Chapter 2

From Hodge theory to Hermitian symmetric domains

2.1 Basic background knowledge on reductive groups

Let k be a field. Let G be a connected linear group defined over k. Let $\overline{k}$ be an algebraic closure of k.

Denote by $\mathbb{G}_{\mathrm{a},k}$ the group defined by: for any k-algebra R, we have $\mathbb{G}_{\mathrm{a}}(R) = R$. When k is clear in the context, we simply write $\mathbb{G}_{\mathrm{a}}$.

Definition 2.1.1. *G is called a* **reductive group** *if $G_{\overline{k}}$ does not contain a normal subgroup isomorphic to $\mathbb{G}_{\mathrm{a}}$.*

A notion closely related to reductive groups is the *unipotent radical*. Let us briefly recall the definition. Recall that G can be embedded as a closed subgroup scheme of GL_N for some N. An element $g \in G$ is said to be *unipotent* if $(I_N - g)^N = 0$ (as matrix). A subgroup of G is said to be *unipotent* if all its elements are unipotent. As an example, U_N (consisting of upper triangular matrices whose diagonal entries are 1) is a unipotent subgroup of GL_N. Moreover, it is known that any unipotent subgroup of GL_N is a subgroup of gU_Ng^{-1} for some $g \in \mathrm{GL}_N$.

Definition 2.1.2. *The* **unipotent radical** *of G, denoted by $R_u(G)$, is the identity component of its maximal normal unipotent subgroup.*

As an example, $R_u(\mathrm{GL}_N) = 1$. Moreover, any algebraic torus has trivial unipotent radical.

Since $\mathbb{G}_{\mathrm{a}}$ is a unipotent subgroup of GL_N via $x \mapsto \begin{bmatrix} 1 & 0 & \cdots & 0 & x \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$, we have:

Lemma 2.1.3. *G is a reductive group if and only if $R_u(G_{\overline{k}}) = 1$.*

For any reductive group G, its connected center $Z(G)^\circ$ is an algebraic torus. Among reductive groups, those with trivial connected center are of particular importance.

Definition 2.1.4. *A reductive group G is called* **semi-simple** *if its connected center $Z(G)^\circ$ is trivial. It is called* **simple** *if its only connected normal subgroups are 1 and G.*

Clearly, simple groups are semi-simple because $Z(G)$ is a normal subgroup of G. Given a reductive group G, one can naturally construct two semi-simple subgroups:

(i) the *derived subgroup* $G^{\mathrm{der}} := [G, G]$ which is a normal subgroup of G,

(ii) the *adjoint* $G^{\mathrm{ad}} := G/Z(G)$ which is a quotient of G.

The composite $G^{\mathrm{der}} \to G \to G^{\mathrm{ad}}$ is a central isogeny, *i.e.* it is surjective and has finite kernel contained in $Z(G)$. As an example, $\mathrm{GL}_N^{\mathrm{der}} = \mathrm{SL}_N$ and $\mathrm{GL}_N^{\mathrm{ad}} = \mathrm{PGL}_N$, and the kernel of $\mathrm{SL}_N \to \mathrm{PGL}_N$ is $\{\pm I_N\}$.

Next we recall the following structural theorem of reductive groups.

Theorem 2.1.5 (Structural theorem of reductive groups)**.** *Let G be a reductive group. Then there are only finitely many non-trivial simple normal subgroups $G_1, \ldots, G_n$ of G, and*

$$G = Z(G)G_1 \cdots G_n$$

with the intersections $G_i \cap G_j < Z(G)$.

We end this revision by a characterization of a $\mathbb{C}$-group to be reductive.

Proposition 2.1.6. *Assume* $\mathrm{char} k = 0$*. Then the followings are equivalent:*

(i) G is a reductive group;

(ii) Any representation V of G can be decomposed into the direct sum of irreducible ones.

