An Introduction to Arakelov Geometry

Ziyang Gao
UCLA

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Chapter 0

Quick Summary on the Height Machine

0.1 Weil height on projective spaces

Let us start with the simplest case. Let $x \in \mathbb{P}^1(\mathbb{Q})$. There is a unique way to write x as $[a : b]$ with $a, b \in \mathbb{Z}$ and $\gcd(a, b) = 1$. Set

$$h(x) := \log \max\{|a|, |b|\}.$$

For a general number field K, we use the following *normalized valuations* at places of K:

(i) For $v \in M_{K,f}$ a non-archimedean place, v is above a prime number $p \in \mathbb{Z}$. We take the absolute value $\|\cdot\|_v \colon K \to \mathbb{R}$ such that $\|p\|_v = p^{-1}$;

(ii) For $v \in M_{K,\infty} = \mathrm{Hom}(K, \mathbb{C})$ an archimedean place, v corresponds to an embedding $\sigma \colon K \to \mathbb{C}$. We take $\|\cdot\|_v \colon K \to \mathbb{R}$ to be $\|x\|_v := |\sigma(x)|^{[K_v:\mathbb{R}]}$.

Notice that $\|\cdot\|_v$ is an absolute value unless v is a complex place, *i.e.* $K_v = \mathbb{C}$.

Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$.

Definition 0.1.1. *Let $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}^n(\overline{\mathbb{Q}})$. The **(absolute logarithmic Weil) height** of x is defined to be*

$$h(\mathbf{x}) := \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} \log \max\{\|x_0\|_v, \ldots, \|x_n\|_v\},$$

where $K \subseteq \overline{\mathbb{Q}}$ is a number field such that $x_j \in K$ for all j.

*We also set $H(\mathbf{x}) := e^{h(\mathbf{x})}$ to be the **multiplicative height**.*

The Weil height is a well-defined function on $\mathbb{P}^n(\overline{\mathbb{Q}})$, *i.e.* it is independent of the choice of K and independent of the choice of the homogeneous coordinates. This can be proved using the product formula. Also one can check that this definition coincides with the one for $\mathbb{P}^1(\mathbb{Q})$ above.

The following properties are of fundamental importance for the Height Machine.

Theorem 0.1.2. *We have:*

- *(Positivity/Lower Bound) $h(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}})$;*

- *(Northcott Property) For each $B \geq 0$ and $D \geq 1$, the following set is a finite set*

$$\{\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}}) : h(\mathbf{x}) \leq B,\ [\mathbb{Q}(\mathbf{x}) : \mathbb{Q}] \leq D\}.$$

Lemma 0.1.3. *The action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathbb{P}^n(\overline{\mathbb{Q}})$ leaves the height invariant. More precisely, for any $\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}})$ and any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we have $h(\sigma(\mathbf{x})) = h(\mathbf{x})$.*

0.2 Height Machine

Let X be an irreducible *projective* variety defined over $\overline{\mathbb{Q}}$. Denote by $\mathbb{R}^{X(\overline{\mathbb{Q}})}$ the set of functions $X(\overline{\mathbb{Q}}) \to \mathbb{R}$, and by $O(1)$ the subset of bounded functions.

The ***Height Machine*** associates to each line bundle $L \in \mathrm{Pic}(X)$ a unique class of functions $\mathbb{R}^{X(\overline{\mathbb{Q}})}/O(1)$, *i.e.* a map

$$\mathbf{h}_X \colon \mathrm{Pic}(X) \to \mathbb{R}^{X(\overline{\mathbb{Q}})}/O(1), \quad L \mapsto \mathbf{h}_{X,L}. \tag{0.2.1}$$

Let $h_{X,L} \colon X(\overline{\mathbb{Q}}) \to \mathbb{R}$ a representative of the class $\mathbf{h}_{X,L}$; it is called a *height function associated with* (X, L).

Construction 0.2.1. *One can construct $h_{X,L}$ as follows. In each case below, $h_{X,L}$ depends on some extra data and hence is not unique. However, it can be shown that any two choices differ by a bounded functions on $X(\overline{\mathbb{Q}})$, and thus the class of $h_{X,L}$ is well-defined.*

(i) If L is very ample, then the global sections of L give rise to a closed immersion $\iota \colon X \to \mathbb{P}^n$ for some n, such that $\iota^*\mathcal{O}(1) \simeq L$. Set $h_{X,L} = h \circ \iota$, with h the Weil height on $\mathbb{P}^n$ from Definition 0.1.1.

(ii) If L is ample, then $L^{\otimes m}$ is very ample for some $m \gg 1$. Set $h_{X,L} = (1/m)h_{X,L^{\otimes m}}$.

(iii) For an arbitrary L, there exist ample line bundles L_1 and L_2 on X such that $L \simeq L_1 \otimes L_2^{\otimes -1}$ by general theory of Algebraic Geometry. Set $h_{X,L} = h_{X,L_1} - h_{X,L_2}$.

Here are some basic properties of the Height Machine. Moreover, the construction (0.2.1) is also uniquely determined by the normalization, additivity, and functoriality.

Proposition 0.2.2. *We have*

- *(Normalization) Let h be the Weil height from Definition 0.1.1. Then for all $\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}})$, we have*

$$h_{\mathbb{P}^n,\mathcal{O}(1)}(\mathbf{x}) = h(\mathbf{x}) + O(1).$$

- *(Additivity) Let L and M be two line bundles on X. Then for all $x \in X(\overline{\mathbb{Q}})$, we have*

$$h_{X,L\otimes M}(x) = h_{X,L}(x) + h_{X,M}(x) + O(1).$$

- *(Functoriality) Let $\phi \colon X \to Y$ be a morphism of irreducible projective varieties and let L be a line bundle on Y. Then for all $x \in X(\overline{\mathbb{Q}})$, we have*

$$h_{X,\phi^*L}(x) = h_{Y,L}(\phi(x)) + O(1).$$

- *(Lower Bound) If $s \in H^0(X, L)$ is a global section, then for all $x \in (X \setminus \mathrm{div}(s))(\overline{\mathbb{Q}})$ we have*

$$h_{X,L}(x) \geq O(1).$$

- *(Northcott property) Assume L is ample. Let K_0 be a number field on which X is defined. Then for any $d \geq 1$ and any constant B, the set*

$$\{x \in X(K) : [K : K_0] \leq d,\ h_{X,L}(x) \leq B\}$$

is a finite set.

The $O(1)$'s that appear in the proposition depend on the varieties, line bundles, morphisms, and the choices of the representatives in the classes of height functions. But they are independent of the points on the varieties.

A natural question arises at this point:

Question: What should one do to get a genuine function $X(\overline{\mathbb{Q}}) \to \mathbb{R}$ from a line bundle L? Or, in other words, to choose a nice representative $h_{X,L}$?

Here is a naive way: one can always fix a representative by fixing every operation needed to define h_L (for example, the basis of $H^0(X, L)$ giving the embedding of X into some $\mathbb{P}^N$ if L is very ample).

In the next section, we will see that a canonical choice of $h_{X,L}$ exists when (X, L) defines a polarized dynamical system, after Néron and Tate.

In general, we use *Arakelov Geometry* for this purpose. This is the main content of this course.

0.3 Normalized height function, after Néron and Tate

Let X be an irreducible projective variety defined over $\overline{\mathbb{Q}}$. Let $L \in \mathrm{Pic}(X)$.

Assume there exists $\phi \colon X \to X$ is a morphism satisfying $\phi^* L \simeq L^{\otimes \alpha}$ for some integer $\alpha > 1$. The following theorem gives a canonical representative of $\mathbf{h}_{X,L}$.

Theorem 0.3.1. *There exists a unique height function*

$$\widehat{h}_{X,\phi,L} \colon X(\overline{\mathbb{Q}}) \to \mathbb{R}$$

with the following properties.

(i) $\widehat{h}_{X,\phi,L}(x) = h_{X,L}(x) + O(1)$ *for all* $x \in X(\overline{\mathbb{Q}})$,

(ii) $\widehat{h}_{X,\phi,L}(\phi(x)) = \alpha \widehat{h}_{X,\phi,L}(x)$ *for all* $x \in X(\overline{\mathbb{Q}})$.

The height function $\widehat{h}_{X,\phi,L}$ depends only on the isomorphism class of L. Moreover, it can be computed as the limit

$$\widehat{h}_{X,\phi,L}(x) = \lim_{n \to \infty} \frac{1}{\alpha^n} h_{X,L}(\phi^n(x)) \tag{0.3.1}$$

with ϕ^n the n-fold iterate of ϕ.

Before moving on to the proof, let us have a digest. The morphism ϕ induces a $\mathbb{Z}$-linear map $\phi^* \colon \mathrm{Pic}(X) \to \mathrm{Pic}(X)$.[1] Tensoring with $\mathbb{R}$ gives a linear map $\phi^* \colon \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ of real vector spaces of finite dimension. Say L is non-trivial. Then the assumption $\phi^* L \simeq L^{\otimes \alpha}$ implies that L is an eigenvector for the eigenvalue α. The assumption $\alpha > 1$ guarantees that the *Tate Limit Process* (0.3.1) will work in the end.

We finish this section by two examples of normalized height.

[1]The "addition" on the group $\mathrm{Pic}(X)$ is $\otimes$.

Example 0.3.2. *Let $X = \mathbb{P}^n$ and $L = \mathcal{O}(1)$. Let $\phi\colon \mathbb{P}^n \to \mathbb{P}^n$ be given by homogeneous polynomials of degree $d > 1$, then $\phi^*\mathcal{O}(1) \simeq \mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$. If $\phi([x_0 : \cdots : x_n]) = [x_0^d : \cdots : x_n^d]$, then one can check that $\widehat{h}_{\mathbb{P}^n,\phi,\mathcal{O}(1)}$ is precisely the Weil height h.*

Notice that ϕ restricted to the algebraic torus $\mathbb{G}_{\mathrm{m}}^n \subseteq \mathbb{P}^n$ is precisely the multiplication-by-d morphism. Using this observation, one can prove the following Kronecker's Theorem: For any $\zeta := (\zeta_1, \ldots, \zeta_n) \in \mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}}) = (\overline{\mathbb{Q}}^*)^n$, we have $h(\zeta) = 0$ if and only if each component ζ_j is a root of unity.

A more important example for the Tate Limit Process (0.3.1) is the definition of the *Néron–Tate heights on abelian varieties.* Let $X = A$ be an abelian variety and L be a symmetric line bundle, *i.e.* $[-1]^*L \simeq L$. Then $[n]^*L \simeq L^{\otimes n^2}$ for the multiplication-by-n map. Taking $n = 2$ gives the *Néron–Tate height on* A, which we denote by $\widehat{h}_{A,L}$.

The following theorem summarizes some important properties of $\widehat{h}_{A,L}$. Notice that by (i), in the definition of the Néron–Tate height we can replace the morphism $[2]\colon A \to A$ by $[n]$ for any $n \geq 2$.

Theorem 0.3.3. *Assume L is ample.*

(i) For each $N \in \mathbb{Z}$, we have $\widehat{h}_{A,L}([N]x) = N^2\widehat{h}_{A,L}(x)$ for all $x \in A(\overline{\mathbb{Q}})$.

(ii) $\widehat{h}_{A,L}(x) \geq 0$ for all $x \in A(\overline{\mathbb{Q}})$, and $\widehat{h}_{A,L}(x) = 0$ if and only if x is a torsion point.

(iii) For each finitely generated subgroup Γ of $A(\overline{\mathbb{Q}})$, the $\mathbb{R}$-linearly extension of $\widehat{h}_{A,L}$ is a quadratic form on $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ which is furthermore positive definite.

Chapter 1

Hermitian line bundles on $\mathrm{Spec}\mathcal{O}_K$ and positivity

In the whole chapter, let K be a number field and $\mathcal{O}_K$ be its ring of integers. It is known that $\mathrm{Spec}\mathcal{O}_K$ is not a projective scheme. A key idea in Arakelov Geometry is to identify $\mathrm{Spec}\mathcal{O}_K$ with the set of finite places of K and then compactify $\mathrm{Spec}\mathcal{O}_K$ by adding the archimedean places.

1.1 Hermitian line bundles and arithmetic divisors on $\mathrm{Spec}\mathcal{O}_K$

Definition 1.1.1. *A* **Hermitian line bundle** *on* $\mathrm{Spec}\mathcal{O}_K$ *is a pair* $\overline{\mathcal{L}} := (\mathcal{L}, \|\cdot\|)$*, where* $\mathcal{L}$ *is a line bundle on* $\mathrm{Spec}\mathcal{O}_K$ *and* $\|\cdot\| = \{\|\cdot\|_\sigma\}_{\sigma\colon K\hookrightarrow\mathbb{C}}$ *is a collection of Hermitian metrics* $\|\cdot\|_\sigma$ *on each* $\mathcal{L}_\sigma = H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L}) \otimes_\sigma \mathbb{C}$ *satisfying* $\|s\|_\sigma = \|s\|_{\overline{\sigma}}$ *for all* $s \in H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})$*.*

We say that such collections of metrics are *invariant under complex conjugation*. Notice that $H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})$ is a projective $\mathcal{O}_K$-module of rank 1, and each $\mathcal{L}_\sigma$ is a $\mathbb{C}$-vector space of dimension 1. Thus $\|\cdot\|_\sigma$ is determined by $\|s\|_\sigma$ for any non-zero rational section s of $\mathcal{L}$.

Next we introduce the *group of isometric classes of Hermitian line bundles* on $\mathrm{Spec}\mathcal{O}_K$, denoted by $\widehat{\mathrm{Pic}}(\mathrm{Spec}\mathcal{O}_K)$. The identity element is the trivial Hermitian line bundle, the multiplication is the tensor product, and the inverse is the dual.

Definition 1.1.2. *(i) An* **isometry** *between two Hermitian line bundles* $\overline{\mathcal{L}}$ *and* $\overline{\mathcal{L}}'$ *on* $\mathrm{Spec}\mathcal{O}_K$ *is an isomorphism*

$$i\colon \mathcal{L} \to \mathcal{L}'$$

of line bundles on $\mathrm{Spec}\mathcal{O}_K$ *satisfying*

$$\|s\|_\sigma = \|i(s)\|_\sigma, \qquad \forall s \in H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L}),\ \forall\sigma\colon K \hookrightarrow \mathbb{C}.$$

(ii) The **trivial Hermitian line bundle** *on* $\mathrm{Spec}\mathcal{O}_K$ *is* $(\mathcal{O}_{\mathrm{Spec}\mathcal{O}_K}, |\cdot|)$ *where* $|\cdot|_\sigma$ *is the absolute value at each archimedean place* σ*.*

(iii) The **tensor product** *of two Hermitian line bundles* $\overline{\mathcal{L}}$ *and* $\overline{\mathcal{L}}'$ *on* $\mathrm{Spec}\mathcal{O}_K$ *is defined to be*

$$\overline{\mathcal{L}} \otimes \overline{\mathcal{L}}' := (\mathcal{L} \otimes \mathcal{L}', \|\cdot\|\|\cdot\|').$$

(iv) The **dual** *of a Hermitian line bundle* $\overline{\mathcal{L}}$ *on* $\mathrm{Spec}\mathcal{O}_K$ *is defined to be*

$$\overline{\mathcal{L}}^\vee := (\mathcal{L}^\vee, \|\cdot\|^\vee)$$

where $\mathcal{L}^\vee := \mathrm{Hom}(\mathcal{L}, \mathcal{O}_{\mathrm{Spec}\mathcal{O}_K})$ *and, for each* $t \in H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L}^\vee)$,

$$\|t\|_\sigma^\vee := \frac{|t(s)|}{\|s\|_\sigma} \quad \text{for any non-zero } s \in \mathcal{L}_\sigma.$$

Definition-Lemma 1.1.3. *Let* $\overline{\mathcal{L}}$ *be a Hermitian line bundle on* $\mathrm{Spec}\mathcal{O}_K$. *For any non-zero* $s \in H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})$, *the number*

$$\widehat{\deg}(\overline{\mathcal{L}}) := \log \# \left(H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})/s\mathcal{O}_K\right) - \sum_{\sigma \colon K \hookrightarrow \mathbb{C}} \log \|s\|_\sigma \tag{1.1.1}$$

does not depend on the choice of s.

This number $\widehat{\deg}(\overline{\mathcal{L}})$ *is called the* **arithmetic degree** *of* $\overline{\mathcal{L}}$.

The proof is an application of the product formula. We will postpone it to Proposition 1.1.7, using the relation between Hermitian line bundles and arithmetic divisors introduced below.

In Algebraic Geometry, line bundles and (Cartier) divisors are closely related. In Arakelov Geometry, we also have the notion of arithmetic divisors.

Definition 1.1.4. *An* **arithmetic divisor** *is a formal finite sum*

$$\overline{D} = \sum_{\mathfrak{p} \in M_{K,f}} n_\mathfrak{p}[\mathfrak{p}] + \sum_{\sigma \colon K \hookrightarrow \mathbb{C}} n_\sigma[\sigma] \tag{1.1.2}$$

with $n_\mathfrak{p} \in \mathbb{Z}$, $n_\sigma \in \mathbb{R}$ *and* $n_\sigma = n_{\overline{\sigma}}$.

A **principal arithmetic divisor** *is of the form*

$$\widehat{\mathrm{div}}(\alpha) := \sum_{\mathfrak{p} \in M_{K,f}} \mathrm{ord}_\mathfrak{p}(\alpha)[\mathfrak{p}] - \sum_{\sigma \colon K \hookrightarrow \mathbb{C}} \log |\sigma(\alpha)|[\sigma]$$

for some $\alpha \in K^*$.

In (1.1.2), we usually denote by $D_{\mathrm{f}} := \sum_{\mathfrak{p} \in M_{K,f}} n_\mathfrak{p}[\mathfrak{p}]$ the *finite part of* $\overline{D}$ and by $D_\infty := \sum_{\sigma \colon K \hookrightarrow \mathbb{C}} n_\sigma[\sigma]$ the *infinite part of* $\overline{D}$.

We will also introduce the following groups, where the group law is clear:

$$\begin{aligned}
\widehat{\mathrm{Div}}(\mathrm{Spec}\mathcal{O}_K) &:= \{\text{arithmetic divisors on } \mathrm{Spec}\mathcal{O}_K\}, \\
\widehat{\mathrm{Prin}}(\mathrm{Spec}\mathcal{O}_K) &:= \{\text{principal arithmetic divisors on } \mathrm{Spec}\mathcal{O}_K\}, \\
\widehat{\mathrm{Cl}}(\mathrm{Spec}\mathcal{O}_K) &:= \widehat{\mathrm{Div}}(\mathrm{Spec}\mathcal{O}_K)/\widehat{\mathrm{Prin}}(\mathrm{Spec}\mathcal{O}_K).
\end{aligned}$$

Definition 1.1.5. *The* **arithmetic degree** *of an arithmetic divisor* $\overline{D}$ *of the form* (1.1.2) *is defined to be*

$$\widehat{\deg}(\overline{D}) := \sum_{\mathfrak{p} \in M_{K,f}} n_\mathfrak{p} \log \#(\mathcal{O}_K/\mathfrak{p}) + \sum_{\sigma \colon K \hookrightarrow \mathbb{C}} n_\sigma.$$

The product formula immediately implies that any principal arithmetic divisor has arithmetic degree 0. Thus we get a group homomorphism

$$\widehat{\deg} \colon \widehat{\mathrm{Cl}}(\mathrm{Spec}\mathcal{O}_K) \to \mathbb{R}. \tag{1.1.3}$$

Proposition 1.1.6. *We have a group homomorphism*

$$\widehat{\mathrm{Div}}(\mathrm{Spec}\mathcal{O}_K) \to \widehat{\mathrm{Pic}}(\mathrm{Spec}\mathcal{O}_K), \quad \overline{D} \mapsto \mathcal{O}(\overline{D}), \tag{1.1.4}$$

where for $\overline{D} = D_{\mathrm{f}} + \sum_\sigma n_\sigma[\sigma]$, the Hermitian line bundle $\mathcal{O}(\overline{D})$ is defined to be $(\mathcal{O}(D_{\mathrm{f}}), \|\cdot\|_\sigma)$ with $\|1\|_\sigma := \exp(-n_\sigma)$ for the canonical 1 of $\mathcal{O}(D_{\mathrm{f}})$ (i.e. the divisor of 1 is D_{f}).

And this group homomorphism induces a group isomorphism

$$\widehat{\mathrm{Cl}}(\mathrm{Spec}\mathcal{O}_K) \xrightarrow{\sim} \widehat{\mathrm{Pic}}(\mathrm{Spec}\mathcal{O}_K). \tag{1.1.5}$$

The inverse of (1.1.5) is called the *arithmetic first Chern class* and is denoted by $\widehat{c}_1$.

By constructions, the group homomorphism (1.1.4) is compatible with the forgetful maps $\widehat{\mathrm{Div}}(\mathrm{Spec}\mathcal{O}_K) \to \mathrm{Div}(\mathrm{Spec}\mathcal{O}_K)$, $\overline{D} \mapsto D_{\mathrm{f}}$, and $\widehat{\mathrm{Pic}}(\mathrm{Spec}\mathcal{O}_K) \to \mathrm{Pic}(\mathrm{Spec}\mathcal{O}_K)$, $\overline{\mathcal{L}} \mapsto \mathcal{L}$. Thus the isomorphism (1.1.5) is an extension of the natural isomorphism $\mathrm{Cl}(\mathcal{O}_K) \simeq \mathrm{Pic}(\mathrm{Spec}\mathcal{O}_K)$.

Proof. It is easy to check that (1.1.4) is a group homomorphism.

For any $\widehat{\mathrm{div}}(\alpha) \in \widehat{\mathrm{Prin}}(\mathrm{Spec}\mathcal{O}_K)$, it is not hard to check that the isomorphism $\alpha\colon \mathcal{O}_{\mathrm{Spec}\mathcal{O}_K} \to \mathcal{O}(\mathrm{div}(\alpha))$ induces an isometry between the trivial Hermitian line bundle on $\mathrm{Spec}\mathcal{O}_K$ and $\widehat{\mathrm{div}}(\alpha)$. Thus we have a group homomorphism $\widehat{\mathrm{Cl}}(\mathrm{Spec}\mathcal{O}_K) \to \widehat{\mathrm{Pic}}(\mathrm{Spec}\mathcal{O}_K)$.

The inverse is defined as follows. For any $\overline{\mathcal{L}} \in \widehat{\mathrm{Pic}}(\mathrm{Spec}\mathcal{O}_K)$, let s be a non-zero rational section of $\mathcal{L}_K$ and set

$$\widehat{\mathrm{div}}(s) := \mathrm{div}(s) + \sum_\sigma (-\log \|s\|_\sigma)[\sigma]. \tag{1.1.6}$$

If we have two non-zero rational sections s and s', then $s = \alpha s'$ for some $\alpha \in K^*$. Then $\widehat{\mathrm{div}}(s) - \widehat{\mathrm{div}}(s')$ is a principal arithmetic divisor. Thus we obtain a group homomorphism

$$\widehat{\mathrm{Pic}}(\mathrm{Spec}\mathcal{O}_K) \to \widehat{\mathrm{Cl}}(\mathrm{Spec}\mathcal{O}_K), \quad \overline{L} \mapsto \widehat{\mathrm{div}}(s).$$

It is not hard to check that this is the desired inverse. □

Proposition 1.1.7. *The following diagram of group homomorphisms commutes:*

$$\begin{tikzcd} \widehat{\mathrm{Cl}}(\mathrm{Spec}\mathcal{O}_K) \arrow[r, "\sim"] \arrow[d, "\widehat{\deg}"'] & \widehat{\mathrm{Pic}}(\mathrm{Spec}\mathcal{O}_K) \arrow[d, "\widehat{\deg}"] \\ \mathbb{R} \arrow[r, "="] & \mathbb{R}, \end{tikzcd} \tag{1.1.7}$$

where the top arrow is the one induced by (1.1.4).

Proof. By the definitions of the arithmetic degrees ((1.1.1) and Definition 1.1.5) and the inverse of the top arrow (1.1.6), it suffices to prove the following claim. For any non-zero $s \in H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})$, we have

$$\#\left(H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})/s\mathcal{O}_K\right) = \prod_{\mathfrak{p}} \#(\mathcal{O}_K/\mathfrak{p})^{\mathrm{ord}_{\mathfrak{p}}(s)}.$$

Write $M := H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})$. Then for each $\mathfrak{p}$, the localization $M_{\mathfrak{p}}$ is a free $\mathcal{O}_{K,\mathfrak{p}}$-module of rank 1 and $M/s\mathcal{O}_K \simeq \oplus_{\mathfrak{p}} M_{\mathfrak{p}}/s\mathcal{O}_{K,\mathfrak{p}}$. Thus the desired equality holds true. We are done. □

We finish this section by stating a lemma which compares $\widehat{\mathrm{Pic}}(\mathrm{Spec}\mathcal{O}_K) \simeq \widehat{\mathrm{Cl}}(\mathrm{Spec}\mathcal{O}_K)$ and $\mathrm{Pic}(\mathrm{Spec}\mathcal{O}_K) \simeq \mathrm{Cl}(\mathcal{O}_K)$. The proof is easy.

Lemma 1.1.8. *Let $\rho_1, \dots, \rho_{r_1}$ be the real embeddings of K and $\sigma_1, \overline{\sigma}_1, \dots, \sigma_{r_2}, \overline{\sigma}_{r_2}$ be the complex embeddings. Then We have the following exact sequence:*

$$1 \to \mu_K \to \mathcal{O}_K^* \xrightarrow{\log_K} \mathbb{R}^{r_1+r_2} \xrightarrow{\ell} \widehat{\mathrm{Cl}}(\mathrm{Spec}\mathcal{O}_K) \to \mathrm{Cl}(\mathcal{O}_K) \to 1,$$

where μ_K is the group of roots of unities contained in K, $\log_K$ is given by $\alpha \mapsto (\log|\sigma(\alpha)|)_{\sigma\colon K\hookrightarrow\mathbb{C}}$,

$$\ell\colon (a_1, \dots, a_{r_1}, b_1, \dots, b_{r_2}) \mapsto \sum_{i=1}^{r_1} a_i[\rho_i] + \sum_{i=1}^{r_2} b_i([\sigma_i] + [\overline{\sigma}_i]),$$

and $\widehat{\mathrm{Cl}}(\mathrm{Spec}\mathcal{O}_K) \to \mathrm{Cl}(\mathcal{O}_K)$ is the forgetful map.

1.2 Hermitian vector bundles on $\mathrm{Spec}\mathcal{O}_K$

Hermitian vector bundles are higher rank generalizations of Hermitian line bundles, for which there is a rich theory. In this course, we focus on: Even to study Hermitian line bundles on $\mathrm{Spec}\mathcal{O}_K$, it turns out to be sometimes helpful to study the more general Hermitian vector bundles as will be shown in §1.3.

Definition 1.2.1. *A* **Hermitian coherent sheaf** *on $\mathrm{Spec}\mathcal{O}_K$ is a pair $\overline{\mathcal{E}} := (\mathcal{E}, \|\cdot\|)$, where $\mathcal{E}$ is an $\mathcal{O}_K$-module of finite type and $\|\cdot\| = \{\|\cdot\|_\sigma\}_{\sigma\colon K\hookrightarrow\mathbb{C}}$ is a collection of Hermitian metrics $\|\cdot\|_\sigma$ on each $\mathcal{E}_\sigma := \mathcal{E} \otimes_\sigma \mathbb{C}$ such that $\|e\|_\sigma = \|e\|_{\overline{\sigma}}$ for all $e \in \mathcal{E}$ and all $\sigma\colon K \hookrightarrow \mathbb{C}$.*

If moreover $\mathcal{E}$ is a projective $\mathcal{O}_K$-module, then $\overline{\mathcal{E}}$ is called a **Hermitian vector bundle***.*

We define the rank of $\overline{\mathcal{E}}$, denoted by $\mathrm{rk}(\overline{\mathcal{E}})$, to be the rank of $\mathcal{E}$ as an $\mathcal{O}_K$-module. A Hermitian coherent sheaf $\overline{\mathcal{E}}$ on $\mathrm{Spec}\mathcal{O}_K$ is a Hermitian vector bundle if and only if $\mathcal{E}$ is torsion-free.

The category of vector bundles on $\mathrm{Spec}\mathcal{O}_K$ is equivalent to the category projective $\mathcal{O}_K$-modules of finite rank. Using this one sees that any Hermitian line bundle on $\mathrm{Spec}\mathcal{O}_K$ is a Hermitian vector bundle on $\mathrm{Spec}\mathcal{O}_K$.

Definition 1.2.2. *Let $\overline{\mathcal{E}}$ and $\overline{\mathcal{F}}$ be Hermitian coherent sheaves (or Hermitian vector bundles) on $\mathrm{Spec}\mathcal{O}_K$. A* **morphism**

$$\varphi\colon \overline{\mathcal{E}} \to \overline{\mathcal{F}}$$

is a morphism between the underlying projective $\mathcal{O}_K$-modules such that $\|\varphi(e)\|_\sigma \le \|e\|_\sigma$ for all $\sigma\colon K \hookrightarrow \mathbb{C}$ and all $e \in \mathcal{E}_\sigma$.

Thus we can define the category of Hermitian coherent sheaves on $\mathrm{Spec}\mathcal{O}_K$, and the full sub-category of Hermitian vector bundles on $\mathrm{Spec}\mathcal{O}_K$.

1.2.1 Several constructions on $\mathrm{Spec}\mathcal{O}_K$

<u>**Short exact sequence**</u> Let $\overline{\mathcal{E}}$ be a Hermitian coherent sheaf on $\mathrm{Spec}\mathcal{O}_K$.

Let $\mathcal{F}$ be a submodule of $\mathcal{E}$ and consider the quotient $\mathcal{E} \to \mathcal{G} := \mathcal{E}/\mathcal{F}$. The restriction of the Hermitian metrics $\|\cdot\|_\sigma$ to $\mathcal{F}_\sigma$ for each $\sigma\colon K \hookrightarrow \mathbb{C}$ gives rise to a *Hermitian sub-coherent sheaf* $\overline{\mathcal{F}}$ of $\overline{\mathcal{E}}$. The quotient metrics, *i.e.* for each σ and each $g \in \mathcal{G}_\sigma$,

$$\|g\|_{\mathcal{G},\sigma} := \inf_{e\in\mathcal{E}_\sigma,\ e\mapsto g} \|e\|_\sigma,$$

define a *quotient Hermitian coherent sheaf* $\overline{\mathcal{G}}$ of $\overline{\mathcal{E}}$. We have a short exact sequence in the category of Hermitian coherent sheaves on $\mathrm{Spec}\mathcal{O}_K$

$$0 \to \overline{\mathcal{F}} \to \overline{\mathcal{E}} \to \overline{\mathcal{G}} \to 0.$$

If $\mathcal{F} = \mathcal{E}_{\mathrm{tor}}$, then $\overline{\mathcal{G}}$ is a Hermitian vector bundle.

Direct sum Let $\overline{\mathcal{E}}$ and $\overline{\mathcal{F}}$ be Hermitian coherent sheaves on $\mathrm{Spec}\mathcal{O}_K$. The *direct sum* $\overline{\mathcal{E}} \oplus \overline{\mathcal{F}}$ is defined to be $(\mathcal{E} \oplus \mathcal{F}, \|\cdot\|_{\mathcal{E}} + \|\cdot\|_{\mathcal{F}})$. It is a Hermitian vector bundle if both $\overline{\mathcal{E}}$ and $\overline{\mathcal{F}}$ are Hermitian vector bundles. The rank is $\mathrm{rk}(\overline{\mathcal{E}}) + \mathrm{rk}(\overline{\mathcal{F}})$.

Tensor product Let $\overline{\mathcal{E}}$ and $\overline{\mathcal{F}}$ be Hermitian coherent sheaves on $\mathrm{Spec}\mathcal{O}_K$. The *tensor product* $\overline{\mathcal{E}} \otimes \overline{\mathcal{F}}$ is defined to be $(\mathcal{E} \otimes \mathcal{F}, \|\cdot\|_{\mathcal{E}}\|\cdot\|_{\mathcal{F}})$. It is a Hermitian vector bundle if both $\overline{\mathcal{E}}$ and $\overline{\mathcal{F}}$ are Hermitian vector bundles. The rank is $\mathrm{rk}(\overline{\mathcal{E}})\mathrm{rk}(\overline{\mathcal{F}})$.

Dual and homomorphism Let $\overline{\mathcal{E}}$ be a Hermitian coherent sheaf on $\mathrm{Spec}\mathcal{O}_K$. Its *dual* $\overline{\mathcal{E}}^{\vee}$ is defined to be $(\mathcal{E}^{\vee}, \|\cdot\|^{\vee})$, where $\mathcal{E}^{\vee} := \mathrm{Hom}(\mathcal{E}, \mathcal{O}_K)$ and

$$\|v\|_{\sigma} := \sup_{e \in \mathcal{E}_{\sigma}} \frac{|v(e)|_{\sigma}}{\|e\|_{\sigma}} \quad \text{for all } \sigma \in M_{K,\infty} \text{ and all } v \in \mathcal{E}_{\sigma}^{\vee}.$$

It is a Hermitian vector bundle if $\overline{\mathcal{E}}$ is a Hermitian vector bundle. The rank is $\mathrm{rk}(\overline{\mathcal{E}})$.

More generally, let $\overline{\mathcal{E}}$ and $\overline{\mathcal{F}}$ be Hermitian coherent sheaves on $\mathrm{Spec}\mathcal{O}_K$. Then the *homomorphism* $\mathrm{Hom}(\overline{\mathcal{E}}, \overline{\mathcal{F}})$ is defined to be $\overline{\mathcal{E}}^{\vee} \otimes \overline{\mathcal{F}}$. It is a Hermitian vector bundle if both $\overline{\mathcal{E}}$ and $\overline{\mathcal{F}}$ are Hermitian vector bundles. The rank is $\mathrm{rk}(\overline{\mathcal{E}})\mathrm{rk}(\overline{\mathcal{F}})$.

Determinant Let $\overline{\mathcal{E}}$ be a Hermitian vector bundle on $\mathrm{Spec}\mathcal{O}_K$ of rank n. The *determinant* of $\overline{\mathcal{E}}$ is defined to be $\det \overline{\mathcal{E}} := (\bigwedge^n \mathcal{E}, \|\cdot\|_{\det})$, where for each $\sigma\colon K \hookrightarrow \mathbb{C}$, the metric $\|\cdot\|_{\det,\sigma}$ is the unique metric on $(\bigwedge^n \mathcal{E})_{\sigma}$ such that

$$\|e_1 \wedge \cdots \wedge e_n\|_{\det,\sigma} = 1$$

for any orthonormal basis $\{e_1, \ldots, e_n\}$ of the normed Euclidean space $(\mathcal{E}_{\sigma}, \|\cdot\|_{\sigma})$.

Notice that the determinant is always a Hermitian line bundle on $\mathrm{Spec}\mathcal{O}_K$. Now we can define:

Definition 1.2.3. *Let $\overline{\mathcal{E}}$ be a Hermitian vector bundle on* $\mathrm{Spec}\mathcal{O}_K$. *The* **arithmetic degree** *of $\overline{\mathcal{E}}$ is defined to be*

$$\widehat{\deg}(\overline{\mathcal{E}}) := \widehat{\deg}(\det \overline{\mathcal{E}}).$$

Let us look at the example and particularly important case where $K = \mathbb{Q}$. Since the class number of $\mathbb{Q}$ is 1, any projective module of finite rank is a free module. Consider a Hermitian vector bundle $\overline{\mathcal{E}} = (\mathcal{E}, \|\cdot\|)$. Let $\{v_1, \ldots, v_n\}$ be a $\mathbb{Z}$-basis of $\mathcal{E}$. Then $v := v_1 \wedge \cdots \wedge v_n$ is a $\mathbb{Z}$-basis of $\det \mathcal{E} := \bigwedge^n \mathcal{E}$. Thus

$$\widehat{\deg}(\overline{\mathcal{E}}) = \log \#(\det \mathcal{E}/\mathbb{Z}v) - \log \|v\| = -\log \|v\| = -\frac{1}{2} \log \det \left(h(v_i, v_j)\right),$$

where $h(\cdot,\cdot)$ is the Hermitian form on $\mathcal{E}_{\mathbb{C}}$, *i.e.* $h(v', v') = \|v'\|^2$ for all $v' \in \mathcal{E}_{\mathbb{C}}$.

On the other hand, let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathcal{E}_{\mathbb{R}}$. Then we have an isomorphism $\mathcal{E}_{\mathbb{R}} \xrightarrow{\sim} \mathbb{R}^n$ with $\mathcal{E}$ identified with a lattice in $\mathbb{R}^n$. Let $\mathrm{covol}(\mathcal{E}_{\mathbb{R}}/\mathcal{E})$ denote the co-volume of this lattice, namely the volume of any fundamental domain of this lattice for the Lebesgue measure on $\mathcal{E}_{\mathbb{R}}$.

For each i, we have $v_i = \sum_j a_{ij} e_j$ for some $a_{ij} \in \mathbb{R}$. Then $h(v_i, v_j) = \sum_k a_{ik} a_{jk}$. Thus $\det\left(h(v_i, v_j)\right) = \det(A^{\mathrm{t}} A) = \det(A)^2$ for the matrix $A = (a_{ij})$. Therefore we have

$$\widehat{\deg}(\overline{\mathcal{E}}) = -\log \mathrm{covol}(\mathcal{E}_{\mathbb{R}}/\mathcal{E}). \tag{1.2.1}$$

1.2.2 Pullback, pushforward, norm

Let $K \subseteq K'$ be an inclusion of number fields, and $f\colon \mathrm{Spec}\mathcal{O}_{K'} \to \mathrm{Spec}\mathcal{O}_K$ the induced morphism; then f is finite of degree $d := [K' : K]$.

Pullback Let $\overline{\mathcal{E}}$ be a Hermitian vector bundle on $\mathrm{Spec}\mathcal{O}_K$ of rank n. Define its pullback $f^*\overline{\mathcal{E}}$ as follows. First, set $f^*\mathcal{E} := \mathcal{E} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$; then $f^*\mathcal{E}$ is a projective $\mathcal{O}_{K'}$-module of rank n. Next, for any embedding $\sigma'\colon K' \hookrightarrow \mathbb{C}$, its restriction to K (denoted by σ) is an embedding of K into $\mathbb{C}$, and the canonical isomorphism $(f^*\mathcal{E}) \otimes_{\sigma'} \mathbb{C} = \mathcal{E} \otimes_\sigma \mathbb{C}$ gives the desired metric $\|\cdot\|_{\sigma'}$ on $(f^*\mathcal{E})_{\sigma'} = (f^*\mathcal{E}) \otimes_{\sigma'} \mathbb{C}$.

Proposition 1.2.4. *The pullback f^* commutes with direct sums, tensor products, and taking determinants. Moreover,*

$$\widehat{\deg} f^*\overline{\mathcal{E}} = [K' : K]\widehat{\deg}\overline{\mathcal{E}}.$$

Proof. The first claim is easy to check and we leave it as an exercise. To prove the second claim, it then suffices to check for Hermitian line bundles.

Let $\ell \in \mathcal{E} \setminus \{0\}$. Then

$$\widehat{\deg}(\overline{\mathcal{E}}) = \log \#(\mathcal{E}/\ell\mathcal{O}_K) - \sum_{\sigma\colon K\hookrightarrow\mathbb{C}} \log \|\ell\|_\sigma = \sum_{\mathfrak{p}\in M_{K,f}} \mathrm{ord}_\mathfrak{p}(\ell) \log \#(\mathcal{O}_K/\mathfrak{p}) - \sum_{\sigma\colon K\hookrightarrow\mathbb{C}} \log \|\ell\|_\sigma.$$

Thus

$$\begin{aligned}
\widehat{\deg}(f^*\overline{\mathcal{E}}) &= \sum_{\mathfrak{p}'\in M_{K',f}} \mathrm{ord}_{\mathfrak{p}'}(\ell) \log \#(\mathcal{O}_{K'}/\mathfrak{p}') - \sum_{\sigma'\colon K'\hookrightarrow\mathbb{C}} \log \|\ell\|_{\sigma'} \\
&= \sum_{\mathfrak{p}\in M_{K,f}} \sum_{\mathfrak{p}'|\mathfrak{p}} \mathrm{ord}_{\mathfrak{p}'}(\ell) \log \#(\mathcal{O}_{K'}/\mathfrak{p}') - \sum_{\sigma\colon K\hookrightarrow\mathbb{C}} \sum_{\sigma'|\sigma} \log \|\ell\|_{\sigma'} \\
&= \sum_{\mathfrak{p}\in M_{K,f}} d \cdot \mathrm{ord}_\mathfrak{p}(\ell) \log \#(\mathcal{O}_K/\mathfrak{p}) - \sum_{\sigma\colon K\hookrightarrow\mathbb{C}} d \log \|\ell\|_\sigma \\
&= d \cdot \widehat{\deg}(\overline{\mathcal{E}}).
\end{aligned}$$

We are done. ☐

Pushforward Let $\overline{\mathcal{E}}'$ be a Hermitian vector bundle on $\mathrm{Spec}\mathcal{O}_{K'}$ of rank n. Define its pushforward $f_*\overline{\mathcal{E}}'$ as follows. First, the underlying projective module $f_*\mathcal{E}'$ is set to be $\mathcal{E}'$, viewed as an $\mathcal{O}_K$-module of rank dn which is again projective (locally free). Next, for any embedding $\sigma\colon K \hookrightarrow \mathbb{C}$, the tensor product $\mathcal{O}_{K'} \otimes_{\mathcal{O}_K,\sigma} \mathbb{C}$ is canonically isomorphic to

$$\bigoplus\nolimits_{\sigma'|\sigma} \mathbb{C} := \bigoplus\nolimits_{\sigma'\colon K'\hookrightarrow\mathbb{C},\ \sigma'|_K=\sigma} \mathbb{C}.$$

Thus we have a canonical isomorphism

$$(f_*\mathcal{E}')_\sigma = \mathcal{E}' \otimes_{\mathcal{O}_K,\sigma} \mathbb{C} = \mathcal{E}' \otimes_{\mathcal{O}_{K'}} (\mathcal{O}_{K'} \otimes_{\mathcal{O}_K,\sigma} \mathbb{C}) = \bigoplus\nolimits_{\sigma'|\sigma} \mathcal{E}'_{\sigma'}.$$

Thus the desired Hermitian metric is given by: for any $e = (e'_{\sigma'})_{\sigma'|\sigma} \in (f_*\mathcal{E}')_\sigma$, set

$$\|e\|_\sigma^2 := \sum_{\sigma'|\sigma} d_{\sigma'/\sigma} \|e'_{\sigma'}\|_{\sigma'}^2, \tag{1.2.2}$$

where $d_{\sigma'/\sigma} = 2$ if σ' is a complex place and σ is a real place, and $d_{\sigma'/\sigma} = 1$ otherwise.

Sometimes, it is more convenient to put a singular metric, by changing (1.2.2) to

$$\|e\|_{\max,\sigma} := \max_{\sigma'|\sigma} \|e'_{\sigma'}\|_{\sigma'}. \tag{1.2.3}$$

We denote by $f_{\max,*}\overline{\mathcal{E}}' := (f_*\mathcal{E}', \|\cdot\|_{\max})$.

Example 1.2.5. *A particularly important case is when $f\colon \mathrm{Spec}\mathcal{O}_K \to \mathrm{Spec}\mathbb{Z}$ is induced by the inclusion $\mathbb{Q} \subseteq K$ (we changed our notation for this particular case). Let $\overline{\mathcal{L}}$ be a Hermitian line bundle on $\mathrm{Spec}\mathcal{O}_K$. Then $f_*\mathcal{L}$ is a vector bundle on $\mathrm{Spec}\mathbb{Z}$ which must be trivial since the class number of $\mathbb{Q}$ is 1. Under the identification of vector bundles and projective modules, this is equivalent to say that $H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})$ is a projective $\mathcal{O}_K$-module of rank 1, and is free if viewed as a $\mathbb{Z}$-module. Moreover, we have*

$$H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L}) \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus\nolimits_{\sigma\colon K\hookrightarrow\mathbb{C}} \mathcal{L}_\sigma.$$

For any $s = (s_\sigma)_\sigma \in H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L}) \otimes_{\mathbb{Z}} \mathbb{C}$, we then have

$$\|s\|_{\max} = \max_\sigma\{\|s_\sigma\|\}.$$

Set

$$H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})_{\mathbb{R}} := \{s = (s_\sigma)_\sigma \in H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L}) \otimes_{\mathbb{Z}} \mathbb{C} : s_\sigma = \overline{s}_{\overline{\sigma}} \text{ for all } \sigma\}. \tag{1.2.4}$$

Then $H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})$ is a lattice in $H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})_{\mathbb{R}}$, and $\|\cdot\|_{\max}$ induces a norm on $H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})_{\mathbb{R}}$.

We will come back to this example later.

Norm of Hermitian line bundles Let $\overline{\mathcal{L}}'$ be a Hermitian line bundle on $\mathrm{Spec}\mathcal{O}_{K'}$. We wish to define the *norm* $\mathrm{Norm}_{K'/K}(\overline{\mathcal{L}}') \in \widehat{\mathrm{Pic}}(\mathrm{Spec}\mathcal{O}_K)$, which corresponds to the pushforward of the arithmetic class group (even though we have not defined what it means), *i.e.*

$$f_*\widehat{c}_1(\overline{\mathcal{L}}') = \widehat{c}_1(\mathrm{Norm}_{K'/K}(\overline{\mathcal{L}}')).$$

for the arithmetic first Chern class $\widehat{c}_1$ (the inverse of (1.1.5)).

Let (U_i) be an open cover of $\mathrm{Spec}\mathcal{O}_K$ such that $\mathcal{L}'|_{f^{-1}(U_i)}$ is trivial for each i. Choose a section $\epsilon_i \in H^0(f^{-1}(U_i), \mathcal{L}')$ which generates $\mathcal{L}'$ everywhere on $f^{-1}(U_i)$. Then the line bundle $\mathcal{L}'$ is represented by the 1-cocycle (f_{ij}) defined as follows: for each pair (i,j) and $U_{ij} := U_i \cap U_j$, $f_{ij} \in H^0(f^{-1}(U_{ij}), \mathcal{O}^\times_{\mathrm{Spec}\mathcal{O}_{K'}})$ is the unique invertible function on $f^{-1}(U_{ij})$ such that $\epsilon_i = f_{ij}\epsilon_j$.

The underlying line bundle $\mathrm{Norm}_{K'/K}(\mathcal{L}')$ is then defined to be the line bundle on $\mathrm{Spec}\mathcal{O}_K$ determined by the 1-cocycle $\mathrm{Norm}_{K'/K}(f_{ij})$, relative to the open cover (U_i). It admits a canonical trivialization over U_i with generator $\mathrm{Norm}_{K'/K}(\epsilon_i)$.

The Hermitian metrics are defined as follows. Let $\sigma\colon K \hookrightarrow \mathbb{C}$. Then we have a canonical isomorphism

$$\mathrm{Norm}_{K'/K}(\mathcal{L}')_\sigma = \bigotimes\nolimits_{\sigma'|\sigma} \mathcal{L}'_{\sigma'}.$$

This defines a canonical Hermitian metric on $\mathrm{Norm}_{K'/K}(\mathcal{L}')_\sigma$.

