

# Chapter 6

## Baily–Borel compactification

Let  $(\mathbf{G}, X)$  be a Shimura datum. By abuse of notation use  $X$  to denote a connected component. Let  $\Gamma < \mathbf{G}(\mathbb{Q})$  be an arithmetic subgroup.

For simplicity, we will assume  $\mathbf{G}$  to be quasi-simple, *i.e.*  $\mathbf{G}^{\text{der}}$  is a simple group. For the purpose of compactifying  $\Gamma \backslash X$  we can easily reduce to this case. Notice that  $\mathbf{G}_{\mathbb{R}}^{\text{der}}$  may not be simple as an  $\mathbb{R}$ -group, so that  $X$  is not necessarily irreducible.

### 6.1 Baily–Borel compactification as a topological space

Consider the Harish–Chandra embedding

$$X \simeq \mathcal{D} \subseteq \mathfrak{m}^+ \simeq \mathbb{C}^N$$

with  $\overline{\mathcal{D}}$  the closure of  $\mathcal{D}$  in  $\mathfrak{m}^+$ . Let  $F$  be an analytic boundary component of  $X$ , with normalizer  $N(F) = \{g \in \mathbf{G}^{\text{der}}(\mathbb{R})^+ : gF = F\}$ .

Recall [5.1.3] the fundamental set  $\Sigma$  constructed from Siegel sets associated with a minimal rational proper parabolic subgroup  $\mathbf{P}_0$  of  $\mathbf{G}^{\text{der}}$ .

Let  $\overline{\Sigma}$  be the closure of  $\Sigma \subseteq X \simeq \mathcal{D} \subseteq \mathfrak{m}^+$ . Then  $\overline{\Sigma} \subseteq \overline{\mathcal{D}}$ , with an induced topology.

**Theorem 6.1.1.** *The followings are equivalent:*

- (1)  $\Gamma F \cap \overline{\Sigma} \neq \emptyset$ ,
- (2)  $\mathbf{P}_{0,\mathbb{R}} < N(F)$ , and  $F$  is a rational analytic boundary component (*i.e.*  $N(F)$  equals  $\mathbf{P}_F(\mathbb{R})$  for a parabolic subgroup  $\mathbf{P}_F$  of  $\mathbf{G}^{\text{der}}$ ).

Moreover in these cases,  $\mathbf{P}_F$  is a maximal proper parabolic subgroup of  $\mathbf{G}^{\text{der}}$ .

Theorem 6.1.1 indicates that we can do the following compactification of  $\Gamma \backslash X$ :

- (i) Define  $\overline{X}^{\text{BB}} := \mathcal{D} \cup \bigsqcup_{\mathbf{P}} F_{\mathbf{P}} \subseteq \overline{\mathcal{D}}$ , where  $\mathbf{P}$  runs over all *maximal* proper parabolic subgroup of  $\mathbf{G}^{\text{der}}$  and  $F_{\mathbf{P}}$  is the rational analytic boundary component  $F_{\mathbf{P}}$ .
- (ii) Endow  $\overline{X}^{\text{BB}}$  with the Satake topology.
- (iii) The space  $\Gamma \backslash \overline{X}^{\text{BB}}$  is then a compact Hausdorff space containing  $\Gamma \backslash X$  as an open dense subset.

Then  $(\Gamma \backslash \overline{X})^{\text{BB}} := \Gamma \backslash \overline{X}^{\text{BB}}$  is called the *Baily–Borel compactification* of  $\Gamma \backslash X$ , and

$$(\Gamma \backslash \overline{X})^{\text{BB}} = \Gamma \backslash X \cup \bigsqcup_{j=1}^m \Gamma_{F_j} \backslash F_j, \quad (6.1.1)$$

where  $F_1, \dots, F_m$  are rational analytic boundary components such that  $\{\mathbf{P}_{F_1}, \dots, \mathbf{P}_{F_m}\}$  is a set of representatives of  $\Gamma$ -conjugacy classes of maximal proper parabolic subgroups of  $\mathbf{G}^{\text{der}}$ , with  $\Gamma_{F_j} := \Gamma \cap \mathbf{P}_{F_j}(\mathbb{Q})$ .

