

Chapter 5

Adelic line bundles

In §0.3, we have seen that polarized dynamical systems can sometimes give normalized height functions, which are genuine functions in contrast to the abstract height machine. The Weil height on \mathbb{P}^N can be obtained in this way. Another important case is the Néron–Tate height on abelian varieties.

In §2.3.2, we explained how to use arithmetic models (with Hermitian line bundles) to find representatives of each class of height functions constructed by the height machine.

It is desirable to express each normalized height in §0.3 in the framework of §2.3.2. This is the case for the Weil height, as shown in Example 2.3.8. When an abelian variety has good reduction everywhere, it is also possible to do so using the Néron model and the *cubist metric*. However, if the abelian variety does not have good reduction everywhere, it is not possible to define the Néron–Tate height using arithmetic models as in §2.3.2.

To solve this problem, S. Zhang defined and studied *adelically metrized line bundles* (*adelic line bundles* for short) over projective varieties, by putting suitable metrics at the places of bad reduction. All the normalized heights from §0.3 can be defined in this framework. This tool is fundamental in the solution of the famous Bogomolov Conjecture by Ullmo and S. Zhang.

More recently, Yuan and S. Zhang generalized this framework to *adelic line bundles over quasi-projective varieties*. On the one hand, this allows to study the normalized height functions *in family*. On the other hand, it turns out that many other height functions can be defined in this framework, for example the Faltings height as a function on the moduli space of principally polarized abelian varieties. This powerful theory opens another chapter of Arakelov Geometry.

In the whole chapter, we take K to be a number field, and X to be an irreducible quasi-projective variety defined over K .

5.1 Limit construction for the geometric setting

Via $\mathbb{Q} \subseteq K$, we can see X as a quasi-projective variety over $\text{Spec}\mathbb{Q}$.

In this section, we construct the category of *geometric adelic line bundles* on X , denoted by $\mathcal{P}\text{ic}(X/\mathbb{Q})$. Roughly speaking, they are line bundles on X which can be extended to a line bundle on “some compatification” of X .

If X itself is projective, then the construction is void. Nevertheless, in practice we often need to work with quasi-projective varieties which are not projective, for example moduli spaces.

5.1.1 \mathbb{Q} -line bundles

We define the category of \mathbb{Q} -line bundles on X , denoted by $\mathcal{P}\text{ic}(X)_{\mathbb{Q}}$, as follows:

Definition 5.1.1. A \mathbb{Q} -line bundle on X is a pair (a, L) (often written as aL) with $a \in \mathbb{Q}$ and L a line bundle on X . A **morphism** of two \mathbb{Q} -line bundles aL and $a'L'$ is defined to be

$$\mathrm{Hom}(aL, a'L') := \lim_{m \rightarrow \infty} \mathrm{Hom}(amL, a'mL')$$

where m runs over all positive integers such that $am, a'm \in \mathbb{Z}$.

Denote by $\mathrm{Pic}(X)_{\mathbb{Q}}$ the group of isomorphism classes of \mathbb{Q} -line bundles on X . We can define *nef*, *ample*, *big* \mathbb{Q} -line bundles on projective varieties.

Definition 5.1.2. A \mathbb{Q} -line bundle aL on X is said to be **nef (ample, big)** if amL is for some positive integer m such that amL is a usual line bundle on X .

Next we define sections of \mathbb{Q} -line bundles.

Definition 5.1.3. Let $aL \in \mathrm{Pic}(X)_{\mathbb{Q}}$.

- (i) A **(global) section** of aL on X is an element of $H^0(X, aL) := \mathrm{Hom}(\mathcal{O}_X, aL) = \varinjlim_m H^0(X, amL)$ where m runs over all positive integers with $am \in \mathbb{Z}$.
- (ii) A **rational section** of aL on X is an element of $\mathrm{Hom}(\mathcal{O}_{\eta}, aL_{\eta}) = \varinjlim_m H^0(\eta, amL)$, where η is the generic point of X and m runs over all positive integers with $am \in \mathbb{Z}$.
- (iii) For a (rational) section s of aL on X , represented by $(s_m)_m$, define

$$\mathrm{div}(s) := (1/m)\mathrm{div}(s_m) \in \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q} =: \mathrm{Div}(X)_{\mathbb{Q}}.$$

For two elements $D_1, D_2 \in \mathrm{Div}(X)_{\mathbb{Q}}$, we write $D_1 \leq D_2$ if $m(D_2 - D_1)$ is a usual effective Cartier divisor for some positive integer m .

5.1.2 Model geometric adelic line bundles and boundary norm/topology

Now we are ready to define *model geometric adelic line bundles* on X as follows.

Definition 5.1.4. The category of model geometric adelic line bundles on X , denoted by $\widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathrm{mod}}$, is defined to be the category of pairs (X', L') with

- X' is a projective variety defined over K which contains X as an open subset (called a projective model of X);
- L' is a \mathbb{Q} -line bundle on X' , such that $L'|_X$ is a usual line bundle on X .