1.3 Positivity of Hermitian line bundles on $\mathrm{Spec}\mathcal{O}_K$

Let $\overline{\mathcal{L}}$ be a Hermitian line bundle on $\mathrm{Spec}\mathcal{O}_K$.

Definition 1.3.1. *The Hermitian line bundle $\overline{\mathcal{L}}$ is said to be* **ample** *(resp.* **nef***) if $\widehat{\deg}(\overline{\mathcal{L}}) > 0$ (resp. $\widehat{\deg}(\overline{\mathcal{L}}) \geq 0$).*

We will prove a criterion for ampleness (Theorem 1.3.8) which is the arithmetic version of the criterion for ample line bundles over curves. For this we need to introduce the sets of effective sections and of strictly effective sections of $\overline{\mathcal{L}}$.

Definition 1.3.2. *Define*

$$\widehat{H}^0(\overline{\mathcal{L}}) := \{s \in H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L}) : \|s\|_\sigma \leq 1,\ \forall \sigma\},$$
$$\widehat{H}^0_{\mathrm{s}}(\overline{\mathcal{L}}) := \{s \in H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L}) : \|s\|_\sigma < 1,\ \forall \sigma\}.$$

Lemma 1.3.3. *Both $\widehat{H}^0(\overline{\mathcal{L}})$ and $\widehat{H}^0_{\mathrm{s}}(\overline{\mathcal{L}})$ are finite sets.*

Proof. It suffices to prove the result for $\widehat{H}^0(\overline{\mathcal{L}})$. By Example 1.2.5, $\widehat{H}^0(\overline{\mathcal{L}})$ is the set of lattice points in $H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})_{\mathbb{R}}$ contained in the unit ball defined by the norm induced by $\|\cdot\|_{\max}$. Thus it is a finite set. □

Definition 1.3.4. *Define*

$$\widehat{h}^0(\overline{\mathcal{L}}) := \log \#\widehat{H}^0(\overline{\mathcal{L}}),$$
$$\widehat{h}^0_{\mathrm{s}}(\overline{\mathcal{L}}) := \log \#\widehat{H}^0_{\mathrm{s}}(\overline{\mathcal{L}}).$$

By definition of arithmetic degree (1.1.1), $\overline{\mathcal{L}}$ is ample if $\widehat{h}^0_{\mathrm{s}}(\overline{\mathcal{L}}) > 0$ and is nef if $\widehat{h}^0(\overline{\mathcal{L}}) > 0$.

As indicated by the proof of Lemma 1.3.3, we are interested in counting the number of lattice points in a unit ball, both contained in a Euclidean space. In general this is not an easy task. But there are tools in the theory of geometry of numbers which we can use.

1.3.1 Geometry of numbers

Consider the pairs $\overline{M} = (M, \|\cdot\|)$ where M is a free $\mathbb{Z}$-module of finite rank of $r \geq 1$ and $\|\cdot\|$ is a norm on $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. Thus the natural map $M \to M_{\mathbb{R}}$ makes M into a lattice in $M_{\mathbb{R}}$. An example is the one obtained from $H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})$ and $\|\cdot\|_{\max}$ from Example 1.2.5.

Set

$$\widehat{H}^0(\overline{M}) := \{m \in M : \|m\| \leq 1\}, \qquad \widehat{h}^0(\overline{M}) := \log \#\widehat{H}^0(\overline{M});$$
$$\widehat{H}^0_{\mathrm{s}}(\overline{M}) := \{m \in M : \|m\| < 1\}, \qquad \widehat{h}^0_{\mathrm{s}}(\overline{M}) := \log \#\widehat{H}^0_{\mathrm{s}}(\overline{M}).$$

Minkowski's First Theorem is a tool to prove the existence of a non-zero small lattice point, via the quantity $\chi(\overline{M})$ defined as below. Denote by $B(\overline{M}) := \{m \in M_{\mathbb{R}} : \|m\| \leq 1\}$ the unit ball in $M_{\mathbb{R}}$. Fix a Haar measure on $M_{\mathbb{R}}$ and let

$$\chi(\overline{M}) := \log \frac{\mathrm{vol}(B(\overline{M}))}{\mathrm{covol}(M_{\mathbb{R}}/M)}, \tag{1.3.1}$$

which is independent of the choice of the Haar measure. This is an arithmetic analogue of the Euler characteristic.

By Minkowski's First Theorem, $\widehat{h}^0(\overline{M}) > 0$ if $\chi(\overline{M}) > r \log 2$. The following is a quantitative version:

Theorem 1.3.5. *We have*

$$\widehat{h}^0(\overline{M}) \geq \chi(\overline{M}) - r \log 2. \tag{1.3.2}$$

Moreover, there exists a non-zero $m \in M$ such that

$$-\log \|m\| \geq \frac{\chi(\overline{M})}{r} - \log 2.$$

To prove Theorem 1.3.5, we use a common trick called *Variational Principle* in Arakelov Geometry. For any real number c, set

$$\overline{M}(c) := (M, e^{-c}\|\cdot\|).$$

It is not hard to check that

$$\chi(\overline{M}(c)) = \chi(\overline{M}) + cr.$$

Proof. Consider the universal covering

$$u\colon M_{\mathbb{R}} \to M_{\mathbb{R}}/M.$$

For $c \in \mathbb{R}$, there exists a point $y \in M_{\mathbb{R}}/M$ such that

$$\#\left(u^{-1}(y) \cap B(\overline{M}(c))\right) \geq \mathrm{vol}(B(\overline{M}(c)))/\mathrm{covol}(M_{\mathbb{R}}/M).$$

Otherwise we would have $\mathrm{vol}(B(\overline{M}(c))) < \mathrm{covol}(M_{\mathbb{R}}/M) \cdot \mathrm{vol}(B(\overline{M}(c)))/\mathrm{covol}(M_{\mathbb{R}}/M)$.

Take $m_0 \in u^{-1}(y) \cap B(\overline{M}(c))$. For any $m \in u^{-1}(y) \cap B(\overline{M}(c))$, we have $m - m_0 \in M$ and $\|m - m_0\| \leq 2e^c$, and therefore $m - m_0 \in \widehat{H}^0(\overline{M}(c + \log 2))$. Hence

$$\widehat{h}^0(\overline{M}(c + \log 2)) \geq \log \#\left(u^{-1}(y) \cap B(\overline{M}(c))\right).$$

The two inequalities above together with the definition of $\chi(\overline{M}(c))$ yield

$$\widehat{h}^0(\overline{M}(c + \log 2)) \geq \chi(\overline{M}(c)) = \chi(\overline{M}) + cr.$$

Thus we get (1.3.2) by letting $c = -\log 2$.

Now for any $c \in \mathbb{R}$, we have

$$\widehat{h}^0(\overline{M}(-c)) \geq \chi(\overline{M}(-c)) - r\log 2 = \chi(\overline{M}) - rc - r\log 2.$$

Thus for all $c < \chi(\overline{M})/r - \log 2$, there exists a non-zero $m \in M$ such that $e^c\|m\| \leq 1$. In other words, for any $\epsilon > 0$, there exists a non-zero $m_\epsilon \in M$ with

$$-\log\|m_\epsilon\| \geq \frac{\chi(\overline{M})}{r} - \log 2 - \epsilon.$$

Taking a sequence $\{\epsilon_n\}_{n\geq 1}$ decreasing to 0, the corresponding sequence $\{m_n\}_{n\geq 1}$ takes finitely many values since m_n are lattice points in a bounded ball. Thus we find an $m \in M$ with

$$-\log\|m\| \geq \frac{\chi(\overline{M})}{r} - \log 2 - \epsilon_n$$

with $\epsilon_n \to 0$. It suffices to take $n \to \infty$. □

Proposition 1.3.6. *We have*

$$\widehat{h}^0_{\mathrm{s}}(\overline{M}) \leq \widehat{h}^0(\overline{M}) \leq \widehat{h}^0_{\mathrm{s}}(\overline{M}) + r\log 3.$$

Proof. We will prove the desired comparison by the following: For any $c > 0$, we have

$$\widehat{h}^0(\overline{M}(-c)) \leq \widehat{h}^0(\overline{M}) \leq \widehat{h}^0(\overline{M}(-c)) + rc + r\log 3. \tag{1.3.3}$$

In fact, the desired inequality follows directly from (1.3.3) by letting $c \to 0$.

Let us prove (1.3.3). We only need to prove the second inequality. For any $t > 0$, set $B(t) := \{m \in M_{\mathbb{R}} : \|m\| \leq t\}$ the ball of radius t centered at 0. Then $\mathrm{vol}(B(t)) = t^r \mathrm{vol}(B(1))$. Notice that $B(t) = B(\overline{M}(\log t))$.

Now consider $B(1 + 2^{-1}e^{-c})$. Let us also consider all the balls of radius $2^{-1}e^{-c}$ centered at points in $M \cap B(1) = \widehat{H}^0(\overline{M})$; all these small balls are contained in $B(1 + 2^{-1}e^{-c})$. Thus there exists a point $m \in B(1 + 2^{-1}e^{-c})$ which is contained in N of these small balls, with

$$N \geq \frac{\#\widehat{H}^0(\overline{M}) \cdot \mathrm{vol}(B(2^{-1}e^{-c}))}{\mathrm{vol}(B(1 + 2^{-1}e^{-c}))} = \#\widehat{H}^0(\overline{M}) \frac{1}{(1 + 2e^c)^r}.$$

Thus

$$\log N \geq \widehat{h}^0(\overline{M}) - r(c + \log 3).$$

Let $x_1, \ldots, x_N$ be the centers of these small balls. Then $x_i - m \in B(2^{-1}e^{-c})$. Hence $x_i - x_1 \in B(e^{-c})$ for all i. In particular we find N points in $\widehat{H}^0(\overline{M}(-c))$. Therefore we can conclude. □

1.3.2 Ampleness and nefness

Let $\overline{\mathcal{L}}$ be a Hermitian line bundle on Spec$\mathcal{O}_K$.

We relate Hermitian line bundles with the theory of geometry of numbers as follows. Let $\overline{M} = (M, \|\cdot\|)$ be the pair as in §1.3.1 obtained from $H^0(\mathrm{Spec}\mathcal{O}_K, \mathcal{L})$ and $\|\cdot\|_{\max}$ from Example 1.2.5. Then by definition, we have

$$\widehat{H}^0(\overline{\mathcal{L}}) = \widehat{H}^0(\overline{M}), \qquad \widehat{h}^0(\overline{\mathcal{L}}) = \widehat{h}^0(\overline{M});$$
$$\widehat{H}^0_{\mathrm{s}}(\overline{\mathcal{L}}) = \widehat{H}^0_{\mathrm{s}}(\overline{M}), \qquad \widehat{h}^0_{\mathrm{s}}(\overline{\mathcal{L}}) = \widehat{h}^0_{\mathrm{s}}(\overline{M}).$$

We also set

$$\chi(\overline{\mathcal{L}}) := \chi(\overline{M}). \tag{1.3.4}$$

The following arithmetic Riemann–Roch theorem is not hard to prove.

Theorem 1.3.7 (Arithmetic Riemann–Roch over Spec$\mathcal{O}_K$)**.** $\chi(\overline{\mathcal{L}}) = \widehat{\deg}\overline{\mathcal{L}} + \chi(\mathcal{O}_{\mathrm{Spec}\mathcal{O}_K}, |\cdot|)$. *Here $|\cdot|$ is the trivial norm on* Spec$\mathcal{O}_K$.

Theorem 1.3.8. *The followings are equivalent:*

(i) $\overline{\mathcal{L}}$ is ample,

(ii) $\widehat{h}^0_{\mathrm{s}}(\overline{\mathcal{L}}^{\otimes m}) > 0$ for $m \gg 1$,

(iii) for any Hermitian line bundle $\overline{\mathcal{M}}$ on Spec$\mathcal{O}_K$*, we have $\widehat{h}^0_{\mathrm{s}}(\overline{\mathcal{L}}^{\otimes m} \otimes \overline{\mathcal{M}}) > 0$ for $m \gg 1$.*

Proof. (iii) clearly implies (ii).

(ii) implies (i): Take a non-zero $s \in \widehat{H}^0_{\mathrm{s}}(\overline{\mathcal{L}}^{\otimes m})$. Then by definition of arithmetic degree (1.1.1), we have $\widehat{\deg}(\overline{\mathcal{L}}^{\otimes m}) > 0$. But $\widehat{\deg}(\overline{\mathcal{L}}^{\otimes m}) = m\widehat{\deg}(\overline{\mathcal{L}})$ by Proposition 1.1.7. Thus $\overline{\mathcal{L}}$ is ample.

(i) implies (iii): By Theorem 1.3.7, we have

$$\chi(\overline{\mathcal{L}}^{\otimes m} \otimes \overline{\mathcal{M}}) = m\widehat{\deg}(\overline{\mathcal{L}}) + \widehat{\deg}(\overline{\mathcal{M}}) + \chi(\mathcal{O}_{\mathrm{Spec}\mathcal{O}_K}, |\cdot|).$$

Since $\widehat{\deg}(\overline{\mathcal{L}}) > 0$, for $m \gg 1$ we have $\chi(\overline{\mathcal{L}}^{\otimes m} \otimes \overline{\mathcal{M}}) > [K : \mathbb{Q}] \log 6$. Thus $\widehat{h}^0_{\mathrm{s}}(\overline{\mathcal{L}}^{\otimes m} \otimes \overline{\mathcal{M}}) > 0$ by Theorem 1.3.5 and Proposition 1.3.6. □

Chapter 2

Hermitian line bundles on projective arithmetic varieties

In this chapter, we define Hermitian line bundles on arithmetic varieties, explain how to use them to define the height machine, and discuss about their positivity (nefness, ampleness, bigness).

2.1 Review on complex geometry

2.1.1 Complex spaces (complex analytic varieties)

Definition 2.1.1. *Let Ω be a connected open subset of $\mathbb{C}^n$ for some $n \geq 1$. A* **complex analytic subset** *V of Ω is the vanishing locus $V = V(f_1, \ldots, f_m)$ of holomorphic function $f_1, \ldots, f_m$ on Ω.*

For Ω and V as in the definition, let $\mathcal{O}_\Omega$ be the sheaf of holomorphic functions on Ω, and set

$$\mathcal{O}_V := (\mathcal{O}_\Omega/(f_1, \ldots, f_m))\,|_V. \tag{2.1.1}$$

This makes $(V, \mathcal{O}_V)$ a locally ringed space. We call such pairs $(V, \mathcal{O}_V)$ *local models* of complex spaces.

Definition 2.1.2. *A* **complex space** *(or* **complex analytic variety***) is a locally ringed space $(X, \mathcal{O}_X)$ where*

- *X is a locally compact Hausdorff space,*
- *$\mathcal{O}_X$ is a structure sheaf*

such that $(X, \mathcal{O}_X)$ is locally isomorphic to a local model $(V, \mathcal{O}_V)$ defined above.

When the structure sheaf is clear, we by abuse of notation write X for the complex space.

With this definition, one can define morphisms between complex spaces, holomorphic functions on complex spaces, *etc.*

Notice that complex manifolds are precisely complex spaces which are smooth. Moreover, for any complex space X, its *regular locus* X^{reg} is open and dense in X, and is naturally a complex manifold. The *singular locus* $X^{\mathrm{sing}} = X \setminus X^{\mathrm{reg}}$ is a closed complex subspace of X.

Definition 2.1.3. *Let X be a complex space. A* **smooth function** *on X is a continuous function $f\colon X \to \mathbb{R}$ such that for any $x \in X$, there exists an open neighborhood U_x of x in X and an analytic map $i\colon U_x \to \Omega$ (with Ω open in $\mathbb{C}^n$ for some $n \geq 1$) satisfying the following property: $i(U_x)$ is closed in Ω and $f|_{U_x} = \widetilde{f}|_{i(U_x)} \circ i$ with $\widetilde{f}$ a smooth function on Ω.*

2.1.2 Forms and currents

Let us start with the case of *complex manifolds* (smooth complex spaces) M.

We start with the real forms and currents. For each $r \geq 0$, let

$$\begin{aligned} A^r(M) &:= \text{space of smooth complex valued } r\text{-forms on } M, \\ A^r_{\mathrm{c}}(M) &:= \text{spaced of compactly supported smooth complex valued } r\text{-forms on } M. \end{aligned}$$

The topology on $A^r(M)$ is defined using the following semi-norms (with s, Ω, L varying for all posibilities): For any $\Omega \subseteq M$ a coordinate open subset, and any compact subset $L \subseteq \Omega$ and any $s \in \mathbb{Z}_{\geq 0}$, define the semi-norm

$$p^s_L(u) := \sup_{x \in L} \max_{|I|=r,\ |\alpha| \leq s} |D^\alpha u_I(x)| \tag{2.1.2}$$

for any r-form $u = \sum_I u_I \mathrm{d}x_I$ on Ω. In other words, a sequence $\{u_n\}$ in $A^r(M)$ converges to a form $u \in A^r(M)$ if and only if the following holds true: for each compact subset of every coordiante neighborhood, the sequence $\{u - u_n\}$ and the sequences of higher derivatives converge to 0 uniformly.

The topology on $A^r_{\mathrm{c}}(M)$ is simply the sub-space topology induced by $A^r_{\mathrm{c}}(M) \subseteq A^r(M)$.

Definition 2.1.4. *A* **current** *of dimension r on M is a complex linear functional $T \colon A^r_{\mathrm{c}}(M) \to \mathbb{C}$ which is continuous in the topology on $A^r_{\mathrm{c}}(M)$ defined above.*

We use $D_r(M)$ to denote the space of currents of dimension r, and

$$D^{\dim_{\mathbb{R}} M - r}(M) := D_r(M). \tag{2.1.3}$$

We call $\dim_{\mathbb{R}} M - r$ the *degree* of a current in this space. For $T \in D_r(M)$ and $\alpha \in A^r_{\mathrm{c}}$, write

$$\langle T, \alpha \rangle := T(\alpha) \in \mathbb{C}. \tag{2.1.4}$$

Example 2.1.5. *(i) Let $Z \subseteq M$ be a complex subspace of M with $\dim_{\mathbb{C}} Z = r$. Then the Dirac operator*

$$\delta_Z := (u \mapsto \int_Z u)$$

is an element in $D_{2r}(M)$.

(ii) For any $f \in A^r(M)$ with L^1_{loc}-coefficients, we have

$$T_f := (u \mapsto \int_M f \wedge u) \in D_{\dim_{\mathbb{R}} M - r}(M) = D^r(M).$$

The map $f \mapsto T_f$ then makes $A^r(M)$ into a subspace of $D^r(M)$.

This explains the terminology of "degree" of a current: a degree r current can be written as $\sum_{|I|=r} u_I \mathrm{d}x_I$ with each u_I a distribution.

Next we separate the holomorphic and anti-holomorphic parts. For each $r \geq 0$, we have a decomposition into (p, q)-forms $A^r(M) = \bigoplus_{p+q=r} A^{p,q}(M)$. Define

$$\begin{aligned} A^{p,q}_{\mathrm{c}}(M) &:= A^{p,q}(M) \cap A^r_{\mathrm{c}}(M) \\ D_{p,q}(M) &:= \{T \in D_{p+q}(M) : T(u) = 0 \text{ for all } u \in A^{r,s}_{\mathrm{c}}(M) \text{ with } r \neq p\} \\ D^{\dim_{\mathbb{C}} M - p, \dim_{\mathbb{C}} M - q}(M) &:= D_{p,q}(M). \end{aligned} \tag{2.1.5}$$

Example 2.1.6. *(i) In Example 2.1.5.(i), we have furthermore*

$$\delta_Z \in D_{r,r}(M) = D^{\dim M - r, \dim M - r}(M).$$

If Z is a divisor, i.e. $\mathrm{codim}_M Z = 1$*, then we get a $(1,1)$-current δ_Z.*

(ii) In Example 2.1.5.(ii), if we furthermore assume $f \in A^{p,q}(M)$, then $T_f \in D^{p,q}(M)$. Thus $f \mapsto T_f$ makes $A^{p,q}(M)$ into a subspace of $D^{p,q}(M)$.

Now we are ready to discuss the general case of complex spaces $(X, \mathcal{O}_X)$.

At each $x \in X$, we have a local model $(V, \mathcal{O}_V)$ with $x \in V$ closed in some connected open subset Ω of $\mathbb{C}^n$ for some $n \geq 1$. Recall that $\mathcal{O}_V$ is a quotient of $\mathcal{O}_\Omega$.

Definition 2.1.7. *A* **smooth** (p,q)**-form** *on X is a smooth (p,q)-form α on X^{reg} such that for any $x \in X$ and the local model above, α extends to a smooth (p,q)-form on Ω.*

Let $\mathcal{A}_X^{p,q}$ be the sheaf of smooth (p,q)-forms on X. Then on each local model V, we have

$$\mathcal{A}_X^{p,q}|_V = \mathcal{A}_\Omega^{p,q}/\{u : i^*u = 0\}$$

where i is $X^{\mathrm{reg}} \cap V \subseteq V \subseteq \Omega$.

For each $n \geq 0$, define $\mathcal{A}_X^n := \bigoplus_{p+q=n} \mathcal{A}_X^{p,q}$. There are natural differential operators

$$\begin{aligned} \partial\colon \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p+1,q}, \qquad & \overline{\partial}\colon \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p,q+1} \\ \mathrm{d} = \partial + \overline{\partial}\colon & \mathcal{A}_X^n \to \mathcal{A}_X^{n+1} \end{aligned}$$

for all $p, q, n \geq 0$. We have $\partial^2 = \overline{\partial}^2 = \mathrm{d}^2 = 0$ and thus $\partial\overline{\partial} = -\overline{\partial}\partial$. We furthermore introduce

$$\mathrm{d^c} := \frac{1}{2\pi\sqrt{-1}}(\partial - \overline{\partial}). \tag{2.1.6}$$

Then $\mathrm{dd^c} = \frac{\sqrt{-1}}{\pi}\partial\overline{\partial}$.

Denote by $A^{p,q}(X) := \mathcal{A}_X^{p,q}(X)$. Denote by $A_{\mathrm{c}}^{p,q}(X) \subseteq A^{p,q}(X)$ the subspace of compactly supported (p,q)-forms. A (p,q)-form α on X is said to be *closed* if $\mathrm{d}\alpha = 0$.

Currents on X are defined in a similar way to the smooth case. We omit it here. The differential operators above can also be applied to currents by considering the duality. More precisely, $\mathrm{d} = \partial + \overline{\partial}$ where

$$\partial\colon D^{p,q}(X) \to D^{p+1,q}(X), \qquad \overline{\partial}\colon D^{p,q}(X) \to D^{p,q+1}(X)$$

are defined according to the formulae:

$$\begin{aligned} \langle \partial T, \alpha\rangle &:= (-1)^{p+q+1}\langle T, \partial\alpha\rangle \quad \text{for all } \alpha \in A_{\mathrm{c}}^{\dim X - p - 1, \dim X - q}(X) \\ \langle \overline{\partial} T, \alpha\rangle &:= (-1)^{p+q+1}\langle T, \overline{\partial}\alpha\rangle \quad \text{for all } \alpha \in A_{\mathrm{c}}^{\dim X - p, \dim X - q - 1}(X). \end{aligned}$$

A (p,q)-current T on X is said to be *closed* if $\mathrm{d}T = 0$.

2.1.3 Positivity and the Poincaré–Lelong Formula

Definition 2.1.8. *On an open subset $\Omega \subseteq \mathbb{C}^n$, a $(1,1)$-current $u = \sqrt{-1}\sum u_{jk}\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k$ (with each u_{jk} a distribution) is said to be* **(semi-)positive** *if the associated Hermitian form $\xi \mapsto \sum u_{jk}\xi_j\overline{\xi}_k$ is (semi-)positivie on $\mathbb{C}^n$.*

If each u_{jk} is a smooth function, then we recover the definition of (semi-)positive $(1,1)$-forms. Let $(X, \mathcal{O}_X)$ be a complex space.

Definition 2.1.9. *(i) A smooth $(1,1)$-form on X is said to be* **(semi-)positive** *if locally it is (semi-)positive.*

(ii) A $(1,1)$-current $T \in D^{1,1}(X)$ is said to be **(semi-)positive** *if locally it is (semi-)positive.*

An equivalent way to define semi-positive $(1,1)$-current is to use the duality: $T \in D^{1,1}(X)$ is semi-positive if and onoy if $T(\eta \wedge \overline{\eta}) \geq 0$ for all $\eta \in A_{\mathrm{c}}^{n-1,0}(X)$.

Proposition 2.1.10. *Let $T \in D^{1,1}(X)$ be a closed $(1,1)$-current. Then T is semi-positive if and only if locally T can be written as $\sqrt{-1}\partial\overline{\partial}\log|u|$ for some plurisubharmonic function u.*

We end this section with the following result.

Theorem 2.1.11 (Poincaré–Lelong Formula for meromorphic functions)**.** *Let X be a complex space and let f be a meromorphic function. Then as $(1,1)$-currents on X, we have*

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log|f|^2 = \delta_{\mathrm{div}(f)}.$$

2.2 Hermitian line bundles in complex geometry

Let X be a complex space.

2.2.1 Hermitian metrics on holomorphic line bundles

Let L be a holomorphic line bundle on X.

Definition 2.2.1. *A* **smooth (resp. continuous) Hermitian metric** $\|\cdot\|$ *of L on X is the assignment of a $\mathbb{C}$-metric $\|\cdot\|$ to the fiber $L(x)$ above each point $x \in X$, which varies smoothly (resp. continuously). More precisely, for any open subset U of X and any section s of $L|_U \to U$, the function $\|s(x)\|^2$ is smooth (resp. continuous) in $x \in U$.*

We call $(L, \|\cdot\|)$ a smooth/continuous *Hermitian line bundle* on X.

Next we define the *curvature form/current* of the Hermitian line bundle L on X. We need the following preparation. The line bundle L is determined by: (i) an open cover $\{U_\alpha\}$ of X with $L|_{U_\alpha} \simeq U_\alpha \times \mathbb{C}$, (ii) 1-cocyles $\{g_{\alpha\beta}\}$ which are nowhere-zero holomorphic functions on $U_\alpha \cap U_\beta$. The Hermitian metric corresponds to the collection $(U_\alpha, h_\alpha)_\alpha$ with $h_\alpha \colon U_\alpha \to \mathbb{R}_{>0}$, with $h_\beta|g_{\alpha\beta}|^2 = h_\alpha$ on $U_\alpha \cap U_\beta$; indeed h_α is $\|\cdot\|^2$ locally on U_α.

Now consider the $(1,1)$-current $-\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h_\alpha$ on U_α; if the Hermitian metric is smooth then it is a $(1,1)$-form. Since $h_\beta|g_{\alpha\beta}|^2 = h_\alpha$ on $U_\alpha \cap U_\beta$, we have $\log h_\alpha + \log g_{\alpha\beta} + \log\overline{g}_{\alpha\beta} = \log h_\beta$ for some local branch of $\log g_{\alpha\beta}$. But $g_{\alpha\beta}$ is holomorphic, so $\overline{\partial}\log g_{\alpha\beta} = \partial\log\overline{g}_{\alpha\beta} = 0$. Thus $-\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h_\alpha = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h_\beta$ on $U_\alpha \cap U_\beta$. In other words, these local $(1,1)$-currents patch together to a $(1,1)$-current on the whole X, and it is a $(1,1)$-form if the Hermitian metric is smooth. Sometimes we also use $-\frac{\sqrt{-1}}{\pi}\partial\overline{\partial}\log\|\cdot\|$ to denote this current.

Definition 2.2.2. *The* **curvature current** *of* $(L, \|\cdot\|)$, *denoted by* $c_1(L, \|\cdot\|)$, *is the* $(1,1)$-*current on* X *defined above. It is called the* **curvature form** *if the Hermitian metric is smooth.*

It is clear that $c_1(L, \|\cdot\|)$ is a closed since $\mathrm{d} = \partial + \overline{\partial}$ and $\partial^2 = \overline{\partial}^2 = 0$.

Theorem 2.2.3 (Poincaré–Lelong Formula for Hermitian line bundles)**.** *As* $(1,1)$-*currents, we have*

$$c_1(L, \|\cdot\|) = -\frac{\sqrt{-1}}{\pi}\partial\overline{\partial}\log\|s\| + \delta_{\mathrm{div}(s)}$$

for any non-zero meromorphic section s *of* L.

Proof. Let s be a non-zero meromorphic section of L over X. Then s corresponds to $(U_\alpha, s_\alpha)_\alpha$ with $s_\alpha \colon U_\alpha \to \mathbb{C}$ with $s_\alpha = g_{\alpha\beta}s_\beta$. Then $\|s\| = \sqrt{h_\alpha}|s_\alpha|$ on U_α. Thus $\log\|s\|^2 = \log h_\alpha + \log|s_\alpha|^2$. The conclusion then follows by definition of $c_1(L, \|\cdot\|)$ and Theorem 2.1.11. ☐

Definition 2.2.4. *A Hermitian metric* $\|\cdot\|$ *on* L *is said to be* **(semi-)positive** *if* $c_1(L, \|\cdot\|)$ *is a (semi-)positive current.*

By Proposition 2.1.10, $\|\cdot\|$ is semi-positive if and only if the following holds true: For any local section s of L which is everywhere non-vanishing over an open subset U of X, the function $-2\log\|s(x)\|$ is plurisubharmonic.

We close this subsection by stating the following results when X is projective, *i.e.* X is the analytification of a projective variety.

Proposition 2.2.5. *Let* $(L, \|\cdot\|)$ *be a Hermitian line bundle on* X. *Then*

(i) $c_1(L, \|\cdot\|)$ *represents the cohomology class of* L *in* $H^2(X, \mathbb{C})$ *under the natural map* $H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{C})$;

(ii) we have

$$\int_X c_1(L, \|\cdot\|)^{\wedge \dim X} = \deg_L(X).$$

Moreover if X is furthermore smooth, then *Kodaira's embedding theorem* asserts the following: a holomorphic line bundle L on X is ample if and only if L has a positive metric.

2.2.2 Green's functions

Let D be a Cartier divisor on X. Denote by $|D|$ the support of D.

Definition 2.2.6. *A* **smooth (resp. continuous) Green's function** g_D *of* D *over* X *is a function*

$$g_D \colon X \setminus |D| \to \mathbb{R}$$

such that the following holds true: for any meromorphic function f *over an open subset* U *of* X *with* $\mathrm{div}(f) = D|_U$, *the function* $g_D + \log|f|$ *can be extended to a smooth (resp. continuous) on* U.

We say that such a function g_D has *logarithmic singularity along* D.

It is well-known that line bundles and Cartier divisors are closely related. The correspondence can be extended to:

1. Given a smooth/continuous Hermitian line bundle $(L, \|\cdot\|)$ on X, for any meromorphic section s of L on X, we obtain a pair
$$\widehat{\mathrm{div}}(s) := (\mathrm{div}(s), -\log\|s\|)$$
with $-\log\|s\|$ clearly a smooth/continuous Green's function of $\mathrm{div}(s)$ over X.

2. Conversely given a pair (D, g_D) consisting of a Cartier divisor and a smooth/continuous Green's function, we can associated a smooth/continuous Hermitian line bundle $(\mathcal{O}(D), \|\cdot\|_{g_D})$ where $\|\cdot\|_{g_D}$ is defined by $\|s_D\|_{g_D} := e^{-g_D}$ for the canonical section s_D of $\mathcal{O}(D)$ (*i.e.* $\mathrm{div}(s_D) = D$).

By this correspondence, we can make the following definitions.

Definition 2.2.7. *The* **Chern current** *of the pair (D, g_D), where g_D is a Green's function of D over X, is defined to be $c_1(\mathcal{O}(D), \|\cdot\|_{g_D})$. We denote it by $c_1(D, g_D)$.*

Definition 2.2.8. *A Green's function g_D of D over X is said to be* **(semi-)positive** *if $c_1(D, g_D)$ is a (semi-)positive current.*

We close this subsection by stating the following *Stokes' Formula* which allows logarithmic singularity.

Theorem 2.2.9. *Let X be an irreducible projective complex space of dimension n. Let α be a closed $(n-1, n-1)$-form on X. Let $\overline{L}, \overline{M}$ be Hermitian line bundles on X. Let l (resp. m) be a non-zero rational section of L (resp. of M) such that their divisors intersect properly. Then*
$$\int_X (\log\|l\|) c_1(\overline{M}) \wedge \alpha - \int_{[\mathrm{div}(m)]} (\log\|l\|)\alpha = \int_X (\log\|m\|) c_1(\overline{L}) \wedge \alpha - \int_{[\mathrm{div}(l)]} (\log\|m\|)\alpha \quad (2.2.1)$$
and both equal
$$2\int_{X\setminus(|\mathrm{div}(l)|\cup|\mathrm{div}(m)|)} (\mathrm{d}\log\|l\|) \wedge (\mathrm{d^c}\log\|m\|) \wedge \alpha. \quad (2.2.2)$$

Here the divisors in (2.2.1) are the Weil divisors, and the integrals on $\mathrm{div}(l)$ and on $\mathrm{div}(m)$ are induced from those on prime Weil divisors by linearity. The supports of the divisors in (2.2.2) are supports of Cartier divisors.

2.3 Height via Hermitian line bundles on arithmetic varieties

2.3.1 Hermitian line bundles on projective arithmetic varieties

Definition 2.3.1. *An* **arithmetic variety** *is an integral scheme $\mathcal{X}$ which is flat, separated, and of finite type over* Spec$\mathbb{Z}$. *It is said to be* **(quasi-)projective** *if the structure morphism $\mathcal{X} \to \mathrm{Spec}\mathbb{Z}$ is (quasi-)projective.*

From an arithmetic variety $\mathcal{X}$, we obtain a complex space
$$\mathcal{X}(\mathbb{C}) := \mathrm{Hom}_{\mathrm{Spec}\mathbb{Z}}(\mathrm{Spec}\mathbb{C}, \mathcal{X}),$$
with the complex conjugation acting on $\mathcal{X}(\mathbb{C})$ via its action on Spec$\mathbb{C}$. Moreover, if $\mathcal{X} \to \mathrm{Spec}\mathbb{Z}$ factors through Spec$R$ for an order R in a number field K, then $\mathcal{X}(\mathbb{C}) = \coprod_{\sigma\colon K\hookrightarrow\mathbb{C}} \mathcal{X}_\sigma(\mathbb{C})$, with $\mathcal{X}_\sigma(\mathbb{C}) = \mathrm{Hom}_{\mathrm{Spec}\sigma(K)}(\mathrm{Spec}\mathbb{C}, \mathcal{X})$.

Let $\mathcal{X}$ be a *projective* arithmetic variety.

Definition 2.3.2. *A* **Hermitian line bundle** *on $\mathcal{X}$ is a pair $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ consisting of a line bundle $\mathcal{L}$ on $\mathcal{X}$ and a Hermitian metric $\|\cdot\|$ of $\mathcal{L}(\mathbb{C})$ on $\mathcal{X}(\mathbb{C})$ which is invariant under the complex conjugation, i.e. $\|s(x)\| = \|s(\overline{x})\|$ for all local sections s of $\mathcal{L}$ and all $x \in \mathcal{X}(\mathbb{C})$ at which s is defined.*

We make the following assumption on the Hermitian metric $\|\cdot\|$: it is the quotient of two semi-positive metrics (called *integrable*). This automatically holds true for smooth metrics, by using the Fubini–Study metric which will be introduced in (3.1.1).

We can also define the group of isomorphism classes of Hermitian line bundles on $\mathcal{X}$, which will be denote by $\widehat{\mathrm{Pic}}(\mathcal{X})$. The identity element is the trivial Hermitian line bundle, the multiplication is the tensor product, and the inverse is the dual.

Definition 2.3.3. *(i) An* **isomorphism** *(or* **isometry***) between two Hermitian line bundles $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ and $\overline{\mathcal{L}}' = (\mathcal{L}', \|\cdot\|')$ on $\mathcal{X}$ is an isomorphism $i\colon \mathcal{L} \to \mathcal{L}'$ such that $\|\cdot\| = i^*\|\cdot\|'$.*

(ii) The **trivial Hermitian line bundle** *on $\mathcal{X}$ is defined to be $\overline{\mathcal{O}}_{\mathcal{X}} := (\mathcal{O}_{\mathcal{X}}, |\cdot|)$ where $|\cdot|$ is the usual absolute value.*

(iii) The **tensor product** *of two Hermitian line bundles $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ and $\overline{\mathcal{L}}' = (\mathcal{L}', \|\cdot\|')$ on $\mathcal{X}$ is $\overline{\mathcal{L}} \otimes \overline{\mathcal{L}}' := (\mathcal{L} \otimes \mathcal{L}', \|\cdot\|\|\cdot\|')$.*

(iv) The **dual** *of a Hermitian line bundle $\overline{\mathcal{L}}$ on $\mathcal{X}$ is defined to be $\overline{\mathcal{L}}^\vee := (\mathcal{L}^\vee, \|\cdot\|^\vee)$, where $\mathcal{L}^\vee := \mathrm{Hom}(\mathcal{L}, \mathcal{O}_{\mathcal{X}})$ and $\|\cdot\|^\vee$ is the dual metric.*

We also have the definition of *arithmetic divisors*.

Definition 2.3.4. *An* **arithmetic divisor** *on $\mathcal{X}$ is a pair $\overline{D} = (D, g_D)$ consisting of a Cartier divisor D on $\mathcal{X}$ and a Green's function g_D of $D(\mathbb{C})$ on $\mathcal{X}(\mathbb{C})$ which is invariant under the complex conjugation, i.e. $g_D(x) = g_D(\overline{x})$ for all $x \in \mathcal{X}(\mathbb{C}) \setminus |D(\mathbb{C})|$.*

A **principal arithmetic divisor** *is of the form*

$$\widehat{\mathrm{div}}(f) := (\mathrm{div}(f), -\log|f|)$$

where $f \in \mathbb{Q}(\mathcal{X})^*$ is a non-zero rational function on $\mathcal{X}$.

We make the following assumption on the Green's function g_D: it is the quotient of two semi-positive Green's functions (called *integrable*). This automatically holds true if g_D is smooth, by the result for Hermitian line bundles and Proposition 2.3.5 below.

We have the following groups, where the group laws are clear:

$$\begin{aligned}
\widehat{\mathrm{Div}}(\mathcal{X}) &:= \{\text{arithmetic divisors on } \mathcal{X}\},\\
\widehat{\mathrm{Prin}}(\mathcal{X}) &:= \{\text{principal arithmetic divisors on } \mathcal{X}\},\\
\widehat{\mathrm{Cl}}(\mathcal{X}) &:= \widehat{\mathrm{Div}}(\mathcal{X})/\widehat{\mathrm{Prin}}(\mathcal{X}).
\end{aligned}$$

Proposition 2.3.5. *We have a group homomorphism*

$$\widehat{\mathrm{Div}}(\mathcal{X}) \to \widehat{\mathrm{Pic}}(\mathcal{X}), \quad \overline{D} = (D, g_D) \mapsto \mathcal{O}(\overline{D}) = (\mathcal{O}(D), \|\cdot\|_{\overline{D}}) \tag{2.3.1}$$

where $\|\cdot\|_{\overline{D}}$ is defined by $\|s_D\|_{\overline{D}} = e^{-g_D}$ with s_D the canonical section of $\mathcal{O}(D)$ (i.e. $\mathrm{div}(s_D) = D$). Moreover this group homomorphism induces a canonical isomorphism

$$\widehat{\mathrm{Cl}}(\mathcal{X}) \xrightarrow{\sim} \widehat{\mathrm{Pic}}(\mathcal{X}). \tag{2.3.2}$$

Proof. The proof is similar to Proposition 1.1.6. Let us write down the inverse map $\widehat{\mathrm{Pic}}(\mathcal{X}) \to \widehat{\mathrm{Cl}}(\mathcal{X})$. For each $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$, let s be a non-zero rational section of $\mathcal{L}_{\mathbb{Q}}$ and set

$$\widehat{\mathrm{div}}(s) := (\mathrm{div}(s), -\log\|s\|). \tag{2.3.3}$$

Then the inverse is $\overline{\mathcal{L}} \mapsto \widehat{\mathrm{div}}(s)$. □

2.3.2 Height machine via Hermitian line bundles

Let X be a projective variety over $\overline{\mathbb{Q}}$, and let $L \in \mathrm{Pic}^1(X)$. Then X and L are defined over some number field K, with $X \to \mathrm{Spec}K$ the structural morphism.

Definition 2.3.6. *We say that a pair $(\mathcal{X}, \overline{\mathcal{L}})$ is an* **arithmetic model** *of (X, L) over $\mathcal{O}_K$ if*

(i) $\mathcal{X}$ is an integral model of X, i.e $\mathcal{X}$ is an integral scheme, projective and flat over $\mathrm{Spec}\mathcal{O}_K$, such that $\mathcal{X}_K := \mathcal{X} \times_{\mathrm{Spec}\mathcal{O}_K} \mathrm{Spec}K \simeq X$ (notice that $\mathcal{X}$ is naturally an arithmetic variety via $\mathbb{Z} \subseteq \mathcal{O}_K$);

(ii) $\overline{\mathcal{L}}$ is a Hermitian line bundle on $\mathcal{X}$ extending L, i.e. $\mathcal{L}_K \simeq L$ under the identification $\mathcal{X}_K \simeq X$.

Fix an arithmetic model $(\mathcal{X}, \overline{\mathcal{L}})$ of (X, L) over $\mathcal{O}_K$. Let us construct the height on X associated with $(\mathcal{X}, \overline{\mathcal{L}})$, denoted by

$$h_{\overline{\mathcal{L}}}\colon X(\overline{\mathbb{Q}}) \to \mathbb{R} \tag{2.3.4}$$

as follows.

Consider a point $x \in X(K')$ with K'/K a finite extension. Then $x\colon \mathrm{Spec}K' \to X$. The valuative criterion of properness thus gives rise to a unique morphism $\overline{x}\colon \mathrm{Spec}\mathcal{O}_{K'} \to \mathcal{X}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Spec}K' \arrow[r, "x"] \arrow[d] & X = \mathcal{X}_K \arrow[d] \\ \mathrm{Spec}\mathcal{O}_{K'} \arrow[r, "\overline{x}"] & \mathcal{X} \end{tikzcd}$$

where the vertical maps are induced by the inclusions $\mathcal{O}_{K'} \subseteq K'$ and $\mathcal{O}_K \subseteq K$.

Define

$$h_{\overline{\mathcal{L}}}(x) := \frac{1}{[K':K]}\widehat{\deg}\overline{x}^*\overline{\mathcal{L}}. \tag{2.3.5}$$

Definition-Lemma 2.3.7. *Let K''/K' be a finite extension. Let $\overline{x}_0\colon \mathrm{Spec}\mathcal{O}_{K''} \to \mathcal{X}$ be the morphism determined by $x \in X(K'')$. Then*

$$\frac{1}{[K':K]}\widehat{\deg}\overline{x}^*\overline{\mathcal{L}} = \frac{1}{[K'':K]}\widehat{\deg}\overline{x}_0^*\overline{\mathcal{L}}.$$

Thus $h_{\overline{\mathcal{L}}}(x)$ in (2.3.5) extends to a well-defined function $X(\overline{\mathbb{Q}}) \to \mathbb{R}$, which is the desired height function (2.3.4).

Proof. This follows easily from Proposition 1.1.7, the definition of the arithmetic degrees of arithmetic divisors on $\mathrm{Spec}\mathcal{O}_{K'}$ and on $\mathrm{Spec}\mathcal{O}_{K''}$ (Definition 1.1.5), and the fact that $\sum_{v\in M_{K''}} e_{v/v_0}f_{v/v_0} = [K'':K']$ with (in the sum) $v_0 \in M_{K'}$ the place below v. □

Example 2.3.8. *Let* $(X, L) = (\mathbb{P}^N, \mathcal{O}(1))$ *be defined over* $\mathbb{Q}$*, and take the arithmetic model* $(\mathcal{X}, \overline{\mathcal{L}}) = (\mathbb{P}^N_{\mathbb{Z}}, \overline{\mathcal{O}(1)})$ *with the metric on* $\mathcal{O}(1)$ *as follows: For each* $s = a_0X_0 + \ldots + a_NX_N \in H^0(\mathbb{P}^N_{\mathbb{C}}, \mathcal{O}(1))$*, set*

$$\|s(x)\| := \frac{|a_0x_0 + \ldots + a_Nx_N|}{\max\{|x_0|, \ldots, |x_N|\}}$$

for any $x = [x_0 : \cdots : x_N] \in \mathbb{P}^N(\mathbb{C})$*. Then it is not hard to check that* $h_{\overline{\mathcal{O}(1)}}$ *is precisely the Weil height on* $\mathbb{P}^N(\overline{\mathbb{Q}})$*.*

Proposition 2.3.9. *For each arithmetic model* $(\mathcal{X}, \overline{\mathcal{L}})$ *of* (X, L) *over* $\mathcal{O}_K$*, the function* $h_{\overline{\mathcal{L}}}$ *is a height function associated with* (X, L)*.*

Proof. We start by showing that $h_{\overline{\mathcal{L}}_1} - h_{\overline{\mathcal{L}}_2}$ is bounded on $X(\overline{\mathbb{Q}})$ for any two arithmetic models $(\mathcal{X}_1, \overline{\mathcal{L}}_1)$ and $(\mathcal{X}_2, \overline{\mathcal{L}}_2)$ of (X, L). Let $\mathcal{X}$ be the Zariski closure of

$$X \xrightarrow{\Delta} X \times_{\mathrm{Spec}K} X \to \mathcal{X}_1 \times_{\mathrm{Spec}\mathcal{O}_K} \mathcal{X}_2.$$

Write $f_i \colon \mathcal{X} \to \mathcal{X}_i$ for the i-th projection. Then by definition we have $h_{f_i^*\overline{\mathcal{L}}_i} = h_{\overline{\mathcal{L}}_i}$ for $i \in \{1, 2\}$. On the other hand, $f_1^*\overline{\mathcal{L}}_1 - f_2^*\overline{\mathcal{L}}_2$ is trivial on the generic fiber $X = \mathcal{X}_K$. Thus $h_{f_1^*\overline{\mathcal{L}}_1 - f_2^*\overline{\mathcal{L}}_2}$ is bounded on $X(\overline{\mathbb{Q}})$ since we can take the global section to be 1 in the computation of $\widehat{\deg}$. Hence

$$h_{\overline{\mathcal{L}}_1} - h_{\overline{\mathcal{L}}_2} = h_{f_1^*\overline{\mathcal{L}}_1} - h_{f_2^*\overline{\mathcal{L}}_2} = h_{f_1^*\overline{\mathcal{L}}_1 - f_2^*\overline{\mathcal{L}}_2}$$

is bounded on $X(\overline{\mathbb{Q}})$.

