

Chapter 6

Baily–Borel compactification

Let (\mathbf{G}, X) be a Shimura datum. By abuse of notation use X to denote a connected component. Let $\Gamma < \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup.

For simplicity, we will assume \mathbf{G} to be quasi-simple, *i.e.* \mathbf{G}^{der} is a simple group. For the purpose of compactifying $\Gamma \backslash X$ we can easily reduce to this case. Notice that $\mathbf{G}_{\mathbb{R}}^{\text{der}}$ may not be simple as an \mathbb{R} -group, so that X is not necessarily irreducible.

6.1 Baily–Borel compactification as a topological space

Consider the Harish–Chandra embedding

$$X \simeq \mathcal{D} \subseteq \mathfrak{m}^+ \simeq \mathbb{C}^N$$

with $\overline{\mathcal{D}}$ the closure of \mathcal{D} in \mathfrak{m}^+ . Let F be an analytic boundary component of X , with normalizer $N(F) = \{g \in \mathbf{G}^{\text{der}}(\mathbb{R})^+ : gF = F\}$.

Recall (5.1.3) the fundamental set Σ constructed from Siegel sets associated with a minimal rational proper parabolic subgroup \mathbf{P}_0 of \mathbf{G}^{der} .

Let $\overline{\Sigma}$ be the closure of $\Sigma \subseteq X \simeq \mathcal{D} \subseteq \mathfrak{m}^+$. Then $\overline{\Sigma} \subseteq \overline{\mathcal{D}}$, with an induced topology.

Theorem 6.1.1. *The followings are equivalent:*

- (1) $\Gamma F \cap \overline{\Sigma} \neq \emptyset$,
- (2) $\mathbf{P}_{0, \mathbb{R}} < N(F)$, and F is a rational analytic boundary component (*i.e.* $N(F)$ equals $\mathbf{P}_F(\mathbb{R})$ for a parabolic subgroup \mathbf{P}_F of \mathbf{G}^{der}).

Moreover in these cases, \mathbf{P}_F is a maximal proper parabolic subgroup of \mathbf{G}^{der} .

Theorem 6.1.1 indicates that we can do the following compactification of $\Gamma \backslash X$:

- (i) Define $\overline{X}^{\text{BB}} := \mathcal{D} \cup \bigsqcup_{\mathbf{P}} F_{\mathbf{P}} \subseteq \overline{\mathcal{D}}$, where \mathbf{P} runs over all *maximal* proper parabolic subgroup of \mathbf{G}^{der} and $F_{\mathbf{P}}$ is the rational analytic boundary component $F_{\mathbf{P}}$.
- (ii) Endow \overline{X}^{BB} with the Satake topology.
- (iii) The space $\Gamma \backslash \overline{X}^{\text{BB}}$ is then a compact Hausdorff space containing $\Gamma \backslash X$ as an open dense subset.

Then $(\Gamma \backslash \overline{X})^{\text{BB}} := \Gamma \backslash \overline{X}^{\text{BB}}$ is called the *Baily-Borel compactification* of $\Gamma \backslash X$, and

$$(\Gamma \backslash \overline{X})^{\text{BB}} = \Gamma \backslash X \cup \bigsqcup_{j=1}^m \Gamma_{F_j} \backslash F_j, \quad (6.1.1)$$

where F_1, \dots, F_m are rational analytic boundary components such that $\{\mathbf{P}_{F_1}, \dots, \mathbf{P}_{F_m}\}$ is a set of representatives of Γ -conjugacy classes of maximal proper parabolic subgroups of \mathbf{G}^{der} , with $\Gamma_{F_j} := \Gamma \cap \mathbf{P}_{F_j}(\mathbb{Q})$.

6.1.1 Satake topology on \overline{X}^{BB}

The *Satake topology* on \overline{X}^{BB} is defined as follows. For each $x \in \overline{X}^{\text{BB}} \subseteq \overline{\mathcal{D}}$, the neighborhoods of any point $x \in X^*$ is the saturations of the neighborhoods of the corresponding points in $\overline{\Sigma}$ under the action of $\Gamma_x := \{\gamma \in \Gamma : \gamma \cdot x = x\}$. More precisely, a fundamental system of neighborhoods of x is given by all subsets $U \subseteq \overline{\mathcal{D}}$ such that

$$\Gamma_x \cdot U = U,$$

and such that $\gamma U \cap \overline{\Sigma}$ is a neighborhood of $\gamma \cdot x$ in $\overline{\Sigma}$ whenever $\gamma \cdot x \in \overline{\Sigma}$.

