HEIGHTS AND PERIODS OF ALGEBRAIC CYCLES IN FAMILIES

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Dedicated to Gerd Faltings's 70th birthday

Abstract. We consider the Beilinson–Bloch heights and Abel–Jacobian periods of homologically trivial Chow cycles in families. For the Beilinson–Bloch heights, we show that for any $g \geq 3$, we can construct a Zariski open dense subset $\mathcal{M}_g^{\mathrm{amp}}$ of $\mathcal{M}_g$, the coarse moduli of curves of genus g over $\mathbb{Q}$, such that the heights of Ceresa cycles and Gross–Schoen cycles over $\mathcal{M}_g^{\mathrm{amp}}$ satisfy the Northcott property. For the Abel–Jacobi periods, we provide an algebraic criterion for the existence of a Zariski open dense subset of any family such that all cycles not defined over $\overline{\mathbb{Q}}$ are non-torsion and verify that this criterion holds for Ceresa cycles and Gross–Schoen cycles.

Contents

1. Introduction

For a curve over $\mathbb{C}$, the classical Abel–Jacobi map is a bijection

$$\mathrm{AJ}\colon \mathrm{Pic}^0(C) \xrightarrow{\sim} \mathrm{Jac}(C).$$

When C is defined over a number field k, one can define the Néron–Tate height on $\mathrm{Jac}(C)(\bar{k})$ using the canonical polarization. This height has a Northcott property, which gives positivity of the height pairing, which in turn is an important ingredient for the proof of the Mordell–Weil theorem and the formulation of the Birch and Swinnerton-Dyer conjecture. Notice that by the Hodge index theorem of Faltings and Hriljac, these heights can also be defined as arithmetic intersections on integral models of C.

In the 1950s-1960s, Weil and Griffiths extended the Abel–Jacobi map to homologically trivial cycles (say of codimension i) on smooth and projective varieties X over $\mathbb{C}$

$$\mathrm{AJ}^i\colon \mathrm{Ch}^i(X)_{\mathrm{hom}} \longrightarrow \mathrm{Jac}^i(X)$$

where $\mathrm{Jac}^i(X)$ are called *intermediate Jacobians* and are only complex tori in general. In the 1980s, Beilinson and Bloch independently proposed a conditional definition of heights for these cycles using integral models. They conjectured the positivity of their heights, an extension of the Mordell–Weil theorem, and an extension of the Birch and Swinnerton-Dyer conjecture. However, the positivity cannot be proved via a Northcott property because of the lack of algebraicity of $\mathrm{Jac}^i(X)$.

This paper aims to initiate a project to study Beilinson–Bloch heights and Abel–Jacobi periods for homologically trivial cycles in a family of smooth and projective varieties $f\colon X \longrightarrow S$ parametrized by a quasi-projective variety S. We have achieved two goals:

(1) The generic positivity and Northcott property for the Beilinson–Bloch heights of the Gross–Schoen cycles and Cerasa cycles parametrized by moduli of curves. These are the simplest situations where the Beilinson–Bloch heights are unconditionally defined.
(2) An algebraic criterion for the non-degeneracy of Abel–Jacobi periods for general families of cycles.

Our method of proof, roughly speaking, is to utilize the intermediate Jacobians to get some geometric positivity and then extend it to some arithmetic positivity. In the following, we describe the main results and the methods of proof in detail.

1.1. **Main results on Gross–Schoen and Ceresa cycles.** For a smooth projective irreducible curve C defined over a field k and a class $\xi \in \mathrm{Pic}^1(C)$, we have two homologically trivial 1-cycles:

- the Gross–Schoen cycle based at ξ defined by $\mathrm{GS}_\xi(C) := \Delta_{\mathrm{GS},\xi}(C) \in \mathrm{Ch}_1(C^3)$ obtained by modified the diagonal cycle in C^3 using base class ξ ([GS95, Zha10]);
- the Ceresa cycle based at ξ defined by $\mathrm{Ce}_\xi(C) := i_\xi(C) - [-1]_* i_\xi(C) \in \mathrm{Ch}_1(\mathrm{Jac}(C))$ defined by embedding $i_\xi\colon C \longrightarrow \mathrm{Jac}(C)$ via ξ ([Cer83]).

Let $\mathcal{M}_g$ be the moduli space of smooth projective curves of genus g, let $\mathcal{C}_g/\mathcal{M}_g$ be the universal curve, and let $\pi\colon \mathrm{Pic}^1(\mathcal{C}_g/\mathcal{M}_g) \longrightarrow \mathcal{M}_g$ be the relative Picard group of degree 1 on $\mathcal{C}_g/\mathcal{M}_g$. For each $s \in \mathcal{M}_g(\mathbb{C})$, denote by $\mathcal{C}_s$ the curve parametrized by s.

For each $\xi \in \mathrm{Pic}^1(\mathcal{C}_g/\mathcal{M}_g)(\mathbb{C})$, to ease notation denote by

$$\mathrm{GS}(\xi) := \mathrm{GS}_\xi(\mathcal{C}_s) \quad \text{and} \quad \mathrm{Ce}(\xi) := \mathrm{Ce}_\xi(\mathcal{C}_s)$$

for $s = \pi(\xi) \in \mathcal{M}_g(\mathbb{C})$. For each $s \in \mathcal{M}_g(\mathbb{C})$, a particularly important element $\xi_s \in \mathrm{Pic}^1(\mathcal{C}_s)$ is such that the divisor class of $(2g-2)\xi_s$ is the canonical divisor class on $\mathcal{C}_s$; it is well-defined up to $(2g-2)$-torsions.

Here are our main results.

Theorem 1.1. *For each $g \geq 3$, there exist a Zariski open dense subset $\mathcal{M}_g^{\mathrm{amp}}$ of $\mathcal{M}_g$ defined over $\mathbb{Q}$ and positive numbers ϵ and c, such that for any $s \in \mathcal{M}_g^{\mathrm{amp}}(\overline{\mathbb{Q}})$ and any $\xi \in \mathrm{Pic}^1(\mathcal{C}_s)$, we have*

$$\langle \mathrm{GS}(\xi), \mathrm{GS}(\xi) \rangle_{\mathrm{BB}} \geq \epsilon(h_{\mathrm{Fal}}(s) + h_{\mathrm{NT}}(\xi - \xi_s)) - c, \tag{1.1}$$

$$\langle \mathrm{Ce}(\xi), \mathrm{Ce}(\xi)\rangle_{\mathrm{BB}} \geq \epsilon(h_{\mathrm{Fal}}(s) + h_{\mathrm{NT}}(\xi - \xi_s)) - c,$$

where h_{Fal} *is the Faltings height on* $\mathcal{M}_g(\overline{\mathbb{Q}})$ *and* h_{NT} *is the canonical Néron–Tate height on* $\mathrm{Jac}(\mathcal{C}_s)(\overline{\mathbb{Q}})$.

The proof will be executed in §6.3, in which we give a construction of $\mathcal{M}_g^{\mathrm{amp}}$. Indeed, we will take $\mathcal{M}_g^{\mathrm{amp}}$ to be $\mathcal{M}_g \setminus \mathcal{M}_{g,\mathscr{F}}(1)$, where $\mathcal{M}_{g,\mathscr{F}}(1)$ is the Betti degeneracy locus (1.3) with the normal function ν being the Gross–Schoen normal function (C.5). A priori $\mathcal{M}_{g,\mathscr{F}}(1)$ is only a real analytic set. But we show that $\mathcal{M}_{g,\mathscr{F}}(1)$ is *proper* Zariski closed in $\mathcal{M}_{g,\mathbb{C}}$ by applying Ax-Schanuel multiple times; see Theorem 1.4 and Corollary 1.8. Finally, we show that it is defined over $\mathbb{Q}$ by volume calculation for adelic line bundles; see Corollary 6.4. Because of the importance of this Zariski open dense set $\mathcal{M}_g^{\mathrm{amp}}$ on which (1.1) holds, we will make the following definition: We call

(i) $\mathcal{M}_g^{\mathrm{amp}}$ the *ample locus* of the Gross–Schoen and Ceresa cycles;
(ii) $\mathcal{M}_g^{\mathrm{slim}} := \mathcal{M}_g \setminus \mathcal{M}_g^{\mathrm{amp}}$ the *slim locus* of the Gross–Schoen and Ceresa cycles. By the explanation above, it is the $\mathcal{M}_{g,\mathscr{F}}(1)$.

Theorem 1.1 yields the following Northcott property immediately.

Corollary 1.2 (Northcott property)**.** *For each* $g \geq 3$ *and the Zariski open dense subset* $\mathcal{M}_g^{\mathrm{amp}}$ *of the* $\mathcal{M}_g$ *defined over* $\mathbb{Q}$ *as in Theorem 1.1, we have the following Northcott property: For any* $H, D \in \mathbb{R}$,

$$\#\{\xi \in \pi^{-1}U(\overline{\mathbb{Q}}) : \quad \deg[\mathbb{Q}(\xi) : \mathbb{Q}] < D, \quad \langle \mathrm{GS}(\xi), \mathrm{GS}(\xi)\rangle_{\mathrm{BB}} < H\} < \infty,$$
$$\#\{\xi \in \pi^{-1}U(\overline{\mathbb{Q}}) : \quad \deg[\mathbb{Q}(\xi) : \mathbb{Q}] < D, \quad \langle \mathrm{Ce}(\xi), \mathrm{Ce}(\xi)\rangle_{\mathrm{BB}} < H\} < \infty.$$

In the course of the proof, we also establish the following geometric result on the torsion locus. For each $s \in \mathcal{M}_g(\mathbb{C})$, denote by $\mathrm{GS}(\mathcal{C}_s) := \mathrm{GS}(\xi_s)$ and $\mathrm{Ce}(\mathcal{C}_s) := \mathrm{Ce}(\xi_s)$.

Theorem 1.3. *For each* $g \geq 3$ *and the Zariski open dense subset* $\mathcal{M}_g^{\mathrm{amp}}$ *of the* $\mathcal{M}_g$ *defined over* $\mathbb{Q}$ *as in Theorem 1.1, the followings hold true:*

(i) $\mathrm{GS}(\mathcal{C}_s)$ *and* $\mathrm{Ce}(\mathcal{C}_s)$ *are both non-torsion in the Chow groups for all* $s \in \mathcal{M}_g^{\mathrm{amp}}(\mathbb{C}) \setminus \mathcal{M}_g^{\mathrm{amp}}(\overline{\mathbb{Q}})$.
(ii) *there exist at most countably many* $s \in \mathcal{M}_g^{\mathrm{amp}}(\mathbb{C})$ *such that* $\mathrm{AJ}(\mathrm{GS}(\mathcal{C}_s))$ *or* $\mathrm{AJ}(\mathrm{Ce}(\mathcal{C}_s))$ *is torsion in the intermediate Jacobians.*

Independely, Hain [Hai24] and Kerr–Tayou [KT24] proved weaker forms of Theorem with different methods, where $\mathcal{M}_g^{\mathrm{GS}}(\mathbb{C})$ is replaced by an open subset U of $\mathcal{M}_g(\mathbb{C})$ which is real analytic and Zariski open over $\mathbb{C}$ respectively.

The proofs of Theorem 1.1, Corollary 1.2, and Theorem 1.1 have two parts: the *arithmetic part* and the *geometric part*. Both parts are needed to prove Theorem 1.1, Corollary 1.2, and Theorem 1.1. We will explain each part in more detail in §1.3.

Before moving on, let us mention that the geometric part of our paper works for any family of homologically trivial cycles and, more generally, for any admissible normal function. We prove a formula for its Betti rank and prove the Zariski closedness of the Betti strata. A simple application is the following result on the torsion locus: For any smooth projective morphism $f\colon X \to S$ of algebraic varieties with irreducible fibers and any family of homologically trivial cycles Z, all defined over $\overline{\mathbb{Q}}$, we show that there exists a *Zariski closed subset* $S_{\mathscr{F}}(1)$ of $S_{\mathbb{C}}$ such that: (i) $[Z_s]$ is non-torsion in $\mathrm{Ch}^*(X_s)$ for every transcendental point s of $S \setminus S_{\mathscr{F}}(1)$; (ii) there are at most countably many $s \in S(\mathbb{C})$ outside $S_{\mathscr{F}}(1)$ such that $\mathrm{AJ}(Z_s)$ is torsion in the intermediate Jacobian. We then prove a checkable criterion for $S_{\mathscr{F}}(1) \neq S$. A simple case will be presented in Corollary 1.9 and Remark 1.10. In the case of Gross–Schoen or Ceresa cycles, we will show that $\mathcal{M}_{g,\mathscr{F}}(1)$ is defined over $\mathbb{Q}$.

1.2. **Normal functions.** Now, we turn to the geometric part of the framework and explain our results. Let S be a quasi-projective variety. Let $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet) \to S$ be a VHS of weight -1 over S,[1] and consider the intermediate Jacobian (write $\mathcal{V}$ for the holomorphic vector associated with $\mathbb{V}_{\mathbb{Z}}$)

$$\pi \colon \mathcal{J}(\mathbb{V}_{\mathbb{Z}}) = \mathbb{V}_{\mathbb{Z}} \backslash \mathcal{V}/\mathcal{F}^0\mathcal{V} \longrightarrow S.$$

Let $\nu \colon S \longrightarrow \mathcal{J}(\mathbb{V}_{\mathbb{Z}})$ be an admissible normal function; then ν defines an admissible variation of mixed Hodge structures (VMHS) $\mathbb{E}_\nu$ on S which is an extension of $\mathbb{Z}(0)_S$ by $\mathbb{V}_{\mathbb{Z}}$. To any family of homologically trivial cycles, we can associate such an admissible normal function; see §C.5.

The fiberwise isomorphism $\mathbb{V}_{\mathbb{R},s} \xrightarrow{\sim} \mathbb{V}_{\mathbb{C},s}/\mathcal{F}^0_s = \mathcal{V}_s/\mathcal{F}^0_s$ makes $\mathcal{J}(\mathbb{V}_{\mathbb{Z}})$ into a local system of real tori

$$\mathcal{J}(\mathbb{V}_{\mathbb{Z}}) \xrightarrow{\sim} \mathbb{V}_{\mathbb{R}}/\mathbb{V}_{\mathbb{Z}}.$$

Let $\mathscr{F}_{\mathrm{Betti}}$ denote the induced foliation, which we call the *Betti foliation*. More precisely, for any point $x \in \mathcal{J}(\mathbb{V}_{\mathbb{Z}})$, there is a local section $\sigma \colon U \longrightarrow \mathcal{J}(\mathbb{V}_{\mathbb{Z}})$ from a neighborhood U of $\pi(x)$ in S^{an}, with $x \in \sigma(U)$, represented by a section of the local system $\mathbb{V}_{\mathbb{R}}$. The manifolds $\sigma(U)$ gluing together to a foliation $\mathscr{F}_{\mathrm{Betti}}$ on $\mathcal{J}(\mathbb{V}_{\mathbb{Z}})$. See §C.2 for more details. In particular, all torsion multi-sections are leaves of $\mathscr{F}_{\mathrm{Betti}}$.

