

Chapter 3

Shimura data and Shimura varieties

3.1 Basic definitions

3.1.1 Shimura data

Definition 3.1.1. A Shimura datum is a pair (\mathbf{G}, X) where

- \mathbf{G} is a reductive group defined over \mathbb{Q} ,
- X is a $\mathbf{G}(\mathbb{R})$ -orbit in $\mathrm{Hom}(\mathbb{S}, \mathbf{G}_{\mathbb{R}})$

such that for one (and hence all) $h \in X$, we have

(SV1) the Hodge structure $\mathrm{Ad} \circ h$ on $\mathrm{Lie} \mathbf{G}$ has type $(-1, 1) + (0, 0) + (1, -1)$,

(SV2) $\mathrm{Int}(h(\sqrt{-1}))$ is a Cartan involution of $\mathbf{G}_{\mathbb{R}}^{\mathrm{ad}}$,

(SV3) for every \mathbb{Q} -simple factor \mathbf{H} of \mathbf{G}^{ad} , the morphism $\mathbb{S} \xrightarrow{h} \mathbf{G}_{\mathbb{R}} \rightarrow \mathbf{H}_{\mathbb{R}}$ is non-trivial.

A (Shimura) morphism between two Shimura data $\rho: (\mathbf{G}', X') \rightarrow (\mathbf{G}, X)$ is a morphism ρ on the underlying groups such that $\rho \circ h \in X$ for all $h \in X'$. In particular, we call the image of such a Shimura morphism to be a **sub-Shimura datum** of (\mathbf{G}, X) .

The main difference of a Shimura datum and the pair (G, X^+) from §2.3 is the definition field of the group (over \mathbb{Q} or over \mathbb{R}). A similar assumption to (SV3) for (G, X^+) has been discussed in Remark 2.3.3. By Theorem 2.3.1, each connected component X^+ of X is a Hermitian symmetric domain (and the complex structure on X is $\mathbf{G}(\mathbb{R})$ -invariant). By Proposition 1.3.5, each representation V of \mathbf{G} gives rise to a \mathbb{Q} -VHS on X^+ by (SV1), which furthermore carries \mathbb{R} -polarization by Proposition 2.2.6 and (SV2).^[1]

The following two further assumptions guarantee that this \mathbb{Q} -VHS carries a \mathbb{Q} -polarization. Notice that they may not be satisfied by an arbitrary Shimura datum.

(SV4) the morphism $w_h: \mathbb{G}_{\mathrm{m}, \mathbb{R}} \rightarrow Z(\mathbf{G})_{\mathbb{R}}$ is defined over \mathbb{Q} ,

(SV2') $\mathrm{Int}(h(\sqrt{-1}))$ is a Cartan involution of $\mathbf{G}_{\mathbb{R}}/w_h(\mathbb{G}_{\mathrm{m}, \mathbb{R}})$.

Example 3.1.2 (0-dimensional Shimura datum). *The set X is a finite set if and only if \mathbf{G} is abelian (and hence an algebraic torus). This case shows up when we study CM abelian varieties.*

^[1](SV1) implies that $w_h: \mathbb{G}_{\mathrm{m}} \xrightarrow{w} \mathbb{S} \xrightarrow{h} \mathbf{G}_{\mathbb{R}}$ factors through $Z(\mathbf{G})_{\mathbb{R}}$, so we can apply Proposition 2.2.6

Example 3.1.3 (Siegel Shimura datum). *Let us take the example of the Siegel case from Example 2.3.1, except that the vector space and the groups are defined over \mathbb{Q} . More precisely, $V = \mathbb{Q}^{2d}$ and $\psi: V \times V \rightarrow \mathbb{Q}$ is $(x, y) \mapsto x^t J y$ with $J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}$. The \mathbb{Q} -group is*

$$\begin{aligned} \mathbf{GSp}(\psi) &= \mathbf{GSp}_{2d} := \{g \in \mathrm{GL}(V) : \psi(gx, gy) = c\psi(x, y) \text{ for some } c \in \mathbb{Q}^\times\} \\ &= \{g \in \mathrm{GL}_{2d, \mathbb{Q}} : gJg^t = cJ \text{ for some } c \in \mathbb{Q}^\times\}, \end{aligned}$$

and $h_0: \mathbb{S} \rightarrow \mathbf{GSp}_{2d, \mathbb{R}}$ maps $a + b\sqrt{-1} \mapsto aI_{2d} + bJ$. The derived subgroup is $\mathbf{Sp}(\psi) = \mathbf{Sp}_{2d}$ by requesting the $c \in \mathbb{Q}^\times$ in the definition to be 1.

The $\mathbf{G}(\mathbb{R})$ -orbit is $\mathbf{GSp}_{2d}(\mathbb{R})h_0 \subseteq \mathrm{Hom}(\mathbb{S}, \mathbf{GSp}_{2d, \mathbb{R}})$. Under the identification similar to 2.3.1, we have

$$\mathbf{GSp}_{2d}(\mathbb{R})h_0 \xrightarrow{\sim} \mathfrak{H}_d^\pm := \{\tau \in \mathrm{Mat}_{d \times d}(\mathbb{C}) : \tau = \tau^t, \text{ either } \mathrm{Im} \tau > 0 \text{ or } \mathrm{Im} \tau < 0\}. \quad (3.1.1)$$

Then $(\mathbf{GSp}_{2d}, \mathfrak{H}_d^\pm)$ is a Shimura datum by Example 2.3.1 (see Remark 2.3.3 for (SV3)). It is called the **Siegel Shimura datum**. Moreover, (SV4) and (SV2') are easily seen to be also satisfied. In fact, V is a representation of \mathbf{GSp}_{2d} , and ψ is the desired \mathbb{Q} -polarization on the induced \mathbb{Q} -VHS.

Next we present an example where (SV4) and (SV2') are not satisfied. We also see in this example that two Shimura data may have the same underlying \mathbb{R} -group and the same underlying space, but the \mathbb{Q} -groups are different.

