Chapter 3

Shimura data and Shimura varieties

3.1 Basic definitions

3.1.1 Shimura data

Definition 3.1.1. *A* **Shimura datum** *is a pair* $(\mathbf{G}, X)$ *where*

- $\mathbf{G}$ *is a reductive group defined over* $\mathbb{Q}$*,*
- X *is a* $\mathbf{G}(\mathbb{R})$*-orbit in* $\mathrm{Hom}(\mathbb{S}, \mathbf{G}_{\mathbb{R}})$

such that for one (and hence all) $h \in X$*, we have*

(SV1) the Hodge structure $\mathrm{Ad} \circ h$ *on* $\mathrm{Lie}\mathbf{G}$ *has type* $(-1,1)+(0,0)+(1,-1)$*,*

(SV2) $\mathrm{Int}(h(\sqrt{-1}))$ *is a Cartan involution of* $\mathbf{G}^{\mathrm{ad}}_{\mathbb{R}}$*,*

(SV3) for every $\mathbb{Q}$*-simple factor* $\mathbf{H}$ *of* $\mathbf{G}^{\mathrm{ad}}$*, the morphism* $\mathbb{S} \xrightarrow{h} \mathbf{G}_{\mathbb{R}} \to \mathbf{H}_{\mathbb{R}}$ *is non-trivial.*

A **(Shimura) morphism** *between two Shimura data* $\rho\colon (\mathbf{G}', X') \to (\mathbf{G}, X)$ *is a morphism* ρ *on the underlying groups such that* $\rho \circ h \in X$ *for all* $h \in X'$*. In particular, we call the image of such a Shimura morphism to be a* **sub-Shimura datum** *of* $(\mathbf{G}, X)$*.*

The main difference of a Shimura datum and the pair (G, X^+) from §2.3 is the definition field of the group (over $\mathbb{Q}$ or over $\mathbb{R}$). A similar assumption to (SV3) for (G, X^+) has been discussed in Remark 2.3.3. By Theorem 2.3.1, each connected component X^+ of X is a Hermitian symmetric domain (and the complex structure on X is $\mathbf{G}(\mathbb{R})$-invariant). By Proposition 1.3.5, each representation V of $\mathbf{G}$ gives rise to a $\mathbb{Q}$-VHS on X^+ by (SV1), which furthermore carries $\mathbb{R}$-polarization by Proposition 2.2.6 and (SV2).[1]

The following two further assumptions guarantee that this $\mathbb{Q}$-VHS carries a $\mathbb{Q}$-polarization. Notice that they may not be satisfied by an arbitrary Shimura datum.

(SV4) the morphism $w_h\colon \mathbb{G}_{\mathrm{m},\mathbb{R}} \to Z(\mathbf{G})_{\mathbb{R}}$ is defined over $\mathbb{Q}$,

(SV2') $\mathrm{Int}(h(\sqrt{-1}))$ is a Cartan involution of $\mathbf{G}_{\mathbb{R}}/w_h(\mathbb{G}_{\mathrm{m},\mathbb{R}})$.

Example 3.1.2 (0-dimensional Shimura datum)**.** *The set* X *is a finite set if and only if* $\mathbf{G}$ *is abelian (and hence an algebraic torus). This case shows up when we study CM abelian varieties.*

[1] (SV1) implies that $w_h\colon \mathbb{G}_{\mathrm{m}} \xrightarrow{w} \mathbb{S} \xrightarrow{h} \mathbf{G}_{\mathbb{R}}$ factors through $Z(\mathbf{G})_{\mathbb{R}}$, so we can apply Proposition 2.2.6.

Example 3.1.3 (Siegel Shimura datum). *Let us take the example of the Siegel case from Example 2.3.1, except that the vector space and the groups are defined over $\mathbb{Q}$. More precisely, $V = \mathbb{Q}^{2d}$ and $\psi\colon V \times V \to \mathbb{Q}$ is $(x,y) \mapsto x^{\mathrm{t}} J y$ with $J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}$. The $\mathbb{Q}$-group is*

$$\begin{aligned}\mathbf{GSp}(\psi) = \mathbf{GSp}_{2d} &:= \left\{ g \in \mathrm{GL}(V) : \psi(gx, gy) = c\psi(x,y) \text{ for some } c \in \mathbb{Q}^{\times} \right\} \\ &= \left\{ g \in \mathrm{GL}_{2d,\mathbb{Q}} : gJg^{\mathrm{t}} = cJ \text{ for some } c \in \mathbb{Q}^{\times} \right\},\end{aligned}$$

and $h_0\colon \mathbb{S} \to \mathbf{GSp}_{2d,\mathbb{R}}$ maps $a + b\sqrt{-1} \mapsto aI_{2d} + bJ$. The derived subgroup is $\mathbf{Sp}(\psi) = \mathbf{Sp}_{2d}$ by requesting the $c \in \mathbb{Q}^{\times}$ in the definition to be 1.

The $\mathbf{G}(\mathbb{R})$-orbit is $\mathbf{GSp}_{2d}(\mathbb{R})h_0 \subseteq \mathrm{Hom}(\mathbb{S}, \mathbf{GSp}_{2d,\mathbb{R}})$. Under the identification similar to (2.3.1), we have

$$\mathbf{GSp}_{2d}(\mathbb{R})h_0 \xrightarrow{\sim} \mathfrak{H}_d^{\pm} := \left\{ \tau \in \mathrm{Mat}_{d\times d}(\mathbb{C}) : \tau = \tau^{\mathrm{t}},\ \text{either } \mathrm{Im}\tau > 0 \text{ or } \mathrm{Im}\tau < 0 \right\}. \tag{3.1.1}$$

Then $(\mathbf{GSp}_{2d}, \mathfrak{H}_d^{\pm})$ is a Shimura datum by Example 2.3.1 (see Remark 2.3.3 for (SV3)). It is called the **Siegel Shimura datum**. *Moreover, (SV4) and (SV2') are easily seen to be also satisfied. In fact, V is a representation of $\mathbf{GSp}_{2d}$, and ψ is the desired $\mathbb{Q}$-polarization on the induced $\mathbb{Q}$-VHS.*

Next we present an example where (SV4) and (SV2') are not satisfied. We also see in this example that two Shimura data may have the same underlying $\mathbb{R}$-group and the same underlying space, but the $\mathbb{Q}$-groups are different.