1.2.1. *Betti strata.* The Betti foliation defines a strata on S as follows. For each integer $t \geq 0$, set

$$S_{\mathscr{F}}(t) := \left\{ s \in S(\mathbb{C}) : \dim_{\nu(s)}(\nu(S) \cap \mathscr{F}_{\mathrm{Betti}}) \geq t \right\} \tag{1.2}$$

where by abuse of notation $\nu(S) \cap \mathscr{F}_{\mathrm{Betti}}$ means the intersection with the leaves. This subset is, by definition, real-analytic. We then have the following *Betti strata* on S

$$\emptyset = S_{\mathscr{F}}(\dim S + 1) \subseteq S_{\mathscr{F}}(\dim S) \subseteq \cdots \subseteq S_{\mathscr{F}}(1) \subseteq S_{\mathscr{F}}(0) = S. \tag{1.3}$$

A main theorem of the geometric part is that the Betti strata are algebraic:

Theorem 1.4 (Theorem 3.2)**.** *For each $t \geq 0$, $S_{\mathscr{F}}(t)$ is Zariski closed in S.*

Remark 1.5. *In the Betti strata, $S_{\mathscr{F}}(1)$ plays a particularly important role. For example, $S_{\mathscr{F}}(1)$ contains any analytic curve $C \subseteq S^{\mathrm{an}}$ such that $\nu(C)$ is torsion because all torsion multi-sections are leaves of the Betti foliation. In particular, there are at most countably many $s \in S(\mathbb{C})$ outside $S_{\mathscr{F}}(1)$ such that $\nu(s)$ is torsion.*

Finally, let us point out that $S_{\mathscr{F}}(t)$ is closely related to the degeneracy loci defined by the first-named author. Indeed, when $\mathcal{J}(\mathbb{V}_{\mathbb{Z}}) \to S$ is an abelian scheme, then $S_{\mathscr{F}}(t)$ is the $\nu(S)^{\mathrm{deg}}(-t+1)$ in [Gao20a, Defn. 1.6] for each $t \geq 1$.

[1]Any VHS of odd weight can be reduced to weight -1 by applying a suitable twist; see §C.1.

1.2.2. *Betti rank.* Now assume that S is smooth. Then the Betti foliation induces a decomposition $T_x\mathcal{J}(\mathbb{V}_{\mathbb{Z}}) = T_x\mathscr{F}_{\mathrm{Betti}} \oplus T_x\mathcal{J}(\mathbb{V}_{\mathbb{Z}})_{\pi(x)}$ for each $x \in \mathcal{J}(\mathbb{V}_{\mathbb{Z}})$. Thus for each $s \in S(\mathbb{C})$ we have a linear map

$$\nu_{\mathrm{Betti},s}\colon T_sS \xrightarrow{\mathrm{d}\nu} T_{\nu(s)}\mathcal{J}(\mathbb{V}_{\mathbb{Z}}) \to T_{\nu(s)}\mathcal{J}(\mathbb{V}_{\mathbb{Z}})_s. \tag{1.4}$$

The *Betti rank* of ν is defined to be:

$$r(\nu) := \max_{s\in S(\mathbb{C})} \dim \nu_{\mathrm{Betti},s}(T_sS). \tag{1.5}$$

A trivial upper bound is $r(\nu) \le \min\{\dim S, \frac{1}{2}\dim \mathbb{V}_{\mathbb{Q},s}\}$ for any $s \in S(\mathbb{C})$. One can also easily improve this trivial upper bound, as explained below.

The Betti rank is easily seen to be related to the Betti strata in the following way: $\dim S - r(\nu)$ is the maximum of $t \ge 0$ such that $S_{\mathscr{F}}(t)$ contains a non-empty open subset of S^{an}. Thus Theorem 1.4 implies that $r(\nu) = \dim S - \min\{t \ge 0 : S_{\mathscr{F}}(t) = S\}$. However, this equality is, in general, not applicable to compute $r(\nu)$. But at this stage, we have

$$S_{\mathscr{F}}(1) \ne S \Longleftrightarrow r(\nu) = \dim S. \tag{1.6}$$

The second main theorem for the geometric part is the following formula for $r(\nu)$, which is often computable in practice. The VMHS $\mathbb{E}_\nu$ on S induces a period map $\varphi = \varphi_\nu\colon S \longrightarrow \Gamma\backslash\mathcal{D}$, with $\mathcal{D}$ a mixed Mumford–Tate domain whose Mumford–Tate group we denote by $\mathbf{G}$ (so $\mathcal{D}$ is a $\mathbf{G}(\mathbb{R})^+$-orbit and $\mathbf{G}$ is the generic Mumford–Tate group of the VMHS $\mathbb{E}_\nu$); we refer to §2.1–2.2 for the construction of φ and $\mathcal{D}$.

Denote by V the unipotent radical of $\mathbf{G}$. It equals $\mathbb{V}'_{\mathbb{Q},s}$ for any $s \in S(\mathbb{C})$, where

(∗) $\mathbb{V}'_{\mathbb{Z}}$ is the largest sub-VHS of $\mathbb{V}_{\mathbb{Z}}$ such that ν becomes torsion under the projection $\mathcal{J}(\mathbb{V}_{\mathbb{Z}}) \to \mathcal{J}(\mathbb{V}_{\mathbb{Z}}/\mathbb{V}'_{\mathbb{Z}})$.

See Remark 2.2. Now the trivial upper bound on $r(\nu)$ can be easily improved to be $r(\nu) \le \min\{\dim \varphi(S), \frac{1}{2}\dim V\}$.

Here is our formula to compute $r(\nu)$. The trivial upper bound on $r(\nu)$ above is recovered by taking $N = \{1\}$ and $N = \mathbf{G}$.

Theorem 1.6 (Theorem 3.1)**.** *The Betti rank is given by*

$$r(\nu) = \min_N \left(\dim \varphi_{/N}(S) + \frac{1}{2}\dim_{\mathbb{Q}}(V \cap N)\right), \tag{1.7}$$

where N runs through the set of normal subgroup of $\mathbf{G}$, and $\varphi_{/N}$ is the induced period map

$$\varphi_{/N} : S \xrightarrow{\varphi} \Gamma\backslash\mathcal{D} \xrightarrow{[p_N]} \Gamma_{/N}\backslash(\mathcal{D}/N)$$

with $[p_N]$ the quotient by N (see §B.3).

We will give two applications of Theorem 1.6 on two cases where the trivial upper bound on $r(\nu)$ is attained: Theorem 5.1 when $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet) \to S$ is irreducible, and Theorem 5.2 when the *algebraic monodromy group of* $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet) \to S$ *is simple* (or when the Mumford–Tate group of $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet) \to S$ is simple).

To highlight these applications, we state Theorem 5.1 in the introduction. We also point out that Theorem 5.2 could be more applicable in geometric situations.

Corollary 1.7 (Theorem 5.1)**.** *Assume: (i)* $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet) \to S$ *is irreducible, i.e. the only sub-VHSs are trivial or itself; (ii)* $\nu(S)$ *is not a torsion section. Then* $r(\nu) = \min\{\dim \varphi(S), \frac{1}{2}\dim_{\mathbb{Q}} \mathbb{V}_{\mathbb{Q},s}\}$ *for one (and hence all)* $s \in S(\mathbb{C})$.

It is easy to see that (i) and (ii) of Corollary 1.7 imply $\mathbb{V}'_{\mathbb{Z}} = \mathbb{V}_{\mathbb{Z}}$, and hence $V = \mathbb{V}_{\mathbb{Q},s}$. Moreover, if $r(\nu) = \frac{1}{2}\dim_{\mathbb{Q}} \mathbb{V}_{\mathbb{Q},s}$, then $\mathcal{J}(\mathbb{V}_{\mathbb{Z}}) \to S$ must be an abelian scheme by *Griffiths' transversality*; see the comment below (2.8).

Applying Corollary 1.7 to the Gross–Schoen and the Ceresa normal functions, we obtain the following corollary will be used to prove Theorem 1.1, Corollary 1.2 and Theorem 1.1.

Corollary 1.8 (Corollary 5.3)**.** *If* $g \geq 3$*, then the Betti rank of the Gross-Schoen (resp. of the Ceresa) normal function over* $\mathcal{M}_g$ *is* $3g-3$.

Hain [Hai24] gave a different proof of Corollary 1.8. Our Theorem 1.6 can also be applied to any subvariety $S \subseteq \mathcal{M}_g$.

1.2.3. *Consequence on torsion loci.* An immediate corollary on the torsion locus, combining (1.6) and Remark 1.5 and Theorem 1.4, is:

Corollary 1.9. *Under the assumptions of either Corollary 1.7 or Theorem 5.2. Assume that* φ *is generically finite and that* $\dim S \leq \frac{1}{2}\dim_{\mathbb{Q}} V$*. Then there exists a Zariski open dense subset* U *of* S *such that* $\nu(s)$ *is torsion for at most countably many* $s \in U(\mathbb{C})$*. Indeed, one can take* $U = S \setminus S_{\mathscr{F}}(1)$.

Remark 1.10. *In general* φ *is not necessarily generically finite. Then the conclusion becomes: there exists a Zariski open dense subset* U *of* S *such that* $\{s \in U(\mathbb{C}) : \nu(s) \text{ is torsion}\}$ *is contained in at most countably many fibers of* φ*. Indeed,* $\varphi(S)$ *is an algebraic variety by* [BBT23]*. Then we can conclude by applying Corollary 1.9 to* $\varphi(S)$ *and the induced normal function* ν' *(see §3.1 for the construction of* ν'*).*

In Corollary 1.9 and Remark 1.10, the condition "$\dim S \leq \frac{1}{2}\dim_{\mathbb{Q}} V$" can be removed if $\mathcal{J}(\mathbb{V}'_{\mathbb{Z}}) \to S$ is not an abelian scheme, with $\mathbb{V}'_{\mathbb{Z}}$ defined above Theorem 1.6. This follows from the comments below Corollary 1.7 (or Theorem 5.2) and is ultimately a consequence of *Griffiths' transversality.*

Kerr–Tayou [KT24] also proved Corollary 1.9 and Remark 1.10, with a different method, under the assumption of Theorem 5.2.

1.3. **Plan and Ingredients of proofs.** Our plan to prove results like Theorem 1.1, Corollary 1.2 and Theorem 1.1 has two parts: the *arithmetic part* and the *geometric part.* Let $f\colon X \to S$ be a smooth projective morphism of algebraic varieties with irreducible fibers and let Z be a family of homologically trivial cycles, all defined over $\overline{\mathbb{Q}}$

The *arithmetic part* requires the construction of a suitable adelic line bundle $\overline{\mathscr{L}}$ over the base, a theory initiated by the second-named author [Zha95] and developed in further joint work with Yuan [YZ21], such that the height function $h_{\overline{\mathscr{L}}}$ is the Beilinsin–Bloch height. Then, the desired height inequalities will follow from suitable bigness properties of $\overline{\mathscr{L}}$. In the context of Theorem 1.1 and Corollary 1.2, by the second-named author's [Zha10, Thm. 2.5.5] the desired $\overline{\mathscr{L}}$ can be obtained from a suitable Deligne pairing. This $\overline{\mathscr{L}}$ has been constructed by Yuan [Yua21, §3.3.4] is slightly different steps. The required bigness property is the bigness of the generic fiber $\tilde{\mathscr{L}}$ of $\overline{\mathscr{L}}$. We furthermore show in Proposition 6.3 that the bigness of $\tilde{\mathscr{L}}$ will follow from the non-vanishing of $c_1(\overline{\mathscr{L}})^{\wedge \dim S}$, with $c_1(\overline{\mathscr{L}})$ the curvature form. The arithmetic part of this paper is confined to the Gross-Shoen and the Ceresa cycles.

The *geometric part* studies the (admissible) normal function ν associated with Z. The main results are the Zariski closedness of the Betti strata and the formula for the Betti rank $r(\nu)$. This part has been explained in §1.2. The proofs of Theorem 1.4 and Theorem 1.6 are simultaneous and follow the guideline of the first-named author's [Gao20a] on the generic rank of the Betti map for abelian schemes. A core of our proof is Ax–Schanuel for VMHS independently proved by Chiu [Chi24] and Gao–Klingler [GK24], which will be used multiple times.

Hain's works on the Hodge-theoretic computation of the Archimedean local height pairing and the Betti form are the key ingredients to bridging the geometric and arithmetic parts. Indeed, Hain in [Hai90] proved that the archimedean local height pairing could be computed using the metrized biextension line bundle on S, and in [HR04] computed (joint with Reed) the curvature form of the metrized biextension line bundle. In our situation, we work with the height pairing of Z_s with itself for any $s \in S(\overline{\mathbb{Q}})$, and the metrized biextension line bundle in question is the pullback of the metrized tautological line bundle under ν. Hain in [Hai13] proved that its curvature form β_ν, called the *Betti form*, is a semi-positive $(1,1)$-form. We show in Corollary D.7 that $\beta_\nu^{\wedge \dim S} \not\equiv 0$ if and only if $r(\nu) = \dim S$. Finally, back to the situation of Theorem 1.1, Corollary 1.2, and Theorem 1.1. Hain's formula [Hai90, Prop. 3.3.12] and the second-named author's [Zha10, Thm. 2.5.5] together imply that the curvature $c_1(\overline{\mathscr{L}})$ is precisely the Betti form. This finishes the bridge between the geometric part and the arithmetic part. Again, the bridge between the geometric part and the arithmetic part works in much larger generality, provided that the arithmetic part is solved.

1.4. **Organization of the paper.** We will work on the geometric part of our proof in §2–5. More precisely, we will define the period map associated with a normal function in §2 and explain how to see the Betti foliation in this language, prove the formula to compute the Betti rank in §3 assumption a finiteness result *à la Ullmo* which will be proved in §4, and then explain two applications of this formula in geometric situations (including the Gross–Schoen and Ceresa normal functions as particular cases) in §5. Then in §6, we will explain the arithmetic part on how to construct the desired adelic line bundle, and prove that this construction together with the conclusion of the geometric part imply our main theorem.

For readers' convenience, we include several appendices, for which we do not claim originality, to make our paper as self-contained as possible.

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2. Period map associated with normal functions

Let S be a smooth irreducible quasi-projective variety. Let $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet)$ be a polarized VHS on S of weight -1. Let $\nu\colon S \to \mathcal{J}(\mathbb{V}_{\mathbb{Z}})$ be an admissible normal function.

The first goal of this section is to define the *period map* $\varphi_\nu : S \to \Gamma\backslash\mathcal{D}$ associated with ν (when ν is clearly in context, we simply denote it by φ), where $\mathcal{D}$ is a (mixed) Mumford–Tate domain with generic Mumford–Tate group $\mathbf{G}$ (see §B.2 for definition), and Γ is a suitable arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$.

The second goal of this section is to explain how to see the Betti foliation on $\mathcal{J}(\mathbb{V}_{\mathbb{Z}})$ in terms of the period map and the fibered structure of the Mumford–Tate domain $\mathcal{D}$, or as called in references, how to see the fibration (B.6) of the classifying space $\mathcal{M} \to \mathcal{M}_0$ as the *universal intermediate Jacobians*. This is done in §2.4.

We also recall in §2.3 the o-minimal structure attached to the period map.

2.1. **Universal period map to the classifying space.** Recall that ν induces a VMHS $(\mathbb{E}_\nu, W_\bullet, \mathcal{F}^\bullet_{\mathbb{E}})$ on S of weight -1 and 0 which is graded-polarized (better, admissible), fitting into the short exact sequence in the category of graded-polarized VMHS

$$0 \to \mathbb{V}_{\mathbb{Z}} \to \mathbb{E}_\nu \to \mathbb{Z}(0)_S \to 0$$

with the canonical polarization on $\mathbb{Z}(0)_S$.