So the conclusion of the proposition does not depend on the choice of the arithmetic model. By linearity/additivity, we may and do assume that L is very ample on X, *i.e.* there exists an embedding $i \colon X \hookrightarrow \mathbb{P}^N_K$ such that $i^*\mathcal{O}(1) \simeq L$. Then i extends to $i \colon \mathcal{X} \to \mathbb{P}^N_{\mathcal{O}_K}$ for the Zariski closure $\mathcal{X}$ of X in $\mathbb{P}^N_{\mathcal{O}_K}$. Then the conclusion follows from Example 2.3.8. We are done. □

2.4 Self-intersection of Hermitian line bundles on arithmetic varieties

2.4.1 Review on intersection of line bundles in algebraic geometry

Let X be a projective variety defined over an algebraically closed field k. Let $\mathrm{Pic}(X)$ be the Picard group, *i.e.* the isomorphism classes of line bundles on X.

Definition 2.4.1 (multiplicity in complete intersection)**.** *Let* R *be a noetherian local domain of Krull dimension* n*. For* $f_1, \ldots, f_n \in R \setminus \{0\}$ *such that* $|\mathrm{div}(f_1)| \cap \cdots \cap |\mathrm{div}(f_n)|$ *has dimension* 0 *in* $\mathrm{Spec}R$*, define*

$$\mathrm{ord}_R(f_1, \ldots, f_n) = \mathrm{length}_R R/(f_1, \ldots, f_n).$$

By linearity, this definition extends to, for $K = \mathrm{Frac}(R)$*,*

$$\mathrm{ord}_R \colon (K^*)^n \to \mathbb{Z}$$

for $f_1, \ldots, f_n \in K^*$ *such that* $|\mathrm{div}(f_1)| \cap \cdots \cap |\mathrm{div}(f_n)|$ *has dimension* 0 *in* $\mathrm{Spec}R$*.*

Definition 2.4.2. *Let $D_1, \ldots, D_r$ be Cartier divisors on X which intersect properly, i.e. $|D_1| \cap \cdots \cap |D_r|$ is pure of codimension r in X. Define the r-cocycle of X*

$$D_1 \cdots D_r := \sum_{\substack{Y \subseteq X \text{ integral} \\ \operatorname{codim}_X Y = r}} \operatorname{ord}_{\mathcal{O}_{X,\eta_Y}}(D_1, \ldots, D_r)[Y],$$

where η_Y is the generic point of Y.

Notice that when $r = 1$, the right hand side is just the Weil divisor associated with D_1. To distinguish Cartier and Weil divisors, we use $[D]$ to denote the Weil divisor associated with the Cartier divisor D.

On the other hand, for $r = \dim X$, we can furthermore define the *degree of* $D_1 \cdots D_{\dim X}$ to be

$$\deg(D_1 \cdots D_{\dim X}) := \sum_{P \in X(k)} \operatorname{ord}_{\mathcal{O}_{X,P}}(D_1, \ldots, D_{\dim X}).$$

Lemma 2.4.3. *Let $d = \dim X$. Let $L_1, \ldots, L_d \in \operatorname{Pic}(X)$. There exist rational sections s_i of L_i on X for each $i \in \{1, \ldots, d\}$ such that $\operatorname{div}(s_1), \ldots, \operatorname{div}(s_d)$ intersect properly.*

Notice that $\operatorname{div}(s_i) \in \operatorname{Div}(X)$ is mapped to L_i under $\operatorname{Div}(X) \to \operatorname{Cl}(X) = \operatorname{Div}(X)/\operatorname{Prin}(X) \xrightarrow{\sim} \operatorname{Pic}(X)$, where $\operatorname{Div}(X)$ is the group of Cartier divisors on X and $\operatorname{Prin}(X)$ is the subgroup of principal Cartier divisors.

Definition 2.4.4. *Let $d = \dim X$. The* **intersection pairing**

$$\operatorname{Pic}(X)^d \to \mathbb{Z}$$

is defined to be define

$$L_1 \cdots L_d := \deg(\operatorname{div}(s_1) \cdots \operatorname{div}(s_d)) \tag{2.4.1}$$

for the rational sections $s_1, \ldots, s_d$ obtained from Lemma 2.4.3, where the right hand side is Definition 2.4.2 with $r = d$.

Lemma 2.4.5. *The intersection pairing $\operatorname{Pic}(X)^d \to \mathbb{Z}$ can equivalently defined inductively as follows. When $d = 1$, it is the composite*

$$\operatorname{Pic}(X) \xrightarrow{\sim} \operatorname{Cl}(X) = \operatorname{Div}(X)/\operatorname{Prin}(X) \xrightarrow{\deg} \mathbb{Z}.$$

For general $d \geq 2$, we have

$$L_1 \cdots L_d = \sum_i m_i L_1|_{Y_i} \cdots L_{d-1}|_{Y_i} \tag{2.4.2}$$

where $\sum_i m_i [Y_i]$ is the Weil divisor for any rational section s_d of L_d on X.

Proof. When $d = 1$, this is immediately true by the discussion below Definition 2.4.2.

For general $d \geq 2$, by multi-linearity (definition of ord) we can reduce to the case where $L_1, \ldots, L_d$ are all very ample. Then both sides of (2.4.2) equal

$$\dim_k \mathcal{O}_{\operatorname{div}(s_1) \cap \cdots \cap \operatorname{div}(s_d)}$$

for some global sections $s_i \in H^0(X, L_i)$ such that $\dim |\operatorname{div}(s_1)| \cap \cdots \cap |\operatorname{div}(s_d)| = 0$, and $\operatorname{div}(s_1) \cap \cdots \cap \operatorname{div}(s_d)$ is the scheme-theoretic intersection in X. We can replace s_d by any rational section (which is $f s_d$ for some $f \in K(X)^*$) since $L_1 \cdots L_{d-1} \cdot \mathcal{O}_X = 0$. $\square$

Proposition 2.4.6 (Projection Formula). *Let $f\colon X' \to X$ be a surjective morphism of projective varieties over a field. Assume* $\dim X' = d$. *Then for any $L_1, \ldots, L_d \in \mathrm{Pic}(X)$, we have*

$$f^*L_1 \cdots f^*L_d = \deg(f) L_1 \cdots L_d.$$

Here we use the convention that

$$\deg(f) = \begin{cases} 0 & \text{if } \dim X < \dim X' \\ [K(X') : K(X)] & \text{if } \dim X = \dim X'. \end{cases}$$

As suggested by (2.4.2), it is convenient to define the intersection pairing restricted to integral closed subschemes of X. Let Y be a closed subvariety of X of dimension r, and let $L_1, \ldots, L_r \in \mathrm{Pic}(X)$. Define

$$L_1 \cdots L_r \cdot Y := L_1|_Y \cdots L_r|_Y.$$

By linearity, this definition extends to a map

$$\mathrm{Pic}(X)^r \times Z_r(X) \to \mathbb{Z} \tag{2.4.3}$$

with $Z_r(X)$ the group of r-cycles on X, *i.e.* the abelian group generated by integral closed subschemes of X of dimension r. In stating the Projection Formula, it is then convenient to introduce

$$f_*\colon Z_r(X') \to Z_r(X), \tag{2.4.4}$$

where for Y' an integral closed subscheme of X we have

$$f_*([Y']) = \begin{cases} 0 & \text{if } \dim f(Y') < \dim Y' \\ \deg(Y' \to f(Y'))[f(Y')] & \text{if } \dim f(Y') = \dim Y'. \end{cases}$$

In particular, if $f\colon X' \to X$ is generically finite, then $f_*([X']) = (\deg f)[X]$.

2.4.2 Top intersection number of Hermitian line bundles on projective arithmetic varieties

Let $\mathcal{X}$ be a projective arithmetic variety, with $\mathcal{X} \to \mathrm{Spec}\mathbb{Z}$ the structural morphism. Now we turn to the intersection theory of Hermitian line bundles on $\mathcal{X}$.

Definition 2.4.7. *An integral closed subscheme $\mathcal{Y}$ of $\mathcal{X}$ is said to be:*

(i) **horizontal** *if $\mathcal{Y}$ is flat over* $\mathrm{Spec}\mathbb{Z}$ *(notice that $\mathcal{Y} \to \mathbb{Z}$ is then surjective),*

(ii) **vertical** *if the image of $\mathcal{Y} \to$* $\mathrm{Spec}\mathbb{Z}$ *is a point.*

Let $n + 1 = \dim \mathcal{X}$. Let $Z_r(\mathcal{X})$ be the group of r-cycles on $\mathcal{X}$, *i.e.* *i.e.* the abelian group generated by integral closed subschemes of $\mathcal{X}$ of dimension r.

To define the arithmetic version of the top self-intersection, we start with the definition of the arithmetic degree for $n = 0$. When $n = 0$, we have $\mathcal{X} = \mathrm{Spec}R$ for some order R of a number field K. If $R = \mathcal{O}_K$, then the we have the arithmetic degree $\widehat{\deg}\colon \widehat{\mathrm{Pic}}(\mathrm{Spec}\mathcal{O}_K) \to \mathbb{R}$ from (1.1.1). For general R, we take the same definition with $\mathcal{O}_K$ replaced by R.

Definition 2.4.8. *Define the intersection pairing*

$$\widehat{\mathrm{Pic}}(\mathcal{X})^{n+1} \to \mathbb{R}$$

and, more generally (for $r \leq n+1$)

$$\widehat{\mathrm{Pic}}(\mathcal{X})^{r} \times Z_r(\mathcal{X}) \to \mathbb{R},$$

as follows.

(i) When $n = 0$, this is precisely $\widehat{\deg}$. For $n \geq 1$ and $\overline{\mathcal{L}}_1, \ldots, \overline{\mathcal{L}}_{n+1} \in \widehat{\mathrm{Pic}}(\mathcal{X})$, define

$$\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n+1} := \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_n \cdot [\mathrm{div}(s_{n+1})] - \int_{\mathcal{X}(\mathbb{C})} \log \|s_{n+1}\| c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_n), \tag{2.4.5}$$

with s_n an arbitrary rational section of $\mathcal{L}_{n+1}$ on $\mathcal{X}$ (and $[\mathrm{div}(s_{n+1})]$ is the Weil divisor);

(ii) For $\overline{\mathcal{L}}_1, \ldots, \overline{\mathcal{L}}_r \in \widehat{\mathrm{Pic}}(\mathcal{X})$ and an integral closed subscheme $\mathcal{Y}$ of $\mathcal{X}$ of dimension r, define $\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_r \cdot \mathcal{Y}$ inductively on r according to:

(a) If $\mathcal{Y}$ is horizontal, then set

$$\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_r \cdot \mathcal{Y} := \overline{\mathcal{L}}_1|_{\mathcal{Y}} \cdots \overline{\mathcal{L}}_r|_{\mathcal{Y}}. \tag{2.4.6}$$

(b) If $\mathcal{Y}$ is vertical, then the image of $\mathcal{Y} \to \mathrm{Spec}\mathbb{Z}$ is (p) for some prime number p and hence we view $\mathcal{Y}$ as a scheme over $\mathrm{Spec}\mathbb{F}_p$ (and hence over $\mathrm{Spec}\overline{\mathbb{F}}_p$). Set

$$\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_r \cdot \mathcal{Y} := (\mathcal{L}_1|_{\mathcal{Y}} \cdots \mathcal{L}_r|_{\mathcal{Y}}) \log p. \tag{2.4.7}$$

Theorem 2.4.9. *The pairing $\widehat{\mathrm{Pic}}(\mathcal{X})^{n+1} \to \mathbb{R}$ is well-defined, multi-linear and symmetric.*

Proof. Take s_i to be a rational section of $\mathcal{L}_i$ such that $\mathrm{div}(s_1), \ldots, \mathrm{div}(s_{n+1})$ intersect properly in $\mathcal{X}$. Set

$$\begin{aligned}\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} \cdot \widehat{\mathrm{div}}(s_n) \cdot \widehat{\mathrm{div}}(s_{n+1}) &:= \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} \left(\mathrm{div}(s_n) \cdot \mathrm{div}(s_{n+1})\right) \\ &\quad - \int_{[\mathrm{div}(s_{n+1})](\mathbb{C})} \log \|s_n\| c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{n-1}) - \int_{\mathcal{X}(\mathbb{C})} \log \|s_{n+1}\| c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_n).\end{aligned}$$

By induction on n, we then get the definition of $\widehat{\mathrm{div}}(s_1) \cdots \widehat{\mathrm{div}}(s_{n+1})$ and have

$$\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n+1} = \widehat{\mathrm{div}}(s_1) \cdots \widehat{\mathrm{div}}(s_{n+1}).$$

By Stokes' Formula (Theorem 2.2.9), we have

$$\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} \cdot \widehat{\mathrm{div}}(s_n) \cdot \widehat{\mathrm{div}}(s_{n+1}) = \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} \cdot \widehat{\mathrm{div}}(s_{n+1}) \cdot \widehat{\mathrm{div}}(s_n).$$

Thus we obtain

$$\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} \cdot \overline{\mathcal{L}}_n \cdot \overline{\mathcal{L}}_{n+1} = \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} \cdot \overline{\mathcal{L}}_{n+1} \cdot \overline{\mathcal{L}}_n.$$

This proves the symmetry by induction on n. The multi-linearity then follows easily. Moreover, the symmetry and induction on n implies that $\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_n \cdot \widehat{\mathrm{div}}(f) = 0$ for all $f \in K(\mathcal{X})^*$. Hence well-defined. □

We also have the Projection Formula for the arithmetic case.

Proposition 2.4.10. *Let $f\colon \mathcal{X}' \to \mathcal{X}$ be a morphism of projective arithmetic varieties. For $[\mathcal{Y}'] \in Z_r(\mathcal{X}')$ and $\overline{\mathcal{L}}_1, \ldots, \overline{\mathcal{L}}_r \in \widehat{\mathrm{Pic}}(\mathcal{X})$, we have*

$$f^*\overline{\mathcal{L}}_1 \cdots f^*\overline{\mathcal{L}}_r \cdot [\mathcal{Y}'] = \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_r \cdot f_*[\mathcal{Y}'],$$

where $f_*\colon Z_r(\mathcal{X}') \to Z_r(\mathcal{X})$ is defined in the same way as in the geometric case (2.4.4)*.*

2.5 Positivity of Hermitian line bundles on projective arithmetic varieties

2.5.1 Review on nef and big line bundles in algebraic geometry

Let X be a projective variety defined over a field k, and let $L \in \mathrm{Pic}(X)$. Let $d = \dim X$.

Definition 2.5.1. *The line bundle L is called* **nef** *(numerically effective) if $L \cdot C \geq 0$ for any closed subcurve $C \subseteq X$.*

In fact, if L is nef, then $L^{\dim Y} \cdot Y \geq 0$ for any irreducible closed subvariety Y of X. Thus, nef line bundles are the boundary of the cone of ample line bundle because, by the criterion of Nakai–Moishezon, L is ample if and only if $L^{\dim Y} \cdot Y > 0$ for any irreducible closed subvariety Y of X.

Use the symbol $+$ to denote the binary operation on the group $\mathrm{Pic}(X)$ (so $L + L'$ means $L \otimes L'$). For $n \in \mathbb{Z}_{\geq 1}$, write nL for $L^{\otimes n}$. Denote by $h^0(nL) := \dim_k H^0(X, nL)$.

Definition-Lemma 2.5.2. *The limit*

$$\mathrm{vol}(L) := \lim_{n \to \infty} \frac{d!}{n^d} h^0(nL)$$

exists, and is called the **volume** *of L.*

Definition 2.5.3. *The line bundle L is said to be* **big** *if* $\mathrm{vol}(L) > 0$.

Both definitions are stable under base change, *i.e.*

Lemma 2.5.4. *Assume $k \subseteq k'$ is an inclusion of fields. Then L is a nef (resp. big) line bundle on X if and only if $L_{k'}$ is a nef (resp. big) line bundle on $X_{k'}$.*

In height theory, if we have a big line bundle L on X, then by definition there exists a global section s of nL on X for some $n \gg 1$. Thus the height function h_L has a lower bound outside $|\mathrm{div}(s)|$ by "Lower Bound" of Proposition 0.2.2. In fact, in algebraic geometry, we furthermore have:

Theorem 2.5.5. *The line bundle L is big if and only if $mL = A + \mathcal{O}(E)$ for some $m > 1$, some ample line bundle A and some effective divisor E on X.*

Here are two important theorems to check the bigness of certain line bundles under suitable nefness assumption.

Theorem 2.5.6 (Hilbert–Samuel)**.** *Assume L is nef. Then* $\mathrm{vol}(L) = L^d$.

Theorem 2.5.7 (Siu's inequality)**.** *If L and M are nef line bundles, then*

$$\mathrm{vol}(L - M) \geq L^d - dL^{d-1} \cdot M.$$

In particular, if L is nef and big, then $mL - M$ is big for $m \gg 1$.

If $k = \mathbb{C}$ and L carries a smooth Hermitian metric $\|\cdot\|$, then we can use the curvature form $c_1(L, \|\cdot\|)$ to check the nefness and bigness. Indeed, in this case for any irreducible closed subvariety Y of X, we have

$$L^{\dim Y} \cdot Y = \int_{Y^{\mathrm{reg}}(\mathbb{C})} c_1(L, \|\cdot\|)^{\wedge \dim Y},$$

where the integral is on the regular locus of Y (or equivalently, the desingularization of Y and then take the pullback of $c_1(L, \|\cdot\|)$). Hence we have:

(i) L is nef if $c_1(L, \|\cdot\|) \geq 0$;

(ii) if $c_1(L, \|\cdot\|) \geq 0$, then L is big if and only if $c_1(L, \|\cdot\|)^{\wedge d} \not\equiv 0$.

2.5.2 Arithmetic volumes

Let $\mathcal{X}$ be a projective arithmetic variety. Let $n+1 = \dim \mathcal{X}$.

Let $\overline{\mathcal{L}} \in \widehat{\mathrm{Pic}}(\mathcal{X})$ be a Hermitian line bundle. Define

$$H^0(\mathcal{X}, \overline{\mathcal{L}}) := \{s \in H^0(\mathcal{X}, \mathcal{L}) : \|s\|_{\sup} \leq 1\}, \tag{2.5.1}$$

where $\|s\|_{\sup} = \sup_{x \in \mathcal{X}(\mathbb{C})} \|s(x)\|$ is the usual supremum norm on $H^0(\mathcal{X}, \mathcal{L})_{\mathbb{C}}$.

The proof of the following lemma is important, especially the construction of the real Euclidean space $H^0(\mathcal{X}, \mathcal{L})_{\mathbb{R}}$ (which generalizes Example 1.2.5).

Lemma 2.5.8. *$H^0(\mathcal{X}, \overline{\mathcal{L}})$ is a finite set.*

Proof. The structural morphism $\mathcal{X} \to \mathrm{Spec}\mathbb{Z}$ factors through $\mathrm{Spec}R$ for an order R in a number field K, such that the generic fiber $\mathcal{X}_K$ is irreducible. We have

$$H^0(\mathcal{X}, \mathcal{L})_{\mathbb{C}} = H^0(\mathcal{X}(\mathbb{C}), \mathcal{L}(\mathbb{C})) = \bigoplus\nolimits_{\sigma \colon K \hookrightarrow \mathbb{C}} H^0(\mathcal{X}_\sigma, \mathcal{L}_\sigma),$$

with $\mathcal{X}_\sigma = \mathrm{Hom}_{\mathrm{Spec}\sigma(K)}(\mathrm{Spec}\mathbb{C}, \mathcal{X})$ and $\mathcal{L}_\sigma$ defined similarly. Set

$$H^0(\mathcal{X}, \mathcal{L})_{\mathbb{R}} := \{s = (s_\sigma)_\sigma \in H^0(\mathcal{X}, \mathcal{L})_{\mathbb{C}} : s_\sigma = \overline{s_{\overline{\sigma}}} \text{ for all } \sigma\}. \tag{2.5.2}$$

Then $\|\cdot\|_{\sup}$ induces a norm on $H^0(\mathcal{X}, \mathcal{L})_{\mathbb{R}}$, and $H^0(\mathcal{X}, \overline{\mathcal{L}}) = H^0(\mathcal{X}, \mathcal{L}) \cap B(\overline{\mathcal{L}})$ with

$$B(\overline{\mathcal{L}}) = \{s \in H^0(\mathcal{X}, \mathcal{L})_{\mathbb{R}} : \|s\|_{\sup} \leq 1\}.$$

So $H^0(\mathcal{X}, \overline{\mathcal{L}})$ is the set of lattice points contained in the unit ball, which is a finite set. $\square$

Notice that we are again back in the context of Geometry of Numbers discussed in §1.3.1, with $M = H^0(\mathcal{X}, \mathcal{L})$ and $\|\cdot\|_{\sup}$.

Now define

$$h^0(\overline{\mathcal{L}}) := \log \# H^0(\mathcal{X}, \overline{\mathcal{L}}). \tag{2.5.3}$$

Elements in $H^0(\mathcal{X}, \overline{\mathcal{L}})$ are usually called *small sections* or *effective sections* (we will explain this second terminology at the end of this section).

Definition-Proposition 2.5.9. *The sup-limit*

$$\mathrm{vol}(\overline{\mathcal{L}}) := \limsup_{N \to \infty} \frac{h^0(N\overline{\mathcal{L}})}{N^{n+1}/(n+1)!}$$

exists, and is called the **(arithmetic) volume** *of* $\overline{\mathcal{L}}$.

In practice, it is not easy to count the number of lattice points. Instead, here is a number which approximates this number in an asymptotic way and is easier to handle. Fix any Haar measure on $H^0(\mathcal{X}, \mathcal{L})_{\mathbb{R}}$, and set

$$\chi(\overline{\mathcal{L}}) := \log \frac{\mathrm{vol}(B(\overline{\mathcal{L}}))}{\mathrm{covol}(H^0(\mathcal{X}, \mathcal{L})_{\mathbb{R}}/H^0(\mathcal{X}, \mathcal{L}))}, \tag{2.5.4}$$

which is independent of the choice of the Haar measure (not hard to check). The quantitative version of Minkowski's first theorem (Theorem 1.3.5) then yields

$$h^0(\overline{\mathcal{L}}) \geq \chi(\overline{\mathcal{L}}) - h^0(\mathcal{L}_{\mathbb{Q}}) \log 2. \tag{2.5.5}$$

Thus we can make the following definition:

Definition 2.5.10. *The* **χ-volume** *of* $\overline{\mathcal{L}}$ *is defined to be the sup-limit*

$$\mathrm{vol}_\chi(\overline{\mathcal{L}}) := \limsup_{N \to \infty} \frac{\chi(N\overline{\mathcal{L}})}{N^{n+1}/(n+1)!}.$$

(2.5.5) furthermore implies that $\mathrm{vol}_\chi(\overline{\mathcal{L}}) \leq \mathrm{vol}(\overline{\mathcal{L}})$.

2.5.3 Arithmetic nefness, bigness, and ampleness

Let $\mathcal{X}$ be a projective arithmetic variety, and let $n+1 = \dim \mathcal{X}$.

Definition 2.5.11. *A Hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|) \in \widehat{\mathrm{Pic}}(\mathcal{X})$ is said to be:*

(1) **nef** *if*

  (i) $c_1(\overline{\mathcal{L}}, \|\cdot\|) \geq 0$*;*

  (ii) $\overline{\mathcal{L}} \cdot \mathcal{Y} \geq 0$ *for any integral* 1*-dimensional subscheme* $\mathcal{Y}$ *of* $\mathcal{X}$*.*

(2) **weakly ample** *if* $\overline{\mathcal{L}}$ *is nef and* $\mathcal{L}_{\mathbb{Q}}$ *is ample.*

(3) **ample** *if* $\overline{\mathcal{L}}$ *is weakly ample and* $\overline{\mathcal{L}}^{\dim \mathcal{Y}} \cdot \mathcal{Y} > 0$ *for any integral subscheme* $\mathcal{Y}$ *of* $\mathcal{X}$*.*

Definition 2.5.12. *A Hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|) \in \widehat{\mathrm{Pic}}(\mathcal{X})$ is said to be* **big** *if* $\mathrm{vol}(\overline{\mathcal{L}}) > 0$*.*

In height theory, suppose $(\mathcal{X}, \overline{\mathcal{L}})$ is an arithmetic model of (X, L) with X a projective variety. If theHermitian line bundle $\overline{\mathcal{L}}$ is big, then by definition there exists a global section s of $N\overline{\mathcal{L}}$ on $\mathcal{X}$ with $\|s\|_{\sup} \leq 1$ for some $N \gg 1$. Thus the height function $h_{\overline{\mathcal{L}}}$ is bounded below by 0 outside the generic fiber of $|\mathrm{div}(s)|$, by the definition of $h_{\overline{\mathcal{L}}}$ (2.3.5). Thus instead of having only a lower bound, we have *positivity*.

Theorem 2.5.13 (Arithmetic Hilbert–Samuel)**.** *Assume $\overline{\mathcal{L}}$ is nef. Then* $\mathrm{vol}(\overline{\mathcal{L}}) = \overline{\mathcal{L}}^{n+1}$.

Theorem 2.5.14 (Arithmetic Siu)**.** *Assume $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$ are nef Hermitian line bundles on $\mathcal{X}$. Then*

$$\mathrm{vol}(\overline{\mathcal{L}} - \overline{\mathcal{M}}) \geq \overline{\mathcal{L}}^{n+1} - (n+1)\overline{\mathcal{L}}^{n} \cdot \overline{\mathcal{M}}.$$

Indeed, both theorems still hold true with vol replaced by vol_{χ}. For vol_{χ} and for weakly ample $\overline{\mathcal{L}}$, the Arithmetic Hilbert–Samuel Formula is a consequence of Gillet–Soulé's arithmetic Riemann–Roch theorem and an estimate of analytic torsions by Bismut–Vasserot (with refinement by Zhang); a direct proof was later on given by Abbès–Bouche. For vol and $\overline{\mathcal{L}}$ ample, the Arithmetic Hilbert–Samuel Formula by Zhang by furthermore using his arithmetic Nakai–Moishezon theorem. Moriwaki extended these results to nef Hermitian line bundle (with continuous metrics). Arithmetic Siu is a result of Yuan.

In the next chapters, we will present the proof of Abbès–Bouche of the Arithmetic Hilbert–Samuel Formula.

We close this section with the following discussion on the effectiveness of arithmetic divisors. Let $\overline{D} = (D, g_D)$ be an arithmetic divisor on $\mathcal{X}$.

Definition 2.5.15. *We say that $\overline{D}$ is* **effective** *(resp.* **strictly effective***) if $D \geq 0$ and $g_D \geq 0$ (resp. $D \geq 0$ and $g_D > 0$).*

Recall that $\mathcal{O}(\overline{D})$ is the Hermitian line bundle on $\mathcal{X}$ with the metric $\|\cdot\|$ determined by $\|s_D\| = e^{-g_D}$. Thus if $\overline{D}$ is effective, then $h^0(\mathcal{O}(\overline{D})) > 0$. Conversely, if a Hermitian line bundle $\overline{\mathcal{L}}$ on $\mathcal{X}$ satisfies $h^0(\overline{\mathcal{L}}) > 0$, then there exists a non-zero $s \in H^0(\mathcal{X}, \mathcal{L})$ such that $\|s(x)\| \leq 1$ for all $x \in \mathcal{X}(\mathbb{C})$, and hence the arithmetic divisor $\widehat{\mathrm{div}}(s) = (\mathrm{div}(s), -\log \|s\|)$ is effective.

For this reason, we sometimes call elements in $H^0(\mathcal{X}, \overline{\mathcal{L}})$ *effective sections*, and say that $\overline{\mathcal{L}}$ is *effective* if $h^0(\overline{\mathcal{L}}) > 0$.

Chapter 3

Preparation on analysis for the proof of Arithmetic Hilbert–Samuel

The goal of this chapter is to discuss about some analytic tools and results which will be used to prove the Arithmetic Hilbert–Samuel Theorem (for which we follow the approach of Abbès–Bouche) in the next chapter.

3.1 Distortion function

3.1.1 Fubini–Study metric

Let X be a connected complex manifold of dimension n, endowed with a smooth Hermitian metric (*i.e.* a J-invariant positive-definite Hermitian inner product $h(\cdot,\cdot)$ on T_X where J is the complex structure on X). This Hermitian metric induces a positive $(1,1)$-form $\omega = -\mathrm{Im}h$ on X, and hence a volume form $\mathrm{d}V := \omega^{\wedge n}/n!$ on X. Notice that h can be recovered from ω and J via the formula $h(u,v) = \omega(u,Jv) - \sqrt{-1}\omega(u,v)$.

Definition 3.1.1. *Such a complex manifold X is called a* **Kähler manifold** *if ω is closed.*

If X is a Kähler manifold, we usually call ω its *Kähler form.*

Example 3.1.2. *For $X = \mathbb{P}^n$, the* **Fubini–Study metric** *is defined as follows. We have the standard projection $\mathbb{C}^{n+1}\setminus\{0\}\to\mathbb{P}^n$ by viewing $\mathbb{P}^n$ as the space consisting of all complex lines in $\mathbb{C}^{n+1}$. The standard Hermitian metric on $\mathbb{C}^{n+1}$ defines the following $(1,1)$-form on $\mathbb{P}^n$*

$$\omega_{\mathrm{FS}} = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(|z_0|^2+\cdots+|z_n|^2)$$

with $(z_0,\ldots,z_n)$ the standard coordinate of $\mathbb{C}^{n+1}$. To see this, consider any open subset $U\subseteq\mathbb{P}^n$ such that natural projection admits a lifting $Z\colon U\to\mathbb{C}^{n+1}\setminus\{0\}$. Thenany other lifting Z' differs from Z by a non-zero holomorphic function f, and hence $\partial\overline{\partial}\log|Z'|^2 = \partial\overline{\partial}\log|fZ|^2 = \partial\overline{\partial}\log|Z|^2 + \partial\overline{\partial}\log(f\overline{f}) = \partial\overline{\partial}\log|Z|^2$. Thus the local $(1,1)$-forms $\partial\overline{\partial}\log|Z|^2$, with U varying, patch together to a global $(1,1)$-form, which is exactly $(2\pi/\sqrt{-1})\omega_{\mathrm{FS}}$.

Notice that $\mathrm{d}\omega_{\mathrm{FS}} = 0$, i.e. ω_{FS} is closed.

To see that ω_{FS} is a positive $(1,1)$-form, it suffices to prove that it is positive at one point since ω is invariant under the group action of $U(n+1)$ on $\mathbb{P}^n$ (which is transitive). Use $\{w_1,\ldots,w_n\}$

to denote the standard coordinate on the open subset $U_0 := \{z_0 \neq 0\} \subseteq \mathbb{P}^n$, i.e. $w_j = z_j/z_0$. Then

$$\omega_{\mathrm{FS}}|_{U_0} = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(1+\sum w_j\overline{w}_j) = \frac{\sqrt{-1}}{2\pi}\left(\frac{\sum \mathrm{d}w_j\wedge\mathrm{d}\overline{w}_j}{1+\sum w_j\overline{w}_j} - \frac{(\sum \overline{w}_j\mathrm{d}w_j)\wedge(\sum w_j\wedge\mathrm{d}\overline{w}_j)}{(1+\sum w_j\overline{w}_j)^2}\right),$$

which is $\frac{\sqrt{-1}}{2\pi}\sum \mathrm{d}w_j\wedge\mathrm{d}\overline{w}_j$ at $[1:0:\cdots:0]$. Thus ω_{FS} is positive.

By the discussion above, ω_{FS} defines a Hermitian metric on $\mathbb{P}^n$, which is called the Fubini–Study metric.

By Example 3.1.2, the analytification of any smooth quasi-projective variety is a Kähler manifold.

Another way to see the Fubini–Study metric on $\mathbb{P}^n$ is as via a suitable Hermitian metric $\|\cdot\|_{\mathrm{FS}}$ on $\mathcal{O}_{\mathbb{P}^n}(1)$ as follows. The coordinate functions $X_0,\ldots,X_n$ form a basis of $H^0(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(1))$. At each point $x=[x_0:\cdots:x_n]\in\mathbb{P}^n$, define for a global section $s=a_0X_0+\cdots+a_Nx_N$

$$\|s(x)\|_{\mathrm{FS}} := \frac{|a_0x_0+\cdots+a_Nx_N|}{\sqrt{|x_0|^2+\cdots+|x_N|^2}}. \tag{3.1.1}$$

Then one can check that $c_1(\mathcal{O}_{\mathbb{P}^n}(1),\|\cdot\|_{\mathrm{FS}}) = \omega_{\mathrm{FS}}$.

3.1.2 Distortion function

Let X be a *compact* Kähler manifold. Let L be a line bundle on X, endowed with a smooth Hermitian metric $\|\cdot\|$ which is positive, *i.e.* $c_1(L,\|\cdot\|)$ is a positive $(1,1)$-form on X. By the Kodaira embedding theorem, L is an ample line bundle on X (and hence X is projective). Now for each $k\geq 0$, denote by $kL := L^{\otimes k}$, $V_k := H^0(X,kL)$ the space of holomorphic sections of kL on X, and

$$\Phi_k\colon X\to\mathbb{P}(V_k^\vee), \qquad x\mapsto H_x=\{\sigma\in V_k : \sigma(x)=0\}. \tag{3.1.2}$$

Then Φ_k is a closed immersion with $\Phi_k^*\mathcal{O}_{\mathbb{P}(V_k^\vee)}(1)\simeq kL$ for all $k\gg 1$.

On kL, we have the natural Hermitian metric $\|\cdot\|_k$, which is the metric of $(L,\|\cdot\|)^{\otimes k}$. On the other hand, we have the Fubini–Study metric on $\mathcal{O}_{\mathbb{P}(V_k^\vee)}(1)$ as defined by (3.1.1). Thus its pullback via Φ_k defines a Hermitian metric on kL, which we call $\|\cdot\|_{k^{\mathrm{FS}}}$.

Thus on kL, we have two Hermitian metrics: $\|\cdot\|_k$ and $\|\cdot\|_{k^{\mathrm{FS}}}$.

Definition 3.1.3. *The k-th* **distortion function** *is*

$$b_k\colon X\to\mathbb{R}, \qquad x\mapsto \frac{\|\xi\|_k^2}{\|\xi\|_{k^{\mathrm{FS}}}^2}$$

for any $\xi\in(kL)_x\setminus\{0\}$.

Here is a more explicit expression of the distortion function. On V_k we have the L^2-norm defined by

$$\|s\|_{L^2}^2 = \int_X \|s(x)\|_k^2\mathrm{d}V \qquad \text{for all } s\in V_k=H^0(X,kL).$$

Then V_k is canonically isomorphic to $V_k^\vee$, by sending $v\mapsto\langle v,-\rangle_{L^2}$ for the inner product determined by the L^2-norm. Let $s_1,\ldots,s_N$ be an orthonormal basis of $V_k=H^0(X,kL)$ for this L^2-norm. Then it is not hard to compute that $\Phi_k(x)=[s_1(x):\cdots:s_N(x)]$ under $V_k=\bigoplus_{j=1}^N\mathbb{C}s_j$. Then $\|\xi\|_{k^{\mathrm{FS}}}^2 = (\|s_1(x)\|_k^2+\cdots+\|s_N(x)\|_k^2)^{-1}\|\xi\|_k^2$ by (3.1.1). Thus

$$b_k(x)=\sum_{j=1}^N\|s_j(x)\|_k^2. \tag{3.1.3}$$

3.1.3 Main result on the distortion function

The main result about the distortion function is the following:

Theorem 3.1.4. *The function $(b_k)^{1/k}$ converges to 1 uniformly on X. Namely for any $\epsilon > 0$, there exists k_0 such that $|b_k(x)^{1/k} - 1| < \epsilon$ for all $k \geq k_0$ and all $x \in X$.*

In other terminology, the Fubini–Study metric on L flattens uniformly into the initial metric $\|\cdot\|$.

We shall prove a more precise version of this theorem. For the statement we need to introduce the following notion. Locally on X we can find a suitable complex coordinate $(z_1, \ldots, z_n)$ of X such that: (i) $\omega = \frac{\sqrt{-1}}{2}\sum_{j=1}^n \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$ (in other words, $\frac{1}{\sqrt{2}}(\mathrm{d}z_1, \mathrm{d}\overline{z}_1 \ldots, \mathrm{d}z_n, \mathrm{d}\overline{z}_n)$ is an orthonormal frame of T_X^* with respect to the Hermitian metric), (ii) the $(1,1)$-form $c_1(L, \|\cdot\|)$ equals $\frac{\sqrt{-1}}{2}\sum_{j=1}^n \alpha_j(x)\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$ with $\alpha_j(x) > 0$.

Definition 3.1.5. *The functions $\alpha_1, \ldots, \alpha_n$ are called the* **eigenfunctions** *of $c_1(L, \|\cdot\|)$ with respect to ω (or with respect to the Hermitian metric on X). The* **determinant** *is defined to be the smooth function on X*

$$\det c_1(L, \|\cdot\|) := \alpha_1 \cdots \alpha_n.$$

Theorem 3.1.6. *When $k \to \infty$, the function*

$$\frac{b_k}{k^n \det c_1(L, \|\cdot\|)}$$

converges to 1 uniformly on X.

Theorem 3.1.6 implies Theorem 3.1.4 immediately.

3.2 Proof of the main theorem on the distortion function via heat kernel

Let X be a connected *compact* Kähler manifold of dimension n, and let $\mathrm{d}V$ be the volume form on X. Let L be a line bundle on X, endowed with a smooth Hermitian metric $\|\cdot\|$.

3.2.1 Anti-holomorphic Kodaira Laplacian and Harmonic forms

For any $k \geq 1$, denote by $A^{0,q}(X, kL)$ the space of smooth global $(0,q)$-forms with values in $kL := L^{\otimes k}$ (*i.e.* global sections of $(T_X^{0,q})^* \otimes L^{\otimes k}$). If $q = 0$, notice that $A^{0,0}(X, kL)$ is precisely the space of smooth (real) sections of kL over X.

The Hermitian metric on X and the Hermitian metric on L together induce a Hermitian metric on $(T_X^{0,q})^* \otimes L^{\otimes k}$ which we denote by $\|\cdot\|_{k,q}$. Then we can endow $A^{0,q}(X, kL)$ with norms, for example the L^2-norm

$$\|\sigma\|_{L^2} := \left(\int_X \|\sigma(x)\|_{k,q}^2 \mathrm{d}V\right)^{1/2}, \qquad \forall \sigma \in A^{0,q}(X, kL).$$

Each such norm defines a sesquilinear pairing $(\cdot,\cdot)_q$ on $A^{0,q}(X, kL)$. Denote by $L_q^2(X, kL)$ the completion of $A^{0,q}(X, kL)$ with respect to the L^2-norm. It is a Hilbert space.

The differential operator $\overline{\partial}\colon (T_X^{0,q})^* \to (T_X^{0,q+1})^*$ induces a differential operator $\overline{\partial}_{k,q}\colon A^{0,q}(X, kL) \to A^{0,q+1}(X, kL)$. And $\overline{\partial}_{k,q}$ has an adjoint $\overline{\partial}_{k,q}^*\colon A^{0,q+1}(X, kL) \to A^{0,q}(X, kL)$ with respect to the given norms, determined by $(\overline{\partial}_{k,q}u, u')_{q+1} = (u, \overline{\partial}_{k,q}^* u')_q$.

Definition 3.2.1. *The* **anti-holomorphic Kodaira Laplacian** *is*

$$\Delta''_{k,q} := \overline{\partial}_{k,q-1}\overline{\partial}^*_{k,q-1} + \overline{\partial}^*_{k,q}\overline{\partial}_{k,q}$$

with the first term being 0 *if* $q = 0$.

A smooth $(0, q)$*-form* u *is called a* **harmonic form** *if* $\Delta''_{k,q}u = 0$.

In our case, we are interested in the operator

$$\overline{\Box}^q_k := \frac{2}{k}\Delta''_{k,q}. \tag{3.2.1}$$

Notice that $\mathrm{Ker}\overline{\Box}^q_k = \mathrm{Ker}\Delta''_{k,q}$.

The cohomology of the Dolbeault complex $\cdots \to A^{0,q}(X, kL) \xrightarrow{\overline{\partial}} A^{0,q+1}(X, kL) \to \cdots$ gives $H^{0,q}(X, kL) \simeq H^q(X, \Omega^0_X \otimes L^{\otimes k}) = H^q(X, kL)$.

We state the following lemma without proof (the proof is not hard).

Lemma 3.2.2. *A* $\overline{\partial}$*-closed form* $u \in A^{0,q}(X, kL)$ *is of minimal norm in* $u + \mathrm{Im}\overline{\partial}$ *if and only if* $\overline{\partial}^* u = 0$.

This lemma (formally) implies that the Dolbeault cohomology group $H^{0,q}(X, kL)$ is represented exactly by solutions of two first-order equations

$$\overline{\partial}u = 0, \quad \overline{\partial}^* u = 0,$$

which can be replaced by the single second-order equation

$$\Delta''_{k,q}u = 0.$$

Thus we have

$$H^q(X, kL) \simeq \mathrm{Ker}\Delta''_{k,q} = \mathrm{Ker}\overline{\Box}^q_k.$$

In particular if $q = 0$, then this realizes $H^0(X, kL)$ as the subspace $\mathrm{Ker}\overline{\Box}^0_k$ of $A^{0,0}(X, kL)$.

In general, we have an L^2-orthogonal decomposition

$$A^{0,q}(X, kL) = \mathrm{Ker}\overline{\Box}^q_k \oplus \mathrm{Im}\overline{\partial}_{k,q-1} \oplus \mathrm{Im}\overline{\partial}^*_{k,q+1}.$$

Recall that X is compact. We state the following (special case of a) theorem on the spectrum of any self-adjoint elliptic operator which is semi-bounded.

Theorem 3.2.3 (Spectral theorem)**.** *The operator* $\overline{\Box}^q_k$ *has discrete spectrum (of eigenvalues)*

$$0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_m \le \cdots \to \infty$$

and there exists a corresponding orthonormal basis consisting of smooth eigenforms $\{\psi_m\}$*, i.e.* $\overline{\Box}^q_k\psi_m = \lambda_m\psi_m$ *for non-zero* ψ_m.

In general, this theorem can be applied to any self-adjoint elliptic operator P which is semi-bounded (*i.e.* $(Pu, u)_{L^2} \ge -c\|u\|^2_{L^2}$ for some fixed $c \in \mathbb{R}$) and with 0 replaced by $-c$.

3.2.2 Heat kernel associated with the anti-holomorphic Kodaira Laplacian

We shall assume the following proposition which claims the existence of the heat kernel, which is our main tool to prove Theorem 3.1.6.

Proposition 3.2.4. *The operator $\overline{\Box}_k^q$ admits a smooth (heat) kernel $e_k^q(t,x,y)$, uniquely determined by the following properties:*

(i) It is a smooth function on $\mathbb{R}_{>0} \times X \times X$ taking values in $\mathrm{End}((T_X^{0,q})^* \otimes L^{\otimes k})$.

(ii) $(\frac{\partial}{\partial t} + \overline{\Box}_k)e_k^q = 0$ *with $\overline{\Box}_k^q$ acting on the x-variable.*

(iii) $e_k^q(t,x,y) \to \delta_y$ *(Dirac function) when $t \to 0^+$.*

More concretely, (ii) and (iii) mean the following: For each $u_0(x)$, there exists a unique smooth solution $u = u(t,x) \colon \mathbb{R}_{\geq 0} \times X \to \mathrm{End}((T_X^{0,q})^* \otimes L^{\otimes k})$ to the heat equation

$$\begin{cases} (\frac{\partial}{\partial t} + \overline{\Box}_k^q)u = 0 \\ u(0,x) = u_0(x), \end{cases}$$

which can be obtained as

$$u(t,x) = \int_X e_k^q(t,x,y)u_0(y)\mathrm{d}y. \tag{3.2.2}$$

We sometimes call $e_k^q(t,x,y)$ the *fundamental solution* of $(\frac{\partial}{\partial t} + \overline{\Box}_k)u = 0$. It is known that under the eigenbasis given by Theorem 3.2.3, we have

$$e_k^q(t,x,y) = \sum_{m\geq 1} e^{-\lambda_m t}\psi_m(x) \otimes \psi_m^*(y).$$

We shall be interested in the *diagonal* of the heat kernel, which for simplicity we denote by

$$e_k^q(t,x) := e_k^q(t,x,x) = \sum_{\lambda} e^{-\lambda t}\|\psi_\lambda(x)\|_{k,q}^2 \tag{3.2.3}$$

for the L^2-orthonormal eigenbasis $(\lambda, \psi_\lambda)_\lambda$ given by Theorem 3.2.3; here we abuse the notation since there can be more than 1 eigenforms for each λ.

The following theorem is the main theorem on heat kernel expansion and is of fundamental importance. We state the theorem without proof.

Let $\alpha_1, \ldots, \alpha_n$ be the eigenfunctions of $c_1(L, \|\cdot\|)$ with respect to the Hermitian metric on X. For any multi-index J, set $\overline{\alpha}_J := \sum_{j\notin J} \alpha_j - \sum_{j\in J}\alpha_j$. Define

$$e_\infty^q(t,x) := \alpha_1(x)\cdots\alpha_n(x)\frac{\sum_{|J|=q} e^{t\overline{\alpha}_J(x)}}{\prod_{j=1}^n (e^{t\alpha_j(x)} - e^{-t\alpha_j(x)})}. \tag{3.2.4}$$

Theorem 3.2.5. *There exists a real number $\epsilon > 0$ with the following property. When $k \to \infty$, the function $k^{-n}e_k^q(t,x)$ converges to $e_\infty^q(t,x)$ uniformly with respect to $x \in X$ and $t \in (0, k^{2\epsilon})$.*

3.2.3 Application to the proof of Theorem 3.1.6

Let us prove Theorem 3.1.6 by using the results on heat kernel above.

Let $(\lambda, \psi_\lambda)_\lambda$ be an L^2-orthonormal eigenbasis for the operator $\overline{\square}_k^0$ from Theorem 3.2.3. Recall that $H^0(X, kL)$ is precisely the subspace $\mathrm{Ker}\overline{\square}_k^0$ of $A^{0,0}(X, kL)$. Thus

$$e_k^0(t, x) = \sum_\lambda e^{-t\lambda} \|\psi_\lambda(x)\|_{k,0}^2 = b_k(x) + \sum_{\lambda>0} e^{-t\lambda} \|\psi_\lambda(x)\|_{k,0}^2 \tag{3.2.5}$$

where the second equality follows from (3.1.3).

We will study the asymptotic behavior of $e_k^0(t, x)$ and of $e_k^0(t, x) - b_k(x) = \sum_{\lambda>0} e^{-t\lambda} \|\psi_\lambda(x)\|_{k,0}^2$ separately.

