

Chapter 6

Baily–Borel compactification

Let (\mathbf{G}, X) be a Shimura datum. By abuse of notation use X to denote a connected component. Let $\Gamma < \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup.

Throughout the whole chapter, we will assume \mathbf{G} to be quasi-simple, *i.e.* \mathbf{G}^{der} is a simple group. For the purpose of compactifying $\Gamma \backslash X$ we can easily reduce to this case. Notice that $\mathbf{G}_{\mathbb{R}}^{\text{der}}$ may not be simple as an \mathbb{R} -group, so that X is not necessarily irreducible.

6.1 Baily–Borel compactification as a topological space

Consider the Harish–Chandra embedding

$$X \simeq \mathcal{D} \subseteq \mathfrak{m}^+ \simeq \mathbb{C}^N$$

with $\overline{\mathcal{D}}$ the closure of \mathcal{D} in \mathfrak{m}^+ . Let F be an analytic boundary component of X , with normalizer $N(F) = \{g \in \mathbf{G}^{\text{der}}(\mathbb{R})^+ : gF = F\}$.

Recall [5.1.3] the fundamental set Σ constructed from Siegel sets associated with a minimal rational proper parabolic subgroup \mathbf{P}_0 of \mathbf{G}^{der} .

Let $\overline{\Sigma}$ be the closure of $\Sigma \subseteq X \simeq \mathcal{D} \subseteq \mathfrak{m}^+$. Then $\overline{\Sigma} \subseteq \overline{\mathcal{D}}$, with an induced topology.

Theorem 6.1.1. *The followings are equivalent:*

- (1) $\Gamma F \cap \overline{\Sigma} \neq \emptyset$,
- (2) $\mathbf{P}_{0,\mathbb{R}} < N(F)$, and F is a rational analytic boundary component (*i.e.* $N(F)$ equals $\mathbf{P}_F(\mathbb{R})$ for a parabolic subgroup \mathbf{P}_F of \mathbf{G}^{der}).

Moreover in these cases, \mathbf{P}_F is a maximal proper parabolic subgroup of \mathbf{G}^{der} .

Theorem 6.1.1 indicates that we can do the following compactification of $\Gamma \backslash X$:

- (i) Define $\overline{X}^{\text{BB}} := \mathcal{D} \cup \bigsqcup_{\mathbf{P}} F_{\mathbf{P}} \subseteq \overline{\mathcal{D}}$, where \mathbf{P} runs over all *maximal* proper parabolic subgroup of \mathbf{G}^{der} and $F_{\mathbf{P}}$ is the rational analytic boundary component $F_{\mathbf{P}}$.
- (ii) Endow \overline{X}^{BB} with the Satake topology.
- (iii) The space $\Gamma \backslash \overline{X}^{\text{BB}}$ is then a compact Hausdorff space containing $\Gamma \backslash X$ as an open dense subset.

Then $(\Gamma \backslash \overline{X})^{\text{BB}} := \Gamma \backslash \overline{X}^{\text{BB}}$ is called the *Baily–Borel compactification* of $\Gamma \backslash X$, and

$$(\Gamma \backslash \overline{X})^{\text{BB}} = \Gamma \backslash X \cup \bigsqcup_{j=1}^m \Gamma_{F_j} \backslash F_j, \quad (6.1.1)$$

where F_1, \dots, F_m are rational analytic boundary components such that $\{\mathbf{P}_{F_1}, \dots, \mathbf{P}_{F_m}\}$ is a set of representatives of Γ -conjugacy classes of maximal proper parabolic subgroups of \mathbf{G}^{der} , with $\Gamma_{F_j} := \Gamma \cap \mathbf{P}_{F_j}(\mathbb{Q})$.

6.1.1 Satake topology on \overline{X}^{BB}

The *Satake topology* on \overline{X}^{BB} is defined as follows. For each $x \in \overline{X}^{\text{BB}} \subseteq \overline{\mathcal{D}}$, the neighborhoods of any point $x \in X^*$ is the saturations of the neighborhoods of the corresponding points in $\overline{\Sigma}$ under the action of $\Gamma_x := \{\gamma \in \Gamma : \gamma \cdot x = x\}$. More precisely, a fundamental system of neighborhoods of x is given by all subsets $U \subseteq \overline{\mathcal{D}}$ such that

$$\Gamma_x \cdot U = U,$$

and such that $\gamma U \cap \overline{\Sigma}$ is a neighborhood of $\gamma \cdot x$ in $\overline{\Sigma}$ whenever $\gamma \cdot x \in \overline{\Sigma}$.

Proposition 6.1.2. *The Satake topology is the unique topology on \overline{X}^{BB} such that the followings hold:*

(i) *it induces the original topologies on $\overline{\Sigma}$ and on X ,*

(ii) *the $\mathbf{G}^{\text{der}}(\mathbb{Q})$ -action on \overline{X}^{BB} is continuous,*

(iii) *for any $x \in \overline{X}^{\text{BB}}$, there exists a fundamental system of neighborhoods $\{U\}$ of x such that*

$$\gamma U = U \text{ for all } \gamma \in \Gamma_x \quad \text{and} \quad \gamma U \cap U = \emptyset \text{ for all } \gamma \notin \Gamma_x,$$

(iv) *if $x, x' \in \overline{X}^{\text{BB}}$ are not in one Γ -orbit, then there exist neighborhoods U of x and U' of x' such that*

$$\Gamma U \cap U' = \emptyset.$$

Corollary 6.1.3. $\Gamma \backslash \overline{X}^{\text{BB}}$ is compact and Hausdorff.