Example 3.1.4 (Shimura curves). *Let B be a simple quaternion algebra over a totally real number field F . Assume that B is split at exactly one real place of F , i.e. there exists a real embedding $\sigma: K \rightarrow \mathbb{R}$ such that*

$$B_\sigma \simeq \begin{cases} M_2(\mathbb{R}) & \text{if } \sigma = \sigma_0 \\ \mathbb{H} & \text{otherwise} \end{cases}$$

for all real embeddings $\sigma: K \rightarrow \mathbb{R}$, where \mathbb{H} is the Hamilton quaternion algebra over \mathbb{R} .

Define the \mathbb{Q} -group \mathbf{G}

$$\mathbf{G}(R) := (B \otimes_{\mathbb{Q}} R)^\times \quad \text{for all } \mathbb{Q}\text{-algebra } R,$$

and let

$$h_0: \mathbb{S} \rightarrow \mathbf{G}_{\mathbb{R}} \simeq \mathrm{GL}_{2, \mathbb{R}} \times \mathbb{H}^\times \times \cdots \times \mathbb{H}^\times, \quad a + b\sqrt{-1} \mapsto \left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}, 1, \dots, 1 \right).$$

Thus all but the first components of $\mathbf{G}(\mathbb{R})h_0$ are the identity map, and so $\mathbf{G}(\mathbb{R})h_0 \subseteq \mathrm{Hom}(\mathbb{S}, \mathbf{G}_{\mathbb{R}})$ is isomorphic to \mathfrak{H}_1^\pm , via an isomorphism similar to (3.1.1) (with $d = 1$). Both (SV1) and (SV2) hold true for the pair $(\mathbf{G}, \mathfrak{H}_1^\pm)$ similarly to the Siegel case. To see (SV3), it suffices to observe that \mathbf{G}^{ad} is a simple group because B is a simple quaternion algebra over F .

So $(\mathbf{G}, \mathfrak{H}_1^\pm)$ is a Shimura datum. However, if $[F : \mathbb{Q}] > 1$, then (SV4) and (SV2') do not hold true, by looking at the action of $\mathrm{Aut}(\mathbb{R}/\mathbb{Q})$.

And even in the case $F = \mathbb{Q}$, the group \mathbf{G} is not necessarily \mathbf{GL}_2 . So $(\mathbf{G}, \mathfrak{H}_1^\pm)$ needs not be the Siegel Shimura datum in this case.

3.1.2 Shimura varieties

Denote by $\mathbb{A}_f \subseteq \prod_{p \in M_{\mathbb{Q},f}} \mathbb{Q}_p$ the ring of finite adèles over \mathbb{Q} , and by $\widehat{\mathbb{Z}} := \prod \mathbb{Z}_p$. Then $\widehat{\mathbb{Z}}$ is a (maximal) compact open subgroup of \mathbb{A}_f , and \mathbb{Q} is dense in \mathbb{A}_f .

Let (\mathbf{G}, X) be a Shimura datum. Then $\mathbf{G}(\mathbb{Q})$ acts on X by definition of Shimura data, and consider the action of $\mathbf{G}(\mathbb{Q})$ on $\mathbf{G}(\mathbb{A}_f)$ by multiplication on the left.

Definition 3.1.5. Let (\mathbf{G}, X) be a Shimura datum. A **Shimura variety** is a double coset

$$\mathrm{Sh}_K(\mathbf{G}, X) := \mathbf{G}(\mathbb{Q}) \backslash X \times \mathbf{G}(\mathbb{A}_f) / K$$

where $K \subseteq \mathbf{G}(\mathbb{A}_f)$ is a compact open subset. Here $\mathbf{G}(\mathbb{Q})$ acts on both X and $\mathbf{G}(\mathbb{A}_f)$ on the left as above, and K acts on $\mathbf{G}(\mathbb{A}_f)$ by the multiplication on the right; i.e. $q(x, g)k = (q \cdot x, qgk)$ for all $q \in \mathbf{G}(\mathbb{Q})$, $(x, g) \in X \times \mathbf{G}(\mathbb{A}_f)$ and $k \in K$.

We will prove in this course that the double coset $\mathrm{Sh}_K(\mathbf{G}, X)$ is the set of \mathbb{C} -points of an algebraic variety. This justifies the name of Shimura variety.

Example 3.1.6. In the Siegel example above, the group \mathbf{GSp}_{2d} is defined over \mathbb{Z} ; indeed we can take V to be \mathbb{Z}^{2d} and ψ maps $V \times V$ to \mathbb{Z} . Then $\mathbf{GSp}_{2d}(\widehat{\mathbb{Z}})$ is a compact open subgroup of $\mathbf{GSp}_{2d}(\mathbb{A}_f)$. Other compact open subgroups include gKg^{-1} for any $g \in \mathbf{GSp}_{2d}(\mathbb{A}_f)$ and any finite-indexed subgroup K of $\mathbf{GSp}_{2d}(\widehat{\mathbb{Z}})$. We will come back to this example in §3.3 and prove that the Siegel Shimura varieties are moduli spaces of abelian varieties.

Definition 3.1.7. A **(Shimura) morphism** $[\rho]: \mathrm{Sh}_{K'}(\mathbf{G}', X') \rightarrow \mathrm{Sh}_K(\mathbf{G}, X)$ between two Shimura varieties is a morphism of Shimura data $\rho: (\mathbf{G}', X') \rightarrow (\mathbf{G}, X)$ such that $\rho(K') \subseteq K$.

Example 3.1.8. Let $\mathrm{Sh}_K(\mathbf{G}, X)$ be a Shimura variety.

Let $K' \subseteq K$ be another compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$. Then the identity map on (\mathbf{G}, X) induces a Shimura morphism $\mathrm{Sh}_{K'}(\mathbf{G}, X) \rightarrow \mathrm{Sh}_K(\mathbf{G}, X)$, with finite fibers since K' must have finite index in K . In fact, this is finite morphism in the category of algebraic varieties.

