Chapter 2

Hermitian line bundles on projective arithmetic varieties

In this chapter, we define Hermitian line bundles on arithmetic varieties, explain how to use them to define the height machine, and discuss about their positivity (nefness, ampleness, bigness).

2.1 Review on complex geometry

2.1.1 Complex spaces (complex analytic varieties)

Definition 2.1.1. *Let Ω be a connected open subset of $\mathbb{C}^n$ for some $n \geq 1$. A* **complex analytic subset** *V of Ω is the vanishing locus $V = V(f_1, \ldots, f_m)$ of holomorphic function $f_1, \ldots, f_m$ on Ω.*

For Ω and V as in the definition, let $\mathcal{O}_\Omega$ be the sheaf of holomorphic functions on Ω, and set

$$\mathcal{O}_V := (\mathcal{O}_\Omega/(f_1, \ldots, f_m))\,|_V. \tag{2.1.1}$$

This makes $(V, \mathcal{O}_V)$ a locally ringed space. We call such pairs $(V, \mathcal{O}_V)$ *local models* of complex spaces.

Definition 2.1.2. *A* **complex space** *(or* **complex analytic variety***) is a locally ringed space $(X, \mathcal{O}_X)$ where*

- *X is a locally compact Hausdorff space,*
- *$\mathcal{O}_X$ is a structure sheaf*

such that $(X, \mathcal{O}_X)$ is locally isomorphic to a local model $(V, \mathcal{O}_V)$ defined above.

When the structure sheaf is clear, we by abuse of notation write X for the complex space.

With this definition, one can define morphisms between complex spaces, holomorphic functions on complex spaces, *etc.*

Notice that complex manifolds are precisely complex spaces which are smooth. Moreover, for any complex space X, its *regular locus* X^{reg} is open and dense in X, and is naturally a complex manifold. The *singular locus* $X^{\mathrm{sing}} = X \setminus X^{\mathrm{reg}}$ is a closed complex subspace of X.

Definition 2.1.3. *Let X be a complex space. A* **smooth function** *on X is a continuous function $f\colon X \to \mathbb{R}$ such that for any $x \in X$, there exists an open neighborhood U_x of x in X and an analytic map $i\colon U_x \to \Omega$ (with Ω open in $\mathbb{C}^n$ for some $n \geq 1$) satisfying the following property: $i(U_x)$ is closed in Ω and $f|_{U_x} = \widetilde{f}|_{i(U_x)} \circ i$ with $\widetilde{f}$ a smooth function on Ω.*

2.1.2 Forms and currents

Let us start with the case of *complex manifolds* (smooth complex spaces) M.

We start with the real forms and currents. For each $r \geq 0$, let

$$A^r(M) := \text{space of smooth complex valued } r\text{-forms on } M,$$
$$A^r_{\mathrm{c}}(M) := \text{spaced of compactly supported smooth complex valued } r\text{-forms on } M.$$

The topology on $A^r(M)$ is defined using the following semi-norms (with s, Ω, L varying for all posibilities): For any $\Omega \subseteq M$ a coordinate open subset, and any compact subset $L \subseteq \Omega$ and any $s \in \mathbb{Z}_{\geq 0}$, define the semi-norm

$$p^s_L(u) := \sup_{x \in L} \max_{|I|=r,\ |\alpha| \leq s} |D^\alpha u_I(x)| \tag{2.1.2}$$

for any r-form $u = \sum_I u_I \mathrm{d}x_I$ on Ω. In other words, a sequence $\{u_n\}$ in $A^r(M)$ converges to a form $u \in A^r(M)$ if and only if the following holds true: for each compact subset of every coordiante neighborhood, the sequence $\{u - u_n\}$ and the sequences of higher derivatives converge to 0 uniformly.

The topology on $A^r_{\mathrm{c}}(M)$ is simply the sub-space topology induced by $A^r_{\mathrm{c}}(M) \subseteq A^r(M)$.

Definition 2.1.4. *A* **current** *of dimension r on M is a complex linear functional $T \colon A^r_{\mathrm{c}}(M) \to \mathbb{C}$ which is continuous in the topology on $A^r_{\mathrm{c}}(M)$ defined above.*

We use $D_r(M)$ to denote the space of currents of dimension r, and

$$D^{\dim_{\mathbb{R}} M - r}(M) := D_r(M). \tag{2.1.3}$$

We call $\dim_{\mathbb{R}} M - r$ the *degree* of a current in this space. For $T \in D_r(M)$ and $\alpha \in A^r_{\mathrm{c}}$, write

$$\langle T, \alpha \rangle := T(\alpha) \in \mathbb{C}. \tag{2.1.4}$$

Example 2.1.5. *(i) Let $Z \subseteq M$ be a complex subspace of M with $\dim_{\mathbb{C}} Z = r$. Then the Dirac operator*

$$\delta_Z := (u \mapsto \int_Z u)$$

is an element in $D_{2r}(M)$.

(ii) For any $f \in A^r(M)$ with L^1_{loc}-coefficients, we have

$$T_f := (u \mapsto \int_M f \wedge u) \in D_{\dim_{\mathbb{R}} M - r}(M) = D^r(M).$$

The map $f \mapsto T_f$ then makes $A^r(M)$ into a subspace of $D^r(M)$.

This explains the terminology of "degree" of a current: a degree r current can be written as $\sum_{|I|=r} u_I \mathrm{d}x_I$ with each u_I a distribution.

Next we separate the holomorphic and anti-holomorphic parts. For each $r \geq 0$, we have a decomposition into (p,q)-forms $A^r(M) = \bigoplus_{p+q=r} A^{p,q}(M)$. Define

$$A^{p,q}_{\mathrm{c}}(M) := A^{p,q}(M) \cap A^r_{\mathrm{c}}(M)$$
$$D_{p,q}(M) := \{T \in D_{p+q}(M) : T(u) = 0 \text{ for all } u \in A^{r,s}_{\mathrm{c}}(M) \text{ with } r \neq p\} \tag{2.1.5}$$
$$D^{\dim_{\mathbb{C}} M - p, \dim_{\mathbb{C}} M - q}(M) := D_{p,q}(M).$$

Example 2.1.6. *(i) In Example 2.1.5.(i), we have furthermore*

$$\delta_Z \in D_{r,r}(M) = D^{\dim M - r, \dim M - r}(M).$$

If Z is a divisor, i.e. $\operatorname{codim}_M Z = 1$*, then we get a $(1,1)$-current δ_Z.*

(ii) In Example 2.1.5.(ii), if we furthermore assume $f \in A^{p,q}(M)$, then $T_f \in D^{p,q}(M)$. Thus $f \mapsto T_f$ makes $A^{p,q}(M)$ into a subspace of $D^{p,q}(M)$.

Now we are ready to discuss the general case of complex spaces $(X, \mathcal{O}_X)$.

At each $x \in X$, we have a local model $(V, \mathcal{O}_V)$ with $x \in V$ closed in some connected open subset Ω of $\mathbb{C}^n$ for some $n \geq 1$. Recall that $\mathcal{O}_V$ is a quotient of $\mathcal{O}_\Omega$.

Definition 2.1.7. *A* **smooth** (p,q)**-form** *on X is a smooth (p,q)-form α on X^{reg} such that for any $x \in X$ and the local model above, α extends to a smooth (p,q)-form on Ω.*

Let $\mathcal{A}_X^{p,q}$ be the sheaf of smooth (p,q)-forms on X. Then on each local model V, we have

$$\mathcal{A}_X^{p,q}|_V = \mathcal{A}_\Omega^{p,q}/\{u : i^* u = 0\}$$

where i is $X^{\mathrm{reg}} \cap V \subseteq V \subseteq \Omega$.