Example 3.1.4 (Shimura curves). *Let B be a simple quaternion algebra over a totally real number field F. Assume that B is split at exactly one real place of F, i.e. there exists a real embedding $\sigma\colon K \to \mathbb{R}$ such that*

$$B_\sigma \simeq \begin{cases} \mathrm{M}_2(\mathbb{R}) & \text{if } \sigma = \sigma_0 \\ \mathbb{H} & \text{otherwise} \end{cases}$$

for all read embeddings $\sigma\colon K \to \mathbb{R}$, where $\mathbb{H}$ is the Hamilton quaternion algebra over $\mathbb{R}$.

Define the $\mathbb{Q}$-group $\mathbf{G}$

$$\mathbf{G}(R) := (B \otimes_{\mathbb{Q}} R)^{\times} \quad \text{for all } \mathbb{Q}\text{-algebra } R,$$

and let

$$h_0\colon \mathbb{S} \to \mathbf{G}_{\mathbb{R}} \simeq \mathrm{GL}_{2,\mathbb{R}} \times \mathbb{H}^{\times} \times \cdots \times \mathbb{H}^{\times}, \qquad a + b\sqrt{-1} \mapsto \left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}, 1, \ldots, 1 \right).$$

Thus all but the first components of $\mathbf{G}(\mathbb{R})h_0$ are the identity map, and so $\mathbf{G}(\mathbb{R})h_0 \subseteq \mathrm{Hom}(\mathbb{S}, \mathbf{G}_{\mathbb{R}})$ is isomorphic to $\mathfrak{H}_1^{\pm}$, via an isomorphism similar to (3.1.1) (with $d = 1$). Both (SV1) and (SV2) hold true for the pair $(\mathbf{G}, \mathfrak{H}_1^{\pm})$ similarly to the Siegel case. To see (SV3), it suffices to observe that $\mathbf{G}^{\mathrm{ad}}$ is a simple group because B is a simple quaternion algebra over F.

So $(\mathbf{G}, \mathfrak{H}_1^{\pm})$ is a Shimura datum. However, if $[F : \mathbb{Q}] > 1$, then (SV4) and (SV2') do not hold true, by looking at the action of $\mathrm{Aut}(\mathbb{R}/\mathbb{Q})$.

And even in the case $F = \mathbb{Q}$, the group $\mathbf{G}$ is not necessarily $\mathbf{GL}_2$. So $(\mathbf{G}, \mathfrak{H}_1^{\pm})$ needs not be the Siegel Shimura datum in this case.

3.1.2 Shimura varieties

Denote by $\mathbb{A}_{\mathrm{f}} \subseteq \prod_{p\in M_{\mathbb{Q},\mathrm{f}}} \mathbb{Q}_p$ the ring of finite adèles over $\mathbb{Q}$, and by $\widehat{\mathbb{Z}} := \prod \mathbb{Z}_p$. Then $\widehat{\mathbb{Z}}$ is a (maximal) compact open subgroup of $\mathbb{A}_{\mathrm{f}}$, and $\mathbb{Q}$ is dense in $\mathbb{A}_{\mathrm{f}}$.

Let $(\mathbf{G}, X)$ be a Shimura datum. Then $\mathbf{G}(\mathbb{Q})$ acts on X by definition of Shimura data, and consider the action of $\mathbf{G}(\mathbb{Q})$ on $\mathbf{G}(\mathbb{A}_{\mathrm{f}})$ by multiplication on the left.

Definition 3.1.5. *Let $(\mathbf{G}, X)$ be a Shimura datum. A* **Shimura variety** *is a double coset*

$$\mathrm{Sh}_K(\mathbf{G}, X) := \mathbf{G}(\mathbb{Q})\backslash X \times \mathbf{G}(\mathbb{A}_{\mathrm{f}})/K$$

where $K \subseteq \mathbf{G}(\mathbb{A}_{\mathrm{f}})$ is a compact open subset. Here $\mathbf{G}(\mathbb{Q})$ acts on both X and $\mathbf{G}(\mathbb{A}_{\mathrm{f}})$ on the left as above, and K acts on $\mathbf{G}(\mathbb{A}_{\mathrm{f}})$ by the multiplication on the right; i.e. $q(x, g)k = (q \cdot x, qgk)$ for all $q \in \mathbf{G}(\mathbb{Q})$, $(x, g) \in X \times \mathbf{G}(\mathbb{A}_{\mathrm{f}})$ and $k \in K$.

We will prove in this course that the double coset $\mathrm{Sh}_K(\mathbf{G}, X)$ is the set of $\mathbb{C}$-points of an algebraic variety. This justifies the name of Shimura variety.

Example 3.1.6. *In the Siegel example above, the group $\mathbf{GSp}_{2d}$ is defined over $\mathbb{Z}$; indeed we can take V to be $\mathbb{Z}^{2d}$ and ψ maps $V \times V$ to $\mathbb{Z}$. Then $\mathbf{GSp}_{2d}(\widehat{\mathbb{Z}})$ is a compact open subgroup of $\mathbf{GSp}_{2d}(\mathbb{A}_{\mathrm{f}})$. Other compact open subgroups include gKg^{-1} for any $g \in \mathbf{GSp}_{2d}(\mathbb{A}_{\mathrm{f}})$ and any finite-indexed subgroup K of $\mathbf{GSp}_{2d}(\widehat{\mathbb{Z}})$. We will come back to this example in §3.3 and prove that the Siegel Shimura varieties are moduli spaces of abelian varieties.*

Definition 3.1.7. *A* **(Shimura) morphism** *$[\rho]\colon \mathrm{Sh}_{K'}(\mathbf{G}', X') \to \mathrm{Sh}_K(\mathbf{G}, X)$ between two Shimura varieties is a morphism of Shimura data $\rho\colon (\mathbf{G}', X') \to (\mathbf{G}, X)$ such that $\rho(K') \subseteq K$.*

Example 3.1.8. *Let $\mathrm{Sh}_K(\mathbf{G}, X)$ be a Shimura variety.*

Let $K' \subseteq K$ be another compact open subgroup of $\mathbf{G}(\mathbb{A}_{\mathrm{f}})$. Then the identity map on $(\mathbf{G}, X)$ induces a Shimura morphism $\mathrm{Sh}_{K'}(\mathbf{G}, X) \to \mathrm{Sh}_K(\mathbf{G}, X)$, with finite fibers since K' must have finite index in K. In fact, this is finite morphism in the category of algebraic varieties.

