$.*

Proof. For $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$, we can do the following computation. Let $(z_1, z_2) \in \mathbb{S}(\mathbb{C})$ and $v = (v_{p,q})_{p,q} \in V_{\mathbb{C}}$.

Recall that the Hodge decomposition satisfies $\overline{V^{p,q}} = V^{q,p}$. So $\overline{v_{p,q}} \in \overline{V^{p,q}} = V^{q,p}$. Hence the decomposition of $\overline{v} = \sigma(v)$ under $V_{\mathbb{C}} = \bigoplus V^{p,q}$ is $\overline{v} = (\overline{v_{q,p}})_{p,q}$. In particular, $\overline{v}_{p,q} = \overline{v_{q,p}}$.

Now we have

$$h(\sigma(z_1, z_2))v = (\overline{z}_2, \overline{z}_1) \cdot v = (\overline{z}_2^{-p} \overline{z}_1^{-q} v_{p,q})_{p,q}$$

and

$$\sigma\left(h(z_1, z_2)\right) v = \sigma\left(h(z_1, z_2)\overline{v}\right) = \sigma\left((z_1, z_2) \cdot \overline{v}\right) = \sigma((z_1^{-p} z_2^{-q} \overline{v}_{p,q})_{p,q}) = \sigma((z_1^{-p} z_2^{-q} \overline{v_{q,p}})_{p,q}) = (\overline{z}_1^{-q} \overline{z}_2^{-p} v_{p,q})_{p,q}.$$

Hence h is $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant, and therefore descends to $\mathbb{R}$. □

Thus from any R-Hodge structure V of weight n, we have constructed a morphism $\mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$. Conversely given any $h\colon \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$, we can define $V^{p,q}$ to be the eigenspace of the character $(p, q) \in X^*(\mathbb{S})$ of $\mathbb{S}_{\mathbb{C}}$. Then $V = \bigoplus V^{p,q}$, and $\overline{V^{q,p}} = V^{p,q}$ because h is defined over $\mathbb{R}$. Hence we have the following proposition.

Proposition 1.2.4. *Let $R = \mathbb{Z}, \mathbb{Q}$ and let V be a torsion-free R-module of finite type.*

Then there are bijections between the following sets of:

(i) Hodge structures of weight n on V;

(ii) morphisms $h\colon \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ such that the eigenspace of $(p, q) \in X^*(\mathbb{S})$ is 0 unless $p+q = n$.

(iii) morphisms $h\colon \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ such that the composite $h \circ w\colon \mathbb{G}_{\mathrm{m},\mathbb{R}} \to \mathrm{GL}(V_{\mathbb{R}})$ sends each $z \in \mathbb{R}^{\times}$ to the multiplication by z^{-n}.

If a Hodge structure on V corresponds to $h\colon \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$, by abuse of notation we use (V, h) to denote this Hodge structure. In this terminology, the Weil operator C of the Hodge structure (V, h) in the definition of polarizations (1.1.3) is simply $h(\sqrt{-1})$.

Proposition 1.2.5. *Let (V, h) and (W, h') be two R-Hodge structures of weight n, and let $\varphi\colon V \to W$ be an R-linear map.*

Then φ is a morphism of Hodge structures if and only if $\varphi(h(z)v) = h'(z)\varphi(v)$ for all $z \in \mathbb{S}(\mathbb{R})$ and $v \in V_{\mathbb{R}}$.

The proof of Lemma 1.1.7.(i) can be much simplified by this proposition: $\psi(y, x) = \psi(Cy, Cx) = (2\pi\sqrt{-1})^{-2n}\psi_C(Cy, x) = (2\pi\sqrt{-1})^{-2n}\psi_C(x, Cy) = \psi(x, C^2y) = (-1)^n\psi(x, y)$, and hence ψ is $(-1)^n$-symmetric.

Proof. Write $v = (v_{p,q})_{p,q} \in V_{\mathbb{C}} = \bigoplus V^{p,q}$. Then $h(z)v = (z^{-p}\overline{z}^{-q}v_{p,q})_{p,q}$. So $\varphi(h(z)v) = (z^{-p}\overline{z}^{-q}\varphi(v_{p,q}))_{p,q}$ by linearity of φ.