Let $g \in \mathbf{G}(\mathbb{A}_f)$. Then gKg^{-1} is a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$, and we have a Shimura morphism $[g]: \mathrm{Sh}_{gKg^{-1}}(\mathbf{G}, X) \rightarrow \mathrm{Sh}_K(\mathbf{G}, X)$, sending $[x, g'] \mapsto [x, gg']$. More generally, if K' is a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$ such that $K' \subseteq gKg^{-1}$, then we have a Shimura morphism $[g]: \mathrm{Sh}_{K'}(\mathbf{G}, X) \rightarrow \mathrm{Sh}_K(\mathbf{G}, X)$ which is a finite morphism.

Example 3.1.9 (Hecke operator). Let $\mathrm{Sh}_K(\mathbf{G}, X)$ be a Shimura variety.

Any $g \in \mathbf{G}(\mathbb{A}_f)$ induces a correspondence on $\mathrm{Sh}_K(\mathbf{G}, X)$ as follows. Write $K' := K \cap gKg^{-1}$ for simplicity; it is a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$ and $K' \subseteq gKg^{-1}$. We have Shimura morphisms

$$\begin{array}{ccc} & \mathrm{Sh}_{K'}(\mathbf{G}, X) & \\ \swarrow [g] & & \searrow [1] \\ \mathrm{Sh}_K(\mathbf{G}, X) & & \mathrm{Sh}_K(\mathbf{G}, X) \end{array}$$

where the right one is induced by identity on (\mathbf{G}, X) . Both are finite morphisms, so we have a correspondence on $\mathrm{Sh}_K(\mathbf{G}, X)$, which is called the **Hecke correspondence/operator** and denoted by T_g .

Definition 3.1.10. Let $\mathrm{Sh}_K(\mathbf{G}, X)$ be a Shimura variety. We call any irreducible component of $(T_g \circ [\rho])(\mathrm{Sh}_{K'}(\mathbf{G}', X'))$, where $[\rho]$ is a Shimura morphism and $g \in \mathbf{G}(\mathbb{A}_f)$, to be a **special subvariety** of $\mathrm{Sh}_K(\mathbf{G}, X)$. A special subvariety of dimension 0 is called a **special point**.

Of course in the definition of special subvarieties, it suffices to consider the Shimura morphisms arising from sub-Shimura data of (\mathbf{G}, X) . Thus special points arise from sub-Shimura data $(\mathbf{T}, X_{\mathbf{T}})$ of (\mathbf{G}, X) where \mathbf{T} is an algebraic torus.

3.2 Decomposition of Shimura varieties into Hermitian locally symmetric domains

Let (\mathbf{G}, X) be a Shimura datum. Then any connected component X is a Hermitian symmetric domain. Fix one such X^+ .

Let K be a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$. Then we have a Shimura variety $\mathrm{Sh}_K(\mathbf{G}, X)$ defined as the double coset $\mathbf{G}(\mathbb{Q}) \backslash X \times \mathbf{G}(\mathbb{A}_f) / K$. We wish to prove that this double coset is the \mathbb{C} -points of an algebraic variety.

In this section, we start with the first step, by endowing $\mathrm{Sh}_K(\mathbf{G}, X)$ with a structure of complex varieties.

Theorem 3.2.1. *There exists a finite-indexed subgroup K' of K such that*

$$\mathrm{Sh}_{K'}(\mathbf{G}, X) \simeq \bigsqcup_{g \in \mathcal{C}} \Gamma_g \backslash X^+, \quad (3.2.1)$$

for a finite set $\mathcal{C} \subseteq \mathbf{G}(\mathbb{A}_f)$, with each Γ_g a torsion-free discrete group acting on X^+ .

The actual decomposition will be given later on (3.2.3), where the definitions of \mathcal{C} and Γ_g are given. At this stage, let us make the following observation: since Γ_g is torsion-free discrete, the quotient $\Gamma_g \backslash X^+$ has a natural structure of complex manifolds and even more is a Hermitian locally symmetric domain. So $\mathrm{Sh}_{K'}(\mathbf{G}, X)$ is a finite disjoint union of Hermitian locally symmetric domains. As for $\mathrm{Sh}_K(\mathbf{G}, X)$, the finite-to-1 map $\mathrm{Sh}_{K'}(\mathbf{G}, X) \rightarrow \mathrm{Sh}_K(\mathbf{G}, X)$ then makes $\mathrm{Sh}_K(\mathbf{G}, X)$ into a finite union of complex orbifolds.

3.2.1 Two approximation theorems for algebraic groups

Let \mathbf{H} be an algebraic group defined over \mathbb{Q} . We will use the following approximation theorems.

- (*Real Approximation*) $\mathbf{H}(\mathbb{Q})$ is dense in $\mathbf{H}(\mathbb{R})$.
- (*Strong Approximation*) If \mathbf{H} is semi-simple and simply-connected, then $\mathbf{H}(\mathbb{Q})$ is dense in $\mathbf{H}(\mathbb{A}_f)$.

The definition of “simply-connected” will be recalled later in §3.2.5.

3.2.2 Preparation and adjoint Shimura data

Now let us introduce the *adjoint Shimura datum* $(\mathbf{G}^{\mathrm{ad}}, \overline{X})$ of (\mathbf{G}, X) . Take $h \in X^+$. Then h induces a morphism

$$\overline{h}: \mathbb{S} \xrightarrow{h} \mathbf{G}_{\mathbb{R}} \rightarrow \mathbf{G}_{\mathbb{R}}^{\mathrm{ad}}.$$

Hence we obtain a $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})$ -orbit $\overline{X} := \mathbf{G}^{\mathrm{ad}}(\mathbb{R})\overline{h}$ in $\mathrm{Hom}(\mathbb{S}, \mathbf{G}_{\mathbb{R}}^{\mathrm{ad}})$, with a natural map $X \rightarrow \overline{X}$. The image of X^+ in \overline{X} is connected, and the following lemma (applied to $G = \mathbf{G}(\mathbb{R})$)^[2] easily implies that this image is again a connected component of \overline{X} . So by abuse of notation, we will also use X^+ to denote a connected component of \overline{X} .