For each $n \geq 0$, define $\mathcal{A}_X^n := \bigoplus_{p+q=n} \mathcal{A}_X^{p,q}$. There are natural differential operators

$$\partial \colon \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p+1,q}, \qquad \overline{\partial} \colon \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p,q+1}$$
$$\mathrm{d} = \partial + \overline{\partial} \colon \mathcal{A}_X^n \to \mathcal{A}_X^{n+1}$$

for all $p, q, n \geq 0$. We have $\partial^2 = \overline{\partial}^2 = \mathrm{d}^2 = 0$ and thus $\partial\overline{\partial} = -\overline{\partial}\partial$. We furthermore introduce

$$\mathrm{d}^{\mathrm{c}} := \frac{1}{2\pi\sqrt{-1}}(\partial - \overline{\partial}). \tag{2.1.6}$$

Then $\mathrm{dd}^{\mathrm{c}} = \frac{\sqrt{-1}}{\pi}\partial\overline{\partial}$.

Denote by $A^{p,q}(X) := \mathcal{A}_X^{p,q}(X)$. Denote by $A_{\mathrm{c}}^{p,q}(X) \subseteq A^{p,q}(X)$ the subspace of compactly supported (p,q)-forms. A (p,q)-form α on X is said to be *closed* if $\mathrm{d}\alpha = 0$.

Currents on X are defined in a similar way to the smooth case. We omit it here. The differential operators above can also be applied to currents by considering the duality. More precisely, $\mathrm{d} = \partial + \overline{\partial}$ where

$$\partial \colon D^{p,q}(X) \to D^{p+1,q}(X), \qquad \overline{\partial} \colon D^{p,q}(X) \to D^{p,q+1}(X)$$

are defined according to the formulae:

$$\langle \partial T, \alpha \rangle := (-1)^{p+q+1} \langle T, \partial\alpha \rangle \quad \text{for all } \alpha \in A_{\mathrm{c}}^{\dim X - p - 1, \dim X - q}(X)$$
$$\langle \overline{\partial} T, \alpha \rangle := (-1)^{p+q+1} \langle T, \overline{\partial}\alpha \rangle \quad \text{for all } \alpha \in A_{\mathrm{c}}^{\dim X - p, \dim X - q - 1}(X).$$

A (p,q)-current T on X is said to be *closed* if $\mathrm{d}T = 0$.

2.1.3 Positivity and the Poincaré–Lelong Formula

Definition 2.1.8. *On an open subset $\Omega \subseteq \mathbb{C}^n$, a $(1,1)$-current $u = \sqrt{-1}\sum u_{jk}\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k$ (with each u_{jk} a distribution) is said to be* **(semi-)positive** *if the associated Hermitian form $\xi \mapsto \sum u_{jk}\xi_j\overline{\xi}_k$ is (semi-)positivie on $\mathbb{C}^n$.*

If each u_{jk} is a smooth function, then we recover the definition of (semi-)positive $(1,1)$-forms. Let $(X, \mathcal{O}_X)$ be a complex space.

Definition 2.1.9. *(i) A smooth $(1,1)$-form on X is said to be* **(semi-)positive** *if locally it is (semi-)positive.*

(ii) A $(1,1)$-current $T \in D^{1,1}(X)$ is said to be **(semi-)positive** *if locally it is (semi-)positive.*

An equivalent way to define semi-positive $(1,1)$-current is to use the duality: $T \in D^{1,1}(X)$ is semi-positive if and onoy if $T(\eta \wedge \overline{\eta}) \geq 0$ for all $\eta \in A_{\mathrm{c}}^{n-1,0}(X)$.

Proposition 2.1.10. *Let $T \in D^{1,1}(X)$ be a closed $(1,1)$-current. Then T is semi-positive if and only if locally T can be written as $\sqrt{-1}\partial\overline{\partial}\log|u|$ for some plurisubharmonic function u.*

We end this section with the following result.

Theorem 2.1.11 (Poincaré–Lelong Formula for meromorphic functions)**.** *Let X be a complex space and let f be a meromorphic function. Then as $(1,1)$-currents on X, we have*

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log|f|^2 = \delta_{\mathrm{div}(f)}.$$

2.2 Hermitian line bundles in complex geometry

Let X be a complex space.

2.2.1 Hermitian metrics on holomorphic line bundles

Let L be a holomorphic line bundle on X.

Definition 2.2.1. *A* **smooth (resp. continuous) Hermitian metric** *$\|\cdot\|$ of L on X is the assignment of a $\mathbb{C}$-metric $\|\cdot\|$ to the fiber $L(x)$ above each point $x \in X$, which varies smoothly (resp. continuously). More precisely, for any open subset U of X and any section s of $L|_U \to U$, the function $\|s(x)\|^2$ is smooth (resp. continuous) in $x \in U$.*

We call $(L, \|\cdot\|)$ a smooth/continuous *Hermitian line bundle* on X.

Next we define the *curvature form/current* of the Hermitian line bundle L on X. We need the following preparation. The line bundle L is determined by: (i) an open cover $\{U_\alpha\}$ of X with $L|_{U_\alpha} \simeq U_\alpha \times \mathbb{C}$, (ii) 1-cocyles $\{g_{\alpha\beta}\}$ which are nowhere-zero holomorphic functions on $U_\alpha \cap U_\beta$. The Hermitian metric corresponds to the collection $(U_\alpha, h_\alpha)_\alpha$ with $h_\alpha \colon U_\alpha \to \mathbb{R}_{>0}$, with $h_\beta|g_{\alpha\beta}|^2 = h_\alpha$ on $U_\alpha \cap U_\beta$; indeed h_α is $\|\cdot\|^2$ locally on U_α.

Now consider the $(1,1)$-current $-\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h_\alpha$ on U_α; if the Hermitian metric is smooth then it is a $(1,1)$-form. Since $h_\beta|g_{\alpha\beta}|^2 = h_\alpha$ on $U_\alpha \cap U_\beta$, we have $\log h_\alpha + \log g_{\alpha\beta} + \log\overline{g}_{\alpha\beta} = \log h_\beta$ for some local branch of $\log g_{\alpha\beta}$. But $g_{\alpha\beta}$ is holomorphic, so $\overline{\partial}\log g_{\alpha\beta} = \partial\log\overline{g}_{\alpha\beta} = 0$. Thus $-\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h_\alpha = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h_\beta$ on $U_\alpha \cap U_\beta$. In other words, these local $(1,1)$-currents patch together to a $(1,1)$-current on the whole X, and it is a $(1,1)$-form if the Hermitian metric is smooth. Sometimes we also use $-\frac{\sqrt{-1}}{\pi}\partial\overline{\partial}\log\|\cdot\|$ to denote this current.

Definition 2.2.2. *The* **curvature current** *of* $(L, \|\cdot\|)$*, denoted by* $c_1(L, \|\cdot\|)$*, is the* $(1,1)$*-current on* X *defined above. It is called the* **curvature form** *if the Hermitian metric is smooth.*

It is clear that $c_1(L, \|\cdot\|)$ is a closed since $\mathrm{d} = \partial + \overline{\partial}$ and $\partial^2 = \overline{\partial}^2 = 0$.