Assume φ is a morphism of Hodge structures. Then $\varphi(V^{p,q}) \subseteq W^{p,q}$ for all p, q, and hence $\varphi(v_{p,q}) = \varphi(v)_{p,q}$ for all p, q. So $\varphi(h(z)v) = (z^{-p}\overline{z}^{-q}\varphi(v)_{p,q})_{p,q} = h'(z)\varphi(v)$.

Conversely assume $\varphi(h(z)v) = h'(z)\varphi(v)$ for all $z \in \mathbb{S}(\mathbb{R})$ and $v \in V_{\mathbb{R}}$. Let $v \in V^{p,q}$. By considering $v + \overline{v}$ and $(v - \overline{v})/\sqrt{-1}$, we have $\varphi(h(z)v) = h'(z)\varphi(v)$ for all $z \in \mathbb{S}(\mathbb{R})$. So $h'(z)\varphi(v) = \varphi(h(z)v) = \varphi(z^{-p}\overline{z}^{-q}v) = z^{-p}\overline{z}^{-q}\varphi(v)$ for all $z \in \mathbb{S}(\mathbb{R})$. Therefore $\varphi(v) \in W^{p,q}$. $\square$

This proposition has the following immediate corollary.

Corollary 1.2.6. *Let (V, h) be an R-Hodge structure of weight n, and let W be a torsion-free R-submodule of V.*

Then $h|_W$ is an R-Hodge structure if and only if $W_{\mathbb{R}}$ is an $h(\mathbb{S})$-submodule of V.

In this case, we call the Hodge structure $(W, h|_W)$ a *sub-R-Hodge structure* of (V, h).

Another corollary is:

Corollary 1.2.7. *Let $Q\colon V \times V \to R$ induce a polarization on (V, h). Then $h(\mathbb{S}) \subseteq \mathrm{Aut}(V, Q)$.*

Proof. By definition, Q induces a morphism of Hodge structures between $V \otimes V$ and $R(-n)$. Thus the conclusion follows immediately from Proposition 1.2.5. $\square$

1.2.4 Mumford–Tate group

In this subsection, assume $R = \mathbb{Z}$ or $\mathbb{Q}$. Let (V, h) be an R-Hodge structure.

Definition 1.2.8. *The* **Mumford–Tate group** *of (V, h) is the smallest $\mathbb{Q}$-subgroup $\mathrm{MT}(h)$ of $\mathrm{GL}(V_{\mathbb{Q}})$ such that $h(\mathbb{S}) \subseteq \mathrm{MT}(h)(\mathbb{R})$.*

It is easy to check that $\mathrm{MT}(h)$ is connected since $\mathbb{S}$ is, and $\mathrm{MT}(h)(\mathbb{C})$ is the subgroup of $\mathrm{GL}(V(\mathbb{C}))$ generated by $\sigma(h(\mathbb{S}(\mathbb{C})))$ for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. We also have the following characterization of $\mathrm{MT}(h)$ using the principal cocharacter μ defined above (1.2.4).

Lemma 1.2.9. *$\mathrm{MT}(h)$ is the smallest $\mathbb{Q}$-subgroup of $\mathrm{GL}(V_{\mathbb{Q}})$ such that $\mu_h := h \circ \mu\colon \mathbb{G}_{\mathrm{m},\mathbb{C}} \to \mathrm{GL}(V_{\mathbb{C}})$ factors through $\mathrm{MT}(h)_{\mathbb{C}}$.*