Let $u_S\colon \tilde{S} \to S^{\mathrm{an}}$ be the universal covering map. Then the pullback $u_S^*\mathbb{E}_\nu$ (resp. $u_S^*\mathbb{V}_{\mathbb{Z}}$) is canonically trivialized as a local system $u_S^*\mathbb{E}_\nu \simeq \tilde{S} \times E_\nu$ (resp. $u_S^*\mathbb{V}_{\mathbb{Z}} \simeq \tilde{S} \times V_{\mathbb{Z}}$), with $E_\nu = H^0(\tilde{S}, u_S^*\mathbb{E}_\nu)$ (resp. with $V_{\mathbb{Z}} = H^0(\tilde{S}, u_S^*\mathbb{V}_{\mathbb{Z}})$). Then for each $s \in S(\mathbb{C})$, the fiber $\mathbb{E}_{\nu,s}$ can be canonically identified with E_ν and $\mathbb{V}_{\mathbb{Z},s}$ can be canonically identified with $V_{\mathbb{Z}}$. The pullback under u_S of the short exact sequence of VMHS defined by ν becomes split after $\otimes\mathbb{Q}$ since $\tilde{S}$ is simply connected.

Each $\tilde{s} \in \tilde{S}$ gives rise to a polarized mixed Hodge structure $(E_{\nu,\mathbb{Q}}, (W_\bullet)_{\tilde{s}}, (\mathcal{F}^\bullet_{\mathbb{E}})_{\tilde{s}})$ on $E_{\nu,\mathbb{Q}}$, And this induces the universal period map.

$$\tilde{\varphi} = \tilde{\varphi}_\nu\colon \tilde{S} \longrightarrow \mathcal{M}$$

to the classifying space $\mathcal{M}$ defined in §B.1.2, and $\mathcal{M}$ is a $\mathbf{G}^{\mathcal{M}}(\mathbb{R})^+$-orbit for some $\mathbb{Q}$-group $\mathbf{G}^{\mathcal{M}}$, whose unipotent radical is $V_{\mathbb{Q}} = V_{\mathbb{Z}} \otimes \mathbb{Q}$; see (B.3).

Similarly each $\tilde{s}$ gives rise to a polarized pure Hodge structure of weight -1 on $V_{\mathbb{Q}}$, and this induces a universal period map $\tilde{\varphi}_0\colon \tilde{S} \to \mathcal{M}_0$ to the classifying space defined in §B.1.1. Notice that $\tilde{\varphi}_0$ does not depend on the choice of ν, in contrast to $\tilde{\varphi}$. We have a commutative diagram, with p the projection from (B.6)

(2.1)
$$\begin{tikzcd} & \mathcal{M} \arrow[d, "p"] \\ \tilde{S} \arrow[ur, "\tilde{\varphi}"] \arrow[r, "\tilde{\varphi}_0"'] & \mathcal{M}_0. \end{tikzcd}$$

In practice, we need to refine this period map $\tilde{\varphi}$ and replace the classifying space $\mathcal{M}$ by the smallest Mumford–Tate domain $\mathcal{D}$ which contains $\tilde{\varphi}(\tilde{S})$. This construction is done in the next subsection.

2.2. **Construction of the Mumford–Tate domain.** By the first part of the proof of [And92, §4, Lemma 4], the Mumford-Tate group $\mathrm{MT}_{\tilde{s}} \subseteq \mathrm{GL}(E_{\nu,\mathbb{Q}})$ of the Hodge structure $(E_{\nu,\mathbb{Q}}, (W_\bullet)_{\tilde{s}}, (\mathcal{F}^\bullet_{\mathbb{E}})_{\tilde{s}})$ is locally constant on $\tilde{S}^\circ = \tilde{S} \setminus \Sigma$ for a meager subset Σ of $\tilde{S}$. We call this group the *generic Mumford–Tate group* of $(\mathbb{E}_\nu, W_\bullet, \mathcal{F}^\bullet) \to S$ and denote it by $\mathbf{G}$. It is known that $\mathrm{MT}_{\tilde{s}} \subseteq \mathbf{G}$ for all $\tilde{s} \in \tilde{S}$.

Fix $\tilde{s} \in \tilde{S}$. Define

$$\mathcal{D} := \mathbf{G}(\mathbb{R})^+ \cdot \tilde{\varphi}(\tilde{s}) \subseteq \mathcal{M}. \tag{2.2}$$

Then $\mathcal{D}$ is the smallest Mumford–Tate domain which contains $\tilde{\varphi}(\tilde{S})$; see [GK24, §7.1].

Finally, since S is a quasi-projective variety, there exists an arithmetic subgroup Γ of $\mathbf{G}(\mathbb{Q})$ such that $\tilde{\varphi}$ descends to a morphism $\varphi\colon S^{\mathrm{an}} \to \Gamma\backslash\mathcal{D}$.

2.3. **Setup for o-minimality.** Let $\mathfrak{F} \subseteq \mathcal{D}$ be a fundamental set for the quotient $u\colon \mathcal{D} \to \Gamma\backslash\mathcal{D}$, *i.e.* $u|_{\mathfrak{F}}$ is surjective and $(u|_{\mathfrak{F}})^{-1}(\bar{x})$ is finite for each $\bar{x} \in \Gamma\backslash\mathcal{D}$. If $\mathfrak{F}$ is a semi-algebraic subset of $\mathcal{D}$, then we have a semi-algebraic structure on $\Gamma\backslash\mathcal{D}$ induced by $u|_{\mathfrak{F}}$.

By the main result of [BBKT24], there exists a semi-algebraic fundamental set $\mathfrak{F} \subseteq \mathcal{D}$ for the quotient $u\colon \mathcal{D} \to \Gamma\backslash\mathcal{D}$ with the following properties: φ is $\mathbb{R}_{\mathrm{an,exp}}$-definable for the semi-algebraic structure on $\Gamma\backslash\mathcal{D}$ defined by $\mathfrak{F}$.

2.4. **Relating the Betti foliation and the period map.** Denote by $\mathcal{D}_0 := p(\mathcal{D}) \subseteq \mathcal{M}_0$ for the projection to the pure part p from (2.1), and by abuse of notation $p = p|_{\mathcal{D}}\colon \mathcal{D} \to \mathcal{D}_0$. Recalling the semi-algebraic structure $\mathcal{D} = V(\mathbb{R}) \times \mathcal{D}_0$ as in Proposition B.3.(i), the period maps φ and $\widetilde{\varphi}$ can be complete into:

$$\begin{tikzcd}
\tilde{S} \arrow[r, "\tilde{\varphi}"'] \arrow[d, "u_S"'] \arrow[rr, bend left, "\tilde{\varphi}_0"] & \mathcal{D} = V(\mathbb{R}) \times \mathcal{D}_0 \arrow[r, "p"'] \arrow[d, "u"] & \mathcal{D}_0 \arrow[d, "u_0"] \\
S \arrow[r, "\varphi"] \arrow[rr, bend right, "\varphi_0"'] & \Gamma\backslash\mathcal{D} \arrow[r, "{[p]}"] & \Gamma_0\backslash\mathcal{D}_0
\end{tikzcd} \tag{2.3}$$

and $\varphi_0 = p \circ \varphi$ is the period map for the VHS $\mathbb{V}_{\mathbb{Z}} \to S$.

For each $\tilde{s} \in \tilde{S}$, write $\tilde{s}_V$ for the image $\tilde{\varphi}(\tilde{s})$ under the natural projection $\mathcal{D} = V(\mathbb{R}) \times \mathcal{D}_0 \to V(\mathbb{R})$. This gives a map $\tilde{\varphi}_V\colon \tilde{S} \to V(\mathbb{R})$ sending $\tilde{s} \mapsto \tilde{s}_V$. Notice that

$$\tilde{\varphi} = (\tilde{\varphi}_V, \tilde{\varphi}_0)\colon \tilde{S} \longrightarrow V(\mathbb{R}) \times \mathcal{D}_0 = \mathcal{D}. \tag{2.4}$$

Recall $S_{\mathscr{F}}(t) = \{s \in S(\mathbb{C}) : \dim_{\nu(s)}(\nu(S) \cap \mathscr{F}_{\mathrm{Betti}}) \geq t\}$ and define

$$S_{\mathrm{rk}}(t) = \{s \in S(\mathbb{C}) : \dim \nu_{\mathrm{Betti},s}(T_s S) \leq t\}$$

for $\nu_{\mathrm{Betti},s}\colon T_s S \to T_{\nu(s)}\mathcal{J}(\mathbb{V}_{\mathbb{Z},s})$ defined in (1.4).

Lemma 2.1. *For each $t \geq 0$, we have*

$$u_S^{-1}(S_{\mathscr{F}}(t)) = \{\tilde{s} \in \tilde{S} : \dim_{\tilde{s}} \tilde{\varphi}^{-1}(\{\tilde{s}_V\} \times \mathcal{D}_0) \geq t\}$$

$$(2.5) \qquad = \bigcup_{r \geq 0} \{\tilde{s} \in \tilde{S} : \dim_{\tilde{s}} \tilde{\varphi}^{-1}(\tilde{\varphi}(\tilde{s})) = r,\ \{\tilde{s}_V\} \times \tilde{C} \subseteq \tilde{\varphi}(\tilde{S}) \text{ for some complex analytic } \tilde{C} \text{ with } \dim \tilde{C} \geq t - r\}.$$

and

$$(2.6) \qquad u_S^{-1}(S_{\mathrm{rk}}(t)) = \{\tilde{s} \in \tilde{S} : \mathrm{rank}(\mathrm{d}\tilde{\varphi}_V)_{\tilde{s}} \leq t\}.$$

Notice that in the union in (2.5), the second condition $\{\tilde{s}_V\} \times \tilde{C} \subseteq \tilde{\varphi}(\tilde{S})$ always holds for $r > t$.

Proof. Consider $\pi \colon \mathcal{J}(\mathbb{V}_{\mathbb{Z}}) \to S$. Each $x \in \mathcal{J}(\mathbb{V}_{\mathbb{Z}})$ lies in $\mathcal{J}(\mathbb{V}_{\mathbb{Z}})_s = \mathcal{J}(\mathbb{V}_{\mathbb{Z},s})$ for $s = \pi(x)$, which is canonically isomorphic to $\mathrm{Ext}_{\mathrm{MHS}}(\mathbb{Z}(0), \mathbb{V}_{\mathbb{Z},s})$ by Carlson [Car85]. Hence each x gives rise to a $\mathbb{Z}$-mixed Hodge structure of weight -1 and 0, and for the universal covering map $u_{\mathcal{J}} \colon \tilde{\mathcal{J}} \to \mathcal{J}(\mathbb{V}_{\mathbb{Z}})$ we obtain a period map $\tilde{\varphi}_{\mathcal{J}} \colon \tilde{\mathcal{J}} \to \mathcal{M}$. The map $\nu \circ u_S \colon \tilde{S} \to S \to \mathcal{J}(\mathbb{V}_{\mathbb{Z}})$ lifts to $\tilde{\nu} \colon \tilde{S} \to \tilde{\mathcal{J}}$, and $\tilde{\varphi} = \tilde{\varphi}_{\mathcal{J}} \circ \tilde{\nu}$.

Recall that $V = H^0(\tilde{S}, u_S^*\mathbb{V}_{\mathbb{Q}})$. Thus $V_{\mathbb{Z}} := H^0(\tilde{S}, u_S^*\mathbb{V}_{\mathbb{Z}})$ is a lattice in $\mathbb{V}(\mathbb{R})$ and $u_S^*\mathbb{V}_{\mathbb{Z}} \simeq V_{\mathbb{Z}} \times \tilde{S}$. So $\mathcal{J}(\mathbb{V}_{\mathbb{Z}}) = \mathbb{V}_{\mathbb{R}}/\mathbb{V}_{\mathbb{Z}}$ induces $\mathcal{J}(\mathbb{V}_{\mathbb{Z}}) \times_S \tilde{S} = (V_{\mathbb{R}}/V_{\mathbb{Z}}) \times \tilde{S}$ and hence $\tilde{\mathcal{J}} = V(\mathbb{R}) \times \tilde{S}$. By definition of the Betti foliation in §C.2, the leaves of $\mathscr{F}_{\mathrm{Betti}}$ are precisely $u_{\mathcal{J}}(\{a\} \times \tilde{S})$ for all $a \in V(\mathbb{R})$.

The zero section of $\pi \colon \mathcal{J}(\mathbb{V}_{\mathbb{Z}}) \to S$ gives rise to a Levi decomposition of $\mathbf{G}^{\mathcal{M}}$ and hence an identification $\mathcal{M} = V(\mathbb{R}) \times \mathcal{M}_0$ as in Proposition B.3.(i) applied to $\mathcal{M}$, and $\tilde{\varphi}_{\mathcal{J}}$ becomes

$$(2.7) \qquad \tilde{\varphi}_{\mathcal{J}} \colon \tilde{\mathcal{J}} = V(\mathbb{R}) \times \tilde{S} \xrightarrow{(1,\tilde{\varphi}_0)} V(\mathbb{R}) \times \mathcal{D}_0 \subseteq V(\mathbb{R}) \times \mathcal{M}_0.$$

By the last paragraph, the leaves of $\mathscr{F}_{\mathrm{Betti}}$ are precisely $u_{\mathcal{J}}\left(\tilde{\varphi}_{\mathcal{J}}^{-1}(\{a\} \times \mathcal{M}_0)\right)$ for all $a \in V(\mathbb{R})$.

Thus $u_{\mathcal{J}}^{-1}(\nu(S_{\mathscr{F}}(t))) = \{\tilde{s}' \in \tilde{\nu}(\tilde{S}) : \dim_{\tilde{s}'} \tilde{\varphi}_{\mathcal{J}}^{-1}(\{\tilde{s}'_V\} \times \mathcal{M}_0\}) \geq t\}$, where $\tilde{s}'_V$ is the image of $\tilde{s} \in \tilde{\mathcal{J}} \xrightarrow{\tilde{\varphi}_{\mathcal{J}}} V(\mathbb{R}) \times \mathcal{M}_0 \to V(\mathbb{R})$ with the last map being the natural projection. Applying $\tilde{\nu}^{-1}$ (whose fibers are of dimension 0 because ν is injective) to the set above and noticing that $u_{\mathcal{J}} \circ \tilde{\nu} = \nu \circ u_S$, we have

$$u_S^{-1}(S_{\mathscr{F}}(t)) = \{\tilde{s} \in \tilde{S} : \dim_{\tilde{s}} (\tilde{\varphi}_{\mathcal{J}} \circ \tilde{\nu})^{-1}(\{\tilde{s}'_V\} \times \mathcal{M}_0\}) \geq t\}.$$

Hence the first equality in (2.5) holds because $\tilde{\varphi}_{\mathcal{J}} \circ \tilde{\nu} = \tilde{\varphi}$ and by Lemma B.5 applied to $\mathcal{D} \subseteq \mathcal{M}$. Similarly, we get (2.6).

The second equality in (2.5) clearly holds. Hence, we are done. $\square$

Remark 2.2. *The proof of Lemma 2.1 (with Lemma B.5 applied to $\mathcal{D} \subseteq \mathcal{M}$ and the identification $\tilde{\mathcal{J}} = V(\mathbb{R}) \times \tilde{S}$) also yields the following assertion: $\mathcal{J}_\nu := S \times_{\Gamma_0 \backslash \mathcal{D}_0} (\Gamma \backslash \mathcal{D}) \subseteq \mathcal{J}(\mathbb{V}_{\mathbb{Z}})$ is the intermediate Jacobian of a sub-VHS translated by a torsion multisection; it contains $\nu(S)$ and is the minimal one containing $\nu(S)$ with respect to inclusion. The relative dimension $\dim \mathcal{J}_\nu - \dim S$ equals $\frac{1}{2}\dim V$.*

Similarly, V equals $\mathbb{V}'_{\mathbb{Q},s}$ for any $s \in S(\mathbb{C})$, where $\mathbb{V}'_{\mathbb{Z}}$ is the largest sub-VHS of $\mathbb{V}_{\mathbb{Z}}$ such that ν is torsion under the projection $\mathcal{J}(\mathbb{V}_{\mathbb{Z}}) \to \mathcal{J}(\mathbb{V}_{\mathbb{Z}}/\mathbb{V}'_{\mathbb{Z}})$.