Let $g \in \mathbf{G}(\mathbb{A}_{\mathrm{f}})$. Then the Shimura morphism $(\mathbf{G}, X) \to (\mathbf{G}, X)$, $(g', x) \mapsto (gg'g^{-1}, g \cdot x)$, induces a Shimura morphism $[g\cdot]\colon \mathrm{Sh}_K(\mathbf{G}, X) \to \mathrm{Sh}_{gKg^{-1}}(\mathbf{G}, X)$ which is an isomorphism. More generally, if K' is a compact open subgroup of $\mathbf{G}(\mathbb{A}_{\mathrm{f}})$ such that $gK'g^{-1} \subseteq K$, then we have a Shimura morphism $[g\cdot]\colon \mathrm{Sh}_{K'}(\mathbf{G}, X) \to \mathrm{Sh}_K(\mathbf{G}, X)$ which is a finite morphism.

Example 3.1.9 (Hecke operator)**.** *Let $\mathrm{Sh}_K(\mathbf{G}, X)$ be a Shimura variety.*

Any $g \in \mathbf{G}(\mathbb{A}_{\mathrm{f}})$ induces a correspondence on $\mathrm{Sh}_K(\mathbf{G}, X)$ as follows. Write $K' := K \cap g^{-1}Kg$ for simplicity; it is a compact open subgroup of $\mathbf{G}(\mathbb{A}_{\mathrm{f}})$ and $gK'g^{-1} \subseteq K$. We have Shimura morphisms

$$\begin{tikzcd} & \mathrm{Sh}_{K'}(\mathbf{G}, X) \arrow[dl, "{[1]}"'] \arrow[dr, "{[g\cdot]}"] & \\ \mathrm{Sh}_K(\mathbf{G}, X) & & \mathrm{Sh}_K(\mathbf{G}, X) \end{tikzcd}$$

where the left one is induces by identity on $(\mathbf{G}, X)$. Both are finite morphisms, so we have a correspondence on $\mathrm{Sh}_K(\mathbf{G}, X)$, which is called the **Hecke correspondence/operator** *and denoted by T_g.*

Definition 3.1.10. *Let $\mathrm{Sh}_K(\mathbf{G}, X)$ be a Shimura variety. We call any irreducible component of $(T_g \circ [\rho])(\mathrm{Sh}_{K'}(\mathbf{G}', X'))$, where $[\rho]$ is a Shimura morphism and $g \in \mathbf{G}(\mathbb{A}_{\mathrm{f}})$, to be a* **special subvariety** *of $\mathrm{Sh}_K(\mathbf{G}, X)$. A special subvariety of dimension 0 is called a* **special point***.*

Of course in the definition of special subvarieties, it suffices to consider the Shimura morphisms arising from sub-Shimura data of $(\mathbf{G}, X)$. Thus special points arise from sub-Shimura data $(\mathbf{T}, X_{\mathbf{T}})$ of $(\mathbf{G}, X)$ where $\mathbf{T}$ is an algebraic torus.

3.2 Decomposition of Shimura varieties into Hermitian locally symmetric domains

Let $(\mathbf{G}, X)$ be a Shimura datum. Then any connected component X is a Hermitian symmetric domain. Fix one such X^+.

Let K be a compact open subgroup of $\mathbf{G}(\mathbb{A}_\mathrm{f})$. Then we have a Shimura variety $\mathrm{Sh}_K(\mathbf{G}, X)$ defined as the double coset $\mathbf{G}(\mathbb{Q})\backslash X \times \mathbf{G}(\mathbb{A}_\mathrm{f})/K$. We wish to prove that this double coset is the $\mathbb{C}$-points of an algebraic variety.

In this section, we start with the first step, by endowing $\mathrm{Sh}_K(\mathbf{G}, X)$ with a structure of complex varieties.

Theorem 3.2.1. *There exists a finite-indexed subgroup K' of K such that*

$$\mathrm{Sh}_{K'}(\mathbf{G}, X) \simeq \bigsqcup_{g \in \mathcal{C}} \Gamma_g \backslash X^+, \tag{3.2.1}$$

for a finite set $\mathcal{C} \subseteq \mathbf{G}(\mathbb{A}_\mathrm{f})$, with each Γ_g a torsion-free discrete group acting on X^+.

The actual decomposition will be given later on (3.2.3), where the definitions of $\mathcal{C}$ and Γ_g are given. At this stage, let us make the following observation: since Γ_g is torsion-free discrete, the quotient $\Gamma_g\backslash X^+$ has a natural structure of complex manifolds and even more is a Hermitian locally symmetric domain. So $\mathrm{Sh}_{K'}(\mathbf{G}, X)$ is a finite disjoint union of Hermitian locally symmetric domains. As for $\mathrm{Sh}_K(\mathbf{G}, X)$, the finite-to-1 map $\mathrm{Sh}_{K'}(\mathbf{G}, X) \to \mathrm{Sh}_K(\mathbf{G}, X)$ then makes $\mathrm{Sh}_K(\mathbf{G}, X)$ into a finite union of complex orbifolds.

3.2.1 Two approximation theorems for algebraic groups

Let $\mathbf{H}$ be an algebraic group defined over $\mathbb{Q}$. We will use the following approximation theorems.

- (*Real Approximation*) $\mathbf{H}(\mathbb{Q})$ is dense in $\mathbf{H}(\mathbb{R})$.
- (*Strong Approximation*) If $\mathbf{H}$ is semi-simple and simply-connected, then $\mathbf{H}(\mathbb{Q})$ is dense in $\mathbf{H}(\mathbb{A}_\mathrm{f})$.