Corollary 2.1.7. *Let G be a connected linear algebraic group defined over $\mathbb{C}$. Then G is reductive if and only if G has a real form $G_{\mathbb{R}}$ (i.e. $G_{\mathbb{R}} \otimes \mathbb{C} \simeq G$) such that $G_{\mathbb{R}}(\mathbb{R})$ is compact.*

Proof. We only sketch for $\Leftarrow$. By definition it is enough to prove that $G_{\mathbb{R}}$ is reductive. For any representation V of $G_{\mathbb{R}}$, define an inner product on V induced by $\|v\| := \int_{G_{\mathbb{R}}(\mathbb{R})} gv$ with respect to a Haar measure on $G_{\mathbb{R}}(\mathbb{R})$. Then this inner product is $G_{\mathbb{R}}$-invariant. Thus V can be decomposed into the direct sum of irreducible sub-representations of $G_{\mathbb{R}}$. $\square$

Example 2.1.8. *Let $G = \mathrm{GL}_{N,\mathbb{C}}$. Then $\mathrm{GL}_{N,\mathbb{R}}$ and (write $J_{p,q} = \mathrm{diag}\{I_p, -I_q\}$ and denote for simplicity by $J = J_{p,q}$)*

$$U(p,q) := \left\{ g \in \mathrm{GL}_{N,\mathbb{C}} : \overline{g}^{\mathrm{t}} J g = J \right\}$$

are $\mathbb{R}$-forms of G, with all $p + q = N$. The associated complex conjugation for $U(p,q)$ is $\sigma\colon g \mapsto J(\overline{g}^{\mathrm{t}})^{-1}J$. A compact $\mathbb{R}$-form is $U(N)$.

2.2 Polarization on families and reductive groups

Recall the setting of §1.3.2: V is a finite-dimensional $\mathbb{R}$-vector space, $n \in \mathbb{Z}$, $G < \mathrm{GL}(V)$ and $X^+ \subseteq \mathrm{Hom}(\mathbb{S}, G)$ is a G^+-orbit. We know that X^+ parametrizes certain Hodge structures on V of weight n, and hence has carries a family of Hodge structures. By Proposition 1.3.5, X^+ has a unique complex structure such that this family of Hodge structures varies holomorphically.

Better, we have fixed a $(-1)^n$-symmetric pairing $Q\colon V \times V \to \mathbb{R}$ which induces a polarization for the Hodge structure on V given by each $h \in X^+$. In this section, we prove that this extra information forces G to be a reductive group.

2.2.1 Cartan involution

We need some background knowledge on Cartan involutions.

Let $G_{\mathbb{R}}$ be a linear algebraic group defined over $\mathbb{R}$. Let $\sigma\colon G_{\mathbb{C}} \to G_{\mathbb{C}}$ be the associated conjugation.

Definition 2.2.1. *A* **Cartan involution** *is a morphism $\theta\colon G_{\mathbb{R}} \to G_{\mathbb{R}}$ such that $\theta^2 = 1$ and that $(G_{\mathbb{C}})^{\tau} := \{g \in G_{\mathbb{C}} : \tau(g) = g\}$ is a compact real form of $G_{\mathbb{C}}$, where $\tau = \theta_{\mathbb{C}} \circ \sigma = \sigma \circ \theta_{\mathbb{C}}$.*

Example 2.2.2. *Let us look at the following examples with $G_{\mathbb{C}} = \mathrm{GL}_{N,\mathbb{C}}$.*

(a) $G_{\mathbb{R}} = U(N)$, *with* $\theta = 1$.

(b) $G_{\mathbb{R}} = U(p,q)$, *with* $\theta(g) = JgJ$ *where* $J = J_{p,q}$.

(c) $G_{\mathbb{R}} = \mathrm{GL}_{N,\mathbb{R}}$, *with* $\theta(g) = (g^{\mathrm{t}})^{-1}$.

Proposition 2.2.3. *$G_{\mathbb{R}}$ is reductive if and only if $G_{\mathbb{R}}$ admits a Cartan involution. And any two Cartan involutions of $G_{\mathbb{R}}$ are conjugate.*

In Example 2.2.2, the Cartan involutions in (a) and (b) are induced by an element of $G(\mathbb{R})$, while in (c) it is not. The first kind is called *inner Cartan involution* and is of particular importance because of its relation with polarizations explained by the following lemma.

