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Bigness of the tautological line bundle and degeneracy loci in families of abelian varieties

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Abstract

This note is a compte-rendu of my talks at the conference "Degeneracy of Algebraic Points" in Spring 2023 at the SLMath (previously known as the MSRI) and at the ICCM 2023. It aims to give a quick summary on the degeneracy loci in families of abelian varieties defined and studied in [Gao20a], without going into detailed proofs. Degeneracy loci play a crucial role in the recent solutions of Mazur's Conjecture B, the Uniform Mordell–Lang Conjecture, and the Relative Manin–Mumford Conjecture.

This note is divided into three parts: motivation (why do we study the degeneracy loci), definition (what the degeneracy loci are), and some applications (how the degeneracy loci are applied). For applications, we also explain how degeneracy loci are used to study the denseness of torsion points in families of abelian varieties.

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1 Motivation and bigness of the tautological line bundle

1.1 Background

Let C be a smooth projective geometrically irreducible curve defined over a number field K. In 1983, Faltings proved the famous Mordell conjecture.

Theorem 1.1 ([Fal83]) *If the genus of the curve C is at least 2, then $C(K)$ is a finite set.*

Recently, the following rather uniform bound on $\#C(K)$ was obtained by Dimitrov–Gao–Habegger, which gives an affirmative answer to Mazur's Conjecture B [Maz00].

Theorem 1.2 ([DGH21]) *Assume $g \geq 2$. Then there exists a constant $c = c(g, [K:\mathbb{Q}]) > 0$ such that*

$$\#C(K) \leq c^{1+\mathrm{rank} J(K)}.$$

In fact, [DGH21] proves more: the constant c depends only on g when the curve has large height (in terms of g). Later on, Kühne [Küh21] proved the same bound for curves of small height. Combining these two results, Theorem 1.2 can be improved such that c depends only on g. But we remark that $\#C(K)$ must depend on $[K:\mathbb{Q}]$, and in the improved bound $\mathrm{rank} J(K)$ still depends on K. We do not further recall the historical development of Mazur's Conjecture B in this note but refer to the survey [Gao21b].

In the proof of Theorem 1.2 and in Kühne's result mentioned above, a crucial notion of *non-degenerate subvarieties* in families of abelian varieties, introduced by Habegger [Hab13a] and extensively studied by Gao [Gao20a]. Degeneracy loci were defined in [Gao20a] in the purpose of studying non-degenerate subvarieties and beyond.

1.2 Basic setup

In this subsection, we set up the basic notation. Let $\mathbb{A}_g$ be the moduli space of principally polarized abelian varieties of dimension g with level-4-structure. Then each point $s \in \mathbb{A}_g(\mathbb{C})$ parametrizes a principally polarized abelian variety (A_s, λ_s). Let $\pi\colon \mathfrak{A}_g \to \mathbb{A}_g$ be the universal abelian variety, *i.e.* $\pi^{-1}(s) \cong A_s$ for each $s \in \mathbb{A}_g(\mathbb{C})$. By [Pin89, Chap. 10], there exists a symmetric line bundle $\mathfrak{L}_g$, called the *tautological line bundle* on $\mathfrak{A}_g$ with the following properties:

(i) $\mathfrak{L}_g|_{\pi^{-1}(s)}$ gives the polarization λ_s on A_s, for each $s \in \mathbb{A}_g(\mathbb{C})$;
(ii) $\mathfrak{L}_g$ becomes trivial when restricted to the 0-section of π.

In a more recent work of Yuan–Zhang [YZ21, §6.1], it is proved that $\mathfrak{L}_g$ extends to an *adelic line bundle* $\widetilde{\mathfrak{L}}_g$, which we call the *tautological adelic line bundle*. The authors also defined ampleness, nefness, and bigness for adelic line bundles in the same paper.

At this stage, we are ready to define non-degenerate subvarieties of $\mathfrak{A}_g$.

1.3 Betti map and Betti form

This subsection can be skipped first. However, it is helpful to understand the definition and basic properties of non-degenerate subvarieties. Moreover, non-degenerate subvarieties were first defined using the Betti map [GH19, DGH21], and then using the Betti form [DGH21, Prop. 2.3.(iii)], and now there is a new way using the adelic line bundle [YZ21] which will be presented in the next subsection. And in studying non-degenerate subvarieties, one needs to shift back and forth among all these definitions.