3.2.2 Preparation and adjoint Shimura data

Now let us introduce the *adjoint Shimura datum* $(\mathbf{G}^{\mathrm{ad}}, \overline{X})$ of $(\mathbf{G}, X)$. Take $h \in X^+$. Then h induces a morphism

$$\overline{h} \colon \mathbb{S} \xrightarrow{h} \mathbf{G}_\mathbb{R} \to \mathbf{G}^{\mathrm{ad}}_\mathbb{R}.$$

Hence we obtain a $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})$-orbit $\overline{X} := \mathbf{G}^{\mathrm{ad}}(\mathbb{R})\overline{h}$ in $\mathrm{Hom}(\mathbb{S}, \mathbf{G}^{\mathrm{ad}}_\mathbb{R})$, with a natural map $X \to \overline{X}$. The image of X^+ in $\overline{X}$ is connected, and the following lemma (applied to $G = \mathbf{G}(\mathbb{R})$)[2] easily implies that this image is again a connected component of $\overline{X}$. So by abuse of notation, we will also use X^+ to denote a connected component of $\overline{X}$.

Lemma 3.2.2. *For any algebraic group G over $\mathbb{R}$, the adjoint quotient $G^+ \to (G^{\mathrm{ad}})^+$ is surjective when restricted to the identity component.*

[2]Here is a background for this lemma. Let $\varphi \colon H \to H'$ be a morphism of algebraic groups defined over k. Assume $\mathrm{char}(k) = 0$. Then φ is called *surjective* if $\varphi(H(\overline{k})) = H'(\overline{k})$. If φ is surjective, it may happen that $\varphi(H(k)) \neq H'(k)$!

We omit the proof of this lemma. Define

$$\begin{aligned}\mathbf{G}(\mathbb{R})_+ &:= \text{inverse image of } \mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+ \text{ in } \mathbf{G}(\mathbb{R})\\ \mathbf{G}(\mathbb{Q})_+ &:= \mathbf{G}(\mathbb{Q}) \cap \mathbf{G}(\mathbb{R})_+.\end{aligned} \tag{3.2.2}$$

Lemma 3.2.3. $\mathbf{G}(\mathbb{R})_+$ *is the stabilizer of* X^+, *i.e.* $\mathbf{G}(\mathbb{R})_+ = \{g \in \mathbf{G}(\mathbb{R}) : gX^+ = X^+\}$.

With Lemma 3.2.3, we can complete our more precise version of (3.2.1):

$$\mathrm{Sh}_K(\mathbf{G}, X) \simeq \coprod_{[g]\in \mathbf{G}(\mathbb{Q})_+\backslash \mathbf{G}(\mathbb{A}_{\mathrm{f}})/K} \Gamma_g \backslash X^+, \tag{3.2.3}$$

with $\Gamma_g := gKg^{-1} \cap \mathbf{G}(\mathbb{Q})_+$; replacing K by a suitable finite-indexed subgroup K' guarantees that Γ_g is torsion-free, see §3.2.4. The finiteness of the double coset $\mathbf{G}(\mathbb{Q})_+\backslash\mathbf{G}(\mathbb{A}_{\mathrm{f}})/K$ will be proved in §3.2.5; the proof uses the *Strong Approximation Theorem*.

Proof of Lemma 3.2.3. Consider the action of $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})$ on $\overline{X}$, and recall that X^+ is a connected component of $\overline{X}$. It suffices to prove that $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+ = \{g \in \mathbf{G}^{\mathrm{ad}}(\mathbb{R}) : gX^+ = X^+\}$. This follows from general theory of Hermitian symmetric domains (and some knowledge on $\mathbb{R}$-algebraic groups *v.s.* real Lie groups) which we will not cover in this course. □

3.2.3 Proof of (3.2.3)

We start by showing that there is a bijection

$$\mathbf{G}(\mathbb{Q})_+\backslash X^+ \times \mathbf{G}(\mathbb{A}_{\mathrm{f}}) \xrightarrow{\sim} \mathbf{G}(\mathbb{Q})\backslash X \times \mathbf{G}(\mathbb{A}_{\mathrm{f}}), \qquad [x, g] \mapsto [x, g]. \tag{3.2.4}$$

Injectivity: Assume $(x, g), (x', g') \in X^+ \times \mathbf{G}(\mathbb{A}_{\mathrm{f}})$ are mapped to the same point on the right hand side. Then there exists $q \in \mathbf{G}(\mathbb{Q})$ such that $(x', g') = q(x, g) = (qx, qg)$. Hence $qX^+ \cap X^+$ is non-empty as it contains $qx = x'$. So $qX^+ = X^+$. So $q \in \mathbf{G}(\mathbb{R})_+ \cap \mathbf{G}(\mathbb{Q}) = \mathbf{G}(\mathbb{Q})_+$. This proves the injectivity of the map above.

Surjectivity: Assume $(x, g) \in X \times \mathbf{G}(\mathbb{A}_{\mathrm{f}})$. By the *Real Approximation* in §3.2.1, $\mathbf{G}(\mathbb{Q})x$ is dense in $\mathbf{G}(\mathbb{R})x = X$. So $\mathbf{G}(\mathbb{Q})x \cap X^+ \neq \emptyset$, and hence there exists $q \in \mathbf{G}(\mathbb{Q})$ such that $qx \in X^+$. Then $(qx, qg) \in X^+ \times \mathbf{G}(\mathbb{A}_{\mathrm{f}})$, and its image under (3.2.4) is $[x, g]$. We are done for the surjectivity of (3.2.3).

Now let us prove the bijectivity of the map

$$\coprod_{[g]\in \mathbf{G}(\mathbb{Q})_+\backslash \mathbf{G}(\mathbb{A}_{\mathrm{f}})/K} \Gamma_g\backslash X^+ \to \mathbf{G}(\mathbb{Q})_+\backslash X^+ \times \mathbf{G}(\mathbb{A}_{\mathrm{f}})/K, \qquad \Gamma_g x \mapsto [x, g]. \tag{3.2.5}$$

Injectivity: If $[x', g'] = [x, g]$, then $(qx, qgk) = (x', g')$ for some $q \in \mathbf{G}(\mathbb{Q})_+$ and $k \in K$. So $[g] = [g']$ in $\mathbf{G}(\mathbb{Q})_+\backslash\mathbf{G}(\mathbb{A}_{\mathrm{f}})/K$. Hence it suffices to prove the injectivity of $\Gamma_g\backslash X^+ \to \mathbf{G}(\mathbb{Q})_+\backslash X^+ \times \mathbf{G}(\mathbb{A}_{\mathrm{f}})/K$. Now if $[x', g] = [x, g]$, then $(qx, qgk) = (x', g)$ for some $q \in \mathbf{G}(\mathbb{Q})_+$ and $k \in K$. So $q = gk^{-1}g^{-1} \in gKg^{-1}$. So $q \in \Gamma_g = gKg^{-1} \cap \mathbf{G}(\mathbb{Q})_+$. Thus we have proved the injectivity of (3.2.5).