Adelic line bundles are, roughly speaking, limits of sequences of model adelic line bundles. In order for the limit process to make sense, we need to introduce a suitable *Cauchy condition* for a sequence of model adelic line bundles. Let us explain it now.

For each compactification X' of X , denote by $\mathrm{Div}(X', X) := \mathrm{Div}(X')_{\mathbb{Q}} \otimes_{\mathrm{Div}(X)_{\mathbb{Q}}} \mathrm{Div}(X)$, i.e. the group of \mathbb{Q} -divisors on X' whose restriction to X is a usual Cartier divisor.

Definition 5.1.5. The group of **model geometric adelic divisors** is defined to be

$$\widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathrm{mod}} := \varinjlim_{X'} \mathrm{Div}(X', X) \quad (5.1.1)$$

with X' running over all projective models of X .

Notice that there is a partial order \leq on $\widehat{\text{Div}}(X/\mathbb{Q})_{\text{mod}}$. For any $D_1, D_2 \in \widehat{\text{Div}}(X/\mathbb{Q})_{\text{mod}}$, there exists a projective model X' such that both D_1 and D_2 can be represented by elements (by abuse of notation still denoted by D_1, D_2) in $\text{Div}(X', X) \subseteq \text{Div}(X')_{\mathbb{Q}}$. We say that $D_1 \leq D_2$ as elements in $\widehat{\text{Div}}(X/\mathbb{Q})_{\text{mod}}$ if this is the case in $\text{Div}(X')_{\mathbb{Q}}$. It is not hard to check that this partial ordering on $\widehat{\text{Div}}(X/\mathbb{Q})_{\text{mod}}$ is well-defined.

With this in hand, we can define the *boundary topology* on $\widehat{\text{Div}}(X/\mathbb{Q})_{\text{mod}}$ as follows. Fix a projective model X_0 of X such that $X_0 \setminus X$ is a divisor, which we call D_0 . Then $D_0 \in \text{Div}(X_0, X)$, which gives rise to an element in $\widehat{\text{Div}}(X/\mathbb{Q})_{\text{mod}}$ which is still denoted by D_0 . The following *boundary norm* on $\widehat{\text{Div}}(X/\mathbb{Q})_{\text{mod}}$ (we use the convention $\inf(\emptyset) = \infty$)

$$\|\cdot\|_{D_0} : \widehat{\text{Div}}(X/\mathbb{Q})_{\text{mod}} \rightarrow [0, \infty], \quad D \mapsto \inf\{\epsilon \in \mathbb{Q}_{>0} : -\epsilon D_0 \leq D \leq \epsilon D_0\} \quad (5.1.2)$$

then induces a topology on $\widehat{\text{Div}}(X/\mathbb{Q})_{\text{mod}}$, by defining a neighborhood basis at 0. This is the boundary topology.

Here is an easy lemma on the properties of the boundary norm. The “Moreover” part implies that the boundary topology does not depend on the choice of X_0 .

Lemma 5.1.6. *For any $D, D' \in \widehat{\text{Div}}(X/\mathbb{Q})_{\text{mod}}$, we have*

- (i) $\|D\|_{D_0} = 0$ if and only if $D = 0$,
- (ii) $\|D + D'\|_{D_0} \leq \|D\|_{D_0} + \|D'\|_{D_0}$,
- (iii) $\|aD\|_{D_0} \leq |a| \cdot \|D\|_{D_0}$ for any $a \in \mathbb{Z} \setminus \{0\}$, with $<$ if and only if $D \neq 0$ and $aD = 0$ both hold in $\text{Div}(X)$.

Moreover, if X'_0 is another projective model of X such that $D'_0 := X'_0 \setminus X$ is a divisor, then there exists a real number $r > 1$ such that $r^{-1}\|\cdot\|_{D_0} \leq \|\cdot\|_{D'_0} \leq r\|\cdot\|_{D_0}$.

5.1.3 Geometric adelic line bundles and adelic divisors

Definition 5.1.7. A **geometric adelic divisor** on X is a Cauchy sequence in $\widehat{\text{Div}}(X/\mathbb{Q})_{\text{mod}}$, Cauchy for the boundary topology on $\widehat{\text{Div}}(X/\mathbb{Q})_{\text{mod}}$.

The group of geometric adelic divisors on X is denoted by $\widehat{\text{Div}}(X/\mathbb{Q})$, with the obvious binary operation.

Definition 5.1.8. A **geometric adelic line bundle** on X is a pair $(L, (X_i, L_i, \ell_i)_{i \geq 1})$ with

- L is a line bundle on X ;
- $(X_i, L_i) \in \mathcal{P}\text{ic}(X/\mathbb{Q})_{\text{mod}}$;
- $\ell_i : L \rightarrow L_i|_X$ is an isomorphism in $\mathcal{P}\text{ic}(X)_{\mathbb{Q}}$;

such that the sequence $\{\text{div}(\ell_i \ell_1^{-1})\}_{i \geq 1}$ satisfies the Cauchy condition defined using the boundary topology on $\widehat{\text{Div}}(X/\mathbb{Q})_{\text{mod}}$.