Moreover for the natural projection $\varphi_{\mathcal{J}_\nu} \colon \mathcal{J}_\nu \to \Gamma \backslash \mathcal{D}$, we have $\varphi(S) = \varphi_{\mathcal{J}_\nu}(\nu(S))$.

Since $\tilde{\varphi}_V$ factors through $\tilde{\varphi}$ and has target $V(\mathbb{R})$, (2.6) immediately yields the following trivial upper bound

$$\nu_{\mathrm{Betti},s}(T_sS) \leq \min\left\{\dim \varphi(S), \frac{1}{2}\dim V\right\} \qquad \text{for all } s \in S(\mathbb{C}). \tag{2.8}$$

We can say more. By *Griffiths' transversality*, we must have $\nu_{\mathrm{Betti},s}(T_sS) < \frac{1}{2}\dim V$ if $\mathcal{J}(\mathbb{V}'_{\mathbb{Z}}) \to S$ is not an abelian scheme (or equivalently, the Hodge type of $\mathbb{V}'_{\mathbb{Z}} \to S$ is not $(-1,0)+(0,-1)$); see above (D.3).

3. The Betti rank and Zariski closedness of the Betti strata

The goal of this section is to prove one of the main theorems of this paper. We will prove a formula to compute the Betti rank $r(\nu)$, and in the process, we show that the Betti foliation on the intermediate Jacobian defines Zariski closed strata.

Let S be a smooth irreducible quasi-projective variety. Let $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet)$ be a polarized VHS on S of weight -1. Let $\nu\colon S \to \mathcal{J}(\mathbb{V}_{\mathbb{Z}})$ be an admissible normal function.

Recall the Betti foliation $\mathscr{F}_{\mathrm{Betti}}$ on $\mathcal{J}(\mathbb{V}_{\mathbb{Z}})$ defined in §C.2, the linear map (1.4)

$$\nu_{\mathrm{Betti},s}\colon T_sS \to T_{\nu(s)}\mathcal{J}(\mathbb{V}_{\mathbb{Z},s})$$

at each $s \in S(\mathbb{C})$, and the Betti rank (1.5)

$$r(\nu) = \max_{s\in S(\mathbb{C})} \dim \nu_{\mathrm{Betti},s}(T_sS).$$

Retain the notation from (2.3). In particular, we have the period map $\varphi = \varphi_\nu\colon S \to \Gamma\backslash\mathcal{D}$ for the (mixed) Mumford–Tate domain $\mathcal{D}$, the $\mathbb{Q}$-group $\mathbf{G}$ and its unipotent radical V which is a vector group. We emphasize that V is, in general, not a fiber of $\mathbb{V}_{\mathbb{Q}}$.

The main result of this section is the following formula for $r(\nu)$. The advantage is that it is often computable in practice.

Theorem 3.1. *The following formula gives the Betti rank:*

$$r(\nu) = \min_N \left\{\dim \varphi_{/N}(S) + \frac{1}{2}\dim_{\mathbb{Q}}(V\cap N)\right\}, \tag{3.1}$$

where N runs through the set of normal subgroups of $\mathbf{G}$, and $\varphi_{/N}$ is the induced map

$$\varphi_{/N}\colon S \xrightarrow{\varphi} \Gamma\backslash\mathcal{D} \xrightarrow{[p_N]} \Gamma_{/N}\backslash(\mathcal{D}/N)$$

with $[p_N]$ the quotient defined in §B.3.

Taking $N = \{1\}$ and $N = \mathbf{G}$, we recover the trivial upper bound (2.8) of $r(\nu)$.

We also show that the Betti foliation defines Zariski's closed strata on S.

Theorem 3.2. *For each $t \geq 0$, the set $S_{\mathscr{F}}(t) := \{s \in S(\mathbb{C}) : \dim_{\nu(s)}(\nu(S)\cap \mathscr{F}_{\mathrm{Betti}}) \geq t\}$ is Zariski closed in S. In particular, $r(\nu) = \dim S - \min\{t \geq 0 : S_{\mathscr{F}}(t) = S\}$.*

The proofs of Theorem 3.1 and Theorem 3.2 are simultaneous and follow the guideline of the first-named author's [Gao20a] on the case when $\mathcal{J}(\mathbb{V}_{\mathbb{Z}})$ is polarizable. A key ingredient for our proof is the mixed Ax–Schanuel theorem, which is used multiple times. In [Gao20a], the version for universal abelian varieties [Gao20b, Thm. 1.1] was used. In the current paper, we need the version for admissible VMHS independently proved by

Chiu [Chi24] and Gao–Klingler [GK24]. We also invoke [BBT23] on the algebraicity of $\varphi(S)$, whose proof builds up on o-minimal GAGA, to ease the notation for the proof.

Another crucial input for the proof is the o-minimal structure associated with the period map [BBKT24], which we will recall in §2.3. This allows us to apply (o-minimal) definable Chow.

3.1. **Replacing** S **by** $\varphi(S)$**.** We shall replace S by $\varphi(S)$ in the proof using [BBT23]. This largely eases the notation.

By the main result of [BBT23], the period map $\varphi = \varphi_\nu\colon S \to \Gamma\backslash\mathfrak{D}$ factors as $S \to S' \xrightarrow{\iota} \Gamma\backslash\mathfrak{D}$, with $S \to S'$ a dominant morphism between algebraic varieties and ι an immersion in the category of complex varieties. Then ι induces an integral admissible VMHS on S' for which ι is the period map. By abuse of notation, we use $\varphi\colon S \to S'$ and see ι as an inclusion. We have the following diagram.

$$\begin{tikzcd} & S' \arrow[r, hook] \arrow[dr, dashed] & \Gamma\backslash\mathfrak{D} \arrow[d, "{[p]}"] \\ S \arrow[ur, "\varphi"] \arrow[rr, "\varphi_0"'] & & \Gamma_0\backslash\mathfrak{D}_0 \end{tikzcd}$$

with the dotted arrow being the restriction $[p]|_{S'}$.

Recall from Remark 2.2 that $\mathcal{J}_\nu := S \times_{\Gamma_0\backslash\mathfrak{D}_0} (\Gamma\backslash\mathfrak{D})$ is an intermediate Jacobian over S. Set $\mathcal{J}' := S' \times_{\Gamma_0\backslash\mathfrak{D}_0} (\Gamma\backslash\mathfrak{D})$; it is an intermediate Jacobian over S'. Then the inclusion $S' \subseteq \Gamma\backslash\mathfrak{D}$ yields a section ν' of $\mathcal{J}' \to S'$, and thus we can define $S'_{\mathscr{F}}(t)$ for each $t \geq 0$ with respect to ν'. We have the following commutative diagram, with $\varphi_{\mathcal{J}}$ induced by φ, such that $\varphi_{\mathcal{J}} \circ \nu = \nu' \circ \varphi$:

$$\begin{tikzcd} \mathcal{J}_\nu \arrow[r, "\varphi_{\mathcal{J}}"] \arrow[d] & \mathcal{J}' \arrow[d] \\ S \arrow[u, bend left, "\nu"] \arrow[r, "\varphi"] & S'. \arrow[u, bend left, "\nu'"] \end{tikzcd}$$

For each $r \geq 0$, denote by

$$S_{\geq r} := \{s \in S(\mathbb{C}) : \dim_s \varphi^{-1}(\varphi(s)) \geq r\}. \tag{3.2}$$

It is a closed algebraic subset of S by upper semi-continuity.

By (the proof of) Lemma 2.1, more precisely the second equality of (2.5), we have

$$S_{\mathscr{F}}(t) = S_{\geq t} \cup \bigcup\nolimits_{0 \leq r \leq t-1} S_{\geq r} \cap \varphi^{-1}\left(S'_{\mathscr{F}}(t-r)\right). \tag{3.3}$$

This equality allows us to replace S by $\varphi(S)$ to study the Betti rank and the Betti strata.

3.2. **Bi-algebraic system and Ax–Schanuel.** From now on, in the whole section, we replace S by $\varphi(S)$ and view S as an algebraic subvariety of the complex analytic space $\Gamma\backslash\mathfrak{D}$, unless otherwise stated.

Recall that weak Mumford–Tate domain is defined in Definition B.6. The following proposition [BBKT24, Cor. 6.7] follows from the o-minimal setup explained in §2.3 and definable Chow.

Proposition 3.3. *Let $\mathfrak{D}_N$ be a weak Mumford–Tate domain. Then $u(\mathfrak{D}_N) \cap S$ is a closed algebraic subset of S.*

Definition 3.4. (i) *Let* $\tilde{Y} \subseteq \mathcal{D}$ *be a complex analytic irreducible subset. The* ***weakly special closure*** *of* $\tilde{Y}$*, denoted by* $\tilde{Y}^{\mathrm{ws}}$*, is the smallest weak Mumford–Tate domain in* $\mathcal{D}$ *which contains* $\tilde{Y}$*.*

(ii) *Let* $Y \subseteq S$ *be an irreducible subvariety. The* ***weakly special closure*** *of* Y*, denoted by* Y^{ws}*, is* $u(\widetilde{Y}^{\mathrm{ws}})$ *for one (hence any) complex analytic irreducible component* $\widetilde{Y}$ *of* $u^{-1}(Y)$*.*

The following Ax–Schanuel theorem for VMHS was independently proved by Chiu [Chi24] and Gao–Klingler [GK24]. We refer to Definition B.2 for the algebraic structure on $\mathcal{D}$.

Theorem 3.5 (weak Ax–Schanuel for VMHS)**.** *Let* $\tilde{Z} \subseteq u^{-1}(S)$ *be a complex analytic irreducible subset. Then*

$$\dim \tilde{Z}^{\mathrm{Zar}} + \dim u(\tilde{Z})^{\mathrm{Zar}} \geq \dim \tilde{Z}^{\mathrm{ws}} + \dim \tilde{Z}, \tag{3.4}$$

where $\tilde{Z}^{\mathrm{ws}}$ *is the smallest weak Mumford–Tate domain which contains* $\tilde{Z}$*.*

Proof. Let $Y := u(\tilde{Z})^{\mathrm{Zar}}$. Let $\mathcal{Z} := \{(z, y) \in \tilde{Z} \times Y(\mathbb{C}) : u(z) = y\}$, then $\mathcal{Z}$ is a complex analytic irreducible subset of $\mathcal{D} \times_{\Gamma\backslash\mathcal{D}} Y'$. The Zariski closure of $\mathcal{Z}$ in $\mathcal{D} \times Y$ is contained in $\tilde{Z}^{\mathrm{Zar}} \times Y$, and $\dim \mathcal{Z} = \dim \tilde{Z}$. Then (3.4) is a direct consequence of the mixed Ax–Schanuel theorem [GK24, Thm. 1.1] applied to $\mathcal{Z}$. □

We close this introductory subsection with the following definition. In practice, we often need to work with algebraic subvarieties $Y \subseteq S$, which are not weak Mumford–Tate domains, and the following number measures how far it is from being one.

$$\delta_{\mathrm{ws}}(Y) := \dim Y^{\mathrm{ws}} - \dim Y. \tag{3.5}$$

If we do not replace S by $\varphi(S)$, then each Y on the right-hand side should be replaced by $\varphi(Y)$.

Definition 3.6. *An irreducible algebraic subvariety* Y *of* S *is called* weakly optimal *if the following holds:* $Y \subsetneq Y' \subseteq S \Rightarrow \delta_{\mathrm{ws}}(Y) < \delta_{\mathrm{ws}}(Y')$*, for any* $Y' \subseteq S$ *irreducible.*

3.3. **Applications of Ax–Schanuel.** Retain the notation in (2.3). We start with the following application of mixed Ax–Schanuel.

Proposition 3.7. *For each* $t \geq 0$*,* $S_{\mathscr{F}}(t)$ *is contained in the union of weakly optimal subvarieties* $Y \subseteq S$ *satisfying*

$$\dim Y \geq \dim Y^{\mathrm{ws}} - \dim[p](Y^{\mathrm{ws}}) + t. \tag{3.6}$$

Proof. It suffices to prove two things:

(i) $S_{\mathscr{F}}(t)$ is covered by the union of irreducible subvarieties $Y \subseteq S$ satisfying (3.6) (without requiring Y to be weakly optimal);

(ii) If $Y \subseteq S$ is an irreducible subvariety satisfying (3.6) and is maximal for this property with respect to inclusions, then Y is weakly optimal.

Let us prove (i). By Lemma 2.1, more precisely the first equality of (2.5), $S_{\mathscr{F}}(t)$ is covered by irreducible subvarieties $Y \subseteq S$ such that

$$Y := \overline{u(\{a\} \times \tilde{C})}^{\mathrm{Zar}},$$

for some complex analytic irreducible $\tilde{C} \subseteq \mathcal{D}_0$ with $\dim \tilde{C} = t$ and some $a \in V(\mathbb{R})$.

Apply mixed Ax–Schanuel in this context (Theorem 3.5 to $\{a\} \times \tilde{C}$). Then we get

$$\dim \overline{\{a\} \times \tilde{C}}^{\mathrm{Zar}} + \dim Y \geq \dim(\{a\} \times \tilde{C})^{\mathrm{ws}} + t.$$

By Lemma B.4, $\overline{\{a\} \times \tilde{C}}^{\mathrm{Zar}} = \{a\} \times \overline{\tilde{C}}^{\mathrm{Zar}}$. Hence

$$\dim Y \geq \dim(\{a\} \times \tilde{C})^{\mathrm{ws}} - \dim \overline{\tilde{C}}^{\mathrm{Zar}} + t \geq \dim(\{a\} \times \tilde{C})^{\mathrm{ws}} - \dim \tilde{C}^{\mathrm{ws}} + t.$$

The last inequality holds because $\overline{\tilde{C}}^{\mathrm{Zar}} \subseteq \tilde{C}^{\mathrm{ws}}$. So

$$\dim Y \geq \dim(\{a\} \times \tilde{C})^{\mathrm{ws}} - \dim \tilde{C}^{\mathrm{ws}} + t.$$

Now, to prove (i), it suffices to prove $\dim(\{a\} \times \tilde{C})^{\mathrm{ws}} = \dim Y^{\mathrm{ws}}$ and $\dim \tilde{C}^{\mathrm{ws}} = \dim[p](Y^{\mathrm{ws}})$.

Let us prove $u((\{a\} \times \tilde{C})^{\mathrm{ws}}) = Y^{\mathrm{ws}}$; the upshot is $\dim(\{a\} \times \tilde{C})^{\mathrm{ws}} = \dim Y^{\mathrm{ws}}$. By definition of Y, we have $u(\{a\} \times \tilde{C}) \subseteq Y$. Hence $u((\{a\} \times \tilde{C})^{\mathrm{ws}}) \subseteq Y^{\mathrm{ws}}$. On the other hand, $u((\{a\} \times \tilde{C})^{\mathrm{ws}})$ is closed algebraic by Proposition 3.3, so $Y \subseteq u((\{a\} \times \tilde{C})^{\mathrm{ws}})$. So $Y^{\mathrm{ws}} \subseteq u((\{a\} \times \tilde{C})^{\mathrm{ws}})$. Now we have established $u((\{a\} \times \tilde{C})^{\mathrm{ws}}) = Y^{\mathrm{ws}}$.