6.1.2 \mathbb{Q} -roots vs \mathbb{R} -roots, and \mathbb{Q} -polydisc

Let $\mathbf{S} < \mathbf{P}_0$ be a maximal \mathbb{Q} -split torus, and let K_∞ be a maximal compact subgroup of $\mathbf{G}_{\mathbb{R}}^{\text{der}}$ such that $\text{Lie}K_\infty \cap \text{Lie}\mathbf{S}_{\mathbb{R}} = 0$ ^[1]. Then there exists $x_0 \in X$ such that $\text{Stab}_{\mathbf{G}^{\text{der}}(\mathbb{R})}(x_0) = K_\infty$.

In §4.6.2 and §4.6.3, we discussed the complex roots and real roots respectively. Now we need to analyze the \mathbb{Q} -roots of \mathbf{G}^{der} . First, we can make an appropriate choice of K_∞ such that $\mathbf{S}_{\mathbb{R}} < A$ with A from §4.6.3.

Let ${}_{\mathbb{R}}\Psi = \{\gamma_1, \dots, \gamma_r\}$ be as in (4.6.13); it is obtained from a set of strongly orthogonal complex roots $\Psi = \{\alpha_1, \dots, \alpha_r\}$ from (4.6.4). If $\mathbf{G}_{\mathbb{R}}^{\text{der}}$ is simple, we described the real roots ${}_{\mathbb{R}}\Phi = \Phi(A, \mathbf{G}_{\mathbb{R}}^{\text{der}})$ in terms of $\gamma_1, \dots, \gamma_r$ in Proposition 4.6.12. It turns out that one can also do this for the rational roots ${}_{\mathbb{Q}}\Phi := \Phi(\mathbf{S}, \mathbf{G}^{\text{der}})$ when \mathbf{G}^{der} is simple.

^[1]Even strongly, $\text{Lie}K_\infty$ is orthogonal to $\text{Lie}\mathbf{S}_{\mathbb{R}}$ for the Killing form on $\text{Lie}\mathbf{G}_{\mathbb{R}}^{\text{der}}$.

Proposition 6.1.4. *Let $s = \dim \mathbf{S}$. There is a partition*

$$\{1, \dots, r\} = I_0 \cup I_1 \cup \dots \cup I_s \quad (6.1.2)$$

such that

$$\begin{aligned} X(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{Q} &\simeq \frac{X(A) \otimes_{\mathbb{Z}} \mathbb{Q}}{\{subspace \text{ spanned by } \gamma_i \text{ for } i \in I_0 \text{ and } \gamma_i - \gamma_j \text{ for } i, j \in I_\ell, \ell > 0\}} \\ &\simeq \sum_{\ell=1}^s \mathbb{Q}\beta_\ell, \quad \text{where } \beta_\ell = \text{image of any } \gamma_j \text{ with } j \in I_\ell. \end{aligned}$$

In particular, $\mathbf{S} = \{\gamma_i = 1 \text{ for } i \in I_0; \gamma_i = \gamma_j \text{ for } i, j \in I_\ell, \text{ where } \ell > 0\}$.

Corollary 6.1.5. *Recall our assumption that \mathbf{G}^{der} is simple. One of the two cases occurs:*

- (Type C_s) ${}_{\mathbb{Q}}\Phi = \{\pm \frac{1}{2}(\beta_i + \beta_j) \text{ for } i \geq j, \pm \frac{1}{2}(\beta_i - \beta_j) \text{ for } i > j\}$.
- (Type BC_s) ${}_{\mathbb{Q}}\Phi = \{\pm \frac{1}{2}(\beta_i + \beta_j) \text{ for } i \geq j, \pm \frac{1}{2}(\beta_i - \beta_j) \text{ for } i > j, \pm \frac{1}{2}\beta_i\}$.

If we order the roots such that $\beta_1 > \dots > \beta_s$, then the set of simple roots is:

- (Type C_s) ${}_{\mathbb{Q}}\Delta = \{\nu_1 := \frac{1}{2}(\beta_1 - \beta_2), \dots, \nu_{s-1} := \frac{1}{2}(\beta_{s-1} - \beta_s), \nu_s := \beta_s\}$.
- (Type BC_s) ${}_{\mathbb{Q}}\Delta = \{\nu_1 := \frac{1}{2}(\beta_1 - \beta_2), \dots, \nu_{s-1} := \frac{1}{2}(\beta_{s-1} - \beta_s), \nu_s := \frac{1}{2}\beta_s\}$.

The proof goes as follows: We have the group-theoretic result that $\mathbf{G}^{\text{der}} = \text{Res}_{k/\mathbb{Q}} \mathbf{G}'$ for some absolutely simple k -group \mathbf{G}' with k a totally real number field. Then $\mathbf{G}_{\mathbb{R}}^{\text{der}} = \prod_{\sigma: k \hookrightarrow \mathbb{R}} \mathbf{G}'_\sigma$ with each \mathbf{G}'_σ a group defined over $\sigma(k) \subseteq \mathbb{R}$. Then one analyzes each factor and use the Galois action.