The category of geometric adelic line bundles is denoted by $\widehat{\mathcal{P}\text{ic}}(X/\mathbb{Q})$. The group of isomorphism classes of geometric adelic line bundles, with \otimes being the binary operation, is denoted by $\widehat{\text{Pic}}(X/\mathbb{Q})$.

We need to explain that the sequence $\{\operatorname{div}(\ell_i \ell_1^{-1})\}_{i \geq 1}$ is indeed a sequence in $\widehat{\operatorname{Div}}(X/\mathbb{Q})_{\text{mod}}$. For each $i \geq 1$, we have an isomorphism $\ell_i \ell_1^{-1}: L_1|_X \rightarrow L_i|_X$ of \mathbb{Q} -line bundles on X , and hence a rational map $\ell_i \ell_1^{-1}: L_1 \rightarrow L_i$. Hence $\operatorname{div}(\ell_i \ell_1^{-1})$ is a model adelic divisor for each $i \geq 1$, i.e. $\operatorname{div}(\ell_i \ell_1^{-1}) \in \widehat{\operatorname{Div}}(X/\mathbb{Q})_{\text{mod}}$.

Next we will establish a canonical isomorphism between $\widehat{\operatorname{Pic}}(X/\mathbb{Q})$ with $\widehat{\operatorname{Cl}}(X/\mathbb{Q})$, the group of geometric adelic divisor classes. We should first of all define $\widehat{\operatorname{Cl}}(X/\mathbb{Q})$. We start by defining $\widehat{\operatorname{Prin}}(X/\mathbb{Q})_{\text{mod}} := \varinjlim_{X'} \operatorname{Prin}(X')$, where X' runs over all the projective models of X and $\operatorname{Prin}(X')$ is the group of principal divisors on X' . Then we can define

$$\widehat{\operatorname{Cl}}(X/\mathbb{Q}) := \widehat{\operatorname{Div}}(X/\mathbb{Q}) / \widehat{\operatorname{Prin}}(X/\mathbb{Q})_{\text{mod}}. \quad (5.1.3)$$

Lemma 5.1.9. *The group $\widehat{\operatorname{Prin}}(X/\mathbb{Q})_{\text{mod}}$ is discrete in $\widehat{\operatorname{Div}}(X/\mathbb{Q})_{\text{mod}}$ under the boundary topology.*

Before moving on to the proof, let us see an immediate corollary. If we let

$$\widehat{\operatorname{Cl}}(X/\mathbb{Q})_{\text{mod}} := \widehat{\operatorname{Div}}(X/\mathbb{Q})_{\text{mod}} / \widehat{\operatorname{Prin}}(X/\mathbb{Q})_{\text{mod}} = \lim_{X'} (\operatorname{Div}(X', X) / \operatorname{Prin}(X')), \quad (5.1.4)$$

then $\widehat{\operatorname{Cl}}(X/\mathbb{Q})_{\text{mod}}$ is dense in $\widehat{\operatorname{Cl}}(X/\mathbb{Q})$ by Lemma 5.1.9. In other words, $\widehat{\operatorname{Cl}}(X/\mathbb{Q})$ is the completion of $\widehat{\operatorname{Cl}}(X/\mathbb{Q})_{\text{mod}}$.

Proof of Lemma 5.1.9. Assume that there exists a sequence $\{D_i\}_{i \geq 1}$ in $\widehat{\operatorname{Prin}}(X/\mathbb{Q})_{\text{mod}}$ converging to 0. Then there exists a sequence $\{\epsilon_i \in \mathbb{Q}_{>0}\}_{i \geq 1}$ such that $\epsilon_i \rightarrow 0$ and $\epsilon D_0 \pm D_i \geq 0$ in $\widehat{\operatorname{Div}}(X/\mathbb{Q})_{\text{mod}}$ for all $i \geq 1$. Assume D_i is represented by $\operatorname{div}(f_i)$ for a projective model X_i of X and a rational function $f_i \in \mathbb{Q}(X_i)^* = \mathbb{Q}(X)^*$. Recall the projective model X_0 used to define the boundary topology. Then $\epsilon_i D_0 \pm \operatorname{div}(f_i) \geq 0$ in $\operatorname{Div}(X_0)_{\mathbb{Q}}$. Hence $\operatorname{Div}(f_i) = 0$ on X_0 by taking ϵ_i to be small enough. We are done. \square

Proposition 5.1.10. *There is a canonical isomorphism*

$$\widehat{\operatorname{Cl}}(X/\mathbb{Q}) \xrightarrow{\sim} \widehat{\operatorname{Pic}}(X/\mathbb{Q}).$$

Proof. We write the two morphisms.