### 6.1.1 Satake topology on $\overline{X}^{\text{BB}}$

The *Satake topology* on  $\overline{X}^{\text{BB}}$  is defined as follows. For each  $x \in \overline{X}^{\text{BB}} \subseteq \overline{\mathcal{D}}$ , the neighborhoods of any point  $x \in X^*$  is the saturations of the neighborhoods of the corresponding points in  $\overline{\Sigma}$  under the action of  $\Gamma_x := \{\gamma \in \Gamma : \gamma \cdot x = x\}$ . More precisely, a fundamental system of neighborhoods of  $x$  is given by all subsets  $U \subseteq \overline{\mathcal{D}}$  such that

$$\Gamma_x \cdot U = U,$$

and such that  $\gamma U \cap \overline{\Sigma}$  is a neighborhood of  $\gamma \cdot x$  in  $\overline{\Sigma}$  whenever  $\gamma \cdot x \in \overline{\Sigma}$ .

**Proposition 6.1.2.** *The Satake topology is the unique topology on  $\overline{X}^{\text{BB}}$  such that the followings hold:*

(i) *it induces the original topologies on  $\overline{\Sigma}$  and on  $X$ ,*

(ii) *the  $\mathbf{G}^{\text{der}}(\mathbb{Q})$ -action on  $\overline{X}^{\text{BB}}$  is continuous,*

(iii) *for any  $x \in \overline{X}^{\text{BB}}$ , there exists a fundamental system of neighborhoods  $\{U\}$  of  $x$  such that*

$$\gamma U = U \text{ for all } \gamma \in \Gamma_x \quad \text{and} \quad \gamma U \cap U = \emptyset \text{ for all } \gamma \notin \Gamma_x,$$

(iv) *if  $x, x' \in \overline{X}^{\text{BB}}$  are not in one  $\Gamma$ -orbit, then there exist neighborhoods  $U$  of  $x$  and  $U'$  of  $x'$  such that*

$$\Gamma U \cap U' = \emptyset.$$

**Corollary 6.1.3.**  $\Gamma \backslash \overline{X}^{\text{BB}}$  is Hausdorff.

### 6.1.2 $\mathbb{Q}$ -roots vs $\mathbb{R}$ -roots, and $\mathbb{Q}$ -polydisc

Let  $\mathbf{S} < \mathbf{P}_0$  be a maximal  $\mathbb{Q}$ -split torus, and let  $K_\infty$  be a maximal compact subgroup of  $\mathbf{G}_\mathbb{R}^{\text{der}}$  such that  $\text{Lie}K_\infty \cap \text{Lie}\mathbf{S}_\mathbb{R} = 0$ <sup>[1]</sup>. Then there exists  $x_0 \in X$  such that  $\text{Stab}_{\mathbf{G}^{\text{der}}(\mathbb{R})}(x_0) = K_\infty$ .

In §4.6.2 and §4.6.3, we discussed the complex roots and real roots respectively. Now we need to analyze the  $\mathbb{Q}$ -roots of  $\mathbf{G}^{\text{der}}$ . First, we can make an appropriate choice of  $K_\infty$  such that  $\mathbf{S}_\mathbb{R} < A$  with  $A$  from §4.6.3.