Similarly, we have $\dim \tilde{C}^{\mathrm{ws}} = \dim[p](Y^{\mathrm{ws}})$. Hence, we are done for (i).

For (ii), let $Y \subseteq Y' \subseteq X$. Assume $\delta_{\mathrm{ws}}(Y) \geq \delta_{\mathrm{ws}}(Y')$, *i.e.*

$$\dim Y^{\mathrm{ws}} - \dim Y \geq \dim Y'^{,\mathrm{ws}} - \dim Y'.$$

The assumption on Y implies $\dim Y^{\mathrm{ws}} - \dim Y \leq \dim[p](Y^{\mathrm{ws}}) - t$. Combined with the inequality above, we obtain $\dim Y'^{,\mathrm{ws}} - \dim Y' \leq \dim[p](Y'^{,\mathrm{ws}}) - t$ because $Y \subseteq Y'$. Therefore $Y = Y'$ by maximality of Y. Hence, (ii) is established. □

Next, we state a finiteness proposition for weakly optimal subvarieties in S *à la Ullmo*, which we give a proof using twice mixed Ax–Schanuel in the next section. When $\mathcal{J}(\mathbb{V}_{\mathbb{Z}})$ is polarizable, *i.e.* in the mixed Shimura case, this is [Gao20b, Thm. 1.4].

Proposition 3.8. *There exist finitely many pairs $(\mathcal{D}'_1, N_1), \ldots, (\mathcal{D}'_k, N_k)$, with each $\mathcal{D}'_j$ a Mumford–Tate domain contained in $\mathcal{D}$ and N_j a normal subgroup of $\mathrm{MT}(\mathcal{D}'_j)$, such that the following holds. For each weakly optimal subvariety $Y \subseteq S$, Y^{ws} is the image of an $N_j(\mathbb{R})^+$-orbit contained in $\mathcal{D}'_j$ under $u\colon \mathcal{D} \to \Gamma\backslash\mathcal{D}$ for some $j \in \{1, \ldots, k\}$.*

The same statement (with Y^{ws} replaced by $\varphi(Y)^{\mathrm{ws}}$) still holds without replacing S by $\varphi(S)$.

Denote by $\Gamma_j = \Gamma \cap \mathrm{MT}(\mathcal{D}'_j)(\mathbb{Q})$ and $\Gamma_{j,/N_j} = \Gamma_j/(\Gamma_j \cap N_j(\mathbb{Q}))$. Then equivalently, each such Y^{ws} is a fiber of the quotient $[p_{N_j}]\colon u(\mathcal{D}'_j) = \Gamma_j\backslash\mathcal{D}'_j \to \Gamma_{j,/N_j}\backslash(\mathcal{D}'_j/N_j)$.

In §2.3, we have endowed $\Gamma\backslash\mathcal{D}$ with a semi-algebraic structure, and hence $\Gamma_j\backslash\mathcal{D}'_j$ with a semi-algebraic structure. In a similar way, we can endow $\Gamma_{j,/N_j}\backslash(\mathcal{D}'_j/N_j)$ with a semi-algebraic structure. Then $[p_N]$ is semi-algebraic because the quotient map $\mathcal{D}'_j \to \mathcal{D}'_j/N_j$ is; see §B.3.

3.4. **Proof of Theorem 3.1.** For each $j \in \{1, \ldots, k\}$, Proposition 3.3 says that $u(\mathcal{D}'_j) \cap S$ is a closed algebraic subset of S. The restriction

$$[p_{N_j}]|_S\colon u(\mathcal{D}'_j) \cap S \to \Gamma_{j,/N_j}\backslash(\mathcal{D}'_j/N_j)$$

is both complex analytic and definable; see §2.3.

For each $t \geq 0$, the subset

$$E_j(t) := \left\{ s \in u(\mathcal{D}'_j) \cap S : \dim_s [p_{N_j}]|_S^{-1}([p_{N_j}](s)) \geq \frac{1}{2}\dim(V \cap N_j) + t \right\} \tag{3.7}$$

is both definable and complex analytic in $u(\mathcal{D}'_j) \cap S$. Hence, $E_j(t)$ is algebraic by definable Chow. Moreover, it is closed in $u(\mathcal{D}'_j) \cap S$ by the upper semi-continuity of fiber dimensions. So $E_j(t)$ is a closed algebraic subset of S.

Proposition 3.9. *For each $t \geq 0$, we have*

$$S_{\mathscr{F}}(t) \subseteq \bigcup\nolimits_{j=1}^{k} E_j(t).$$

Proof. Let $t \geq 0$. By Proposition 3.7, $S_{\mathscr{F}}(t)$ is covered by weakly optimal $Y \subseteq S$ such that $\dim Y \geq \dim Y^{\mathrm{ws}} - \dim[p](Y^{\mathrm{ws}}) + t$. Then by Proposition 3.8, Y^{ws} is a fiber $[p_{N_j}]$ for some $j \in \{1, \ldots, k\}$, and hence $\dim Y^{\mathrm{ws}} - \dim[p](Y^{\mathrm{ws}}) = \frac{1}{2}\dim(V \cap N_j)$. So $\dim Y \geq \frac{1}{2}\dim(V \cap N_j) + t$. So $Y \subseteq E_j(t)$ because $[p_{N_j}](Y)$ is a point. $\square$

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. In this proof, we go back to our original setting and do not replace S by $\varphi(S)$. We have $S \xrightarrow{\varphi} S' \subseteq \Gamma\backslash\mathcal{D}$ with $S' = \varphi(S)$ an algebraic subvariety of $\Gamma\backslash\mathcal{D}$.

Let us prove "$\leq$". By (2.6) we have $r(\nu) = \max_{\tilde{s}\in\tilde{S}}(\mathrm{d}\tilde{\varphi}_V)_{\tilde{s}}$. Hence $r(\nu) \leq \dim \varphi_{/N}(S) + \frac{1}{2}\dim(V \cap N)$ for any normal subgroup $N \lhd \mathbf{G}$.

Let us prove "$\geq$". Let $t = \dim S - r(\nu)$. Then $S_{\mathscr{F}}(t)$ contains a non-empty open subset of S^{an}. By (3.3) and Proposition 3.9 (which should be applied to $S'_{\mathscr{F}}(t-r)$ for each $0 \leq r \leq t-1$), we have

$$S_{\mathscr{F}}(t) \subseteq S_{\geq t} \cup \bigcup\nolimits_{0\leq r\leq t-1,\ 1\leq j\leq k} S_{\geq r} \cap \varphi^{-1}\left(E_j(t-r)\right).$$

Each $S_{\geq r}$ is Zariski closed in S, and each $E_j(t-r)$ is Zariski closed in $S' = \varphi(S)$. Hence, each member in the union on the right-hand side is Zariski closed in S. Taking the Zariski closure of both sides, we then have S equal to a member on the right-hand side.

If $S = S_{\geq t}$, then "$\geq$" holds already for $N = \{1\}$.

Assume $S = S_{\geq r} \cap \varphi^{-1}\left(E_j(t-r)\right)$ for some $0 \leq r \leq t-1$ and some j. Then $S = S_{\geq r} = \varphi^{-1}\left(E_j(t-r)\right)$, So each fiber of φ has dimension $\geq r$, and $S' = E_j(t-r)$. Moreover, $\mathrm{MT}(\mathcal{D}'_j) = \mathbf{G}$ because $S' = \varphi(S)$ is Hodge generic in $\Gamma\backslash\mathcal{D}$. Set $N = N_j$. Each fiber of the map

$$\varphi_{/N} \colon S \xrightarrow{\varphi} S' \subseteq \Gamma\backslash\mathcal{D} \xrightarrow{[p_N]} \Gamma_{/N}\backslash(\mathcal{D}/N),$$

has $\mathbb{C}$-dimension $\geq r + \left(\frac{1}{2}\dim(V \cap N) + (t-r)\right) = \frac{1}{2}\dim(V \cap N) + t$ by definition of $E_j(t-r)$. So

$$r(\nu) = \dim S - t \geq \left(\dim \varphi_{/N}(S) + \frac{1}{2}\dim(V \cap N) + t\right) - t = \dim \varphi_{/N}(S) + \frac{1}{2}\dim(V \cap N).$$

So, "$\geq$" is established. $\square$

3.5. **Zariski closedness of the degeneracy loci.** We start with the following lemma, which is the converse of Proposition 3.9.

Lemma 3.10. *For each $t \geq 0$ and each $j \in \{1, \ldots, k\}$, we have $E_j(t) \subseteq S_{\mathscr{F}}(t)$.*

Proof. Fix j. Denote by $\mathbf{H}_j = \mathrm{MT}(\mathcal{D}'_j)$, $V_j := V \cap \mathbf{H}_j$, and $\mathbf{H}_{j,0} := \mathbf{H}_j/V_j$. Under the identification $\mathcal{D} = V(\mathbb{R}) \times \mathcal{D}_0$ in Proposition B.3.(i), we have $\mathcal{D}'_j = (V_j(\mathbb{R}) + v_0) \times p(\mathcal{D}'_j)$ by Lemma B.5 (applied to $\mathcal{D}'_j \subseteq \mathcal{D}$).

Because $N_j \lhd \mathbf{H}_j$, we have: (i) $V \cap N_j = V_j \cap N_j$ is a $\mathbf{H}_{j,0}$-module; (ii) the action of $p(N_j) \lhd \mathbf{H}_{j,0}$ on $V_j/(V_j \cap N_j)$ is trivial. Let $x \in \mathcal{D}'_j$. Under $\mathcal{D}'_j = (V_j(\mathbb{R}) + v_0) \times p(\mathcal{D}'_j)$, write $x = (v, x_0)$. Then $N_j(\mathbb{R})^+x$ becomes $((V \cap N_j)(\mathbb{R}) + v) \times p(N_j)(\mathbb{R})^+x_0$. Notice that this $v \in V(\mathbb{R})$ is fixed.

For each $s \in E_j(t)$, by definition there exist an irreducible $\tilde{Y} \subseteq u^{-1}(S) \cap \mathcal{D}'_j$ such that $s \in u(\tilde{Y})$, $\dim \tilde{Y} \geq \frac{1}{2}\dim(V \cap N_j) + t$, and that $\tilde{Y}$ is contained in a fiber of the quotient $\mathcal{D}'_j \to \mathcal{D}'_j/N_j$. The last condition implies that $\tilde{Y} \subseteq N_j(\mathbb{R})^+x$ for some $x \in \mathcal{D}'_j$. Hence by the discussion above, $\tilde{Y} \subseteq ((V \cap N_j)(\mathbb{R}) + v) \times \mathcal{D}_0$ for a fixed $v \in V(\mathbb{R})$. Now that $\dim_{\mathbb{C}} \tilde{Y} \geq \frac{1}{2}\dim(V \cap N_j) + t$, the following property holds: For each $(a, x_0) \in \tilde{Y} \subseteq ((V \cap N_j)(\mathbb{R}) + v) \times \mathcal{D}_0$, there exists a complex analytic subset $\tilde{C} \subseteq \mathcal{D}_0$ with $\dim \tilde{C} \geq t$ such that $\{a\} \times \tilde{C} \subseteq \tilde{Y}$. Hence $s \in S_{\mathscr{F}}(t)$ by Lemma 2.1 (more precisely, the first equality in (2.5)). Now, the conclusion of the lemma holds as s runs over $E_j(t)$. $\square$

Now we are ready to prove Theorem 3.2.

Proof of Theorem 3.2. In this proof, we go back to our original setting and do not replace S by $\varphi(S)$. We have $S \xrightarrow{\varphi} S' \subseteq \Gamma\backslash\mathcal{D}$ with $S' = \varphi(S)$ an algebraic subvariety of $\Gamma\backslash\mathcal{D}$.

By (3.3), Proposition 3.9 and Lemma 3.10 (both applied to $S'_{\mathscr{F}}(t-r)$ for each $0 \leq r \leq t-1$), we have

$$S_{\mathscr{F}}(t) = S_{\geq t} \cup \bigcup\nolimits_{0 \leq r \leq t-1,\ 1 \leq j \leq k} S_{\geq r} \cap \varphi^{-1}\left(E_j(t-r)\right). \tag{3.8}$$

So $S_{\mathscr{F}}(t)$ is Zariski closed in S because each member in the union on the right-hand side is. The "In particular" part is easy to check once we have established the Zariski closedness of $S_{\mathscr{F}}(t)$. $\square$

4. Proof of the finiteness result à la Ullmo

The goal of this section is to prove Proposition 3.8, the finiteness result regarding weakly optimal subvarieties for admissible VMHS of weight -1 and 0.

Before moving on to our proof, let us take a step back and look at this proposition from a historical point of view. In studying the André–Oort conjecture, Ullmo [Ull14, Thm. 4.1] proved this finiteness result for *maximal weakly special subvarieties* (a particular kind of weakly optimal subvarieties) in the case of pure Shimura varieties as an application of the pure Ax–Lindemann theorem (a special case of Ax–Schanuel). The finiteness is ultimately obtained by the following fact: any countable set which is definable in an o-minimal structure is finite.

Ullmo's result should be seen as the analog of the classical result [Bog81, Thm. 1] in the Shimura case. His proof laid a blueprint for later generalizations: in the pure case and for weakly optimal subvarieties by Daw–Ren [DR18, Prop. 6.3] for pure Shimura varieties and by Baldi–Klingler–Ullmo [BKU24, §6] for VHS, and in the mixed Shimura case by the first-named author [Gao17, Thm. 12.2] for maximal weakly special subvarieties and [Gao20b, Thm. 1.4] for weakly optimal subvarieties when the mixed Shimura variety is of Kuga type. Our proof of Proposition 3.8 follows this blueprint. While the method also works for admissible VMHS of general weights if one considers the successive fibered structure of Mumford–Tate domain [GK24, §6], we focus on our case for our application and to ease notation.

Recently, Baldi–Urbanik [BU24, Thm. 7.1] proved Proposition 3.8 for general admissible VMHS in a different way (but also uses Ax–Schanuel as a core). Weakly optimal subvarieties are called *monodromically atypical maximal*, and Proposition 3.8 is called *Geometric Zilber–Pink* as in [BKU24]. Their proof does not use o-minimality and gives some effective results.

Retain the notation in (2.3). In this section, we also replace S by $\varphi(S)$ to ease notation. The proof also works without doing so, except that when we apply Ax–Schanuel for Theorem 4.2, we need to combine it with [BBT23].

4.1. **Zariski optimal subsets and an application of mixed Ax–Schanuel.**

Definition 4.1. *A complex analytic irreducible subset $\tilde{Y}$ of $u^{-1}(S)$ is called* ***Zariski optimal*** *if the following holds: $\tilde{Y} \subsetneq \tilde{Y}' \subseteq u^{-1}(S) \Rightarrow \delta_{\mathrm{Zar}}(\tilde{Y}) < \delta_{\mathrm{Zar}}(\tilde{Y}')$, with $\tilde{Y}'$ complex analytic irreducible.*

The following theorem is, in fact, an equivalent statement to the weak mixed Ax–Schanuel theorem (Theorem 3.5). In this paper, we only need one deduction.