Next we turn to the \mathbb{Q} -polydisc. Recall from the Polydisc Theorem (Theorem 4.6.7) that we have a totally geodesic embedding $D^r \rightarrow X$ (with $D = \{z \in \mathbb{C} : |z| < 1\}$ the Poincaré unit disc) arising from a group morphism

$$\varphi: \text{SL}_2(\mathbb{R})^r \rightarrow \mathbf{G}^{\text{der}}(\mathbb{R}), \quad (6.1.3)$$

and $X = K_\infty \cdot D^r$. This embedding gives rise to the analytic boundary components as in the diagram (4.6.15). Let us rephrase it here. Recall $\mathfrak{H} \simeq D$ with the Cayley transformation sending $\sqrt{-1} \mapsto 0$ and $\infty \mapsto 1$. Then we have the diagram

$$\begin{array}{ccc} \mathfrak{H}^r & \xrightarrow{f_1} & \mathcal{D} \\ \subseteq \downarrow & & \downarrow \subseteq \\ (\mathbb{P}^1)^r & \xrightarrow{f_3} & X^\vee \end{array} \quad (6.1.4)$$

where f_1 is the natural composite $\mathfrak{H}^r \simeq D^r \rightarrow X \simeq \mathcal{D}$, with $D^r \rightarrow X$ the geodesic embedding as above and $X \simeq \mathcal{D}$ the Harish–Chandra realization, and $\mathcal{D} \subseteq X^\vee$ from (4.6.3). Then for any subset $S \subseteq \{1, \dots, r\}$, the unique standard analytic boundary component containing the point $f_3((\sqrt{-1})_{j \notin S}, (\infty)_{j \in S})$ is F_S . In general, an analytic boundary component of X is of the form $g \cdot F_S$ for some $g \in \mathbf{G}^{\text{der}}(\mathbb{R})$.

We wish to do this discussion and obtain the relevant results over \mathbb{Q} . First of all, any rational analytic boundary component is easily seen to be of the form $g \cdot F_S$, with $g \in \mathbf{G}^{\text{der}}(\mathbb{Q})$ and F_S rational. Next we prove the following lemma.

Lemma 6.1.6. *For $S \subseteq \{1, \dots, r\}$, the standard analytic boundary component F_S is rational if and only if $S = I_{\ell_1} \cup \dots \cup I_{\ell_t}$, where $1 \leq \ell_1 < \dots < \ell_t \leq s$, for the partition (6.1.2).*

Proof. For the proof, it is more convenient to use the description of parabolic subgroups given by Theorem 4.2.5. In §4.6.6, we explained that the normalizer $P_{F_S} = P(\lambda_S)$, with $\lambda_S: \mathbb{G}_{m,\mathbb{R}} \rightarrow \mathbf{G}_\mathbb{R}^{\text{der}}$ sending

$$t \mapsto \varphi(\underbrace{\dots, 1, \dots, \dots}_{j \notin S}, \underbrace{\left[\begin{array}{cc} t & 0 \\ 0 & t^{-1} \end{array} \right], \dots}_{j \in S}),$$

with φ from (6.1.3). By Proposition 6.1.4, λ is defined over \mathbb{Q} if and only if $S = I_{\ell_1} \cup \dots \cup I_{\ell_t}$ for some $1 \leq \ell_1 < \dots < \ell_t \leq s$. We are done. \square

With this lemma in hand, we obtain the \mathbb{Q} -version of (6.1.4)

$$\begin{array}{ccc} \mathfrak{H}^s & \xrightarrow{f_{1,\mathbb{Q}}} & \mathcal{D} \\ \subseteq \downarrow & & \downarrow \subseteq \\ (\mathbb{P}^1)^s & \xrightarrow{f_{3,\mathbb{Q}}} & X^\vee \end{array} \quad (6.1.5)$$

arising from

$$\varphi_\mathbb{Q}: \text{SL}_2(\mathbb{R})^s \rightarrow \mathbf{G}_\mathbb{R}^{\text{der}} \quad (6.1.6)$$

such that $\varphi_\mathbb{Q}$ (diagonal matrices) is the maximal \mathbb{Q} -split torus \mathbf{S} of \mathbf{G}^{der} . We can renumber the factors of \mathfrak{H}^s and $\text{SL}_2(\mathbb{R})^s$ such that: For the $\beta_1, \dots, \beta_s \in \mathbb{Q}\Phi$ from Proposition 6.1.4, we have

$$\beta_\ell: \varphi_\mathbb{Q} \left(\left[\begin{array}{cc} t_1 & 0 \\ 0 & t_1^{-1} \end{array} \right], \dots, \left[\begin{array}{cc} t_s & 0 \\ 0 & t_s^{-1} \end{array} \right] \right) \mapsto t_\ell^2. \quad (6.1.7)$$

Now for each subset $S_\mathbb{Q} \subseteq \{1, \dots, s\}$, the unique standard analytic boundary component which contains the point

$$f_{3,\mathbb{Q}}(\underbrace{\dots, \sqrt{-1}, \dots, \dots, \infty, \dots}_{\ell \notin S_\mathbb{Q}}, \underbrace{\dots, \infty, \dots}_{\ell \in S_\mathbb{Q}})$$

is F_S with $S = \bigcup_{\ell \in S_\mathbb{Q}} I_\ell$. In particular, F_S is rational.