Lemma 2.2.4 (Deligne). *Let $C \in G(\mathbb{R})$ with $C^2 = 1$. Then the followings are equivalent:*

(i) $\mathrm{Int}(C)$ *is a Cartan involution of* $G_{\mathbb{R}}$,

(ii) any $G_{\mathbb{R}}$-representation V is C-polarizable, i.e. there exists a $G_{\mathbb{R}}$-invariant bi-linear map $\phi\colon V \times V \to \mathbb{R}$ such that $(x,y) \mapsto \phi_{\mathbb{C}}(x, C\overline{y})$ is Hermitian and positive-definite (equivalently, $(x,y) \mapsto \phi(x,Cy)$ is symmetric and positive-definite),

(iii) $G_{\mathbb{R}}$ admits one faithful representation which is C-polarizable.

Proof. Let ϕ be a bi-linear map. Observe that the followings are equivalent:

- ϕ is G-invariant;
- $\phi_{\mathbb{C}}(gx, \sigma(g)\overline{y}) = \phi_{\mathbb{C}}(x, \overline{y})$ for all $g \in G_{\mathbb{C}}$ and $x, y \in V_{\mathbb{C}}$;
- $\phi_{\mathbb{C}}(gx, \sigma(g)C\overline{y}) = \phi_{\mathbb{C}}(x, C\overline{y})$ for all $g \in G_{\mathbb{C}}$ and $x, y \in V_{\mathbb{C}}$;
- $(x,y) \mapsto \phi_{\mathbb{C}}(x, C\overline{y})$ is U-invariant, where $U = (G_{\mathbb{C}})^{\tau}$ with $\tau = \mathrm{Int}(C) \circ \sigma$.

The last equivalence follows from $\phi_{\mathbb{C}}(gx, \sigma(g)C\overline{y}) = \phi_{\mathbb{C}}(gx, C\tau(g)\overline{y})$.

Now let us go back to the proof of the lemma. (ii) implying (iii) is trivial. (iii) implies that U is compact, and hence implies (i). It remains to show that (i) implies (ii).

Assume (i). Then U is compact. So there exists a U-invariant positive-definite symmetric bi-linear map $\phi\colon V \times V \to \mathbb{R}$. Hence $\phi_{\mathbb{C}}$ is $G_{\mathbb{C}}$-invariant, and so $\phi_{\mathbb{C}}(gx, \tau(g)\overline{y}) = \phi_{\mathbb{C}}(x, \overline{y})$ for all $g \in G_{\mathbb{C}}$. But $\tau(g) = C\sigma(g)C^{-1} = C\sigma(g)C$, hence $\phi_{\mathbb{C}}(gx, \sigma(g)C\overline{y}) = \phi_{\mathbb{C}}(x, C\overline{y})$ for all $g \in G_{\mathbb{C}}$. Thus ϕ is also $G_{\mathbb{R}}$-invariant. This establishes (ii). □

Here is a corollary on the Mumford–Tate group.

Corollary 2.2.5. *Let (V,h) be a $\mathbb{Q}$-Hodge structure of weight n with a polarization ψ. Then $\mathrm{MT}(h)$ is a reductive group.*

Proof. Let $G_{\mathbb{R}} := \mathrm{MT}(h)_{\mathbb{R}}$ and $C := h(\sqrt{-1})$. Then $C^2 = 1$, and $V_{\mathbb{R}}$ is a faithful representation of $G_{\mathbb{R}}$ which is C-polarization. Hence $\mathrm{Int}(C)$ is a Cartan involution of $G_{\mathbb{R}}$ by Lemma 2.2.4. So $G_{\mathbb{R}}$ is reductive by Proposition 2.2.3. Hence $\mathrm{MT}(h)$ is a reductive group. □

2.2.2 Polarization on parametrizing space

Now let us go back to our setting at the beginning of this section.

Let $h \in X^+$. Let G_1 be the subgroup of G generated by $h(\mathbb{S})$ for all $h \in X^+$. In other words, G_1 is the smallest subgroup of G which contains $h(\mathbb{S})$ for all $h \in X^+$. It is easy to check that G_1 is a normal subgroup of G, and that X^+ is a G_1^+-orbit.