Proposition 6.1.2. *The Satake topology is the unique topology on \overline{X}^{BB} such that the followings hold:*

- (i) *it induces the original topologies on $\overline{\Sigma}$ and on X ,*
- (ii) *the $\mathbf{G}^{\text{der}}(\mathbb{Q})$ -action on \overline{X}^{BB} is continuous,*
- (iii) *for any $x \in \overline{X}^{\text{BB}}$, there exists a fundamental system of neighborhoods $\{U\}$ of x such that*

$$\gamma U = U \text{ for all } \gamma \in \Gamma_x \quad \text{and} \quad \gamma U \cap U = \emptyset \text{ for all } \gamma \notin \Gamma_x,$$

- (iv) *if $x, x' \in \overline{X}^{\text{BB}}$ are not in one Γ -orbit, then there exist neighborhoods U of x and U' of x' such that*

$$\Gamma U \cap U' = \emptyset.$$

Corollary 6.1.3. $\Gamma \backslash \overline{X}^{\text{BB}}$ *is Hausdorff.*

6.1.2 \mathbb{Q} -roots vs \mathbb{R} -roots, and \mathbb{Q} -polydisc

Let $\mathbf{S} < \mathbf{P}_0$ be a maximal \mathbb{Q} -split tours, and let K_∞ be a maximal compact subgroup of $\mathbf{G}_{\mathbb{R}}^{\text{der}}$ such that $\text{Lie} K_\infty \cap \text{Lie} \mathbf{S}_{\mathbb{R}} = 0$.^[1] Then there exists $x_0 \in X$ such that $\text{Stab}_{\mathbf{G}^{\text{der}}(\mathbb{R})}(x_0) = K_\infty$.

In §4.6.2 and §4.6.3, we discussed the complex roots and real roots respectively. Now we need to analyze the \mathbb{Q} -roots of \mathbf{G}^{der} . First, we can make an appropriate choice of K_∞ such that $\mathbf{S}_{\mathbb{R}} < A$ with A from §4.6.3.

Let ${}_{\mathbb{R}}\Psi = \{\gamma_1, \dots, \gamma_r\}$ be as in (4.6.13); it is obtained from a set of strongly orthogonal complex roots $\Psi = \{\alpha_1, \dots, \alpha_r\}$ from (4.6.4). If $\mathbf{G}_{\mathbb{R}}^{\text{der}}$ is simple, we described the real roots ${}_{\mathbb{R}}\Phi = \Phi(A, \mathbf{G}_{\mathbb{R}}^{\text{der}})$ in terms of $\gamma_1, \dots, \gamma_r$ in Proposition 4.6.12. It turns out that one can also do this for the rational roots ${}_{\mathbb{Q}}\Phi := \Phi(\mathbf{S}, \mathbf{G}^{\text{der}})$ when \mathbf{G}^{der} is simple.

^[1]Even strongly, $\text{Lie} K_\infty$ is orthogonal to $\text{Lie} \mathbf{S}_{\mathbb{R}}$ for the Killing form on $\text{Lie} \mathbf{G}_{\mathbb{R}}^{\text{der}}$.

Proposition 6.1.4. *There is a partition*

$$\{1, \dots, s = \dim \mathbf{S}\} = I_0 \cup I_1 \cup \dots \cup I_r$$

such that

$$\begin{aligned} X(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{Q} &\simeq \frac{X(A) \otimes_{\mathbb{Z}} \mathbb{Q}}{\{\text{subspace spanned by } \gamma_i \text{ for } i \in I_0 \text{ and } \gamma_i - \gamma_j \text{ for } i, j \in I_\ell, \ell > 0\}} \\ &\simeq \sum_{j=1}^s \mathbb{Q}\beta_j, \quad \text{where } \beta_j = \text{image of any } \gamma_m \text{ with } m \in I_j. \end{aligned}$$

In particular, $\mathbf{S} = \{\gamma_i = 1 \text{ for } i \in I_0; \gamma_i = \gamma_j \text{ for } i, j \in I_\ell, \text{ where } \ell > 0\}$.

Corollary 6.1.5. *Recall our assumption that \mathbf{G}^{der} is simple. One of the two cases occurs:*

- (Type C_s) ${}_{\mathbb{Q}}\Phi = \{\pm \frac{1}{2}(\beta_i + \beta_j) \text{ for } i \geq j, \pm \frac{1}{2}(\beta_i - \beta_j) \text{ for } i > j\}$.
- (Type BC_s) ${}_{\mathbb{Q}}\Phi = \{\pm \frac{1}{2}(\beta_i + \beta_j) \text{ for } i \geq j, \pm \frac{1}{2}(\beta_i - \beta_j) \text{ for } i > j, \pm \frac{1}{2}\beta_i\}$.

If we order the roots such that $\beta_1 > \dots > \beta_s$, then the set of simple roots is:

- (Type C_s) ${}_{\mathbb{R}}\Delta = \{\mu_1 := \frac{1}{2}(\beta_1 - \beta_2), \dots, \mu_{s-1} := \frac{1}{2}(\beta_{s-1} - \beta_s), \mu_s := \beta_s\}$.
- (Type BC_s) ${}_{\mathbb{R}}\Delta = \{\mu_1 := \frac{1}{2}(\beta_1 - \beta_2), \dots, \mu_{s-1} := \frac{1}{2}(\beta_{s-1} - \beta_s), \mu_s := \frac{1}{2}\beta_s\}$.

The proof goes as follows: We have the group-theoretic result that $\mathbf{G}^{\text{der}} = \text{Res}_{k/\mathbb{Q}} \mathbf{G}'$ for some absolutely simple k -group \mathbf{G}' with k a totally real number field. Then $\mathbf{G}_{\mathbb{R}}^{\text{der}} = \prod_{\sigma: k \hookrightarrow \mathbb{R}} \mathbf{G}'_{\sigma}$ with each \mathbf{G}'_{σ} a group defined over $\sigma(k) \subseteq \mathbb{R}$. Then one analyzes each factor and use the Galois action.