Theorem 2.2.3 (Poincaré–Lelong Formula for Hermitian line bundles)**.** *As* $(1,1)$*-currents, we have*

$$c_1(L, \|\cdot\|) = -\frac{\sqrt{-1}}{\pi}\partial\overline{\partial}\log\|s\| + \delta_{\mathrm{div}(s)}$$

for any non-zero meromorphic section s *of* L*.*

Proof. Let s be a non-zero meromorphic section of L over X. Then s corresponds to $(U_\alpha, s_\alpha)_\alpha$ with $s_\alpha \colon U_\alpha \to \mathbb{C}$ with $s_\alpha = g_{\alpha\beta}s_\beta$. Then $\|s\| = \sqrt{h_\alpha}|s_\alpha|$ on U_α. Thus $\log\|s\|^2 = \log h_\alpha + \log|s_\alpha|^2$. The conclusion then follows by definition of $c_1(L, \|\cdot\|)$ and Theorem 2.1.11. ☐

Definition 2.2.4. *A Hermitian metric* $\|\cdot\|$ *on* L *is said to be* **(semi-)positive** *if* $c_1(L, \|\cdot\|)$ *is a (semi-)positive current.*

By Proposition 2.1.10, $\|\cdot\|$ is semi-positive if and only if the following holds true: For any local section s of L which is everywhere non-vanishing over an open subset U of X, the function $-2\log\|s(x)\|$ is plurisubharmonic.

We close this subsection by stating the following results when X is projective, *i.e.* X is the analytification of a projective variety.

Proposition 2.2.5. *Let* $(L, \|\cdot\|)$ *be a Hermitian line bundle on* X*. Then*

(i) $c_1(L, \|\cdot\|)$ *represents the cohomology class of* L *in* $H^2(X, \mathbb{C})$ *under the natural map* $H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{C})$*;*

(ii) we have

$$\int_X c_1(L, \|\cdot\|)^{\wedge \dim X} = \deg_L(X).$$

Moreover if X is furthermore smooth, then *Kodaira's embedding theorem* asserts the following: a holomorphic line bundle L on X is ample if and only if L has a positive metric.

2.2.2 Green's functions

Let D be a Cartier divisor on X. Denote by $|D|$ the support of D.

Definition 2.2.6. *A* **smooth (resp. continuous) Green's function** g_D *of* D *over* X *is a function*

$$g_D \colon X \setminus |D| \to \mathbb{R}$$

such that the following holds true: for any meromorphic function f *over an open subset* U *of* X *with* $\mathrm{div}(f) = D|_U$*, the function* $g_D + \log|f|$ *can be extended to a smooth (resp. continuous) on* U*.*

We say that such a function g_D has *logarithmic singularity along* D.

It is well-known that line bundles and Cartier divisors are closely related. The correspondence can be extended to:

1. Given a smooth/continuous Hermitian line bundle $(L, \|\cdot\|)$ on X, for any meromorphic section s of L on X, we obtain a pair
$$\widehat{\mathrm{div}}(s) := (\mathrm{div}(s), -\log\|s\|)$$
with $-\log\|s\|$ clearly a smooth/continuous Green's function of $\mathrm{div}(s)$ over X.

2. Conversely given a pair (D, g_D) consisting of a Cartier divisor and a smooth/continuous Green's function, we can associated a smooth/continuous Hermitian line bundle $(\mathcal{O}(D), \|\cdot\|_{g_D})$ where $\|\cdot\|_{g_D}$ is defined by $\|s_D\|_{g_D} := e^{-g_D}$ for the canonical section s_D of $\mathcal{O}(D)$ (*i.e.* $\mathrm{div}(s_D) = D$).

By this correspondence, we can make the following definitions.

Definition 2.2.7. *The* **Chern current** *of the pair (D, g_D), where g_D is a Green's function of D over X, is defined to be $c_1(\mathcal{O}(D), \|\cdot\|_{g_D})$. We denote it by $c_1(D, g_D)$.*

Definition 2.2.8. *A Green's function g_D of D over X is said to be* **(semi-)positive** *if $c_1(D, g_D)$ is a (semi-)positive current.*

We close this subsection by stating the following *Stokes' Formula* which allows logarithmic singularity.

Theorem 2.2.9. *Let X be an irreducible projective complex space of dimension n. Let α be a closed $(n-1, n-1)$-form on X. Let $\overline{L}, \overline{M}$ be Hermitian line bundles on X. Let l (resp. m) be a non-zero rational section of L (resp. of M) such that their divisors intersect properly. Then*
$$\int_X (\log\|l\|)c_1(\overline{M}) \wedge \alpha - \int_{[\mathrm{div}(m)]} (\log\|l\|)\alpha = \int_X (\log\|m\|)c_1(\overline{L}) \wedge \alpha - \int_{[\mathrm{div}(l)]} (\log\|m\|)\alpha \quad (2.2.1)$$
and both equal
$$2\int_{X\setminus(|\mathrm{div}(l)|\cup|\mathrm{div}(m)|)} (\mathrm{d}\log\|l\|) \wedge (\mathrm{d^c}\log\|m\|) \wedge \alpha. \quad (2.2.2)$$

Here the divisors in (2.2.1) are the Weil divisors, and the integrals on $\mathrm{div}(l)$ and on $\mathrm{div}(m)$ are induced from those on prime Weil divisors by linearity. The supports of the divisors in (2.2.2) are supports of Cartier divisors.

2.3 Height via Hermitian line bundles on arithmetic varieties

2.3.1 Hermitian line bundles on projective arithmetic varieties

Definition 2.3.1. *An* **arithmetic variety** *is an integral scheme $\mathcal{X}$ which is flat, separated, and of finite type over* $\mathrm{Spec}\mathbb{Z}$. *It is said to be* **(quasi-)projective** *if the structure morphism $\mathcal{X} \to \mathrm{Spec}\mathbb{Z}$ is (quasi-)projective.*

From an arithmetic variety $\mathcal{X}$, we obtain a complex space
$$\mathcal{X}(\mathbb{C}) := \mathrm{Hom}_{\mathrm{Spec}\mathbb{Z}}(\mathrm{Spec}\mathbb{C}, \mathcal{X}),$$
with the complex conjugation acting on $\mathcal{X}(\mathbb{C})$ via its action on $\mathrm{Spec}\mathbb{C}$. Moreover, if $\mathcal{X} \to \mathrm{Spec}\mathbb{Z}$ factors through $\mathrm{Spec}R$ for an order R in a number field K, then $\mathcal{X}(\mathbb{C}) = \coprod_{\sigma\colon K\hookrightarrow\mathbb{C}} \mathcal{X}_\sigma(\mathbb{C})$, with $\mathcal{X}_\sigma(\mathbb{C}) = \mathrm{Hom}_{\mathrm{Spec}\sigma(K)}(\mathrm{Spec}\mathbb{C}, \mathcal{X})$.

Let $\mathcal{X}$ be a *projective* arithmetic variety.

Definition 2.3.2. *A* **Hermitian line bundle** *on $\mathcal{X}$ is a pair $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ consisting of a line bundle $\mathcal{L}$ on $\mathcal{X}$ and a Hermitian metric $\|\cdot\|$ of $\mathcal{L}(\mathbb{C})$ on $\mathcal{X}(\mathbb{C})$ which is invariant under the complex conjugation, i.e. $\|s(x)\| = \|s(\overline{x})\|$ for all local sections s of $\mathcal{L}$ and all $x \in \mathcal{X}(\mathbb{C})$ at which s is defined.*

We can also define the group of isomorphism classes of Hermitian line bundles on $\mathcal{X}$, which will be denote by $\widehat{\mathrm{Pic}}(\mathcal{X})$. The identity element is the trivial Hermitian line bundle, the multiplication is the tensor product, and the inverse is the dual.