Surjectivitity: Let $[x, g]$ be an element of the right hand side. Then it is the image of $\Gamma_g x$.

We have thus proved (3.2.3). □

3.2.4 Torsion-free subgroup

Here is a choice of K' so that $gK'g^{-1} \cap \mathbf{G}(\mathbb{Q})_+$ is torsion-free for all $g \in \mathbf{G}(\mathbb{A}_f)$. Take a faithful representation V of $\mathbf{G}$. Then there exists a lattice L in V such that $\widehat{L} := L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ is fixed by K. Equivalently, we are embedding $\mathbf{G}$ as a closed subgroup of $\mathbf{GL}_N$ over $\mathbb{Q}$ such that K is a subgroup of $\mathbf{GL}_N(\widehat{\mathbb{Z}})$. Let $\ell \geq 3$ be an integer. Take K' to be the subgroup of K which acts trivially on $\widehat{L}/\ell\widehat{L}$, or equivalently

$$K' := \{g \in K < \mathbf{GL}_N(\widehat{\mathbb{Z}}) : g \equiv I_N \bmod \ell\}.$$

Then any element $\gamma \in \Gamma_g := gK'g^{-1} \cap \mathbf{G}(\mathbb{Q})_+ < \mathrm{GL}(V)$ acts trivially on $\widehat{gL}/\ell\widehat{gL}$, so all the eigenvalues of γ are 1 (as they are 1 modulo $\ell \geq 3$). So $\gamma = 1$ if γ is torsion. So Γ_g is torsion-free.

3.2.5 The group of connected components of a Shimura variety

In this subsection, we prove the finiteness of the double coset $\mathbf{G}(\mathbb{Q})_+\backslash\mathbf{G}(\mathbb{A}_f)/K$. This finishes the proof of Theorem 3.2.1, and shows that $\pi_0(\mathrm{Sh}_K(\mathbf{G}, X)) \simeq \mathbf{G}(\mathbb{Q})_+\backslash\mathbf{G}(\mathbb{A}_f)/K$.

Case: simply-connected derived subgroup

The result in this case is better, with a clear understanding of the group $\pi_0(\mathrm{Sh}_K(\mathbf{G}, X))$. Consider the short exact sequence of $\mathbb{Q}$-groups

$$1 \to \mathbf{G}^{\mathrm{der}} \to \mathbf{G} \to \mathbf{T} := \mathbf{G}/\mathbf{G}^{\mathrm{der}} \to 1$$

with $\mathbf{T}$ an algebraic torus defined over $\mathbb{Q}$.

Definition 3.2.4. *An algebraic group H defined over a field k of characteristic 0 is said to be* **simply-connected** *if any central isogeny $H' \to H$ (i.e. a surjective morphism whose kernel is finite and contained in the center of H') is an isomorphism.*

Theorem 3.2.5. *Assume $\mathbf{G}^{\mathrm{der}}$ is simply-connected. Then $\nu(\mathbf{G}(\mathbb{Q})_+)$ has finite index in $\mathbf{G}(\mathbb{Q})$, $\nu(K)$ is a compact open subgroup of $\mathbf{T}(\mathbb{A}_f)$, and $\nu(\mathbf{G}(\mathbb{Q})_+)\backslash\mathbf{T}(\mathbb{A}_f)/\nu(K)$ is a finite abelian group. Moreover, ν induces a natural isomorphism of groups*

$$\pi_0(\mathrm{Sh}_K(\mathbf{G}, X)) \simeq \nu(\mathbf{G}(\mathbb{Q})_+)\backslash\mathbf{T}(\mathbb{A}_f)/\nu(K).$$

Before proving this theorem, we point out without proof that

$$\nu(\mathbf{G}(\mathbb{Q})_+) = \mathbf{T}(\mathbb{Q}) \cap \nu(Z(\mathbf{G})(\mathbb{R})) =: \mathbf{T}(\mathbb{Q})^{\dagger}. \tag{3.2.6}$$

Proof. General theory of semi-simple simply-connected $\mathbb{Q}$-groups asserts that $\mathbf{G}^{\mathrm{der}}(\mathbb{R})$ is connected. Therefore $\mathbf{G}^{\mathrm{der}}(\mathbb{R})$ stabilizes X^+ and hence is contained in $\mathbf{G}(\mathbb{R})_+$ by Lemma 3.2.3. So $\mathbf{G}^{\mathrm{der}}(\mathbb{Q}) \subseteq \mathbf{G}(\mathbb{Q})_+$. By the *Strong Approximation Theorem* from §3.2.1, $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$ is dense in $\mathbf{G}^{\mathrm{der}}(\mathbb{A}_f)$. Hence

$$\mathbf{G}^{\mathrm{der}}(\mathbb{A}_f) = \mathbf{G}^{\mathrm{der}}(\mathbb{Q}) \cdot (K \cap \mathbf{G}^{\mathrm{der}}(\mathbb{A}_f)) \subseteq \mathbf{G}(\mathbb{Q})_+ \cdot (K \cap \mathbf{G}^{\mathrm{der}}(\mathbb{A}_f)). \tag{3.2.7}$$

Because $\mathbf{G}^{\mathrm{der}}$ is simply-connected, the short exact sequence of groups above Theorem 3.2.5 induces a short exact sequence

$$1 \to \mathbf{G}^{\mathrm{der}}(\mathbb{A}_f) \to \mathbf{G}(\mathbb{A}_f) \xrightarrow{\nu} \mathbf{T}(\mathbb{A}_f) \to 1.$$

Here we use the knowledge on semi-simple simply-connected $\mathbb{Q}$-groups that $H^1(\mathbb{Q}_p, \mathbf{G}^{\mathrm{der}}) = 0$ for any prime p.

Now ν induces a map

$$\mathbf{G}(\mathbb{Q})_+ \backslash \mathbf{G}(\mathbb{A}_{\mathrm{f}})/K \to \nu(\mathbf{G}(\mathbb{Q})_+) \backslash \mathbf{T}(\mathbb{A}_{\mathrm{f}})/\nu(K), \tag{3.2.8}$$

which, by (3.2.7), is a bijection. The right hand side is an abelian group because $\mathbf{T}$ is an algebraic torus (hence abelian).