In the course of studying the relative Manin–Mumford conjecture, Masser and Zannier introduced the following *Betti map*; we refer to §3.2 for their work. We give a short explanation here and refer to [Gao20a, §3] for a precise definition. For any $s \in \mathbb{A}_g(\mathbb{C})$, there exists an open neighborhood $\Delta \subseteq (\mathbb{A}_g)^{\mathrm{an}}$ of s with a real-analytic map

$$b_\Delta \colon \mathfrak{A}_g|_\Delta = \pi^{-1}(\Delta) \to \mathbb{T}^{2g}, \tag{1.1}$$

where $\mathbb{T}^{2g}$ is the real torus of dimension $2g$. Up to shrinking Δ we may assume that it is simply-connected. Then one can define a basis $\omega_1(s), \ldots, \omega_{2g}(s)$ of the period lattice of each fiber $s \in \Delta$ as holomorphic functions of s. Now each fiber $\mathfrak{A}_g|_s = \pi^{-1}(s)$ can be identified with the complex torus $\mathbb{C}^g/\mathbb{Z}\omega_1(s) \oplus \cdots \oplus \mathbb{Z}\omega_{2g}(s)$, and each point $x \in \mathfrak{A}_g|_s(\mathbb{C})$ can be expressed as the class of $\sum_{i=1}^{2g} b_i(x)\omega_i(s)$ for real numbers $b_1(x), \ldots, b_{2g}(x)$. Then $b_\Delta(x)$ is defined to be the class of the $2g$-tuple $(b_1(x), \ldots, b_{2g}(x)) \in \mathbb{R}^{2g}$ modulo $\mathbb{Z}^{2g}$.

Let X be an irreducible subvariety of $\mathfrak{A}_g$, and let $x \in X^{\mathrm{sm}}(\mathbb{C})$. The *Betti rank of X at x*, denoted by $\mathrm{rank}(X, x)$, is defined to be the $\mathbb{R}$–rank of the tangent of b_Δ at x, for any Δ contains $\pi(x)$. It is not hard to show that the Betti rank does not depend on the choice of Δ. The following upper bound is trivially true: For each $x \in X^{\mathrm{sm}}(\mathbb{C})$, we have

$$\mathrm{rank}(X, x) \leq 2 \min\{g, \dim X\}. \tag{1.2}$$

Another useful tool is the *Betti form* ω on $\mathfrak{A}_g$, which is a semi-positive $(1,1)$-form; see for example [Mok91, pp. 374]. Its cohomology class $[\omega]$ represents $c_1(\mathfrak{L}_g)$.

Proposition 1.3 *Let X be an irreducible subvariety of $\mathfrak{A}_g$. The followings are equivalent:*

(i) There exists a point $x \in X^{\mathrm{sm}}(\mathbb{C})$ such that $\mathrm{rank}(X, x) = 2 \dim X$*;*
(ii) $\omega|_X^{\wedge \dim X} \not\equiv 0$;
(iii) $\widetilde{\mathfrak{L}}_g|_X$ is big as an adelic line bundle.

The equivalence of (i) and (ii) is by [DGH21, Prop. 2.2.(iii)], and the equivalence of (ii) and (iii) is established in [YZ21, §6.2.2].

1.4 Non-degenerate subvarieties and bigness of the adelic tautological line bundle

Let X be an irreducible subvariety of $\mathfrak{A}_g$.

Definition 1.4 *The subvariety X is said to be* ***non-degenerate*** *if $\widetilde{\mathfrak{L}}_g|_X$ is big as an adelic line bundle.*

We make the following remark before moving on. If X is projective, then $\widetilde{\mathfrak{L}}_g|_X = \mathfrak{L}_g|_X$, and in this case X is non-degenerate if and and only if $\mathfrak{L}_g|_X$ is a big line bundle. In particular, if X is contained in one fiber of $\mathfrak{A}_g \to \mathbb{A}_g$, then X is non-degenerate.

Notice that any X with $\dim X > g$ must be degenerate. This follows immediately from (1.2) and Proposition 1.3 (more precisely, the equivalence of (i) and (iii)). Inspired by this observation, we make the following definition.