Definition 2.3.3. *(i) An* **isomorphism** *(or* **isometry***) between two Hermitian line bundles $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ and $\overline{\mathcal{L}}' = (\mathcal{L}', \|\cdot\|')$ on $\mathcal{X}$ is an isomorphism $i\colon \mathcal{L} \to \mathcal{L}'$ such that $\|\cdot\| = i^*\|\cdot\|'$.*

(ii) The **trivial Hermitian line bundle** *on $\mathcal{X}$ is defined to be $\overline{\mathcal{O}}_{\mathcal{X}} := (\mathcal{O}_{\mathcal{X}}, |\cdot|)$ where $|\cdot|$ is the usual absolute value.*

(iii) The **tensor product** *of two Hermitian line bundles $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ and $\overline{\mathcal{L}}' = (\mathcal{L}', \|\cdot\|')$ on $\mathcal{X}$ is $\overline{\mathcal{L}} \otimes \overline{\mathcal{L}}' := (\mathcal{L} \otimes \mathcal{L}', \|\cdot\|\|\cdot\|')$.*

(iv) The **dual** *of a Hermitian line bundle $\overline{\mathcal{L}}$ on $\mathcal{X}$ is defined to be $\overline{\mathcal{L}}^\vee := (\mathcal{L}^\vee, \|\cdot\|^\vee)$, where $\mathcal{L}^\vee := \mathrm{Hom}(\mathcal{L}, \mathcal{O}_{\mathcal{X}})$ and $\|\cdot\|^\vee$ is the dual metric.*

We also have the definition of *arithmetic divisors*.

Definition 2.3.4. *An* **arithmetic divisor** *on $\mathcal{X}$ is a pair $\overline{D} = (D, g_D)$ consisting of a Cartier divisor D on $\mathcal{X}$ and a Green's function g_D of $D(\mathbb{C})$ on $\mathcal{X}(\mathbb{C})$ which is invariant under the complex conjugation, i.e. $g_D(x) = g_D(\overline{x})$ for all $x \in \mathcal{X}(\mathbb{C}) \setminus |D(\mathbb{C})|$.*

A **principal arithmetic divisor** *is of the form*

$$\widehat{\mathrm{div}}(f) := (\mathrm{div}(f), -\log|f|)$$

where $f \in \mathbb{Q}(\mathcal{X})^*$ is a non-zero rational function on $\mathcal{X}$.

We have the following groups, where the group laws are clear:

$$\begin{aligned}
\widehat{\mathrm{Div}}(\mathcal{X}) &:= \{\text{arithmetic divisors on } \mathcal{X}\},\\
\widehat{\mathrm{Prin}}(\mathcal{X}) &:= \{\text{principal arithmetic divisors on } \mathcal{X}\},\\
\widehat{\mathrm{Cl}}(\mathcal{X}) &:= \widehat{\mathrm{Div}}(\mathcal{X})/\widehat{\mathrm{Prin}}(\mathcal{X}).
\end{aligned}$$

Proposition 2.3.5. *We have a group homomorphism*

$$\widehat{\mathrm{Div}}(\mathcal{X}) \to \widehat{\mathrm{Pic}}(\mathcal{X}), \quad \overline{D} = (D, g_D) \mapsto \mathcal{O}(\overline{D}) = (\mathcal{O}(D), \|\cdot\|_{\overline{D}}) \tag{2.3.1}$$

where $\|\cdot\|_{\overline{D}}$ is defined by $\|s_D\|_{\overline{D}} = e^{-g_D}$ with s_D the canonical section of $\mathcal{O}(D)$ (i.e. $\mathrm{div}(s_D) = D$). Moreover this group homomorphism induces a canonical isomorphism

$$\widehat{\mathrm{Cl}}(\mathcal{X}) \xrightarrow{\sim} \widehat{\mathrm{Pic}}(\mathcal{X}). \tag{2.3.2}$$

Proof. The proof is similar to Proposition 1.1.6. Let us write down the inverse map $\widehat{\mathrm{Pic}}(\mathcal{X}) \to \widehat{\mathrm{Cl}}(\mathcal{X})$. For each $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$, let s be a non-zero rational section of $\mathcal{L}_{\mathbb{Q}}$ and set

$$\widehat{\mathrm{div}}(s) := (\mathrm{div}(s), -\log\|s\|). \tag{2.3.3}$$

Then the inverse is $\overline{\mathcal{L}} \mapsto \widehat{\mathrm{div}}(s)$. $\square$

2.3.2 Height machine via Hermitian line bundles

Let X be a projective variety over $\overline{\mathbb{Q}}$, and let $L \in \mathrm{Pic}^1(X)$. Then X and L are defined over some number field K, with $X \to \mathrm{Spec}K$ the structural morphism.

Definition 2.3.6. *We say that a pair $(\mathcal{X}, \overline{\mathcal{L}})$ is an* **arithmetic model** *of (X, L) over $\mathcal{O}_K$ if*

(i) $\mathcal{X}$ is an integral model of X, i.e $\mathcal{X}$ is an integral scheme, projective and flat over $\mathrm{Spec}\mathcal{O}_K$, such that $\mathcal{X}_K := \mathcal{X} \times_{\mathrm{Spec}\mathcal{O}_K} \mathrm{Spec}K \simeq X$ (notice that $\mathcal{X}$ is naturally an arithmetic variety via $\mathbb{Z} \subseteq \mathcal{O}_K$);

(ii) $\overline{\mathcal{L}}$ is a Hermitian line bundle on $\mathcal{X}$ extending L, i.e. $\mathcal{L}_K \simeq L$ under the identification $\mathcal{X}_K \simeq X$.

Fix an arithmetic model $(\mathcal{X}, \overline{\mathcal{L}})$ of (X, L) over $\mathcal{O}_K$. Let us construct the height on X associated with $(\mathcal{X}, \overline{\mathcal{L}})$, denoted by

$$h_{\overline{\mathcal{L}}} \colon X(\overline{\mathbb{Q}}) \to \mathbb{R} \tag{2.3.4}$$

as follows.

Consider a point $x \in X(K')$ with K'/K a finite extension. Then $x \colon \mathrm{Spec}K' \to X$. The valuative criterion of properness thus gives rise to a unique morphism $\overline{x} \colon \mathrm{Spec}\mathcal{O}_{K'} \to \mathcal{X}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Spec}K' \arrow[r, "x"] \arrow[d] & X = \mathcal{X}_K \arrow[d] \\
\mathrm{Spec}\mathcal{O}_{K'} \arrow[r, "\overline{x}"] & \mathcal{X}
\end{tikzcd}$$

where the vertical maps are induced by the inclusions $\mathcal{O}_{K'} \subseteq K'$ and $\mathcal{O}_K \subseteq K$.

Define

$$h_{\overline{\mathcal{L}}}(x) := \frac{1}{[K' : K]} \widehat{\deg} \overline{x}^* \overline{\mathcal{L}}. \tag{2.3.5}$$

Definition-Lemma 2.3.7. *Let K''/K' be a finite extension. Let $\overline{x}_0 \colon \mathrm{Spec}\mathcal{O}_{K''} \to \mathcal{X}$ be the morphism determined by $x \in X(K'')$. Then*

$$\frac{1}{[K' : K]} \widehat{\deg} \overline{x}^* \overline{\mathcal{L}} = \frac{1}{[K'' : K]} \widehat{\deg} \overline{x}_0^* \overline{\mathcal{L}}.$$

Thus $h_{\overline{\mathcal{L}}}(x)$ in (2.3.5) extends to a well-defined function $X(\overline{\mathbb{Q}}) \to \mathbb{R}$, which is the desired height function (2.3.4).