Proof of Theorem 6.1.1. Assume F meets $\bar{\Sigma}$.

Order the roots such that $\beta_1 > \dots > \beta_s$, then $\mathbf{S}(\mathbb{R})^+$ consists of

$$\varphi_\mathbb{Q} \left(\left[\begin{array}{cc} t_1 & 0 \\ 0 & t_1^{-1} \end{array} \right], \dots, \left[\begin{array}{cc} t_s & 0 \\ 0 & t_s^{-1} \end{array} \right] \right)$$

where $t_1 \geq \dots \geq t_s \geq 1$. Hence

$$\overline{\mathbf{S}(\mathbb{R})^+ x_0} = f_{3,\mathbb{Q}} \left(\{(\sqrt{-1}x_1, \dots, \sqrt{-1}x_s) : \infty \geq x_1 \geq \dots \geq x_s \geq 1\} \right).$$

Hence $\overline{\mathbf{S}(\mathbb{R})^+ x_0}$ meets the boundary components F_1, \dots, F_s with

$$f_{3,\mathbb{Q}}(\infty, \sqrt{-1}, \dots, \sqrt{-1}) \in F_1, f_3(\infty, \infty, \sqrt{-1}, \dots, \sqrt{-1}) \in F_2, \dots, f_3(\sqrt{-1}, \sqrt{-1}, \dots, \sqrt{-1}) \in F_s.$$

So $F = F_\ell$ for some $\ell \in \{1, \dots, s\}$. We can compute the normalizer of each F_ℓ as in Theorem 4.6.19, and get

$$N(F_\ell) = \mathbf{P}_{\mathbb{Q}\Delta \setminus \{\nu_\ell\}}(\mathbb{R})$$

for each $\ell \in \{1, \dots, s\}$. Hence we are done. \square

6.2 First step towards the complex structure

6.2.1 A general criterion for a topological space to be complex analytic

Assume V is a compact Hausdorff space which can be written as a disjoint union

$$V = V_0 \sqcup V_1 \sqcup \cdots \sqcup V_m$$

with each V_j an irreducible normal complex analytic space. Assume that $\dim V_0 > \dim V_j$ for all $j \geq 1$, and that V_0 is open dense in V .

Define a sheaf \mathcal{F} of \mathcal{A} -functions on V as follows. For any open subset $U \subseteq V$, a complex-valued continuous function on U is an \mathcal{A} -function if its restriction to each $U \cap V_j$ ($0 \leq j \leq m$) is complex analytic.

Proposition 6.2.1. *Assume:*

- (i) *For each integer $d \geq 1$, the union $V_{(d)} := \bigcup_{\dim V_j \leq d} V_j$ is closed.*
- (ii) *Any $v \in V$ has a countable fundamental set of open neighborhoods $\{U_\ell\}$ such that $U_\ell \cap V_0$ is connected for all ℓ .*
- (iii) *The restriction to V_j of local \mathcal{A} -functions define the structure sheaf of V_j , for all $j \geq 0$.*
- (iv) *Any $v \in V$ has a neighborhood U_v whose points are separated by the \mathcal{A} -functions defined on U .*

Then V is an irreducible normal complex analytic space with structure sheaf \mathcal{F} . For each $d \leq \dim V_0$, the union $V_{(d)}$ is an analytic subspace of V with dimension $\max\{\dim V_j : V_j \subseteq V_{(d)}\}$.

6.2.2 Application to the Baily–Borel compactification

We shall apply Proposition 6.2.1 to the Baily–Borel compactification (6.1.1) (which is compact Hausdorff space by Corollary 6.1.3), with $V_0 = \Gamma \backslash X$ and $V_j = \Gamma_{F_j} \backslash F_j$ for $1 \leq j \leq m$.

Conditions (i) and (ii) can be shown to hold by checking with the Satake topology from §6.1.1.

To check condition (iii), we define the projection

$$\pi_F: X \rightarrow F \tag{6.2.1}$$

for each analytic boundary component F . We focus on the rational ones. The example of the Siegel case will be presented in Example 6.3.6.

Recall our choice of a maximal \mathbb{Q} -split torus \mathbf{S} (from §6.1.2) in our minimal parabolic subgroup \mathbf{P}_0 of \mathbf{G}^{der} (see above Theorem 6.1.1), and the basis ${}_{\mathbb{Q}}\Delta = \{\nu_1, \dots, \nu_s\}$ (see Corollary 6.1.5) of the relative root system ${}_{\mathbb{Q}}\Phi := \Phi(\mathbf{S}, \mathbf{G}^{\text{der}})$. The root ν_s is called the *distinguished root* because it has different length.

Over \mathbb{R} , we explained the relation between F and the boundary symmetric domain associated with $P_F = N(F)$; see below Theorem 4.6.19. The discussion can be carried over \mathbb{Q} .