For any $\{D_i\}_{i \geq 1} \in \widehat{\operatorname{Div}}(X/\mathbb{Q})$, assume each D_i is defined on the projective model X_i . Then $L_i := \mathcal{O}(D_i)$ is a \mathbb{Q} -line bundle on X_i . Notice that $D_i|_X = D_1|_U$ for all $i \geq 1$. Hence we get a line bundle $L := \mathcal{O}(D_1|_X)$ on X and isomorphisms $\ell_i: L \rightarrow L_i|_X$ for each $i \geq 1$. It is not hard to check the Cauchy condition for the sequence $\operatorname{div}(\ell_i \ell_1^{-1}) = D_i - D_1$. This defines the desired homomorphism

$$\widehat{\operatorname{Div}}(X/\mathbb{Q}) \rightarrow \widehat{\operatorname{Pic}}(X/\mathbb{Q}).$$

It is not hard to check that $\widehat{\operatorname{Prin}}(X/\mathbb{Q})_{\text{mod}}$ is in the kernel.

To see the surjectivity: given any $(L, (X_i, L_i, \ell_i)_{i \geq 1})$ in $\widehat{\operatorname{Pic}}(X/\mathbb{Q})$, take a nonzero rational section s of L on X , and set

$$\widehat{\operatorname{div}}(s) := \{\operatorname{div}_{(X_1, L_1)}(s) + \operatorname{div}(\ell_i \ell_1^{-1})\}_{i \geq 1}, \quad (5.1.5)$$

where $\operatorname{div}_{(X_1, L_1)}(s)$ means to see s as a rational section of L_1 on X_1 , and take the corresponding divisor. This defines the desired element in $\widehat{\operatorname{Cl}}(X/\mathbb{Q})$. \square

5.1.4 Positivity

Definition 5.1.11. An adelic line bundle $\tilde{L} \in \widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})$ is said to be:

- (i) **strongly nef** if it is isomorphic to an object $(L, (X_i, L_i, \ell_i)_{i \geq 1})$ where each L_i is nef on X_i ;
- (ii) **nef** if there exists a strongly nef $\tilde{M} \in \widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})$ such that $a\tilde{L} + \tilde{M}$ is strongly nef for all positive integers a ;
- (iii) **integrable** if it is isomorphic to the difference of two strongly nef adelic line bundles.

We will use $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})_{\mathrm{snef}}$ (resp. $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})_{\mathrm{nef}}$, $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})_{\mathrm{int}}$) to denote the full subcategories of $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})$ of strongly nef (resp. nef, integrable) ones. We will use $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})_{\mathrm{snef}}$ (resp. $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})_{\mathrm{nef}}$, $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})_{\mathrm{int}}$) to denote the corresponding subsets of $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})$. It is a semi-subgroup (resp. semi-subgroup, subgroup).

For any $\tilde{L} = (L, \{X_i, L_i, \ell_i\}_{i \geq 1}) \in \widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})$, we define

$$\widehat{H}^0(X, \tilde{L}) := \{s \in H^0(X, L) : \mathrm{div}(s) \geq 0\} \quad (5.1.6)$$

This is a vector space. We shall prove later on () that $\dim \widehat{H}^0(X, \tilde{L}) < \infty$.

In height theory, elements in (5.1.6) play the same role as global sections on X when X is projective. Indeed, given a non-zero element $s \in \widehat{H}^0(X, \tilde{L})$, then roughly speaking the height function defined by \tilde{L} has a lower bound outside $\widehat{\mathrm{div}}(s)$ which is proper Zariski closed.

Definition-Theorem 5.1.12. The following limit exists and is defined to be the **volume** of $\tilde{L} = (L, \{X_i, L_i, \ell_i\}_{i \geq 1})$:

$$\mathrm{vol}(X, \tilde{L}) := \lim_{m \rightarrow \infty} \frac{(\dim X)!}{m^{\dim X}} \dim \widehat{H}^0(X, m\tilde{L}). \quad (5.1.7)$$

Moreover,

$$\mathrm{vol}(X, \tilde{L}) = \lim_{i \rightarrow \infty} \mathrm{vol}(X_i, L_i).$$

Definition 5.1.13. An adelic line bundle $\tilde{L} \in \widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})$ is said to be **big** if $\mathrm{vol}(X, \tilde{L}) > 0$.

Let $d = \dim X$. We also have an intersection pairing in this situation

$$\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Q})_{\mathrm{int}}^d \rightarrow \mathbb{R}, \quad (\tilde{L}_1, \dots, \tilde{L}_d) \mapsto \tilde{L}_1 \cdots \tilde{L}_d. \quad (5.1.8)$$

Theorem 5.1.14 (Hilbert–Samuel). Assume \tilde{L} is nef. Then $\widehat{\mathrm{vol}}(X, \tilde{L}) = \tilde{L}^d$.