Proof. This follows easily from Proposition 1.1.7, the definition of the arithmetic degrees of arithmetic divisors on $\mathrm{Spec}\mathcal{O}_{K'}$ and on $\mathrm{Spec}\mathcal{O}_{K''}$ (Definition 1.1.5), and the fact that $\sum_{v \in M_{K''}} e_{v/v_0} f_{v/v_0} = [K'' : K']$ with (in the sum) $v_0 \in M_{K'}$ the place below v. $\square$

Example 2.3.8. *Let $(X, L) = (\mathbb{P}^N, \mathcal{O}(1))$ be defined over $\mathbb{Q}$, and take the arithmetic model $(\mathcal{X}, \overline{\mathcal{L}}) = (\mathbb{P}^N_{\mathbb{Z}}, \overline{\mathcal{O}(1)})$ with the metric on $\mathcal{O}(1)$ as follows: For each $s = a_0 X_0 + \ldots + a_N X_N \in H^0(\mathbb{P}^N_{\mathbb{C}}, \mathcal{O}(1))$, set*

$$\|s(x)\| := \frac{|a_0 x_0 + \ldots + a_N x_N|}{\max\{|x_0|, \ldots, |x_N|\}}$$

for any $x = [x_0 : \cdots : x_N] \in \mathbb{P}^N(\mathbb{C})$. Then it is not hard to check that $h_{\overline{\mathcal{O}(1)}}$ is precisely the Weil height on $\mathbb{P}^N(\overline{\mathbb{Q}})$.

Proposition 2.3.9. *For each arithmetic model $(\mathcal{X}, \overline{\mathcal{L}})$ of (X, L) over $\mathcal{O}_K$, the function $h_{\overline{\mathcal{L}}}$ is a height function associated with (X, L).*

Proof. We start by showing that $h_{\overline{\mathcal{L}}_1} - h_{\overline{\mathcal{L}}_2}$ is bounded on $X(\overline{\mathbb{Q}})$ for any two arithmetic models $(\mathcal{X}_1, \overline{\mathcal{L}}_1)$ and $(\mathcal{X}_2, \overline{\mathcal{L}}_2)$ of (X, L). Let $\mathcal{X}$ be the Zariski closure of

$$X \xrightarrow{\Delta} X \times_{\mathrm{Spec}K} X \to \mathcal{X}_1 \times_{\mathrm{Spec}\mathcal{O}_K} \mathcal{X}_2.$$

Write $f_i \colon \mathcal{X} \to \mathcal{X}_i$ for the i-th projection. Then by definition we have $h_{f_i^*\overline{\mathcal{L}}_i} = h_{\overline{\mathcal{L}}_i}$ for $i \in \{1, 2\}$. On the other hand, $f_1^*\overline{\mathcal{L}}_1 - f_2^*\overline{\mathcal{L}}_2$ is trivial on the generic fiber $X = \mathcal{X}_K$. Thus $h_{f_1^*\overline{\mathcal{L}}_1 - f_2^*\overline{\mathcal{L}}_2}$ is bounded on $X(\overline{\mathbb{Q}})$ since we can take the global section to be 1 in the computation of $\widehat{\deg}$. Hence

$$h_{\overline{\mathcal{L}}_1} - h_{\overline{\mathcal{L}}_2} = h_{f_1^*\overline{\mathcal{L}}_1} - h_{f_2^*\overline{\mathcal{L}}_2} = h_{f_1^*\overline{\mathcal{L}}_1 - f_2^*\overline{\mathcal{L}}_2}$$

is bounded on $X(\overline{\mathbb{Q}})$.

So the conclusion of the proposition does not depend on the choice of the arithmetic model. By linearity/additivity, we may and do assume that L is very ample on X, *i.e.* there exists an embedding $i \colon X \hookrightarrow \mathbb{P}^N_K$ such that $i^*\mathcal{O}(1) \simeq L$. Then i extends to $i \colon \mathcal{X} \to \mathbb{P}^N_{\mathcal{O}_K}$ for the Zariski closure $\mathcal{X}$ of X in $\mathbb{P}^N_{\mathcal{O}_K}$. Then the conclusion follows from Example 2.3.8. We are done. □

2.4 Self-intersection of Hermitian line bundles on arithmetic varieties

2.4.1 Review on intersection of line bundles in algebraic geometry

Let X be a projective variety defined over an algebraically closed field k. Let $\mathrm{Pic}(X)$ be the Picard group, *i.e.* the isomorphism classes of line bundles on X.

Definition 2.4.1 (multiplicity in complete intersection)**.** *Let R be a noetherian local domain of Krull dimension n. For $f_1, \ldots, f_n \in R \setminus \{0\}$ such that $|\mathrm{div}(f_1)| \cap \cdots \cap |\mathrm{div}(f_n)|$ has dimension 0 in $\mathrm{Spec}R$, define*

$$\mathrm{ord}_R(f_1, \ldots, f_n) = \mathrm{length}_R R/(f_1, \ldots, f_n).$$

By linearity, this definition extends to, for $K = \mathrm{Frac}(R)$,

$$\mathrm{ord}_R \colon (K^*)^n \to \mathbb{Z}$$

for $f_1, \ldots, f_n \in K^*$ such that $|\mathrm{div}(f_1)| \cap \cdots \cap |\mathrm{div}(f_n)|$ has dimension 0 in $\mathrm{Spec}R$.

Definition 2.4.2. *Let $D_1, \ldots, D_r$ be Cartier divisors on X which intersect properly, i.e. $|D_1| \cap \cdots \cap |D_r|$ is pure of codimension r in X. Define the r-cocycle of X*

$$D_1 \cdots D_r := \sum_{\substack{Y \subseteq X \text{ integral} \\ \mathrm{codim}_X Y = r}} \mathrm{ord}_{\mathcal{O}_{X,\eta_Y}}(D_1, \ldots, D_r)[Y],$$

where η_Y is the generic point of Y.

Notice that when $r = 1$, the right hand side is just the Weil divisor associated with D_1. To distinguish Cartier and Weil divisors, we use $[D]$ to denote the Weil divisor associated with the Cartier divisor D.

On the other hand, for $r = \dim X$, we can furthermore define the *degree of* $D_1 \cdots D_{\dim X}$ to be

$$\deg(D_1 \cdots D_{\dim X}) := \sum_{P \in X(k)} \mathrm{ord}_{\mathcal{O}_{X,P}}(D_1, \ldots, D_{\dim X}).$$

Lemma 2.4.3. *Let $d = \dim X$. Let $L_1, \ldots, L_d \in \mathrm{Pic}(X)$. There exist rational sections s_i of L_i on X for each $i \in \{1, \ldots, d\}$ such that $\mathrm{div}(s_1), \ldots, \mathrm{div}(s_d)$ intersect properly.*

Notice that $\mathrm{div}(s_i) \in \mathrm{Div}(X)$ is mapped to L_i under $\mathrm{Div}(X) \to \mathrm{Cl}(X) = \mathrm{Div}(X)/\mathrm{Prin}(X) \xrightarrow{\sim} \mathrm{Pic}(X)$, where $\mathrm{Div}(X)$ is the group of Cartier divisors on X and $\mathrm{Prin}(X)$ is the subgroup of principal Cartier divisors.