Let F be a rational analytic boundary component which meets $\bar{\Sigma}$. We have shown in the proof of Theorem 6.1.1 that $\mathbf{P}_F = \mathbf{P}_{\mathbb{Q}\Delta \setminus \{\nu_\ell\}}$ for some $\ell \in \{1, \dots, s\}$. Let $I_h := \{\nu_{\ell+1}, \dots, \nu_s\}$ and $I_l := \{\nu_1, \dots, \nu_{\ell-1}\}$. We thus have the refined rational horospherical decomposition

$$X \simeq N_{P_F}(\mathbb{R}) \times A_{\mathbf{P}_F}(\mathbb{R})^+ \times X_{\mathbf{P}_{I_h}} \times X_{\mathbf{P}_{I_l}}.$$

Moreover, the proof of Theorem 6.1.1 exhausts the possibilities of all F 's, and hence implies that F can be identified with the boundary symmetric domain associated with \mathbf{P}_{I_h} . Thus the refined rational horospherical decomposition above becomes

$$X \simeq N_{P_F}(\mathbb{R}) \times A_{\mathbf{P}_F}(\mathbb{R})^+ \times F \times X_{\mathbf{P}_{I_l}}, \quad (6.2.2)$$

and it induces a natural projection $X \rightarrow F$, which is our desired π_F . Although the decomposition is only real semi-algebraic, the projection π_F is also holomorphic.

If F is contained in \bar{F}' for another rational boundary component F' , then F is a rational boundary component of F' , and one gets a projection $\pi_{F',F}: F' \rightarrow F$. It is not hard to check that π_F is the composite of $\pi_{F',F} \circ \pi_{F'}$.

Now to check condition (iii) of Proposition 6.2.1, we only need to work locally and hence on the universal covering. But now for any rational boundary component F of X , any complex analytic function near F can be extended to an \mathcal{A} -function on a neighborhood of F in \overline{X}^{BB} by the discussion above. This establishes (iii).

Proving condition (iv) is the hardest part. We need to realize X as a Siegel domain of the third kind^[2] and define the Poincaré–Eisenstein series.

6.3 X as a Siegel domain of the third kind

Continue to use the notation from §6.1.2. In particular, \mathbf{S} is a maximal \mathbb{Q} -split torus, \mathbf{P}_0 is a minimal parabolic subgroup of \mathbf{G}^{der} which contains \mathbf{S} . We have the relative root system ${}_{\mathbb{Q}}\Phi = \Phi(\mathbf{S}, \mathbf{G}^{\text{der}})$, the roots $\beta_1 > \dots > \beta_s$ which arise from the set of strongly orthogonal roots, and the basis ${}_{\mathbb{Q}}\Delta = \{\nu_1 = \frac{1}{2}(\beta_1 - \beta_2), \dots, \nu_{s-1} = \frac{1}{2}(\beta_{s-1} - \beta_s)\} \cup \{\nu_s\}$; ν_s is the distinguish roots which is either β_s or $\frac{1}{2}\beta_s$.

Let \mathbf{P} be a standard maximal proper parabolic subgroup of \mathbf{G}^{der} , i.e. $\mathbf{P} > \mathbf{P}_0$. Then $\mathbf{P} = \mathbf{P}_{\mathbb{Q}\Delta \setminus \{\nu_\ell\}}$ for some $\ell \in \{1, \dots, s\}$. We have seen in the proof of Theorem 6.1.1 that $\mathbf{P}(\mathbb{R})$ is the normalizer of the standard rational analytic boundary component $F = F_S$ with $S = I_1 \cup \dots \cup I_\ell$.

The non-standard maximal proper parabolic subgroup of \mathbf{G}^{der} are $\mathbf{G}^{\text{der}}(\mathbb{Q})$ -conjugates of the standard ones, and the rational analytic boundary components are all of the form $g \cdot F_S$ with $g \in \mathbf{G}^{\text{der}}(\mathbb{Q})$ and F_S as above. So all the discussion in this section applies to an arbitrary rational analytic boundary component by applying suitable $\mathbf{G}^{\text{der}}(\mathbb{Q})$ -conjugation.

6.3.1 Rational 5-group decomposition and refined horospherical decomposition

Let $I_h := \{\nu_{\ell+1}, \dots, \nu_s\}$ and $I_l := \{\nu_1, \dots, \nu_{\ell-1}\}$.

Above Example 4.6.20, we defined several subgroups of $\mathbf{P}_{\mathbb{R}}$. We can also define the following \mathbb{Q} -subgroups of \mathbf{P} similarly:

- $\mathbf{W}(F) := \mathcal{R}_u(\mathbf{P})$,
- $\mathbf{L}(F)$ is a suitable Levi subgroup of \mathbf{P} which we shall define later using Lie algebra,
- $\mathbf{G}_h(F)$ is the semi-simple group whose \mathbb{Q} -root system is spanned by I_h . It is normal subgroup of $\mathbf{L}(F)$, has no compact \mathbb{Q} -factors, and F is a $\mathbf{G}_h(F)(\mathbb{R})^+$ -orbit,

^[2]In the original paper of Baily–Borel, this was done using partial Cayley transformation and the map η from Theorem 2.3.5. We will directly introduce the more explicit version using the 5-decomposition of the parabolic subgroup. This more explicit version will be crucial for the toroidal compactifications.