Definition 1.5 *The subvariety X is called* ***naively degenerate*** *if $\dim X > g$.*

At this stage, a natural question to ask is whether all degenerate subvarieties of $\mathfrak{A}_g$ are naively degenerate. The answer is *no*, and an example is given by [Gao20a, Eg. 8.2]. Nevertheless, the following theorem asserts that all degenerate subvarieties are built up from naive ones. To ease notation, denote by $S := \pi(X) \subseteq \mathbb{A}_g$ and $\mathcal{A} := \mathfrak{A}_g \times_{\mathbb{A}_g} S$.

Theorem 1.6 ([Gao20a, Thm. 1.1 with $t = 0$]) *Assume $\mathbb{Z}X$ is Zariski dense in $\mathcal{A}$. Then the followings are equivalent:*

(i) X is degenerate, i.e. $\widetilde{\mathfrak{L}}_g|_X$ is not big;

(ii) there exists a quotient abelian scheme $\mathcal{B}$ of $\mathcal{A} \to S$, of relative dimension $g' < g$, such that

$$\dim X - \dim(\iota \circ p)(X) > g - g' \tag{1.3}$$

where $p\colon \mathcal{A} \to \mathcal{B}$ is the quotient and ι is the modular map

$$\begin{tikzcd} \mathcal{A} \arrow[r, two heads, "p"] \arrow[d] & \mathcal{B} \arrow[r, "\iota"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{A}_{g'} \arrow[d] \\ S \arrow[r, "="] & S \arrow[r] & \mathbb{A}_{g'}. \end{tikzcd}$$

An equivalent formulation of (1.3) is: a generic fiber of $(\iota \circ p)|_X$ is naively degenerate in the abelian scheme $\ker p$ over S.

The assumption $\mathbb{Z}X$ being Zariski dense in $\mathcal{A}$ is a mild condition and often we can reduce to this case without loss of generality. It can be checked over the geometric generic fiber of $\mathcal{A} \to S$. Indeed, let η be the generic point of S and fix an algebraic closure of the function field of S. Write $X_{\overline{\eta}}$ for the geometric generic fiber of $\pi|_X$. Then $X_{\overline{\eta}}$ is non-empty if and only if $\pi|_X \colon X \to S$ is dominant. In particular, $\mathcal{A}_{\overline{\eta}}$ is an abelian variety over an algebraically closed field containing the possible reducible $X_{\overline{\eta}}$. Then $\mathbb{Z}X$ is Zariski dense in $\mathcal{A}$ if and only if $X_{\overline{\eta}}$ is non-empty and not contained in a finite union of proper algebraic subgroups of $\mathcal{A}_{\overline{\eta}}$.

Theorem 1.6 is used to construct non-degenerate subvarieties in abelian schemes. Two such constructions will be given at the end of §3.1 and are the ones which are used to prove Mazur's Conjecture B (Theorem 1.2) and beyond and the full *Uniform Mordell–Lang Conjecture* [GGK21].

2 Definition of degeneracy loci

2.1 Weakly special subvarieties

Let A be an abelian variety defined over $\mathbb{C}$, and let $Y \subseteq A$ be an irreducible subvariety. We say that Y is *weakly special* if Y is a coset of A, *i.e.* the translate of an abelian subvariety by a point.

We wish to generalize this definition to families of abelian varieties.

Let $\pi\colon \mathcal{A} \to S$ be an abelian scheme defined over $\mathbb{C}$. Let $\mathcal{C}$ be the maximal isotrivial abelian subscheme, *i.e.* the maximal abelian subscheme such that $\mathcal{C} \times_S S'$

becomes a trivial product $C \times S'$ (with C an abelian variety defined over $\mathbb{C}$) for some finite covering $S' \to S$. By an *iso-constant* of $\mathcal{A}/S$, we mean $\rho(\{c\} \times S')$ for some $c \in C$, where $\rho\colon \mathcal{A} \times_S S' \to \mathcal{A}$ is the natural projection.

Let $Y \subseteq \mathcal{A}$ by an irreducible subvariety. Assume that $\pi|_Y$ is dominant; this assumption can be achieved by shrinking the base S. We say that Y is *weakly special* if the following condition holds true: there exists a finite covering $S' \to S$ such that $Y = \rho(Y')$, where $\rho\colon \mathcal{A}' := \mathcal{A} \times_S S' \to \mathcal{A}$ is the natural projection, where Y' is the translate of an abelian subscheme of $\mathcal{A}'/S'$ by a torsion section and then by an iso-constant section.