Recall the weight cocharacter $w\colon \mathbb{G}_{\mathrm{m}} \to \mathbb{S}$ induced by $\mathbb{R}^\times \subseteq \mathbb{C}^\times$.

Proposition 2.2.6. *Assume $h \circ w$ factors through $Z(G)$ for one (and hence all) $h \in X^+$. Then the followings are equivalent:*

(1) There exists $\psi\colon V \otimes V \to \mathbb{R}(-n)$ which is a polarization for the Hodge structure determined by each $h \in X^+$;

(2) G_1 is a reductive group for one (and hence all) $h \in X^+$, and $\mathrm{Int}(h(\sqrt{-1}))$ is a Cartan involution of G_1^{ad}.

In our setting, ψ is induced by Q. But this proposition also gives an abstract way of showing the existence of a polarization on a family of Hodge structures, which will be useful in §**??**.

Proof. By assumption, the subgroup $(h \circ w)(\mathbb{G}_{\mathrm{m}})$ of G_1 is independent of the choice of $h \in X^+$, and we denote it by W. Then $W < Z(G_1)$.

Recall the short exact sequence of group over $\mathbb{R}$

$$1 \to U(1) \to \mathbb{S} \xrightarrow{\mathrm{Nm}} \mathbb{G}_{\mathrm{m}} \to 1.$$

Let G_2 be the subgroup of G_1 generated by $h(U(1))$ for all $h \in X^+$. Then $G_1 = W \cdot G_2$. Moreover since $W < Z(G_1)$ and $W \simeq \mathbb{G}_{\mathrm{m}}$, the inclusion $G_2 < G_1$ induces $G_2^{\mathrm{ad}} \simeq G_1^{\mathrm{ad}}$. So (2) is equivalent to:

(*) G_2 is a reductive group for $h \in X^+$, and $\mathrm{Int}(h(\sqrt{-1}))$ is a Cartan involution of G_2^{ad}.

Take a map $\psi\colon V \otimes V \to \mathbb{R}$. Then

$$\begin{aligned}
&\psi\colon V \otimes V \to \mathbb{R}(-n) \text{ is a morphism of Hodge structures for all } h \in X^+\\
\Leftrightarrow &\psi \text{ is } h(\mathbb{S})\text{-equivariant for all } h \in X^+\\
\Leftrightarrow &\psi \text{ is } h(U(1))\text{-invariant for all } h \in X^+ \quad \text{because } \mathbb{S} = w(\mathbb{G}_{\mathrm{m}}) \cdot U(1)\\
\Leftrightarrow &\psi \text{ is } G_2\text{-invariant.}
\end{aligned}$$

Thus $\psi\colon V \otimes V \to \mathbb{R}(-n)$ is a polarization for all $h \in X^+$ if and only if the G_2-equivariant map $(x, y) \mapsto \psi(x, h(\sqrt{-1})\overline{y})$ is Hermitian and positive-definite. Hence by Lemma 2.2.4, (1) is equivalent to $\mathrm{Int}(h(\sqrt{-1}))$ being a Cartan involution of G_2. Hence by (*), it suffices to prove that $\mathrm{Int}(h(\sqrt{-1}))$ is a Cartan involution of G_2 if and only if it is a Cartan involution of G_2^{ad}. So it remains to prove that $Z(G_2)$ is compact. This is true because G_2 is generated by compact subgroups (since $U(1)$ is compact). □

2.3 Hermitian symmetric domains

Motivated by Proposition 1.3.5 and 2.2.6, we shall study pairs (G, X^+) where

- G is a reductive group defined over $\mathbb{R}$,
- X^+ is a G^+-orbit contained in $\mathrm{Hom}(\mathbb{S}, G)$, with G acting on $\mathrm{Hom}(\mathbb{S}, G)$ via conjugation

satisfying the following properties:

(i) For any $h \in X^+$, the Hodge structure $(\mathrm{Lie}G, h)$ has type $(-1,1)+(0,0)+(1,-1)$,

(ii) For any $h \in X^+$, $\mathrm{Int}(h(\sqrt{-1}))$ is a Cartan involution for G^{ad}.