- $\mathbf{G}_l(F)\mathbf{M}(F)$ is the normal reductive subgroup of $\mathbf{L}(F)$ which is the complement to $\mathbf{G}_h(F)$, with $\mathbf{G}_l(F)$ with no compact \mathbb{Q} -factors and $\mathbf{M}(F)(\mathbb{R})$ compact.

Then $\mathbf{P} = \mathbf{W}(F) \rtimes (\mathbf{G}_h(F) \cdot \mathbf{G}_l(F) \cdot \mathbf{M}(F))$.

Remark 6.3.1. Let us compare them with the subgroups of $\mathbf{P}_{\mathbb{R}}$ defined above Example 4.6.20. We have $W(F) = \mathbf{W}(F)_{\mathbb{R}}$ and $L(F) = \mathbf{L}(F)_{\mathbb{R}}$ by definition. Later on, we will see that $G_l(F) = \mathbf{G}_l(F)_{\mathbb{R}}$ in Corollary 6.3.4. However, in general $\mathbf{G}_h(F)_{\mathbb{R}}$ is not $G_h(F)$. In other words, $G_h(F)$ may not be defined over \mathbb{Q} , in which case we have $G_h(F) = \mathbf{G}_h(F)_{\mathbb{R}} \cdot M'$ for some factor M' of $\mathbf{M}(F)_{\mathbb{R}}$.

Denote by $\mathfrak{g} := \text{Lie}\mathbf{G}^{\text{der}}$. Then one can compute that

$$\text{Lie}\mathbf{P} = \mathfrak{g}^S \oplus \sum_{\varphi = \frac{\pm\beta_i \pm \beta_j}{2} \text{ or } \frac{\pm\beta_i}{2}, \ell+1 \leq i, j \leq s} \mathfrak{g}_{\varphi} \oplus \sum_{\varphi = \frac{\gamma_i \pm \gamma_j}{2} \text{ or } \frac{\gamma_i}{2}, 1 \leq i \leq \ell} \mathfrak{g}_{\varphi}.$$

Now we define $\mathbf{L}(F)$ to be the subgroup of \mathbf{P} whose Lie algebra is the direct sum of the first two factors. Then by construction, $\mathbf{L}(F)$ is defined over \mathbb{Q} and $\mathbf{L}(F)_{\mathbb{R}}$ is precisely the $L(F)$ defined above Example 4.6.20.

The Lie algebra of $\mathbf{W}(F) = \mathcal{R}_u(\mathbf{P})$ is, by computation, the direct sum of

$$\mathfrak{u} := \sum_{\substack{\varphi = \frac{\gamma_i + \gamma_j}{2} \\ 1 \leq i, j \leq \ell}} \mathfrak{g}_{\varphi}$$

and

$$\mathfrak{v} := \sum_{\substack{\varphi = \frac{\gamma_i \pm \gamma_j}{2} \text{ or } \frac{\gamma_i}{2} \\ 1 \leq i \leq \ell, \ell+1 \leq j \leq s}} \mathfrak{g}_{\varphi}.$$

Let $\mathbf{U}(F)$ be the exponent of \mathfrak{u} . One can prove:

Lemma 6.3.2. $\mathbf{U}(F)$ is the center of $\mathbf{W}(F)$, and $\mathbf{V}(F) := \mathbf{W}(F)/\mathbf{U}(F)$ is a vector group (i.e. abelian and diffeomorphic to its Lie algebra).

Clearly, $\text{Lie}\mathbf{V}(F)$ can be canonically identified with \mathfrak{v} .

Write $U(F) = \mathbf{U}(F)_{\mathbb{R}}$ and $V(F) = \mathbf{V}(F)_{\mathbb{R}}$. Then the refined rational horospherical decomposition (6.2.2) can be furthermore refined to be

$$X \simeq U(F)(\mathbb{R}) \times V(F)(\mathbb{R}) \times A_{\mathbf{P}}(\mathbb{R})^+ \times F \times X_{\mathbf{P}, l}. \quad (6.3.1)$$

Notice that the \mathbb{R} -split torus $A_{\mathbf{P}}$ is contained in $\mathbf{G}_l(F)$ by definition of $\mathbf{G}_l(F)$, and $X_{\mathbf{P}, l} \simeq (\mathbf{G}_l(F)/A_{\mathbf{P}})(\mathbb{R})^+$. Denote by $K_{l, \infty}$ this maximal compact subgroup of $\mathbf{G}_l(F)(\mathbb{R})^+$, then we have

$$X_{\mathbf{P}, l} \simeq \mathbf{G}_l(F)(\mathbb{R})^+ / K_{l, \infty} A_{\mathbf{P}}(\mathbb{R})^+. \quad (6.3.2)$$

6.3.2 Cone in $U(F)(\mathbb{R})$

We start with the following proposition. Let us go back to \mathbb{R} -groups, and recall the subgroups $G_h(F)$, $G_l(F)$, $M(F)$ of $\mathbf{P}_{\mathbb{R}}$ defined above Example 4.6.20. We have $\mathbf{L}(F)_{\mathbb{R}} = G_h(F) \cdot G_l(F) \cdot M(F)$ as almost direct product.