Theorem 4.2. *Assume $\tilde{Y} \subseteq u^{-1}(S)$ is Zariski optimal. Then $\tilde{Y}^{\mathrm{Zar}} = \tilde{Y}^{\mathrm{ws}}$ and $\tilde{Y}$ is a complex analytic irreducible component of $\tilde{Y}^{\mathrm{ws}} \cap u^{-1}(S)$.*

Proof. Let $\tilde{Y} \subseteq u^{-1}(S)$ be Zariski optimal. Then $\tilde{Y}$ is a complex analytic irreducible component of $\tilde{Y}^{\mathrm{Zar}} \cap u^{-1}(S)$. Hence it suffices to prove $\tilde{Y}^{\mathrm{Zar}} = \tilde{Y}^{\mathrm{ws}}$.

Let $\tilde{Y}'$ be a complex analytic irreducible component of $\tilde{Y}^{\mathrm{ws}} \cap u^{-1}(S)$ which contains $\tilde{Y}$. Then $u(\tilde{Y}')$ is closed algebraic in S because $u(\tilde{Y}^{\mathrm{ws}}) \cap S$ is a closed algebraic subset of S by Proposition 3.3.

Assume $\tilde{Y} \neq \tilde{Y}'$. Then $\delta_{\mathrm{Zar}}(\tilde{Y}) < \delta_{\mathrm{Zar}}(\tilde{Y}')$ by the Zariski optimality of $\tilde{Y}$. So

$$\dim \tilde{Y}^{\mathrm{Zar}} - \dim \tilde{Y} < \dim \tilde{Y}'^{\mathrm{Zar}} - \dim \tilde{Y}' \leq \dim \tilde{Y}^{\mathrm{ws}} - \dim \tilde{Y}'.$$

Now that $u(\tilde{Y}) \subseteq u(\tilde{Y}')$, we have $u(\tilde{Y})^{\mathrm{Zar}} \subseteq u(\tilde{Y}')$ because the right hand side is closed algebraic in S. So

$$\dim u(\tilde{Y})^{\mathrm{Zar}} \leq \dim u(\tilde{Y}') = \dim \tilde{Y}'.$$

These two inequalities together yield.

$$\dim \tilde{Y}^{\mathrm{Zar}} + \dim u(\tilde{Y})^{\mathrm{Zar}} < \dim \tilde{Y}^{\mathrm{ws}} + \dim \tilde{Y}.$$

This contradicts the weak mixed Ax–Schanuel theorem (Theorem 3.5) for $\tilde{Y}$.

Hence we must have $\tilde{Y} = \tilde{Y}'$. In particular, $u(\tilde{Y})$ is algebraic. So $\tilde{Y}^{\mathrm{Zar}} = \tilde{Y}^{\mathrm{ws}}$. □

For $Y \subseteq S$ irreducible, let $\tilde{Y}$ be a complex analytic irreducible component of $u^{-1}(Y)$.

Corollary 4.3. *If $Y \subseteq X$ is weakly optimal, then $\tilde{Y}$ is Zariski optimal in $u^{-1}(S)$.*

Proof. Let $\tilde{Y}' \supseteq \tilde{Y}$ be a complex analytic irreducible subset of $u^{-1}(S)$ such that $\delta_{\mathrm{Zar}}(\tilde{Y}) \geq \delta_{\mathrm{Zar}}(\tilde{Y}')$. We may and do assume that $\tilde{Y}'$ is Zariski optimal.

Set $Y'' := u(\tilde{Y}')^{\mathrm{Zar}}$. Then $Y'' \subseteq u(\tilde{Y}'^{,\mathrm{ws}}) \cap S$ because $u(\tilde{Y}'^{,\mathrm{ws}}) \cap S$ is closed and algebraic. Thus

$$\delta_{\mathrm{ws}}(Y'') = \dim Y''^{,\mathrm{ws}} - \dim Y'' \leq \dim \tilde{Y}'^{,\mathrm{ws}} - \dim \tilde{Y}'.$$

Since $\tilde{Y}'$ is Zariski optimal, we have $\tilde{Y}'^{,\mathrm{Zar}} = \tilde{Y}'^{,\mathrm{ws}}$ by Theorem 4.2. So, the inequality above further implies

$$\delta_{\mathrm{ws}}(Y'') \leq \dim \tilde{Y}'^{,\mathrm{Zar}} - \dim \tilde{Y}' = \delta_{\mathrm{Zar}}(\tilde{Y}') \leq \delta_{\mathrm{Zar}}(\tilde{Y}) \leq \dim \tilde{Y}^{\mathrm{ws}} - \dim \tilde{Y} = \delta_{\mathrm{ws}}(Y).$$

So $Y'' = Y$ because Y is weakly optimal. Thus $\tilde{Y} = \tilde{Y}'$ is Zariski optimal. $\square$

4.2. **A parametrization of Zariski optimal subsets.** The goal of this subsection is to construct a space $\mathscr{N}^0$ which parametrizes Zariski optimal subsets of $u^{-1}(S)$.

Fix a Levi decomposition $\mathbf{G} = V \rtimes \mathbf{G}_0$. Then for the identification $\mathcal{D} = V(\mathbb{R}) \times \mathcal{D}_0$ in Proposition B.3.(i), the action of $\mathbf{G}(\mathbb{R})$ on $\mathcal{D}$ is given by

$$(v, g_0) \cdot (x_V, x_0) = (v + g_0 x_V, g_0 x_0).$$

Let $\mathfrak{F}$ be the semi-algebraic fundamental set for $u \colon \mathcal{D} \to \Gamma\backslash\mathcal{D}$ from §2.3.

Consider the set $\mathscr{N}$ of pairs (x, H) consisting of $x \in (u|_{\mathfrak{F}})^{-1}(S)$ and H a connected subgroup of $\mathbf{G}_{\mathbb{R}}$ with the following properties: $H_0 := H/(V_{\mathbb{R}} \cap H)$ is a semi-simple subgroup of $\mathbf{G}_{0,\mathbb{R}}$, and $h_x(\mathbb{S}) \subseteq \mathrm{N}_{\mathbf{G}_{\mathbb{R}}}(H)$.

For each $(x, H) \in \mathscr{N}$, denote by $V_H := V_{\mathbb{R}} \cap H$. Then V_H is the unipotent radical of H. It can be easily shown that each weak Mumford–Tate domain is $N(\mathbb{R})x$ for some $(x, N_{\mathbb{R}}) \in \mathscr{N}$ (with $N_{\mathbb{R}}$ defined over $\mathbb{Q}$).

Lemma 4.4. *Let $(x, H) \in \mathscr{N}$. Then $H(\mathbb{R})x$ is:*

(i) *an algebraic subset of $\mathcal{D}$;*
(ii) *$(v' + V_H(\mathbb{R})) \times H_0(\mathbb{R})x_0 \subseteq V(\mathbb{R}) \times \mathcal{D}_0 = \mathcal{D}$ for some $v' \in V(\mathbb{R})$ and $x_0 = p(x)$.*

Proof. $H(\mathbb{R})x$ is by definition semi-algebraic, and is complex analytic since $h_x(\mathbb{S}) \subseteq \mathrm{N}_{\mathbf{G}_{\mathbb{R}}}(H)$. So (i) holds.

Any two Levi decompositions differ from the conjugation by an element of the unipotent radical. Hence there exists a $v \in V(\mathbb{R})$ such that

$$H = (v, 1)(V_H \rtimes H_0)(-v, 1).$$

Thus the condition $h_x(\mathbb{S}) \subseteq \mathrm{N}_{\mathbf{G}_{\mathbb{R}}}(H)$ implies $\mathrm{Int}(-v)(h_x(\mathbb{S})) \subseteq \mathrm{N}_{\mathbf{G}_{\mathbb{R}}}(V_H \rtimes H_0) = V' \rtimes G_0'$ for some $V' < V_{\mathbb{R}}$ and $G_0' < \mathbf{G}_{0,\mathbb{R}}$. Then H_0 acts trivially on V'/V_H by the normality condition.

Write $x = (x_V, x_0) \in V(\mathbb{R}) \times \mathcal{D}_0 = \mathcal{D}$, then $\mathrm{Int}(-v)(h_x(\mathbb{S})) = h_{(x_V - v, x_0)}(\mathbb{S})$ by Proposition B.3.(i). So by the last paragraph $x_V - v \in V_H(\mathbb{R}) + v''$ for some $v'' \in V(\mathbb{R})$ with $H_0(\mathbb{R}) \cdot v'' = v''$. Thus $(v, 1)(V_H \rtimes H_0)(\mathbb{R})(-v, 1) \cdot x = (V_H(\mathbb{R}) + v'' + v) \times H_0(\mathbb{R})x_0$. So (ii) holds. $\square$

Define the following two functions on $\mathscr{N}$:

$$d \colon \mathscr{N} \to \mathbb{R}, \quad (x, H) \mapsto \dim_x \left(u^{-1}(S) \bigcap H(\mathbb{R})x\right), \tag{4.1}$$

$$\delta \colon \mathscr{N} \to \mathbb{R}, \quad (x, H) \mapsto \dim_x H(\mathbb{R})x - \dim_x \left(u^{-1}(S) \bigcap H(\mathbb{R})x\right).$$

Finally, we are ready to define

$$\begin{aligned}(4.2)\qquad \mathscr{N}^0 := \big\{(x,H) \in \mathscr{N} : \ &\text{for any } (x,H') \in \mathscr{N}, \text{ we have} \\ &H'(\mathbb{R})x \subsetneq H(\mathbb{R})x \Rightarrow d(x,H) > d(x,H'), \\ &H(\mathbb{R})x \subsetneq H'(\mathbb{R})x \Rightarrow \delta(x,H) < \delta(x,H')\big\}.\end{aligned}$$

The proof of the following proposition uses Theorem 4.2 (twice) and hence mixed Ax–Schanuel.

Proposition 4.5. *The following two sets are equal:*

- *the set of orbits* $\{H(\mathbb{R})x : (x,H) \in \mathscr{N}^0\}$*;*
- $\{\tilde{Y}^{\mathrm{Zar}} = \tilde{Y}^{\mathrm{ws}} : \tilde{Y} \subseteq u^{-1}(S)$ *Zariski optimal, with* $\tilde{Y} \cap \mathfrak{F} \neq \emptyset\}$.

Moreover, $H(\mathbb{R})x$ *is a weak Mumford–Tate domain for each* $(x,H) \in \mathscr{N}^0$.

Proof. Take $(x,H) \in \mathscr{N}^0$. We wish to prove that $H(\mathbb{R})x$ equals $\tilde{Y}^{\mathrm{Zar}}$ for some Zariski optimal $\tilde{Y} \subseteq u^{-1}(S)$ which passes through $x \in \mathfrak{F}$.

Let $\tilde{Y}'$ be a complex analytic irreducible component of $u^{-1}(S) \cap H(\mathbb{R})x$ which passes through x with $\dim \tilde{Y}' = d(x,H)$.

Take $\tilde{Y} \supseteq \tilde{Y}'$ with $\tilde{Y} \subseteq u^{-1}(S)$ complex analytic irreducible and $\delta_{\mathrm{Zar}}(\tilde{Y}) \leq \delta_{\mathrm{Zar}}(\tilde{Y}')$. We may and do assume that $\tilde{Y}$ is Zariski optimal. Then $\tilde{Y}^{\mathrm{Zar}} = \tilde{Y}^{\mathrm{ws}}$ is a weak Mumford–Tate domain by Theorem 4.2, and hence $\tilde{Y}^{\mathrm{Zar}} = H'(\mathbb{R})x$ for some $(x,H') \in \mathscr{N}$. Thus $\delta_{\mathrm{Zar}}(\tilde{Y}) = \delta(x,H')$.

Thus $\tilde{Y}'$ is contained in some irreducible component of $H(\mathbb{R})x \cap H'(\mathbb{R})x$, which by Lemma 4.4.(ii) is $H''(\mathbb{R})x$ for some $(x,H'') \in \mathscr{N}$. Hence $d(x,H) = d(x,H'')$, so by definition of $\mathscr{N}^0$ we have $H(\mathbb{R})x = H''(\mathbb{R})x$. So $H(\mathbb{R})x \subseteq H'(\mathbb{R})x$. But $\delta(x,H) = \delta(x,H')$. So by definition of $\mathscr{N}^0$ we have $H(\mathbb{R})x = H'(\mathbb{R})x$, and hence $H(\mathbb{R})x = \tilde{Y}^{\mathrm{Zar}}$. Now we have proved that the first set is contained in the second set and the "Moreover" part of the proposition.

Conversely let $\tilde{Y} \subseteq u^{-1}(S)$ be Zariski optimal with $\tilde{Y} \cap \mathfrak{F} \neq \emptyset$. By Theorem 4.2, $\tilde{Y}^{\mathrm{Zar}} = \tilde{Y}^{\mathrm{ws}}$ is a weak Mumford–Tate domain, and hence equals $H(\mathbb{R})x$ for some $(x,H) \in \mathscr{N}$. We may furthermore choose this x to be a smooth point of $H(\mathbb{R})x \cap u^{-1}(S) \cap \mathfrak{F}$; then $\dim \tilde{Y} = d(x,H)$.

We wish to prove $(x,H) \in \mathscr{N}^0$. Let us check the two properties which define $\mathscr{N}^0$ by contradiction.

Assume there exists $(x,H') \in \mathscr{N}$ such that $H'(\mathbb{R})x \subsetneq H(\mathbb{R})x$ and $d(x,H) = d(x,H')$. Then $\dim \tilde{Y} = d(x,H')$ and hence $\tilde{Y} \subseteq H'(\mathbb{R})x$ by analytic continuation. But then $\tilde{Y}^{\mathrm{Zar}} \subseteq H'(\mathbb{R})x$ by Lemma 4.4.(i). So $\tilde{Y}^{\mathrm{Zar}} \subseteq H'(\mathbb{R})x \subsetneq H(\mathbb{R})x = \tilde{Y}^{\mathrm{Zar}}$, which is impossible. So for any $(x,H') \in \mathscr{N}$, we have $H'(\mathbb{R})x \subsetneq H(\mathbb{R})x \Rightarrow d(x,H) > d(x,H')$.

Next, assume there exists $(x,H') \in \mathscr{N}$ such that $H(\mathbb{R})x \subsetneq H'(\mathbb{R})x$ and $\delta(x,H) \geq \delta(x,H')$. Take a complex analytic irreducible component $\tilde{Y}'$ of $u^{-1}(S) \bigcap H'(\mathbb{R})x$ such that $\dim \tilde{Y}' = d(x,H')$. Then $\tilde{Y} \subseteq \tilde{Y}'$ by analytic continuation. By Lemma 4.4.(i), we have $\tilde{Y}'^{,\mathrm{Zar}} \subseteq H'(\mathbb{R})x$. So $\delta_{\mathrm{Zar}}(\tilde{Y}') \leq \delta(x,H') \leq \delta(x,H) = \delta_{\mathrm{Zar}}(\tilde{Y})$. Hence $\tilde{Y}' = \tilde{Y}$ because $\tilde{Y}$ is Zariski optimal. So $\delta(x,H') = \delta(x,H)$ and $d(x,H') = d(x,H)$, and so

$$\dim H(\mathbb{R})x = d(x,H) + \delta(x,H) = d(x,H') + \delta(x,H') = \dim H'(\mathbb{R})x.$$

This contradicts $H(\mathbb{R})x \subsetneq H'(\mathbb{R})x$. So for any $(x, H') \in \mathscr{N}$, we have $H(\mathbb{R})x \subsetneq H'(\mathbb{R})x \Rightarrow \delta(x, H) < \delta(x, H')$.