We remark that this is not the same definition as weakly special subvarieties of $\mathfrak{A}_g$ by Pink [Pin05a, Defn. 4.1.(b)]. Nevertheless these two definitions are closed related by [Gao17a, Prop. 3.3]; see also [Gao20a, Prop. 5.3].

2.2 Definition and properties

Let X be an irreducible subvariety of $\mathfrak{A}_g$.

Definition 2.1 *For each $t \in \mathbb{Z}$, define*

$$X^{\deg}(t) := \bigcup_{\substack{Y \subseteq X \textit{ irreducible}, \ \dim Y > 0 \\ \dim Y > \dim \langle Y \rangle_{\mathrm{ws}} - \dim \pi(Y) - t}} Y.$$

Here $\langle Y \rangle_{\mathrm{ws}}$ is the smallest weakly special subvariety of $\mathfrak{A}_g \times_{\mathbb{A}_g} \pi(Y) \to \pi(Y)$ which contains Y.

Observe that $X^{\deg}(0) \subseteq X^{\deg}(1)$ immediately by definition.

Remark 2.2 *Assume that $\pi(X)$ is a point, so that X is contained in an abelian variety A. Then the condition $\dim Y > \dim \langle Y \rangle_{\mathrm{ws}} - \dim \pi(Y) - t$ becomes $\dim Y > \dim \langle Y \rangle_{\mathrm{ws}} - t$, because $\dim \pi(Y) = 0$ for any $Y \subseteq X$.*

When $t = 0$, then $\dim Y > \dim \langle Y \rangle_{\mathrm{ws}}$ cannot hold true for any Y because $Y \subseteq \langle Y \rangle_{\mathrm{ws}}$. So in this case $X^{\deg}(0) = \emptyset$. The same is true for any $t \leq 0$.

When $t = 1$, then $\dim Y > \dim \langle Y \rangle_{\mathrm{ws}} - 1$ forces $Y = \langle Y \rangle_{\mathrm{ws}}$ because $Y \subseteq \langle Y \rangle_{\mathrm{ws}}$. So $X^{\deg}(1)$ is the union of all positive dimensional cosets of A which are contained in X. It is then the Ueno locus (or Kawamata locus) of X.

Theorem 2.3 ([Gao20a, Thm. 1.8] and [Gao21a, Prop. 4.2.4]) *The subset $X^{\deg}(t)$ is Zariski closed in X for every $t \in \mathbb{Z}$. Moreover, if X is defined over an algebraically closed field $K \subseteq \mathbb{C}$, so is $X^{\deg}(t)$.*

The following theorem characterizes when the t-th degeneracy locus in large. Denote by $S := \pi(X) \subseteq \mathbb{A}_g$ and $\mathcal{A} := \mathfrak{A}_g \times_{\mathbb{A}_g} S$.

Theorem 2.4 ([Gao20a, Thm. 8.1] for $t \leq 1$ and [GH23a, Prop. 4.3] for general t) *Assume that $\mathbb{Z}X$ is Zariski dense in $\mathcal{A}$. Then the followings are equivalent:*

(i) $X^{\deg}(t) = X$*;*

(ii) there exists a quotient abelian scheme $\mathcal{B}$ of $\mathcal{A} \to S$, of relative dimension $g' < g$, such that

$$\dim X - \dim(\iota \circ p)(X) > \max\{0, g - g' - t\}, \tag{2.1}$$

where $p \colon \mathcal{A} \to \mathcal{B}$ is the quotient and ι is the modular map

$$\begin{tikzcd} \mathcal{A} \arrow[r, two heads, "p"] \arrow[d] & \mathcal{B} \arrow[r, "\iota"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{A}_{g'} \arrow[d] \\ S \arrow[r, "="] & S \arrow[r] & \mathbb{A}_{g'}. \end{tikzcd}$$

Notice that (ii) of Theorem 2.4 in the case $t = 0$ is the same as (ii) of Theorem 1.6. This is not a coincidence. In fact, Theorem 1.6 is proved as a combination of Theorem 2.4 and Proposition 3.1 below.