Proof. By definition $\mu_h(\mathbb{G}_{\mathrm{m},\mathbb{C}}) \subseteq \mathrm{MT}(h)_{\mathbb{C}}$. Conversely let M be a $\mathbb{Q}$-subgroup of $\mathrm{GL}(V_{\mathbb{Q}})$ which contains $\mu_h(\mathbb{G}_{\mathrm{m},\mathbb{C}}) = h(\mu(\mathbb{G}_{\mathrm{m},\mathbb{C}}))$. Then $M(\mathbb{C})$ contains $h(z, 1) \in \mathrm{GL}(V(\mathbb{C}))$ for all $z \in \mathbb{C}^\times$. Since M is defined over $\mathbb{Q}$ and h is defined over $\mathbb{R}$, we have that $M(\mathbb{C})$ contains $\sigma(h(z, 1)) = h(\sigma(z, 1)) = h(1, \overline{z})$ for all $z \in \mathbb{C}^\times$, where $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$. Hence $M(\mathbb{C})$, as a group, contains $h(z, 1)h(1, \overline{z}') = h(z, \overline{z}')$ for all $z, z' \in \mathbb{C}^\times$. Hence $h(\mathbb{S}_{\mathbb{C}}) \subseteq M_{\mathbb{C}}$, so $\mathrm{MT}(h) \subseteq M$. $\square$

It is not hard to check that the Mumford–Tate of the dual Hodge structure of (V, h) is still $\mathrm{MT}(h)$.

Now assume $R = \mathbb{Q}$. For $m, n \in \mathbb{Z}_{\geq 0}$, we have a Hodge structure $T^{m,n}V := V^{\otimes m} \otimes (V^\vee)^{\otimes n}$, and $\mathrm{MT}(h)$ acts on $T^{m,n}V$ componentwise. The following proposition is an immediate consequence of Corollary 1.2.6 (applied to $T^{m,n}V$).

Proposition 1.2.10. *Let W be a $\mathbb{Q}$-subspace of $T^{m,n}V$. Then W is a sub-$\mathbb{Q}$-Hodge structure of $T^{m,n}V$ if and only if W is a $\mathrm{MT}(h)$-submodule of $T^{m,n}V$.*

This proposition gives rise to another useful characterization of $\mathrm{MT}(h)$, which is important in the study of (sub-)Shimura varieties. We make the following definition.

Definition 1.2.11. *The elements of $(T^{m,n}V_{\mathbb{C}})^{0,0} \cap T^{m,n}V$, with m and n running over all non-negative integers, are called the* **Hodge tensor** *for (V, h).*

Denote by Hdg_h the set of all Hodge tensors for (V, h).

Proposition 1.2.12. *We have $\mathrm{MT}(h) = Z_{\mathrm{GL}(V)}(\mathrm{Hdg}_h)$.*

In particular by dimension reasons, $\mathrm{MT}(h) = Z_{\mathrm{GL}(V)}(\mathfrak{I})$ for some finite set $\mathfrak{I} \subseteq \mathrm{Hdg}_h$.

Proof. Take $t \in \mathrm{Hdg}_h$. For any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, we have $\sigma(t) = t$ since t is a $\mathbb{Q}$-element. By (1.2.5) we have $h(z_1, z_2)t = z_1^0 \overline{z_2}^0 t = t$ for any $(z_1, z_2) \in \mathbb{S}(\mathbb{C})$. Applying the action of any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ and recalling that $\mathrm{MT}(h)(\mathbb{C})$ is generated by the $\sigma(h(\mathbb{S}(\mathbb{C})))$'s, we have that t is fixed by $\mathrm{MT}(h)(\mathbb{Q})$. This establishes "$\subseteq$".

To get $\mathrm{MT}(h) = Z_{\mathrm{GL}(V)}(\mathrm{Hdg}_h)$, notice that $\mathrm{MT}(h)$ is a closed subgroup of $\mathrm{GL}(V)$. By theory of algebraic groups, $\mathrm{MT}(h)$ is thus the stabilizer of some 1-dimensional $\mathbb{Q}$-subspace L in $\bigoplus_{(m,n)\in I} T^{m,n}V$ for some finite subset $I \subseteq \mathbb{Z}^2_{\geq 0}$. Now that L is a 1-dimensional $\mathrm{MT}(h)$-submodule of $\bigoplus_{(m,n)\in I} T^{m,n}V$, Proposition 1.2.10 implies that L is a 1-dimensional $\mathbb{Q}$-Hodge structure, and hence $L_{\mathbb{C}} = L^{p,q}$ for some p and q. But then $p = q$ since $L^{p,q} = \overline{L^{q,p}}$.[2] In other words, $L \simeq \mathbb{Q}(-p)$ has weight $2p$.