This finishes the proof of $(x, H) \in \mathscr{N}^0$. So, the second set is contained in the first set. Now we are done. □

4.3. **A finiteness result for $\mathscr{N}^0$.** Let $\mathscr{N}^0$ be as defined in (4.2). In this subsection, we prove that the following finiteness result. The proof relies on o-minimality.

Proposition 4.6. *There are only finitely many subgroups H of $\mathbf{G}_{\mathbb{R}}$ such that $(x, H) \in \mathscr{N}^0$ for some $x \in (u|_{\mathfrak{F}})^{-1}(S)$.*

We start with the following classical result.

Lemma 4.7. (i) *There exists a finite set $\Omega_V = \{V_1, \ldots, V_n\}$ of subspaces of $V_{\mathbb{R}}$ with the following property: Each subspace of $V_{\mathbb{R}}$ equals $g_V V_j$ for some $g_V \in \mathrm{GL}(V_{\mathbb{R}})$ and some $j \in \{1, \ldots, n\}$.*
(ii) *There exists a finite set $\Omega_0 = \{G_1, \ldots, G_n\}$ of semi-simple subgroups of $\mathbf{G}_{0,\mathbb{R}}$, with no compact factors, such that the following holds: Each semi-simple subgroups of $\mathbf{G}_{0,\mathbb{R}}$ equals $g_0 G_j g_0^{-1}$ for some $g_0 \in \mathbf{G}_0(\mathbb{R})$ and some $j \in \{1, \ldots, n\}$.*

Consider the set Υ consisting of elements $(x, g_V, g_0, V_j, G_j, v) \in (u|_{\mathfrak{F}})^{-1}(S) \times \mathrm{GL}(V_{\mathbb{R}}) \times \mathbf{G}_0(\mathbb{R}) \times \Omega_V \times \Omega_0 \times V(\mathbb{R})$ satisfying the following properties:

(a) $g_0 G_j g_0^{-1}$ stabilizes $g_V V_j$; hence $g_V V_j \rtimes g_0 G_j g_0^{-1}$ is a subgroup of $\mathbf{G}_{\mathbb{R}} = (V \rtimes \mathbf{G}_0)_{\mathbb{R}}$;
(b) $h_x(\mathbb{S}) \subseteq \mathrm{N}_{\mathbf{G}(\mathbb{R})}\big((v, 1)(g_V V_j \rtimes g_0 G_j g_0^{-1})(-v, 1)\big)$.

Recall from §2.3 that $(u|_{\mathfrak{F}})^{-1}(S)$ is a definable subset of $\mathcal{D}$. Hence, Υ is a definable set.

For each $(x, g_V, g_0, V_j, G_j, v) \in \Upsilon$, denote by $H_v^{(j)} := (v, 1)(g_V V_j \rtimes g_0 G_j g_0^{-1})(-v, 1)$. Then we obtain a map that is surjective by Lemma 4.7

$$\psi\colon \Upsilon \to \mathscr{N}, \quad (x, g_V, g_0, V_j, G_j, v) \mapsto \big(x, H_v^{(j)}\big)\,. \tag{4.3}$$

Composing this map with the functions $d\colon \mathscr{N} \to \mathbb{R}$ and $\delta\colon \mathscr{N} \to \mathbb{R}$ from (4.1), we obtain two functions on Υ

$$d_{\Upsilon}\colon \Upsilon \to \mathbb{R}, \quad (x, g_V, g_0, V_j, G_j, v) \mapsto \dim_x \Big((u|_{\mathfrak{F}})^{-1}(S) \bigcap H_v^{(j)}(\mathbb{R})x\Big)\,,$$

$$\delta_{\Upsilon}\colon \Upsilon \to \mathbb{R}, \quad (x, g_V, g_0, V_j, G_j, v) \mapsto \dim_x H_v^{(j)}(\mathbb{R})x - \dim_x \Big((u|_{\mathfrak{F}})^{-1}(S) \bigcap H_v^{(j)}(\mathbb{R})x\Big)\,.$$

The general theory of o-minimal geometry says that $x \mapsto \dim_x X$ is a definable function for any definable set X. Hence d_{Υ} and δ_{Υ} are definable functions on Υ.

For the subset $\mathscr{N}^0 \subseteq \mathscr{N}$, the inverse $\psi^{-1}(\mathscr{N}^0) \subseteq \Upsilon$ is

$$\begin{aligned}\Xi = \big\{(x, g_V, g_0, V_j, G_j, v) \in \Upsilon :\ & \text{for any } (x, g'_V, g'_0, V_{j'}, G_{j'}, v') \in \Upsilon,\\ & H_{v'}^{(j')}(\mathbb{R})x \subsetneq H_v^{(j)}(\mathbb{R})x \Rightarrow d_{\Upsilon}(x, g_V, g_0, V_j, G_j, v) > d_{\Upsilon}(x, g'_V, g'_0, V_{j'}, G_{j'}, v'),\\ & H_v^{(j)}(\mathbb{R})x \subsetneq H_{v'}^{(j')}(\mathbb{R})x \Rightarrow \delta_{\Upsilon}(x, g_V, g_0, V_j, G_j, v) < \delta_{\Upsilon}(x, g'_V, g'_0, V_{j'}, G_{j'}, v')\big\},\end{aligned}$$

which is a definable subset of Υ since both d_{Υ} and δ_{Υ} are definable functions.

With these preparations, we are ready to prove Proposition 4.6.

Proof of Proposition 4.6. Consider the map

$$\begin{array}{rcl} \rho\colon \Xi & \to & \bigcup_{i=1}^{n} \big(\mathrm{GL}(V_{\mathbb{R}})/\mathrm{Stab}_{\mathrm{GL}(V_{\mathbb{R}})}(V_j)\big) \times \big(\mathbf{G}_0(\mathbb{R})/N_{\mathbf{G}_0(\mathbb{R})}(G_j)\big) \times V(\mathbb{R}) \\ (x, g_V, g_0, V_j, G_j, v) & \mapsto & (g_V V_j, g_0 G_j g_0^{-1}, v). \end{array}$$

By the surjectivity of ψ, for any $(x, H) \in \mathscr{N}^0$, the group H equals $H_v^{(j)} = (v, 1)(g_V V_j \rtimes g_0 G_j g_0^{-1})(-v, 1)$ for some $(g_V V_j, g_0 G_j g_0^{-1}, v) \in \rho(\Xi)$.

So, it suffices to prove that $\rho(\Xi)$ is finite.

First, the map ρ is clearly definable. Hence $\rho(\Xi)$ is definable.

Next, Proposition 4.5 says that $H_v^{(j)}(\mathbb{R})x$ is a weak Mumford–Tate domain. Hence there exists a $\mathbb{Q}$-subgroup N of $\mathbf{G}$ such that $N_{\mathbb{R}} = (v, 1)(g_V V_j \rtimes g_0 G_j g_0^{-1})(-v, 1)$. This implies that $g_V V_j = (V \cap N)_{\mathbb{R}}$ and $g_0 G_j g_0^{-1} = N_{0,\mathbb{R}}$. Moreover, $(v, 1)\left((V \cap N) \rtimes N_0\right)(-v, 1) = N$ and hence $v \in V(\mathbb{Q})$. So $\rho(\Xi)$ is countable since $\mathbb{Q}$ is countable.

Therefore, $\rho(\Xi)$ is finite because it is definable and countable. □

4.4. **Proof of Proposition 3.8.** Let Y be a weakly optimal subvariety of S. Take a complex analytic irreducible component $\tilde{Y}$ of $u^{-1}(Y)$ such that $\tilde{Y} \cap \mathfrak{F} \neq \emptyset$. Then $\tilde{Y}$ is Zariski optimal in $u^{-1}(S)$ by Corollary 4.3. Then by Proposition 4.5, we have $\tilde{Y}^{\mathrm{ws}} = H(\mathbb{R})x$ for some $(x, H) \in \mathscr{N}^0$.

Hence by the finiteness result Proposition 4.6, there exist finitely many $\mathbb{Q}$-groups $N_1, \ldots, N_k < \mathbf{G}$ satisfying the following property: For each weakly optimal subvariety $Y \subseteq S$, $Y^{\mathrm{ws}} = u(\tilde{Y}^{\mathrm{ws}})$ equals $u(N_j(\mathbb{R})^+x)$ for some $j \in \{1, \ldots, k\}$ and some $x \in \mathcal{D}$ with $h_x(\mathbb{S}) \subseteq \mathrm{N}_{\mathbf{G}}(N_j)(\mathbb{R})$. Then $\mathcal{D}'_j := \mathrm{N}_{\mathbf{G}}(N_j)(\mathbb{R})^+x$ is a Mumford–Tate domain and is independent of the choice of such x. We are done. □

5. Application to non-degeneracy in some geometric cases

In this section, we give two applications of Theorem 3.1: when the VHS is irreducible or when it has a simple algebraic monodromy group. Both cases apply to the Gross-Shoen and the Ceresa normal functions.

Let S be a smooth irreducible quasi-projective variety. Let $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet)$ be a polarized VHS on S of weight -1. Let $\nu\colon S \to \mathcal{J}(\mathbb{V}_{\mathbb{Z}})$ be an admissible normal function. Then we have the associated Betti form β_ν defined in Definition D.5; it is a semi-positive $(1,1)$-form on S.

We wish to check whether $\beta_\nu^{\wedge \dim S} \not\equiv 0$, which by Corollary D.7 becomes $r(\nu) = \dim S$ for the Betti rank $r(\nu)$ defined in (1.5). Now Theorem 3.1 gives a checkable criterion. Recall the period map $\varphi = \varphi_\nu\colon S \to \Gamma\backslash\mathcal{D}$, the $\mathbb{Q}$-group $\mathbf{G}$ and its unipotent radical V which is a vector group from (2.3). We emphasize that V is, in general, not a fiber of $\mathbb{V}_{\mathbb{Q}}$ and its geometric meaning is given by Remark 2.2: $\frac{1}{2}\dim V$ is the relative dimension of $\mathcal{J}_\nu$, the smallest intermediate Jacobian of sub-VHS $\mathbb{V}'_{\mathbb{Z}}$ of $\mathbb{V}_{\mathbb{Z}} \to S$ translated by a torsion multisection which contains $\nu(S)$.

Notice that the trivial upper bound (2.8) yields the following necessary condition for $r(\nu) = \dim S$:

$$(\mathtt{Hyp}): \qquad \dim \varphi(S) = \dim S \leq \frac{1}{2}\dim_{\mathbb{Q}} V.$$

Theorem 5.1. *Assume: (i)* $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet) \to S$ *is irreducible, i.e. the only sub-VHSs are trivial or itself; (ii)* $\nu(S)$ *is not a torsion section. Then*

$$r(\nu) = \min\left\{\dim \varphi(S), \frac{1}{2}\dim \mathbb{V}_{\mathbb{Q},s}\right\}$$

for one (and hence for all) $s \in S(\mathbb{C})$.

In particular, if futhermore $\dim \varphi(S) = \dim S$ *and* $\dim \mathbb{V}_{\mathbb{Q},s} \geq 2\dim S$, *then we have* $\beta_\nu^{\wedge \dim S} \not\equiv 0$.

We can say more in the situation of Theorem 5.1. In fact in this case $\mathbb{V}'_{\mathbb{Z}} = \mathbb{V}_{\mathbb{Z}}$. By the comment below (2.8), the equality $r(\nu) = \frac{1}{2}\dim \mathbb{V}_{\mathbb{Q},s}$ can be attained *only* if $\mathcal{J}(\mathbb{V}_{\mathbb{Z}}) \to S$ is an abelian scheme. Hence $\dim \mathbb{V}_{\mathbb{Q},s} > 2\dim \varphi(S)$ in the situation of Theorem 5.1 if $\mathcal{J}(\mathbb{V}_{\mathbb{Z}}) \to S$ is not an abelian scheme.

Proof. Set $\mathbf{G}_0 = \mathbf{G}/V$. Then V is a $\mathbf{G}_0$-submodule of $\mathbb{V}_{\mathbb{Q},s}$ for one (and hence all) $s \in S(\mathbb{C})$, and $\mathbf{G}_0$ is by definition a subgroup of $\mathrm{GL}(\mathbb{V}_{\mathbb{Q},s})$.

By (i), $\mathbb{V}_{\mathbb{Q},s}$ is irreducible as a $\mathbf{G}_0$-module. By (ii), $V \neq \{0\}$. Hence $V = \mathbb{V}_{\mathbb{Q},s}$. Thus we have $\mathbf{G}_0 \subset \mathrm{GL}(V)$. Let N be a normal subgroup of $\mathbf{G}$. Then $V \cap N$ is a $\mathbf{G}_0$-submodule of V, and hence gives rise to a sub-VHS of $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet) \to S$. Hence by (i), either $V \cap N = \{0\}$ or $V \cap N = V$.

In the first case, we have that the image N_0 of N acts trivially on V. Thus $N_0 = 0$. This shows that N is trivial. So $r(\nu) = \dim \varphi(S)$.

In the second case, $V \cap N = V$. It is clearly true that

$$\min_{N,\ V\cap N = V} \{\dim \varphi_{/N}(S)\} + \frac{1}{2}\dim V$$

is attained at $N = \mathbf{G}$ and hence equals $\frac{1}{2}\dim V$. $\square$

Theorem 5.2. *Assume: (i) the connected algebraic monodromy group* H *of* $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet) \to S$ *is simple; (ii)* $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet) \to S$ *has no isotrivial sub-VHS, i.e. locally constant VHS. Then*

$$r(\nu) = \min\left\{\dim \varphi(S), \frac{1}{2}\dim V\right\}.$$

An important case where assumption (i) is satisfied is when the Mumford–Tate group of $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet) \to S$ is quasi-simple; this follows from Deligne [Del87, §7.5]. And similar as the comment below Theorem 5.1, we must have $\dim V > 2\dim \varphi(S)$ in the situation of Theorem 5.2 if $\mathcal{J}(\mathbb{V}'_{\mathbb{Z}}) \to S$ is not an abelian scheme.

Proof. Set $\mathbf{G}_0 = \mathbf{G}/V$. By Deligne [Del87, §7.5], $H \lhd \mathbf{G}_0^{\mathrm{der}}$.

Let N be a normal subgroup of $\mathbf{G}$. The reductive part $N_0 := N/(V \cap N)$ is a normal subgroup of $\mathbf{G}_0$. Now that $N_0 \cap H$ is a normal subgroup of H, by (i) we have that either $N_0 \cap H$ is finite or $H < N_0$.

Assume $H < N_0$. Since $\mathbf{G}_0$ is reductive, we can decompose $V = (V \cap N) \bigoplus (V \cap N)^\perp$ as $\mathbf{G}_0$-modules. Now $(V \cap N)^\perp$ gives rise to a sub-VHS of $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet)$. But H acts trivially on $(V \cap N)^\perp$ because N_0 acts trivially on $V/(V \cap N)$. So $(V \cap N)^\perp = \{0\}$ by (ii). Hence $V \cap N = V$, and hence $V < N$. So in this case we have

$$\dim \varphi_{/N}(S) + \frac{1}{2}\dim(V \cap N) \geq \frac{1}{2}\dim V.$$

The equality obtains when $N = \mathbf{G}$.

Assume that $N_0 \cap H$ is finite. To handle this case, we again use the universal period map of the VHS $\tilde{\varphi}_0 \colon \tilde{S} \to \mathcal{D}_0$ associated with the VHS $\mathbb{V}_{\mathbb{Z}} \to S$ from (2.3), where $\mathcal{D}_0$ is a $\mathbf{G}_0(\mathbb{R})^+$-orbit and the natural projection $p \colon \mathcal{D} \to \mathcal{D}_0$ is induced by the quotient $\mathbf{G} \to \mathbf{G}_0$. Moreover, we have $\tilde{\varphi}_0 = p \circ \tilde{\varphi}$.