Definition 2.4.4. *Let $d = \dim X$. The* **intersection pairing**

$$\mathrm{Pic}(X)^d \to \mathbb{Z}$$

is defined to be define

$$L_1 \cdots L_d := \deg(\mathrm{div}(s_1) \cdots \mathrm{div}(s_d)) \tag{2.4.1}$$

for the rational sections $s_1, \ldots, s_d$ obtained from Lemma 2.4.3, where the right hand side is Definition 2.4.2 with $r = d$.

Lemma 2.4.5. *The intersection pairing $\mathrm{Pic}(X)^d \to \mathbb{Z}$ can equivalently defined inductively as follows. When $d = 1$, it is the composite*

$$\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Cl}(X) = \mathrm{Div}(X)/\mathrm{Prin}(X) \xrightarrow{\deg} \mathbb{Z}.$$

For general $d \geq 2$, we have

$$L_1 \cdots L_d = \sum_i m_i L_1|_{Y_i} \cdots L_{d-1}|_{Y_i} \tag{2.4.2}$$

where $\sum_i m_i[Y_i]$ is the Weil divisor for any rational section s_d of L_d on X.

Proof. When $d = 1$, this is immediately true by the discussion below Definition 2.4.2.

For general $d \geq 2$, by multi-linearity (definition of ord) we can reduce to the case where $L_1, \ldots, L_d$ are all very ample. Then both sides of (2.4.2) equal

$$\dim_k \mathcal{O}_{\mathrm{div}(s_1) \cap \cdots \cap \mathrm{div}(s_d)}$$

for some global sections $s_i \in H^0(X, L_i)$ such that $\dim |\mathrm{div}(s_1)| \cap \cdots \cap |\mathrm{div}(s_d)| = 0$, and $\mathrm{div}(s_1) \cap \cdots \cap \mathrm{div}(s_d)$ is the scheme-theoretic intersection in X. We can replace s_d by any rational section (which is $f s_d$ for some $f \in K(X)^*$) since $L_1 \cdots L_{d-1} \cdot \mathcal{O}_X = 0$. $\square$

Proposition 2.4.6 (Projection Formula)**.** *Let $f \colon X' \to X$ be a surjective morphism of projective varieties over a field. Assume $\dim X' = d$. Then for any $L_1, \ldots, L_d \in \mathrm{Pic}(X)$, we have*

$$f^* L_1 \cdots f^* L_d = \deg(f) L_1 \cdots L_d.$$

Here we use the convention that

$$\deg(f) = \begin{cases} 0 & \text{if } \dim X < \dim X' \\ [K(X') : K(X)] & \text{if } \dim X = \dim X'. \end{cases}$$

As suggested by (2.4.2), it is convenient to define the intersection pairing restricted to integral closed subschemes of X. Let Y be a closed subvariety of X of dimension r, and let $L_1, \ldots, L_r \in \mathrm{Pic}(X)$. Define

$$L_1 \cdots L_r \cdot Y := L_1|_Y \cdots L_r|_Y.$$

By linearity, this definition extends to a map

$$\mathrm{Pic}(X)^r \times Z_r(X) \to \mathbb{Z} \tag{2.4.3}$$

with $Z_r(X)$ the group of r-cycles on X, *i.e.* the abelian group generated by integral closed subschemes of X of dimension r. In stating the Projection Formula, it is then convenient to introduce

$$f_*\colon Z_r(X') \to Z_r(X), \tag{2.4.4}$$

where for Y' an integral closed subscheme of X we have

$$f_*([Y']) = \begin{cases} 0 & \text{if } \dim f(Y') < \dim Y' \\ \deg(Y' \to f(Y'))[f(Y')] & \text{if } \dim f(Y') = \dim Y'. \end{cases}$$

In particular, if $f\colon X' \to X$ is generically finite, then $f_*([X']) = (\deg f)[X]$.

2.4.2 Top intersection number of Hermitian line bundles on projective arithmetic varieties

Let $\mathcal{X}$ be a projective arithmetic variety, with $\mathcal{X} \to \mathrm{Spec}\mathbb{Z}$ the structural morphism. Now we turn to the intersection theory of Hermitian line bundles on $\mathcal{X}$.

Definition 2.4.7. *An integral closed subscheme $\mathcal{Y}$ of $\mathcal{X}$ is said to be:*

(i) **horizontal** *if $\mathcal{Y}$ is flat over* $\mathrm{Spec}\mathbb{Z}$ *(notice that $\mathcal{Y} \to \mathbb{Z}$ is then surjective),*

(ii) **vertical** *if the image of $\mathcal{Y} \to \mathrm{Spec}\mathbb{Z}$ is a point.*

Let $n+1 = \dim \mathcal{X}$. Let $Z_r(\mathcal{X})$ be the group of r-cycles on $\mathcal{X}$, *i.e.* *i.e.* the abelian group generated by integral closed subschemes of $\mathcal{X}$ of dimension r.

To define the arithmetic version of the top self-intersection, we start with the definition of the arithmetic degree for $n = 0$. When $n = 0$, we have $\mathcal{X} = \mathrm{Spec}R$ for some order R of a number field K. If $R = \mathcal{O}_K$, then the we have the arithmetic degree $\widehat{\deg}\colon \widehat{\mathrm{Pic}}(\mathrm{Spec}\mathcal{O}_K) \to \mathbb{R}$ from (1.1.1). For general R, we take the same definition with $\mathcal{O}_K$ replaced by R.

Definition 2.4.8. *Define the intersection pairing*

$$\widehat{\mathrm{Pic}}(\mathcal{X})^{n+1} \to \mathbb{R}$$

and, more generally (for $r \le n+1$)

$$\widehat{\mathrm{Pic}}(\mathcal{X})^r \times Z_r(\mathcal{X}) \to \mathbb{R},$$

as follows.

(i) When $n = 0$, this is precisely $\widehat{\deg}$. For $n \ge 1$ and $\overline{\mathcal{L}}_1, \ldots, \overline{\mathcal{L}}_{n+1} \in \widehat{\mathrm{Pic}}(\mathcal{X})$, define

$$\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n+1} := \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_n \cdot [\mathrm{div}(s_{n+1})] - \int_{\mathcal{X}(\mathbb{C})} \log \|s_{n+1}\| c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_n), \tag{2.4.5}$$

with s_n an arbitrary rational section of $\mathcal{L}_{n+1}$ on $\mathcal{X}$ (and $[\mathrm{div}(s_{n+1})]$ is the Weil divisor);

(ii) For $\overline{\mathcal{L}}_1, \ldots, \overline{\mathcal{L}}_r \in \widehat{\mathrm{Pic}}(\mathcal{X})$ and an integral closed subscheme $\mathcal{Y}$ of $\mathcal{X}$ of dimension r, define $\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_r \cdot \mathcal{Y}$ inductively on r according to:

(a) If $\mathcal{Y}$ is horizontal, then set

$$\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_r \cdot \mathcal{Y} := \overline{\mathcal{L}}_1|_{\mathcal{Y}} \cdots \overline{\mathcal{L}}_r|_{\mathcal{Y}}. \tag{2.4.6}$$

(b) If $\mathcal{Y}$ is vertical, then the image of $\mathcal{Y} \to \mathrm{Spec}\mathbb{Z}$ is (p) for some prime number p and hence we view $\mathcal{Y}$ as a scheme over $\mathrm{Spec}\mathbb{F}_p$ (and hence over $\mathrm{Spec}\overline{\mathbb{F}}_p$). Set

$$\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_r \cdot \mathcal{Y} := (\mathcal{L}_1|_{\mathcal{Y}} \cdots \mathcal{L}_r|_{\mathcal{Y}}) \log p. \tag{2.4.7}$$