If $p = 0$, take a $\mathbb{Q}$-generator ℓ of L. Then $\mathrm{MT}(h)(\mathbb{Q})$ fixes ℓ by the same argument on proving "$\subseteq$". So $\mathrm{MT}(h)$, being the stabilizer of $\mathbb{Q}\ell$, equals $Z_{\mathrm{GL}(V)}(\ell)$. If $p \neq 0$, then the weight of (V, h) is not zero, and hence the weight r of the Hodge structure $\det V := \bigwedge^{\dim V} V$ is non-zero (since $\det V$ can be realized as a $\mathrm{MT}(h)$-submodule of $V^{\otimes \dim V}$). We may assume $r > 0$ up to replacing V by $V^\vee$. The 1-dimensional $\mathbb{Q}$-space $L^{\otimes r} \otimes (\det V)^{\otimes -2p}$ is a Hodge structure of weight 0 and hence equals its $(0,0)$-piece. Let ℓ be a generator of $L^{\otimes r} \otimes (\det V)^{\otimes -2p}$. Then ℓ is fixed by $\mathrm{MT}(h)(\mathbb{Q})$ by the same argument on proving "$\subseteq$". Hence $\mathrm{MT}(h) = Z_{\mathrm{GL}(V)}(\ell)$ as in the case of $p = 0$.

To summarize, there exists a finite sum of Hodge tensors $t_1 + \cdots + t_N$ such that $\mathrm{MT}(h) = Z_{\mathrm{GL}(V)}(t_1 + \cdots + t_N)$. So $\mathrm{MT}(h) \subseteq \bigcap_{i=1}^N Z_{\mathrm{GL}(V)}(t_i) \subseteq Z_{\mathrm{GL}(V)}(t_1 + \cdots + t_N)$ becomes an equality. We are done. □

Finally, we point out that the Mumford–Tate group of any *polarized* $\mathbb{Q}$-Hodge structure of weight n is a reductive group. A detailed discussion on this will be given in the next chapter (Corollary 2.2.5).

1.3 Passing to families

In practice it is important for us to work with families. We discuss two aspects, and end up with a question to relate them.

[2]To make the argument in this paragraph vigorous, we need to argue with *mixed* Hodge structures because $\bigoplus_{(m,n)\in I} T^{m,n}V$ may have more than one weight. However, since $\bigoplus_{(m,n)\in I} T^{m,n}V$ is a direct sum of (pure) Hodge structures and $\dim L = 1$, we are essentially working with a pure Hodge structure.

1.3.1 Variation of Hodge structures

Let S be a complex manifold.

Definition 1.3.1. *A $\mathbb{Z}$-**variation of Hodge structures** ($\mathbb{Z}$-VHS) of weight n on S is* $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet)$ *where*

- $\mathbb{V}_{\mathbb{Z}}$ *is a local system of free $\mathbb{Z}$-modules on S of finite rank,*
- $\mathcal{F}^\bullet$ *is a finite decreasing filtration (called the **Hodge filtration**) of the holomorphic vector bundle $\mathcal{V} := \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}_S} \mathcal{O}_S$ by holomorphic subbundles,*

such that

(i) $(\mathbb{V}_{\mathbb{Z},s}, \mathcal{F}^\bullet_s)$ *is a $\mathbb{Z}$-Hodge structure of weight n for each $s \in S$,*

*(ii) the connection $\nabla \colon \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}_S} \Omega^1_S$ whose sheaf of horizontal sections is $\mathbb{V}_{\mathbb{C}} := \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ satisfies the **Griffiths' transversality** condition*

$$\nabla(\mathcal{F}^p) \subseteq \mathcal{F}^{p-1} \otimes \Omega^1_S \qquad \text{for all } p. \tag{1.3.1}$$

*A **polarization** on $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet)$ is a morphism of local systems*

$$\mathbb{V}_{\mathbb{Q}} \otimes \mathbb{V}_{\mathbb{Q}} \to \mathbb{Q}_S$$

inducing on each fiber a polarization of the corresponding $\mathbb{Q}$-Hodge structure.