By Theorem 3.2.5 with $q = 0$, we get

$$e_k^0(t, x) = \left(\alpha_1(x) \cdots \alpha_n(x) \prod_{j=1}^n \frac{1}{1 - e^{-2t\alpha_j(x)}} \right) k^n + o(k^n)$$

uniformly in $x \in X$ and in $t \in (0, k^{2\epsilon})$ for a fixed ϵ. Taking $t = k^\epsilon \to \infty$, we get

$$e_k^0(k^\epsilon, x) \sim \alpha_1(x) \cdots \alpha_n(x) k^n. \tag{3.2.6}$$

On the other hand for each $\lambda > 0$, we have $e^{-t\lambda/2} \|\psi_\lambda(x)\|_{k,0}^2 < e_k^0(t/2, x)$ by (3.2.5). Thus

$$\sum_{\lambda>0} e^{-t\lambda} \|\psi_\lambda(x)\|_{k,0}^2 < e_k^0(t/2, x) \sum_{\lambda>0} e^{-t\lambda/2} \tag{3.2.7}$$

Lemma 3.2.6. *Let $\lambda > 0$ be an eigenvalue of $\overline{\square}_k^0$. For any eigenfunction ψ_λ associated with λ, the $(0,1)$-form $\overline{\partial}\psi_\lambda$ is an eigenform for $\overline{\square}_k^1$ associated with λ.*

Sometimes we say that *the positive spectrum of $\overline{\square}_k^0$ injects into the positive spectrum of $\overline{\square}_k^1$*. Notice that this lemma immediately implies that $\overline{\partial}\psi_\lambda = \overline{\partial}\psi'_\lambda$ if and only if $\psi_\lambda = \psi'_\lambda$.

Proof. We have $\overline{\square}_k^0 \psi_\lambda = \lambda\psi_\lambda$. Applying $\overline{\partial}$ to both sides, we get $\overline{\partial}\overline{\partial}^* \overline{\partial}\psi_\lambda = (k/2)\lambda\overline{\partial}\psi_\lambda$. Thus $\overline{\square}_k^1(\overline{\partial}\psi_\lambda) = \lambda\overline{\partial}\psi_\lambda$.

It remains to show that $\overline{\partial}\psi_\lambda \neq 0$. Suppose $\overline{\partial}\psi_\lambda = 0$. Then ψ_λ is a holomorphic function on X, and hence is constant since X is compact. But then $\overline{\square}_k^0 \psi_\lambda = 0$, so $\psi_\lambda = 0$, which is a contradiction. $\square$

These $(0,1)$-forms $\overline{\partial}\psi_\lambda$ are still orthogonal to each other, but they do not necessary have L^2-norm 1 (and hence should be normalized).

By Lemma 3.2.6 and (3.2.3), we have

$$\sum_{\lambda>0} e^{-t\lambda} \frac{\|\overline{\partial}\psi_\lambda(x)\|_{k,1}^2}{\|\overline{\partial}\psi_\lambda\|_{L^2}^2} < e_k^1(t, x).$$

Integrating on X and by the definition of the L^2-norm, we get

$$\sum_{\lambda>0} e^{-t\lambda} < \int_X e_k^1(t, x) \mathrm{d}V. \tag{3.2.8}$$

Now (3.2.7) and (3.2.8) together yield

$$\sum_{\lambda>0} e^{-t\lambda}\|\psi_\lambda(x)\|^2_{k,0} < e^0_k(t/2,x)\int_X e^1_k(t/2,x)\mathrm{d}V. \tag{3.2.9}$$

By Theorem 3.2.5 with $q=1$, we get

$$e^1_k(t,x) = \left(\alpha_1(x)\cdots\alpha_n(x)\sum_{j=1}^n \frac{1}{e^{2t\alpha_j(x)}-1}\prod_{j'\neq j}\frac{1}{1-e^{-2t\alpha_{j'}(x)}}\right)k^n + o(k^n)$$

uniformly in $x\in X$ and in $t\in(0,k^{2\epsilon})$. Set $\alpha_0 := 2\inf_{j,x}\alpha_j(x) > 0$. Then $e^1_k(t,x)$ is uniformly bounded above by $Ce^{-\alpha_0 t}k^n$ for some real number $C>0$. Letting $t=k^\epsilon$, we get

$$e^0_k(k^\epsilon/2,x)\int_X e^1_k(k^\epsilon/2,x)\mathrm{d}V \le C' e^{-\alpha_0 k^\epsilon}k^{2n}$$

which converges to 0 uniformly in $x\in X$ when $k\to\infty$. Thus by (3.2.9) we have

$$\sum_{\lambda>0} e^{-k^\epsilon\lambda}\|\psi_\lambda(x)\|^2_{k,0}\to 0 \quad \text{when } k\to\infty \tag{3.2.10}$$

uniformly in $x\in X$.

Let $t=k^\epsilon$ in (3.2.5), Theorem 3.1.6 immediately follows from (3.2.6) and (3.2.10). □

3.2.4 Application to a lower bound of the smallest non-zero eigenvalue

Lemma 3.2.7. *Let μ_k be the smallest non-zero eigenvalue of $\overline{\square}^0_k$ on X. Then*

$$\liminf_k \mu_k \ge \alpha_0$$

where $\alpha_0 := 2\inf_{j,x}\alpha_j(x) > 0$ for the eigenfunctions $\alpha_1,\ldots,\alpha_n$ of $c_1(L,\|\cdot\|)$ with respect to the Hermitian metric on X.

Proof. By (3.2.8), we have $e^{-t\mu_k} < \int_X e^1_k(t,x)\mathrm{d}V$. By Theorem 3.2.5 with $q=1$, we get that $e^1_k(k^\epsilon,x)$ is uniformly bounded above in $x\in X$ by $Ce^{-\alpha_0 t}k^n$ for some real numbers $C>0$ and $\epsilon>0$ by the argument as above. Thus $e^{t\mu_k} < Ce^{-\alpha_0 k^\epsilon}k^n$. Taking the log of both sides and letting $k\to\infty$, we can conclude. □

3.3 L^2-existence

Let X be a connected (not necessarily compact) Kähler manifold of dimension n with Kähler form ω, and let $\mathrm{d}V_\omega = \omega^{\wedge n}/n!$ be the volume form on X.

Let L be a line bundle on X, endowed with a smooth Hermitian metric $\|\cdot\|$.

3.3.1 Setup

Denote by $A^{p,q}_{\mathrm{c}}(X,L)$ the space of compactly supported smooth global (p,q)-forms with values in L (*i.e.* global sections of $(T^{p,q}_X)^*\otimes L$ which are compactly supported). The Hermitian metric on X and the Hermitian metric on L together induce a Hermitian metric on $(T_X^{p,q})^* \otimes L$ which we denote by $|\cdot|_{p,q}$. Then we can endow $A_{\mathrm{c}}^{p,q}(X,L)$ with the L^2-norm

$$\|\sigma\|_{L^2} := \left(\int_X |\sigma(x)|_{p,q}^2 \mathrm{d}V_\omega \right)^{1/2}, \qquad \forall \sigma \in A_{\mathrm{c}}^{p,q}(X,L).$$

This norm defines a sesquilinear pairing $\langle \cdot, \cdot \rangle_{L^2}$ on $A^{p,q}(X,L)$.

Denote by $L^2_{p,q}(X,L)$ the completion of $A_{\mathrm{c}}^{p,q}(X,L)$ with respect to the L^2-norm. It is a Hilbert space.

Let $\Lambda := \Lambda_\omega$ be the adjoint of the operator $\omega\wedge\colon A_{\mathrm{c}}^{p,q}(X,L) \to A_{\mathrm{c}}^{p+1,q+1}(X,L)$ with respect to the L^2-norm. Then we have a differential operator

$$A_\omega := [2\pi c_1(L, \|\cdot\|)\wedge, \Lambda] = 2\pi c_1(L, \|\cdot\|) \wedge \circ \Lambda - \Lambda \circ 2\pi c_1(L, \|\cdot\|)\wedge \tag{3.3.1}$$

on $A_{\mathrm{c}}^{p,q}(X,L)$ for all $p, q \geq 1$.

Example 3.3.1. *Consider $X = \mathbb{C}^n$ with the standard metric, and $L = \mathcal{O}_X$ with the trivial metric (i.e. $(\mathcal{O}_X, \|\cdot\|)$ is the trivial Hermitian line bundle on $\mathbb{C}^n$). Then $\omega = 2\pi c_1(\mathcal{O}_X, \|\cdot\|) = \frac{\sqrt{-1}}{2}\sum_{j=1}^n \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$. For each j, denote by $e_j\colon A_{\mathrm{c}}^{p,q}(\mathbb{C}^n) \to A_{\mathrm{c}}^{p+1,q}(\mathbb{C}^n)$ the operator $\mathrm{d}z_j\wedge$ (resp. $\overline{e}_j\colon A_{\mathrm{c}}^{p,q}(\mathbb{C}^n) \to A_{\mathrm{c}}^{p,q+1}(\mathbb{C}^n)$ the operator $\mathrm{d}\overline{z}_j\wedge$). Then their adjoints satisfy $e_j^*(\mathrm{d}z_J \wedge \mathrm{d}\overline{z}_{J'}) = 0$ if $j \notin J$ and $e_j^*(\mathrm{d}z_j \wedge \mathrm{d}z_J \wedge \mathrm{d}\overline{z}_{J'}) = 2\mathrm{d}z_J \wedge \mathrm{d}\overline{z}_{J'}$ (since the length of $\mathrm{d}z_j$ is 2), and $\overline{e}_j^*(\mathrm{d}z_J \wedge \mathrm{d}\overline{z}_{J'}) = 0$ if $j \notin J'$ and $\overline{e}_j^*(\mathrm{d}\overline{z}_j \wedge \mathrm{d}z_J \wedge \mathrm{d}\overline{z}_{J'}) = 2\mathrm{d}z_J \wedge \mathrm{d}\overline{z}_{J'}$. In this case, $\omega\wedge = \frac{\sqrt{-1}}{2}\sum e_j\overline{e}_j$ and $\Lambda = -\frac{\sqrt{-1}}{2}\sum \overline{e}_j^* e_j^*$. Thus $A_\omega = \frac{1}{4}\sum(\overline{e}_j\overline{e}_j^* - e_j^* e_j)$.*

Also we have $\overline{\partial} = \sum \overline{\partial}_j \overline{e}_j = \sum \overline{e}_j \overline{\partial}_j$, where $\overline{\partial}_j(\sum f_{JJ'}\mathrm{d}z_J \wedge \mathrm{d}\overline{z}_{J'}) = \sum \frac{\partial f_{JJ'}}{\partial \overline{z}_j}\mathrm{d}z_J \wedge \mathrm{d}\overline{z}_{J'}$. Then $\overline{\partial}^* = -\sum \partial_j \overline{e}_j^*$.

We need to extend the differential operators $\overline{\partial}$ and A_ω to $L^2_{p,q}(X,L)$. First, notice that A_ω extends to an operator on the whole $L^2_{p,q}(X,L)$ because both $2\pi c_1(L, \|\cdot\|)\wedge$ and Λ_ω do. Next, the differential operator $\overline{\partial}\colon A_{\mathrm{c}}^{p,q}(X,L) \to A_{\mathrm{c}}^{p,q+1}(X,L)$ then has an adjoint $\overline{\partial}^*\colon A_{\mathrm{c}}^{p,q+1}(X,L) \to A_{\mathrm{c}}^{p,q}(X,L)$ with respect to the L^2-norm. Let $\mathrm{dom}\overline{\partial} \subseteq L^2_{p,q}(X,L)$ consist of those u for which $\overline{\partial}u$, computed in the sense of distribution (*i.e.* using $\langle \overline{\partial}u, v\rangle_{L^2} := \langle u, \overline{\partial}^* v\rangle_{L^2}$ for all $v \in A_{\mathrm{c}}^{p,q+1}(X,L)$), is in $L^2_{p,q+1}(X,L)$. Similarly we can define $\mathrm{dom}\overline{\partial}^*$.

3.3.2 Classical L^2-existence

Theorem 3.3.2 (Classical L^2-existence)**.** *Assume X is geodesic complete for the Riemannian metric determined by ω.*

Assume that the operator A_ω is positive definite everywhere in $L^2_{p,q}(X,L)$. Assume $p \geq 0$, $q \geq 1$ and $u \in L^2_{p,q}(X,L)$ satisfies $\overline{\partial}u = 0$ (in the sense of distributions) and $\langle A_\omega^{-1}u, u\rangle_{L^2} < \infty$.

Then there exists $f \in L^2_{p,q-1}(X,L)$ such that $\overline{\partial}f = u$ and $\|f\|^2_{L^2} \leq \langle A_\omega^{-1}u, u\rangle_{L^2}$.

We shall assume the following lemma, which is an easy application of the Bochner–Kodaira–Nakano identity (which itself is an easy computation via the Hodge identities).

Lemma 3.3.3. *For any $v \in A_{\mathrm{c}}^{p,q}(L)$ with $q \geq 1$, we have*

$$\|\overline{\partial}v\|^2_{L^2} + \|\overline{\partial}^* v\|^2_{L^2} \geq \langle A_\omega v, v\rangle_{L^2}.$$

Proof of Theorem 3.3.2. Both $\mathrm{Ker}\overline{\partial}$ and $\mathrm{Im}\overline{\partial}^*$ are closed subspaces of $L^2_{p,q}(X,L)$. General theory of Hilbert spaces gives the orthogonal decomposition $L^2_{p,q}(X,L) = \mathrm{Ker}\overline{\partial} \oplus \mathrm{Im}\overline{\partial}^*$.

Denote for simplicity by $C := \langle A_\omega^{-1}u, u\rangle_{L^2} < \infty$. Consider the linear functional

$$\mathrm{Im}\overline{\partial}^* \subseteq L^2_{p,q}(X,L) \to \mathbb{C}, \qquad \overline{\partial}^* v \mapsto \langle v, u\rangle_{L^2}. \tag{3.3.2}$$

We shall prove that the norm of this linear functional is bounded by $\sqrt{C}$, *i.e.*

$$\frac{|\langle v, u\rangle_{L^2}|^2}{\|\overline{\partial}^* v\|^2_{L^2}} \leq C \qquad \text{for all } v \in \mathrm{dom}\overline{\partial}^*. \tag{3.3.3}$$

We start with $v \in A_c^{p,q+1}(X,L)$, and write $v = v_1 + v_2$ according to the decomposition $L^2_{p,q}(X,L) = \mathrm{Ker}\overline{\partial} \oplus \mathrm{Im}\overline{\partial}^*$. Then Lemma 3.3.3 applied to v_1 implies

$$\|\overline{\partial}^* v\|^2_{L^2} = \|\overline{\partial}^* v_1\|^2_{L^2} \geq \langle A_\omega v_1, v_1\rangle_{L^2}.$$

On the other hand, Cauchy–Schwarz yields

$$|\langle v, u\rangle_{L^2}|^2 = |\langle v_1, u\rangle_{L^2}|^2 \leq \langle A_\omega v_1, v_1\rangle_{L^2} \langle A_\omega^{-1} u, u\rangle_{L^2}.$$

Thus (3.3.3) holds true for all $v \in A_c^{p,q+1}(X,L)$.

To claim (3.3.3) for all $v \in \mathrm{dom}\overline{\partial}^*$, we need to use the geodesic completeness of ω. Indeed, under this assumption, the Andreotti–Vesentini lemma says that $A_c^{p,q+1}(X,L)$ is dense in $\mathrm{Im}\overline{\partial}^*$ (for the graph norm of $\overline{\partial}^*$, *i.e* the graph norm of v is $\|v\|_{L^2} + \|\overline{\partial}^* v\|_{L^2}$), and hence we can conclude for (3.3.3).

Thus we can apply the Riesz representation theorem to the *continuous* linear functional (3.3.2) to conclude that (3.3.2) is represented by an element $f \in L^2_{p,q-1}(X,L)$ of L^2-norm $\leq \sqrt{C}$, *i.e.* $\langle v, u\rangle_{L^2} = \langle \overline{\partial}^* v, f\rangle_{L^2}$ for all $v \in \mathrm{dom}\overline{\partial}^*$. Therefore $\overline{\partial} f = u$ as distributions. We are done. $\square$

3.3.3 Hörmander's L^2-existence theorem

Theorem 3.3.4. *Assume X carries a Kähler form $\widehat{\omega}$ such that X is geodesic complete for the Riemannian metric determined by $\widehat{\omega}$.*

Assume $c_1(L, \|\cdot\|) > 0$. Assume $q \geq 1$ and $u \in L^2_{n,q}(X,L)$ satisfies $\overline{\partial} u = 0$ (in the sense of distributions) and $\langle A_\omega^{-1} u, u\rangle_{L^2} < \infty$.

Then there exists $f \in L^2_{n,q-1}(X,L)$ such that $\overline{\partial} f = u$ and $\|f\|^2_{L^2} \leq \langle A_\omega^{-1} u, u\rangle_{L^2}$.

Remark 3.3.5. *(i) A particularly important case for which X carries such a complete Kähler form $\widehat{\omega}$ is as follows: $X = X' \setminus Z$ where X' is a compact Kähler manifold and Z is an analytic subvariety.*

(ii) Since $c_1(L, \|\cdot\|) > 0$, locally on X we can find a suitable complex coordinate $(z_1, \ldots, z_n)$ of X such that: (i) $\omega = \frac{\sqrt{-1}}{2}\sum_{j=1}^n \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$, (ii) the $(1,1)$-form $c_1(L, \|\cdot\|)$ equals $\frac{\sqrt{-1}}{2}\sum_{j=1}^n \alpha_j(x)\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$ with $\alpha_j(x) > 0$. By the computation from Example 3.3.1, we have then $A_\omega = \frac{\pi}{2}\sum_j \alpha_j(\overline{e}_j\overline{e}_j^* - e_j^* e_j)$, which simplifies to $\frac{\pi}{2}\sum_j \alpha_j \overline{e}_j\overline{e}_j^*$ for (n,q)-forms (this is why we are constraint to (n,q)-forms!). Thus A_ω is positive definite.

With this observation, we shall reduce Theorem 3.3.4 to Theorem 3.3.2 using the following monotonicity result.

Proposition 3.3.6 (Monotonicity)**.** *Assume X has two Kähler metrics ω, ω' such that $\omega' \geq \omega$ pointwise. Then for any positive $(1,1)$-form β, we have*

$$|u|^2_{\omega'} \mathrm{d}V_{\omega'} \leq |u|^2_{\omega} \mathrm{d}V_{\omega}, \quad \langle [\beta, \Lambda_{w'}]u, u\rangle_{L^2,\omega'} \mathrm{d}V_{\omega'} \geq \langle [\beta, \Lambda_{w}]u, u\rangle_{L^2,\omega} \mathrm{d}V_{\omega}$$

for all (n,q)-form u.

Here is a brief explanation on the proof of Proposition 3.3.6. The conclusion can be checked locally, and hence it suffices to check for $X = \mathbb{C}^n$, ω the standard Kähler form, and $\omega' = (\sqrt{-1}/2)\sum \gamma_j \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$ for $\gamma_j \geq 1$. The proof is then a direct computation.

Proof of Theorem 3.3.4. For every $\epsilon > 0$, set $\omega_\epsilon := \omega + \epsilon\widehat{\omega}$. Since $c_1(L, \|\cdot\|) \geq 0$, we can apply Proposition 3.3.6 to $\beta = 2\pi c_1(L, \|\cdot\|)$ and to ω and $\omega' = \omega_\epsilon$ to get that u is L^2 with respect to ω_ϵ and

$$\langle A^{-1}_{\omega_\epsilon} u, u\rangle_{L^2,\omega_\epsilon} \leq \langle A^{-1}_{\omega} u, u\rangle_{L^2,\omega}.$$

It is known that ω_ϵ is complete (because $\widehat{\omega}$ is), *i.e.* X is geodesically complete for the Riemannian metric determined by ω_ϵ. The argument of Remark 3.3.5.(ii) shows that A_{ω_ϵ} is positive definite. Thus we can apply Theorem 3.3.2 to the Kähler manifold (X, ω_ϵ). So we obtain an $f_\epsilon \in L^2_{n,q-1}(X, L)$ (with L^2 with respect to ω_ϵ) satisfying $\overline{\partial} f_\epsilon = u$ and $\|f_\epsilon\|^2_{L^2,\omega_\epsilon} \leq \langle A^{-1}_{\omega_\epsilon} u, u\rangle_{L^2,\omega_\epsilon}$. In particular, the family (f_ϵ) is locally bounded in the L^2-norm, and hence we can extract a weal limit f in L^2_{loc} (locally L^2-coefficients), which is the required f. □

3.3.4 Weighted L^2-existence

To prove the L^2-extension theorem in the next section, we need a fancier version of Hörmander's L^2-existence theorem by introducing weights on the operator A_ω. Let us explain this.

Let $\eta, \lambda \colon X \to \mathbb{R}_{>0}$ be smooth functions. Define

$$B_{\eta,\lambda,\omega} := [\big(\eta 2\pi c_1(L, \|\cdot\|) - \sqrt{-1}\partial\overline{\partial}\eta - \sqrt{-1}\lambda^{-1}\partial\eta \wedge \overline{\partial}\eta\big) \wedge,\ \Lambda_\omega]. \tag{3.3.4}$$

Theorem 3.3.7. *Assume X carries a Kähler form $\widehat{\omega}$ such that X is geodesic complete for the Riemannian metric determined by $\widehat{\omega}$.*

Assume that the $(1,1)$-form $\eta 2\pi c_1(L, \|\cdot\|) - \sqrt{-1}\partial\overline{\partial}\eta - \sqrt{-1}\lambda^{-1}\partial\eta \wedge \overline{\partial}\eta$ is positive.

Assume $q \geq 1$ and $u \in L^2_{n,q}(X, L)$ satisfies $\overline{\partial} u = 0$ (in the sense of distributions) and $\langle B^{-1}_{\eta,\lambda,\omega} u, u\rangle_{L^2} < \infty$.

Then there exists $f \in L^2_{n,q-1}(X, L)$ such that $\overline{\partial} f = u$ and

$$\left\| \frac{f}{\sqrt{\eta + \lambda}} \right\|^2_{L^2} \leq 2 \left\| \frac{f}{\eta^{1/2} + \lambda^{1/2}} \right\|^2_{L^2} \leq 2\langle B^{-1}_{\eta,\lambda,\omega} u, u\rangle_{L^2}.$$

The proof follows the same line as Theorem 3.3.4. The extra information needed is the following estimate: For all (n,q)-forms u, we have

$$\langle B^{-1}_{\eta,\lambda,\omega} u, u\rangle_{L^2} \leq \|(\eta^{1/2} + \lambda^{1/2})\overline{\partial}^* u\|^2_{L^2} \|\eta^{1/2}\overline{\partial} u\|^2_{L^2}.$$

We close this section with the following variant of Theorem 3.3.7 which applies to *singular* Hermitian metric on L, *i.e.* in the following theorem we do not assume the Hermitian metric $\|\cdot\|$ on L to be smooth in contrast to the general setting of this section.

Theorem 3.3.7′. *Assume that X is compact. Assume that the Hermitian metric $\|\cdot\|$ on L is smooth outside a proper analytic subset Z of X. Assume that the $(1,1)$-form $\eta 2\pi c_1(L, \|\cdot\|) - \sqrt{-1}\partial\overline{\partial}\eta - \sqrt{-1}\lambda^{-1}\partial\eta \wedge \overline{\partial}\eta$ is positive on $X \setminus Z$.*

The conclusion of Theorem 3.3.7 still holds true in this setting.

Proof. By a result of Demailly (*c.f.* *"Estimations L^2 pour l'opérateur $\overline{\partial}$ d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète"*), $X \setminus Z$ carries a Kähler form for which $X \setminus Z$ is geodesic complete. Hence we can apply Theorem 3.3.7 to $X \setminus Z$ to get an L^2-solution f. Then f extends to the whole X by a lemma of Demailly (Lemma 6.9 of *loc.cit.*). □

3.4 L^2-extension

Let X be a connected compact Kähler manifold of dimension n with Kähler form ω, and let $\mathrm{d}V_{X,\omega} = \omega^{\wedge n}/n!$ be the volume form on X.

Let $K_X := \bigwedge^n T_X^*$ be the canonical line bundle on X.

Theorem 3.4.1 (L^2-extension)**.** *Let L be a line bundle on X, endowed with a smooth Hermitian metric $\|\cdot\|$.*

Let Y be the zero of a holomorphic section $s \in H^0(X, L_0)$ of another Hermitian line bundle $(L_0, \|\cdot\|_0)$ on X. Assume $c_1(L, \|\cdot\|) - (1+\delta)c_1(L_0, \|\cdot\|_0) > 0$ for a positive rational number $\delta > 0$.

Then for any $f \in H^0(Y, L + K_X)$, there exists $F \in H^0(X, L + K_X)$ such that $F|_Y = f$ and

$$\int_X \frac{\|F\|^2}{\|s\|_0^2(\log\|s\|_0)^2}\mathrm{d}V_{X,\omega} \leq 72 \cdot 32\pi \int_Y \frac{\|f\|^2}{\|\mathrm{d}s\|_0^2}\mathrm{d}V_{Y,\omega}. \tag{3.4.1}$$

Here we use the following abuse of notation: use $\|\cdot\|$ (resp. $\|\cdot\|_0$) to denote the Hermitian metric on $L + K_X$ induced by $\|\cdot\|$ on L and ω on X (resp. on $L_0 \otimes T_X^*$ induced by $\|\cdot\|_0$ on L_0 and ω on X). Moreover, $\mathrm{d}s$ induces a vector bundle isomorphism $T_X/T_Y \xrightarrow{\sim} L_0$ along Y, and hence is a section of $((T_X|_Y)/T_Y)^* \otimes L_0|_Y (\simeq \mathcal{O}_Y) \subseteq T_X^* \otimes L_0$.

Remark 3.4.2. *There are more general versions of L^2-extension. One can replace the line bundle L_0 by a vector bundle of rank r (and hence Y has codimension r) and modify the assumptions accordingly. The Hermitian metric on L_0 does not play an important role. We refer to Demailly's paper "On the Ohsawa–Takegoshi–Manivel L^2 extension theorem".*

In the proof of arithmetic Hilbert–Samuel, we will take L to be $L' - K_X$ and L_0 to be $(1/N)L'$ for a very ample line bundle L' and an integer $N \gg 1$.

The whole section is divided into steps of the proof of Theorem 3.4.1.

3.4.1 Construction of a smooth extension $\widetilde{f}_\infty$ and truncation

By partition of unity, we can find a smooth section

$$\widetilde{f}_\infty \in C^\infty(X, L + K_X) = A^{0,0}(X, L + K_X) \simeq A^{n,0}(X, L)$$

such that

(i) $\widetilde{f}_\infty|_Y = f$,

(ii) $\overline{\partial}\widetilde{f}_\infty = 0$ on Y.

Since we do not know about $\widetilde{f}_\infty$ far away from Y, we will consider a truncation $\widetilde{f}_\epsilon$ of $\widetilde{f}_\infty$ with support in a small tubular neighborhood $\|s\|_0 < \epsilon$ of Y as follows. Take a bumping function $\theta\colon \mathbb{R} \to [0,1]$ satisfying the following properties: θ is smooth, $|\theta'| \leq 3$ and

$$\theta(t) = \begin{cases} 1 & \text{for } t \leq 1/2 \\ 0 & \text{for } t \geq 1. \end{cases}$$

For $\epsilon > 0$ small, consider the truncation

$$\widetilde{f}_\epsilon := \theta(\epsilon^{-2}\|s\|_0^2)\widetilde{f}_\infty.$$

Then $\widetilde{f}_\epsilon|_Y = f$, and $\widetilde{f}(x) = 0$ for all $x \in X$ with $\|s(x)\|_0 \geq \epsilon$.

3.4.2 Construction of weights

We make use of the standard subharmonic function

$$\sigma_\epsilon = \log(\|s\|_0^2 + \epsilon^2). \tag{3.4.2}$$

and the following inequality (we omit this computation using the Chern connection and Lagrange inequality) to compute the twisted curvature:

$$\sqrt{-1}\partial\overline{\partial}\sigma_\epsilon \geq \sqrt{-1}\frac{\epsilon^2}{\|s\|_0^2}\partial\sigma_\epsilon \wedge \overline{\partial}\sigma_\epsilon - \frac{\|s\|_0^2}{\|s\|_0^2+\epsilon^2}2\pi c_1(L_0, \|\cdot\|_0). \tag{3.4.3}$$

Recall that $\|s\|_0^2 \colon X \to \mathbb{R}$ is a smooth function. Hence

$$e^{-2\alpha} := \sup_{x\in X} \|s(x)\|_0^2 < \infty \tag{3.4.4}$$

since X is compact. We may rescale the metric $\|\cdot\|_0$ so that $\alpha \in (0, 1/\delta)$, because the conclusion (3.4.1) is unchanged under this operation.

Let $\chi_0 \colon (-\infty, 0] \to (-\infty, 0]$, $t \mapsto t - \log(1-t)$. Then $2t \leq \chi_0(t) \leq t$, $1 \leq \chi_0' \leq 2$, and $\chi_0''(t) = 1/(1-t)^2$.

Let $\eta_\epsilon := \epsilon - \chi_0(\sigma_\epsilon)$. Then $\eta_\epsilon \geq \epsilon - \log(e^{-2\alpha} + \epsilon^2)$. For $\epsilon > 0$ small enough, we thus have $\eta_\epsilon \geq 2\alpha$. We can compute

$$-\partial\overline{\partial}\eta_\epsilon = \chi_0'(\sigma_\epsilon)\partial\overline{\partial}\sigma_\epsilon + \chi_0''(\sigma_\epsilon)\partial\sigma_\epsilon \wedge \overline{\partial}\sigma_\epsilon, \qquad \partial\eta_\epsilon \wedge \overline{\partial}\eta_\epsilon = \chi_0'(\sigma_\epsilon)^2\partial\sigma_\epsilon \wedge \overline{\partial}\sigma_\epsilon.$$

Let $\lambda_\epsilon := \chi_0'(\sigma_\epsilon)^2/\chi_0''(\sigma_\epsilon)$. Then

$$-\sqrt{-1}\partial\overline{\partial}\eta_\epsilon - \sqrt{-1}\lambda_\epsilon^{-1}\partial\eta_\epsilon \wedge \overline{\partial}\eta_\epsilon = \sqrt{-1}\chi_0'(\sigma_\epsilon)\partial\overline{\partial}\sigma_\epsilon$$

for which we have a lower bound from (3.4.3).

We are interested in the metric on L defined by $\|\cdot\|\|s\|_0^{-2}$, for a reason which will be explained in the next step. By the Poincaré–Lelong formula (Theorem 2.2.3), we have $\sqrt{-1}\partial\overline{\partial}\log\|s\|_0^2 \geq -2\pi c_1(L_0, \|\cdot\|_0)$ with equality on $X \setminus Y$. So on $X \setminus Y$, we have that

$$\begin{aligned}
\Theta(L, \epsilon, s) :=&\eta_\epsilon 2\pi c_1(L, \|\cdot\|\|s\|_0^{-2}) - \sqrt{-1}\partial\overline{\partial}\eta_\epsilon - \sqrt{-1}\lambda_\epsilon^{-1}\partial\eta_\epsilon \wedge \overline{\partial}\eta_\epsilon \\
\geq&2\alpha(2\pi c_1(L, \|\cdot\|) - 2\pi c_1(L_0, \|\cdot\|_0)) - \sqrt{-1}\partial\overline{\partial}\eta_\epsilon - \sqrt{-1}\lambda_\epsilon^{-1}\partial\eta_\epsilon \wedge \overline{\partial}\eta_\epsilon \\
\geq&2\pi\left(2\alpha c_1(L, \|\cdot\|) - \left(2\alpha + \chi_0'(\sigma_\epsilon)\frac{\|s\|_0^2}{\|s\|_0^2+\epsilon^2}\right)c_1(L_0, \|\cdot\|_0)\right) + \sqrt{-1}\frac{\epsilon^2}{\chi_0'(\sigma_\epsilon)\|s\|_0^2}\partial\eta_\epsilon \wedge \overline{\partial}\eta_\epsilon \\
\geq&4\pi\alpha\left(c_1(L, \|\cdot\|) - (1 + \frac{1}{\alpha})c_1(L_0, \|\cdot\|_0)\right) + \sqrt{-1}\frac{\epsilon^2}{\chi_0'(\sigma_\epsilon)\|s\|_0^2}\partial\eta_\epsilon \wedge \overline{\partial}\eta_\epsilon \\
\geq&\sqrt{-1}\frac{\epsilon^2}{\chi_0'(\sigma_\epsilon)\|s\|_0^2}\partial\eta_\epsilon \wedge \overline{\partial}\eta_\epsilon
\end{aligned} \tag{3.4.5}$$

is positive, where the last inequality follows from $\alpha \in (0, 1/\delta)$ and the assumption that $c_1(L, \|\cdot\|) - (1+\delta)c_1(L_0, \|\cdot\|_0) > 0$. Notice that Example 3.3.1 then implies

$$B_\epsilon := [\Theta(L, \epsilon, s)\wedge, \Lambda_\omega] \geq \frac{\epsilon^2}{\chi_0'(\sigma_\epsilon)\|s\|_0^2}(\overline{\partial}\eta_\epsilon\wedge) \circ (\overline{\partial}\eta_\epsilon\wedge)^* \tag{3.4.6}$$

as an operator on (n, q)-forms.

3.4.3 Estimate the partial derivative

Next we wish to construct a *holomorphic* extension from the smooth extension $\widetilde{f}_\epsilon$. For this purpose we wish to solve the equation $\overline{\partial} u_\epsilon = \overline{\partial}\widetilde{f}_\epsilon$, *with the constraint* $u_\epsilon|_Y = 0$, so that $\widetilde{f}_\epsilon - u_\epsilon$ will be a desired holomorphic extension. Our tool to solve this differential equation is the L^2-existence theorem discussed in the last section (notice that $\overline{\partial}\widetilde{f}_\epsilon$ is a $\overline{\partial}$-closed smooth $(n,1)$-form). Since $\mathrm{codim} Y = 1$, the extra constraint $u_\epsilon|_Y = 0$ will be satisfied if $\|u_\epsilon\|^2\|s\|_0^{-2}$ is locally integrable near Y. *This is why we change the metric on L to be* $\|\cdot\|\|s\|_0^{-2}$. Notice that this new metric is singular along Y, so we need to apply the version of Theorem 3.3.7′ for the L^2-existence.

We start by computing $\overline{\partial}\widetilde{f}_\epsilon$. Observe that $1+\epsilon^{-2}\|s\|_0^2 = \epsilon^{-2}e^{\sigma_\epsilon}$. Thus we have

$$\overline{\partial}\widetilde{f}_\epsilon = \epsilon^{-2}\theta'(\epsilon^{-2}\|s\|_0^2)\overline{\partial}\|s\|_0^2\wedge\widetilde{f}_\infty+\theta(\epsilon^{-2}\|s\|_0^2)\overline{\partial}\widetilde{f}_\infty = (1+\epsilon^{-2}\|s\|_0^2)\theta'(\epsilon^{-2}\|s\|_0^2)\overline{\partial}\sigma_\epsilon\wedge\widetilde{f}_\infty+\theta(\epsilon^{-2}\|s\|_0^2)\overline{\partial}\widetilde{f}_\infty.$$

Both terms on the RHS have supports in $\|s\|_0 \le \epsilon$.

The first term, which is the main term, can be written as

$$g_\epsilon^{(1)} := (1+\epsilon^{-2}\|s\|_0^2)\theta'(\epsilon^{-2}\|s\|_0^2)\chi_0'(\sigma_\epsilon)^{-1}\overline{\partial}\eta_\epsilon\wedge\widetilde{f}_\infty.$$

To estimate $\langle B_\epsilon^{-1}g_\epsilon^{(1)}, g_\epsilon^{(1)}\rangle_{L^2}$, notice that (3.4.6) implies

$$|\langle\overline{\partial}\eta_\epsilon\wedge u, v\rangle|^2 = |\langle u, (\overline{\partial}\eta_\epsilon)^* v\rangle|^2 \le |u|^2|(\overline{\partial}\eta_\epsilon)^* v|^2 = |u|^2\langle(\overline{\partial}\eta_\epsilon)(\overline{\partial}\eta_\epsilon)^* v, v\rangle \le \frac{\chi_0'(\sigma_\epsilon)\|s\|_0^2}{\epsilon^2}|u|^2\langle B_\epsilon v, v\rangle.$$

Thus by letting $v = B_\epsilon^{-1}(\overline{\partial}\eta_\epsilon\wedge u) = B_\epsilon^{-1}g_\epsilon^{(1)}$ and $u = (1+\epsilon^{-2}\|s\|_0^2)\theta'(\epsilon^{-2}\|s\|_0^2)\chi_0'(\sigma_\epsilon)^{-1}\widetilde{f}_\infty$, pointwise on X we get

$$\langle B_\epsilon^{-1}g_\epsilon^{(1)}, g_\epsilon^{(1)}\rangle \le \|s\|_0^2\epsilon^{-2}(1+\epsilon^{-2}\|s\|_0^2)^2\theta'(\epsilon^{-2}\|s\|_0^2)^2\|\widetilde{f}_\infty\|^2 \le 36\|\widetilde{f}_\infty\|^2$$

because $\chi_0'(\epsilon) \ge 1$ on $\mathrm{Supp} g_\epsilon^{(1)} \subseteq \{\|s\|_0 \le \epsilon\}$. So

$$\langle B_\epsilon^{-1}g_\epsilon^{(1)}, g_\epsilon^{(1)}\rangle_{L^2} = \int_X \langle B_\epsilon^{-1}g_\epsilon^{(1)}, g_\epsilon^{(1)}\rangle\|s\|_0^{-2}\mathrm{d}V_{X,\omega} \le 36\int_{\|s\|_0\le\epsilon}\|\widetilde{f}_\infty\|^2\|s\|_0^{-2}\mathrm{d}V_{X,\omega}.$$

When $\epsilon \to 0+$, this integral becomes

$$8\pi\int_Y \|f\|^2\|\mathrm{d}s\|_0^{-2}\mathrm{d}V_{Y,\omega}.$$

Thus

$$\limsup_{\epsilon\to 0^+}\langle B_\epsilon^{-1}g_\epsilon^{(1)}, g_\epsilon^{(1)}\rangle_{L^2} \le 36\cdot 8\pi\int_Y \|f\|^2\|\mathrm{d}s\|_0^{-2}\mathrm{d}V_{Y,\omega}.$$

The second term on the RHS in the expression of $\overline{\partial}\widetilde{f}_\epsilon$ converges uniformly to 0 on every compact set when $\epsilon \to 0^+$ and hence has no contribution in the limit. More precisely, write $g_\epsilon^{(2)} := \theta(\epsilon^{-2}\|s\|_0^2)\overline{\partial}\widetilde{f}_\infty$. Then $g_\epsilon^{(2)} = O(\|s\|_0)$ since $\widetilde{f}_\infty|_Y = 0$. Thus $\langle B_\epsilon^{-1}g_\epsilon^{(1)}, g_\epsilon^{(2)}\rangle_{L^2}$, $\langle g_\epsilon^{(1)}, B_\epsilon^{-1}g_\epsilon^{(2)}\rangle_{L^2}$, $\langle B_\epsilon^{-1}g_\epsilon^{(2)}, g_\epsilon^{(2)}\rangle_{L^2}$ are $O(\epsilon)$ because they are all integrals over $\|s\|_0 \le \epsilon$. Hence

$$\limsup_{\epsilon\to 0^+}\langle B_\epsilon^{-1}\overline{\partial}\widetilde{f}_\epsilon, \overline{\partial}\widetilde{f}_\epsilon\rangle_{L^2} = \limsup_{\epsilon\to 0^+}\langle B_\epsilon^{-1}(g_\epsilon^{(1)}+g_\epsilon^{(2)}), g_\epsilon^{(1)}+g_\epsilon^{(2)}\rangle_{L^2} \le 36\cdot 8\pi\int_Y \|f\|^2\|\mathrm{d}s\|_0^{-2}\mathrm{d}V_{Y,\omega} < \infty.$$

3.4.4 Conclusion by L^2-existence

Apply Theorem 3.3.7′ to the Hermitian metric $\|\cdot\|\|s\|_0^{-2}$ on L, $Z = Y$, $q = 1$ and $u = \overline{\partial}\widetilde{f}_\epsilon$. We then obtain g_ϵ such that $\overline{\partial} g_\epsilon = \overline{\partial}\widetilde{f}_\epsilon$ and

$$\int_X \frac{\|g_\epsilon\|^2\|s\|_0^{-2}}{\eta_\epsilon + \lambda_\epsilon} \mathrm{d}V_{X,\omega} \leq 72\cdot 8\pi \int_Y \|f\|^2\|\mathrm{d}s\|_0^{-2}\mathrm{d}V_{Y,\omega}$$

In particular, $g_\epsilon|_Y = 0$ since $\|g_\epsilon\|^2\|s\|_0^{-2}$ is locally integrable. Set

$$F_\epsilon := \widetilde{f}_\epsilon - g_\epsilon.$$

Then F_ϵ is an L^2-extension of f to the whole X such that $\overline{\partial} F_\epsilon = 0$ on $X \setminus Y$.

We have $\eta_\epsilon = \epsilon - \chi_0(\sigma_\epsilon) \geq \epsilon - \sigma_\epsilon$ and $\lambda_\epsilon = (1-\sigma_\epsilon)^2 + (1-\sigma_\epsilon)$. Thus $\eta_\epsilon + \lambda_\epsilon \geq \sigma_\epsilon^2 - 4\sigma_\epsilon + 2 + \epsilon$ with $\sigma_\epsilon = \log(\|s\|_0^2 + \epsilon^2)$. So

$$\int_X \frac{\|\widetilde{f}_\epsilon\|^2\|s\|_0^{-2}}{\eta_\epsilon + \lambda_\epsilon} \mathrm{d}V_{X,\omega} \leq \frac{M}{(\log\epsilon)^2}$$

because $\widetilde{f}_\epsilon$ is uniformly bounded with support in $\|s\|_0 \leq \epsilon$. Therefore, by using $|t+u|^2 \leq (1+k)|t|^2 + (1+k^{-1})|u|^2$, with $k = |\log\epsilon|$, we obtain

$$\int_X \frac{\|F_\epsilon\|^2}{\|s\|_0^2(\log(\|s_0\|^2+\epsilon^2))^2} \mathrm{d}V_{X,\omega} \leq (1+|\log\epsilon|^{-1})72\cdot 8\pi \int_Y \|f\|^2\|\mathrm{d}s\|_0^{-2}\mathrm{d}V_{Y,\omega} + O(|\log\epsilon|^{-1}).$$

Similarly we can show that $\|F_\epsilon\|_{L^2}$ is bounded above by a constant independent of ϵ (when $\epsilon > 0$ is small enough). Thus we can extract a weak limit F of the family $\{F_\epsilon\}_\epsilon$. Then

$$\int_X \frac{\|F\|^2}{\|s\|_0^2(\log\|s_0\|)^2} \mathrm{d}V_{X,\omega} \leq 72\cdot 32\pi \int_Y \|f\|^2\|\mathrm{d}s\|_0^{-2}\mathrm{d}V_{Y,\omega}.$$

It remains to prove that F is holomorphic. *Since we are applying Theorem 3.3.7′ to $q = 1$, g_ϵ is smooth* (because $\overline{\partial}$ is elliptic in bidegree $(0,0)$). Hence F_ϵ is smooth. Notice that $\overline{\partial} F_\epsilon = 0$ on $X \setminus Y$. So F_ϵ is holomorphic on $X \setminus Y$, and hence is holomorphic on the whole X because F_ϵ is L^2 near Y. Therefore the weak limit F is holomorphic. We are done. □

Chapter 4

Proof of Arithmetic Hilbert–Samuel

With the preparation in the previous chapter, we prove the arithmetic Hilbert–Samuel theorem for vol_χ and $\overline{\mathcal{L}}$ in the following setup in this chapter. We follow the approach of Abbès–Bouche.

Let K be a number field and let $\mathcal{O}_K$ be its ring of integers.

Let $\mathcal{X}$ be a projective arithmetic variety of dimension $n+1$ and let $\overline{\mathcal{L}}$ be a smooth Hermitian line bundle. We furthermore consider the case where $\mathcal{X} \to \mathrm{Spec}\mathbb{Z}$ factors through $\mathrm{Spec}\mathcal{O}_K$ and that the generic fiber $\mathcal{X}_K$ is smooth and irreducible.

Theorem 4.0.1. *Assume $\mathcal{L}$ is very ample on $\mathcal{X}$ and $c_1(\overline{\mathcal{L}}) > 0$. Then*

$$\lim_{k\to\infty} \frac{\chi(k\overline{\mathcal{L}})}{k^{n+1}/(n+1)!} \longrightarrow \overline{\mathcal{L}}^{n+1} \tag{4.0.1}$$

when $k \to \infty$.

In other words, the sup-limit in the definition of $\mathrm{vol}_\chi(\overline{\mathcal{L}})$ (Definition 2.5.10) is an actually limit under the assumption of the theorem, and $\mathrm{vol}_\chi(\overline{\mathcal{L}}) = \overline{\mathcal{L}}^{n+1}$.

In the proof, we will use the Hilbert–Samuel theorem in algebraic geometry. Let P be the Hilbert polynomial of $\mathcal{L}_K$ on $\mathcal{X}_K$, *i.e.* $P(k) = \dim H^0(\mathcal{X}_K, k\mathcal{L}_K)$ for $k \gg 1$. It is known that $\deg P = n$ with leading coefficient $\mathcal{L}_K^n/n!$.

4.1 Framework of the proof

4.1.1 Revision on the statement

We start by recalling the objects appearing in the statement of Theorem 6.2.4.

First we have the embedding

$$0 \to H^0(\mathcal{X}, \mathcal{L}) \to H^0(\mathcal{X}, \mathcal{L})_{\mathbb{R}}, \tag{4.1.1}$$

with $H^0(\mathcal{X}, \mathcal{L})_{\mathbb{R}}$ a real vector space of finite dimension defined in (2.5.2) and $H^0(\mathcal{X}, \mathcal{L})$ a lattice. In fact, the structural morphism $\mathcal{X} \to \mathrm{Spec}\mathbb{Z}$ factors through $\mathrm{Spec}R$ for an order R in a number field K, such that the generic fiber $\mathcal{X}_K$ is irreducible, and

$$H^0(\mathcal{X}, \mathcal{L})_{\mathbb{R}} := \{s = (s_\sigma)_\sigma \in H^0(\mathcal{X}, \mathcal{L})_{\mathbb{C}} = \bigoplus\nolimits_{\sigma\colon K \hookrightarrow \mathbb{C}} H^0(\mathcal{X}_\sigma, \mathcal{L}_\sigma) : s_\sigma = \overline{s_{\overline{\sigma}}} \text{ for all } \sigma\}.$$

We shall use the *sup-norm* on $H^0(\mathcal{X}, \mathcal{L})_{\mathbb{R}}$ defined as follows:

- For any $s = (s_\sigma)_\sigma \in H^0(\mathcal{X}, \mathcal{L})_{\mathbb{R}}$, define $\|s\|_{\sup} := \sup_{\sigma, x \in \mathcal{X}_\sigma}(\|s_\sigma(x)\|)$.

Set $B(\overline{\mathcal{L}})$ to be the unit ball in $H^0(\mathcal{X},\mathcal{L})_{\mathbb{R}}$ defined by the sup-norm $\|\cdot\|_{\sup}$. Then

$$\chi(\overline{\mathcal{L}}) := \log \frac{\mathrm{vol}(B(\overline{\mathcal{L}}))}{\mathrm{covol}(H^0(\mathcal{X},\mathcal{L})_{\mathbb{R}}/H^0(\mathcal{X},\mathcal{L}))} \tag{4.1.2}$$

is independent of the choice of the Haar measure on $H^0(\mathcal{X},\mathcal{L})_{\mathbb{R}}$. This finishes the explanation of the limit in (4.0.1).