In fact, it is enough to require (i) and (ii) for one $h \in X^+$. And condition (i) implies that $h \circ w \colon \mathbb{G}_{\mathrm{m}} \to G$ factors through $Z(G)$. Indeed by (i), $\mathrm{Ad} \circ h \circ w \colon \mathbb{G}_{\mathrm{m}} \to \mathrm{GL}(\mathrm{Lie}G)$ sends z to the multiplication by $z^0 = 1$, and hence is trivial. So $\mathrm{im}(h \circ w) \subseteq \mathrm{Ker}(\mathrm{Ad}) = Z(G)$.

Now take any representation V of G. Then $X^+ \times V \to X^+$ is a family of $\mathbb{R}$-Hodge structures, with the Hodge structure on $h \in X^+$ determined by $\mathbb{S} \xrightarrow{h} G \to \mathrm{GL}(V)$. By Proposition 1.3.5 and 2.2.6, this family is an $\mathbb{R}$-variation of Hodge structures endowed with a polarization.

Theorem 2.3.1. *X^+ is a Hermitian symmetric domain. More precisely, this means:*

(1) $X^+ \simeq X_1^+ \times \cdots \times X_k^+$;

(2) Each X_i^+ is a Riemannian symmetric space of non-compact type, i.e. $X_i^+ \simeq G_i/K_{i,\infty}$ where G_i is a simple group defined over $\mathbb{R}$ and $K_{i,\infty}$ is a maximal compact subgroup of G_i;

(3) For each $i \in \{1, \ldots, k\}$, X_i^+ has a G_i-invariant complex structure.

Conversely, any Hermitian symmetric domain can be obtained as X^+ for a pair (G, X^+) as above. But we will not prove this in this course.

2.3.1 The example of Siegel case

Let $V = \mathbb{R}^{2d}$. Let $\psi \colon V \times V \to \mathbb{R}$ be $(x, y) \mapsto x^{\mathrm{t}} J y$ with $J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}$.

Define the $\mathbb{R}$-group

$$G_{\mathbb{R}} = \mathrm{GSp}(\psi) = \mathrm{GSp}_{2d} := \left\{ g \in \mathrm{GL}(V) : \psi(gx, gy) = c\psi(x,y) \text{ for some } c \in \mathbb{R}^\times \right\}$$
$$= \left\{ g \in \mathrm{GL}_{2n} : gJg^{\mathrm{t}} = cJ \text{ for some } c \in \mathbb{R}^\times \right\}.$$

The derived subgroup $G_{\mathbb{R}}^{\mathrm{der}} = \mathrm{Sp}_{2d} = \{g \in \mathrm{GL}(V) : \psi(gx, gy) = \psi(x, y)\} = \left\{ g \in \mathrm{GL}_{2d} : gJg^{\mathrm{t}} = J \right\}$.

Define

$$h_0 \colon \mathbb{S} \to \mathrm{GSp}_{2n}, \qquad a + b\sqrt{-1} \mapsto aI_{2d} + bJ.$$

Indeed, this map is well-defined since $(aI_{2d} + bJ)J(aI_{2d} + bJ)^{\mathrm{t}} = (a^2 + b^2)J$. Notice that $h_0 \circ w \colon \mathbb{G}_{\mathrm{m}} \to \mathrm{GSp}_{2d}$ sends $r \in \mathbb{R}^\times$ to multiplication on V by r. Hence the Hodge structure (V_0, h_0) has weight -1.

The eigenvalues for J are $\pm\sqrt{-1}$. Let $V^{-1,0}$ (resp. $V^{0,-1}$) be the eigenspace of $\sqrt{-1}$ (resp. of $-\sqrt{-1}$). Then one can check that each $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ acts on $V^{-1,0}$ as multiplication by z and on $V^{0,-1}$ as multiplication by $\bar{z}$. Thus (V, h_0) is a Hodge structure of type $(-1,0)+(0,-1)$, and ψ is a polarization.