Theorem 5.1.15 (Siu). If \tilde{L} and \tilde{M} are nef adelic line bundles, then

$$\mathrm{vol}(X, \tilde{L} - \tilde{M}) \geq \tilde{L}^d - d\tilde{L}^{d-1}\tilde{M}.$$

5.2 Adelic line bundles as limits of the model ones

Next we turn to the arithmetic setting and try to find arithmetic objects which will define the height functions as desired. We will do the limit construction in the following steps.

- (i) Consider all the quasi-projective models \mathcal{U} of X , i.e. \mathcal{U} is an integral scheme which is quasi-projective over $\mathrm{Spec} \mathcal{O}_K$ such that X is open in the generic fiber \mathcal{U}_K . These quasi-projective models form an inverse system.

- (ii) Define for each quasi-projective model \mathcal{U} the category of *adelic line bundles* $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$ and the group of *adelic divisors* $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$.
- (iii) Define $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z})$ to be $\varinjlim_{\mathcal{U}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$ where the limit is taken on the inverse system of quasi-projective models of X . Similarly define $\widehat{\mathrm{Div}}(X/\mathbb{Z}) := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$.

We call $\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z})$ the category of adelic line bundles on X , and $\widehat{\mathrm{Div}}(X/\mathbb{Z})$ the group of adelic divisors on X . The group of isomorphism classes of adelic line bundles on X will be denoted by $\widehat{\mathrm{Pic}}(X/\mathbb{Z})$, which \otimes being the binary operation. Similarly, we use the notation $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$ and $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$ to denote the groups of isomorphism classes of (model) adelic line bundles on \mathcal{U} .

Steps (i) and (iii) are formal. Step (ii) is the crucial step. In this section, we will define $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})$ as a suitable completion of *model adelic line bundles* $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$ and define $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$ as a suitable completion of *model adelic divisors* $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$.

5.2.1 Model adelic line bundles on \mathcal{U} and boundary topology

Let \mathcal{U} be an integral scheme which is quasi-projective over $\mathrm{Spec}\mathcal{O}_K$.

Definition 5.2.1. A **model adelic line bundle** on \mathcal{U} is a pair $(\mathcal{X}, \overline{\mathcal{L}})$ consisting of:

- a projective model \mathcal{X} of \mathcal{U} , i.e. an integral scheme which is projective over $\mathrm{Spec}\mathcal{O}_K$ and which contains \mathcal{U} as an open subscheme;
- a \mathbb{Q} -Hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ on \mathcal{X} such that $\mathcal{L}|_{\mathcal{U}}$ is a usual line bundle on \mathcal{U} .

Here, \mathbb{Q} -Hermitian line bundles are defined in the same way to the geometric case (Definition 5.1.1) with L replaced by $\overline{\mathcal{L}}$, and we also have the corresponding nefness, ampleness, and bigness for \mathbb{Q} -Hermitian line bundles.

The category of model adelic line bundles on \mathcal{U} is denoted by $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$, and the group of isomorphism classes of model adelic line bundles is denoted by $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$.

To define model adelic divisors, we need to first of all define arithmetic (\mathbb{Q}, \mathbb{Z}) -divisors. Let \mathcal{X} be a projective model of \mathcal{U} .

Definition 5.2.2. An **arithmetic (\mathbb{Q}, \mathbb{Z}) -divisors** on $(\mathcal{X}, \mathcal{U})$ is a \mathbb{Q} -arithmetic divisor $\overline{D} = (D, g_D) \in \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$ such that $D|_{\mathcal{U}}$ is a usual divisor on \mathcal{U} . It is said to be **nef** if \overline{D} is nef in $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$.

The group of arithmetic (\mathbb{Q}, \mathbb{Z}) -divisors on $(\mathcal{X}, \mathcal{U})$ is denoted by $\widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$. There is a partial ordering \leq on $\widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$: $\overline{D} \leq \overline{D}'$ if $\overline{D}' - \overline{D}$ is effective as a \mathbb{Q} -arithmetic divisor on \mathcal{X} and $D'|_{\mathcal{U}} - D|_{\mathcal{U}} \geq 0$ on \mathcal{U} .

Definition 5.2.3. The group of **model adelic divisors** on \mathcal{U} is defined to be

$$\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}} := \varinjlim_{\mathcal{X}} \widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U}) \quad (5.2.1)$$

where \mathcal{X} runs over all projective models of \mathcal{U} .

Notice that the partial ordering on $\widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$ defined above induces a partial ordering \leq on $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$.