Now to prove the theorem, it remains to prove:

(i) $\nu(\mathbf{G}(\mathbb{Q}))$ has finite index in $\mathbf{T}(\mathbb{Q})$.

(ii) $\nu(K)$ is a compact open subgroup of $\mathbf{T}(\mathbb{A}_{\mathrm{f}})$.

(iii) The right hand side of (3.2.8) is finite.

Let us prove (i). The Hasse Principle for simply-connected $\mathbb{Q}$-groups says that the natural map $H^1(\mathbb{Q}, \mathbf{G}^{\mathrm{der}}) \to \prod_{p \leq \infty} H^1(\mathbb{Q}_p, \mathbf{G}^{\mathrm{der}}) = H^1(\mathbb{R}, \mathbf{G}^{\mathrm{der}})$ is injective; here we used again the fact that $H^1(\mathbb{Q}_p, \mathbf{G}^{\mathrm{der}}) = 0$ for any prime number p (as $\mathbf{G}^{\mathrm{der}}$ is furthermore semi-simple). So by the diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathbf{G}^{\mathrm{der}}(\mathbb{Q}) \arrow[r] \arrow[d] & \mathbf{G}(\mathbb{Q}) \arrow[r] \arrow[d] & \mathbf{T}(\mathbb{Q}) \arrow[r] \arrow[d] & H^1(\mathbb{Q}, \mathbf{G}^{\mathrm{der}}) \arrow[d, hook] \\
1 \arrow[r] & \mathbf{G}^{\mathrm{der}}(\mathbb{R}) \arrow[r] & \mathbf{G}(\mathbb{R}) \arrow[r] & \mathbf{T}(\mathbb{R}) \arrow[r] & H^1(\mathbb{R}, \mathbf{G}^{\mathrm{der}})
\end{tikzcd}$$

we get that $\mathbf{T}(\mathbb{Q})/\nu(\mathbf{G}(\mathbb{Q})) \to \mathbf{T}(\mathbb{R})/\nu(\mathbf{G}(\mathbb{R}))$ is injective. But $\nu(\mathbf{G}(\mathbb{R})^+) = \mathbf{T}(\mathbb{R})^+$. So $\mathbf{T}(\mathbb{R})/\nu(\mathbf{G}(\mathbb{R}))$ is finite, and hence $\mathbf{T}(\mathbb{Q})/\nu(\mathbf{G}(\mathbb{Q}))$ is finite. This establishes the claim.

For (ii), we extend $\mathbf{G} \to \mathbf{T}$ to a morphism of group schemes over $\mathbb{Z}[1/N]$ for some integer N, and prove that $\mathbf{G}(\mathbb{Z}_p) \to \mathbf{T}(\mathbb{Z}_p)$ is surjective for almost all prime p. We first work on $\mathbb{F}_p$ and then list using an argument similar to Newton's Lemma. We omit this proof.

Now we prove (iii). It suffices to prove that $\mathbf{T}(\mathbb{Q}) \backslash \mathbf{T}(\mathbb{A}_{\mathrm{f}})/\nu(K)$ is finite, and up to replacing $\nu(K)$ by a smaller compact open subgroup we may assume $\nu(K) \subseteq \mathbf{T}(\widehat{\mathbb{Z}})$. As $[\mathbf{T}(\widehat{\mathbb{Z}}) : \nu(K)]$ is finite (since $\mathbf{T}(\widehat{Z})$ is compact and $\nu(K)$ is open), it suffices to prove that

$$\mathbf{T}(\mathbb{Q}) \backslash \mathbf{T}(\mathbb{A}_{\mathrm{f}})/\mathbf{T}(\widehat{\mathbb{Z}})$$

is finite. This is exactly the *class group* of the algebraic torus $\mathbf{T}$ which is known to be finite by classical theory (and this number is called the *class number* of $\mathbf{T}$). In the case where $\mathbf{T} = \mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_{\mathrm{m}}$ for a number field K, this is exactly the class group of K. $\square$

General case

Let $\widetilde{\mathbf{G}}$ be the universal cover of $\mathbf{G}^{\mathrm{der}}$, *i.e.* $\widetilde{\mathbf{G}}$ is simply-connected with a central isogeny (surjective with finite kernel contained in the center) $u \colon \widetilde{\mathbf{G}} \to \mathbf{G}^{\mathrm{der}}$. Then we have a surjective morphism of $\mathbb{Q}$-groups

$$\varphi \colon \mathbf{G}' := Z(\mathbf{G}) \times \widetilde{\mathbf{G}} \to \mathbf{G}, \qquad (z, g) \mapsto zu(g)$$

which is a central isogeny. Thus to prove the finiteness of $\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}_{\mathrm{f}})/\nu(K)$, it suffices to prove the finiteness of

$$\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_{\mathrm{f}})/K'$$

for K' a compact open subgroup of $\mathbf{G}'(\mathbb{A}_{\mathrm{f}})$. But the derived subgroup of $\mathbf{G}'$ is $\widetilde{\mathbf{G}}$ which is simply-connected. So we are back to the previous case, and hence $\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_{\mathrm{f}})/K'$ is finite. So $\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}_{\mathrm{f}})/\nu(K)$ is finite.

3.3 Siegel modular variety

Take the example of Siegel case in Example 3.1.3 and Example 3.1.6. In particular $V = \mathbb{Q}^{2d}$, $\psi\colon V \times V \to \mathbb{Q}$ is $(x, y) \mapsto x^{\mathrm{t}} J y$ with $J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}$. Thus there is a lattice L in V such that ψ restricts to $L \times L \to \mathbb{Z}$. To simplify notation, denote by $L = V(\mathbb{Z})$.

The Siegel Shimura datum is $(\mathbf{GSp}_{2d}, \mathfrak{H}_d^{\pm})$. For each N, set

$$K(N) := \left\{ g \in \mathbf{GSp}_{2d}(\mathbb{A}_{\mathrm{f}}) : gV(\widehat{\mathbb{Z}}) \subseteq V(\widehat{\mathbb{Z}}) \text{ and acts trivially on } V(\widehat{\mathbb{Z}})/NV(\widehat{\mathbb{Z}}) \right\}.$$

Then we have the Shimura variety $\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d}, \mathfrak{H}_d^{\pm})$.