We also need to define an L^2-norm on $H^0(\mathcal{X},\mathcal{L})_{\mathbb{R}}$ for the proof of Theorem 6.2.4. For this, consider the positive $(1,1)$-form $c_1(\overline{\mathcal{L}})$ on $\mathcal{X}(\mathbb{C}) = \coprod_\sigma \mathcal{X}_\sigma$. For each $\sigma\colon K \hookrightarrow \mathbb{C}$, the positive $(1,1)$-form $c_1(\overline{\mathcal{L}}_\sigma)$ is a Kähler form on $\mathcal{X}_\sigma$. We normalize it to

$$\omega_\sigma := \frac{c_1(\overline{\mathcal{L}}_\sigma)}{(\int_{\mathcal{X}_\sigma} c_1(\overline{\mathcal{L}}_\sigma)^{\wedge n}/n!)^{1/n}},$$

which is still a Kähler form on $\mathcal{X}_\sigma$, and the volume of $\mathcal{X}_\sigma$ for the associated volume form $\mathrm{d}V_\sigma$ is 1.

Now we define the L^2-*norm* on $H^0(\mathcal{X},\mathcal{L})_{\mathbb{R}}$ to be:

- For any $s = (s_\sigma)_\sigma \in H^0(\mathcal{X},\mathcal{L})_{\mathbb{R}}$, define $\|s\|_{L^2} := \sup_\sigma(\int_{\mathcal{X}_\sigma} \|s(x)\|^2 \mathrm{d}V_\sigma)^{1/2}$.

It is a fundamental question in Arakelov Geometry to compare the sup-norm and the L^2-norm. We shall prove later on, using the distortion function discussed in §3.1, the following result.

Proposition 4.1.1. *There exists a constant $c > 0$ such that for all $N \geq 1$ and all $s \in H^0(\mathcal{X}, k\mathcal{L})$, we have*

$$\|s\|_{L^2} \leq \|s\|_{\sup} \leq cP(k)^{1/2}\|s\|_{L^2}.$$

In fact, this c can be chosen to be $\sqrt{\sup_{x\in\mathcal{X}(\mathbb{C})} b_k(x)/P(k)}$, where b_k is the distortion function. Theorem 3.1.6 guarantees $c > 0$.

4.1.2 A tale of three volumes

Consider the embedding (4.1.1). We shall define three volume forms on $H^0(\mathcal{X}, k\mathcal{L})_{\mathbb{R}}$ for each $k \geq 1$:

(i) $V^k_{X,\sup}$ such that the volume of $B(k\overline{\mathcal{L}})$ has volume 1;

(ii) V^k_{X,L^2} such that the volume of the unit ball in $H^0(\mathcal{X}, k\mathcal{L})_{\mathbb{R}}$ defined by the L^2-norm $\|\cdot\|_{L^2}$ has volume 1;

(iii) $V^k_{X,\alpha}$ for each real number $\alpha \in \mathbb{R}$ defined below by Definition 4.1.2 (which we will call $\eta_{k,\alpha}$), with $M = \bigoplus_{k\geq 0} H^0(\mathcal{X}, k\mathcal{L})$.

A key point to prove arithmetic Hilbert–Samuel is to compare $V^k_{X,\sup}$ with $V^k_{X,\alpha}$, and the comparison is done via V^k_{X,L^2}. The statements of these comparisons and their consequence on arithmetic Hilbert–Samuel will be discussed in the next subsection.

In this subsection, we give the definition of $V^k_{X,\alpha}$. We start by defining the following generalization of the *arithmetic Euler characteristic* (1.3.1) in the context of geometry of numbers. For any M a finitely generated $\mathbb{Z}$-module of rank $r \geq 1$, define for each volume form η on $M_{\mathbb{R}}$ (*i.e.* an element $\eta \in \det_{\mathbb{R}}(M_{\mathbb{R}})$) the following

$$\chi(M,\eta) := -\log(\mathrm{covol}_\eta(M_{\mathbb{R}}/\overline{M})) + \log(\#M_{\mathrm{tor}}) \tag{4.1.3}$$

where $\overline{M} := M/M_{\mathrm{tor}}$. In fact, (4.1.2) can be defined in this context for $M = H^0(\mathcal{X}, \mathcal{L})$ and η the volume form determined by the sup-norm. The function χ is additive: for $0 \to M_1 \to M_2 \to M_3 \to 0$ an exact sequence and $\eta_j \in \det_{\mathbb{R}}(M_{j,\mathbb{R}})$ ($j \in \{1, 2, 3, \}$) such that $\eta_2 = \eta_1 \otimes \eta_3$ in the induced isomorphism $\det_{\mathbb{R}}(M_{2,\mathbb{R}}) \simeq \det_{\mathbb{R}}(M_{1,\mathbb{R}}) \otimes \det_{\mathbb{R}}(M_{3,\mathbb{R}})$, we have

$$\chi(M_2, \eta_2) = \chi(M_1, \eta_1) + \chi(M_3, \eta_3). \tag{4.1.4}$$

Back to our case. Let $M = \bigoplus_k M_k$ be a graduated $\mathcal{O}_K$-module of finite type, and let P_M be the Hilbert polynomial. A typical case for us is when $M = \bigoplus_{k \geq 0} H^0(\mathcal{X}, k\mathcal{L})$ with Hilbert polynomial P.

Definition 4.1.2. *Let $\alpha \in \mathbb{R}$ be a real number. Define $\eta_{k,\alpha} \in \det_{\mathbb{R}}(M_{\mathbb{R}})$ to be the volume form determined by the equation*

$$\chi(M_k, \eta_{k,\alpha}) = \alpha \sum_{j=0}^{k-1} P_M(j) + \chi(\mathcal{O}_K) P_M(k)$$

with the canonical volume form on $\mathcal{O}_K$.

The following lemma is easy to prove. It is the reason that the $\eta_{k,\alpha}$ is of interest to us. *It does not hold for the L^2-volume forms.*

Lemma 4.1.3. *Let $0 \to M^{(1)} \to M^{(2)} \to M^{(3)} \to 0$ be an exact sequence of graduated $\mathcal{O}_K$-modules of finite type, with $\Phi_k \colon \det(M^{(2)}_{k,\mathbb{R}}) \simeq \det(M^{(1)}_{k,\mathbb{R}}) \otimes \det(M^{(3)}_{k,\mathbb{R}})$ the induced isomorphism. Then for each $\alpha \in \mathbb{R}$, we have $\Phi_k(\eta^{(2)}_{k,\alpha}) = \eta^{(1)}_{k,\alpha} \otimes \eta^{(3)}_{k,\alpha}$ for the volume forms defined in Definition 4.1.2.*

4.1.3 Comparison of the three volumes and consequence on arithmetic Hilbert–Samuel

We need to compare the three volume forms on $H^0(\mathcal{X}, k\mathcal{L})_{\mathbb{R}}$. Define the positive functions

$$f_X(k, \alpha) := \frac{V^k_{X,L^2}}{V^k_{X,\alpha}}, \qquad h_X(k) := \frac{V^k_{X,\sup}}{V^k_{X,L^2}} \tag{4.1.5}$$

with $k \geq 1$ and $\alpha \in \mathbb{R}$.

Proposition 4.1.4. $\log h_X(k) = o(k^{n+1})$.

Proof. This follows immediately from Proposition 4.1.4. □

The following proposition will be proved in the next section.

Proposition 4.1.5. *There exists an affine function $\eta \colon \mathbb{R} \to \mathbb{R}$ such that*

$$\log f_X(k, \alpha) = \eta(\alpha) k^{n+1} + o(k^{n+1}). \tag{4.1.6}$$

In particular, there exists a unique real number α_0 such that $\log f_X(k, \alpha_0) = o(k^{n+1})$.

Proof of Theorem 6.2.4 assuming Proposition 4.1.5. By (4.1.2) and the definition of $V^k_{X,\sup}$, we have

$$\chi(k\overline{\mathcal{L}}) = -\log \mathrm{covol}_{V^k_{X,\sup}}(H^0(\mathcal{X}, k\mathcal{L})_{\mathbb{R}}/H^0(\mathcal{X}, k\mathcal{L})).$$

Thus Definition 4.1.2, Proposition 4.1.4, and Proposition 4.1.5 together yield

$$\chi(k\overline{\mathcal{L}}) = \chi(H^0(\mathcal{X}, k\mathcal{L}), V^k_{X,\alpha_0}) + \log h_X(k) + \log f_X(k, \alpha_0) = \alpha_0 \sum_{j=0}^{k-1} P(j) + o(k^{n+1})$$

Since $\deg P = n$ and P has leading coefficient $\mathcal{L}^n_K/n!$, we then have

$$\chi(k\overline{\mathcal{L}}) = \frac{\alpha_0 \mathcal{L}^n_K}{(n+1)!} k^{n+1} + o(k^{n+1}).$$

Thus the LHS converges to $\alpha_0 \mathcal{L}^n_K$ when $k \to \infty$. The real number α_0 can be read off in the proof of Proposition 4.1.5, where we will see that $\alpha_0 \mathcal{L}^n_K = \overline{\mathcal{L}}^{n+1}$. We are done. $\square$

4.2 Algebraic part of the proof of Proposition 4.1.5

The goal of this section is to prove Proposition 4.1.5, assuming an analytic result which will be proved in the next section.

4.2.1 Fundamental short exact sequence

Recall our assumption that $\mathcal{L}$ is very ample on $\mathcal{X}$. Hence there exists a closed immersion $\iota\colon \mathcal{X} \to \mathbb{P}^N_{\mathcal{O}_K}$ with $\iota^*\mathcal{O}(1) \simeq \mathcal{L}$. By Bertini's theorem, up to taking a finite extension of K there exists a non-zero global section ℓ of $\mathcal{O}(1)$ on $\mathbb{P}^N_{\mathcal{O}_K}$ such that $\mathrm{div}(\ell) \cap \mathcal{X}_K$ is a subvariety of dimension $n-1$ of $\mathcal{X}_K$, which is furthermore irreducible smooth if $n \geq 2$ (if $n = 1$ we can only guarantee the reducedness).

Set $s := \iota^*(\ell)$. Then $s \in H^0(\mathcal{X}, \mathcal{L})$. The ideal sheaf of $\mathrm{div}(s)$, which is $\mathcal{L}^{\otimes -1}$, admits a primary decomposition $\mathcal{L}^{\otimes -1} = \mathcal{I} \cap \mathcal{J}$ where $\mathcal{J}$ has vertical support and $\mathcal{I}$ defines a flat closed subscheme $\mathcal{Y}$ over $\mathrm{Spec}\mathcal{O}_K$ whose generic fiber is irreducible smooth if $n \geq 2$ and is reduced if $n = 1$. Moreover $\dim \mathcal{Y} = n = \dim \mathcal{X} - 1$.

Thus for $k \gg 1$, we have the following exact sequence:

$$0 \to H^0(\mathcal{X}, k\mathcal{L} + \mathcal{I}) \to H^0(\mathcal{X}, k\mathcal{L}) \to H^0(\mathcal{Y}, k\mathcal{L}|_{\mathcal{Y}}) \to 0. \tag{4.2.1}$$

Tensoring $\mathbb{R}$ yields, by definition of $\mathcal{I}$,

$$0 \to H^0(\mathcal{X}, (k-1)\mathcal{L})_{\mathbb{R}} \xrightarrow{\cdot s} H^0(\mathcal{X}, k\mathcal{L})_{\mathbb{R}} \to H^0(\mathcal{Y}, k\mathcal{L}|_{\mathcal{Y}})_{\mathbb{R}} \to 0. \tag{4.2.2}$$

4.2.2 Volume forms on the spaces

Our goal is to compare the volume forms V^k_{X,L^2} and $V^k_{X,\alpha}$ on $H^0(\mathcal{X}, k\mathcal{L})_{\mathbb{R}}$ for each real number $\alpha \in \mathbb{R}$, *by induction on* $n = \dim \mathcal{X}_K$. Hence it is desirable to study the respective volume forms on $H^0(\mathcal{Y}, k\mathcal{L}|_{\mathcal{Y}})_{\mathbb{R}}$ and on $H^0(\mathcal{X}, (k-1)\mathcal{L})_{\mathbb{R}}$.

On $H^0(\mathcal{Y}, k\mathcal{L}|_{\mathcal{Y}})_{\mathbb{R}}$, we have the volume forms

- V^k_{Y,L^2}, where the L^2-norm is defined using the same construction above Proposition 4.1.4 but with $\overline{\mathcal{L}}|_{\mathcal{Y}}$;

- $V^k_{Y,\alpha}$ defined by Defintion 4.1.2.

They are related by $V^k_{Y,L^2} = f_Y(k,\alpha)V^k_{Y,\alpha}$.

On $H^0(\mathcal{X},(k-1)\mathcal{L})_{\mathbb{R}}$, we have the volume forms

- V^{k-1}_{X,L^2};
- Z^{k-1}_{α}, by applying Definition 4.1.2 to $M = \bigoplus_{k\geq 0} H^0(\mathcal{X},(k+1)\mathcal{L}+\mathcal{I})$.

Set $t_X(k-1,\alpha) := V^{k-1}_{X,L^2}/Z^{k-1}_{\alpha}$.

Apply Lemma 4.1.3 to the exact sequence (4.2.1). Then we get $V^k_{X,\alpha} = V^k_{Y,\alpha}\otimes Z^{k-1}_{\alpha}$. Thus

$$\frac{V^k_{X,L^2}}{f_X(k,\alpha)} = \frac{V^k_{Y,L^2}\otimes V^{k-1}_{X,L^2}}{f_Y(k,\alpha)t_X(k-1,\alpha)}.$$

Denoting by

$$g(k) := \frac{V^k_{X,L^2}}{V^k_{Y,L^2}\otimes V^{k-1}_{X,L^2}}. \tag{4.2.3}$$

Then we have

$$\log f_X(k,\alpha) = \log t_X(k-1,\alpha) + \log f_Y(k,\alpha) - \log g(k). \tag{4.2.4}$$

The second term on the RHS will be handled by induction hypothesis.

The following proposition will be proved in the next section using analytic method. It handles the third term of the RHS of (4.2.4).

Proposition 4.2.1. *When $k\to\infty$, we have*

$$\frac{1}{P(k-1)}\log g(k) \to -\sum_{\sigma\colon K\hookrightarrow\mathbb{C}}\int_{\mathcal{X}_\sigma}\log\|s(x)\|^2 \mathrm{d}V_\sigma$$

with the volume form $\mathrm{d}V_\sigma$ on $\mathcal{X}_\sigma$ defined above Proposition 4.1.4 (via $c_1(\overline{\mathcal{L}}_\sigma)$).

4.2.3 Further treatment

Consider the following exact sequences of sheaves:

$$0\to\mathcal{I}\cdot\mathcal{J}\to\mathcal{I}\to\mathcal{I}\otimes\mathcal{O}_{\mathcal{X}}/\mathcal{J}\to 0,$$
$$0\to\mathcal{I}\cdot\mathcal{J}\to\mathcal{I}\cap\mathcal{J}=\mathcal{L}^{\otimes -1}\to\mathcal{T}:=\mathrm{Tor}^1(\mathcal{O}_{\mathcal{X}}/\mathcal{I},\mathcal{O}_{\mathcal{X}}/\mathcal{J})\to 0.$$

Then $\mathcal{T}$ has support in $\mathrm{Supp}(\mathcal{J})$, which is vertical over $\mathrm{Spec}\mathcal{O}_K$. For $k\gg 1$, we have exact sequences since $\mathcal{L}$ is ample

$$0\to H^0(\mathcal{X},k\mathcal{L}+\mathcal{I}\cdot\mathcal{J})\to H^0(\mathcal{X},k\mathcal{L}+\mathcal{I})\to H^0(\mathcal{X},k\mathcal{L}+\mathcal{I}+\mathcal{O}_{\mathcal{X}}/\mathcal{J})\to 0,$$
$$0\to H^0(\mathcal{X},k\mathcal{L}+\mathcal{I}\cdot\mathcal{J})\to H^0(\mathcal{X},(k-1)\mathcal{L})\to H^0(\mathcal{X},kL+\mathcal{T})\to 0$$

where we write $+$ for $\otimes$ as usual. The last terms in both short exact sequences are torsion. So applying the additivity of the arithmetic Euler characteristic (4.1.4) to both short exact sequences above and taking the difference, we obtain

$$\log t_X(k-1,\alpha)-\log f_X(k-1,\alpha) = \log\#H^0(\mathcal{X},k\mathcal{L}+\mathcal{I}+\mathcal{O}_X/\mathcal{J})-\log\#H^0(\mathcal{X},k\mathcal{L}+\mathcal{T}). \tag{4.2.5}$$

Combining with (4.2.4), we thus obtain

$$\log f_X(k,\alpha)-\log f_X(k-1,\alpha) = \log f_Y(k,\alpha)-\log g(k)+\log\#H^0(\mathcal{X},k\mathcal{L}+\mathcal{I}+\mathcal{O}_X/\mathcal{J})-\log\#H^0(\mathcal{X},k\mathcal{L}+\mathcal{T}). \tag{4.2.6}$$

4.2.4 Proof of Proposition 4.1.5 assuming Proposition 4.2.1

We do induction on $n = \dim \mathcal{X}_K \geq 0$.

$\boxed{\text{Base step}}$ When $n = 0$, we need to do a bit more, *i.e.* we assume $\mathcal{X}_K$ to be reduced but not necessarily irreducible. In this case $\mathcal{X} = \mathrm{Spec} R$ with R a finite $\mathcal{O}_K$-algebra which is reduced. By definition (4.1.5), we have

$$\log f_X(k, \alpha) = -\chi(k\mathcal{L}, V_{X,\alpha}^k) + \chi(k\mathcal{L}, V_{X,L^2}^k).$$

Notice that the Hilbert polynomial of $\mathcal{L}_K$ is constant. Hence Definition 4.1.2 implies that $\chi(k\mathcal{L}, V_{X,\alpha}^k)$ is an affine function in k. The function $\chi(k\mathcal{L}, V_{Y,L^2}^k)$ is also affine in k, by arithmetic Riemann–Roch applied to $\mathrm{Spec} R$ (we have seen this when $R = \mathcal{O}_K$ as Theorem 1.3.7 whose proof is a direct computation; in general we reduce to the case where R is an order of a number field and prove the similar result by computation). Hence we are done in this base step.

$\boxed{\text{Induction}}$ For general $n \geq 1$, we use (4.2.6) to analyze $\log f_X(k, \alpha) - \log f_X(k-1, \alpha)$.

When $n = 1$, recall our choice $s \in H^0(\mathcal{X}, \mathcal{L})$ satisfies that $\mathrm{div}(s_K)$ is reduced. When $n \geq 2$, the generic fiber $\mathcal{Y}_K$ is smooth by choice of the global section $s \in H^0(\mathcal{X}, \mathcal{L})$. In both cases, we can apply our induction hypothesis and get $\log f_Y(k, \alpha) = \eta'(\alpha) k^n + o(k^n)$. And $\log g(k) = \lambda k^n + o(k^n)$ by Proposition 4.2.1.

For $\log \# H^0(X, k\mathcal{L} + \mathcal{I} + \mathcal{O}_X/\mathcal{J})$ and $\log \# H^0(\mathcal{X}, k\mathcal{L} + \mathcal{T})$, decompose $\mathrm{Supp}(\mathcal{J})$ into disjoint union of connected subvarieties of dimension $\leq n$ (they are all contained in vertical fibers). The Hilbert–Samuel formula in algebraic geometry then implies that both terms are of the form $c' k^n + o(k^n)$.

Therefore, $\log f_X(k, \alpha) - \log f_X(k-1, \alpha) = c_0 k^n + o(k^n)$. So there exists a function $\eta \colon \mathbb{R} \to \mathbb{R}$ such that

$$\log f_X(k, \alpha) = \eta(\alpha) k^{n+1} + o(k^{n+1}).$$

It remains to show that η is affine. For this, notice that Definition 4.1.2 implies

$$\log f_X(k, \alpha) - \log f_X(k, \alpha') = c(\alpha - \alpha') k^{n+1} + o(k^{n+1}),$$

where c is the leading coefficient of $\sum_{j=0}^{k-1} P(j)$. Thus $\eta(\alpha) - \eta(\alpha') = c(\alpha - \alpha')$ for all $\alpha, \alpha' \in \mathbb{R}$. So η is affine. Better, we have $c = \mathcal{L}_K^n/(n+1)!$. We are done.

4.3 Analytic part of the proof

We will prove Proposition 4.1.4 and Proposition 4.2.1 in this section. This finishes the proof of Theorem 6.2.4.

Because both results are analytic, we rephrase our setting as follows to ease notation.

Let X be a projective manifold of dimension $n \geq 1$, and let $(L, \|\cdot\|)$ be a smooth Hermitian line bundle on X (so that $c_1(L, \|\cdot\|)$ is a Kähler form on X). Let ω be the scalar of $c_1(L, \|\cdot\|)$ such that $\int_X \mathrm{d}V = 1$ for the volume form $\mathrm{d}V = \omega^n/n!$ on X. Let P be the Hilbert polynomial, *i.e.* $P(k) = \dim H^0(X, kL)$ for $k \gg 1$. In the setting of Theorem 6.2.4 (see §4.1.1), these are $X = \mathcal{X}_\sigma$, $(L, \|\cdot\|) = (\overline{\sigma}, \|\cdot\|_\sigma)$, and $\omega = \omega_\sigma$ for each $\sigma \colon K \hookrightarrow \mathbb{C}$.

By abuse of notation, we still use $\|\cdot\|$ to denote the Hermitian metric on $kL = L^{\otimes k}$ for each $k \geq 1$ (in §3.1 it was denoted by $\|\cdot\|_k$). The norms $\|\cdot\|_{\sup}$ and $\|\cdot\|_{L^2}$ on $H^0(X, kL)$ are defined by $\|s\|_{\sup} := \sup_{x \in X} \|s(x)\|$ and $\|s\|_{L^2}^2 := \int_X \|s(x)\|^2 \mathrm{d}V$ (for $s \in H^0(X, kL)$).

4.3.1 Comparison of $\|\cdot\|_{\sup}$ and $\|\cdot\|_{L^2}$

Let us prove Proposition 4.1.4, *i.e.* there exists a real number $c > 0$ such that

$$\|s\|_{L^2} \leq \|s\|_{\sup} \leq cP(k)^{1/2}\|s\|_{L^2} \tag{4.3.1}$$

for all $k \geq 1$ and all $s \in H^0(X, kL)$.

The first inequality of (4.3.1) is clearly true by definition.

Let $s \in H^0(X, kL)$. Since X is compact, there exists $x \in X$ such that $\|s\|_{\sup} = \|s(x)\|$. Take an orthonormal basis $s_1, \ldots, s_N$ (with $N = P(k)$) of $H^0(X, kL)$ with respect to the L^2-norm. We may choose $s_2, \ldots, s_N$ such that $s_2(x) = \cdots = s_N(x) = 0$. Then we can write

$$s = \sum_{j=1}^{N} a_j s_j$$

with $a_j \in \mathbb{C}$. So $s(x) = a_1 s_1(x)$, $\|s\|_{L^2}^2 = \sum |a_j|^2$. Thus $\|s\|_{\sup}^2 = \|s(x)\|^2 = |a_1|^2\|s_1(x)\|^2$.

Now we use the distortion function $b_k \colon X \to \mathbb{R}$ defined by $b_k(x) = \sum_{j=1}^{N} \|s_j(x)\|^2$ from (3.1.3). Then $\|s\|_{\sup}^2 = |a_1|^2 b_k(x)$. Therefore

$$\|s\|_{\sup}^2 \leq \|s\|_{L^2} b_k(x).$$

Let $c := \sqrt{\sup_{x\in X} b_k(x)/P(k)}$. Notice that $\sup_{x\in X} b_k(x) < \infty$ since X is compact. So $c < \infty$. Moreover, $c > 0$ by the main theorem on the distortion function (Theorem 3.1.6). Hence we are done.

4.3.2 Setup and first estimates to prove Proposition 4.2.1

Let $s \in H^0(X, L)$ such that $Y := \mathrm{div}(s)$ is connected smooth if $n \geq 2$ (reduced if $n = 1$). For $k \gg 1$, we have the following exact sequence

$$0 \to H^0(X, (k-1)L) \xrightarrow{s} H^0(X, kL) \to H^0(Y, kL|_Y) \to 0. \tag{4.3.2}$$

For the L^2-volumes forms V_{X,L^2}^{k-1}, V_{X,L^2}^{k} and V_{Y,L^2}^{k} induced by the L^2-norms on the three spaces in the exact sequence, define the comparison function

$$g(k) := \frac{V_{X,L^2}^{k}}{V_{Y,L^2}^{k} \otimes V_{X,L^2}^{k-1}}.$$

We shall prove Proposition 4.2.1, *i.e.*

$$\frac{1}{P(k-1)} \log g(k) \to -\int_X \log\|s(x)\|^2 \mathrm{d}V \qquad \text{when } k \to \infty. \tag{4.3.3}$$

We shall make use of (4.3.2). The volume form V_{X,L^2}^{k} induces a quotient volume form V_{q,L^2}^{k} on $H^0(Y, kL|_Y)$, via the quotient L^2-norm $\|\cdot\|_{q,L^2}$ on $H^0(Y, kL|_Y)$. Define

$$\gamma(k) := \frac{V_{Y,L^2}^{k}}{V_{q,L^2}^{k}} > 0.$$

The volume form V^k_{X,L^2} also induces a subspace volume form V^{k-1}_{s,L^2} on $H^0(X,(k-1)L)$ via the multiplication by s, via the subspace L^2-norm $\|v\|_{s,L^2} := \|sv\|_{L^2}$ for all $v \in H^0(X,(k-1)L)$. Define

$$\delta(k) := \frac{V^k_{X,L^2}}{V^k_{s,L^2}} > 0.$$

Finally define

$$\varphi(k) := \frac{V^{k-1}_{s,L^2} \otimes V^k_{q,L^2}}{V^k_{X,L^2}} > 0.$$

Then we have

$$g(k) = \frac{\varphi(k)}{\delta(k-1)\gamma(k)}. \tag{4.3.4}$$

We prove the following estimates in this subsection.

Proposition 4.3.1. *$\log \varphi(k) = o(k^n)$.*

Proposition 4.3.2. *$\log \gamma(k) = o(k^n)$.*

The estimate of $\log \delta(k)$ will be proved in the next subsection (Proposition 4.3.3). These information will be put together to prove (4.3.3).

Let Q be the Hilbert polynomial of $L|_Y$ on Y, *i.e.* $Q(k) = \dim H^0(Y, kL|_Y)$ for $k \gg 1$. Then $\deg Q = \dim Y = n-1$.

Proof of Proposition 4.3.1. Let $a\colon \mathbb{R} \to \mathbb{R}$ which sends m to the volume of the unit ball of $\mathbb{R}^m$ for the usual volume form.

(4.3.2) is a short exact sequence of $\mathbb{C}$-vector spaces. Take a $\|\cdot\|_{L^2}$-orthonormal basis of $H^0(X,kL)$ over $\mathbb{C}$, $\{s_1,\ldots,s_{P(k-1)},t_1,\ldots,t_{Q(k)}\}$, such that $\{s^{-1}s_1,\ldots,s^{-1}s_{P(k-1)}\}$ is a $\|\cdot\|_{s,L^2}$-orthonormal basis of $H^0(X,(k-1)L)$ and the quotients $\{[t_1],\ldots,[t_{Q(k)}]\}$ is a $\|\cdot\|_{q,L^2}$-orthonormal basis of $H^0(Y,kL|_Y)$. Then

$$V^k_{X,L^2} = \frac{1}{a(P(k-1)+Q(k))} s_1\wedge\cdots\wedge s_{P(k-1)}\wedge t_1\wedge\cdots\wedge t_{Q(k)} \bigwedge \sqrt{-1}s_1\wedge\cdots\wedge\sqrt{-1}s_{P(k-1)}\wedge\sqrt{-1}t_1\wedge\cdots\wedge\sqrt{-1}t_{Q(k)}$$

and similarly for V^{k-1}_{s,L^2} and V^k_{q,L^2}. Thus we get

$$\varphi(k) = \frac{a(P(k-1))\cdot a(Q(k))}{a(P(k))}.$$

We are done. □

To prove Proposition 4.3.2, we need a comparison of $\|\cdot\|_{q,L^2}$ and $\|\cdot\|_{Y,L^2}$.

Proof of Proposition 4.3.2. We claim: there exist $k_0 > 0$ and $B > 0$ such that

$$\|t\|_{q,L^2} \le B\|t\|_{Y,L^2} \tag{4.3.5}$$

for all $k \ge k_0$ and all $t \in H^0(Y,kL|_Y)$. To prove this, we use the L^2-extension Theorem 3.4.1. More precisely, $(L,\|\cdot\|)$ in Theorem 3.4.1 is taken to be $kL - K_X$ endowed with the natural smooth metric for $k \gg 1$ and $(L_0,\|\cdot\|)$ in Theorem 3.4.1 is taken to be $(L,\|\cdot\|)$. Then the assumptions of Theorem 3.4.1 are satisfied for $k \gg 1$. Hence for $t \in H^0(Y, kL|_Y)$, there exists $T \in H^0(X, kL)$ such that $T|_Y = t$ and

$$\int_X \frac{\|T\|_{L^2}^2}{\|s\|_{L^2}^2 (\log \|s\|_{L^2})^2} \mathrm{d}V \leq M \|t\|_{Y,L^2}^2$$

for a constant M depending only on Y. Thus (4.3.5) holds true because the LHS of the inequality above is $\geq c_0 \|T\|_{L^2}^2 \geq c_0 \|t\|_{q,L^2}^2$, with $c_0 := \|s\|_{\sup}^{-2} (\log \|s\|_{\sup})^{-2} > 0$ a positive real number.

On the other hand, $\|\cdot\|_{Y,L^2} \leq \|\cdot\|_{Y,\sup} \leq \|\cdot\|_{q,\sup}$, which by Proposition 4.1.4 is furthermore $\leq cQ(k)^{1/2} \|\cdot\|_{q,L^2}$. Hence

$$(cQ(k)^{1/2})^{-1} \|\cdot\|_{Y,L^2} \leq \|\cdot\|_{q,L^2} \leq B \|\cdot\|_{Y,L^2}.$$

Therefore $\log \gamma(k) = o(k^n)$ since $\deg Q = n - 1$. □

4.3.3 Last estimate for the proof of Proposition 4.2.1

We prove in this subsection the following proposition. Notice that this finishes the proof of (4.3.3) (*i.e.* Proposition 4.2.1) in view of (4.3.4), Proposition 4.3.1, and Proposition 4.3.2.

Proposition 4.3.3. $\log \delta(k) = P(k) \int_X \log \|s(x)\|^2 \mathrm{d}V + o(k^n)$.

We start the proof by giving another expression of $\delta(k)$. Write $m_s \colon H^0(X, kL) \xrightarrow{s} H^0(X, (k+1)L)$ for the first morphism in (4.3.2) with k replaced by $k+1$, and write m_s^* for its dual under the L^2-norms. Set $\phi_{k,s} := m_s^* \circ m_s$. Then

$$\langle u, \phi_{k,s} u \rangle_{L^2} = \int_X \|s\|^2 \|u\|^2 \mathrm{d}V \qquad \text{for all } u \in H^0(X, kL). \tag{4.3.6}$$

The eigenvalues of $\phi_{k,s}$ can be obtained as follows. There exists a $\|\cdot\|_{L^2}$-orthonormal basis $\{\widetilde{s}_1, \ldots, \widetilde{s}_N\}$ of $H^0(X, kL)$ (with $N = P(k)$) which is orthogonal for norm $\|\cdot\|_{s,L^2}$ on $H^0(X, kL)$. Then the eigenvalues of $\phi_{k,s}$ are

$$\lambda_j := \|s\widetilde{s}_j\|_{L^2}^2 = \int_X \|s\|^2 \|\widetilde{s}_j\|^2 \mathrm{d}V$$

with $j \in \{1, \ldots, N\}$.

Lemma 4.3.4. $\delta(k) = \det \phi_{k,s} = \prod_{j=1}^N \lambda_j$.

Proof. By choice of the basis $\{\widetilde{s}_1, \ldots, \widetilde{s}_N\}$, the matrix of the Hermitian pairing $\langle \cdot, \cdot \rangle_{s,L^2}$ obtained from $\|\cdot\|_{s,L^2}$, under this basis of $H^0(X, kL)$, is $\mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. Hence

$$V_{X,L^2}^k = a(N) \widetilde{s}_1 \wedge \cdots \wedge \widetilde{s}_N \wedge \sqrt{-1}\widetilde{s}_1 \wedge \cdots \wedge \sqrt{-1}\widetilde{s}_N$$
$$V_{s,L^2}^k = a(N) \cdot \lambda_1^{-1/2} \widetilde{s}_1 \wedge \cdots \wedge \lambda_N^{-1/2} \widetilde{s}_N \wedge \sqrt{-1}\lambda_1^{-1/2} \widetilde{s}_1 \wedge \cdots \wedge \sqrt{-1}\lambda_N^{-1/2} \widetilde{s}_N.$$

So $\delta(k) = \prod_{j=1}^N \lambda_j = \det \phi_{k,s}$. □

This lemma tells us that $\frac{1}{P(k)} \log \delta(k)$ is precisely the logarithm of the geometric mean of the eigenvalues λ_j of $\phi_{k,s}$. Thus rescaling the metric $\|\cdot\|$ does not change the conclusion of Proposition 4.3.3. So from now on, we may and so assume

$$\|s\|_{\sup} < 1. \tag{4.3.7}$$

Thus $\lambda_j < 1$ for all $j \in \{1, \dots, N\}$.

Now we are ready to proceed to the proof of Proposition 4.3.3.

$\boxed{\text{Proof of } \geq}$ Let us show

$$\liminf_{k\to\infty} P(k)^{-1} \log \delta(k) \geq \int_X \log \|s\|^2 \mathrm{d}V. \tag{4.3.8}$$

By assumption, $\int_X \|\widetilde{s}_j\|^2 \mathrm{d}V = 1$ for all $j \in \{1, \dots, N\}$. So Jensen's inequality implies (where the measure is $\|\widetilde{s}_j\| \mathrm{d}V$)

$$\log \lambda_j = \log \int_X \|s\|^2 \|\widetilde{s}_j\|^2 \mathrm{d}V \geq \int_X (\log \|s\|^2) \|\widetilde{s}_j\|^2 \mathrm{d}V.$$

Taking sum over $j \in \{1, \dots, N = P(k)\}$ and recalling the distortion function $b_k = \sum \|\widetilde{s}_j\|^2$ defined in (3.1.3), we get

$$P(k)^{-1} \log \delta(k) = P(k)^{-1} \sum_{j=1}^{N} \log \lambda_j \geq \int_X \log \|s\|^2 \frac{b_k}{P(k)} \mathrm{d}V. \tag{4.3.9}$$

Hence (4.3.8) follows from the main theorem on the distortion function (Theorem 3.1.6).

$\boxed{\text{Proof of } \leq}$ It remains to prove the hard direction

$$\limsup_{k\to\infty} P(k)^{-1} \log \delta(k) \leq \int_X \log \|s\|^2 \mathrm{d}V. \tag{4.3.10}$$

The proof goes through *tilings* of X, *i.e.* a disjoint union of finitely many connected open subsets of X whose union is dense in X; we will furthermore assume each such connected open subset to have smooth boundary. We have assumed $\|s\|_{\sup} < 1$, so to control $\delta(k) = \det \phi_{k,s}$ it suffices to work on subspaces of $H^0(X, kL)$. Ideally, we would be able prove (4.3.10) if we could find a subspace of $H^0(X, kL)$ of dimension $\sim P(k)$ which has an orthonormal basis with supports in a suitable tiling, so that (4.3.9) eventually becomes an equality. This is not possible in the holomorphic category, and we need to extend our discussion to anti-holomorphic analysis discussed in §3.2.1. All is not lost: we can approximate holomorphic sections by *near holomorphic sections* subcoordinate to finer and finer tilings.

Fix a tiling Ω of X (which is an open subset of X). Then Ω is the disjoint union of finitely many connected open subsets $\Omega_1, \dots, \Omega_l$ of X

Recall the anti-holomorphic Kodaira Laplacian Δ''_k from Definition 3.2.1 (with $q = 0$) and the heat operator $\overline{\Box}_k := (2/k)\Delta''_k$. They acts on the Hilbert space $L^2(X, kL)$, and $H^0(X, kL)$ can be identified with the closed subspace $\mathrm{Ker}\overline{\Box}_k \subseteq L^2(X, kL)$. The L^2-orthogonal decomposition $L^2(X, kL) = \mathrm{Ker}\overline{\Box}_k \oplus \mathrm{Ker}\overline{\Box}_k^{\perp}$ defines the *Bergman projector*

$$\Psi_k \colon L^2(X, kL) \to \mathrm{Ker}\overline{\Box}_k = H^0(X, kL). \tag{4.3.11}$$

Let us consider the differential operator $\overline{\Box}_{k,\Omega}$, which is the restriction of $\overline{\Box}_k$ to Ω with the Dirichlet condition on the boundary $\partial\Omega$. Moreover, $\overline{\Box}_{k,\Omega}$ also have discrete spectrum.

Now we can define near holomorphic sections. For any real number $\mu > 0$, denote by $\mathcal{H}_k(\Omega, \mu)$ the direct sum of eigenspaces of $\overline{\Box}_{k,\Omega}$ associated with eigenvalues $\leq \mu$. We have a canonical way to obtain holomorphic sections from near holomorphic ones via $\Psi_k|_{\mathcal{H}_k(\Omega,\mu)}$.

Similarly we can define $\mathcal{H}_k(\Omega_j, \mu)$ for each $j \in \{1, \dots, l\}$. Then $\mathcal{H}_k(\Omega, \mu) = \bigoplus_{j=1}^{l} \mathcal{H}_k(\Omega_j, \mu)$.

Recall our assumption that $c_1(L, \|\cdot\|) = 2\alpha_0\omega$ for some $\alpha_0 > 0$.

The following lemma says that the Bergman projector injects $\mathcal{H}_k(\Omega, \mu)$ quasi-isometrically into $H^0(X, kL)$, for μ small enough and $k \gg 1$.

Lemma 4.3.5. *Assume $\mu < \alpha_0$. Then for $k \gg 1$, $\Psi_k|_{\mathcal{H}_k(\Omega,\mu)}$ is injective and*

$$\|\Psi_k(u) - u\|_{L^2} \le \frac{2\mu}{\alpha_0}\|u\|_{L^2} \qquad \text{for all } u \in \mathcal{H}_k(\Omega,\mu).$$

Proof. By Lemma 3.2.7, the smallest non-zero eigenvalue of $\overline{\square}_k$ is $\ge \alpha_0$ for $k \gg 1$.

Let $u \in \mathrm{Ker}\Psi_k$. Then $u \in \mathrm{Ker}\overline{\square}_k^{\perp}$, and hence $\|\overline{\square}_k u\|_{L^2} \ge \alpha_0\|u\|_{L^2} \ge \mu\|u\|_{L^2}$ by the previous paragraph. Now if $u \in \mathcal{H}_k(\Omega,\mu) \cap \mathrm{Ker}\Psi_k$, then the definition of $\mathcal{H}_k(\Omega,\mu)$ furthermore implies $u = 0$. This establishes the injectivity.

For $u \in \mathcal{H}_k(\Omega,\mu)$, set $\widetilde{u} := u - \Psi_k(u)$. Then $\widetilde{u} \in \mathrm{Ker}\overline{\square}_k^{\perp}$, and hence $\|\overline{\square}_k\widetilde{u}\|_{L^2} \ge \alpha_0\|\widetilde{u}\|_{L^2}$ by the previous paragraph. So

$$\alpha_0\|\widetilde{u}\|_{L^2} \le \|\overline{\square}_k(\Psi_k u - u)\|_{L^2} \le \|\Psi_k\overline{\square}_k u\|_{L^2} + \|\overline{\square}_k u\|_{L^2} \le 2\|\overline{\square}_k u\|_{L^2} \le 2\mu\|u\|_{L^2},$$

where the second inequality is the triangular inequality, and the last inequality is by definition of $\mathcal{H}_k(\Omega,\mu)$. ☐

The next lemma says that the Bergman projector is also a quasi-isometry for the quadratic form q_k defined by $q_k(u) := \int_X \|s\|^2\|u\|^2 \mathrm{d}V$. Recall that $q_k(u) = \langle u, \phi_{k,s}(u)\rangle_{L^2}$ for $u \in H^0(X,kL)$ by (4.3.6).

Lemma 4.3.6. *Assume $\mu < \alpha_0$. Then for $k \gg 1$, we have*

$$|q_k(u) - q_k(\Psi_k(u))| \le \frac{4\mu}{\alpha_0}\left(\frac{\mu}{\alpha_0} + 1\right)\|u\|_{L^2}^2 < \frac{8\mu}{\alpha_0}\|u\|_{L^2}^2.$$

Proof. We have

$$|q_k(u) - q_k(\Psi_k(u))| = \left|\int_X \|s\|^2(\|u\|^2 - \|\Psi_k(u)\|^2)\mathrm{d}V\right| \le \|s\|_{\sup}\int_X \left|\|u\|^2 - \|\Psi_k(u)\|^2\right| \mathrm{d}V.$$

We have assumed $\|s\|_{\sup} < 1$. So (by $\left||v_1|^2 - |v_2|^2\right| \le |v_1 - v_2|^2 + 2|v_2||v_1 - v_2|$)

$$\begin{aligned}|q_k(u) - q_k(\Psi_k(u))| &\le \|u - \Psi_k(u)\|_{L^2}^2 + 2\int_X \|\Psi_k(u)\| \cdot \left|\|u\| - \|\Psi_k(u)\|\right|\mathrm{d}V\\ &\le \|u - \Psi_k(u)\|_{L^2}^2 + 2\|\Psi_k(u)\|_{L^2}\|u - \Psi_k(u)\|_{L^2}\\ &\le \|u - \Psi_k(u)\|_{L^2}^2 + 2\|u\|_{L^2}\|u - \Psi_k(u)\|_{L^2}\end{aligned}$$

Now the conclusion follows from Lemma 4.3.5. ☐

The last estimate we need is the following:

Lemma 4.3.7. *For each $j \in \{1,\dots,l\}$, we have*

$$\dim \mathcal{H}_k(\Omega_j, k^{-1/6}) = P(k)\mathrm{vol}(\Omega_j) + o(k^n).$$

As a consequence, $\dim \mathcal{H}_k(\Omega, k^{-1/6}) = P(k) + o(k^n)$ *since* $\mathrm{vol}(\Omega) = \mathrm{vol}(X) = 1$.

Proof. The differential operator $\overline{\square}_{k,\Omega_j}$ (restriction of $\overline{\square}_k$ to Ω_j with Dirichlet condition on the boundary) also admits a heat kernel $e_{k,\Omega_j}(t,x,y)$, with condition (iii) of Proposition 3.2.4 replaced by $e_{k,\Omega_j}(t,\partial\Omega_j,y) = 0$ on the boundary $\partial\Omega_j$. Denote by $e_{k,\Omega_j}(t,x) := e_{k,\Omega_j}(t,x,x)$. Then the localization process of proving the heat kernel expansion implies

$$\lim_{k\to\infty} \|e_k(t,x) - e_{k,\Omega_j}(t,x)\| = 0$$

uniformly in $x \in X$ and $t \in [1, T_k]$, as long as $T_k = o(\sqrt{k})$. Consider an L^2-orthonormal eigenbasis $(\lambda, \psi_\lambda)_\lambda$ (resp. $(\widetilde{\lambda}, \psi_{\widetilde{\lambda}})$) for $\overline{\Box}_k$ (resp. for $\overline{\Box}_{k,\Omega_j}$). Integrating over X yields

$$(\sum_{\widetilde{\lambda}} e^{-\widetilde{\lambda} t})\mathrm{vol}(\Omega_j) = \sum_\lambda e^{-\lambda t} + o(1)$$

for $k \gg 1$, uniformly in $t \in [1, T_k]$. The conclusion then follows by setting $t = k^{1/4}$, using an argument similar to the estimate of the second term of (3.2.5) (use $\overline{\Box}^1_k$ and $\overline{\Box}^1_{k,\Omega_j}$, and the injectivity of the positive spectrum of $\overline{\Box}^0_{k(,\Omega_j)}$ into that of $\overline{\Box}^1_{k(,\Omega_j)}$). $\Box$

Now we are ready to finish the proof of (4.3.10), which is what is left to prove Proposition 4.3.3.

Proof of (4.3.10). Since $\|s\|_{\sup} < 1$, we have

$$\delta(k) = \det \phi_{k,s} = \det(q_k|_{H^0(X,kL)}) \leq \det(q_k|_{\Psi_k(\mathcal{H}_k(\Omega,\mu))}).$$

Lemma 4.3.5 implies

$$\det(q_k|_{\Psi_k(\mathcal{H}_k(\Omega,\mu))}) \leq \left(\frac{1}{1 - 2\mu/\alpha_0}\right)^{2\dim \mathcal{H}_k(\Omega,\mu)} \det(q_k \circ \Psi_k|_{\mathcal{H}_k(\Omega,\mu)}).$$

We shall use the following elementary result in linear algebra: the determinant of a positive-definite Hermitian matrix is bounded above by the product of the diagonal entries. Recall $\Omega = \coprod_{j=1}^l \Omega_j$. Now for each k and each $j \in \{1, \ldots, l\}$, take an orthonormal basis $\{h_m^{(j)}\}_m$ of $\mathcal{H}_k(\Omega_j, \mu)$ (hence $\int_{\Omega_j} \|h_m^{(j)}\|^2 \mathrm{d}V = 1$). Then $\{h_m^{(j)}\}_{m,j}$ is an orthonormal basis of $\mathcal{H}_k(\Omega, \mu) = \bigoplus_{j=1}^l \mathcal{H}_k(\Omega_j, \mu)$. Then

$$\det(q_k \circ \Psi_k|_{\mathcal{H}_k(\Omega,\mu)}) \leq \prod_{m,j} q_k \circ \Psi_k(h_m^{(j)}),$$

while Lemma 4.3.6 implies that

$$q_k \circ \Psi_k(h_m^{(j)}) \leq q_k(h_m^{(j)}) + \frac{8\mu}{\alpha_0} = \int_{\Omega_j} \|s\|^2 \|h_m^{(j)}\|^2 \mathrm{d}V + \frac{8\mu}{\alpha_0} \leq \sup_{\Omega_j} \|s\|^2 + \frac{8\mu}{\alpha_0}.$$

Combining the inequalities above, we get

$$\log \delta(k) \leq 2 \dim \mathcal{H}_k(\Omega, \mu) \log(1 - 2\mu/\alpha_0) + \sum_{j=1}^l \log\left(\sup_{\Omega_j} \|s\|^2 + 8\mu/\alpha_0\right) \dim \mathcal{H}_k(\Omega_j, \mu). \quad (4.3.12)$$

Fix $\epsilon \in (0, \frac{1 - \|s\|_{\sup}}{8})$. Take $\mu = \alpha_0 \epsilon$. Then $\|s\|_{\sup} + 8\mu/\alpha_0 < 1$. Now

$$\sum_j \log\left(\sup_{\Omega_j} \|s\|^2 + 8\mu/\alpha_0\right) \mathrm{vol}(\Omega_j) = \sum_j \sup_{\Omega_j} \log(\|s\|^2 + 8\epsilon)\mathrm{vol}(\Omega_j) \to \int_X \log(\|s\|^2 + 8\epsilon)\mathrm{d}V, \quad (4.3.13)$$

where the limit is on taking finer and finer tilings of X. More precisely, by letting the diameter of Ω tend to 0^+.