We also have a *boundary topology* on $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$ defined as follows. Fix a projective model \mathcal{X}_0 of \mathcal{U} and a strictly effective divisor $\overline{D}_0 = (D_0, g_0)$ on \mathcal{X}_0 such that $\mathrm{Supp}(D_0) = \mathcal{X}_0 \setminus \mathcal{U}$. Such

a pair $(\mathcal{X}_0, \overline{D}_0)$ is called a *boundary divisor*. Then \overline{D}_0 gives rise to an element in $\widehat{\text{Div}}(\mathcal{U}/\mathbb{Z})_{\text{mod}}$ which we still denote by \overline{D}_0 . Then the boundary norm is defined to be $(\inf(\emptyset))$ is set to be ∞

$$\|\cdot\|_{\overline{D}_0} : \widehat{\text{Div}}(\mathcal{U}/\mathbb{Z})_{\text{mod}} \rightarrow [0, \infty], \quad \overline{D} \mapsto \inf\{\epsilon \in \mathbb{Q}_{>0} : -\epsilon\overline{D}_0 \leq \overline{D} \leq \epsilon\overline{D}_0\}. \quad (5.2.2)$$

This boundary norm induces a topology on $\widehat{\text{Div}}(X/\mathbb{Z})_{\text{mod}}$, by defining a neighborhood basis at 0. This is the boundary topology.

As in the geometric, we also have the following lemma, which asserts that the boundary topology does not depend on the choice of the pair $(\mathcal{X}_0, \overline{D}_0)$.

Lemma 5.2.4. *For any $\overline{D}, \overline{D}' \in \widehat{\text{Div}}(X/\mathbb{Z})_{\text{mod}}$, we have*

- (i) $\|\overline{D}\|_{\overline{D}_0} = 0$ if and only if $\overline{D} = 0$,
- (ii) $\|\overline{D} + \overline{D}'\|_{\overline{D}_0} \leq \|\overline{D}\|_{\overline{D}_0} + \|\overline{D}'\|_{\overline{D}_0}$,
- (iii) $\|a\overline{D}\|_{\overline{D}_0} \leq |a| \cdot \|\overline{D}\|_{\overline{D}_0}$ for any $a \in \mathbb{Z} \setminus \{0\}$, with $<$ if and only if $D_{\mathcal{U}} \neq 0$ and $aD|_{\mathcal{U}} = 0$ both hold in $\text{Div}(\mathcal{U})$.

Moreover, if $(\mathcal{X}'_0, \overline{D}'_0)$ is another boundary divisor, then there exists a real number $r > 1$ such that $r^{-1}\|\cdot\|_{\overline{D}_0} \leq \|\cdot\|_{\overline{D}'_0} \leq r\|\cdot\|_{\overline{D}_0}$.

5.2.2 Adelic line bundles and adelic divisors on \mathcal{U}

Let \mathcal{U} be an integral scheme which is quasi-projective over $\text{Spec}\mathcal{O}_K$.

Definition 5.2.5. *An adelic divisor on \mathcal{U} is a Cauchy sequence in $\widehat{\text{Div}}(\mathcal{U}/\mathbb{Z})_{\text{mod}}$, Cauchy for the boundary topology.*

The group of adelic divisors on X is denoted by $\widehat{\text{Div}}(\mathcal{U}/\mathbb{Z})$, with the obvious binary operation.

Definition 5.2.6. *An adelic line bundle on \mathcal{U} is a pair $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i \geq 1})$ with*

- \mathcal{L} is a line bundle on \mathcal{U} ;
- $(\mathcal{X}_i, \overline{\mathcal{L}}_i) \in \widehat{\mathcal{P}\text{ic}}(\mathcal{U}/\mathbb{Z})_{\text{mod}}$;
- $\ell_i : \mathcal{L} \rightarrow \mathcal{L}_i|_{\mathcal{U}}$ is an isomorphism in $\text{Pic}(\mathcal{U})_{\mathbb{Q}}$;

such that the sequence $\{\text{div}(\ell_i \ell_1^{-1})\}_{i \geq 1}$ satisfies the Cauchy condition defined using the boundary topology on $\widehat{\text{Div}}(\mathcal{U}/\mathbb{Z})_{\text{mod}}$.

The category of adelic line bundles is denoted by $\widehat{\mathcal{P}\text{ic}}(\mathcal{U}/\mathbb{Z})$. The group of isomorphism classes of geometric adelic line bundles, with \otimes being the binary operation, is denoted by $\widehat{\text{Pic}}(\mathcal{U}/\mathbb{Z})$.

We need to explain that the sequence $\{\text{div}(\ell_i \ell_1^{-1})\}_{i \geq 1}$ is indeed a sequence in $\widehat{\text{Div}}(\mathcal{U}/\mathbb{Z})_{\text{mod}}$. For each $i \geq 1$, we have an isomorphism $\ell_i \ell_1^{-1} : \mathcal{L}_1|_{\mathcal{U}} \rightarrow \mathcal{L}_i|_{\mathcal{U}}$ of \mathbb{Q} -line bundles on \mathcal{U} , and hence a rational map $\ell_i \ell_1^{-1} : \mathcal{L}_1 \rightarrow \mathcal{L}_i$. Hence $\widehat{\text{div}}(\ell_i \ell_1^{-1})$ is a model adelic divisor for each $i \geq 1$, i.e. $\widehat{\text{div}}(\ell_i \ell_1^{-1}) \in \widehat{\text{Div}}(\mathcal{U}/\mathbb{Z})_{\text{mod}}$.