Now that $\mathrm{Lie}G_{\mathbb{R}} \subseteq \mathrm{End}(V) = V \otimes V^\vee$, we know that the Hodge structure $(\mathrm{Lie}G, h)$ has type $(-1,1)+(0,0)+(1,-1)$. So condition (i) holds true.

For condition (ii), apply Lemma 2.2.4 to the group Sp_{2d} and the element $h_0(\sqrt{-1}) = J$. Since ψ is a J-polarization of the Hodge structure (V, h_0), by Lemma 2.2.4 $\mathrm{Int}(h_0(\sqrt{-1}))$ is a Cartan involution for Sp_{2d}. Hence condition (ii) holds true because $G_{\mathbb{R}}^{\mathrm{ad}} = \mathrm{Sp}_{2n}/\{\pm I_{2d}\}$.

Let $X^+ \subseteq \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$ be the G^+-orbit of h_0. Then Sp_{2d} acts transitively on X^+, and $\mathrm{Stab}_{\mathrm{Sp}_{2d}}(h_0) = U(d) = O(2d) \cap \mathrm{Sp}_{2d}$ is a maximal compact subgroup of Sp_{2d}. So

$$X^+ \simeq \mathrm{Sp}_{2d}/(O(2d) \cap \mathrm{Sp}_{2d})$$

with Sp_{2d} a simple group defined over $\mathbb{R}$ which is not compact. To see the complex structure in a more concrete way, let us make the identification

$$\mathrm{Sp}_{2n}/(O(2d)\cap \mathrm{Sp}_{2d}) \xrightarrow{\sim} \mathfrak{H}_d := \left\{\tau \in \mathrm{Mat}_{d\times d}(\mathbb{C}) : \tau = \tau^{\mathrm{t}} \text{ and } \mathrm{Im}\tau > 0\right\}$$

which sends

$$g\cdot h_0 \mapsto g\cdot\tau := (A\tau + B)(C\tau + D)^{-1} \qquad \text{with } g = \begin{bmatrix} A & B \\ C & D\end{bmatrix}.$$

The Sp_{2d}-invariant complex structure on X^+ is the same as the complex structure on $\mathfrak{H}_d$ inherited from the open inclusion $\mathfrak{H}_d \subseteq \left\{\tau \in \mathrm{Mat}_{d\times d}(\mathbb{C}) : \tau = \tau^{\mathrm{t}}\right\} \simeq \mathbb{C}^{d(d+1)/2}$.

2.3.2 Cartan decomposition of semi-simple groups

In this subsection, we review background knowledge (without proof) on the Cartan decomposition of semi-simple groups G defined over $\mathbb{R}$. This is closely related to the Cartan involution from §2.2.1.

Let θ be a Cartan involution of a semi-simple group G defined over $\mathbb{R}$. Composing with the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathrm{Lie}G)$, we get an involution on $\mathfrak{g} := \mathrm{Lie}G$ which we still call a *Cartan involution* and denote by θ. Then θ has eigenvalues ± 1, and let $\mathfrak{k}$ (resp. $\mathfrak{m}$) be the eigenspace for 1 (resp. for -1). Then

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}. \tag{2.3.1}$$

Moreover, $[\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k},\mathfrak{m}] \subseteq \mathfrak{m}$, and $[\mathfrak{m},\mathfrak{m}] \subseteq \mathfrak{k}$ by looking at the eigenvalues. So $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$, while any Lie subalgebra contained in $\mathfrak{m}$ is commutative.

Lemma 2.3.2. $K_\infty := \exp(\mathfrak{k})$ *is a maximal compact subgroup of* G.

We can also recover the compact real form of G as follows. The Cartan involution θ extends to $\mathfrak{g}_{\mathbb{C}}$ and we have a corresponding $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}}$. Let $\mathfrak{g}_c := \mathfrak{k} \oplus \sqrt{-1}\mathfrak{m}$. Then $G_c := \exp(\mathfrak{g}_c)$ is a compact real Lie group and which is a real form of G. Notice that $K_\infty = G \cap G_c$.