Let  ${}_{\mathbb{R}}\Psi = \{\gamma_1, \dots, \gamma_r\}$  be as in (4.6.13); it is obtained from a set of strongly orthogonal complex roots  $\Psi = \{\alpha_1, \dots, \alpha_r\}$  from (4.6.4). If  $\mathbf{G}_\mathbb{R}^{\text{der}}$  is simple, we described the real roots  ${}_{\mathbb{R}}\Phi = \Phi(A, \mathbf{G}_\mathbb{R}^{\text{der}})$  in terms of  $\gamma_1, \dots, \gamma_r$  in Proposition 4.6.12. It turns out that one can also do this for the rational roots  ${}_{\mathbb{Q}}\Phi := \Phi(\mathbf{S}, \mathbf{G}^{\text{der}})$  when  $\mathbf{G}^{\text{der}}$  is simple.

<sup>[1]</sup>Even strongly,  $\text{Lie}K_\infty$  is orthogonal to  $\text{Lie}\mathbf{S}_\mathbb{R}$  for the Killing form on  $\text{Lie}\mathbf{G}_\mathbb{R}^{\text{der}}$ .

**Proposition 6.1.4.** *There is a partition*

$$\{1, \dots, s = \dim \mathbf{S}\} = I_0 \cup I_1 \cup \dots \cup I_r$$

such that

$$\begin{aligned} X(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{Q} &\simeq \frac{X(A) \otimes_{\mathbb{Z}} \mathbb{Q}}{\{subspace \text{ spanned by } \gamma_i \text{ for } i \in I_0 \text{ and } \gamma_i - \gamma_j \text{ for } i, j \in I_\ell, \ell > 0\}} \\ &\simeq \sum_{j=1}^s \mathbb{Q}\beta_j, \quad \text{where } \beta_j = \text{image of any } \gamma_m \text{ with } m \in I_j. \end{aligned}$$

In particular,  $\mathbf{S} = \{\gamma_i = 1 \text{ for } i \in I_0; \gamma_i = \gamma_j \text{ for } i, j \in I_\ell, \text{ where } \ell > 0\}$ .

**Corollary 6.1.5.** *Recall our assumption that  $\mathbf{G}^{\text{der}}$  is simple. One of the two cases occurs:*

- (Type  $C_s$ )  ${}_{\mathbb{Q}}\Phi = \{\pm \frac{1}{2}(\beta_i + \beta_j) \text{ for } i \geq j, \pm \frac{1}{2}(\beta_i - \beta_j) \text{ for } i > j\}$ .
- (Type  $BC_s$ )  ${}_{\mathbb{Q}}\Phi = \{\pm \frac{1}{2}(\beta_i + \beta_j) \text{ for } i \geq j, \pm \frac{1}{2}(\beta_i - \beta_j) \text{ for } i > j, \pm \frac{1}{2}\beta_i\}$ .

If we order the roots such that  $\beta_1 > \dots > \beta_s$ , then the set of simple roots is:

- (Type  $C_s$ )  ${}_{\mathbb{R}}\Delta = \{\mu_1 := \frac{1}{2}(\beta_1 - \beta_2), \dots, \mu_{s-1} := \frac{1}{2}(\beta_{s-1} - \beta_s), \mu_s := \beta_s\}$ .
- (Type  $BC_s$ )  ${}_{\mathbb{R}}\Delta = \{\mu_1 := \frac{1}{2}(\beta_1 - \beta_2), \dots, \mu_{s-1} := \frac{1}{2}(\beta_{s-1} - \beta_s), \mu_s := \frac{1}{2}\beta_s\}$ .

The proof goes as follows: We have the group-theoretic result that  $\mathbf{G}^{\text{der}} = \text{Res}_{k/\mathbb{Q}} \mathbf{G}'$  for some absolutely simple  $k$ -group  $\mathbf{G}'$  with  $k$  a totally real number field. Then  $\mathbf{G}_{\mathbb{R}}^{\text{der}} = \prod_{\sigma: k \hookrightarrow \mathbb{R}} \mathbf{G}'_\sigma$  with each  $\mathbf{G}'_\sigma$  a group defined over  $\sigma(k) \subseteq \mathbb{R}$ . Then one analyzes each factor and use the Galois action.