Theorem 2.4.9. *The pairing $\widehat{\mathrm{Pic}}(\mathcal{X})^{n+1} \to \mathbb{R}$ is well-defined, multi-linear and symmetric.*

Proof. Take s_i to be a rational section of $\mathcal{L}_i$ such that $\mathrm{div}(s_1), \ldots, \mathrm{div}(s_{n+1})$ intersect properly in $\mathcal{X}$. Set

$$\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} \cdot \widehat{\mathrm{div}}(s_n) \cdot \widehat{\mathrm{div}}(s_{n+1}) := \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} (\mathrm{div}(s_n) \cdot \mathrm{div}(s_{n+1})) \\ - \int_{[\mathrm{div}(s_{n+1})](\mathbb{C})} \log \|s_n\| c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{n-1}) - \int_{\mathcal{X}(\mathbb{C})} \log \|s_{n+1}\| c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_n).$$

By induction on n, we then get the definition of $\widehat{\mathrm{div}}(s_1) \cdots \widehat{\mathrm{div}}(s_{n+1})$ and have

$$\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n+1} = \widehat{\mathrm{div}}(s_1) \cdots \widehat{\mathrm{div}}(s_{n+1}).$$

By Stokes' Formula (Theorem 2.2.9), we have

$$\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} \cdot \widehat{\mathrm{div}}(s_n) \cdot \widehat{\mathrm{div}}(s_{n+1}) = \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} \cdot \widehat{\mathrm{div}}(s_{n+1}) \cdot \widehat{\mathrm{div}}(s_n).$$

Thus we obtain

$$\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} \cdot \overline{\mathcal{L}}_n \cdot \overline{\mathcal{L}}_{n+1} = \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} \cdot \overline{\mathcal{L}}_{n+1} \cdot \overline{\mathcal{L}}_n.$$

This proves the symmetry by induction on n. The multi-linearity then follows easily. Moreover, the symmetry and induction on n implies that $\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_n \cdot \widehat{\mathrm{div}}(f) = 0$ for all $f \in K(\mathcal{X})^*$. Hence well-defined. $\square$

We also have the Projection Formula for the arithmetic case.

Proposition 2.4.10. *Let $f \colon \mathcal{X}' \to \mathcal{X}$ be a morphism of projective arithmetic varieties. For $[\mathcal{Y}'] \in Z_r(\mathcal{X}')$ and $\overline{\mathcal{L}}_1, \ldots, \overline{\mathcal{L}}_r \in \widehat{\mathrm{Pic}}(\mathcal{X})$, we have*

$$f^*\overline{\mathcal{L}}_1 \cdots f^*\overline{\mathcal{L}}_r \cdot [\mathcal{Y}'] = \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_r \cdot f_*[\mathcal{Y}'],$$

where $f_* \colon Z_r(\mathcal{X}') \to Z_r(\mathcal{X})$ is defined in the same way as in the geometric case (2.4.4).

2.5 Positivity of Hermitian line bundles on projective arithmetic varieties

2.5.1 Review on nef and big line bundles in algebraic geometry

Let X be a projective variety defined over a field k, and let $L \in \mathrm{Pic}(X)$. Let $d = \dim X$.

Definition 2.5.1. *The line bundle L is called* **nef** *(numerically effective) if $L \cdot C \geq 0$ for any closed subcurve $C \subseteq X$.*

In fact, if L is nef, then $L^{\dim Y} \cdot Y \geq 0$ for any irreducible closed subvariety Y of X. Thus, nef line bundles are the boundary of the cone of ample line bundle because, by the criterion of Nakai–Moishezon, L is ample if and only if $L^{\dim Y} \cdot Y > 0$ for any irreducible closed subvariety Y of X.

Use the symbol $+$ to denote the binary operation on the group $\mathrm{Pic}(X)$ (so $L + L'$ means $L \otimes L'$). For $n \in \mathbb{Z}_{\geq 1}$, write nL for $L^{\otimes n}$. Denote by $h^0(nL) := \dim_k H^0(X, nL)$.

Definition-Lemma 2.5.2. *The limit*

$$\mathrm{vol}(L) := \lim_{n\to\infty} \frac{d!}{n^d} h^0(nL)$$

exists, and is called the **volume** *of* L.

Definition 2.5.3. *The line bundle* L *is said to be* **big** *if* $\mathrm{vol}(L) > 0$.

Both definitions are stable under base change, *i.e.*

Lemma 2.5.4. *Assume* $k \subseteq k'$ *is an inclusion of fields. Then* L *is a nef (resp. big) line bundle on* X *if and only if* $L_{k'}$ *is a nef (resp. big) line bundle on* $X_{k'}$.

In height theory, if we have a big line bundle L on X, then by definition there exists a global section s of nL on X for some $n \gg 1$. Thus the height function h_L has a lower bound outside $|\mathrm{div}(s)|$ by "Lower Bound" of Proposition 0.2.2. In fact, in algebraic geometry, we furthermore have:

Theorem 2.5.5. *The line bundle* L *is big if and only if* $mL = A + \mathcal{O}(E)$ *for some* $m > 1$, *some ample line bundle* A *and some effective divisor* E *on* X.

Here are two important theorems to check the bigness of certain line bundles under suitable nefness assumption.

Theorem 2.5.6 (Hilbert–Samuel)**.** *Assume* L *is nef. Then* $\mathrm{vol}(L) = L^d$.

Theorem 2.5.7 (Siu's inequality)**.** *If* L *and* M *are nef line bundles, then*

$$\mathrm{vol}(L - M) \geq L^d - dL^{d-1} \cdot M.$$

In particular, if L is nef and big, then $mL - M$ is big for $m \gg 1$.

If $k = \mathbb{C}$ and L carries a smooth Hermitian metric $\|\cdot\|$, then we can use the curvature form $c_1(L, \|\cdot\|)$ to check the nefness and bigness. Indeed, in this case for any irreducible closed subvariety Y of X, we have

$$L^{\dim Y} \cdot Y = \int_{Y^{\mathrm{reg}}(\mathbb{C})} c_1(L, \|\cdot\|)^{\wedge \dim Y},$$

where the integral is on the regular locus of Y (or equivalently, the desingularization of Y and then take the pullback of $c_1(L, \|\cdot\|)$). Hence we have:

(i) L is nef if $c_1(L, \|\cdot\|) \geq 0$;

(ii) if $c_1(L, \|\cdot\|) \geq 0$, then L is big if and only if $c_1(L, \|\cdot\|)^{\wedge d} \not\equiv 0$.

2.5.2 Arithmetic volumes

Let $\mathcal{X}$ be a projective arithmetic variety. Let $n + 1 = \dim \mathcal{X}$.

Let $\overline{\mathcal{L}} \in \widehat{\mathrm{Pic}}(\mathcal{X})$ be a Hermitian line bundle. Define

$$H^0(\mathcal{X}, \overline{\mathcal{L}}) := \{s \in H^0(\mathcal{X}, \mathcal{L}) : \|s\|_{\sup} \leq 1\}, \tag{2.5.1}$$

where $\|s\|_{\sup} = \sup_{x\in\mathcal{X}(\mathbb{C})} \|s(x)\|$ is the usual supremum norm on $H^0(\mathcal{X}, \mathcal{L})_{\mathbb{C}}$. It is a finite set: indeed, we have $H^0(\mathcal{X}, \overline{\mathcal{L}}) = H^0(\mathcal{X}, \mathcal{L}) \cap B(\overline{\mathcal{L}})$ with

$$B(\overline{\mathcal{L}}) = \{s \in H^0(\mathcal{X}, \mathcal{L})_{\mathbb{R}} : \|s\|_{\sup} \leq 1\},$$

and hence $H^0(\mathcal{X},\overline{\mathcal{L}})$ is the set of lattice points contained in the unit ball. We thus define

$$h^0(\overline{\mathcal{L}}) := \log \#H^0(\mathcal{X},\overline{\mathcal{L}}). \tag{2.5.2}$$

Elements in $H^0(\mathcal{X},\overline{\mathcal{L}})$ are usually called *small sections* or *effective sections* (we will explain this second terminology at the end of this section).