2.3.3 Proof of Theorem 2.3.1

By definition of X^+, the center $Z(G)$ acts trivially on X^+. Hence the action of G^+ factors through $G^{\mathrm{ad}}(\mathbb{R})^+$. By Theorem 2.1.5, $(G^{\mathrm{ad}})^+$ can be decomposed into a direct product $(G^{\mathrm{ad}})^+ = G_1 \times \cdots \times G_k$ with each G_i a simple group. Fix $h \in X^+$, and let $X_i^+ := G_i \cdot h$. Then the decomposition of the group induces

$$X^+ \simeq X_1^+ \times \cdots \times X_k^+.$$

This establishes (1).

In the rest of proof, to ease notation, use G to denote G_i and X^+ to denote X_i^+. Then G is a simple group with trivial center.

Denote by $\mathfrak{g} := \mathrm{Lie}G$. Consider the action of $h(\sqrt{-1})$ on $\mathfrak{g}$ via the adjoint representation. Then $h(\sqrt{-1})$ acts on $\mathfrak{g}^{0,0}$ as identity and on $\mathfrak{g}^{1,-1} \oplus \mathfrak{g}^{-1,1}$ as multiplication by -1. Thus $X^+ \simeq G/K_\infty$ for the subgroup $K_\infty := \exp(\mathfrak{g}^{0,0})$ of G. Condition (ii) says that the action of $h(\sqrt{-1})$ on $\mathfrak{g}$ is a Cartan involution, and hence we have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ as in (2.3.1). Then condition (i) says that $\mathfrak{k} = \mathfrak{g}^{0,0}$ (and $\mathfrak{m}_{\mathbb{C}} = \mathfrak{g}^{1,-1} \oplus \mathfrak{g}^{-1,1}$). Hence $K_\infty := \exp(\mathfrak{g}^{0,0})$ is maximal compact in G by Lemma 2.3.2. This establishes (2).

Remark 2.3.3. *Assume G is simple with trivial center. If G is compact, we claim that $X^+ = \{\text{trivial map}\}$. Indeed, $\mathrm{Int}(h(\sqrt{-1}))$ is identity because it is a Cartan involution for G. Thus $\mathrm{Ad} \circ h \colon \mathbb{S} \to \mathrm{GL}(\mathfrak{g})$ sends $\sqrt{-1}$ to identity, and hence $(\mathfrak{g}, h)$ has Hodge type $(0,0)$ by assumption (i) (since $\sqrt{-1}$ acts on the complement of $\mathfrak{g}^{0,0}$ by multiplication by -1). But then $\mathrm{Ad} \circ h$ is trivial since $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ acts on $\mathfrak{g}$ as multiplication by $z^0\bar{z}^0 = 1$. Thus $h(\mathbb{S}) \subseteq \mathrm{Ker}(\mathrm{Ad}) = Z(G) = \{1\}$.*

For part (3), notice that $[\mathfrak{g}^{1,-1}, \mathfrak{g}^{1,-1}] \subseteq \mathfrak{g}^{2,-2} = 0$. Hence $\mathfrak{g}^{1,-1}$ is an abelian Lie subalgebra of $\mathfrak{g}_\mathbb{C}$. Same is true for $\mathfrak{g}^{-1,1}$. Thus $F^0\mathfrak{g}_\mathbb{C} = \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{1,-1}$ is a Lie subalgebra of $\mathfrak{g}_\mathbb{C}$. Therefore $P_\mathbb{C} := \exp(F^0\mathfrak{g}_\mathbb{C})$ is a subgroup of $G(\mathbb{C})$, with $P_\mathbb{C} \cap G = K_\infty$. Thus the inclusion $G \subseteq G(\mathbb{C})$ induces an injective morphism of real smooth manifolds

$$X^+ = G/K_\infty \to X^\vee := G(\mathbb{C})/P_\mathbb{C}. \tag{2.3.2}$$

The tangent of this map is an isomorphism as real vector spaces. Hence this map realizes X^+ as an open subset of $X^\vee$. This establishes (3). We are done. □

2.3.4 Borel embedding theorem and Harish–Chandra realization

Replacing G by G^{der} does not change X^+. Hence we may assume that G is semi-simple. Fix $h \in X^+$, and take the inner Cartan involution θ obtained from $h(\sqrt{-1})$. Use the notation from §2.3.2. The real tangent space of X^+ at h, denoted by $T_\mathbb{R}(X^+)$, can be identified as $\mathfrak{m}$.