Next we will establish a canonical isomorphism between $\widehat{\text{Pic}}(\mathcal{U}/\mathbb{Z})$ with $\widehat{\text{Cl}}(\mathcal{U}/\mathbb{Z})$, the group of adelic divisor classes. We should first of all define $\widehat{\text{Cl}}(\mathcal{U}/\mathbb{Z})$. For each projective model \mathcal{X} of

\mathcal{U} , there is a natural homomorphism $\widehat{\text{Div}}(\mathcal{X}) \rightarrow \widehat{\text{Div}}(\mathcal{X}, \mathcal{U})$, which makes $\widehat{\text{Prin}}(\mathcal{X})$ a subgroup of $\widehat{\text{Div}}(\mathcal{X}, \mathcal{U})$. Hence we can define

$$\widehat{\text{Prin}}(\mathcal{U}/\mathbb{Z})_{\text{mod}} := \varinjlim_{\mathcal{X}} \widehat{\text{Prin}}(\mathcal{X}),$$

where \mathcal{X} runs over all the projective models of \mathcal{U} . Then we can define

$$\widehat{\text{Cl}}(\mathcal{U}/\mathbb{Z}) := \widehat{\text{Div}}(\mathcal{U}/\mathbb{Z}) / \widehat{\text{Prin}}(\mathcal{U}/\mathbb{Z})_{\text{mod}}. \quad (5.2.3)$$

Lemma 5.2.7. *The group $\widehat{\text{Prin}}(\mathcal{U}/\mathbb{Z})_{\text{mod}}$ is discrete in $\widehat{\text{Div}}(\mathcal{U}/\mathbb{Z})_{\text{mod}}$ under the boundary topology.*

We omit the proof but state the following immediate corollary. If we let

$$\widehat{\text{Cl}}(\mathcal{U}/\mathbb{Z})_{\text{mod}} := \widehat{\text{Div}}(\mathcal{U}/\mathbb{Z})_{\text{mod}} / \widehat{\text{Prin}}(\mathcal{U}/\mathbb{Z})_{\text{mod}} = \lim_{\mathcal{X}} (\widehat{\text{Div}}(\mathcal{X}, \mathcal{U}) / \widehat{\text{Prin}}(\mathcal{X})), \quad (5.2.4)$$

then $\widehat{\text{Cl}}(\mathcal{U}/\mathbb{Z})_{\text{mod}}$ is dense in $\widehat{\text{Cl}}(\mathcal{U}/\mathbb{Z})$ by Lemma 5.2.7. In other words, $\widehat{\text{Cl}}(\mathcal{U}/\mathbb{Z})$ is the completion of $\widehat{\text{Cl}}(\mathcal{U}/\mathbb{Z})_{\text{mod}}$.

Proposition 5.2.8. *There is a canonical isomorphism*

$$\widehat{\text{Cl}}(\mathcal{U}/\mathbb{Z}) \xrightarrow{\sim} \widehat{\text{Pic}}(\mathcal{U}/\mathbb{Z}).$$

Proof. We write the two morphisms.

For any $\{\overline{D}_i\}_{i \geq 1} \in \widehat{\text{Div}}(\mathcal{U}/\mathbb{Z})$, assume each \overline{D}_i is defined on the projective model \mathcal{X}_i . Then $\overline{\mathcal{L}}_i := \mathcal{O}(\overline{D}_i)$ is a \mathbb{Q} -Hermitian line bundle on \mathcal{X}_i . Notice that $D_i|_{\mathcal{U}} = D_1|_{\mathcal{U}}$ for all $i \geq 1$. Hence we get a line bundle $\mathcal{L} := \mathcal{O}(D_1|_{\mathcal{U}})$ on \mathcal{U} and isomorphisms $\ell_i: \mathcal{L} \rightarrow \mathcal{L}_i|_{\mathcal{U}}$ for each $i \geq 1$. It is not hard to check the Cauchy condition for the sequence $\widehat{\text{div}}(\ell_i \ell_1^{-1}) = \overline{D}_i - \overline{D}_1$. This defines a homomorphism

$$\widehat{\text{Div}}(\mathcal{U}/\mathbb{Z}) \rightarrow \widehat{\text{Pic}}(\mathcal{U}/\mathbb{Z}).$$

Now assume that $\{\overline{D}_i\}_{i \geq 1}$ is in the kernel of this homomorphism. Then there exists an isomorphism from $(\mathcal{O}_{\mathcal{U}}, (\mathcal{X}_0, \overline{\mathcal{O}}_{\mathcal{X}_0}, 1))$ to $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i))$. Hence we have an isomorphism $\mathcal{O}_{\mathcal{U}} \rightarrow \mathcal{O}(D_1|_{\mathcal{U}})$, which is given by the multiplication by some $f \in H^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}})^*$ with $\text{div}(f) = D_1|_{\mathcal{U}} = 0$ on \mathcal{U} . The further properties of the isomorphism are equivalent to that \overline{D}_i converges to $-\widehat{\text{div}}(f)$ in $\widehat{\text{Div}}(\mathcal{U}/\mathbb{Z})_{\text{mod}}$. Hence the kernel of the group homomorphism above is $\widehat{\text{Prin}}(\mathcal{U}/\mathbb{Z})_{\text{mod}}$. So we have an injective group homomorphism

$$\widehat{\text{Cl}}(\mathcal{U}/\mathbb{Z}) \rightarrow \widehat{\text{Pic}}(\mathcal{U}/\mathbb{Z}).$$