We point out that the statement of [Gao20a, Prop. 7.6.(ii)] should be corrected to "$E_{\max\{0, g_N - t\}} \subseteq X^{\deg}(t)$". We need to take $\max\{0, g_N - t\}$ instead of just $g_N - t$ because in the definition of $X^{\deg}(t)$, each Y in the union has $\dim > 0$. This does not affect the results in [Gao20a], *i.e.* Theorem 2.4 for $t \leq 1$ or Theorem 2.3. But for Theorem 2.4 with $t \geq 2$ we need replace $g - g' - t$ by $\max\{0, g - g' - t\}$ in (2.1) (which equals $g - g' - t$ for $t \leq 1$ because $g' < g$).

3 Applications

In this section, we give some applications of $X^{\deg}(t)$, notably when t is 0 and 1. We also briefly explain some other cases.

3.1 Case $t = 0$: bigness of the tautological adelic line bundle

As an application of the mixed Ax–Schanuel theorem for $\mathfrak{A}_g$ [Gao20b], it is proved [Gao20a, Prop. 6.1] that:

Proposition 3.1 *Let X be an irreducible subvariety of $\mathfrak{A}_g$. Then X is degenerate if and only if $X^{\deg}(0) = X$.*

Thus Theorem 1.6 follows from Theorem 2.4 with $t = 0$.

This is also in conformity with our discussions in the case where X is contained in one fixed abelian variety A (equivalently, in a fiber of $\mathfrak{A}_g \to \mathbb{A}_g$). On the one hand, we have seen below Definition 1.4 that such X is always non-degenerate. On the other hand, we have shown that $X^{\deg}(0) = \emptyset$ in Remark 2.2, so in view of Proposition 3.1 such X is always non-degenerate.

Now let us explain the crucial construction for the proof of Mazur's Conjecture B (Theorem 1.2). Let $\mathbb{M}_g$ be the moduli space of pointed curves of genus g with level-4-structure, then each point $s \in \mathbb{M}_g(\mathbb{C})$ parametrizes an irreducible smooth projective genus C_s of genus g and a point $P_s \in C_s(\mathbb{C})$. There exists a universal curve $\mathfrak{C}_g \to \mathbb{M}_g$, *i.e.* the fiber over s is the curve C_s.

Let $\mathfrak{J}_g := \mathrm{Jac}(\mathfrak{C}_g/\mathbb{M}_g)$. Then there is a natural $\mathbb{M}_g$-immersion $\mathfrak{C}_g \to \mathfrak{J}_g$, which over each $s \in \mathbb{M}_g(\mathbb{C})$ is the Abel–Jacobi embedding of C_s into $\mathrm{Jac}(C_s)$ via P_s. For each integer $M \geq 1$, let $\mathfrak{C}_g^{[M]}$ be the M-th fibered power of $\mathfrak{C}_g \to \mathbb{M}_g$. By abuse of notation, we also use $\mathfrak{C}_g^{[M]}$ to denote the image of

$$\mathfrak{C}_g^{[M]} \subseteq \mathfrak{J}_g^{[M]} \to \mathfrak{A}_{gM}, \tag{3.1}$$

where the second morphism is the modular map. Theorem 1.6 yields the following construction.

Theorem 3.2 ([Gao20a, Thm. 1.2', Cor. A.4 and Rem. A.5]) *Assume $g \geq 2$. For each irreducible S in the Torelli locus in $\mathbb{A}_g$, the subvariety $\mathfrak{C}_g^{[M]} \times_{\mathbb{A}_g} S$ of $\mathfrak{A}_{gM}$ is non-degenerate if $M \geq 3g-2$. Moreover, $\mathfrak{C}_g^{[M]}$ is non-degenerate if $M \geq 4$.*

It is this construction which was used in [DGH21] and [Küh21].

Another construction derived from Theorem 1.6 is [GGK21, Prop. 3.4] about the Hilbert scheme and the universal family, inspired by Ge [Ge24, Prop. 3.4]. The non-degenerate subvariety thus obtained plays a crucial role in the solution of the full *Uniform Mordell–Lang Conjecture* by Gao–Ge–Kühne.