Example 1.3.2. *Let $f \colon X \to S$ be a smooth projective morphism. Then $\mathbb{V} := R^n f_* \mathbb{Z}_X$ is a local system of $\mathbb{Z}$-modules on S with fiber $\mathbb{V}_s = H^n(X_s, \mathbb{Z})$. Replace $\mathbb{V}$ by its quotient by torsion. Under the isomorphism $\mathcal{V} \simeq R^n f_* \Omega^\bullet_{X/S}$, the Hodge filtration is $\mathcal{F}^p\mathcal{V} = R^n f_* \Omega^{\geq p}_{X/S}$. Notice that the subbundle of (p,q)-forms is not holomorphic if $q \neq 0$, but $\mathcal{F}^p\mathcal{V}$ is holomorphic. The fiberwise polarization from Example 1.1.9 gives a polarization on $\mathbb{V}$.*

And this example is the geometric origin of the Griffiths' transversality.

1.3.2 Parametrizing space

Next we turn to the following question. Let V be a finite-dimensional $\mathbb{R}$-vector space, and let $n \in \mathbb{Z}$.

Fix a partition $\{h^{p,q}\}_{p,q\in\mathbb{Z}}$ of $\dim V_{\mathbb{C}}$ into non-negative integers with $p + q = n$ such that $h^{p,q} = h^{q,p}$. Consider the set of all Hodge structures on V such that in the Hodge decomposition, we have $\dim V^{p,q} = h^{p,q}$ for all p, q. Equivalently by Proposition 1.2.4, we are considering the subset $\mathcal{M}_0$ of $\mathrm{Hom}(\mathbb{S}, \mathrm{GL}_V)$ such that the eigenspace of $(p,q) \in X^*(\mathbb{S})$ has dimension $h^{p,q}$. Notice that GL_V acts on $\mathrm{Hom}(\mathbb{S}, \mathrm{GL}_V)$, by sending $h \mapsto \mathrm{Int}(g) \circ h$.

Lemma 1.3.3. *$\mathcal{M}_0$ is a GL_V-orbit.*

Proof. Fix $h \in \mathcal{M}_0$. Then $V_h^{p,q} = \{v \in V_{\mathbb{C}} : h(z)v = z^{-p}\overline{z}^{-q}v \text{ for all } z \in \mathbb{S}(\mathbb{R})\}$.

For any $g \in \mathrm{GL}_V$, it is easy to check that $\{v \in V_{\mathbb{C}} : (g\cdot h)(z)v = z^{-p}\overline{z}^{-q}v \text{ for all } z \in \mathbb{S}(\mathbb{R})\}$ equals $gV_h^{p,q}$, and hence has dimension $h^{p,q}$. Hence the Hodge structure on V determined by $g \cdot h$ is in $\mathcal{M}_0$. Namely $\mathrm{GL}_V \cdot h \subseteq \mathcal{M}_0$.

Conversely let $h' \in \mathcal{M}_0$. By assumption $\dim V_{h'}^{p,q} = \dim V_h^{p,q}$ for all p, q. Assume $h^{p,q} = 0$ unless $r \leq p \leq s$. Such r and s exist since $\dim V_{\mathbb{C}} < \infty$. Now there exists a $g_1 \in \mathrm{GL}_V$ such that $V_{h'}^{r,n-r} = g_1 V_h^{r,n-r}$ by dimension reasons. Now we work with h' and $g_1 \cdot h$, and there exists $g_2 \in \mathrm{GL}_V$ such that $g_2 V_{h'}^{r,n-r} = V_{h'}^{r,n-r}$ and $V_{h'}^{r+1,n-r-1} = g_2 V_{g_1\cdot h}^{r+1,n-r-1}$. We continue to work with h' and $g_2 g_1 \cdot h$ and repeat this process which stops after finitely many steps. Hence we find a $g \in \mathrm{GL}_V$ such that $V_{h'}^{p,q} = V_{g\cdot h}^{p,q}$ for all (p,q). So $h' = g \cdot h$. Thus $\mathcal{M}_0 \subseteq \mathrm{GL}_V \cdot h$. $\square$

Next we fix furthermore a non-degenerate $(-1)^n$-symmetric pairing $Q\colon V \times V \to \mathbb{R}$. We furthermore consider the subset $\mathcal{M}$ of $\mathcal{M}_0$ consisting of Hodge structures on V for which Q is a polarization. Then by Corollary 1.2.7, we have $\mathcal{M} \subseteq \mathrm{Hom}(\mathbb{S}, \mathrm{Aut}(V,Q))$. Moreover using (the proof of) Lemma 1.3.3, we see that $\mathcal{M}$ is an $\mathrm{Aut}(V,Q)$-orbit.