Thus the conclusion follows from Lemma 4.3.7, (4.3.12), and (4.3.13), by letting $\epsilon \to 0^+$. $\Box$

Chapter 5

Adelic line bundles

In §0.3, we have seen that polarized dynamical systems can sometimes give normalized height functions, which are genuine functions in contrast to the abstract height machine. The Weil height on $\mathbb{P}^N$ can be obtained in this way. Another important case is the Néron–Tate height on abelian varieties.

In §2.3.2, we explained how to use arithmetic models (with Hermitian line bundles) to find representatives of each class of height functions constructed by the height machine.

It is desirable to express each normalized height in §0.3 in the framework of §2.3.2. This is the case for the Weil height, as shown in Example 2.3.8. When an abelian variety has good reduction everywhere, it is also possible to do so using the Néron model and the *cubist metric*. However, if the abelian variety does not have good reduction everywhere, it is not possible to define the Néron–Tate height using arithmetic models as in §2.3.2.

To solve this problem, S. Zhang defined and studied *adelically metrized line bundles* (*adelic line bundles* for short) over projective varieties, by putting suitable metrics at the places of bad reduction. All the normalized heights from §0.3 can be defined in this framework. This tool is fundamental in the solution of the famous Bogomolov Conjecture by Ullmo and S. Zhang.

More recently, Yuan and S. Zhang generalized this framework to *adelic line bundles over quasi-projective varieties*. On the one hand, this allows to study the normalized height functions *in family*. On the other hand, it turns out that many other height functions can be defined in this framework, for example the Faltings height as a function on the moduli space of principally polarized abelian varieties. This powerful theory opens another chapter of Arakelov Geometry.

In the whole chapter, we take K to be a number field, and X to be an irreducible quasi-projective variety defined over K.

5.1 Limit construction for the geometric setting

Via $\mathbb{Q} \subseteq K$, we can see X as a quasi-projective variety over Spec$\mathbb{Q}$.

In this section, we construct the category of *geometric* adelic line bundles on X, denoted by $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})$. Roughly speaking, they are line bundles on X which can be extended to a line bundle on "some compatification" of X.

If X itself is projective, then the construction is void. Nevertheless, in practice we often need to work with quasi-projective varieties which are not projective, for example moduli spaces.

5.1.1 $\mathbb{Q}$-line bundles

We define the category of $\mathbb{Q}$-line bundles on X, denoted by $\mathcal{P}\mathrm{ic}(X)_{\mathbb{Q}}$, as follows:

Definition 5.1.1. *A* **$\mathbb{Q}$-line bundle** *on X is a pair (a, L) (often written as aL) with $a \in \mathbb{Q}$ and L a line bundle on X. A* **morphism** *of two $\mathbb{Q}$-line bundles aL and $a'L'$ is defined to be*

$$\mathrm{Hom}(aL, a'L') := \varinjlim_{m \to \infty} \mathrm{Hom}(amL, a'mL')$$

where m runs over all positive integers such that $am, a'm \in \mathbb{Z}$.

Denote by $\mathrm{Pic}(X)_{\mathbb{Q}}$ the group of isomorphism classes of $\mathbb{Q}$-line bundles on X. We can define *nef, ample, big* $\mathbb{Q}$-line bundles on projective varieties.

Definition 5.1.2. *A $\mathbb{Q}$-line bundle aL on X is said to be* **nef (ample, big)** *if amL is for some positive integer m such that amL is a usual line bundle on X.*

Next we define sections of $\mathbb{Q}$-line bundles.

Definition 5.1.3. *Let $aL \in \mathcal{P}\mathrm{ic}(X)_{\mathbb{Q}}$.*

(i) A **(global) section** *of aL on X is an element of $H^0(X, aL) := \mathrm{Hom}(\mathcal{O}_X, aL) = \varinjlim_m H^0(X, amL)$ where m runs over all positive integers with $am \in \mathbb{Z}$.*

(ii) A **rational section** *of aL on X is an element of $\mathrm{Hom}(\mathcal{O}_\eta, aL_\eta) = \varinjlim_m H^0(\eta, amL)$, where η is the generic point of X and m runs over all positive integers with $am \in \mathbb{Z}$.*

(iii) For a (rational) section s of aL on X, represented by $(s_m)_m$, define

$$\mathrm{div}(s) := (1/m)\mathrm{div}(s_m) \in \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q} =: \mathrm{Div}(X)_{\mathbb{Q}}.$$

For two elements $D_1, D_2 \in \mathrm{Div}(X)_{\mathbb{Q}}$, we write $D_1 \leq D_2$ if $m(D_2 - D_1)$ is a usual effective Cartier divisor for some positive integer m.

5.1.2 Model geometric adelic line bundles and boundary norm/topology

Now we are ready to define *model geometric adelic line bundles* on X as follows.

Definition 5.1.4. *The category of model geometric adelic line bundles on X, denoted by $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})_{\mathrm{mod}}$, is defined to be the category of pairs (X', L') with*

- *X' is a compactification of X, i.e. a projective variety defined over K which contains X as an open subset;*
- *L' is a $\mathbb{Q}$-line bundle on X', such that $L'|_X$ is isomorphic to a usual line bundle on X.*

Adelic line bundles are, roughly speaking, limits of sequences of model adelic line bundles. In order for the limit process to make sense, we need to introduce a suitable *Cauchy condition* for a sequence of model adelic line bundles. Let us explain it now.

For each compatification X' of X, denote by $\mathrm{Div}(X', X) := \mathrm{Div}(X')_{\mathbb{Q}} \otimes_{\mathrm{Div}(X)_{\mathbb{Q}}} \mathrm{Div}(X)$, *i.e.* the group of $\mathbb{Q}$-divisors on X' whose restriction to X is a usual Cartier divisor.

Definition 5.1.5. *The group of* **model geometric adelic divisors** *is defined to be*

$$\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}} := \varinjlim_{X'} \mathrm{Div}(X', X) \tag{5.1.1}$$

with X' running over all compactifications of X.

Notice that there is a partial order $\leq$ on $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$. For any $D_1, D_2 \in \widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$, there exists a compactification X' such that both D_1 and D_2 can be represented by elements (by abuse of notation still denoted by D_1, D_2) in $\mathrm{Div}(X', X) \subseteq \mathrm{Div}(X')_{\mathbb{Q}}$. We say that $D_1 \leq D_2$ as elements in $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$ if this is the case in $\mathrm{Div}(X')_{\mathbb{Q}}$. It is not hard to check that this partial ordering on $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$ is well-defined.

With this in hand, we can define the *boundary topology* on $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$ as follows. Fix a compactification X_0 of X such that $X_0 \backslash X$ is a divisor, which we call D_0. Then $D_0 \in \mathrm{Div}(X_0, X)$, which gives rise to an element in $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$ which is still denoted by D_0. The following *boundary norm* on $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$ (we use the convention $\inf(\emptyset) = \infty$)

$$\| \cdot \|_{D_0} \colon \widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}} \to [0, \infty], \qquad D \mapsto \inf\{\epsilon \in \mathbb{Q}_{>0} : -\epsilon D_0 \leq D \leq \epsilon D_0\} \tag{5.1.2}$$

then induces a topology on $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$, by defining a neighborhood basis at 0. This is the boundary topology.

Here is an easy lemma on the properties of the boundary norm. The "Moreover" part implies that the boundary topology does not depend on the choice of X_0.

Lemma 5.1.6. *For any $D, D' \in \widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$, we have*

(i) $\|D\|_{D_0} = 0$ *if and only if* $D = 0$,

(ii) $\|D + D'\|_{D_0} \leq \|D\|_{D_0} + \|D'\|_{D_0}$,

(iii) $\|aD\|_{D_0} \leq |a| \cdot \|D\|_{D_0}$ *for any* $a \in \mathbb{Z} \setminus \{0\}$, *with* $<$ *if and only if* $D \neq 0$ *and* $aD = 0$ *both hold in* $\mathrm{Div}(X)$.

Moreover, if X_0' is another compactification of X such that $D_0' := X_0' \backslash X$ is a divisor, then there exists a real number $r > 1$ such that $r^{-1}\| \cdot \|_{D_0} \leq \| \cdot \|_{D_0'} \leq r\| \cdot \|_{D_0}$.

5.1.3 Geometric adelic line bundles and adelic divisors

Definition 5.1.7. *A* **geometric adelic divisor** *on X is an equivalence class of Cauchy sequences in $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$, Cauchy for the boundary topology on $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$.*

The group of geometric adelic divisors on X is denoted by $\widehat{\mathrm{Div}}(X/\mathbb{Q})$, with the obvious binary operation.

Definition 5.1.8. *A* **geometric adelic line bundle** *on X is a pair $(L, (X_i, L_i, \ell_i)_{i \geq 1})$ with*

- *L is a line bundle on X;*
- *$(X_i, L_i) \in \mathcal{P}\mathrm{ic}(X/\mathbb{Q})_{\mathrm{mod}}$;*
- *$\ell_i \colon L \to L_i|_X$ is an isomorphism in $\mathcal{P}\mathrm{ic}(X)_{\mathbb{Q}}$;*

such that the sequence $\{\mathrm{div}(\ell_i \ell_1^{-1})\}_{i \geq 1}$ satisfies the Cauchy condition defined using the boundary topology on $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$.

The category of geometric adelic line bundles is denoted by $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})$. The group of isomorphism classes of geometric adelic line bundles, with $\otimes$ being the binary operation, is denoted by $\widehat{\mathrm{Pic}}(X/\mathbb{Q})$.

We need to explain that the sequence $\{\mathrm{div}(\ell_i\ell_1^{-1})\}_{i\geq 1}$ is indeed a sequence in $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$. For each $i\geq 1$, we have an isomorphism $\ell_i\ell_1^{-1}\colon L_1|_X \to L_i|_X$ of $\mathbb{Q}$-line bundles on X, and hence a rational map $\ell_i\ell_1^{-1}\colon L_1 \dashrightarrow L_i$. Hence $\mathrm{div}(\ell_i\ell_1^{-1})$ is a model adelic divisor for each $i\geq 1$, *i.e.* $\mathrm{div}(\ell_i\ell_1^{-1})\in\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$.

Next we will establish a canonical isomorphism between $\widehat{\mathrm{Pic}}(X/\mathbb{Q})$ with $\widehat{\mathrm{Cl}}(X/\mathbb{Q})$, the group of geometric adelic divisor classes. We should first of all define $\widehat{\mathrm{Cl}}(X/\mathbb{Q})$. We start by defining $\widehat{\mathrm{Prin}}(X/\mathbb{Q})_{\mathrm{mod}} := \varinjlim_{X'}\mathrm{Prin}(X')$, where X' runs over all the compactifications of X and $\mathrm{Prin}(X')$ is the group of principal divisors on X'. Then we can define

$$\widehat{\mathrm{Cl}}(X/\mathbb{Q}) := \widehat{\mathrm{Div}}(X/\mathbb{Q})/\widehat{\mathrm{Prin}}(X/\mathbb{Q})_{\mathrm{mod}}. \tag{5.1.3}$$

Lemma 5.1.9. *The group $\widehat{\mathrm{Prin}}(X/\mathbb{Q})_{\mathrm{mod}}$ is discrete in $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$ under the boundary topology.*

Before moving on to the proof, let us see an immediate corollary. If we let

$$\widehat{\mathrm{Cl}}(X/\mathbb{Q})_{\mathrm{mod}} := \widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}/\widehat{\mathrm{Prin}}(X/\mathbb{Q})_{\mathrm{mod}} = \varinjlim_{X'}(\mathrm{Div}(X',X)/\mathrm{Prin}(X')), \tag{5.1.4}$$

then $\widehat{\mathrm{Cl}}(X/\mathbb{Q})_{\mathrm{mod}}$ is dense in $\widehat{\mathrm{Cl}}(X/\mathbb{Q})$ by Lemma 5.1.9. In other words, $\widehat{\mathrm{Cl}}(X/\mathbb{Q})$ is the completion of $\widehat{\mathrm{Cl}}(X/\mathbb{Q})_{\mathrm{mod}}$.

Proof of Lemma 5.1.9. Assume that there exists a sequence $\{D_i\}_{i\geq 1}$ in $\widehat{\mathrm{Prin}}(X/\mathbb{Q})_{\mathrm{mod}}$ converging to 0. Then there exists a sequence $\{\epsilon_i\in\mathbb{Q}_{>0}\}_{i\geq 1}$ such that $\epsilon_i\to 0$ and $\epsilon D_0 \pm D_i \geq 0$ in $\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}}$ for all $i\geq 1$. Assume D_i is represented by $\mathrm{div}(f_i)$ for a compactification X_i of X and a rational function $f_i\in\mathbb{Q}(X_i)^* = \mathbb{Q}(X)^*$. Recall the compactification X_0 used to define the boundary topology. Then $\epsilon_i D_0 \pm \mathrm{div}(f_i)\geq 0$ in $\mathrm{Div}(X_0)_{\mathbb{Q}}$. Hence $\mathrm{Div}(f_i) = 0$ on X_0 by taking ϵ_i to be small enough. We are done. $\square$

Proposition 5.1.10. *There is a canonical isomorphism*

$$\widehat{\mathrm{Cl}}(X/\mathbb{Q}) \xrightarrow{\sim} \widehat{\mathrm{Pic}}(X/\mathbb{Q}).$$

Proof. We write the two morphisms.

For any $\{D_i\}_{i\geq 1}\in\widehat{\mathrm{Div}}(X/\mathbb{Q})$, assume each D_i is defined on the compactification X_i. Then $L_i := \mathcal{O}(D_i)$ is a $\mathbb{Q}$-line bundle on X_i. Notice that $D_i|_X = D_1|_U$ for all $i\geq 1$. Hence we get a line bundle $L := \mathcal{O}(D_1|_X)$ on X and isomorphisms $\ell_i\colon L\to L_i|_X$ for each $i\geq 1$. It is not hard to check the Cauchy condition for the sequence $\mathrm{div}(\ell_i\ell_1^{-1}) = D_i - D_1$. This defines the desired homomorphism

$$\widehat{\mathrm{Div}}(X/\mathbb{Q}) \to \widehat{\mathrm{Pic}}(X/\mathbb{Q}).$$

It is not hard to check that $\widehat{\mathrm{Prin}}(X/\mathbb{Q})_{\mathrm{mod}}$ is in the kernel.

To see the surjectivity: given any $(L,(X_i,L_i,\ell_i)_{i\geq 1})$ in $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})$, take a nonzero rational section s of L on X, and set

$$\widehat{\mathrm{div}}(s) := \{\mathrm{div}_{(X_1,L_1)}(s) + \mathrm{div}(\ell_i\ell_1^{-1})\}_{i\geq 1}, \tag{5.1.5}$$

where $\mathrm{div}_{(X_1,L_1)}(s)$ means to see s as a rational section of L_1 on X_1, and take the corresponding divisor. This defines the desired element in $\widehat{\mathrm{Cl}}(X/\mathbb{Q})$. $\square$

5.1.4 Positivity

Definition 5.1.11. *An adelic line bundle $\widetilde{L} \in \widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})$ is said to be:*

(i) **strongly nef** *if it is isomorphic to an object $(L, (X_i, L_i, \ell_i)_{i\geq 1})$ where each L_i is nef on X_i;*

(ii) **nef** *if there exists a strongly nef $\widetilde{M} \in \widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})$ such that $a\widetilde{L} + \widetilde{M}$ is strongly nef for all positive integers a;*

(iii) **integrable** *if it is isomorphic to the difference of two strongly nef adelic line bundles.*

We will use $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})_{\mathrm{snef}}$ (resp. $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})_{\mathrm{nef}}$, $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})_{\mathrm{int}}$) to denote the full subcategories of $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})$ of strongly nef (resp. nef, integrable) ones. We will use $\widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathrm{snef}}$ (resp. $\widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathrm{nef}}$, $\widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathrm{int}}$) to denote the corresponding subsets of $\widehat{\mathrm{Pic}}(X/\mathbb{Q})$. It is a semi-subgroup (resp, semi-subgroup, subgroup).

For any $\widetilde{L} = (L, \{X_i, L_i, \ell_i\}_{i\geq 1}) \in \widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})$, we define

$$H^0(X, \widetilde{L}) := \{s \in H^0(X, L) : \widehat{\mathrm{div}}(s) \geq 0\}. \tag{5.1.6}$$

It is known that $H^0(X, \widetilde{L})$ is a finite-dimensional vector space.

In height theory, elements in (5.1.6) play the same role as global sections on X when X is projective. Indeed, given a non-zero element $s \in H^0(X, \widetilde{L})$, then roughly speaking the height function defined by $\widetilde{L}$ has a lower bound outside $\widehat{\mathrm{div}}(s)$ which is proper Zariski closed.

Definition-Theorem 5.1.12. *The following limit exists and is defined to be the* **volume** *of $\widetilde{L} = (L, \{X_i, L_i, \ell_i\}_{i\geq 1})$:*

$$\mathrm{vol}(X, \widetilde{L}) := \lim_{m\to\infty} \frac{(\dim X)!}{m^{\dim X}} \dim H^0(X, m\widetilde{L}). \tag{5.1.7}$$

Moreover,

$$\mathrm{vol}(X, \widetilde{L}) = \lim_{i\to\infty} \mathrm{vol}(X_i, L_i).$$

Definition 5.1.13. *An adelic line bundle $\widetilde{L} \in \widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})$ is said to be* **big** *if* $\mathrm{vol}(X, \widetilde{L}) > 0$.

Let $n = \dim X$. We also have an intersection pairing in this situation (6.1.9)

$$\widehat{\mathrm{Pic}}(X/\mathbb{Q})^n_{\mathrm{int}} \to \mathbb{R}, \qquad (\widetilde{L}_1, \ldots, \widetilde{L}_n) \mapsto \widetilde{L}_1 \cdot \cdots \cdot \widetilde{L}_d. \tag{5.1.8}$$

Theorem 5.1.14 (Hilbert–Samuel)**.** *Assume $\widetilde{L}$ is nef. Then* $\mathrm{vol}(X, \widetilde{L}) = \widetilde{L}^n$.

Theorem 5.1.15 (Siu)**.** *If $\widetilde{L}$ and $\widetilde{M}$ are nef adelic line bundles, then*

$$\mathrm{vol}(X, \widetilde{L} - \widetilde{M}) \geq \widetilde{L}^n - n\widetilde{L}^{n-1}\widetilde{M}.$$

5.2 Adelic line bundles as limits of the model ones

Next we turn to the arithmetic setting and try to find arithmetic objects which will define the height functions as desired. We will do the limit construction in the following steps.

(i) Consider all the quasi-projective models $\mathcal{U}$ of X, *i.e.* $\mathcal{U}$ is an integral scheme which is quasi-projective over $\mathrm{Spec}\mathcal{O}_K$ such that X is open in the generic fiber $\mathcal{U}_K$. These quasi-projective models form an inverse system.

(ii) Define for each quasi-projective model $\mathcal{U}$ the category of *adelic line bundles* $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$ and the group of *adelic divisors* $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$.

(iii) Define $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z})$ to be $\varinjlim_{\mathcal{U}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$ where the limit is taken on the inverse system of quasi-projective models of X. Similarly define $\widehat{\mathrm{Div}}(X/\mathbb{Z}) := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$.

We call $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z})$ the category of adelic line bundles on X, and $\widehat{\mathrm{Div}}(X/\mathbb{Z})$ the group of adelic divisors on X. The group of isomorphism classes of adelic line bundles on X will be denoted by $\widehat{\mathrm{Pic}}(X/\mathbb{Z})$, which $\otimes$ being the binary operation. Similarly, we use the notation $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$ and $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$ to denote the groups of isomorphism classes of (model) adelic line bundles on $\mathcal{U}$.

Steps (i) and (iii) are formal. Step (ii) is the crucial step. In this section, we will define $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$ as a suitable completion of *model adelic line bundles* $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$ and define $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$ as a suitable completion of *model adelic divisors* $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$.

5.2.1 Model adelic line bundles on $\mathcal{U}$ and boundary topology

Let $\mathcal{U}$ be an integral scheme which is quasi-projective over $\mathrm{Spec}\mathcal{O}_K$.

Definition 5.2.1. *A* **model adelic line bundle** *on $\mathcal{U}$ is a pair $(\mathcal{X}, \overline{\mathcal{L}})$ consisting of:*

- *a projective model $\mathcal{X}$ of $\mathcal{U}$, i.e. an integral scheme which is projective over* $\mathrm{Spec}\mathcal{O}_K$ *and which contains $\mathcal{U}$ as an open subscheme;*

- *a $\mathbb{Q}$-Hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ on $\mathcal{X}$ such that $\mathcal{L}|_{\mathcal{U}}$ is a isomorphic to a usual line bundle on $\mathcal{U}$.*

Here, $\mathbb{Q}$-Hermitian line bundles are defined in the same way to the geometric case (Definition 5.1.1) with L replaced by $\overline{\mathcal{L}}$, and we also have the corresponding nefness, ampleness, and bigness for $\mathbb{Q}$-Hermitian line bundles.

The category of model adelic line bundles on $\mathcal{U}$ is denoted by $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$, and the group of isomorphism classes of model adelic line bundles is denoted by $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$.

To define model adelic divisors, we need to first of all define arithmetic $(\mathbb{Q}, \mathbb{Z})$-divisors. Let $\mathcal{X}$ be a projective model of $\mathcal{U}$.

Definition 5.2.2. *An* **arithmetic $(\mathbb{Q}, \mathbb{Z})$-divisors** *on $(\mathcal{X}, \mathcal{U})$ is a $\mathbb{Q}$-arithmetic divisor $\overline{D} = (D, g_D) \in \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$ such that $D|_{\mathcal{U}}$ is a usual divisor on $\mathcal{U}$. It is said to be* **nef** *if $\overline{D}$ is nef in $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$.*

The group of arithmetic $(\mathbb{Q}, \mathbb{Z})$-divisors on $(\mathcal{X}, \mathcal{U})$ is denoted by $\widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$. There is a partial ordering $\leq$ on $\widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$: $\overline{D} \leq \overline{D}'$ if $\overline{D}' - \overline{D}$ is effective as a $\mathbb{Q}$-arithmetic divisor on $\mathcal{X}$ and $D'|_{\mathcal{U}} - D|_{\mathcal{U}} \geq 0$ on $\mathcal{U}$.

Definition 5.2.3. *The group of* **model adelic divisors** *on $\mathcal{U}$ is defined to be*

$$\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}} := \varinjlim_{\mathcal{X}} \widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U}) \tag{5.2.1}$$

where $\mathcal{X}$ runs over all projective models of $\mathcal{U}$.

Notice that the partial ordering on $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$ defined above induces a partial ordering $\leq$ on $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$.

We also have a *boundary topology* on $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$ defined as follows. Fix a projective model $\mathcal{X}_0$ of $\mathcal{U}$ and a strictly effective divisor $\overline{D}_0 = (D_0, g_0)$ on $\mathcal{X}_0$ such that $|D_0| = \mathcal{X}_0 \setminus \mathcal{U}$. Such a pair $(\mathcal{X}_0, \overline{D}_0)$ is called a *boundary divisor*. Then $\overline{D}_0$ gives rise to an element in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$ which we still denote by $\overline{D}_0$. Then the boundary norm is defined to be ($\inf(\emptyset)$ is set to be ∞)

$$\|\cdot\|_{\overline{D}_0} \colon \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}} \to [0,\infty], \qquad \overline{D} \mapsto \inf\{\epsilon \in \mathbb{Q}_{>0} : -\epsilon\overline{D}_0 \leq \overline{D} \leq \epsilon\overline{D}_0\}. \tag{5.2.2}$$

This boundary norm induces a topology on $\widehat{\mathrm{Div}}(X/\mathbb{Z})_{\mathrm{mod}}$, by defining a neighborhood basis at 0. This is the boundary topology.

As in the geometric, we also have the following lemma, which asserts that the boundary topology does not depend on the choice of the pair $(\mathcal{X}_0, \overline{D}_0)$.

Lemma 5.2.4. *For any $\overline{D}, \overline{D}' \in \widehat{\mathrm{Div}}(X/\mathbb{Z})_{\mathrm{mod}}$, we have*

(i) $\|\overline{D}\|_{\overline{D}_0} = 0$ *if and only if* $\overline{D} = 0$,

(ii) $\|\overline{D} + \overline{D}'\|_{\overline{D}_0} \leq \|\overline{D}\|_{\overline{D}_0} + \|\overline{D}'\|_{\overline{D}_0}$,

(iii) $\|a\overline{D}\|_{\overline{D}_0} \leq |a| \cdot \|\overline{D}\|_{\overline{D}_0}$ *for any $a \in \mathbb{Z} \setminus \{0\}$, with $<$ if and only if $D_{\mathcal{U}} \neq 0$ and $aD|_{\mathcal{U}} = 0$ both hold in* $\mathrm{Div}(\mathcal{U})$.

Moreover, if $(\mathcal{X}_0', \overline{D}_0')$ is another boundary divisor, then there exists a real number $r > 1$ such that $r^{-1}\|\cdot\|_{\overline{D}_0} \leq \|\cdot\|_{\overline{D}_0'} \leq r\|\cdot\|_{\overline{D}_0}$.

5.2.2 Adelic line bundles and adelic divisors on $\mathcal{U}$

Let $\mathcal{U}$ be an integral scheme which is quasi-projective over $\mathrm{Spec}\mathcal{O}_K$.

Definition 5.2.5. *An* **adelic divisor** *on $\mathcal{U}$ is an equivalence class of Cauchy sequences in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$, Cauchy for the boundary topology.*

The group of adelic divisors on X is denoted by $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$, with the obvious binary operation.

Definition 5.2.6. *An* **adelic line bundle** *on $\mathcal{U}$ is a pair $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ with*

- *$\mathcal{L}$ is a line bundle on $\mathcal{U}$;*
- *$(\mathcal{X}_i, \overline{\mathcal{L}}_i) \in \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$;*
- *$\ell_i \colon \mathcal{L} \to \mathcal{L}_i|_{\mathcal{U}}$ is an isomorphism in $\mathcal{P}\mathrm{ic}(\mathcal{U})_{\mathbb{Q}}$;*

such that the sequence $\{\widehat{\mathrm{div}}(\ell_i\ell_1^{-1})\}_{i\geq 1}$ satisfies the Cauchy condition defined using the boundary topology on $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$.

The category of adelic line bundles is denoted by $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$. The group of isomorphism classes of geometric adelic line bundles, with $\otimes$ being the binary operation, is denoted by $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$.

We need to explain that the sequence $\{\widehat{\mathrm{div}}(\ell_i\ell_1^{-1})\}_{i\geq 1}$ is indeed a sequence in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$. For each $i \geq 1$, we have an isomorphism $\ell_i\ell_1^{-1} \colon \mathcal{L}_1|_{\mathcal{U}} \to \mathcal{L}_i|_{\mathcal{U}}$ of $\mathbb{Q}$-line bundles on $\mathcal{U}$, and hence a rational map $\ell_i\ell_1^{-1} \colon \mathcal{L}_1 \to \mathcal{L}_i$. Hence $\widehat{\mathrm{div}}(\ell_i\ell_1^{-1})$ is a model adelic divisor for each $i \geq 1$, *i.e.* $\widehat{\mathrm{div}}(\ell_i\ell_1^{-1}) \in \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$.

Next we will establish a canonical isomorphism between $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$ with $\widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z})$, the group of adelic divisor classes. We should first of all define $\widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z})$. For each projective model $\mathcal{X}$ of $\mathcal{U}$, there is a natural homomorphism $\widehat{\mathrm{Div}}(\mathcal{X}) \to \widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$, which makes $\widehat{\mathrm{Prin}}(\mathcal{X})$ a subgroup of $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$. Hence we can define

$$\widehat{\mathrm{Prin}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}} := \varinjlim_{\mathcal{X}} \widehat{\mathrm{Prin}}(\mathcal{X}),$$

where $\mathcal{X}$ runs over all the projective models of $\mathcal{U}$. Then we can define

$$\widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z}) := \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})/\widehat{\mathrm{Prin}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}. \tag{5.2.3}$$

Lemma 5.2.7. *The group $\widehat{\mathrm{Prin}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$ is discrete in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$ under the boundary topology.*

We omit the proof but state the following immediate corollary. If we let

$$\widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}} := \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}/\widehat{\mathrm{Prin}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}} = \varinjlim_{\mathcal{X}}(\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})/\widehat{\mathrm{Prin}}(\mathcal{X})), \tag{5.2.4}$$

then $\widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$ is dense in $\widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z})$ by Lemma 5.2.7. In other words, $\widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z})$ is the completion of $\widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$.

Proposition 5.2.8. *There is a canonical isomorphism*

$$\widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z}) \xrightarrow{\sim} \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z}).$$

Proof. We write the two morphisms.

For any $\{\overline{D}_i\}_{i\geq 1} \in \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$, assume each $\overline{D}_i$ is defined on the projective model $\mathcal{X}_i$. Then $\overline{\mathcal{L}}_i := \mathcal{O}(\overline{D}_i)$ is a $\mathbb{Q}$-Hermitian line bundle on $\mathcal{X}_i$. Notice that $D_i|_{\mathcal{U}} = D_1|_{\mathcal{U}}$ for all $i \geq 1$. Hence we get a line bundle $\mathcal{L} := \mathcal{O}(D_1|_{\mathcal{U}})$ on $\mathcal{U}$ and isomorphisms $\ell_i \colon \mathcal{L} \to \mathcal{L}_i|_{\mathcal{U}}$ for each $i \geq 1$. It is not hard to check the Cauchy condition for the sequence $\widehat{\mathrm{div}}(\ell_i\ell_1^{-1}) = \overline{D}_i - \overline{D}_1$. This defines a homomorphism

$$\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z}) \to \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z}).$$

Now assume that $\{\overline{D}_i\}_{i\geq 1}$ is in the kernel of this homomorphism. Then there exists an isomorphism from $(\mathcal{O}_{\mathcal{U}}, (\mathcal{X}_0, \overline{\mathcal{O}}_{\mathcal{X}_0}, 1))$ to $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i))$. Hence we have an isomorphism $\mathcal{O}_{\mathcal{U}} \to \mathcal{O}(D_1|_{\mathcal{U}})$, which is given by the multiplication by some $f \in H^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}})^*$ with $\mathrm{div}(f) = D_1|_{\mathcal{U}} = 0$ on $\mathcal{U}$. The further properties of the isomorphism are equivalent to that $\overline{D}_i$ converges to $-\widehat{\mathrm{div}}(f)$ in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$. Hence the kernel of the group homomorphism above is $\widehat{\mathrm{Prin}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$. So we have an injective group homomorphism

$$\widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z}) \to \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z}).$$

To see the surjectivity: given any $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$, take a nonzero rational section s of $\mathcal{L}$ on $\mathcal{U}$, and set

$$\widehat{\mathrm{div}}(s) := \{\widehat{\mathrm{div}}_{(\mathcal{X}_1,\overline{\mathcal{L}}_1)}(s) + \widehat{\mathrm{div}}(\ell_i\ell_1^{-1})\}_{i\geq 1}, \tag{5.2.5}$$

where $\widehat{\mathrm{div}}_{(\mathcal{X}_1,\overline{\mathcal{L}}_1)}(s)$ means to see s as a rational section of $\overline{L}_1$ on $\mathcal{X}_1$, and take the corresponding arithmetic divisor. This defines the desired element in $\widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z})$. $\square$

5.2.3 Nefness and integrability

Let $\mathcal{U}$ be an integral scheme which is quasi-projective over $\mathrm{Spec}\mathcal{O}_K$.

Definition 5.2.9. *An adelic line bundle $\overline{\mathcal{L}} \in \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$ is said to be:*

(i) **strongly nef** *if it is isomorphic to an object $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ where each $\overline{\mathcal{L}}_i$ is nef on $\mathcal{X}_i$;*

(ii) **nef** *if there exists a strongly nef $\overline{\mathcal{M}} \in \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$ such that $a\overline{\mathcal{L}} + \overline{\mathcal{M}}$ is strongly nef for all positive integers a;*

(iii) **integrable** *if it is isomorphic to the difference of two strongly nef adelic line bundles.*

We will use $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{snef}}$ (resp. $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{nef}}$, $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{int}}$) to denote the full subcategories of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$ of strongly nef (resp. nef, integrable) ones. We will use $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{snef}}$ (resp. $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{nef}}$, $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{int}}$) to denote the corresponding subsets of $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$. It is a semi-subgroup (resp, semi-subgroup, subgroup).

5.2.4 Generic fiber of adelic line bundles

Now we go back to our original situation, where X is an irreducible quasi-projective variety defined over K.

Recall the definition at the beginning of this section that

$$\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z}) := \varinjlim_{\mathcal{U}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z}), \quad \widehat{\mathrm{Pic}}(X/\mathbb{Z}) := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z}), \quad \widehat{\mathrm{Div}}(X/\mathbb{Z}) := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$$

with $\mathcal{U}$ running over all quasi-projective models of X.

Proposition 5.2.8 implies immediately

Proposition 5.2.10. *There is a canonical isomorphism*

$$\widehat{\mathrm{Cl}}(X/\mathbb{Z}) := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z}) \xrightarrow{\sim} \widehat{\mathrm{Pic}}(X/\mathbb{Z}).$$

For any projective model $\mathcal{X}$ of X, *i.e.* an integral scheme which is projective over $\mathrm{Spec}\mathcal{O}_K$ such that X is open in $\mathcal{X}_K$, the generic fiber $\mathcal{X}_K$ is by definition a projective model of X. Hence the natural map $\widehat{\mathrm{Pic}}(\mathcal{X}) \to \widehat{\mathrm{Pic}}(\mathcal{X}_K)$ induces a group homomorphism

$$\widehat{\mathrm{Pic}}(X/\mathbb{Z}) \to \widehat{\mathrm{Pic}}(X/\mathbb{Q}). \tag{5.2.6}$$

Definition 5.2.11. *For any adelic line bundle $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$, the image under* (5.2.6) *is called the* **generic fiber** *of $\overline{L}$. It is often denoted by $\widetilde{L}$.*

Let $\mathbf{P}$ be one of the symbols {snef, nef, int}. Then we define

$$\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z})_{\mathbf{P}} := \varinjlim_{\mathcal{U}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})_{\mathbf{P}}, \quad \widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathbf{P}} := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathbf{P}}.$$

It is not hard to check that (5.2.6) restricts to

$$\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathbf{P}} \to \widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathbf{P}}.$$

5.3 Metrized line bundles on Berkovich analytification

A second way to understand adelic line bundles is to see them as *metrized line bundles* on the Berkovich analytification of X. In this section, we explain Berkovich analytification and metrized line bundles.

Let k be a Banach ring, *i.e.* a ring with a norm $|\cdot|_{\mathrm{Ban}}$ which is complete for the induced topology. For example, $\mathbb{Z}$ with the archimedean absolute value $|\cdot|_\infty$, $\mathbb{Z}_p$ with the p-adic absolute value $|\cdot|_p$, or *any field* endowed with the trivial absolute value $|\cdot|_0$ ($|a|_0 = 1$ for all $a \neq 0$).

Let Y be a scheme over $\mathrm{Spec}k$. In our discussion, we assume Y to be separated and of finite type.

5.3.1 Berkovich analytification

In this subsection, we explain and recollect some results on Berkovich analytifications.

Definition 5.3.1. *The* **(Berkovich) analytification** *of Y, denoted by $(Y/k)^{\mathrm{an}}$ or Y^{an} for short, is defined as follows.*

If $Y = \mathrm{Spec}A$, then

- *as a set, Y^{an} is defined to be the space $\mathcal{M}(A) = \mathcal{M}(A/k)$ of multiplicative semi-norms on A whose restriction to k is bounded by $|\cdot|_{\mathrm{Ban}}$. For each $y \in \mathcal{M}(A)$, denote the corresponding semi-norm on A by $|\cdot|_y\colon A \to \mathbb{R}$. For any $f \in A$, write $|f|_y$ as $|f(y)|$, which give a real-valued function $|f|$ on $\mathcal{M}(A)$.*

- *the topology on X^{an} is the weakest one such that the function $|f|\colon \mathcal{M}(A) \to \mathbb{R}$ is continuous for all $f \in A$.*

In general, take an affine open cover $\{\mathrm{Spec}A_i\}$ of Y, and define Y^{an} to be the union of $\mathcal{M}(A_i)$, glued canonically. The topology on Y^{an} is the weakest one such that each $\mathcal{M}(A_i)$ is open.

It is known that Y^{an} is locally compact and Hausdorff. If $k = \mathbb{C}$ with the standard absolute value, then Y^{an} is homeomorphic to $Y(\mathbb{C})$ (and so coincides with the usual analytification). If $k = \mathbb{R}$ with the standard absolute value, then Y^{an} is homeomorphic to $Y(\mathbb{C})$ quotient by the complex conjugation.

In general, we have a decomposition

$$Y^{\mathrm{an}} = Y^{\mathrm{an}}[\infty] \cup Y^{\mathrm{an}}[\mathrm{f}] \tag{5.3.1}$$

into the subsets of archimedean and non-archimedean semi-norms. The trivial norm is by definition non-archimedean.

In what follows, when $k = \mathbb{Z}$, we always take $|\cdot|_{\mathrm{Ban}}$ on $\mathbb{Z}$ to be the absolute value $|\cdot|_\infty$

Example 5.3.2. *Let us look at $(\mathrm{Spec}\mathbb{Z})^{\mathrm{an}}$. It is the union of the closed-line segments*

$$[0,1]_\infty := \{|\cdot|_\infty^t : 0 \leq t \leq 1\}$$

and the closed-line segments

$$[0,\infty]_p := \{|\cdot|_p^t : 0 \leq t \leq \infty\}$$

for all finite prime numbers $p > 0$, by identifying the endpoints $|\cdot|_\infty^0$ and $|\cdot|_p^0$ for all p with the trivial norm $|\cdot|_0$ on $\mathbb{Z}$. In particular, $(\mathrm{Spec}\mathbb{Z})^{\mathrm{an}}$ is compact and path-connected.

For convenience, denote by

$$v_0 = |\cdot|_0,\ v_\infty = |\cdot|_\infty,\ v_\infty^t = |\cdot|_\infty^t,\ v_p = |\cdot|_p,\ v_p^t = |\cdot|_p^t,$$

and by

$$(0,1]_\infty,\ (0,1)_\infty,\ (0,\infty]_p,\ [0,\infty)_p,\ (0,\infty)_p$$

the sub-intervals of the line segments obtained by removing one or two endpoints; for example $(0,\infty)_p = \{|\cdot|_p^t : 0 < t < \infty\}$.

And $(\mathrm{Spec}\mathbb{Q}/\mathbb{Z})^{\mathrm{an}}$ *is* $(\mathrm{Spec}\mathbb{Z})^{\mathrm{an}}$ *with* v_p^∞ *removed for all* $p > 0$*, when we see* $\mathrm{Spec}\mathbb{Q}$ *as a scheme over* $\mathrm{Spec}\mathbb{Z}$ *via* $\mathbb{Z} \subseteq \mathbb{Q}$*. If we consider the trivial norm on* $\mathbb{Q}$*, then* $(\mathrm{Spec}\mathbb{Q}/\mathbb{Q})^{\mathrm{an}}$ *is* $\{v_0\}$.

Lemma 5.3.3. *If* Y *is projective over* k*, then* Y^{an} *is compact.*

Here are several basic notions concerning Y^{an}.

Definition 5.3.4. *(i) (Residue field) For each* $y \in Y^{\mathrm{an}}$*, define the* **residue field** H_y *as follows. Take an affine open* $\mathcal{M}(A)$ *such that* $y \in \mathcal{M}(A)$*. The semi-norm* $|\cdot|_y$ *induces a norm on the integral domain* $A/\mathrm{Ker}(|\cdot|_y)$*. Then* H_y *is defined to be the completion of the fraction field of* $A/\mathrm{Ker}(|\cdot|_y)$*. Notice that* $|\cdot|_y\colon A \to \mathbb{R}$ *can be decomposed into*

$$A \to H_y \xrightarrow{|\cdot|} \mathbb{R} \tag{5.3.2}$$

where $|\cdot|$ *is the multiplicative norm on* H_y *induced by* $|\cdot|_y$*. We thus write* $A \to H_y$, $f \mapsto f(y)$*. This notion generalizes to an arbitrary* Y^{an}*. By* (5.3.2)*, each* $y \in Y^{\mathrm{an}}$ *gives rise to a* k*-morphism*

$$\phi_y\colon \mathrm{Spec}H_y \to Y. \tag{5.3.3}$$

(ii) (Contraction) The **contraction map** $\kappa\colon Y^{\mathrm{an}} \to Y$ *is defined as follows. It suffices to define for* $\mathcal{M}(A)$*. For each* $y \in \mathcal{M}(A)$*, define* $\kappa(y) := \mathrm{Ker}(|\cdot|_y) \in \mathrm{Spec}A$.

(iii) (Injection) For each $x \in \mathrm{Spec}k$*, the trivial norm on the integral domain* k/x *induces a semi-norm* $|\cdot|_{x,0}$ *on* k*. Assume that each such* $|\cdot|_{x,0}$ *is bounded by* $|\cdot|_{\mathrm{Ban}}$*. This assumption holds true in the three cases considered at the beginning of this subsection (*$\mathbb{Z}$*,* $\mathbb{Z}_p$*, any field with the trivial norm).*

The **injection map** $\iota\colon Y \to Y^{\mathrm{an}}$ *is defined as follows. It suffices to define for* $Y = \mathrm{Spec}A$*. For* $\mathfrak{p} \in \mathrm{Spec}A$*, denote by* $|\cdot|_{\mathfrak{p},0}$ *the semi-norm on* A *induced by the trivial norm on* $A/\mathfrak{p}$*. Then set* $\iota(y) := |\cdot|_{\mathfrak{p},0}$.

(iv) (Reduction) If Y *is proper over* k*, then we can also define a* **reduction map** $r\colon Y^{\mathrm{an}} \to Y$ *as follows.*

Each $y \in Y^{\mathrm{an}}[\mathrm{f}]$ *gives rise to a* k*-morphism* $\mathrm{Spec}H_y \to Y$ *by* (5.3.3)*, and the valuative criterion of properness gives a uniquely extends it to a* k*-morphism* $\mathrm{Spec}R_y \to Y$ *(where* R_y *is the valuation ring of* H_y*). Then* $r(y)$ *is the image of the unique closed point of* $\mathrm{Spec}R_y$.

For $y \in Y^{\mathrm{an}}[\infty]$*, we still have a morphism* $\mathrm{Spec}H_y \to Y$*. Here* H_y *is isomorphic to* $\mathbb{R}$ *or* $\mathbb{C}$*. Define* $r(y)$ *to be the image of* $\mathrm{Spec}H_y$.

Example 5.3.2′. *In* $(\mathrm{Spec}\mathbb{Z})^{\mathrm{an}}$*. For each finite prime* p*, the residue field of* $v_p^t = |\cdot|_p^t$ *is* $\mathbb{Q}_p$ *when* $t \in (0,\infty)$ *and is* $\mathbb{F}_p$ *when* $t = \infty$*. The residue field of* $v_\infty^t = |\cdot|_\infty^t$ *is* $\mathbb{R}$ *when* $t \in (0,1]$*. The residue field of* v_0 *is* $\mathbb{Q}$.

The contraction map leaves $v_p^\infty = |\cdot|_p^\infty$ *stable and sends all other points to* $v_0 = |\cdot|_0$.

The injection map sends the prime (p) *to* $v_p^\infty = |\cdot|_p^\infty$*, and sends* (0) *to* $v_0 = |\cdot|_0$.

The reduction map sends $(0,\infty]_p$ *to* v_p^∞*, and sends* $[0,1]_\infty$ *to* v_0.

Lemma 5.3.5. *Any morphism $f\colon Y \to Y'$ induces a continuous map $f^{\mathrm{an}}\colon Y^{\mathrm{an}} \to Y'^{\mathrm{an}}$. For any $v \in Y'^{\mathrm{an}}$, the* **fiber** $Y_v^{\mathrm{an}} := (f^{\mathrm{an}})^{-1}(v)$*, defined as a subspace of Y^{an}, is canonically homeomorphic to the Berkovich space $(Y_{H_v}/H_v)^{\mathrm{an}}$.*

With this lemma in hand, we have study the structure of the analytification of Y^{an} for $k = \mathbb{Z}$. This applies to any arithmetic variety $\mathcal{U} \to \mathrm{Spec}\mathbb{Z}$ and also to our quasi-projective variety X (defined over a number field K) via $\mathbb{Z} \subseteq \mathbb{Q} \subseteq K$.

We have a structure map $Y^{\mathrm{an}} = (Y/\mathbb{Z})^{\mathrm{an}} \to \mathcal{M}(\mathbb{Z})$, which gives a disjoint union

$$Y^{\mathrm{an}} = \bigcup\nolimits_{v \in \mathcal{M}(\mathbb{Z})} Y_v^{\mathrm{an}}. \tag{5.3.4}$$

The most distinguished fibers are

$$Y_{v_\infty}^{\mathrm{an}} = Y_{\mathbb{R}}^{\mathrm{an}}, \quad Y_{v_p}^{\mathrm{an}} = Y_{\mathbb{Q}_p}^{\mathrm{an}}.$$

We can furthermore decompose, according to the structure of $\mathcal{M}(\mathbb{Z})$, into

(i) $Y_{\mathrm{triv}}^{\mathrm{an}} := Y_{v_0}^{\mathrm{an}} = (Y_{\mathbb{Q}}/\mathbb{Q})^{\mathrm{an}}$ under the trivial norm of $\mathbb{Q}$;

(ii) $Y_{v_p^\infty}^{\mathrm{an}} = (Y_{\mathbb{F}_p}/\mathbb{F}_p)^{\mathrm{an}}$ under the trivial norm of $\mathbb{F}_p$ for finite primes p;

(iii) $Y_{(0,\infty)_p}^{\mathrm{an}}$, homeomorphic to $Y_{\mathbb{Q}_p}^{\mathrm{an}} \times (0,\infty)$ for finite primes p;

(iv) $Y_{(0,1]_\infty}^{\mathrm{an}}$, homeomorphic to $Y_{\mathbb{R}}^{\mathrm{an}} \times (0,1]$.

Lemma 5.3.6. *The subset $Y^{\mathrm{an}} \setminus Y_{\iota(\mathrm{Spec}\mathbb{Z})}^{\mathrm{an}}$ is dense in Y^{an}.*

Let us go back to our situation where X is a quasi-projective variety defined over a number field K. We close this subsection with the following lemma.