Theorem 3.3 ([GGK21]) *Let $g \geq 1$ and $l \geq 1$ be two integers. Then there exists a constant $c = c(g,l)$ with the following property. For any abelian variety A of dimension g defined over $\mathbb{C}$, any irreducible subvariety $X \subseteq A$ of degree $\leq l$, and any finite rank subgroup $\Gamma < A(\mathbb{C})$, there exists $\leq n := c^{1+\mathrm{rk}\Gamma}$ points $x_1, \ldots, x_n \in A(\mathbb{C})$ and abelian subvarieties $B_1, \ldots, B_n$ such that $x_i + B_i \subseteq X$ for each $i \in \{1, \ldots, n\}$ and*

$$X(\mathbb{C}) \cap \Gamma = \bigcup_{i=1}^{n} (x_i + B_i)(\mathbb{C}) \cap \Gamma.$$

This theorem is a uniform quantitative version of the Mordell–Lang conjecture proved by Faltings [Fal91], and his proof follows the approach of the (second) proof of the Mordell conjecture by Vojta [Voj91]. This uniform version was conjectured by David–Philippons [DP07].

We finish this subsection with the following remark. Yuan [Yua21] gave a second proof to Theorem 1.2 and the more general Uniform Mordell–Lang Conjecture for curves embedded into their Jacobians [DGH21, Küh21] (see [Gao21b] for the precise statement). It shares some similarities with but does not use the construction of Theorem 3.2; instead it relies on previous work of Zhang, Cinkir, and de Jong [Zha93, Zha10, Cin11, dJ18] for lower bounds on the self-intersection numbers of the admissible canonical bundles of curves over global fields, apart from the adelic line bundles by Yuan–Zhang [YZ21]. However, little is known in this direction for subvarieties of higher dimension, and it seems very hard to generalize Yuan's proof to obtain the full Uniform Mordell–Lang Conjecture (Theorem 3.3).

3.2 Case $t = 1$: denseness of torsion points in families of abelian varieties, part I

The following *Relative Manin–Mumford Conjecture* was recently proved, and a key new ingredient compared with previous works is the application of $X^{\deg}(1)$.

Let $\mathcal{A} \to S$ be an abelian scheme of relative dimension g defined over $\overline{\mathbb{Q}}$, and let $X \subseteq \mathcal{A}$ be an irreducible subvariety. Use $\mathcal{A}_{\mathrm{tor}}$ to denote the set of fiberwise torsion points, *i.e.*

$$\mathcal{A}_{\mathrm{tor}} = \left\{ x \in \mathcal{A}(\overline{\mathbb{Q}}) : [N]x \text{ is in the zero section for some } N \in \mathbb{Z} \setminus \{0\} \right\}.$$

Theorem 3.4 ([GH23b, Thm. 1.1]) *Assume that* $\mathbb{Z}X$ *is Zariski dense in* $\mathcal{A}$. *If* $X \cap \mathcal{A}_{\mathrm{tor}}$ *is Zariski dense in* X*, then* $\dim X \geq g$.

This theorem, known as the *Relative Manin–Mumford Conjecture*, was inspired by S. Zhang's ICM talk [Zha98] and proposed by Pink [Pin05b, Conj. 6.2] and Zannier [Zan12]. In the case $\dim X = 1$ it was proved in a series of papers by Corvaja, Masser and Zannier [MZ08, MZ12, MZ14, MZ15, CMZ18, MZ20]. See also Stoll [Sto17] for an explicit case. When $\mathcal{A}$ is a fibered product of families of elliptic curves, for surfaces some results are due to Habegger [Hab13b] and Corvaja–Tsimerman–Zannier [CTZ23], and in general by Kühne [Küh23].

Before moving on to explain the proof of Theorem 3.4, let us briefly explain how it implies the following *Uniform Manin–Mumford Conjecture* for curves embedded into their Jacobians.

Corollary 3.5 *Assume* $g \geq 2$*. There exists a constant* $c(g) > 0$ *with the following property. For each smooth projective curve* C *defined over* $\mathbb{C}$ *and each* $P \in C(\mathbb{C})$*, the size of the torsion packet of* C *containing* P *is at most* $c(g)$.

The torsion packet of C containing P is the set $(C - P)(\mathbb{C}) \cap \mathrm{Jac}(C)_{\mathrm{tor}}$, where $C - P$ is the image of the Abel–Jacobi embedding of C into $\mathrm{Jac}(C)$ via P.