Theorem 3.3.1. *Assume $N \geq 3$. Then $\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d}, \mathfrak{H}_d^{\pm})$ is the fine moduli space of principally polarized abelian varieties of dimension d with a level-N-structure, i.e. there is a canonical bijection between*

- *the $\mathbb{C}$-points of $\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d}, \mathfrak{H}_d^{\pm})$,*
- *and the isomorphism classes of the triples (A, λ, η_N) where A is a complex abelian variety of dimension d, λ is a principal polarization on A, and η_N is a level-N-structure on A.*

When $N = 1, 2$, the Shimura variety is a coarse moduli space.

Let us explain the meaning of this theorem. Let A be an abelian variety defined over $\mathbb{C}$.

(i) A *principal polarization* on A is a polarization on the Hodge structure $H_1(A, \mathbb{Z})$ with determinant 1, *i.e.* an alternating pairing $\lambda\colon H_1(A, \mathbb{Z}) \times H_1(A, \mathbb{Z}) \to \mathbb{Z}$, which under suitable $\mathbb{Z}$-basis of $H_1(A, \mathbb{Z})$ is $\begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}$. In more geometric terms, it is an isomorphism $\lambda\colon A \xrightarrow{\sim} A^{\vee}$.

(ii) A *(symplectic) level-N-structure* on A is a basis of $H_1(A, \mathbb{Z}/N\mathbb{Z})$ which is symplectic with respect to λ. In more geometric terms, it is a basis of the $\mathbb{Z}/N\mathbb{Z}$-module $A[N]$ which is symplectic under $e_N\colon A[N] \times A[N] \xrightarrow{(1,\lambda)} A[N] \times A^{\vee}[N] \to \mu_N$ where last map is the Weil pairing. Or more concretely, it is an isomorphism

$$\eta_N\colon A[N] \xrightarrow{\sim} H_1(A, \mathbb{Z}/N\mathbb{Z})$$

such that the two composites

$$A[N] \times A[N] \xrightarrow{(\eta_N, \eta_N)} H_1(A, \mathbb{Z}/N\mathbb{Z}) \times H_1(A, \mathbb{Z}/N\mathbb{Z}) \xrightarrow{\lambda} \mathbb{Z}/N\mathbb{Z}$$
$$\text{and} \quad A[N] \times A[N] \xrightarrow{e_N} \mu_N \xrightarrow{e^{2\pi\sqrt{-1}a/N} \mapsto [a]} \mathbb{Z}/N\mathbb{Z}$$

differ from the multiplication by an element in $[\ell] \in (\mathbb{Z}/N\mathbb{Z})^{\times}$, and we say that this level-N-structure *has twist* $[\ell]$.

Proof. Recall that each point in $\mathfrak{H}_d^{\pm}$ parametrizes a $\mathbb{Q}$-Hodge structure on V of type $(-1, 0) + (0, -1)$; see §2.3.1.

We shall use Theorem 3.2.1 and the more precise version (3.2.3), and better, Theorem 3.2.5 because $\mathbf{Sp}_{2d}$ is simply-connected. One can compute that $\mathbf{GSp}_{2d}(\mathbb{R})_+ = \mathbf{GSp}_{2d}(\mathbb{R})^+ = \{g \in \mathbf{GSp}_{2d}(\mathbb{R}) : \det(g) > 0\}$. So $\mathbf{GSp}_{2d}(\mathbb{Q})_+ = \{g \in \mathbf{GSp}_{2d}(\mathbb{Q}) : \det(g) > 0\}$. Thus for the quotient

$$1 \to \mathbf{Sp}_{2d} \to \mathbf{GSp}_{2d} \xrightarrow{\nu} \mathbb{G}_{\mathrm{m}} \to 1,$$

we have $\nu(\mathbf{GSp}_{2d}(\mathbb{Q})_+) = \mathbb{Q}_{>0}$.[3] It is not hard to compute that $\nu(K(N)) = \{z \in \widehat{\mathbb{Z}} : z \equiv 1 \pmod{N}\} = 1 + N\widehat{\mathbb{Z}}$. Thus

$$\pi_0\left(\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d}, \mathfrak{H}_d^{\pm})\right) \simeq \mathbb{Q}_{>0}\backslash\mathbb{A}_{\mathrm{f}}^{\times}/(1 + N\widehat{\mathbb{Z}}) \simeq (\mathbb{Z}/N\mathbb{Z})^{\times}.$$

Write $\Gamma_{[\ell]}\backslash\mathfrak{H}_d^+$ for the connected component of $\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d}, \mathfrak{H}_d^{\pm})$ indexed by $[\ell] \in (\mathbb{Z}/N\mathbb{Z})^{\times}$. Below we only give the constructions of the two directions, without proving that they are inverse to each other.

Given a triple (A, λ, η_N). Assume that the level-N-structure has twist $[\ell] \in (\mathbb{Z}/N\mathbb{Z})^{\times}$. First $H_1(A, \mathbb{Z})$ is a $\mathbb{Z}$-Hodge structure of type $(-1,0)+(0,-1)$, and hence under suitable isomorphism $(H_1(A, \mathbb{Z}), \lambda) \simeq (V(\mathbb{Z}), \psi)$ we obtain a point $\tau \in \mathfrak{H}_d^+$. Then we get a point in $\Gamma_{[\ell]}\backslash\mathfrak{H}_d^+$ as the image of τ under $\mathfrak{H}_d^+ \to \Gamma_{[\ell]}\backslash\mathfrak{H}_d^+$.