Lemma 5.3.7. *Let $X \to \mathcal{U}$ be a quasi-projective model of X. Then the induced map $X^{\mathrm{an}} \to \mathcal{U}^{\mathrm{an}}$ is continuous, injective, and with a dense image. Better, the set of $v \in X^{\mathrm{an}}$ corresponding to discrete or archimedean valuations of H_v is dense in $\mathcal{U}^{\mathrm{an}}$.*

5.3.2 Metrized line bundle and arithmetic divisors on Y^{an}

Let L be a line bundle on Y. At each point $y \in Y^{\mathrm{an}}$, denote by $\overline{y} := \kappa(y)$ which is a point of Y. The *fiber* $L^{\mathrm{an}}(y)$ of L at y is defined to be the H_y-line $L(\overline{y}) \otimes_{k(\overline{y})} H_y$, or equivalently the completion of the fiber $L(\overline{y})$ of L on $\overline{y}$ for the semi-norm $|\cdot|_y$. In terms of (5.3.3), $L^{\mathrm{an}}(y) = \phi_y^* L$.

Definition 5.3.8. *A* **metrized line bundle** $\overline{L} = (L, \|\cdot\|)$ *on Y^{an} is a pair where L is a line bundle on Y and $\|\cdot\|$ is a continuous metric on Y^{an}. Here a continuous metric of L on Y^{an} is defined to be a continuous metric on $\coprod_{y \in Y^{\mathrm{an}}} L^{\mathrm{an}}(y)$ which is compatible with the semi-norms on $\mathcal{O}_Y$, i.e. for each $y \in Y^{\mathrm{an}}$, assign a norm $\|\cdot\|_y$ on $L^{\mathrm{an}}(y)$ such that $\|f\ell\|_y = |f|_y \|\ell\|_y$ for all $f \in H_y$ and all $\ell \in L^{\mathrm{an}}(y)$, and that for any local section ℓ of L on Y the function $\|\ell(y)\| := \|\ell(y)\|_y$ is continuous in $y \in Y^{\mathrm{an}}$.*

The category of metrized line bundle on Y^{an} is denoted by $\widehat{\mathcal{P}\mathrm{ic}}(Y^{\mathrm{an}})$, and the group of isomorphism classes of metrized line bundles on Y^{an} is denoted by $\widehat{\mathrm{Pic}}(Y^{\mathrm{an}})$.

Definition 5.3.9. *An* **arithmetic divisor** *on Y^{an} is a pair $\overline{D} = (D, g)$ where D is a Cartier divisor on Y and g is a continuous Green's function of $|D|^{\mathrm{an}}$ on Y^{an}, i.e. a continuous function $g\colon Y^{\mathrm{an}} \setminus |D|^{\mathrm{an}} \to \mathbb{R}$ such that for any rational function f on an open subset V of Y with $\mathrm{div}(f) = D|_V$, the function $g + \log|f|$ extends to a continuous function on V^{an}.*

An arithmetic divisor on Y^{an} is said to be **principal** *if it is of the form $\widehat{\mathrm{div}}(f) := (\mathrm{div}(f), -\log|f|)$ for some non-zero rational function of f on Y.*

The group of arithmetic divisors on Y^{an} is denoted by $\widehat{\mathrm{Div}}(Y^{\mathrm{an}})$, and the subgroup of principal arithmetic divisors is denoted by $\widehat{\mathrm{Prin}}(Y^{\mathrm{an}})$. We also have the following definition of effectiveness.

Definition 5.3.10. *An arithmetic divisor $\overline{D} = (D, g)$ on Y^{an} is called* **effective** *(resp.* **strictly effective***) if D is effective and $g \geq 0$ (resp. $g > 0$) on $Y^{\mathrm{an}} \setminus |D|^{\mathrm{an}}$.*

The Green's function g in this setting contains information not only on $Y^{\mathrm{an}}[\infty]$, but also $Y^{\mathrm{an}}[\mathrm{f}]$. Later on we shall see an example (Lemma 5.4.4) that the effectiveness of D is guaranteed by $g \geq 0$. This is not the case if we do not consider the Berkovich analytification.

In both $\widehat{\mathcal{P}\mathrm{ic}}(Y^{\mathrm{an}})$ and $\widehat{\mathrm{Div}}(Y^{\mathrm{an}})$, there is a distinguished class which is of particular interest.

Definition 5.3.11. *A metrized line bundle $\overline{L} = (L, \|\cdot\|)$ on Y^{an}, or its metric $\|\cdot\|$, is called* **norm-equivariant** *if any points $y, y' \in Y^{\mathrm{an}}$ satisfying $|\cdot|_y = |\cdot|_{y'}^t$ for some $0 \leq t < \infty$ locally on $\mathcal{O}_Y$, we have $\|\cdot\|_y = \|\cdot\|_{y'}^t$ (more precisely, for any rational section s of L on Y^{an} such that these two points y, y' are in $Y^{\mathrm{an}} \setminus |\mathrm{div}(s)|^{\mathrm{an}}$, we have $\|s(y)\| = \|s(y')\|^t$).*

An arithmetic divisor $\overline{D} = (D, g)$ on Y^{an}, or its Green's function g, is called **norm-equivariant** *if for any $y, y' \in Y^{\mathrm{an}} \setminus |D|^{\mathrm{an}}$ satisfying $|\cdot|_y = |\cdot|_{y'}^t$ for some $0 \leq t < \infty$ locally on $\mathcal{O}_Y$, we have $g(y) = tg(y')$.*

By definition, every principal arithmetic divisor is norm-equivariant. Denote by $\widehat{\mathcal{P}\mathrm{ic}}(Y^{\mathrm{an}})_{\mathrm{eqv}}$ the full sub-category of norm-equivariant metrized line bundles on Y^{an}, and $\widehat{\mathrm{Pic}}(Y^{\mathrm{an}})_{\mathrm{eqv}}$ and $\widehat{\mathrm{Div}}(Y^{\mathrm{an}})_{\mathrm{eqv}}$ similarly. We have the following proposition.

Proposition 5.3.12. *There is a natural group isomorphism*

$$\widehat{\mathrm{Cl}}(Y^{\mathrm{an}}) := \widehat{\mathrm{Div}}(Y^{\mathrm{an}})/\widehat{\mathrm{Prin}}(Y^{\mathrm{an}}) \xrightarrow{\sim} \widehat{\mathrm{Pic}}(Y^{\mathrm{an}}).$$

Moreover, it sends restricts to

$$\widehat{\mathrm{Cl}}(Y^{\mathrm{an}})_{\mathrm{eqv}} := \widehat{\mathrm{Div}}(Y^{\mathrm{an}})_{\mathrm{eqv}}/\widehat{\mathrm{Prin}}(Y^{\mathrm{an}}) \xrightarrow{\sim} \widehat{\mathrm{Pic}}(Y^{\mathrm{an}})_{\mathrm{eqv}}.$$

Proof. We write the two group homomorphisms.

Let $\overline{D} = (D, g) \in \widehat{\mathrm{Div}}(Y^{\mathrm{an}})$. Define $\mathcal{O}(\overline{D}) := (\mathcal{O}(D), \|\cdot\|_g)$, with $\|s_D\|_g = e^{-g}$ where s_D is the canonical section of $\mathcal{O}(D)$ (*i.e.* $\mathrm{div}(s_D) = D$). If $\overline{D}$ is principal, then it is not hard to check that $\mathcal{O}(\overline{D})$ is isomorphic to the trivial metrized line bundle.

Conversely let $\overline{L} = (L, \|\cdot\|)$ be a metrized line bundle on Y^{an}. Let s be a rational section of L on Y, and define

$$\widehat{\mathrm{div}}_{Y^{\mathrm{an}}}(s) := (\mathrm{div}(s), -\log\|s\|).$$

This gives the desired inverse. □

When $k = \mathbb{Z}$, a norm-equivariant Green's function or a norm-equivariant metric on a line bundle on Y^{an} is uniquely determined by its restriction to the disjoint union of the distinguished fibers $Y^{\mathrm{an}}_{v_\infty} = Y^{\mathrm{an}}_{\mathbb{R}}$ and $Y^{\mathrm{an}}_{v_p} = Y^{\mathrm{an}}_{\mathbb{Q}_p}$ for all finite primes p. This follows from Lemma 5.3.6.

5.4 Adelic line bundles as metrized line bundles

For our quasi-projective variety X defined over a number field K, we view X as a scheme over $\mathrm{Spec}\mathbb{Z}$ via $\mathbb{Z} \subseteq \mathbb{Q} \subseteq K$. Then we can apply the disjoint union decomposition (5.3.4) to X. In particular, we get a fiber $X^{\mathrm{an}}_{\mathrm{triv}} = X^{\mathrm{an}}_{v_0} = (X/\mathbb{Q})^{\mathrm{an}}$ (under the trivial norm of $\mathbb{Q}$) of $X^{\mathrm{an}} = (X/\mathbb{Z})^{\mathrm{an}}$ (with the Banach norm on $\mathbb{Z}$ being the archimedean absolute value).

The goal of this section is to prove the following theorem.

Theorem 5.4.1. *We have the following commutative diagram of homomorphisms of groups:*

$$\begin{tikzcd} \widehat{\mathrm{Pic}}(X/\mathbb{Z}) \arrow[r, hook] \arrow[d] & \widehat{\mathrm{Pic}}(X^{\mathrm{an}})_{\mathrm{eqv}} \arrow[d] \\ \widehat{\mathrm{Pic}}(X/\mathbb{Q}) \arrow[r, hook] & \widehat{\mathrm{Pic}}(X^{\mathrm{an}}_{\mathrm{triv}})_{\mathrm{eqv}} \end{tikzcd} \tag{5.4.1}$$

where the left vertical arrow is taking the generic fiber $\overline{L} \mapsto \widetilde{L}$, and the right vertical arrow is obtained by pulling back of $X^{\mathrm{an}}_{\mathrm{triv}} \subseteq X^{\mathrm{an}}$.

In the proof we shall see that the top arrow in (5.4.1) exists and is injective with X replaced by any quasi-projective arithmetic variety $\mathcal{U}$ (in fact it is an isomorphism).

We also have the corresponding version for arithmetic divisors and arithmetic divisor classes, in view of Proposition 5.2.10 and Proposition 5.3.12.

Theorem 5.4.1′. *We have the following natural injective group homomorphisms*

$$\widehat{\mathrm{Div}}(X/\mathbb{Z}) \hookrightarrow \widehat{\mathrm{Div}}(X^{\mathrm{an}})_{\mathrm{eqv}}, \qquad \widehat{\mathrm{Cl}}(X/\mathbb{Z}) \hookrightarrow \widehat{\mathrm{Cl}}(X^{\mathrm{an}})_{\mathrm{eqv}}. \tag{5.4.2}$$

Moreover strong results hold true with X replaced by any quasi-projective arithmetic variety $\mathcal{U}$, where the homomorphisms are isomorphisms.

5.4.1 Construction over projective arithmetic varieties

Let $\mathcal{X}$ be a projective arithmetic variety, *i.e.* a separated integral scheme of finite type over $\mathrm{Spec}\mathbb{Z}$ with projective structural morphism. Let us construct a functor

$$\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X}) \to \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X}^{\mathrm{an}})_{\mathrm{eqv}} \tag{5.4.3}$$

where $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})$ is the category of Hermitian line bundles on $\mathcal{X}$.

Let $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ be a Hermitian line bundle on $\mathcal{X}$. We define a metric of $\mathcal{L}$ on $\mathcal{X}^{\mathrm{an}}$ as follows. Recall the decomposition (5.3.1)

$$\mathcal{X}^{\mathrm{an}} = \mathcal{X}^{\mathrm{an}}[\infty] \cup \mathcal{X}^{\mathrm{an}}[\mathrm{f}]$$

and its refinement below (5.3.4). Now $\|\cdot\|$ gives a metric $\|\cdot\|^{\mathrm{an}}$ of $\mathcal{L}$ on $\mathcal{X}^{\mathrm{an}}_{v_\infty} = \mathcal{X}^{\mathrm{an}}_{\mathbb{R}}$, because $\mathcal{X}^{\mathrm{an}}_{\mathbb{R}} = \mathcal{X}(\mathbb{C})/\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and the metric $\|\cdot\|$ is invariant under the complex conjugation. This metric extends to $\mathcal{X}^{\mathrm{an}}[\infty]]$ by norm-equivariance (Definition 5.3.11) as follows: For any $x \in \mathcal{X}^{\mathrm{an}}[\infty]$, write $(x', t) \in \mathcal{X}^{\mathrm{an}}_{v_\infty} \times (0, 1]$ for the coordinate under the homeomorphism $\mathcal{X}^{\mathrm{an}}[\infty] = \mathcal{X}^{\mathrm{an}}_{(0,1]_\infty} \simeq \mathcal{X}^{\mathrm{an}}_{v_\infty} \times (0, 1]$ (with $(0, 1]_\infty = \{v^t_\infty : 0 < t \leq 1\}$), then set $\|\cdot\|^{\mathrm{an}}_x := (\|\cdot\|^{\mathrm{an}}_{x'})^t$. Notice that $\|\cdot\|^{\mathrm{an}}$ is continuous on $\mathcal{X}^{\mathrm{an}}[\infty]$ by construction.

To define the metric of $\mathcal{L}$ at a point $x \in \mathcal{X}^{\mathrm{an}}[\mathrm{f}]$, we use (the construction of the) specialization map $r\colon \mathcal{X}^{\mathrm{an}}[\mathrm{f}] \to \mathcal{X}$ by the properness of $\mathcal{X} \to \mathrm{Spec}\mathbb{Z}$. More precisely, the point x

gives a morphism $\phi_x\colon \mathrm{Spec}H_x \to \mathcal{X}$ which, by the valuative criterion of properness, extends to $\phi_x^\circ\colon \mathrm{Spec}R_x \to \mathcal{X}$ for the valuation ring R_x of H_x. Then $(\phi_x^\circ)^*\mathcal{L}$ is a free module over R_x of rank 1. Let s_x be the basis of this free module. Define the metric $\|\cdot\|_x^{\mathrm{an}}$ on $\mathcal{L}^{\mathrm{an}}(x) = \phi_x^*\mathcal{L}$ by letting $\|s_x\|_x^{\mathrm{an}} = 1$. Notice that this construction does not use the Hermitian metric on $\overline{\mathcal{L}}$.

Lemma 5.4.2. *Let $\mathcal{U}$ be a Zariski open subset of $\mathcal{X}$. Assume $x \in \mathcal{X}^{\mathrm{an}}$ satisfies $r(x) \in \mathcal{U}$. Then $x \in \mathcal{U}^{\mathrm{an}}$.*

Proof. This is clearly true if $x \in \mathcal{X}^{\mathrm{an}}[\infty]$, by definition of r. Now assume $x \in \mathcal{X}^{\mathrm{an}}[\mathrm{f}]$. Recall that $r(x) = \phi_x^\circ(\mathfrak{m}_x)$ where $\mathfrak{m}_x$ is the unique closed point of $\mathrm{Spec}R_x$.

It suffices to prove $\phi_x^\circ(R_x) \subseteq \mathcal{U}$. Assume not. Then $\phi_x\colon \mathrm{Spec}H_x \to \mathcal{X}$ factors through $\mathcal{X} \setminus \mathcal{U}$ which is itself proper, and hence its (unique) extension ϕ_x° also factors through $\mathcal{X} \setminus \mathcal{U}$, contradicting $r(x) \in \mathcal{U}$. □

Lemma 5.4.3. *The metric $\|\cdot\|^{\mathrm{an}}$ is continuous.*

Proof. We first prove the continuity of $\|\cdot\|^{\mathrm{an}}$ on $\mathcal{X}^{\mathrm{an}}[\mathrm{f}]$. Let $r\colon \mathcal{X}^{\mathrm{an}} \to \mathcal{X}$ be the specialization map, *i.e.* $r(x) = \phi_x^\circ(\mathfrak{m}_x)$ where $\mathfrak{m}_x$ is the unique closed point of $\mathrm{Spec}R_x$. Let $\{x_m\}_{m\geq 1}$ be a sequence in $\mathcal{X}^{\mathrm{an}}[\mathrm{f}]$ converging to $x \in \mathcal{X}^{\mathrm{an}}[\mathrm{f}]$, and let ℓ be a local section of $\mathcal{L}$ on $\mathcal{X}$. We need to prove that $\|\ell(x_m)\|^{\mathrm{an}}$ converges to $\|\ell(x)\|^{\mathrm{an}}$.

Take an open cover $\mathcal{U}_1, \ldots, \mathcal{U}_n$ of $\mathcal{X}$ such that each $\mathcal{U}_i$ contains $r(x)$ and trivializes $\mathcal{L}$ (such an open cover exists). Then $x \in \mathcal{U}_i^{\mathrm{an}}$ by Lemma 5.4.2. On the other hand, $\ell|_{\mathcal{U}_i}$ can be seen as a regular function on $\mathcal{U}_i$, which we denote by f_i. Then $\|\ell(x)\|^{\mathrm{an}} = |f_i|_x$.

For each i, denote by I_i the set of $m \geq 1$ such that $r(x_m) \in \mathcal{U}_i$. Then $I_1 \cup \cdots \cup I_n = \mathbb{Z}_{>0}$. Now take $i \in \{1, \ldots, n\}$ and $m \in I_i$. By Lemma 5.4.2, $x_m \in \mathcal{U}_i^{\mathrm{an}}$. And then $\|\ell(x_m)\|^{\mathrm{an}} = |f_i|_{x_m}$ by the discussion in the previous paragraph. So $\lim_{m\in I_i} \|\ell(x_m)\|^{\mathrm{an}} = \lim_{m\in I_i} |f_i|_{x_m} = |f_i|_x = \|\ell(x)\|^{\mathrm{an}}$ for each $i \in \{1, \ldots, n\}$.

Next, we check the continuity of $\|\cdot\|^{\mathrm{an}}$ when $\mathcal{X}^{\mathrm{an}}[\infty]$ approaches $\mathcal{X}_{v_0}^{\mathrm{an}}$. Let $\{x_m\}_{m\geq 1}$ be a sequence in $\mathcal{X}^{\mathrm{an}}[\infty]$ converging to a point $x \in \mathcal{X}_{v_0}^{\mathrm{an}}$ and let ℓ be a local section of $\mathcal{L}$ on $\mathcal{X}$. We need to prove that $\|\ell(x_m)\|^{\mathrm{an}}$ converges to $\|\ell(x)\|^{\mathrm{an}}$.

Write (z_m, t_m) for the point x_m under the homeomorphism $\mathcal{X}^{\mathrm{an}}[\infty] = \mathcal{X}^{\mathrm{an}}_{(0,1]_\infty} \simeq \mathcal{X}_{v_\infty}^{\mathrm{an}} \times (0, 1]$. Then $t_m \to 0$ by assumption on $\{x_m\}_{m\geq 1}$. Assume I is a subsequence of $\mathbb{Z}_{>0}$ such that $\lim_{m\in I} z_m = z \in \mathcal{X}_{v_\infty}^{\mathrm{an}}$. Take an open subset $\mathcal{U}$ of $\mathcal{X}$ which contains $r(x)$ and $r(z)$ such that $\mathcal{L}|_{\mathcal{U}}$ is trivial (such an $\mathcal{U}$ exists). Then $x, z \in \mathcal{U}^{\mathrm{an}}$ by Lemma 5.4.2. Up to removing finitely many elements in I, we may and do assume that $x_m, z_m \in \mathcal{U}^{\mathrm{an}}$ for all $m \in I$. Notice that $\ell|_{\mathcal{U}}$ can be seen as a regular function on $\mathcal{U}$ which we denote by f, and $\|\ell(x)\|^{\mathrm{an}} = |f|_x$ since $x \in \mathcal{U}_{v_0}^{\mathrm{an}} \subseteq \mathcal{U}^{\mathrm{an}}[\mathrm{f}]$. Now f extends to a rational function on $\mathcal{X}$ which we still call f. Then $f^{-1}\ell$ is a rational section of $\mathcal{L}$ on $\mathcal{X}$ such that $\mathcal{U} \cap |\mathrm{div}(f^{-1}\ell)| = \emptyset$. In particular, we have $\|(f^{-1}\ell)(x_m)\|^{\mathrm{an}} = (\|(f^{-1}\ell)(z_m)\|^{\mathrm{an}})^{t_m} = \|(f^{-1}\ell)(z_m)\|^{t_m}$ by definition of $\|\cdot\|^{\mathrm{an}}$ (the first equality is the definition of norm-equivariance Definition 5.3.11), so $\|(f^{-1}\ell)(x_m)\|^{\mathrm{an}} \to 1$ when $m \to \infty$. So $\lim_{m\in I} \|\ell(x_m)\|^{\mathrm{an}} = \lim_{m\in I} |f|_{x_m} = |f|_x = \|\ell(x)\|^{\mathrm{an}}$.

Now the conclusion follows because $\|\cdot\|^{\mathrm{an}}$ is clearly continous on $\mathcal{X}^{\mathrm{an}}[\infty]$. □

The construction (5.4.3) can be translated into a group homomorphism

$$\widehat{\mathrm{Div}}(\mathcal{X}) \to \widehat{\mathrm{Div}}(\mathcal{X}^{\mathrm{an}})_{\mathrm{eqv}}. \tag{5.4.4}$$

Let $\overline{D} = (D, g)$ be an arithmetic divisor on $\mathcal{X}$. The desired Green's function $\widetilde{g}$ of $|D|^{\mathrm{an}}$ on $\mathcal{X}^{\mathrm{an}}$ as follows. Now $g\colon \mathcal{X}(\mathbb{C}) \setminus |D(\mathbb{C})| \to \mathbb{R}$ naturally gives a Green's function on $\mathcal{X}_{v_\infty}^{\mathrm{an}} = \mathcal{X}_{\mathbb{R}}^{\mathrm{an}} = \mathcal{X}(\mathbb{C})/\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ since g is invariant under the complex conjugation. It extends to a

Green's function $\widetilde{g}$ on $\mathcal{X}^{\mathrm{an}}[\infty]$ by norm-equivariance (Definition 5.3.11): For any $x \in \mathcal{X}^{\mathrm{an}}[\infty]$, write $(x', t) \in \mathcal{X}^{\mathrm{an}}_{v_\infty} \times (0,1]$ for the coordinate under the homeomorphism $\mathcal{X}^{\mathrm{an}}[\infty] = \mathcal{X}^{\mathrm{an}}_{(0,1]_\infty} \simeq \mathcal{X}^{\mathrm{an}}_{v_\infty} \times (0,1]$ (with $(0,1]_\infty = \{v_\infty^t : 0 < t \le 1\}$), and then set $\widetilde{g}(x) = tg(x')$.

For $x \in (\mathcal{X} \setminus |D|)^{\mathrm{an}}[\mathrm{f}]$, take a Zariski open $\mathcal{U}$ of $\mathcal{X}$ such that $r(x) \in \mathcal{U}$ and that $D|_{\mathcal{U}} = \mathrm{div}(f)$ for some $f \in \mathbb{Q}(\mathcal{U})^*$. Then $\widetilde{g}(x)$ is defined to be $-\log |f|_x$.

The continuity of $\widetilde{g}$ on $\mathcal{X}^{\mathrm{an}} \setminus |D|^{\mathrm{an}}$ follows from Lemma 5.4.3. It self-improves to that $\widetilde{g}$ is a Green's function of $|D|^{\mathrm{an}}$ on $\mathcal{X}^{\mathrm{an}}$, by applying the continuity to the arithmetic divisor $(D - \mathrm{div}_{\mathcal{X}}(f), g + \log |f|_\infty)$ for any rational function f on an open subset $\mathcal{V}$ of $\mathcal{X}$ with $\mathrm{div}(f) = D|_{\mathcal{V}}$.

The Green's function $\widetilde{g}$ contains much more information than g. As a particular instance, we have the following lemma.

Lemma 5.4.4. *Assume $\mathcal{X}$ is normal. Let $\overline{D} = (D, g)$ be an arithmetic divisor on $\mathcal{X}$ and let $\widetilde{g}$ be the associated Green's function on $\mathcal{X}^{\mathrm{an}}$. Then $\overline{D}$ is effective if and only if $\widetilde{g} \ge 0$ on $\mathcal{X}^{\mathrm{an}} \setminus |D|^{\mathrm{an}}$.*

Proof. Only the "if" part needs to be checked. Assume $\widetilde{g} \ge 0$. We only need to check the effectiveness of D. For any $v \in \mathcal{X}$ of codimension 1, we need to show that the valuation $\mathrm{ord}_v(D)$ in the local ring $\mathcal{O}_{\mathcal{X},v}$ is non-negative. Consider the point $\xi := \exp(-\mathrm{ord}_v)$ of $\mathcal{X}^{\mathrm{an}}$. Let f be a local equation of D in an open neighborhood of v in $\mathcal{X}$, then by definition we have

$$\widetilde{g}(\xi) = -\log |f|_\xi = -\log(\exp(-\mathrm{ord}_v f)) = \mathrm{ord}_v f = \mathrm{ord}_v(D).$$

Hence we are done. □

5.4.2 Construction over quasi-projective arithmetic varieties

Let $\mathcal{U}$ be a quasi-projective arithmetic variety, *i.e.* a separated integral scheme of finite type over $\mathrm{Spec}\mathbb{Z}$ with quasi-projective structural morphism. Now let us construct a functor

$$\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z}) \to \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}} \tag{5.4.5}$$

and prove that it is fully-faithful. Notice that this proves the existence of the top arrow in (5.4.1) and its injectivity, with X replaced by $\mathcal{U}$.

Construction of (5.4.5)

Let $\overline{\mathcal{L}} = (\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i \ge 1}) \in \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$. Each $\overline{\mathcal{L}}_i$ induces a metric $\|\cdot\|_i^{\mathrm{an}}$ of $\mathcal{L}_i$ on $\mathcal{X}_i^{\mathrm{an}}$ as was done in the previous subsection. Then we get a metric $\|\cdot\|_i$ on $\mathcal{L}$ by pulling back via the isomorphism $\ell_i \colon \mathcal{L} \xrightarrow{\sim} \mathcal{L}_i|_{\mathcal{U}}$.

Let us show that $\|\cdot\|_i$ converges pointwise to a metric $\|\cdot\|$ of $\mathcal{L}$ on $\mathcal{U}^{\mathrm{an}}$; then the image of $\overline{\mathcal{L}}$ under (5.4.5) is set to be $\overline{\mathcal{L}}^{\mathrm{an}} := (\mathcal{L}, \|\cdot\|)$.

Let $(\mathcal{X}_0, \overline{D}_0)$ be a boundary divisor. Write $\widetilde{g}_0$ for the Green's function of D_0 on $\mathcal{X}_0^{\mathrm{an}}$ induced by $\overline{D}_0$ via (5.4.4). By the definition of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$, the sequence $\{\widehat{\mathrm{div}}(\ell_i \ell_1^{-1})\}_{i \ge 1}$ is Cauchy in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$, *i.e.* there exists a sequence $\{\epsilon_j\}_{j \ge 1}$ of positive rational numbers tending to 0 such that the following inequality holds true in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$:

$$-\epsilon_j \overline{D}_0 \le \widehat{\mathrm{div}}(\ell_i \ell_1^{-1}) - \widehat{\mathrm{div}}(\ell_j \ell_1^{-1}) \le \epsilon_j \overline{D}_0, \quad \forall i \ge j \ge 1.$$

Write $f_i := \log(\|\cdot\|_i / \|\cdot\|_1)$ as a continuous function on $\mathcal{U}^{\mathrm{an}}$. Then the condition above implies

$$-\epsilon_j \widetilde{g}_0 \le f_i - f_j \le \epsilon_j \widetilde{g}_0, \quad \forall i \ge j \ge 1. \tag{5.4.6}$$

The verification is by a detour of using the constructions of (5.4.3) and (5.4.4), and Proposition 5.2.8 relating $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$ with $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$ (resp. Proposition 5.3.12 ralating $\widehat{\mathrm{Pic}}(\mathcal{U}^{\mathrm{an}})$ with $\widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})$). Thus $\{f_i\}$ is uniformly convergent to a continuous function on any compact subset of $\mathcal{U}^{\mathrm{an}}$.

Recall that $\mathcal{U}^{\mathrm{an}}$ is locally compact. So $\{f_i\}$ converges pointwise to a continuous function f on $\mathcal{U}^{\mathrm{an}}$. Hence $\|\cdot\|_i$ converges pointwise to a continuous metric $\|\cdot\|$ such that

$$-\epsilon_j\widetilde{g}_0 \le \log(\|\cdot\|/\|\cdot\|_j) \le \epsilon_j\widetilde{g}_0, \quad \forall j \ge 1. \tag{5.4.7}$$

Fully-faithfulness of (5.4.5)

Let us show that there exists a canonical isomorphism

$$\Phi\colon \mathrm{Hom}(\overline{\mathcal{O}}_{\mathcal{X}_0}, \overline{\mathcal{L}}) \xrightarrow{\sim} \mathrm{Hom}(\overline{\mathcal{O}}_{\mathcal{U}}, \overline{\mathcal{L}}^{\mathrm{an}}) \tag{5.4.8}$$

where $\overline{\mathcal{O}}_{\mathcal{X}_0} = (\mathcal{O}_{\mathcal{U}}, (\mathcal{X}_0, \overline{\mathcal{O}}_{\mathcal{X}_0}, 1))$ and $\overline{\mathcal{O}}_{\mathcal{U}} = (\mathcal{O}_{\mathcal{U}}, \|\cdot\|_0)$ are the identity elements.

Elements of both sides of Φ are represented by regular sections s of $\mathcal{L}$ which are everywhere non-vanishing on $\mathcal{U}$. Such a section s gives an element of the RHS if and only if $\|s\| = 1$ on $\mathcal{U}^{\mathrm{an}}$, or equivalently if and only if $\widehat{\mathrm{div}}(s) = 0$ in $\widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})$. Such a section s gives an element of the LHS if and only if $\widehat{\mathrm{div}}(s) = 0$ in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$.

Recall that $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}} = \varinjlim_{\mathcal{X}} \widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$ with $\mathcal{X}$ running over all projective models of $\mathcal{U}$. We may assume $\mathcal{X}$ to be normal by taking normalization. Then by Lemma 5.4.4, an element in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$ is effective if and only if its image in $\widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}$ is effective. This gives the desired isomorphism (5.4.8).

Now let $\overline{\mathcal{L}}, \overline{\mathcal{L}}' \in \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$ with images $\overline{\mathcal{L}}^{\mathrm{an}}, \overline{\mathcal{L}}'^{\mathrm{an}} \in \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})$. Applying (5.4.8) to $(\overline{\mathcal{L}}')^\vee \otimes \overline{\mathcal{L}}$, we get a canonical isomorphism

$$\mathrm{Hom}(\overline{\mathcal{O}}_{\mathcal{X}_0}, (\overline{\mathcal{L}}')^\vee \otimes \overline{\mathcal{L}}) \xrightarrow{\sim} \mathrm{Hom}(\overline{\mathcal{O}}_{\mathcal{U}}, (\overline{\mathcal{L}}'^{\mathrm{an}})^\vee \otimes \overline{\mathcal{L}}^{\mathrm{an}}),$$

and hence a canonical isomorphism

$$\mathrm{Hom}(\overline{\mathcal{L}}', \overline{\mathcal{L}}) \xrightarrow{\sim} \mathrm{Hom}(\overline{\mathcal{L}}'^{\mathrm{an}}, \overline{\mathcal{L}}^{\mathrm{an}}).$$

This proves that the functor (5.4.5) is fully-faithful.

In terms of adelic divisors

The construction (5.4.5) can be converted to

$$\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z}) \to \widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}, \qquad \text{and} \quad \widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z}) \to \widehat{\mathrm{Cl}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}. \tag{5.4.9}$$

Here is a more concrete way for this construction of (5.4.9) for which we focus on the first homomorphism. For each projective model $\mathcal{X}$ of $\mathcal{U}$, the analytification map (5.4.4) induces a map

$$\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U}) \to \widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}, \quad \overline{D} = (D, g) \mapsto (D|_{\mathcal{U}}, \widetilde{g}).$$

By direct limit, this map gives $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}} \to \widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}$. Now we wish to extend this map to (5.4.9). Fix a boundary divisor $(\mathcal{X}_0, \overline{D}_0)$ of $\mathcal{U}$. Let $\{(D_i, g_i)\}_{i\ge 1} \in \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$, *i.e.* a Cauchy sequence in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$, with each D_i a divisor of a projective model $\mathcal{X}_i$ of $\mathcal{U}$. Then there exists a sequence $\{\epsilon_j \in \mathbb{Q}_{>0}\}_{j\ge 1}$ with $\epsilon_j \to 0$ and

$$-\epsilon_j(D_0, g_0) \le (D_i - D_j, g_i - g_j) \le \epsilon_j(D_0, g_0), \quad \forall i \ge j \ge 1. \tag{5.4.10}$$

Write $\widetilde{g}_i$ for the Green's function of D_i on $\mathcal{X}_i^{\mathrm{an}}$ induced by $\overline{D}_i = (D_i, g_i)$ via (5.4.4), for each $i \geq 0$ (this includes the boundary divisor). Notice that $D_1|_{\mathcal{U}} = D_2|_{\mathcal{U}} = \cdots$, and we denote by D this divisor on $\mathcal{U}$. Let us show that $\{\widetilde{g}_i\}_{i\geq 1}$ converges to a Green's function of D on $\mathcal{U}^{\mathrm{an}}$. Indeed, (5.4.10) implies that

$$-\epsilon_j \widetilde{g}_0 \leq \widetilde{g}_i - \widetilde{g}_j \leq \epsilon_j \widetilde{g}_0, \quad \forall i \geq j \geq 1. \tag{5.4.11}$$

Thus $\{\widetilde{g}_i\}$ is uniformly convergent to a continuous function on any compact subset of $\mathcal{U}^{\mathrm{an}}$. Recall that $\mathcal{U}^{\mathrm{an}}$ is locally compact. So $\{\widetilde{g}_i\}$ converges pointwise to a continuous function $\widetilde{g}$ on $\mathcal{U}^{\mathrm{an}}$, which is the desired Green's function.

Now (5.4.9) is defined by sending $\{(D_i, g_i)\}_{i\geq 1} \mapsto (D, \widetilde{g})$.

5.4.3 Proof of Theorem 5.4.1

Consider the functor

$$\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z}) = \varinjlim_{\mathcal{U}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z}) \xrightarrow{\lim \text{ of } (5.4.5)} \varinjlim_{\mathcal{U}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}, \tag{5.4.12}$$

which is fully-faithful since (5.4.5) is. For any quasi-projective model $\mathcal{U}$ of X, the map $X^{\mathrm{an}} \to \mathcal{U}^{\mathrm{an}}$ induces a natural map $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}} \to \widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})_{\mathrm{eqv}}$. Thus we have a functor

$$\varinjlim_{\mathcal{U}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}} \to \widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})_{\mathrm{eqv}}. \tag{5.4.13}$$

Now composing the two functors above, we obtain

$$\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z}) \to \widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})_{\mathrm{eqv}} \tag{5.4.14}$$

which gives the top arrow of (5.4.1).

Now let us prove that (5.4.13) is fully-faithful. The upshot is that the top arrow of (5.4.1) is injective.

We start by showing that the natural functor

$$\varinjlim_{\mathcal{U}} \mathcal{P}\mathrm{ic}(\mathcal{U}) \to \mathcal{P}\mathrm{ic}(X) \tag{5.4.15}$$

is fully-faithful. Fix a quasi-projective model $\mathcal{U}_0$ of X. It is not hard to show that the system $\{\mathcal{U}\}$ can be taken to be the inverse system of open subscheme of $\mathcal{U}_0$ containing X. Now take $\mathcal{L}, \mathcal{L}'$ two line bundles on some open neighborhood of X in $\mathcal{U}_0$. Then the map

$$\varinjlim_{\mathcal{U}} H^0(\mathcal{U}, \mathcal{L}^\vee \otimes \mathcal{L}') \to H^0(X, \mathcal{L}^\vee \otimes \mathcal{L}')$$

is injective since both sides are subgroups of rational sections of $\mathcal{L}^\vee \otimes \mathcal{L}'$ on X, and is surjective because any rational section s of $\mathcal{L}^\vee \otimes \mathcal{L}'$ regular and nowhere vanishing on X must be regular and nowhere vanishing on a neighborhood of X in $\mathcal{U}_0$. In other words,

$$\varinjlim_{\mathcal{U}} \mathrm{Hom}(\mathcal{L}|_{\mathcal{U}}, \mathcal{L}'|_{\mathcal{U}}) \simeq \mathrm{Hom}(\mathcal{L}|_X, \mathcal{L}'|_X),$$

whereas the fully-faithfulness of (5.4.15). Hence (5.4.13) is fully-faithful by Lemma 5.3.7.

Next we turn to the bottom arrow of (5.4.1). In fact, we can simply repeat the construction in §5.4.1 if X is projective (replace $\mathcal{X}/\mathbb{Z}$ by $X/\mathbb{Q}$; notice that the construction is easier since

$X^{\mathrm{an}} = X^{\mathrm{an}}[\mathrm{f}]$ in this case), and then pass to quasi-projective X after a similar but easier construction as in §5.4.2. This establishes the bottom arrow of (5.4.1) and proves its injectivity, and at the same time proves the commutativity of the diagram (5.4.1). $\square$

As for Theorem 5.4.1′, the desired homomorphisms are

$$\widehat{\mathrm{Div}}(X/\mathbb{Z}) = \varinjlim_{\mathcal{U}} \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z}) \xrightarrow{\text{lim of (5.4.9)}} \varinjlim_{\mathcal{U}} \widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}} \to \widehat{\mathrm{Div}}(X^{\mathrm{an}})_{\mathrm{eqv}},$$

$$\widehat{\mathrm{Cl}}(X/\mathbb{Z}) = \varinjlim_{\mathcal{U}} \widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z}) \xrightarrow{\text{lim of (5.4.9)}} \varinjlim_{\mathcal{U}} \widehat{\mathrm{Cl}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}} \to \widehat{\mathrm{Cl}}(X^{\mathrm{an}})_{\mathrm{eqv}}.$$

Here the last maps in both compositions are induced by $X \subseteq \mathcal{U}$. Similar to Lemma 5.4.4, we have the following:

Lemma 5.4.5. *An adelic divisor $\overline{D} \in \widehat{\mathrm{Div}}(X/\mathbb{Z})$ is effective if and only if its image $\overline{D}^{\mathrm{an}} \in \widehat{\mathrm{Div}}(X^{\mathrm{an}})_{\mathrm{eqv}}$ is effective.*

5.5 Families of polarized dynamical systems and abelian schemes

Let S be an irreducible quasi-projective variety defined over a number field K. Let (X, f, L) be a *weakly polarized dynamical system* over S, *i.e.*

- X is an integral scheme, projective and flat over S;
- $f : X \to X$ is an S-morphism;
- $L \in \mathcal{P}\mathrm{ic}(X)$ such that $f^*L \simeq qL$ for some integer $q > 1$.

Recall that $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z}) = \varinjlim_{\mathcal{U}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$, and there is a natural functor $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z}) \to \mathcal{P}\mathrm{ic}(X)$, $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1}) \mapsto \mathcal{L}|_X$. Thus we have a natural forgetful functor

$$\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z}) \to \mathcal{P}\mathrm{ic}(X). \tag{5.5.1}$$

Theorem 5.5.1. *There exists an adelic line bundle $\overline{L}_f \in \widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z})$ extending L, i.e. the image of $\overline{L}_f$ under the forgetful functor above is L, satisfying the following properties:*

(i) $\overline{L}_f$ is f-invariant, i.e. $f^*\overline{L}_f \simeq q\overline{L}_f$ in $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z})$.

(ii) Assume L is relatively ample over S. Then $\overline{L}_f$ is nef in $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z})$. If S has an affine quasi-projective model over $\mathbb{Z}$, then $\overline{L}_f$ is strongly nef.

Moreover as an element in $\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathbb{Q}}$, the extension $\overline{L}_f$ is uniquely determined by condition (i).

This theorem in particular applies to any abelian scheme $\mathcal{A} \to S$ with a relatively ample symmetric line bundle $\mathcal{L}$ on $\mathcal{A}$. In this case f can be taken to be $[n]$, and $q = n^2$ $(n \geq 2)$. It turns out that the resulting adelic line bundle does not depend on the choice of n, so it suffices to take $n = 2$.

Remark 5.5.2. *In practice, we sometimes need to work with the following slightly more general situation. Let $L \in \mathcal{P}\mathrm{ic}(X)_{\mathbb{Q}}$ such that $f^*L \simeq qL$ for some rational number $q > 1$. Then in Theorem 5.5.1 we obtain an $\overline{L}_f \in \widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z})_{\mathbb{Q}}$, with $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z})_{\mathbb{Q}}$ defined in the obvious way.*

5.5.1 Construction and f-invariance

Take a projective model $\pi\colon \mathcal{X} \to \mathcal{S}$ of $X \to S$, *i.e.* a projective model $\mathcal{S}$ of S over $\mathbb{Z}$ and a flat morphism $\pi\colon \mathcal{X} \to \mathcal{S}$ of projective schemes over $\mathbb{Z}$ such that the base change $\mathcal{X} \times_{\mathcal{S}} S \to S$ is isomorphic to $X \to S$ (so that we identify $\mathcal{X} \times_{\mathcal{S}} S = X$). Take a Hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|) \in \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})$ such that $\mathcal{L}|_X = L$.

For each positive integer i, we wish to extend the morphism $f^i\colon X \to X$ to integral models, where f^i is the i-fold iterate of f. First, the composite $X \xrightarrow{f^i} X \to \mathcal{X}$ gives a rational map $\mathcal{X} \dashrightarrow \mathcal{X}$. Next after taking normalization we obtain a morphism $f_i\colon \mathcal{X}_i \to \mathcal{X}$. Denote by $\pi_i\colon \mathcal{X}_i \to \mathcal{S}$ the induced map to $\mathcal{S}$. Now we have

$$\begin{tikzcd}
\cdots \arrow[r, "f"] & X \arrow[r, "f"] \arrow[d, hook] & \cdots \arrow[r, "f"] & X \arrow[r, "f"] \arrow[d, hook] & X \arrow[d, hook] \\
\cdots \arrow[r] & \mathcal{X}_i \arrow[r] \arrow[rrr, bend right, "f_i"'] & \cdots \arrow[r] & \mathcal{X}_1 \arrow[r, "f_1"] & \mathcal{X}
\end{tikzcd}$$

with the arrows in the top row being S-morphisms and the arrows in the bottom row being $\mathcal{S}$-morphisms.

Set $\overline{\mathcal{L}}_i := q^{-i} f_i^* \overline{\mathcal{L}} \in \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X}_i)_{\mathbb{Q}}$.

Now let us take an open subscheme $\mathcal{V}$ of $\mathcal{S}$ containing S, such that $\mathcal{U} := \mathcal{X}_{\mathcal{V}}$ is projective and flat over $\mathcal{V}$ and that $f\colon X \to X$ extends to a morphism $f_{\mathcal{V}}\colon \mathcal{U} \to \mathcal{U}$ with $f^*L \simeq qL$ extending to an isomorphism $f_{\mathcal{V}}^* \mathcal{L}_{\mathcal{V}} \xrightarrow{\sim} q\mathcal{L}_{\mathcal{V}}$ in $\mathcal{P}\mathrm{ic}(\mathcal{U})$. Now, we have $\mathcal{X}_{i,\mathcal{V}} = \mathcal{X}_{\mathcal{V}} = \mathcal{U}$ for all $i \geq 1$.

Start with the isomorphism in $\mathcal{P}\mathrm{ic}(\mathcal{U})_{\mathbb{Q}}$

$$\ell\colon \mathcal{L}_{\mathcal{V}} \to q^{-1} f_{\mathcal{V}}^* \mathcal{L}_{\mathcal{V}}.$$

Applying $q^{-1} f_{\mathcal{V}}^*$ to ℓ successively , we obtain canoncial isomorphisms

$$\mathcal{L}_{\mathcal{V}} \to q^{-1} f_{\mathcal{V}}^* \mathcal{L}_{\mathcal{V}} \to q^{-2} (f_{\mathcal{V}}^*)^2 \mathcal{L}_{\mathcal{V}} \to \cdots \to q^{-i} (f_{\mathcal{V}}^*)^i \mathcal{L}_{\mathcal{V}}$$

in $\mathcal{P}\mathrm{ic}(\mathcal{U})_{\mathbb{Q}}$. Notice that $\mathcal{L}_i|_{\mathcal{U}} = q^{-i}(f_{\mathcal{V}}^*)^i \mathcal{L}_{\mathcal{V}}$ by definition of $\overline{\mathcal{L}}_i$. Hence we obtain an isomorphism in $\mathcal{P}\mathrm{ic}(\mathcal{U})_{\mathbb{Q}}$

$$\ell_i\colon \mathcal{L}_{\mathcal{V}} \to \mathcal{L}_i|_{\mathcal{U}}$$

for each $i \geq 1$. Thus we obtain

$$(\mathcal{L}_{\mathcal{V}}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i \geq 1}) \tag{5.5.2}$$

with $\mathcal{L}_{\mathcal{V}} \in \mathcal{P}\mathrm{ic}(\mathcal{U})$, $(\mathcal{X}_i, \overline{\mathcal{L}}_i)$ a model adelic line bundle on $\mathcal{U}$ for each $i \geq 1$, and $\ell_i\colon \mathcal{L}_{\mathcal{V}} \simeq \mathcal{L}_i|_{\mathcal{U}}$ for each $i \geq 1$.

Let us show now that the sequence (5.5.2) converges in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$, *i.e.* $\{\widehat{\mathrm{div}}(\ell_i \ell_1^{-1})\}_{i \geq 1}$ is a Cauchy sequence under the boundary topology. The upshot is that it then gives an object $\overline{L}_f \in \widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z})$ which is f-invariant.

Up to blowing up $\mathcal{S}$ along $\mathcal{S} \setminus \mathcal{V}$, we may and do assume that there is a boundary divisor $(\mathcal{S}, \overline{D}_0)$ of $\mathcal{V}$. Then we get a boundary divisor $(\mathcal{X}, \pi^* \overline{D}_0)$ of $\mathcal{U}$.

View the isomorphism $\ell\colon \mathcal{L}_{\mathcal{V}} \to q^{-1} f_{\mathcal{V}}^* \mathcal{L}_{\mathcal{V}}$ as a rational map $\overline{\mathcal{L}} \to \overline{\mathcal{L}}_1$. This defines a model adelic divisor $\widehat{\mathrm{div}}(\ell)$ in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$ whose image in $\mathrm{Div}(\mathcal{U})$ is 0. Hence there exists $r > 0$ such that

$$-r\pi^* \overline{D}_0 \leq \widehat{\mathrm{div}}(\ell) \leq r\pi^* \overline{D}_0$$

in $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod}}$.