This Uniform Manin–Mumford Conjecture for curves embedded into their Jacobian was proved by Kühne [Küh21], using his equidistribution theorem and our construction Theorem 3.2. A second proof of was given by Yuan in [Yua21], based on the theory of adelic line bundles over quasi-projective varieties of Yuan–Zhang [YZ21]. Prior to Kühne's proof of the full conjecture, DeMarco–Krieger–Ye [DKY20] proved the case where $g = 2$ and C is bi-elliptic, using method of arithmetic dynamical systems. .

The deduction of Corollary 3.5 from Theorem 3.4 [GH23b, §8] is not complicated. Here is a brief sketch. First, Corollary 3.5 can be easily reduced with $\mathbb{C}$ replaced by $\overline{\mathbb{Q}}$ by a specialization argument of Masser [Mas89]. Next, we use (3.1) with $M = 5$. A simple computation shows that $\dim \mathfrak{C}_g^{[5]} = 3g - 2 + 5 < 5g$ if $g \geq 2$. Hence by Theorem 3.4, the fiberwise torsion points are not Zariski dense in $\mathfrak{C}_g^{[5]}$; in other words, the Zariski closure Z of the fiberwise torsion points lying in $\mathfrak{C}_g^{[5]}$ is a proper subvariety of $\mathfrak{C}_g^{[5]}$. Thus using a lemma in flavor of zero estimates [DGH21, Lem. 6.4], one can show that for a generic $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$, the number of fiberwise torsion points on $(\mathfrak{C}_g)_s = C_s - P_s \subseteq \mathrm{Jac}(C_s)$ is bounded above in terms of $\deg Z_s$, which is uniformly bounded for $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$. In the end, we finish the proof by a Noetherian induction on the base $\mathbb{M}_g$.

Before moving on, we remark that the way to apply the degeneracy loci in this problem is genuinely different from the previous applications. In studying Uniform Mordell–Lang related questions, one always started by constructing a non-degenerate subvariety to apply other tools such as height inequality or equidistribution, and only $X^{\deg}(0)$ is used. For Theorem 3.4 we are not allowed to do such a construction. Instead, we study the degeneracy loci more carefully and use both $X^{\deg}(0)$ and $X^{\deg}(1)$.

As a preparation, it is not hard to reduce the theorem to be case where $S \subseteq \mathbb{A}_g$ and $\mathcal{A} = \mathfrak{A}_g \times_{\mathbb{A}_g} S$. Then the proof of Theorem 3.4 has two parts and $X^{\deg}(1)$ is both parts. First, we prove the following proposition.

Proposition 3.6 *Assume that $\mathbb{Z}X$ is Zariski dense in $\mathcal{A}$. If $X \cap \mathcal{A}_{\mathrm{tor}}$ is Zariski dense in X, then $X^{\deg}(1) = X$.*

The proof of Proposition 3.6 is divided into two cases: either X is degenerate or not. If X is degenerate, then $X = X^{\deg}(0)$ by Proposition 3.1, and hence $X^{\deg}(1) = X$ because $X^{\deg}(0) \subseteq X^{\deg}(1)$ by definition. If X is non-degenerate, then we follow the Pila–Zannier strategy to prove Proposition 3.6. The ingredient of this proof includes a quantified version of Masser's result [Mas84] on Galois orbits of torsion points on abelian varieties due to David [Dav93] or Gaudron and Rémond [Rém18, GR22], the height inequality of Dimitrov–Gao–Habegger [DGH21, Thm. 1.6 and B.1], Habegger–Pila's semi-invariant version [HP16, Cor. 7.2] of the the Pila–Wilkie counting theorem, and Gao's mixed Ax–Schanuel theorem [Gao20b].

Next, we use the criterion Theorem 2.4 with $t = 1$ to deduce Theorem 3.4 from Proposition 3.6. The proof is by induction on g. If $g = 0$, then Theorem 3.4 trivially holds true. For arbitrary g, assume Proposition 3.6 holds true for our X in question. Then we apply Theorem 2.4 with $t = 1$ to X, and obtain

$$\begin{tikzcd} \mathcal{A} \arrow[r, two heads, "p"] \arrow[d] & \mathcal{B} \arrow[r, "\iota"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{A}_{g'} \arrow[d] \\ S \arrow[r, "="] & S \arrow[r] & \mathbb{A}_{g'}. \end{tikzcd}$$

with $g' < g$ and

$$\dim X \geq \dim(\iota \circ p)(X) + (g - g'). \tag{3.2}$$

It is not hard to check that the fiberwise torsion points are still Zariski dense in $(\iota \circ p)(X)$. Then we can apply induction hypothesis to $(\iota \circ p)(X)$ since $g' < g$, and obtain $\dim(\iota \circ p)(X) \geq g'$. So $\dim X \geq g$ by (3.2), and we are done.