Conversely let $x \in \Gamma_{[\ell]}\backslash\mathfrak{H}_d^+$. Let τ be a pre-image of x under the quotient $\mathfrak{H}_d^+ \to \Gamma_{[\ell]}\backslash\mathfrak{H}_d^+$. Recall that τ parametrizes a $\mathbb{Q}$-Hodge structure on V of type $(-1,0)+(0,-1)$, and thus we can endow $V(\mathbb{R})$ with a complex structure by the bijection $V(\mathbb{R}) \subseteq V(\mathbb{C}) \to V(\mathbb{C})/V_\tau^{0,-1}$. This makes $A_\tau := V(\mathbb{R})/V(\mathbb{Z})$ into a compact complex torus of dimension d, with $H_1(A_\tau, \mathbb{Z}) = V(\mathbb{Z})$. Thus ψ induces a principle polarization via $H_1(A_\tau, \mathbb{Z})$. Hence A_τ is an abelian variety with a principal polarization which by abuse of notation we still use ψ to denote. The level-N-structure on A_τ is given as follows. We have $A_\tau[N] = \frac{1}{N}V(\mathbb{Z})/V(\mathbb{Z}) = V(\mathbb{Z})/NV(\mathbb{Z}) = V(\mathbb{Z}/N\mathbb{Z})$. Take $g \in \mathbf{GSp}_{2d}(\widehat{\mathbb{Z}})$ such that $\nu(g) \in \widehat{\mathbb{Z}}^{\times}$ is congruent to ℓ modulo $1 + N\widehat{\mathbb{Z}}$. Then g induces an isomorphism $g\colon V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}}) \xrightarrow{\sim} V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}})$. But $V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}}) = V(\mathbb{Z}/N\mathbb{Z}) = H_1(A_\tau, \mathbb{Z}/N\mathbb{Z})$. Thus we have $A_\tau[N] = V(\mathbb{Z}/N\mathbb{Z}) = V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}}) \xrightarrow{g} V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}}) = H_1(A_\tau, \mathbb{Z}/N\mathbb{Z})$. This is the desired level-N-structure because $\psi(gx, gy) = \nu(g)\psi(x, y)$ by definition of $\mathbf{GSp}_{2d}$. □

3.4 CM abelian varieties and special points

Let $\mathrm{Sh}_K(\mathbf{G}, X)$ be a Shimura variety. In Definition 3.1.10 we defined *special points* on $\mathrm{Sh}_K(\mathbf{G}, X)$. They are of particular importance. For example, there exists a natural number field $E(\mathbf{G}, X)$, called the *reflex field* of $(\mathbf{G}, X)$, on which $\mathrm{Sh}_K(\mathbf{G}, X)$ is "naturally" defined (or in more vigorous terms, has a canonical model), characterized by the action of the Galois group of $E(\mathbf{G}, X)$. This action is explicitly defined for special points on $\mathrm{Sh}_K(\mathbf{G}, X)$ via the class field theory, and is uniquely determined in this way by the following theorem whose proof we omit:

Theorem 3.4.1. *The set of special points is dense in* $\mathrm{Sh}_K(\mathbf{G}, X)$.

Here "dense" is true even for the usual topology. The hard part of this theorem is to prove the existence of one special point. Indeed, assume $\mathrm{Sh}_K(\mathbf{G}, X) \simeq \bigsqcup \Gamma_g\backslash X^+$ has a special point $[x]$. Then its inverse image x in X^+ gives rise to a morphism $x\colon \mathbb{S} \to \mathbf{G}_{\mathbb{R}}$ which factors through $\mathbf{T}_{\mathbb{R}}$ for an algebraic torus $\mathbf{T} < \mathbf{G}$. But then the morphism given by $g \cdot x$ for any $g \in \mathbf{G}(\mathbb{Q})$ factors through $(g\mathbf{T}g^{-1})_{\mathbb{R}}$, with $g\mathbf{T}g^{-1}$ clearly an algebraic torus in $\mathbf{G}$ (since it is abelian), and hence defines a Shimura datum $(g\mathbf{T}g^{-1}, g \cdot \mathbf{T}(\mathbb{R})x)$. But $\mathbf{T}(\mathbb{R})x$ is a finite set of points since $\mathbf{T}$ is abelian. So the image of $\mathbf{G}(\mathbb{Q})x$ under the quotient $X^+ \to \Gamma_g\backslash X^+$ consists of special points of $\mathrm{Sh}_K(\mathbf{G}, X)$. Notice that $X^+ = \mathbf{G}(\mathbb{R})^+x$. Now it suffice to use the Real Approximation that $\mathbf{G}(\mathbb{Q})$ is dense in $\mathbf{G}(\mathbb{R})$ to conclude.

For the existence of special points, we shall focus on the Siegel modular variety, for which we have:

[3] In fact $\nu(g) = (\det g)^{1/d}$.

Theorem 3.4.2. *Take $[x] \in \mathrm{Sh}_K(\mathbf{GSp}_{2d}, \mathfrak{H}_d^{\pm})(\mathbb{C})$. Then $[x]$ is a special point if and only if the abelian variety A_x parametrized by $[x]$ is CM, i.e.* $\mathrm{End}(A_x) \otimes_{\mathbb{Z}} \mathbb{Q}$ *contains a commutative $\mathbb{Q}$-subalgebra of dimension $2d$. Equivalently, an abelian variety A defined over $\mathbb{C}$ is CM if and only if the Mumford–Tate group of the $\mathbb{Q}$-Hodge structure $H_1(A, \mathbb{Q})$ is an algebraic torus.*

We will not give a full proof of this theorem, but only recall the definition of CM abelian varieties and give a brief explanation why the associated Mumford–Tate group (which we call the Mumford–Tate group of A) is an algebraic torus.

Assume A is a simple abelian variety. Then A is CM if and only if $E := \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a CM field, *i.e.* there exists a totally real field F such that E/F is a totally imaginary quadratic extension. Write $\overline{(\cdot)}$ for the complex conjugation with respect to E/F. Then there exists an element $\iota \in E$ such that $\bar{\iota} = -\iota$ (totally imaginary element). Then E can be endowed with the $\mathbb{Q}$-symplectic form

$$\langle x, y \rangle := \mathrm{Tr}_{E/\mathbb{Q}}(\overline{x}\iota y).$$

This makes $(E, \langle , \rangle) \simeq (\mathbb{Q}^{2d}, \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix})$ into a symplectic space. Set $\mathbf{GU}_E$ to be the subgroup of $\mathbf{GSp}_{2d}$ generated by $\mathbb{G}_{\mathrm{m}} = Z(\mathbf{GSp}_{2d})$ and

$$\mathbf{U}_E := \{x \in \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{\mathrm{m}} : x\overline{x} = 1\}.$$

Then one can check that $\mathbf{GU}_E$ is an algebraic torus which contains the Mumford–Tate group of A. Thus the Mumford–Tate group of A is abelian, and hence must be an algebraic torus. In fact, one can check that $\mathbf{GU}_E$ is a maximal torus of $\mathbf{GSp}_{2d}$.