Example 1.3.4. *Assume* $\dim V = 2g$ *and let* $Q\colon V \times V \to \mathbb{R}$ *be the standard symplectic pairing. Then* $\mathrm{Aut}(V,Q) = \mathrm{GSp}_{2g}$. *If* $g = 1$, *then* $\mathrm{Aut}(V,Q) = \mathrm{GL}_2$.

Finally fix a collection of tensors $\{s_\alpha\}$ on $T^{m,n} = V^{\otimes m} \otimes (V^\vee)^{\otimes n}$ with m, n running over all non-negative integers. Set

$$G := \mathrm{Aut}(V,Q) \cap \bigcap\nolimits_\alpha \mathrm{Stab}_{\mathrm{GL}_V}(s_\alpha). \tag{1.3.2}$$

Fix $h\colon \mathbb{S} \to \mathrm{Aut}(V,Q)$ such that each s_α is a Hodge tensor for the Hodge structure (V,h). Then the same holds true for the Hodge structure $(V, g\cdot h)$ for all $g \in G^+$. Let $X^+ := G^+ \cdot h \subseteq \mathrm{Hom}(\mathbb{S}, G)$.[3]

Now we have a family of Hodge structures on X^+ as follows: $X^+ \times V \to X^+$, with the Hodge structure on V over each $h \in X^+$ being precisely the one given by h. Now $X^+ \times V$ can be seen as a smooth vector bundle on X^+, and for each p there is a subbundle F^p whose fiber over each $h \in X^+$ is the Hodge filtration F^p_h.

In view of the definition of VHS (Definition 1.3.1), we wish the investigate the following questions:

(i) Is there a complex structure on X^+ for which each subbundle F^p is a holomorphic?

(ii) When does Griffiths' transversality hold true, *i.e.* $\nabla(F^p) \subseteq F^{p-1} \otimes \Omega^1_{X^+}$ for all p?

1.3.3 Constraint on the Hodge type

Continue to use the notation above. Let $h \in X^+ \subseteq \mathrm{Hom}(\mathbb{S}, G)$. Composing with the adjoint representation $G \to \mathrm{GL}(\mathrm{Lie}G)$, we have a Hodge structure on $\mathrm{Lie}G$ by Proposition 1.2.4. By abuse of notation, we use $(\mathrm{Lie}G, h)$ to denote this Hodge structure. Since X^+ is a G^+-orbit, the Hodge type of $(\mathrm{Lie}G, h)$ is independent of the choice of $h \in X^+$.

Moreover h induces a Hodge structure on $\mathrm{End}(V) = V^\vee \otimes V$, which must be of weight 0 and by abuse of notation we denote by $(\mathrm{End}(V), h)$. The inclusion $G \subseteq \mathrm{GL}(V)$ induces $\mathrm{Lie}G \subseteq \mathrm{End}(V) = V^\vee \otimes V$. Hence the weight of $(\mathrm{Lie}G, h)$ is 0.

In what follows, we use $\mathfrak{g}$ to denote $\mathrm{Lie}G$.

Proposition 1.3.5. *There exists a unique complex structure on* X^+ *such that* F^p *is holomorphic for each* p. *Griffiths' transversality holds true if and only if the Hodge structure* $(\mathrm{Lie}G, h)$ *has type* $(-1,1) + (0,0) + (1,-1)$ *for one (and hence all)* $h \in X^+$.