3.3 Case $t = g - \dim X \leq 0$: denseness of torsion points in families of abelian varieties, part II

In this application, we assume $\dim X \geq g$, so the t in question is non-positive.

In this previous subsection, we sketched the key points to prove Theorem 3.4 with $\mathbb{C}$ replaced by $\overline{\mathbb{Q}}$. We wish to do a specialization argument to pass from $\overline{\mathbb{Q}}$ to $\mathbb{C}$. Unlike for Uniform Mordell–Lang, this specialization argument is highly non-trivial. In [GH23b, §10], we solve this problem by a double induction,[1] and a key point is to apply Theorem 2.4 for $t = g - \dim X$. In the end, we obtain a stronger result.

Theorem 3.7 ([GH23b, Thm. 1.3] and [Gao20a, Thm. 1.1 with $l = g$]) *Assume that $\mathbb{Z}X$ is Zariski dense in $\mathcal{A}$. Then the followings are equivalent:*

(i) $X(\mathbb{C}) \cap \mathcal{A}_{\mathrm{tor}}$ *is Zariski dense in* X*;*
(ii) $\mathrm{rank}(X, x) = 2g$ *for some* $x \in X^{\mathrm{sm}}(\mathbb{C})$*;*
(iii) for each quotient abelian subscheme $\mathcal{B}$ *of* $\mathcal{A}/S$*, we have* $\dim(\iota \circ p)(X) \geq g'$ *where*

$$\begin{tikzcd} \mathcal{A} \arrow[r, two heads, "p"] \arrow[d] & \mathcal{B} \arrow[r, "\iota"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{A}_{g'} \arrow[d] \\ S \arrow[r, "="] & S \arrow[r] & \mathbb{A}_{g'}. \end{tikzcd}$$

Notice that Theorem 3.7 implies Theorem 3.4 immediately by (1.2).

The equivalence of (ii) and (iii) is [Gao20a, Thm. 1.1 with $l = g$].

The proof of (ii) implying (i) is not complicated and a proof can be found in [ACZ20, Prop. 2.1.1]. We also point out that [ACZ20] was the first paper which studied this denseness question and obtained some nice results, especially when $\mathrm{End}(\mathcal{A}/S) = \mathbb{Z}$.

Now let us explain how (i) implies (ii). The proof uses Theorem 3.4 and $X^{\mathrm{deg}}(g - \dim X)$. This implication is in fact a step of the specialization argument for Theorem 3.4.

Let X be as in Theorem 3.7 such that $X(\mathbb{C}) \cap \mathcal{A}_{\mathrm{tor}}$ is Zariski dense in X. Thus $\dim X \geq g$ by Theorem 3.4. Assume $\mathrm{rank}(X, x) < 2g$ for all $x \in X^{\mathrm{sm}}(\mathbb{C})$ and we wish to get a contradiction. [Gao20a, Prop. 6.1] implies then $X^{\mathrm{deg}}(g - \dim X) = X$. So by Theorem 2.4 with $t = g - \dim X$, we obtain

$$\begin{tikzcd} \mathcal{A} \arrow[r, two heads, "p"] \arrow[d] & \mathcal{B} \arrow[r, "\iota"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{A}_{g'} \arrow[d] \\ S \arrow[r, "="] & S \arrow[r] & \mathbb{A}_{g'}. \end{tikzcd}$$

with $g' < g$ and $\dim(\iota \circ p)(X) < g'$. On the other hand, it is not hard to check that the fiberwise torsion points are still Zariski dense in $(\iota \circ p)(X)$. Hence $\dim(\iota \circ p)(X) \geq g'$ by Theorem 3.4. This is a contradiction, and we are done.

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[1] A different argument for this specialization is given by [CTZ23, App. A].

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