Lemma 3.2.2. *For any algebraic group G over \mathbb{R} , the adjoint quotient $G^+ \rightarrow (G^{\mathrm{ad}})^+$ is surjective when restricted to the identity component.*

^[2]Here is a background for this lemma. Let $\varphi: H \rightarrow H'$ be a morphism of algebraic groups defined over k . Assume $\mathrm{char}(k) = 0$. Then φ is called *surjective* if $\varphi(H(\overline{k})) = H'(\overline{k})$. If φ is surjective, it may happen that $\varphi(H(k)) \neq H'(k)$!

We omit the proof of this lemma. Define

$$\begin{aligned}\mathbf{G}(\mathbb{R})_+ &:= \text{inverse image of } \mathbf{G}^{\text{ad}}(\mathbb{R})^+ \text{ in } \mathbf{G}(\mathbb{R}) \\ \mathbf{G}(\mathbb{Q})_+ &:= \mathbf{G}(\mathbb{Q}) \cap \mathbf{G}(\mathbb{R})_+.\end{aligned}\tag{3.2.2}$$

Lemma 3.2.3. $\mathbf{G}(\mathbb{R})_+$ is the stabilizer of X^+ , i.e. $\mathbf{G}(\mathbb{R})_+ = \{g \in \mathbf{G}(\mathbb{R}) : gX^+ = X^+\}$.

With Lemma 3.2.3, we can complete our more precise version of 3.2.1:

$$\text{Sh}_K(\mathbf{G}, X) \simeq \bigsqcup_{[g] \in \mathbf{G}(\mathbb{Q})_+ \backslash \mathbf{G}(\mathbb{A}_f)/K} \Gamma_g \backslash X^+, \tag{3.2.3}$$

with $\Gamma_g := gKg^{-1} \cap \mathbf{G}(\mathbb{Q})_+$; replacing K by a suitable finite-indexed subgroup K' guarantees that Γ_g is torsion-free, see 3.2.4. The finiteness of the double coset $\mathbf{G}(\mathbb{Q})_+ \backslash \mathbf{G}(\mathbb{A}_f)/K$ will be proved in 3.2.5; the proof uses the *Strong Approximation Theorem*.

Proof of Lemma 3.2.3. Consider the action of $\mathbf{G}^{\text{ad}}(\mathbb{R})$ on \overline{X} , and recall that X^+ is a connected component of \overline{X} . It suffices to prove that $\mathbf{G}^{\text{ad}}(\mathbb{R})^+ = \{g \in \mathbf{G}^{\text{ad}}(\mathbb{R}) : gX^+ = X^+\}$. This follows from general theory of Hermitian symmetric domains (and some knowledge on \mathbb{R} -algebraic groups *v.s.* real Lie groups) which we will not cover in this course. \square

3.2.3 Proof of 3.2.3

We start by showing that there is a bijection

$$\mathbf{G}(\mathbb{Q})_+ \backslash X^+ \times \mathbf{G}(\mathbb{A}_f) \xrightarrow{\sim} \mathbf{G}(\mathbb{Q}) \backslash X \times \mathbf{G}(\mathbb{A}_f), \quad [x, g] \mapsto [x, g]. \tag{3.2.4}$$

Injectivity: Assume $(x, g), (x', g') \in X^+ \times \mathbf{G}(\mathbb{A}_f)$ are mapped to the same point on the right hand side. Then there exists $q \in \mathbf{G}(\mathbb{Q})$ such that $(x', g') = q(x, g) = (qx, qg)$. Hence $qX^+ \cap X^+$ is non-empty as it contains $qx = x'$. So $qX^+ = X^+$. So $q \in \mathbf{G}(\mathbb{R})_+ \cap \mathbf{G}(\mathbb{Q}) = \mathbf{G}(\mathbb{Q})_+$. This proves the injectivity of the map above.

Surjectivity: Assume $(x, g) \in X \times \mathbf{G}(\mathbb{A}_f)$. By the *Real Approximation* in 3.2.1, $\mathbf{G}(\mathbb{Q})x$ is dense in $\mathbf{G}(\mathbb{R})x = X$. So $\mathbf{G}(\mathbb{Q})x \cap X^+ \neq \emptyset$, and hence there exists $q \in \mathbf{G}(\mathbb{Q})$ such that $qx \in X^+$. Then $(qx, qg) \in X^+ \times \mathbf{G}(\mathbb{A}_f)$, and its image under 3.2.4 is $[x, g]$. We are done for the surjectivity of 3.2.3.

Now let us prove the bijectivity of the map

$$\bigsqcup_{[g] \in \mathbf{G}(\mathbb{Q})_+ \backslash \mathbf{G}(\mathbb{A}_f)/K} \Gamma_g \backslash X^+ \rightarrow \mathbf{G}(\mathbb{Q})_+ \backslash X^+ \times \mathbf{G}(\mathbb{A}_f)/K, \quad \Gamma_g x \mapsto [x, g]. \tag{3.2.5}$$

Injectivity: If $[x', g'] = [x, g]$, then $(qx, qgk) = (x', g')$ for some $q \in \mathbf{G}(\mathbb{Q})_+$ and $k \in K$. So $[g] = [g']$ in $\mathbf{G}(\mathbb{Q})_+ \backslash \mathbf{G}(\mathbb{A}_f)/K$. Hence it suffices to prove the injectivity of $\Gamma_g \backslash X^+ \rightarrow \mathbf{G}(\mathbb{Q})_+ \backslash X^+ \times \mathbf{G}(\mathbb{A}_f)/K$. Now if $[x', g] = [x, g]$, then $(qx, qgk) = (x', g)$ for some $q \in \mathbf{G}(\mathbb{Q})_+$ and $k \in K$. So $q = gk^{-1}g^{-1} \in gKg^{-1}$. So $q \in \Gamma_g = gKg^{-1} \cap \mathbf{G}(\mathbb{Q})_+$. Thus we have proved the injectivity of 3.2.5.

Surjectivity: Let $[x, g]$ be an element of the right hand side. Then it is the image of $\Gamma_g x$.