The element $J := h(e^{\pi\sqrt{-1}/4})$ satisfies $J^2 = 1$. Its action on X^+ induces a decomposition

$$T_\mathbb{R}(X^+) \otimes_\mathbb{R} \mathbb{C} = T^{1,0}(X^+) \oplus T^{0,1}(X^+)$$

where J acts by multiplication by $\sqrt{-1}$ on $T^{1,0}(X^+)$ and by $-\sqrt{-1}$ on $T^{0,1}(X^+)$. Then $T^{1,0}(X^+)$ is the holomorphic tangent space at h. On the other hand, we have $\mathfrak{m}_\mathbb{C} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$ where J acts by multiplication by $\sqrt{-1}$ on $\mathfrak{m}^+$ and by $-\sqrt{-1}$ on $\mathfrak{m}^-$; in fact $\mathfrak{m}^+ = \mathfrak{g}^{-1,1}$ and $\mathfrak{m}^- = \mathfrak{g}^{1,-1}$. Then as we have seen above, both $\mathfrak{m}^+$ and $\mathfrak{m}^-$ are abelian Lie subalgebras of $\mathfrak{g}_\mathbb{C}$.

Let $M^+ := \exp(\mathfrak{m}^+)$, $M^- := \exp(\mathfrak{m}^-)$; both are abelian subgroups of $G_\mathbb{C}$. Let $K_\mathbb{C} := \exp(\mathfrak{k}_\mathbb{C})$ and $P_\mathbb{C} := \exp(\mathfrak{k}_\mathbb{C} + \mathfrak{m}^-) = K_\mathbb{C}M^-$. Then $P_\mathbb{C}$ is a subgroup of $G_\mathbb{C}$.

Here is a more precise version of (2.3.2), with G_c the real form of G from the end of §2.3.2.

Theorem 2.3.4 (Borel Embedding Theorem)**.** *The embedding $G_c < G(\mathbb{C})$ induces an isomorphism of real manifolds $G_c/K_\infty \simeq G(\mathbb{C})/P_\mathbb{C} = X^\vee$. The embedding $G < G(\mathbb{C})$ induces an open embedding*

$$X^+ = G/K_\infty \to X^\vee = G(\mathbb{C})/P_\mathbb{C},$$

realizing X^+ as an open subset (in the usual topology) of $X^\vee$.

We call $X^\vee$ the *compact dual* of X^+.

Theorem 2.3.5 (Harish–Chandra)**.** *The map*

$$F \colon M^+ \times K_\mathbb{C} \times M^- \to G_\mathbb{C}, \qquad (m^+, k, m^-) \mapsto m^+km^-$$

is a biholomorphism of of the left hand side onto an open subset of $G(\mathbb{C})$ containing G. As a consequence, the map

$$\eta \colon \mathfrak{m}^+ \to X^\vee = G(\mathbb{C})/P_\mathbb{C}, \qquad m^+ \mapsto \exp(m^+)P_\mathbb{C}$$

is a biholomorphism onto a dense open subset of $X^\vee$ containing X^+. Futhermore, $\mathcal{D} := \eta^{-1}(X^+)$ is a bounded symmetric domain in $\mathfrak{m}^+ \simeq \mathbb{C}^N$ and $\eta^{-1}(h) = 0$.

Example 2.3.6. *Let us continue with Example 2.3.1. The Harish–Chandra realization of Siegel upper-half space $\mathfrak{H}_d$, based at $\sqrt{-1}I_d$, is*

$$\{Z \in \mathrm{Mat}_{d\times d}(\mathbb{C}) : Z = Z^{\mathrm{t}} \text{ and } I_d - Z\overline{Z} > 0\}.$$