Proof. For each $h \in X^+$, let $F^\bullet_h$ be the Hodge filtration of the Hodge structure (V,h). For each p, write $d_p := \dim F^p_h = \sum_{r \geq p} h^{r,n-r}$ which does not depend on h. We have a flag variety $\mathcal{F}\ell$ parametrizing sequences (called *flags*) $\cdots \supseteq V_p \supseteq V_{p+1} \supseteq \cdots$ of subspaces of $V_{\mathbb{C}}$ with $\dim V_p = d_p$ for each p. By general theory, $\mathcal{F}\ell$ is a complex algebraic variety which is a $\mathrm{GL}(V_{\mathbb{C}})$-orbit. Moreover, the tangent space of $\mathcal{F}\ell$ at the flag $\cdots \supseteq V_p \supseteq V_{p+1} \supseteq \cdots$ is a subspace of

$$\bigoplus_p \mathrm{Hom}(V_p, V_{\mathbb{C}}/V_p). \tag{1.3.3}$$

[3]In fact, X^+ is known to be a connected component of $X \subseteq \mathcal{M}$ which parametrizes all Hodge structures on V for which each s_α is a Hodge tensor.

There is a natural map

$$\varphi\colon X^+ \to \mathcal{F}\ell, \quad h \mapsto F_h^\bullet,$$

which is injective since a Hodge structure is uniquely determined by its Hodge filtration. The group $\mathrm{GL}(V_\mathbb{C})$ naturally acts on $\mathcal{F}\ell$, and it is not hard to check that the stabilizer of $F_h^\bullet$ is $\exp F_h^0\mathrm{End}(V_\mathbb{C})$.

Let us show that φ makes X^+ into a complex subvariety of $\mathcal{F}\ell$. Fix $h_0 \in X^+$ and let $K_\infty := \mathrm{Stab}_{G^+}(h_0)$. Then $X^+ = G^+ \cdot h_0 \simeq G^+/K_\infty$, and $\mathrm{Lie}K_\infty = \mathfrak{g} \cap F_{h_0}^0\mathfrak{g}_\mathbb{C}$ which is the $(0,0)$-constituent of the Hodge structure $(\mathfrak{g}, h_0)$. So φ factors through

$$X^+ = G^+/K_\infty \to X^\vee := G(\mathbb{C})/\exp F_{h_0}^0\mathfrak{g}_\mathbb{C} \to \mathcal{F}\ell \simeq \mathrm{GL}(V_\mathbb{C})/\exp F_{h_0}^0\mathrm{End}(V). \tag{1.3.4}$$

The first map makes X^+ into an open submanifold of $X^\vee$, and the second map is a closed immersion as complex algebraic varieties. So X^+ has a natural complex structure induced from $X^\vee$.

Next we turn to the Griffiths' transversality. The tangent map of φ at h_0 is

$$\mathrm{d}\varphi\colon T_{h_0}X^+ \to T_{h_0}\mathcal{F}\ell \simeq \mathrm{End}(V_\mathbb{C})/F_{h_0}^0\mathrm{End}(V_\mathbb{C}) \subseteq \bigoplus_p \mathrm{Hom}(F_{h_0}^p, V_\mathbb{C}/F_{h_0}^p).$$

Griffiths' transversality holds true if and only if

$$\mathrm{im}(\mathrm{d}\varphi) \subseteq \bigoplus_p \mathrm{Hom}(F_{h_0}^p, F_{h_0}^{p-1}/F_{h_0}^p),$$

and hence if and only if

$$\mathrm{im}(\mathrm{d}\varphi) \subseteq F_{h_0}^{-1}\mathrm{End}(V_\mathbb{C})/F_{h_0}^0\mathrm{End}(V_\mathbb{C}).$$

But $\mathrm{im}(\mathrm{d}\varphi) = \mathrm{Lie}G_\mathbb{C}/F_{h_0}^0\mathfrak{g}_\mathbb{C}$. So Griffiths' transversality holds true if and only if $\mathfrak{g}_\mathbb{C} = F_{h_0}^{-1}\mathfrak{g}_\mathbb{C}$. Therefore we can conclude. $\square$

We yet to understand the polarization attached to this family, for which we need to recall some background knowledge on reductive groups. The full discussion will be carried out in §2.2.