We have thus proved 3.2.3. \square

3.2.4 Torsion-free subgroup

Here is a choice of K' so that $gK'g^{-1} \cap \mathbf{G}(\mathbb{Q})_+$ is torsion-free for all $g \in \mathbf{G}(\mathbb{A}_f)$. Take a faithful representation V of \mathbf{G} . Then there exists a lattice L in V such that $\widehat{L} := L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ is fixed by K . Equivalently, we are embedding \mathbf{G} as a closed subgroup of \mathbf{GL}_N over \mathbb{Q} such that K is a subgroup of $\mathbf{GL}_N(\widehat{\mathbb{Z}})$. Let $\ell \geq 3$ be an integer. Take K' to be the subgroup of K which acts trivially on $\widehat{L}/\ell\widehat{L}$, or equivalently

$$K' := \{g \in K < \mathbf{GL}_N(\widehat{\mathbb{Z}}) : g \equiv I_N \pmod{\ell}\}.$$

Then any element $\gamma \in \Gamma_g := gK'g^{-1} \cap \mathbf{G}(\mathbb{Q})_+ < \mathbf{GL}(V)$ acts trivially on $\widehat{g\widehat{L}}/\ell\widehat{g\widehat{L}}$, so all the eigenvalues of γ are 1 (as they are 1 modulo $\ell \geq 3$). So $\gamma = 1$ if γ is torsion. So Γ_g is torsion-free.

3.2.5 The group of connected components of a Shimura variety

In this subsection, we prove the finiteness of the double coset $\mathbf{G}(\mathbb{Q})_+ \backslash \mathbf{G}(\mathbb{A}_f)/K$. This finishes the proof of Theorem [3.2.1](#), and shows that $\pi_0(\text{Sh}_K(\mathbf{G}, X)) \simeq \mathbf{G}(\mathbb{Q})_+ \backslash \mathbf{G}(\mathbb{A}_f)/K$.

Case: simply-connected derived subgroup

The result in this case is better, with a clear understanding of the group $\pi_0(\text{Sh}_K(\mathbf{G}, X))$. Consider the short exact sequence of \mathbb{Q} -groups

$$1 \rightarrow \mathbf{G}^{\text{der}} \rightarrow \mathbf{G} \rightarrow \mathbf{T} := \mathbf{G}/\mathbf{G}^{\text{der}} \rightarrow 1$$

with \mathbf{T} an algebraic torus defined over \mathbb{Q} .

Definition 3.2.4. *An algebraic group H defined over a field k of characteristic 0 is said to be **simply-connected** if any central isogeny $H' \rightarrow H$ (i.e. a surjective morphism whose kernel is finite and contained in the center of H') is an isomorphism.*

Theorem 3.2.5. *Assume \mathbf{G}^{der} is simply-connected. Then $\nu(\mathbf{G}(\mathbb{Q})_+)$ has finite index in $\mathbf{G}(\mathbb{Q})$, $\nu(K)$ is a compact open subgroup of $\mathbf{T}(\mathbb{A}_f)$, and $\nu(\mathbf{G}(\mathbb{Q})_+) \backslash \mathbf{T}(\mathbb{A}_f)/\nu(K)$ is a finite abelian group. Moreover, ν induces a natural isomorphism of groups*

$$\pi_0(\text{Sh}_K(\mathbf{G}, X)) \simeq \nu(\mathbf{G}(\mathbb{Q})_+) \backslash \mathbf{T}(\mathbb{A}_f)/\nu(K).$$

Before proving this theorem, we point out without proof that

$$\nu(\mathbf{G}(\mathbb{Q})_+) = \mathbf{T}(\mathbb{Q}) \cap \nu(Z(\mathbf{G})(\mathbb{R})) =: \mathbf{T}(\mathbb{Q})^\dagger. \quad (3.2.6)$$

Proof. General theory of semi-simple simply-connected \mathbb{Q} -groups asserts that $\mathbf{G}^{\text{der}}(\mathbb{R})$ is connected. Therefore $\mathbf{G}^{\text{der}}(\mathbb{R})$ stabilizes X^+ and hence is contained in $\mathbf{G}(\mathbb{R})_+$ by Lemma [3.2.3](#). So $\mathbf{G}^{\text{der}}(\mathbb{Q}) \subseteq \mathbf{G}(\mathbb{Q})_+$. By the *Strong Approximation Theorem* from [§3.2.1](#), $\mathbf{G}^{\text{der}}(\mathbb{Q})$ is dense in $\mathbf{G}^{\text{der}}(\mathbb{A}_f)$. Hence

$$\mathbf{G}^{\text{der}}(\mathbb{A}_f) = \mathbf{G}^{\text{der}}(\mathbb{Q}) \cdot (K \cap \mathbf{G}^{\text{der}}(\mathbb{A}_f)) \subseteq \mathbf{G}(\mathbb{Q})_+ \cdot (K \cap \mathbf{G}^{\text{der}}(\mathbb{A}_f)). \quad (3.2.7)$$

Because \mathbf{G}^{der} is simply-connected, the short exact sequence of groups above Theorem [3.2.5](#) induces a short exact sequence

$$1 \rightarrow \mathbf{G}^{\text{der}}(\mathbb{A}_f) \rightarrow \mathbf{G}(\mathbb{A}_f) \xrightarrow{\nu} \mathbf{T}(\mathbb{A}_f) \rightarrow 1.$$

Here we use the knowledge on semi-simple simply-connected \mathbb{Q} -groups that $H^1(\mathbb{Q}_p, \mathbf{G}^{\text{der}}) = 0$ for any prime p .

Now ν induces a map

$$\mathbf{G}(\mathbb{Q})_+ \backslash \mathbf{G}(\mathbb{A}_f) / K \rightarrow \nu(\mathbf{G}(\mathbb{Q})_+) \backslash \mathbf{T}(\mathbb{A}_f) / \nu(K), \quad (3.2.8)$$

which, by (3.2.7), is a bijection. The right hand side is an abelian group because \mathbf{T} is an algebraic torus (hence abelian).

Now to prove the theorem, it remains to prove:

- (i) $\nu(\mathbf{G}(\mathbb{Q}))$ has finite index in $\mathbf{T}(\mathbb{Q})$.
- (ii) $\nu(K)$ is a compact open subgroup of $\mathbf{T}(\mathbb{A}_f)$.
- (iii) The right hand side of (3.2.8) is finite.