By construction, the isomorphism $\ell_{i+1}\ell_i^{-1}\colon \mathcal{L}_i|_{\mathcal{U}} \to \mathcal{L}_{i+1}|_{\mathcal{U}}$ is obtained from $\ell\colon \mathcal{L}_{\mathcal{V}} \to q^{-1}f_{\mathcal{V}}^*\mathcal{L}_{\mathcal{V}}$ by applying $(q^{-1}f_{\mathcal{V}}^*)^i$. Accordingly, *the rational map* $\ell_{i+1}\ell_i^{-1}\colon \overline{\mathcal{L}}_i \dashrightarrow \overline{\mathcal{L}}_{i+1}$ *is obtained from* $\ell\colon \overline{\mathcal{L}} \dashrightarrow \overline{\mathcal{L}}_1$ *by applying* $(q^{-1}f^*)^i$. Hence we have

$$-\frac{r}{q^i}\pi^*\overline{D}_0 \le \widehat{\mathrm{div}}(\ell_{i+1}\ell_i^{-1}) \le \frac{r}{q^i}\pi^*\overline{D}_0 \qquad (5.5.3)$$

in $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod}}$. Thus $\{\widehat{\mathrm{div}}(\ell_i\ell_1^{-1})\}_{i\ge 1}$ is a Cauchy sequence under the boundary topology. We are done for the existence.

Now let us give a vigorous proof of (5.5.3) via a precise explanation of the sentence in italic above. Write $\mathcal{X}_0 = \mathcal{X}$ and $\overline{\mathcal{L}}_0 = \overline{\mathcal{L}}$ for convenience. There exists a projective model $\mathcal{Y}_1$ of $\mathcal{U}$ over $\mathbb{Z}$ and morphisms $\tau_1\colon \mathcal{Y}_1 \to \mathcal{X}_1$ and $\tau_1'\colon \mathcal{Y}_1 \to \mathcal{X}_0$, extending the identity map $\mathcal{U} \to \mathcal{U}$, such that the rational map $\ell\colon \overline{\mathcal{L}}_0 \dashrightarrow \overline{\mathcal{L}}_1$ is given by a morphism

$$\ell'\colon \tau_1'^*\overline{\mathcal{L}}_0 \to \tau_1^*\overline{\mathcal{L}}_1$$

over $\mathcal{Y}_1$. Moreover for each $i \ge 1$, there exists a projective model $\mathcal{Y}_{i+1}$ of $\mathcal{U}$ over $\mathbb{Z}$, together with morphisms

$$\tau_{i+1}\colon \mathcal{Y}_{i+1} \to \mathcal{X}_{i+1},\ \tau_{i+1}'\colon \mathcal{Y}_{i+1} \to \mathcal{X}_i$$

extending the identity map $\mathcal{U} \to \mathcal{U}$, and a morphism

$$g_i\colon \mathcal{Y}_{i+1} \to \mathcal{Y}_1$$

extending the morphism $f_{\mathcal{V}}^i\colon \mathcal{U} \to \mathcal{U}$. Then the rational map $\ell_{i+1}\ell_i^{-1}\colon \overline{\mathcal{L}}_i \dashrightarrow \overline{\mathcal{L}}_{i+1}$ is realized as a morphism $(\ell_{i+1}\ell_i^{-1})'\colon \tau_{i+1}'^*\overline{\mathcal{L}}_i \to \tau_{i+1}^*\overline{\mathcal{L}}_{i+1}$ over $\mathcal{Y}_{i+1}$, by applying $q^{-1}g_i^*$ to ℓ' via g_i. As a consequence, we have the following equality in $\widehat{\mathrm{Div}}(\mathcal{Y}_{i+1})_{\mathbb{Q}}$

$$\widehat{\mathrm{div}}((\ell_{i+1}\ell_i^{-1})') = q^{-i}g_i^*\widehat{\mathrm{div}}(\ell').$$

Denote by $\pi_1'\colon \mathcal{Y}_1 \to \mathcal{S}$ and $\pi_{i+1}'\colon \mathcal{Y}_{i+1} \to \mathcal{S}$ the structural morphisms. Notice that $g_i^*\pi_1'^*\overline{D}_0 = \pi_{i+1}'^*\overline{D}_0$ is equal to $\pi^*\overline{D}_0$ in $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod}}$. Thus (5.5.3) holds true.

5.5.2 Nefness

We shall only focus on the case where S has an affine quasi-projective model $\mathcal{V}$ over $\mathbb{Z}$. In this case, we can assume that S is an open subscheme of $\mathcal{V}$, and then L is in fact ample on X (not just relatively ample). Hence we can start by choosing $(\mathcal{X}, \overline{\mathcal{L}})$ with $\overline{\mathcal{L}}$ nef. Then every $\overline{\mathcal{L}}_i$ is nef on $\mathcal{X}_i$ by pullback, and therefore $\overline{L}_f$ is strongly nef by definition.

5.5.3 Uniqueness

To prove the uniqueness, we start with the case where $L = \mathcal{O}_X$ is the trivial line bundle.

Use Theorem 5.4.1, and in particular the canonical injective map

$$\widehat{\mathrm{Pic}}(X/\mathbb{Z}) \to \widehat{\mathrm{Pic}}(X^{\mathrm{an}})_{\mathrm{eqv}}.$$

The image of $\overline{L}_f$, denoted by $\overline{L}_f^{\mathrm{an}}$, is then represented by an element $(0, g)$ of $\widehat{\mathrm{Div}}(X^{\mathrm{an}})$ because $L = \mathcal{O}_X$. In particular, the Green's function g is actually a continuous function on X^{an}. The condition $f^*\overline{L}_f \simeq q\overline{L}_f$ then implies the following equality in $\widehat{\mathrm{Div}}(X^{\mathrm{an}})$:

$$(0, f^*g - qg) = (\mathrm{div}(\alpha), -\log|\alpha|), \quad \text{for some } \alpha \in \mathbb{Q}(X)^*.$$

Now that $\mathrm{div}(\alpha) = 0$, we have that $\alpha \in H^0(X, \mathcal{O}_X^*)$. Restricted to each fiber of $X^{\mathrm{an}} \to S^{\mathrm{an}}$, the difference

$$f^*g - qg = -\log|\alpha|$$

is then constant.

Let $v \in S^{\mathrm{an}}$ with residue field H_v. The fiber X_v^{an} is the Berkovich space $(X_{H_v}/H_v)^{\mathrm{an}}$. Denote by $g_{\max}$ and $g_{\min}$ the global maximal value and global minimal value of the continuous function g on the compact space X_v^{an}. As $f\colon X_v^{\mathrm{an}} \to X_v^{\mathrm{an}}$ is surjective, the relation $f^*g = qg + c_v$ (for a number c_v) implies $g_{\max} = qg_{\max} + c_v$, whereas $g_{\max} = -c_v/(q-1)$. Similarly $g_{\min} = -c_v/(q-1)$. Hence g is constant on X_v^{an}. So $f^*g = g$ on X^{an}.

Therefore we have $(1-q)\cdot(0,g) = (\mathrm{div}(\alpha), -\log|\alpha|)$. Hence $(1-q)\overline{L}_f$ is trivial in $\widehat{\mathrm{Pic}}(X/\mathbb{Z})$. So $\overline{L}_f$ is 0 in $\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathbb{Q}}$.

Now we turn to an arbitrary L. If $\overline{L}_f$ and $\overline{L}'_f$ are two f-invariant extensions, then $\overline{L}_f - \overline{L}'_f$ is an f-invariant of $\mathcal{O}_X$. Hence the discussion above implies that $\overline{L}_f - \overline{L}'_f$ is 0 in $\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathbb{Q}}$. So $\overline{L}_f = \overline{L}'_f$ in $\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathbb{Q}}$. This establishes the uniqueness.

Chapter 6

Height theory via adelic line bundles

In the whole chapter, we take K to be a number field, and X to be a quasi-projective variety defined over K. Let $n = \dim X$.

6.1 Height via adelic line bundles

In §2.3.2, we defined height functions on projective varieties via Hermitian line bundles, using the arithmetic degree of Hermitian line bundles over $\mathrm{Spec}\mathcal{O}_K$. This degree map was generalized to the intersection pairing (Definition 2.4.8).

In this section, we explain how the definitions extend when we use adelic line bundles on X.

6.1.1 Adelic line bundles on $\mathrm{Spec}K$ and arithmetic degree

Let us start by computing $\widehat{\mathrm{Pic}}(\mathrm{Spec}K/\mathbb{Z})$. It is easier to do the computation with adelic divisors.

Denote by $\mathcal{X} = \mathrm{Spec}\mathcal{O}_K$. For any open subscheme $\mathcal{U}$ of $\mathcal{X}$, we have

$$\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}} = \widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U}), \quad \widehat{\mathrm{Prin}}(\mathcal{U})_{\mathrm{mod}} = \widehat{\mathrm{Prin}}(\mathcal{X})$$

since $\mathcal{X}$ is the only normal projective model of $\mathcal{U}$.

Set $\mathcal{E} := \mathcal{X} \setminus \mathcal{U}$ endowed with the reduced scheme structure. Then we have

$$\begin{aligned}\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U}) &= \{\sum_{v\in|\mathcal{U}|} n_v[v] + \sum_{v'\in|\mathcal{E}|} n_{v'}[v'] + \sum_{\sigma\colon K\hookrightarrow\mathbb{C}} n_\sigma[\sigma] : n_v \in \mathbb{Z},\ n_{v'} \in \mathbb{Q},\ n_\sigma = n_{\overline{\sigma}} \in \mathbb{R} \text{ for all } \sigma\} \\ &\simeq (\bigoplus_{v\in|\mathcal{U}|} \mathbb{Z}) \oplus (\bigoplus_{v'\in|\mathcal{E}|} \mathbb{Q}) \oplus (\bigoplus_{\sigma\in M_{K,\infty}} \mathbb{R}).\end{aligned}$$

Taking the boundary divisor $\overline{\mathcal{E}} := (\mathcal{E}, 1) = \sum_{v\in|\mathcal{E}|\cup M_{K,\infty}}[v]$, we can compute the completion and get

$$\begin{aligned}\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z}) &= \{\sum_{v\in|\mathcal{U}|} n_v[v] + \sum_{v'\in|\mathcal{E}|} n_{v'}[v'] + \sum_{\sigma\colon K\hookrightarrow\mathbb{C}} n_\sigma[\sigma] : n_v \in \mathbb{Z},\ n_{v'} \in \mathbb{R},\ n_\sigma = n_{\overline{\sigma}} \in \mathbb{R} \text{ for all } \sigma\} \\ &\simeq (\bigoplus_{v\in|\mathcal{U}|} \mathbb{Z}) \oplus (\bigoplus_{v\in|\mathcal{E}|\cup M_{K,\infty}} \mathbb{R}).\end{aligned} \tag{6.1.1}$$

Hence we have

$$\widehat{\mathrm{Div}}(\mathrm{Spec}K/\mathbb{Z}) = \varinjlim_{\mathcal{U}}(\bigoplus_{v\in|\mathcal{U}|} \mathbb{Z}) \oplus (\bigoplus_{v\in|\mathcal{E}|\cup M_{K,\infty}} \mathbb{R}), \tag{6.1.2}$$

and an arithmetic degree map

$$\widehat{\deg}\colon \widehat{\mathrm{Div}}(\mathrm{Spec}K/\mathbb{Z}) \to \mathbb{R} \tag{6.1.3}$$

induced by the group homomorphism $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z}) \to \mathbb{R}$, $\sum_{v\in|\mathcal{U}|} n_v[v]+\sum_{v'\in|\mathcal{E}|} n_{v'}[v']+\sum_{\sigma\colon K\hookrightarrow\mathbb{C}} n_\sigma[\sigma] \mapsto \sum_v n_v + \sum_{v'} n_{v'} + \sum_\sigma n_\sigma$.

It is clear that (6.1.3) factors through $\widehat{\mathrm{Cl}}(\mathrm{Spec}K/\mathbb{Z})$, and hence we have the arithmetic degree map via Proposition 5.2.10

$$\widehat{\deg}\colon \widehat{\mathrm{Pic}}(\mathrm{Spec}K/\mathbb{Z}) \to \mathbb{R}. \tag{6.1.4}$$

This arithmetic degree is compatible with the arithmetic degree of Hermitian line bundles on $\mathrm{Spec}\mathcal{O}_K$ (1.1.1) in the following sense. By definition of $\widehat{\mathrm{Div}}(\mathrm{Spec}\mathcal{O}_K)$ and (6.1.2), we have a natural group homomorphism

$$\widehat{\mathrm{Div}}(\mathrm{Spec}\mathcal{O}_K) = (\bigoplus_{v\in|\mathcal{X}|}\mathbb{Z}) \oplus (\bigoplus_{v\in M_{K,\infty}}\mathbb{R}) \longrightarrow \widehat{\mathrm{Div}}(\mathrm{Spec}K/\mathbb{Z}),$$

which induces

$$\iota\colon \widehat{\mathrm{Pic}}(\mathrm{Spec}\mathcal{O}_K) \longrightarrow \widehat{\mathrm{Pic}}(\mathrm{Spec}K/\mathbb{Z}).$$

Then $\widehat{\deg}\circ\iota$ is precisely the arithmetic degree map defined by (1.1.1).

We close this subsection by the following formula for $\widehat{\deg}$. Let $\overline{L} \in \widehat{\mathrm{Pic}}(\mathrm{Spec}K/\mathbb{Z})$. Write $(L, \|\cdot\|_{\mathbb{A}})$ for the image of $\overline{L}$ under the canonical map $\widehat{\mathrm{Pic}}(\mathrm{Spec}K/\mathbb{Z}) \simeq \widehat{\mathrm{Pic}}((\mathrm{Spec}K/\mathbb{Z})^{\mathrm{an}})_{\mathrm{eqv}}$ from (5.4.1). Then $\|\cdot\|_{\mathbb{A}}$ is uniquely determined by the collection of K_v-metrics $\{\|\cdot\|_v$ on $L\otimes_K K_v\}_{v\in M_K}$ by norm-equivariance. Moreover, for any $\ell \in L\setminus\{0\}$, we have $\|\ell\|_v = 1$ for all but finitely many $v \in M_K$. The following lemma is not hard to check and we leave it as an exercise.

Lemma 6.1.1. *Under the notation above, we have*

$$\widehat{\deg}(\overline{L}) = -\sum_{v\in M_K} \log\|\ell\|_v^{\epsilon_v} \qquad \text{for any } \ell\in L\setminus\{0\},$$

where $\epsilon_v = 2$ if v is a complex place and $\epsilon_v = 1$ otherwise. The RHS is well-defined by the Product Formula.

In this terminology, ι sends $(\mathcal{L}, \|\cdot\|)$ to $(\mathcal{L}_K, \|\cdot\|_{\mathbb{A}})$, with $\|\ell\|_v := \inf\{|a| : a \in \mathbb{Q},\ \ell \in a\mathcal{L}\otimes_{\mathcal{O}_K}\mathcal{O}_{K_v}\}$.

6.1.2 Height function defined by adelic line bundles

Let $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$.

Definition 6.1.2. *The* **height function** *defined by $\overline{L}$ is*

$$h_{\overline{L}}\colon X(\overline{\mathbb{Q}}) \to \mathbb{R}, \qquad x \mapsto \frac{\widehat{\deg}(\overline{L}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)x})}{[K(x):K]}.$$

Here, $\overline{L}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)}$ is the image of $\overline{L}$ under $\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{int}} \to \widehat{\mathrm{Pic}}(\mathrm{Gal}(\overline{\mathbb{Q}}/K)x/\mathbb{Z})_{\mathrm{int}}$.

Example 6.1.3. *Assume X is irreducible projective. Assume $L \in \mathcal{P}\mathrm{ic}(X)$ ample such that $f^*L \simeq qL$ for some $f\colon X \to X$ and $q \in \mathbb{Z}_{>1}$. Then by Theorem 5.5.1, there exists $\overline{L}_f \in \widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z})_{\mathrm{nef}}$ extending L such that $f^*\overline{L}_f \simeq q\overline{L}_f$. Then $h_{\overline{L}_f}$ is in the class of the height function* *of $h_{X,L}$ and satisfies $h_{\overline{L}_f}(f(x)) = qh_{\overline{L}_f}(x)$ for all $x \in X(\overline{\mathbb{Q}})$. Hence $h_{\overline{L}_f}$ equals the normalized height function $\widehat{h}_{X,f,L}$ from Theorem 0.3.1.*

This in particular applies to any abelian variety A and any symmetric ample line bundle L on A, both defined over $\overline{\mathbb{Q}}$. So the Néron–Tate height on A is a height function defined by an adelic line bundle $\overline{L}$ on A.

Better, if we have an abelian scheme $\mathcal{A} \to S$ with S an irreducible quasi-projective variety, and $\mathcal{L}$ a relatively ample symmetric line bundle on $\mathcal{A}$; all defined over $\overline{\mathbb{Q}}$. Then Theorem 5.5.1 gives an $\overline{\mathcal{L}} \in \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{A}/\mathbb{Z})_{\mathrm{nef}}$ such that $h_{\overline{\mathcal{L}}}$ is precisely the fiberwise Néron–Tate height defined by $\mathcal{L}$.

As an application of Lemma 6.1.1, we have the following:

Lemma 6.1.4. *Denote by $(L, \|\cdot\|) \in \widehat{\mathrm{Pic}}(X^{\mathrm{an}})_{\mathrm{eqv}}$ the image of $\overline{L}$ under the canonical map $\widehat{\mathrm{Pic}}(X/\mathbb{Z}) \xrightarrow{\sim} \widehat{\mathrm{Pic}}(X^{\mathrm{an}})_{\mathrm{eqv}}$ from* (5.4.1)*. Then for any $x \in X(\overline{\mathbb{Q}})$, we have*

$$h_{\overline{L}}(x) = -\frac{1}{[K(x):K]} \sum_{v \in M_K} \sum_{z \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)x \times_K K_v} \log \|s(z)\|_v^{\deg_{K_v} z} \tag{6.1.5}$$

for any non-zero rational section s of L on X with $x \notin |\mathrm{div}(s)|$.

6.1.3 Top intersection number of adelic line bundles

For any projective arithmetic variety $\mathcal{X}$ of dimension $n+1$, Definition 2.4.8 defines an intersection pairing

$$\widehat{\mathrm{Pic}}(\mathcal{X})_{\mathrm{int}}^{n+1} \to \mathbb{R}. \tag{6.1.6}$$

Here, the index int refers to the integrability condition on the Hermitian metrics we consider; see below Definition 2.3.2.

Let us extend this intersection pairing to adelic line bundles over quasi-projective arithmetic varieties.

Proposition 6.1.5. *Let $\mathcal{U}$ be a quasi-projective arithmetic variety of dimension $n+1$. Then there exists a canonical multi-linear homomorphism, which is symmetric in the $n+1$ variables,*

$$\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{int}}^{n+1} \to \mathbb{R} \tag{6.1.7}$$

extending the intersection pairing (6.1.6) *above. Moreover, if $\overline{\mathcal{L}}_1, \dots, \overline{\mathcal{L}}_{n+1}$ are nef adelic line bundles on $\mathcal{U}$, then the intersection number $\overline{\mathcal{L}}_1 \cdot \overline{\mathcal{L}}_2 \cdot \ldots \cdot \overline{\mathcal{L}}_{n+1} \geq 0$.*

Before moving on to the proof, let us explain how (6.1.7) induces an intersection pairing on $\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{int}}$. Indeed, $\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{int}} = \varinjlim_{\mathcal{U}} \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{int}}$, and hence (6.1.7) gives rise to a canonical multi-linear homomorphism (still called the *intersection pairing*)

$$\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{int}}^{n+1} \to \mathbb{R} \tag{6.1.8}$$

such that $\overline{L}_1 \cdot \ldots \cdot \overline{L}_{n+1} \geq 0$ if all $\overline{L}_i$'s are nef. When $n = 0$, this map is exactly $\widehat{\deg}$.

Similarly we have a canonical multi-linear homomorphism

$$\widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathrm{int}}^{n} \to \mathbb{R}, \tag{6.1.9}$$

such that $\widetilde{L}_1 \cdot \ldots \cdot \widetilde{L}_n \geq 0$ if all $\widetilde{L}_i$'s are nef.

Proof of Proposition 6.1.5. By linearity, it suffices to define (6.1.7) for strongly nef adelic line bundles. The proof is easier to write down in terms of adelic divisors. So we take $\overline{D}_1, \ldots, \overline{D}_{n+1} \in \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$ with each $\mathcal{O}(\overline{D}_j)$ being a strongly nef adelic line bundle. We will define $\overline{D}_1 \cdot \ldots \cdot \overline{D}_{n+1}$.

Fix a boundary divisor $(\mathcal{X}_0, \overline{D}_0)$ of $\mathcal{U}$, which define the boundary topology of $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$. We may furthermore assume $\mathcal{O}(\overline{D}_0)$ to be a nef Hermitian line bundle.

For $j \in \{1, \ldots, n+1\}$, the adelic divisor $\overline{D}_j$ is represented by a Cauchy sequence $\{(\mathcal{X}_i, \overline{D}_{j,i})\}_{i\geq 1}$, where each $\mathcal{O}(\overline{D}_{j,i})$ is a nef Hermitian line bundle on the projective model $\mathcal{X}_i$ dominating $\mathcal{X}_0$. Here, we assume that the model $\mathcal{X}_i$ is independent of j which is always possible. There exists a sequence $\{\epsilon_i \in \mathbb{Q}_{>0}\}_{i\geq 1}$ with $\epsilon_i \to 0$ such that

$$-\epsilon_i \overline{D}_0 \leq \overline{D}_{j,i'} - \overline{D}_{j,i} \leq \epsilon_i \overline{D}_0, \quad \forall i' > i$$

for any $j \in \{1, \ldots, n+1\}$.

For any subset $J \subseteq \{1, \ldots, n+1\}$, consider the intersection number

$$\alpha_{J,i} := \overline{D}_0^{d-|J|} \prod\nolimits_{j\in J} \overline{D}_{j,i}.$$

We shall prove, by induction on $|J|$, that $\{\alpha_{J,i}\}_{i\geq 1}$ is a Cauchy sequence, and hence converges in $\mathbb{R}$. Then the limit of the Cauchy sequence gives our desired definition of $\overline{D}_1 \cdot \ldots \cdot \overline{D}_{n+1}$ when $J = \{1, \ldots, n+1\}$.

The base step $|J| = 0$ is trivial since there is nothing to prove in this case. Assume the claim holds true for any $|J| < r$ for some $r > 0$. We need to prove the result for $|J| = r$. Without loss of generality assume that $J = \{1, 2, \ldots, r\}$. Then

$$\begin{aligned}
\alpha_{J,i'} - \alpha_{J,i} &= \overline{D}_0^{d-r} \overline{D}_{1,i'} \cdots \overline{D}_{r,i'} - \overline{D}_0^{d-r} \overline{D}_{1,i} \cdots \overline{D}_{r,i} \\
&\leq \overline{D}_0^{d-r} (\overline{D}_{1,i} + \epsilon_i \overline{D}_0) \cdots (\overline{D}_{r,i} + \epsilon_i \overline{D}_0) - \overline{D}_0^{d-r} \overline{D}_{1,i} \cdots \overline{D}_{r,i} \\
&= \sum_{J' \subsetneq J} \epsilon_i^{r-|J'|} \alpha_{J',i}
\end{aligned}$$

and similarly

$$\alpha_{J,i} - \alpha_{J,i'} \leq \sum_{J' \subsetneq J} \epsilon_i^{r-|J'|} \alpha_{J',i'}.$$

So

$$|\alpha_{J,i'} - \alpha_{J,i}| \leq \sum_{J' \subsetneq J} \epsilon_i^{r-|J'|} |\alpha_{J',i'} - \alpha_{J',i}|.$$

This shows that $\{\alpha_{J,i}\}_i$ is a Cauchy sequence by induction hypothesis. Hence we are done for the definition of (6.1.7).

The intersection pairing (6.1.7) is symmetric in the $n+1$ variables because (6.1.6) is. Moreover, $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{snef}}^{n+1}$ is mapped to $\mathbb{R}_{\geq 0}$ since (6.1.6) maps $\widehat{\mathrm{Pic}}(\mathcal{X})_{\mathrm{nef}}^{n+1}$ to $\mathbb{R}_{\geq 0}$. Now if $\overline{\mathcal{L}}_1, \ldots, \overline{\mathcal{L}}_{n+1}$ are nef adelic line bundles on $\mathcal{U}$, then there exist strongly nef adelic line bundle $\overline{\mathcal{M}}_j$ ($j \in \{1, \ldots, n+1\}$) such that $a\overline{\mathcal{L}}_j + \overline{\mathcal{M}}_j$ is strongly nef for all $a \in \mathbb{Z}_{>0}$ for all j. Hence

$$(a\overline{\mathcal{L}}_1 + \overline{\mathcal{M}}_1) \cdots (a\overline{\mathcal{L}}_{n+1} + \overline{\mathcal{M}}_{n+1}) \geq 0$$

for all $a \in \mathbb{Z}_{>0}$. And hence the leading coefficient $\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n+1}$ is non-negative. □

We also have the following projection formula, by taking limits of Proposition 2.4.10.

Proposition 6.1.6. *Let $f\colon X' \to X$ be a morphism of varieties defined over K. Assume $\dim X' = \dim X = n$. Then for any $\overline{L}_1, \ldots, \overline{L}_{n+1} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$, we have*

$$f^*\overline{L}_1 \cdot \ldots \cdot f^*\overline{L}_{n+1} = \deg(f) \overline{L}_1 \cdot \ldots \cdot \overline{L}_{n+1}$$

with $\deg(f) = [K(X') : K(X)]$ *if f is dominant and* $\deg(f) = 0$ *otherwise.*

We close this section by defining the height of an arbitrary dimensional closed subvariety of X.

Definition 6.1.7. *Assume $\overline{L}$ is integrable. Let Z be a closed subvariety of X defined over $\overline{\mathbb{Q}}$. Define the* **height of** Z **for** $\overline{L}$ *to be*

$$h_{\overline{L}}(Z) := \frac{(\overline{L}|_{Z'})^{\dim Z+1}}{(\dim Z+1)(\widetilde{L}|_{Z'})^{\dim Z}}. \tag{6.1.10}$$

Here $Z' = \mathrm{Gal}(\overline{\mathbb{Q}}/K)Z$, and $\overline{L} \mapsto \overline{L}|_{Z'} \mapsto \widetilde{L}_{Z'}$ is the image of $\overline{L}$ under $\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{int}} \to \widehat{\mathrm{Pic}}(Z'/\mathbb{Z})_{\mathrm{int}} \to \widehat{\mathrm{Pic}}(Z'/\mathbb{Q})_{\mathrm{int}}$.

On the RHS of (6.1.10), the numerator is the arithmetic intersection pairing (6.1.8), and the second term of the denominator is the geometric intersection pairing (6.1.9). Of course this height is well-defined only if $(\widetilde{L}|_{Z'})^{\dim Z} \neq 0$.

6.2 Volume and bigness of adelic line bundles

We explained in §5.1.4 the volume and bigness of geometric adelic line bundles on X. As for the geometric-arithmetic analogue in the classical situation, we can generalize the discussion to adelic line bundles on X.

6.2.1 Effective/small sections

Let $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$. Denote by

$$\widehat{\mathrm{Pic}}(X/\mathbb{Z}) \to \widehat{\mathrm{Pic}}(X/\mathbb{Q}) \to \mathrm{Pic}(X), \quad \overline{L} \mapsto \widetilde{L} \mapsto L.$$

Define

$$H^0(X,\overline{L}) := \{s \in H^0(X,L) : \widehat{\mathrm{div}}(s) \geq 0\}, \qquad h^0(X,\overline{L}) := \log \# H^0(X,\overline{L}), \tag{6.2.1}$$

and recall

$$H^0(X,\widetilde{L}) = \{s \in H^0(X,L) : \widehat{\mathrm{div}}(s) \geq 0\}, \qquad h^0(X,\widetilde{L}) = \dim H^0(X,\widetilde{L}).$$

In the definition of $H^0(X,\overline{L})$, $\widehat{\mathrm{div}}(s)$ means the *(arithmetic)* adelic divisor (5.2.5), while in the definition of $H^0(X,\widetilde{L})$ it means the geometric adelic divisor (5.1.5).

We state the following lemma without giving the proof. The proof is not too complicated: one first proves the finiteness result for the model case and then passes to Cauchy sequences.

Lemma 6.2.1. *Both $h^0(X,\overline{L})$ and $h^0(X,\widetilde{L})$ are finite numbers.*

Next, recall the diagram (5.4.1). Let $(L, \|\cdot\|) \in \widehat{\mathrm{Pic}}(X^{\mathrm{an}})_{\mathrm{eqv}}$ be the metrized line bundle as the image of $\widehat{\mathrm{Pic}}(X/\mathbb{Z})$. Then for any $s \in H^0(X,L)$ and any $v \in (\mathrm{Spec}\mathbb{Z})^{\mathrm{an}}$, define

$$\|s\|_{\sup} := \sup_{x \in X^{\mathrm{an}}} \|s(x)\|, \qquad \|s\|_{v,\sup} := \sup_{x \in X_v^{\mathrm{an}}} \|s(x)\|. \tag{6.2.2}$$

Then we have by construction of (5.4.1) and Lemma 5.4.5, we have

$$H^0(X,\overline{L}) = \{s \in H^0(X,L) : \|s\|_{\sup} \leq 1\}, \qquad H^0(X,\widetilde{L}) = \{s \in H^0(X,L) : \|s\|_{v_0,\sup} \leq 1\}. \tag{6.2.3}$$

By Lemma 6.1.4, $h_{\overline{L}}$ is non-negative outside $|\mathrm{div}(s)|$ if we can find a non-zero small section $s \in H^0(X,\overline{L})$.

6.2.2 Volume and bigness

Definition-Theorem 6.2.2. *The following limit exists and is defined to be the* **volume** *of* $\overline{L}$*:*

$$\mathrm{vol}(X,\overline{L}) := \lim_{m\to\infty} \frac{(n+1)!}{m^{n+1}} h^0(X, m\overline{L}). \tag{6.2.4}$$

Moreover, assume that $\overline{L}$ *is represented by* $(\mathcal{L}, \{\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i\}_{i\geq 1})$ *on* $\mathcal{U}$ *for a quasi-projective model* $\mathcal{U}$ *of* X*, then*

$$\mathrm{vol}(X,\overline{L}) = \lim_{i\to\infty} \mathrm{vol}(\mathcal{X}_i, \overline{\mathcal{L}}_i).$$

Definition 6.2.3. *An adelic line bundle* $\overline{L} \in \widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z})$ *is said to be* **big** *if* $\mathrm{vol}(X,\overline{L}) > 0$.

Theorem 6.2.4 (Arithmetic Hilbert–Samuel)**.** *Assume* $\overline{L}$ *is nef. Then* $\mathrm{vol}(X,\overline{L}) = \overline{L}^{n+1}$.

Theorem 6.2.5 (Arithmetic Siu)**.** *If* $\overline{L}$ *and* $\overline{M}$ *are nef adelic line bundles on* X*, then*

$$\mathrm{vol}(X, \overline{L} - \overline{M}) \geq \overline{L}^{n+1} - (n+1)\overline{L}^n\overline{M}.$$

All the definitions and results extend to $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathbb{Q}}$, *i.e.* $\mathbb{Q}$-adelic line bundles on X.

Theorem 6.2.6 (continuity)**.** *Let* $\overline{L}, \overline{M}_1, \ldots, \overline{M}_r \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$*. Then*

$$\lim_{t_1,\ldots,t_r\to 0} \mathrm{vol}(\overline{L} + t_1\overline{M}_1 + \cdots + t_r\overline{M}_r) = \mathrm{vol}(\overline{L}),$$

with $t_1, \ldots, t_r$ *rational numbers.*

The following lemma states that the bigness of the generic fiber $\widetilde{L}$ of $\overline{L}$ is not far from the bigness of $\overline{L}$. In view of height theory, this is reasonable: having a non-zero small section yields the non-negativity of $h_{\overline{L}}$, whereas having a non-zero section yields a lower bound of $h_{\overline{L}}$ (both outside the support of the divisor of the section).

Write $f\colon X \to \mathrm{Spec}K$ for the structural morphism.

Lemma 6.2.7. *Let* $\overline{N} \in \widehat{\mathrm{Pic}}(K/\mathbb{Z})$ *be an adelic line bundle with* $\widehat{\deg}(\overline{N}) > 0$*. Let* $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$*.*

Assume that the generic fiber $\widetilde{L}$ *of* $\overline{L}$ *is big (see Definition 5.1.13). Then the adelic line bundle* $\overline{L} + cf^*\overline{N} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$ *is big for all rational numbers* $c \gg 1$.

6.2.3 The height inequality

In this subsection, we prove the following height inequality which plays a significant role in the solution of many problems recently.

Theorem 6.2.8. *Let* $\pi\colon X \to S$ *be a morphism of quasi-projective varieties defined over a number field* K*. Let* $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$ *and* $\overline{M} \in \widehat{\mathrm{Pic}}(S/\mathbb{Z})$*. Denote by* $\widetilde{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Q})$ *the generic fiber of* $\overline{L}$*.*

(i) If $\overline{L}$ *is big, then there exists* $\epsilon > 0$ *and a non-empty Zariski open subset* U *of* X *such that*

$$h_{\overline{L}}(x) \geq \epsilon h_{\overline{M}}(\pi(x)), \quad \forall x \in U(\overline{\mathbb{Q}}).$$

(ii) If $\widetilde{L}$ *is big, then there exist* $c > 0$ *and* $\epsilon > 0$ *and a non-empty Zariski open subset* U *of* X *such that*

$$h_{\overline{L}}(x) \geq \epsilon h_{\overline{M}}(\pi(x)) - c, \quad \forall x \in U(\overline{\mathbb{Q}}).$$

Proof. Let us prove (i). Assume $\overline{L}$ is big. Then $\mathrm{vol}(X, \overline{L}) > 0$.

We claim that there exists $\epsilon \in \mathbb{Q}_{>0}$ such that $\mathrm{vol}(X, \overline{L} - \epsilon\pi^*\overline{M}) > 0$. If $\overline{L}$ and $\overline{M}$ are nef, this follows from Arithmetic Siu (Theorem 6.2.5) and Arithmetic Hilbert–Samuel (Theorem 6.2.4). In general, we use the continuity (Theorem 6.2.6) to get that

$$\lim_{\epsilon \to 0} \mathrm{vol}(X, \overline{L} - \epsilon\pi^*\overline{M}) = \mathrm{vol}(X, \overline{L}) > 0.$$

Hence such an ϵ exists.

Therefore there exists $m \in \mathbb{Z}_{>0}$ and a non-zero $s \in H^0(X, m(\overline{L} - \epsilon\pi^*\overline{M}))$. Hence by (6.2.3), we have

$$h_{\overline{L} - \epsilon\pi^*\overline{M}}(x) \geq 0, \quad \forall x \in (X \setminus |\mathrm{div}(s)|)(\overline{\mathbb{Q}}).$$

Hence (i) holds true because $h_{\overline{L} - \epsilon\pi^*\overline{M}}(x) = h_{\overline{L}}(x) - \epsilon h_{\overline{M}}(\pi(x))$.

Now we prove (ii). Take $\overline{N} \in \widehat{\mathrm{Pic}}(\mathrm{Spec}K/\mathbb{Z})$ with $\widehat{\deg}(\overline{N}) = 1$. For the structural morphism $f \colon X \to \mathrm{Spec}K$, denote by $\overline{L}' = \overline{L} + cf^*\overline{N}$ for a rational number $c > 0$. By Lemma 6.2.7, $\overline{L}'$ is big for $c \gg 1$. Hence we can apply part (i) to $(\overline{L}', \overline{M})$ and conclude. □

6.2.4 A formula to compute the self-intersection of geometric adelic line bundles

Let $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$, and write $(L, \|\cdot\|)$ for its image under the injective homomorphism $\widehat{\mathrm{Pic}}(X/\mathbb{Z}) \to \widehat{\mathrm{Pic}}(X^{\mathrm{an}})_{\mathrm{eqv}}$. Each place v of K can be seen as a point in $(\mathrm{Spec}K/\mathbb{Z})^{\mathrm{an}}$, which is over the point $v_p \in (\mathrm{Spec}\mathbb{Q}/\mathbb{Z})^{\mathrm{an}}$ with $p \leq \infty$. Now $\|\cdot\|_v$ is a metric of $L|_{X_v^{\mathrm{an}}}$, and hence defines a curvature current $c_1(\overline{L})_v$; at archimedean places this is $-\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\|\cdot\|_v$, and we omit the discussion at non-archimedean places.

Lemma 6.2.9. *Let $\overline{L}_1, \ldots, \overline{L}_n \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{int}}$, and let $\widetilde{L}_1, \ldots, \widetilde{L}_n \in \widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathrm{int}}$ be their generic fibers. Then for any place v of K, we have*

$$\widetilde{L}_1 \cdot \ldots \cdot \widetilde{L}_n = \int_{X_v^{\mathrm{an}}} c_1(\overline{L}_1)_v \cdots c_1(\overline{L}_n)_v.$$

In practice, take $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{int}}$ with generic fiber $\widetilde{L}$, and let $\sigma \colon K \hookrightarrow \mathbb{C}$. Then $\widetilde{L}^n = \int_{X_\sigma(\mathbb{C})} c_1(\overline{L}_\sigma)^n$ by Lemma 6.2.9. If $\widetilde{L}$ is known to be nef, then we can use Hilbert–Samuel to get

$$\mathrm{vol}(X, \widetilde{L}) = \int_{X_\sigma(\mathbb{C})} c_1(\overline{L}_\sigma)^n.$$

6.3 A brief discussion on equidistribution

6.3.1 Essential minimum and fundamental inequality

Definition 6.3.1. *Let $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$. Define the* **essential minimum** *to be*

$$e_1(X, \overline{L}) := \sup_{U \subseteq X} \inf_{x \in U(\overline{\mathbb{Q}})} h_{\overline{L}}(x),$$

where U runs over all Zariski open subsets of X.

The *fundamental inequality* is:

Theorem 6.3.2. *Let $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$ be nef such that its generic fiber $\widetilde{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Q})$ is big. Then*

$$e_1(X, \overline{L}) \geq h_{\overline{L}}(X) \geq \frac{1}{n+1} e_1(X, \overline{L}).$$

The second inequality is a weak version of Zhang's *successive minima theorem*. The first inequality is a consequence of the arithmetic Hilbert–Samuel formula with the following lemma, which is an application of the Geometry of Numbers in §1.3.1 and the formula for $h_{\overline{L}}$ from Lemma 6.1.4.

Lemma 6.3.3. *For any positive integer $m > 0$ such that $h^0(X, m\overline{L}) > 0$, we have*

$$e_1(X, \overline{L}) \geq \frac{h^0(X, m\overline{L})}{m h^0(X, m\widetilde{L})} - \frac{2}{m}[K : \mathbb{Q}]$$

if the RHS is > 0.

6.3.2 Equidistribution

Let $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{nef}}$. Assume that $\deg_{\widetilde{L}}(X) = \widetilde{L}^n > 0$. Then the height

$$h_{\overline{L}}(X) = \frac{\overline{L}^{n+1}}{(n+1)\deg_{\widetilde{L}}(X)}$$

from (6.1.10) is well-defined. Define the *equibrilium measure*

$$\mathrm{d}\mu_{\overline{L},v} := \frac{1}{\deg_{\widetilde{L}}(X)} c_1(\overline{L})_v^n. \tag{6.3.1}$$

A sequence $\{x_m\}_{m \geq 1}$ in $X(\overline{\mathbb{Q}})$ is said to be *generic* if any proper closed subvariety of X contains only finitely many terms in the sequence. The sequence is said to be *small* if $h_{\overline{L}}(x_m) \to h_{\overline{L}}(X)$ when $m \to \infty$.

Let $x \in X(\overline{\mathbb{Q}})$. Define $O(x) := \mathrm{Gal}(\overline{\mathbb{Q}}/K)x \subseteq X(\overline{\mathbb{Q}})$, and set

$$\mu_{x,v} := \frac{1}{\#O(x)} \delta_{O(x) \times_K K_v} \tag{6.3.2}$$

where the RHS is the dirac measure.

Theorem 6.3.4. *Let $\{x_m\}_{m\geq 1}$ be a generic small sequence. Then the Galois orbit of $\{x_m\}_{m\geq 1}$ is equidistributed in X_v^{an} for $\mathrm{d}\mu_{\overline{L},v}$ for any place v of K. This means: the weak convergence $\mu_{x_m,v} \to \mathrm{d}\mu_{\overline{L},v}$ holds on X_v^{an}, i.e. for any compactly supported continuous function f on X_v^{an}, we have*

$$\frac{1}{\#O(x_m)} \sum_{y \in O(x_m) \times_K K_v} f(y) \longrightarrow \int_{X_v^{\mathrm{an}}} f \mathrm{d}\mu_{\overline{L},v}. \tag{6.3.3}$$

Proof. The key approach is the *variational principle* of Szpiro–Ullmo–Zhang. The conditions and the result do not change if we replace $\overline{L}$ by $\overline{L} + f^*\overline{N}$ for some $\overline{N} \in \widehat{\mathrm{Pic}}(\mathrm{Spec}K/\mathbb{Z})_{\mathrm{int}}$ with $\widehat{\deg}(\overline{N}) > 0$, where $f \colon X \to \mathrm{Spec}K$ is the structural morphism. So we may assume that $\overline{L}^{n+1} > 0$. Then $\overline{L}$ is big, and hence $\widetilde{L}$ is big.

Take $\overline{M} \in \mathrm{Ker}(\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{int}} \to \widehat{\mathrm{Pic}}(X/\mathbb{Q}))$. Let $\epsilon \in \mathbb{Q}^*$. By the first part of Theorem 6.3.2, we have

$$e_1(X, \overline{L} + \epsilon\overline{M}) \geq \frac{(\overline{L} + \epsilon\overline{M})^{n+1}}{(n+1)\deg_{\widetilde{L}}(X)}$$

if $\overline{L}+\epsilon\overline{M}$ is nef. We have

$$(\overline{L}+\epsilon\overline{M})^{n+1}=\overline{L}^{n+1}+\epsilon(n+1)\overline{L}^{n}\overline{M}+O(\epsilon^2).$$

Hence the RHS is >0 if $|\epsilon|\ll 1$ because $\overline{L}^{n+1}>0$. So

$$e_1(X,\overline{L}+\epsilon\overline{M})\geq \frac{\overline{L}^{n+1}+\epsilon(n+1)\overline{L}^{n}\overline{M}}{(n+1)\deg_{\widetilde{L}}(X)}+O(\epsilon^2).$$

By definitions of e_1 and of generic sequence, we have

$$\liminf_{m\to\infty} h_{\overline{L}+\epsilon\overline{M}}(x_m)\geq \frac{\overline{L}^{n+1}+\epsilon(n+1)\overline{L}^{n}\overline{M}}{(n+1)\deg_{\widetilde{L}}(X)}+O(\epsilon^2).$$

Since $\{x_m\}_{m\geq 1}$ is small, we have

$$\lim_{m\to\infty} h_{\overline{L}}(x_m)=h_{\overline{L}}(X)=\frac{\overline{L}^{n+1}}{(n+1)\deg_{\widetilde{L}}(X)}.$$

Therefore

$$\liminf_{m\to\infty} \epsilon h_{\overline{M}}(x_m)\geq \epsilon\frac{\overline{L}^{n}\overline{M}}{\deg_{\widetilde{L}}(X)}+O(\epsilon^2). \tag{6.3.4}$$

Now letting $\epsilon\to 0^+$ and $\epsilon\to 0^-$, we obtain

$$\lim_{m\to\infty} h_{\overline{M}}(x_m)=\frac{\overline{L}^{n}\overline{M}}{\deg_{\widetilde{L}}(X)}. \tag{6.3.5}$$

Now we wish to translate (6.3.5) into (6.3.3). For this purpose, we would choose an $\overline{M}$ associated with the function f. Roughly speaking, we would take $\overline{M}$ to be the trivial line bundle $\mathcal{O}_X$ endowed with the metric $\|\cdot\|_w$ (for each place w of K), as a line bundle on X^{an}, such that $\|1\|_v=e^{-f}$ and $\|1\|_w=1$ for any $w\neq v$. Then the LHS of (6.3.5) would be the LHS of (6.3.3) and the RHS of (6.3.5) would be the RHS of (6.3.3).

While this idea can be performed directly if X is projective, we need to be more cautious for our quasi-projective X. Now let us explain the execution in more details.

Assume $\overline{L}$ is represented by a Cauchy sequence $(\mathcal{L},(\mathcal{X}_i,\overline{\mathcal{L}}_i,\ell_i)_{i\geq 1})$ in $\widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{mod}}$ for a quasi-projective model $\mathcal{U}$ of X. Assume that $\psi\colon \mathcal{X}_i\to\mathcal{X}_1$ extends the identity morphism on $\mathcal{U}$, and denote by X_i to be the generic fiber of $\mathcal{X}_i$ which contains X as an open subvariety.

Let $\mathcal{X}_1'$ be another projective model of X_1. Let $\overline{\mathcal{M}}\in\widehat{\mathrm{Pic}}(\mathcal{X}_1')_{\mathbb{Q}}$ with a fixed isomorphism $\mathcal{M}_K\to\mathcal{O}_{X_1}$. Then it induces a metric $\|\cdot\|_w$ of $\mathcal{O}_{X_1}$ on $X_{1,w}^{\mathrm{an}}$ for any place w of K. Assume that the metric $\|1\|_w=1$ for any $w\neq v$. Denote by $g=-\log\|1\|_v$; it is continuous on $X_{1,v}^{\mathrm{an}}$. Then by definition, we have

$$h_{\overline{\mathcal{M}}}(x_m)=\frac{1}{\#O(x_m)}\sum_{y\in O(x_m)\times_K K_v} g(y),\qquad \overline{L}^{n}\overline{M}=\lim_{i\to\infty}\overline{\mathcal{L}}_i^{n}\overline{\mathcal{M}}=\lim_{i\to\infty}\int_{X_v^{\mathrm{an}}} gc_1(\overline{\mathcal{L}}_i)_v^{n}.$$

So we get, by (6.3.5),

$$\lim_{m\to\infty}\frac{1}{\#O(x_m)}\sum_{y\in O(x_m)\times_K K_v} g(y)=\frac{1}{\deg_{\widetilde{L}}(X)}\lim_{i\to\infty}\int_{X_v^{\mathrm{an}}} gc_1(\overline{\mathcal{L}}_i)_v^{n}, \tag{6.3.6}$$

with g viewed as a function on $X_{i,v}^{\mathrm{an}}$ by the pullback via $\psi_{i,K}\colon X_i \to X_1$.

Now vary $g = -\log \|1\|_v$, which is a model function on $X_{1,v}^{\mathrm{an}}$ associated with $(\mathcal{X}_1', \overline{\mathcal{M}})$. Gubler's density theorem implies that the space of all such model functions is dense in $C(X_{1,v}^{\mathrm{an}})$ under the topology of uniform convergence. So (6.3.6) holds true for any function in $C(X_{1,v}^{\mathrm{an}})$.

Finally, assume $f \in C_c(X_v^{\mathrm{an}})$, viewed as an element of $C(X_{i,v}^{\mathrm{an}})$ by the open immersion $X \to X_i$. Then

$$\lim_{i\to\infty} \int_{X_v^{\mathrm{an}}} f c_1(\overline{\mathcal{L}}_i)_v^n = \lim_{i\to\infty} \int_{X_v^{\mathrm{an}}} f c_1(\overline{\mathcal{L}}_i)_v^n|_{X_v^{\mathrm{an}}} = \int_{X_v^{\mathrm{an}}} f c_1(\overline{L})_v^n.$$

And we can conclude by (6.3.6) applied to f. $\square$