Definition-Proposition 2.5.8. *The sup-limit*

$$\mathrm{vol}(\overline{\mathcal{L}}) := \limsup_{N\to\infty} \frac{h^0(N\overline{\mathcal{L}})}{N^{n+1}/(n+1)!}$$

exists, and is called the **(arithmetic) volume** *of* $\overline{\mathcal{L}}$.

In practice, it is not easy to count the number of lattice points. Instead, here is a number which approximates this number in an asymptotic way and is easier to handle. Fix any Haar measure on $H^0(\mathcal{X},\mathcal{L})_{\mathbb{R}}$, and set

$$\chi(\overline{\mathcal{L}}) := \log \frac{\mathrm{vol}(B(\overline{\mathcal{L}}))}{\mathrm{covol}(H^0(\mathcal{X},\mathcal{L})_{\mathbb{R}}/H^0(\mathcal{X},\mathcal{L}))}, \tag{2.5.3}$$

which is independent of the choice of the Haar measure (not hard to check). The quantitative version of Minkowski's first theorem (Theorem 1.3.5) then yields

$$h^0(\overline{\mathcal{L}}) \geq \chi(\overline{\mathcal{L}}) - h^0(\mathcal{L}_{\mathbb{Q}})\log 2. \tag{2.5.4}$$

Thus we can make the following definition:

Definition 2.5.9. *The* **χ-volume** *of* $\overline{\mathcal{L}}$ *is defined to be the sup-limit*

$$\mathrm{vol}_{\chi}(\overline{\mathcal{L}}) := \limsup_{N\to\infty} \frac{\chi(N\overline{\mathcal{L}})}{N^{n+1}/(n+1)!}.$$

(2.5.4) furthermore implies that $\mathrm{vol}_{\chi}(\overline{\mathcal{L}}) \leq \mathrm{vol}(\overline{\mathcal{L}})$.

2.5.3 Arithmetic nefness, bigness, and ampleness

Let $\mathcal{X}$ be a projective arithmetic variety, and let $n+1 = \dim \mathcal{X}$.

Definition 2.5.10. *A Hermitian line bundle* $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|) \in \widehat{\mathrm{Pic}}(\mathcal{X})$ *is said to be:*

(1) **nef** *if*

(i) $c_1(\overline{\mathcal{L}}, \|\cdot\|) \geq 0$*;*

(ii) $\overline{\mathcal{L}} \cdot \mathcal{Y} \geq 0$ *for any integral* 1*-dimensional subscheme* $\mathcal{Y}$ *of* $\mathcal{X}$*.*

(2) **weakly ample** *if* $\overline{\mathcal{L}}$ *is nef and* $\mathcal{L}_{\mathbb{Q}}$ *is ample.*

(3) **ample** *if* $\overline{\mathcal{L}}$ *is weakly ample and* $\overline{\mathcal{L}}^{\dim \mathcal{Y}} \cdot \mathcal{Y} > 0$ *for any integral subscheme* $\mathcal{Y}$ *of* $\mathcal{X}$*.*

Definition 2.5.11. *A Hermitian line bundle* $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|) \in \widehat{\mathrm{Pic}}(\mathcal{X})$ *is said to be* **big** *if* $\mathrm{vol}(\overline{\mathcal{L}}) > 0$.

In height theory, suppose $(\mathcal{X}, \overline{\mathcal{L}})$ is an arithmetic model of (X, L) with X a projective variety. If theHermitian line bundle $\overline{\mathcal{L}}$ is big, then by definition there exists a global section s of $N\overline{\mathcal{L}}$ on $\mathcal{X}$ with $\|s\|_{\sup} \leq 1$ for some $N \gg 1$. Thus the height function $h_{\overline{\mathcal{L}}}$ is bounded below by 0 outside the generic fiber of $|\mathrm{div}(s)|$, by the definition of $h_{\overline{\mathcal{L}}}$ (2.3.5). Thus instead of having only a lower bound, we have *positivity*.

Theorem 2.5.12 (Arithmetic Hilbert–Samuel)**.** *Assume $\overline{\mathcal{L}}$ is nef. Then* $\mathrm{vol}(\overline{\mathcal{L}}) = \overline{\mathcal{L}}^{n+1}$.

Theorem 2.5.13 (Arithmetic Siu)**.** *Assume $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$ are nef Hermitian line bundles on $\mathcal{X}$. Then*

$$\mathrm{vol}(\overline{\mathcal{L}} - \overline{\mathcal{M}}) \geq \overline{\mathcal{L}}^{n+1} - (n+1)\overline{\mathcal{L}}^{n} \cdot \overline{\mathcal{M}}.$$

Indeed, both theorems still hold true with vol replaced by vol_χ. For vol_χ and for weakly ample $\overline{\mathcal{L}}$, the Arithmetic Hilbert–Samuel Formula is a consequence of Gillet–Soulé's arithmetic Riemann–Roch theorem and an estimate of analytic torsions by Bismut–Vasserot (with refinement by Zhang); a direct proof was later on given by Abbès–Bouche. For vol and $\overline{\mathcal{L}}$ ample, the Arithmetic Hilbert–Samuel Formula by Zhang by furthermore using his arithmetic Nakai–Moishezon theorem. Moriwaki extended these results to nef Hermitian line bundle (with continuous metrics). Arithmetic Siu is a result of Yuan.

In the next chapters, we will present the proof of Abbès–Bouche of the Arithmetic Hilbert–Samuel Formula.

We close this section with the following discussion on the effectiveness of arithmetic divisors. Let $\overline{D} = (D, g_D)$ be an arithmetic divisor on $\mathcal{X}$.

Definition 2.5.14. *We say that $\overline{D}$ is* **effective** *(resp.* **strictly effective***) if $D \geq 0$ and $g_D \geq 0$ (resp. $D \geq 0$ and $g_D > 0$).*

Recall that $\mathcal{O}(\overline{D})$ is the Hermitian line bundle on $\mathcal{X}$ with the metric $\|\cdot\|$ determined by $\|s_D\| = e^{-g_D}$. Thus if $\overline{D}$ is effective, then $h^0(\mathcal{O}(\overline{D})) > 0$. Conversely, if a Hermitian line bundle $\overline{\mathcal{L}}$ on $\mathcal{X}$ satisfies $h^0(\overline{\mathcal{L}}) > 0$, then there exists a non-zero $s \in H^0(\mathcal{X}, \mathcal{L})$ such that $\|s(x)\| \leq 1$ for all $x \in \mathcal{X}(\mathbb{C})$, and hence the arithmetic divisor $\widehat{\mathrm{div}}(s) = (\mathrm{div}(s), -\log\|s\|)$ is effective.

For this reason, we sometimes call elements in $H^0(\mathcal{X}, \overline{\mathcal{L}})$ *effective sections*, and say that $\overline{\mathcal{L}}$ is *effective* if $h^0(\overline{\mathcal{L}}) > 0$.