Let us prove (i). The Hasse Principle for simply-connected \mathbb{Q} -groups says that the natural map $H^1(\mathbb{Q}, \mathbf{G}^{\text{der}}) \rightarrow \prod_{p \leq \infty} H^1(\mathbb{Q}_p, \mathbf{G}^{\text{der}}) = H^1(\mathbb{R}, \mathbf{G}^{\text{der}})$ is injective; here we used again the fact that $H^1(\mathbb{Q}_p, \mathbf{G}^{\text{der}}) = 0$ for any prime number p (as \mathbf{G}^{der} is furthermore semi-simple). So by the diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbf{G}^{\text{der}}(\mathbb{Q}) & \longrightarrow & \mathbf{G}(\mathbb{Q}) & \longrightarrow & \mathbf{T}(\mathbb{Q}) \longrightarrow H^1(\mathbb{Q}, \mathbf{G}^{\text{der}}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbf{G}^{\text{der}}(\mathbb{R}) & \longrightarrow & \mathbf{G}(\mathbb{R}) & \longrightarrow & \mathbf{T}(\mathbb{R}) \longrightarrow H^1(\mathbb{R}, \mathbf{G}^{\text{der}}) \end{array}$$

we get that $\mathbf{T}(\mathbb{Q})/\nu(\mathbf{G}(\mathbb{Q})) \rightarrow \mathbf{T}(\mathbb{R})/\nu(\mathbf{G}(\mathbb{R}))$ is injective. But $\nu(\mathbf{G}(\mathbb{R})^+) = \mathbf{T}(\mathbb{R})^+$. So $\mathbf{T}(\mathbb{R})/\nu(\mathbf{G}(\mathbb{R}))$ is finite, and hence $\mathbf{T}(\mathbb{Q})/\nu(\mathbf{G}(\mathbb{Q}))$ is finite. This establishes the claim.

For (ii), we extend $\mathbf{G} \rightarrow \mathbf{T}$ to a morphism of group schemes over $\mathbb{Z}[1/N]$ for some integer N , and prove that $\mathbf{G}(\mathbb{Z}_p) \rightarrow \mathbf{T}(\mathbb{Z}_p)$ is surjective for almost all prime p . We first work on \mathbb{F}_p and then list using an argument similar to Newton's Lemma. We omit this proof.

Now we prove (iii). It suffices to prove that $\mathbf{T}(\mathbb{Q}) \backslash \mathbf{T}(\mathbb{A}_f) / \nu(K)$ is finite, and up to replacing $\nu(K)$ by a smaller compact open subgroup we may assume $\nu(K) \subseteq \mathbf{T}(\widehat{\mathbb{Z}})$. As $[\mathbf{T}(\widehat{\mathbb{Z}}) : \nu(K)]$ is finite (since $\mathbf{T}(\widehat{\mathbb{Z}})$ is compact and $\nu(K)$ is open), it suffices to prove that

$$\mathbf{T}(\mathbb{Q}) \backslash \mathbf{T}(\mathbb{A}_f) / \mathbf{T}(\widehat{\mathbb{Z}})$$

is finite. This is exactly the *class group* of the algebraic torus \mathbf{T} which is known to be finite by classical theory (and this number is called the *class number* of \mathbf{T}). In the case where $\mathbf{T} = \text{Res}_{K/\mathbb{Q}} \mathbb{G}_m$ for a number field K , this is exactly the class group of K . \square

General case

Let $\tilde{\mathbf{G}}$ be the universal cover of \mathbf{G}^{der} , i.e. $\tilde{\mathbf{G}}$ is simply-connected with a central isogeny (surjective with finite kernel contained in the center) $u: \tilde{\mathbf{G}} \rightarrow \mathbf{G}^{\text{der}}$. Then we have a surjective morphism of \mathbb{Q} -groups

$$\varphi: \mathbf{G}' := Z(\mathbf{G}) \times \tilde{\mathbf{G}} \rightarrow \mathbf{G}, \quad (z, g) \mapsto zu(g)$$

which is a central isogeny. Thus to prove the finiteness of $\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}_f) / K$, it suffices to prove the finiteness of

$$\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K'$$

for K' a compact open subgroup of $\mathbf{G}'(\mathbb{A}_f)$. But the derived subgroup of \mathbf{G}' is $\tilde{\mathbf{G}}$ which is simply-connected. So we are back to the previous case, and hence $\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}_f) / K'$ is finite. So $\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}_f) / K$ is finite.

3.3 Siegel modular variety

Take the example of Siegel case in Example 3.1.3 and Example 3.1.6. In particular $V = \mathbb{Q}^{2d}$, $\psi: V \times V \rightarrow \mathbb{Q}$ is $(x, y) \mapsto x^t J y$ with $J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}$. Thus there is a lattice L in V such that ψ restricts to $L \times L \rightarrow \mathbb{Z}$. To simplify notation, denote by $L = V(\mathbb{Z})$.

The Siegel Shimura datum is $(\mathbf{GSp}_{2d}, \mathfrak{H}_d^\pm)$. For each N , set

$$\begin{aligned} K(N) &:= \left\{ g \in \mathbf{GSp}_{2d}(\mathbb{A}_f) : gV(\widehat{\mathbb{Z}}) \subseteq V(\widehat{\mathbb{Z}}) \text{ and acts trivially on } V(\widehat{\mathbb{Z}})/NV(\widehat{\mathbb{Z}}) \right\} \\ &= \left\{ g \in \mathbf{GSp}_{2d}(\widehat{\mathbb{Z}}) : g \equiv I_{2d} \pmod{N} \right\}. \end{aligned}$$

Then we have the Shimura variety $\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d}, \mathfrak{H}_d^\pm)$.

Theorem 3.3.1. *Assume $N \geq 3$. Then $\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d}, \mathfrak{H}_d^\pm)$ is the fine moduli space of principally polarized abelian varieties of dimension d with a level- N -structure, i.e. there is a canonical bijection between*

- the \mathbb{C} -points of $\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d}, \mathfrak{H}_d^\pm)$,
- and the isomorphism classes of the triples (A, λ, η_N) where A is a complex abelian variety of dimension d , λ is a principal polarization on A , and η_N is a level- N -structure on A .