The group $\mathbf{L}(F)$ acts naturally on $\mathbf{W}(F)$, and hence on $\mathbf{U}(F)$. So $G_h(F)$, $G_l(F)$, and $M(F)$ act on $U(F)$.

Proposition 6.3.3. *The centralizer of the action of $L(F) = G_h(F) \cdot G_l(F) \cdot M(F)$ on $U(F)$ is $G_h(F) \cdot M(F)$.*

Proof. The subgroups $G_h(F)$, $G_l(F)$, and $M(F)$ can all be defined using the (real) root decomposition (4.6.12), for example (4.6.14) for $G_h(F)$. Hence we can prove this proposition by checking roots and direct computation. \square

This proposition immediately yields the following proposition:

Corollary 6.3.4. *The group $G_l(F)$ is defined over \mathbb{Q} , and hence is precisely $\mathbf{G}_l(F)_{\mathbb{R}}$.*

Proof. Since $U(F)$ is defined over \mathbb{Q} and $L(F)$ is defined over \mathbb{Q} , the centralizer of the action of $L(F)$ on $U(F)$ is also defined over \mathbb{Q} . So $G_h(F) \cdot M(F)$ is defined over \mathbb{Q} by Proposition 6.3.3. But $G_l(F)$ is defined to be the complement of $G_h(F) \cdot M(F)$ in $L(F)$. So we are done. \square

For the morphism $\varphi_{\mathbb{Q}}: \mathrm{SL}_2(\mathbb{R})^s \rightarrow \mathbf{G}_{\mathbb{R}}^{\mathrm{der}}$ from (6.1.6), take the point

$$\Omega_F := \varphi_{\mathbb{Q}} \left(\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \dots, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\ell\text{-components}}, 1, \dots, 1 \right) \in U(F)(\mathbb{R}).$$

Consider the action of $G_l(F)(\mathbb{R})^+$ on $U(F)(\mathbb{R})$.

Proposition 6.3.5. $\mathrm{Stab}_{G_l(F)(\mathbb{R})^+}(\Omega_F) = K_{l,\infty}$.

The orbit

$$C(F) := \{g\Omega_F g^{-1} : g \in G_l(F)(\mathbb{R})^+\}$$

is an open symmetric homogeneous cone in $U(F)(\mathbb{R})$.

By (6.3.2), we have $C(F) \simeq A_{\mathbf{P}}(\mathbb{R})^+ \times X_{\mathbf{P},l}$. Hence (6.3.1) becomes

$$X \simeq U(F)(\mathbb{R}) \times C(F) \times V(F)(\mathbb{R}) \times F. \quad (6.3.3)$$

Denote by

$$\Phi_F: X \rightarrow C(F) \quad (6.3.4)$$

the natural projection.

Example 6.3.6. *In the Siegel case, $s = r$ (i.e. the \mathbb{Q} -rank equals the \mathbb{R} -rank), and the partition (6.1.2) is simply $I_0 = \emptyset$ and $I_\ell = \{\ell\}$.*

Take $\mathbf{P} = \mathbf{P}_{\mathbb{Q}\Delta \setminus \{\nu_{d'}\}}$. Then as in Example 4.6.20, we have

$$\begin{aligned}\mathbf{P} &= \left\{ \begin{bmatrix} A' & 0 & B' & * \\ * & u & * & * \\ C' & 0 & D' & * \\ 0 & 0 & 0 & (u^t)^{-1} \end{bmatrix} \in G : \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \in \mathrm{Sp}_{2d', \mathbb{Q}}, u \in \mathrm{GL}_{d-d', \mathbb{Q}} \right\}, \\ \mathbf{W}(F) &= \left\{ \begin{bmatrix} I_{d'} & 0 & 0 & n \\ m^t & I_{d-d'} & n^t & b \\ 0 & 0 & I_{d'} & -m \\ 0 & 0 & 0 & I_{d-d'} \end{bmatrix} : n^t m + b = m^t n + b^t \right\}, \\ \mathbf{G}_h(F) &= \left\{ \begin{bmatrix} A' & 0 & B' & 0 \\ 0 & I_{d-d'} & 0 & 0 \\ C' & 0 & D' & 0 \\ 0 & 0 & 0 & I_{d-d'} \end{bmatrix} : \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \in \mathrm{Sp}_{2d', \mathbb{Q}} \right\} \simeq \mathrm{Sp}_{2d', \mathbb{Q}}, \\ \mathbf{G}_l(F) &= \left\{ \begin{bmatrix} I_{d'} & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & I_{d'} & 0 \\ 0 & 0 & 0 & (u^t)^{-1} \end{bmatrix} : u \in \mathrm{GL}_{d-d', \mathbb{Q}} \right\} \simeq \mathrm{GL}_{d-d', \mathbb{Q}}, \\ \mathbf{M}(F) &= \{\pm I_{2d}\}.\end{aligned}$$