To see the surjectivity: given any $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i \geq 1})$ in $\widehat{\text{Pic}}(\mathcal{U}/\mathbb{Z})$, take a nonzero rational section s of \mathcal{L} on \mathcal{U} , and set

$$\widehat{\text{div}}(s) := \{\widehat{\text{div}}_{(\mathcal{X}_1, \overline{\mathcal{L}}_1)}(s) + \widehat{\text{div}}(\ell_i \ell_1^{-1})\}_{i \geq 1}, \quad (5.2.5)$$

where $\widehat{\text{div}}_{(\mathcal{X}_1, \overline{\mathcal{L}}_1)}(s)$ means to see s as a rational section of $\overline{\mathcal{L}}_1$ on \mathcal{X}_1 , and take the corresponding arithmetic divisor. This defines the desired element in $\widehat{\text{Cl}}(\mathcal{U}/\mathbb{Z})$. \square

5.2.3 Nefness and integrability

Let \mathcal{U} be an integral scheme which is quasi-projective over $\mathrm{Spec} \mathcal{O}_K$.

Definition 5.2.9. An adelic line bundle $\overline{\mathcal{L}} \in \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$ is said to be:

- (i) **strongly nef** if it is isomorphic to an object $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i \geq 1})$ where each $\overline{\mathcal{L}}_i$ is nef on \mathcal{X}_i ;
- (ii) **nef** if there exists a strongly nef $\overline{\mathcal{M}} \in \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$ such that $a\overline{\mathcal{L}} + \overline{\mathcal{M}}$ is strongly nef for all positive integers a ;
- (iii) **integrable** if it is isomorphic to the difference of two strongly nef adelic line bundles.

We will use $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{snef}}$ (resp. $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{nef}}$, $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{int}}$) to denote the full subcategories of $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$ of strongly nef (resp. nef, integrable) ones. We will use $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{snef}}$ (resp. $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{nef}}$, $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{int}}$) to denote the corresponding subsets of $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$. It is a semi-subgroup (resp. semi-subgroup, subgroup).

5.2.4 Generic fiber of adelic line bundles

Now we go back to our original situation, where X is an irreducible quasi-projective variety defined over K .

Recall the definition at the beginning of this section that

$$\widehat{\mathrm{Pic}}(X/\mathbb{Z}) := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z}), \quad \widehat{\mathrm{Pic}}(X/\mathbb{Z}) := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z}), \quad \widehat{\mathrm{Div}}(X/\mathbb{Z}) := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$$

with \mathcal{U} running over all quasi-projective models of X .

Proposition 5.2.8 implies immediately

Proposition 5.2.10. There is a canonical isomorphism

$$\widehat{\mathrm{Cl}}(X/\mathbb{Z}) := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Cl}}(\mathcal{U}/\mathbb{Z}) \xrightarrow{\sim} \widehat{\mathrm{Pic}}(X/\mathbb{Z}).$$

For any projective model \mathcal{X} of X , i.e. an integral scheme which is projective over $\mathrm{Spec} \mathcal{O}_K$ such that X is open in \mathcal{X}_K , the generic fiber \mathcal{X}_K is by definition a projective model of X . Hence the natural map $\widehat{\mathrm{Pic}}(\mathcal{X}) \rightarrow \widehat{\mathrm{Pic}}(\mathcal{X}_K)$ induces a group homomorphism

$$\widehat{\mathrm{Pic}}(X/\mathbb{Z}) \rightarrow \widehat{\mathrm{Pic}}(X/\mathbb{Q}). \quad (5.2.6)$$

Definition 5.2.11. For any adelic line bundle $\overline{\mathcal{L}} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$, the image under (5.2.6) is called the **generic fiber** of $\overline{\mathcal{L}}$. It is often denoted by $\widetilde{\mathcal{L}}$.

Let \mathbf{P} be one of the symbols {snef, nef, int}. Then we define

$$\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathbf{P}} := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathbf{P}}, \quad \widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathbf{P}} := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathbf{P}}.$$

It is not hard to check that (5.2.6) restricts to

$$\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathbf{P}} \rightarrow \widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathbf{P}}.$$

5.3 Metrized line bundles on Berkovich analytification

A second way to understand adelic line bundles is to see them as *metrized line bundles* on the analytification of X . In this section, we explain this Berkovich analytification of X .