When $N = 1, 2$, the Shimura variety is a coarse moduli space.

Let us explain the meaning of this theorem. Let A be an abelian variety defined over \mathbb{C} .

- (i) A *principal polarization* on A is a polarization on the Hodge structure $H_1(A, \mathbb{Z})$ with determinant 1, i.e. an alternating pairing $\lambda: H_1(A, \mathbb{Z}) \times H_1(A, \mathbb{Z}) \rightarrow \mathbb{Z}$, which under suitable \mathbb{Z} -basis of $H_1(A, \mathbb{Z})$ is $\begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}$. In more geometric terms, it is an isomorphism $\lambda: A \xrightarrow{\sim} A^\vee$.
- (ii) A *(symplectic) level- N -structure* on A is a basis of $H_1(A, \mathbb{Z}/N\mathbb{Z})$ which is symplectic with respect to λ . In more geometric terms, it is a basis of the $\mathbb{Z}/N\mathbb{Z}$ -module $A[N]$ which is symplectic under $e_N: A[N] \times A[N] \xrightarrow{(1, \lambda)} A[N] \times A^\vee[N] \rightarrow \mu_N$ where last map is the Weil pairing. Or more concretely, it is an isomorphism

$$\eta_N: A[N] \xrightarrow{\sim} H_1(A, \mathbb{Z}/N\mathbb{Z})$$

such that the two composites

$$\begin{aligned} A[N] \times A[N] &\xrightarrow{(\eta_N, \eta_N)} H_1(A, \mathbb{Z}/N\mathbb{Z}) \times H_1(A, \mathbb{Z}/N\mathbb{Z}) \xrightarrow{\bar{\lambda}} \mathbb{Z}/N\mathbb{Z} \\ \text{and } A[N] \times A[N] &\xrightarrow{e_N} \mu_N \xrightarrow{e^{2\pi\sqrt{-1}a/N} \mapsto [a]} \mathbb{Z}/N\mathbb{Z} \end{aligned}$$

differ from the multiplication by an element in $[\ell] \in (\mathbb{Z}/N\mathbb{Z})^\times$, and we say that this level- N -structure *has twist* $[\ell]$.

Proof. Recall that each point in \mathfrak{H}_d^\pm parametrizes a \mathbb{Q} -Hodge structure on V of type $(-1, 0) + (0, -1)$; see 2.3.1.

We shall use Theorem 3.2.1 and the more precise version 3.2.3, and better, Theorem 3.2.5 because \mathbf{Sp}_{2d} is simply-connected. One can compute that $\mathbf{GSp}_{2d}(\mathbb{R})_+ = \mathbf{GSp}_{2d}(\mathbb{R})^+ = \{g \in \mathbf{GSp}_{2d}(\mathbb{R}) : \det(g) > 0\}$. So $\mathbf{GSp}_{2d}(\mathbb{Q})_+ = \{g \in \mathbf{GSp}_{2d}(\mathbb{Q}) : \det(g) > 0\}$. Thus for the quotient

$$1 \rightarrow \mathbf{Sp}_{2d} \rightarrow \mathbf{GSp}_{2d} \xrightarrow{\nu} \mathbb{G}_m \rightarrow 1,$$

we have $\nu(\mathbf{GSp}_{2d}(\mathbb{Q})_+) = \mathbb{Q}_{>0}$ [3]. It is not hard to compute that $\nu(K(N)) = \{z \in \widehat{\mathbb{Z}} : z \equiv 1 \pmod{N}\} = 1 + N\mathbb{Z}$. Thus

$$\pi_0(\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d}, \mathfrak{H}_d^\pm)) \simeq \mathbb{Q}_{>0} \backslash \mathbb{A}_f^\times / (1 + N\widehat{\mathbb{Z}}) \simeq (\mathbb{Z}/N\mathbb{Z})^\times.$$

Write $\Gamma_{[\ell]} \backslash \mathfrak{H}_d^+$ for the connected component of $\mathrm{Sh}_{K(N)}(\mathbf{GSp}_{2d}, \mathfrak{H}_d^\pm)$ indexed by $[\ell] \in (\mathbb{Z}/N\mathbb{Z})^\times$. Below we only give the constructions of the two directions, without proving that they are inverse to each other.

Given a triple (A, λ, η_N) . Assume that the level- N -structure has twist $[\ell] \in (\mathbb{Z}/N\mathbb{Z})^\times$. First $H_1(A, \mathbb{Z})$ is a \mathbb{Z} -Hodge structure of type $(-1, 0) + (0, -1)$, and hence under suitable isomorphism $(H_1(A, \mathbb{Z}), \lambda) \simeq (V(\mathbb{Z}), \psi)$ we obtain a point $\tau \in \mathfrak{H}_d^+$. Then we get a point in $\Gamma_{[\ell]} \backslash \mathfrak{H}_d^+$ as the image of τ under $\mathfrak{H}_d^+ \rightarrow \Gamma_{[\ell]} \backslash \mathfrak{H}_d^+$.