Moreover,

$$\mathbf{U}(F) = \left\{ \begin{bmatrix} I_{d'} & 0 & 0 & 0 \\ 0 & I_{d-d'} & 0 & b \\ 0 & 0 & I_{d'} & 0 \\ 0 & 0 & 0 & I_{d-d'} \end{bmatrix} : b = b^t \right\} \simeq \{b \in \mathrm{Mat}_{(d-d') \times (d-d')}(\mathbb{Q}) : b = b^t\} \ni \Omega_F = I_{d-d'},$$

and

$$C(F) = \{b \in \mathrm{Mat}_{(d-d') \times (d-d')}(\mathbb{R}) : b = b^t, b > 0\}.$$

Notice that $F \simeq \mathfrak{H}_{d'}$ in this case. The projections (6.2.1) and (6.3.4) are

$$\begin{aligned}\pi_F: \mathfrak{H}_d &\rightarrow F \simeq \mathfrak{H}_{d'}, & \begin{bmatrix} \tau' & \tau_0 \\ \tau_0^t & \tau'' \end{bmatrix} &\mapsto \tau' \\ \Phi_F: \mathfrak{H}_d &\rightarrow C(F), & \begin{bmatrix} \tau' & \tau_0 \\ \tau_0^t & \tau'' \end{bmatrix} &\mapsto \mathrm{Im}\tau'' - (\mathrm{Im}\tau_0)^t (\mathrm{Im}\tau')^{-1} (\mathrm{Im}\tau_0).\end{aligned}\tag{6.3.5}$$

6.3.3 Fibered structure

Recall the Harish–Chandra embedding together with the Borel embedding $X \simeq \mathcal{D} \subseteq \mathfrak{m}^+ \subseteq X^\vee$, with X^\vee a $\mathbf{G}^{\mathrm{der}}(\mathbb{C})$ -orbit.

Define

$$\mathcal{D}(F) := \mathbf{U}(F)(\mathbb{C}) \cdot \mathcal{D} = \bigcup_{g \in \mathbf{U}(F)(\mathbb{C})} g \cdot \mathcal{D} \subseteq X^\vee.\tag{6.3.6}$$

Then $\mathcal{D}(F)$ has a natural complex structure, and $U(F)(\mathbb{C})$ acts holomorphically on $\mathcal{D}(F)$. So the quotient

$$\mathcal{D}'(F) := \mathbf{U}(F)(\mathbb{C}) \backslash \mathcal{D}(F)\tag{6.3.7}$$

has a complex structure. Moreover since $\mathcal{D}'(F) \simeq V(F)(\mathbb{R}) \times F$ real semi-algebraically, we have that $\mathcal{D}'(F) \rightarrow F$ is a complex vector bundle, *i.e.* each $x \in F$ determines a complex structure on $V(F)(\mathbb{R})$.

Now we have a holomorphic isomorphism $\mathcal{D}(F) \simeq U(F)(\mathbb{C}) \times \mathcal{D}'(F) = (U(F)(\mathbb{R}) \oplus \sqrt{-1}U(F)(\mathbb{R})) \times \mathcal{D}'(F)$. The cone $C(F)$ should be seen as a cone in $\sqrt{-1}U(F)(\mathbb{R})$.

Theorem 6.3.7. *The projection Φ_F in (6.3.4) extends to $\Phi_F: \mathcal{D}(F) \rightarrow U(F)(\mathbb{R})$ such that $X \simeq \mathcal{D} = \Phi_F^{-1}(C(F))$.*

We have the following $\mathbf{P}(\mathbb{R})$ -equivariant commutative diagram of holomorphic maps

$$\begin{array}{ccc}
 C(F) & \subseteq & U(F)(\mathbb{R}) \\
 \Phi_F \uparrow & & \Phi_F \uparrow \\
 X \simeq \mathcal{D} & \subseteq & \mathcal{D}(F) \\
 & \searrow \pi'_F \mod U(F)(\mathbb{C}) & \\
 & \mathcal{D}'(F) & \\
 \pi_F \downarrow & & p_F \downarrow \\
 & & F
 \end{array}$$

The map p_F is a holomorphic vector bundle with each fiber $\simeq V(F)(\mathbb{R})$ (real semi-algebraically).

Example 6.3.8. Continue with the Siegel case in Example 6.3.6. We have

$$\mathcal{D}(F) \simeq \left\{ \tau = \begin{bmatrix} \tau' & \tau_0 \\ \tau_0^t & \tau'' \end{bmatrix} \in \text{Mat}_{d \times d}(\mathbb{C}) : \tau = \tau^t, \text{ Im}\tau' > 0 \right\}.$$

The map $\Phi_F: \mathcal{D}(F) \rightarrow U(F)(\mathbb{R})$ is defined with the same formula as (6.3.5). The map π'_F is $\mod \tau''$, and the map $p_F \circ \pi'_F$ is $\begin{bmatrix} \tau' & \tau_0 \\ \tau_0^t & \tau'' \end{bmatrix} \mapsto \tau'$.

For $\tau' \in F \simeq \mathfrak{H}_{d'}$, the complex structure on $V(F)(\mathbb{R}) = W(F)(\mathbb{R})/U(F)(\mathbb{R})$ determined by τ' is

$$\begin{bmatrix} I_{d'} & 0 & 0 & n \\ m^t & I_{d-d'} & n^t & b \\ 0 & 0 & I_{d'} & -m \\ 0 & 0 & 0 & I_{d-d'} \end{bmatrix} \mod b \mapsto \tau'm + n.$$