Conversely let $x \in \Gamma_{[\ell]} \backslash \mathfrak{H}_d^+$. Let τ be a pre-image of x under the quotient $\mathfrak{H}_d^+ \rightarrow \Gamma_{[\ell]} \backslash \mathfrak{H}_d^+$. Recall that τ parametrizes a \mathbb{Q} -Hodge structure on V of type $(-1, 0) + (0, -1)$, and thus we can endow $V(\mathbb{R})$ with a complex structure by the bijection $V(\mathbb{R}) \subseteq V(\mathbb{C}) \rightarrow V(\mathbb{C})/V_\tau^{0,-1}$. This makes $A_\tau := V(\mathbb{R})/V(\mathbb{Z})$ into a compact complex torus of dimension d , with $H_1(A_\tau, \mathbb{Z}) = V(\mathbb{Z})$. Thus ψ induces a principle polarization via $H_1(A_\tau, \mathbb{Z})$. Hence A_τ is an abelian variety with a principal polarization which by abuse of notation we still use ψ to denote. The level- N -structure on A_τ is given as follows. We have $A_\tau[N] = \frac{1}{N}V(\mathbb{Z})/V(\mathbb{Z}) = V(\mathbb{Z})/NV(\mathbb{Z}) = V(\mathbb{Z}/N\mathbb{Z})$. Take $g \in \mathbf{GSp}_{2d}(\widehat{\mathbb{Z}})$ such that $\nu(g) \in \widehat{\mathbb{Z}}^\times$ is congruent to ℓ modulo $1 + N\widehat{\mathbb{Z}}$. Then g induces an isomorphism $g: V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}}) \xrightarrow{\sim} V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}})$. But $V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}}) = V(\mathbb{Z}/N\mathbb{Z}) = H_1(A_\tau, \mathbb{Z}/N\mathbb{Z})$. Thus we have $A_\tau[N] = V(\mathbb{Z}/N\mathbb{Z}) = V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}}) \xrightarrow{g} V(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}}) = H_1(A_\tau, \mathbb{Z}/N\mathbb{Z})$. This is the desired level- N -structure because $\psi(gx, gy) = \nu(g)\psi(x, y)$ by definition of \mathbf{GSp}_{2d} . \square

3.4 CM abelian varieties and special points

Let $\mathrm{Sh}_K(\mathbf{G}, X)$ be a Shimura variety. In Definition 3.1.10 we defined *special points* on $\mathrm{Sh}_K(\mathbf{G}, X)$. They are of particular importance. For example, there exists a natural number field $E(\mathbf{G}, X)$, called the *reflex field* of (\mathbf{G}, X) , on which $\mathrm{Sh}_K(\mathbf{G}, X)$ is “naturally” defined (or in more vigorous terms, has a canonical model), characterized by the action of the Galois group of $E(\mathbf{G}, X)$. This action is explicitly defined for special points on $\mathrm{Sh}_K(\mathbf{G}, X)$ via the class field theory, and is uniquely determined in this way by the following theorem whose proof we omit:

Theorem 3.4.1. *The set of special points is dense in $\mathrm{Sh}_K(\mathbf{G}, X)$.*

Here “dense” is true even for the usual topology. The hard part of this theorem is to prove the existence of one special point. Indeed, assume $\mathrm{Sh}_K(\mathbf{G}, X) \simeq \bigsqcup \Gamma_g \backslash X^+$ has a special point $[x]$. Then its inverse image x in X^+ gives rise to a morphism $x: \mathbb{S} \rightarrow \mathbf{G}_{\mathbb{R}}$ which factors through $\mathbf{T}_{\mathbb{R}}$ for an algebraic torus $\mathbf{T} < \mathbf{G}$. But then the morphism given by $g \cdot x$ for any $g \in \mathbf{G}(\mathbb{Q})$ factors through $(g\mathbf{T}g^{-1})_{\mathbb{R}}$, with $g\mathbf{T}g^{-1}$ clearly an algebraic torus in \mathbf{G} (since it is abelian), and hence defines a Shimura datum $(g\mathbf{T}g^{-1}, g \cdot \mathbf{T}(\mathbb{R})x)$. But $\mathbf{T}(\mathbb{R})x$ is a finite set of points since \mathbf{T} is abelian. So the image of $\mathbf{G}(\mathbb{Q})x$ under the quotient $X^+ \rightarrow \Gamma_g \backslash X^+$ consists of special points

[3] In fact $\nu(g) = (\det g)^{1/d}$.

of $\mathrm{Sh}_K(\mathbf{G}, X)$. Notice that $X^+ = \mathbf{G}(\mathbb{R})^+x$. Now it suffice to use the Real Approximation that $\mathbf{G}(\mathbb{Q})$ is dense in $\mathbf{G}(\mathbb{R})$ to conclude.

For the existence of special points, we shall focus on the Siegel modular variety, for which we have:

Theorem 3.4.2. *Take $[x] \in \mathrm{Sh}_K(\mathbf{GSp}_{2d}, \mathfrak{H}_d^\pm)(\mathbb{C})$. Then $[x]$ is a special point if and only if the abelian variety A_x parametrized by $[x]$ is CM, i.e. $\mathrm{End}(A_x) \otimes_{\mathbb{Z}} \mathbb{Q}$ contains a commutative \mathbb{Q} -subalgebra of dimension $2d$. Equivalently, an abelian variety A defined over \mathbb{C} is CM if and only if the Mumford–Tate group of the \mathbb{Q} -Hodge structure $H_1(A, \mathbb{Q})$ is an algebraic torus.*

We will not give a full proof of this theorem, but only recall the definition of CM abelian varieties and give a brief explanation why the associated Mumford–Tate group (which we call the Mumford–Tate group of A) is an algebraic torus.

Assume A is a simple abelian variety. Then A is CM if and only if $E := \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a CM field, i.e. there exists a totally real field F such that E/F is a totally imaginary quadratic extension. Write $\bar{(\cdot)}$ for the complex conjugation with respect to E/F . Then there exists an element $\iota \in E$ such that $\bar{\iota} = -\iota$ (totally imaginary element). Then E can be endowed with the \mathbb{Q} -symplectic form

$$\langle x, y \rangle := \mathrm{Tr}_{E/\mathbb{Q}}(\bar{x}\iota y).$$

This makes $(E, \langle \cdot, \cdot \rangle) \simeq (\mathbb{Q}^{2d}, \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix})$ into a symplectic space. Set \mathbf{GU}_E to be the subgroup of \mathbf{GSp}_{2d} generated by $\mathbb{G}_m = Z(\mathbf{GSp}_{2d})$ and

$$\mathbf{U}_E := \{x \in \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m : x\bar{x} = 1\}.$$

Then one can check that \mathbf{GU}_E is an algebraic torus which contains the Mumford–Tate group of A . Thus the Mumford–Tate group of A is abelian, and hence must be an algebraic torus. In fact, one can check that \mathbf{GU}_E is a maximal torus of \mathbf{GSp}_{2d} .