By logarithmic Ax [GK24, Thm. 7.2], we have $\tilde{\varphi}_0(\tilde{S}) \subseteq H(\mathbb{R})^+ x_0$ for some $x_0 \in \mathcal{D}_0$. So $\tilde{\varphi}(\tilde{S}) \subseteq p^{-1}(\tilde{\varphi}_0(\tilde{S})) \subseteq p^{-1}(H(\mathbb{R})^+ x_0)$. Notice that $p^{-1}(H(\mathbb{R})^+ x_0) = (V \rtimes H)(\mathbb{R})^+ x$ for some $x \in \mathcal{D}$. So $\tilde{\varphi}(\tilde{S}) \subseteq (V \rtimes H)(\mathbb{R})^+ x$, and hence $\varphi(S) \subseteq u((V \rtimes H)(\mathbb{R})^+ x)$ for $u \colon \mathcal{D} \to \Gamma \backslash \mathcal{D}$.

Recall the definition $\varphi_{/N} \colon S \xrightarrow{\varphi} \Gamma\backslash\mathcal{D} \xrightarrow{[p_N]} \Gamma_{/N}\backslash(\mathcal{D}/N)$, with $[p_N]$ induced by $p_N \colon \mathbf{G} \to \mathbf{G}/N$. Notice that p_N can be decomposed as the composite

$$p_N \colon \mathbf{G} \xrightarrow{p_{V\cap N}} \mathbf{G}/(V \cap N) \xrightarrow{p_{N_0}} \mathbf{G}/N,$$

and this induces

$$[p_N] \colon \Gamma\backslash\mathcal{D} \xrightarrow{[p_{V\cap N}]} \Gamma_{/V\cap N}\backslash(\mathcal{D}/(V\cap N)) \xrightarrow{[p_{N_0}]} \Gamma_{/N}\backslash(\mathcal{D}/N).$$

It is clearly true that $\dim \varphi(S) = \dim([p_{V\cap N}] \circ \varphi)(S)$. On the other hand, $\dim([p_{V\cap N}] \circ \varphi)(S) = \dim([p_{N_0}] \circ [p_{V\cap N}] \circ \varphi)(S)$ by the conclusion of the previous paragraph. So $\dim \varphi_{/N}(S) = \dim \varphi(S)$ in this case.

Now we are done by combining the two cases above. $\square$

Now, we apply these two theorems to the Gross–Schoen and the Ceresa normal functions. Let ν_{GS} (resp. ν_{Ce}) be the Gross–Schoen (resp. Ceresa) normal function $\mathcal{M}_g \to \mathcal{J}(\mathbb{V}_{\mathbb{Z}})$ defined in (C.5), with $\mathbb{V}_{\mathbb{Z}}$ from Proposition C.10. Let β_{GS} and β_{Ce} be the associated Betti forms.

Corollary 5.3. *Assume $g \geq 3$. The Betti ranks $r(\nu_{\mathrm{GS}}) = r(\nu_{\mathrm{Ce}}) = 3g-3$. Equivalently, $\beta_{\mathrm{GS}}^{\wedge(3g-3)} \not\equiv 0$ and $\beta_{\mathrm{Ce}}^{\wedge(3g-3)} \not\equiv 0$.*

Proof. We shall apply Theorem 5.1 and let us check the assumptions. Assumption (i) holds by part (iii) of Proposition C.10. Assumption (ii) holds by part (ii) of Proposition C.10. Moreover the VHS $\mathbb{V}_{\mathbb{Z}}$ on S has maximal moduli, so $\dim \varphi_{\mathrm{GS}}(\mathcal{M}_g) = \dim \varphi_{\mathrm{Ce}}(\mathcal{M}_g) = \dim \mathcal{M}_g = 3g-3$ where φ_{GS} and φ_{Ce} are the period maps. Finally, $\dim \mathbb{V}_{\mathbb{Q},s} = \frac{2g(2g-1)(2g-2)}{6} - 2g$, which is $> 2(3g-3)$ when $g \geq 3$. Hence, we can conclude by Theorem 5.1. $\square$

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Denote for simplicity by $S = \mathcal{M}_g$. Let ν be either ν_{GS} or ν_{Ce}.

Set $U := S \setminus S_{\mathscr{F}}(1)$, which is Zariski open by Theorem 3.2.

By Corollary 5.3 and the "In particular" part of Theorem 3.2, we have that $\min\{t \geq 0 : S_{\mathscr{F}}(t) = S\} = 0$. Thus $S_{\mathscr{F}}(1) \neq S$. Now, part (i) follows from Proposition C.7.

For (ii), notice that by definition $S_{\mathscr{F}}(1)$ contains any analytic curve $C \subseteq S^{\mathrm{an}}$ such that $\nu(C)$ is torsion. Thus, the set $\{s \in U(\mathbb{C}) : \nu(s) \text{ is torsion}\}$ is discrete, and hence at most countable. $\square$

6. Generic positivity for the Beilinson–Bloch height of Gross–Schoen and Ceresa cycles

In this section, we turn to the arithmetic applications.

6.1. **Construction of the adelic line bundle.** Let $\mathcal{M}_g$ be the moduli scheme of smooth curves of genus g over $\mathbb{Z}$, and let $\mathcal{C}_g \to \mathcal{M}_g$ be the universal curve.

Denote by $\mathfrak{J}_g = \mathrm{Jac}(\mathcal{C}_g/\mathcal{M}_g)$ the relative Jacobian. Identify $\mathfrak{J}_g$ with its dual via the principal polarization given by a suitable theta divisor.

The Poincaré line bundle $\mathscr{P}$ on $\mathfrak{J}_g \times_{\mathcal{M}_g} \mathfrak{J}_g$ extends to an integrable adelic line bundle $\overline{\mathscr{P}}$ as follows. Define $\mathscr{P}^\Delta := \Delta^*\mathscr{P}$ for the diagonal $\Delta\colon \mathfrak{J}_g \to \mathfrak{J}_g \times_{\mathcal{M}_g} \mathfrak{J}_g$. Then $\mathscr{P}^\Delta$ is relatively ample on $\mathfrak{J}_g \to \mathcal{M}_g$, and $[2]^*\mathscr{P}^\Delta = (\mathscr{P}^\Delta)^{\otimes 4}$. So $(\mathfrak{J}_g, [2], \mathscr{P}^\Delta)$ is a polarized dynamical system over $\mathcal{M}_g$ in the sense of [YZ21, §2.6.1]. Thus, Tate's limit process gives a nef adelic line bundle $\overline{\mathscr{P}}^\Delta$ on $\mathfrak{J}_g$, as executed by [YZ21, Thm. 6.1.1]. Now we obtain the desired $\overline{\mathscr{P}} \in \widehat{\mathrm{Pic}}(\mathfrak{J}_g/\mathbb{Z})_{\mathbb{Q}}$ by letting

$$2\overline{\mathscr{P}} := m^*\overline{\mathscr{P}}^\Delta - p_1^*\overline{\mathscr{P}}^\Delta - p_2^*\overline{\mathscr{P}}^\Delta,$$

where $m, p_1, p_2\colon \mathfrak{J}_g \to \mathfrak{J}_g \times_{\mathcal{M}_g} \mathfrak{J}_g$ with m being the addition and p_1 (resp. p_2) being the projection to the first (resp. second) factor.

Take $\xi \in \mathrm{Pic}^1(\mathcal{C}_g/\mathcal{M}_g)$ such that $(2g-2)\xi = \omega_{\mathcal{C}_g/\mathcal{M}_g}$. Let

$$i_\xi\colon \mathcal{C}_g \longrightarrow \mathfrak{J}_g$$

be the Abel–Jacobi map based at ξ. Then we have an $\mathcal{M}_g$-morphism $(i_\xi, i_\xi)\colon \mathcal{C}_g \times_{\mathcal{M}_g} \mathcal{C}_g \to \mathfrak{J}_g \times_{\mathcal{M}_g} \mathfrak{J}_g$, and hence get an integrable adelic line bundle on $\mathcal{C}_g \times_{\mathcal{M}_g} \mathcal{C}_g$

$$\overline{\mathscr{Q}} := (i_\xi, i_\xi)^*\overline{\mathscr{P}} \in \widehat{\mathrm{Pic}}(\mathcal{C}_g \times_{\mathcal{M}_g} \mathcal{C}_g/\mathbb{Z})_{\mathbb{Q}},$$

and we can compute

$$\mathscr{Q} = \mathscr{O}(\Delta) - p_1^*\xi - p_2^*\xi. \tag{6.1}$$

with Δ the diagonal of $\mathcal{C}_g \to \mathcal{C}_g \times_{\mathcal{M}_g} \mathcal{C}_g$, and p_1, p_2 the projections $\mathcal{C}_g \times_{\mathcal{M}_g} \mathcal{C}_g \to \mathcal{C}_g$.

Finally the Deligne pairing gives an adelic line bundle on $\mathcal{M}_g$ by [YZ21, Thm. 4.1.3]

$$\overline{\mathscr{L}} := \langle \overline{\mathscr{Q}}, \overline{\mathscr{Q}}, \overline{\mathscr{Q}} \rangle \in \widehat{\mathrm{Pic}}(\mathcal{M}_g/\mathbb{Z})_{\mathbb{Q}}. \tag{6.2}$$

The line bundle $\overline{\mathscr{L}}$ has been constructed [Yua21, §3.3.4] in slightly different step.

The following theorem is a reformulation of the second-named author's [Zha10, Thm. 2.3.5]. Recall the notation $\mathrm{GS}(\mathcal{C}_s) := \mathrm{GS}(\xi_s)$ with $\xi_s \in \mathrm{Pic}^1(\mathcal{C}_s)$ such that $(2g-2)\xi_s$ being the canonical divisor class on $\mathcal{C}_s$.

Theorem 6.1. *For any $s \in \mathcal{M}_g(\overline{\mathbb{Q}})$, we have*

$$\langle \Delta_{\mathrm{GS}}(\mathcal{C}_s), \Delta_{\mathrm{GS}}(\mathcal{C}_s) \rangle_{\mathrm{BB}} = h_{\overline{\mathscr{L}}}(s).$$

Proof. As notation, for any subset $U \subseteq \mathcal{M}_g$, set $\xi_U \in \mathrm{Pic}^1(\mathcal{C}_g \times_{\mathcal{M}_g} U/U)$ to be such that $(2g-2)\xi_U$ is the canonical divisor class of $\mathcal{C}_g \times_{\mathcal{M}_g} U/U$. In some neighborhood U of s in $\mathcal{M}_g$, $\Delta_{\mathrm{GS}}(\mathcal{C}_U) = \Delta_{\mathrm{GS},\xi_U}(\mathcal{C}_U)$ is an element of $Z^2(\mathcal{C}_U^3)$, *i.e.* it is a 2-cocycle of $\mathcal{C}_U^3$.

Let ℓ be a rational section of $\mathscr{Q}$ over $\mathcal{C}_U^2$. By (6.1), the divisor $\mathrm{div}(\ell_{s'})$ at each $s' \in U(\mathbb{C})$ is not the pullback of a divisor under the two natural projections $\mathcal{C}_U^2 \to \mathcal{C}_U$. So $\mathrm{div}(\ell)$ can be seen as a correspondence of $\mathcal{C}_U$. Up to shrinking U we can take rational sections

ℓ_1, ℓ_2, ℓ_3 of $\mathscr{Q}$ over $\mathcal{C}_U^2$ with the following property: For their divisors $t_1, t_2, t_3 \in \mathrm{Div}(\mathcal{C}_U^2)$, we have

$$|t_1| \cap |t_2| \cap |t_3| = \emptyset, \qquad |\Delta_{\mathrm{GS},\xi_U}(\mathcal{C}_U)| \cap |(t_1 \otimes t_2 \otimes t_3)^* \Delta_{\mathrm{GS},\xi_U}(\mathcal{C}_U)| = \emptyset,$$

where $t_1 \otimes t_2 \otimes t_3$ is seen as a correspondence of $\mathcal{C}_U^3$. Notice that we have a rational section $\langle \ell_1, \ell_2, \ell_3 \rangle$ of $\mathscr{L}$ on U.

We shall apply [Zha10, Thm. 2.3.5] to $\mathcal{C}_s$. We have seen that t_1, t_2, t_3 restricted to the fiber over $s \in U(\overline{\mathbb{Q}})$ satisfies the assumption of [Zha10, Thm. 2.3.5], so

$$\langle \Delta_{\mathrm{GS}}(\mathcal{C}_s), (t_{1,s} \otimes t_{2,s} \otimes t_{3,s})^* \Delta_{\mathrm{GS}}(\mathcal{C}_s) \rangle_{\mathrm{BB}} = \hat{t}_{1,s} \cdot \hat{t}_{2,s} \cdot \hat{t}_{3,s} \tag{6.3}$$

where $\hat{t}_{1,s}$, $\hat{t}_{2,s}$, $\hat{t}_{3,s}$ are suitable arithmetic divisors on some model of $\mathcal{C}_s^2$ extending $t_{1,s}$, $t_{2,s}$, $t_{3,s}$. The right hand side is precisely $\frac{1}{[\mathbb{Q}(s):\mathbb{Q}]} \widehat{\deg \mathscr{L}}|_s = h_{\overline{\mathscr{L}}}(s)$. The left hand side is $\langle \Delta_{\mathrm{GS}}(\mathcal{C}_s), \Delta_{\mathrm{GS}}(\mathcal{C}_s) \rangle_{\mathrm{BB}}$ because, by (6.1), $(t_1 \otimes t_2 \otimes t_3)^* \Delta_{\mathrm{GS},\xi_U}(\mathcal{C}_U)$ is rationally equivalent to $\Delta_{\mathrm{GS},\xi_U}(\mathcal{C}_U)$. Hence, we are done. $\square$

6.2. **Relating with the Gross–Schoen normal function.** Let ν_{GS} be the Gross–Schoen normal function $\mathcal{M}_g \to \mathcal{J}(\mathbb{V}_{\mathbb{Z}})$ defined in (C.5) (with $\mathbb{V}_{\mathbb{Z}}$ from Proposition C.10).

By (6.1), $(t_1 \otimes t_2 \otimes t_3)^* \Delta_{\mathrm{GS},\xi_U}(\mathcal{C}_U)$ is rationally equivalent to $\Delta_{\mathrm{GS},\xi_U}(\mathcal{C}_U)$. Hence, they define the same normal function ν_{GS} on U. Set

$$\overline{\mathcal{P}^{\mathrm{amp}}} := (\nu_{\mathrm{GS}}, \nu_{\mathrm{GS}})^* \overline{\mathcal{P}} = \nu_{\mathrm{GS}}^* \overline{\mathcal{P}^{\Delta}}$$

for the metrized Poincaré bundle $\overline{\mathcal{P}}$ on $\mathcal{J}(\mathbb{V}_{\mathbb{Z}}) \times_{\mathcal{M}_g} \mathcal{J}(\mathbb{V}_{\mathbb{Z}})$ defined in Definition D.1 (or the metrized tautological bundle $\overline{\mathcal{P}^{\Delta}}$ from Definition D.3). Set

$$\beta_{\mathrm{GS}} := c_1(\overline{\mathcal{P}^{\mathrm{amp}}})$$

to be the Betti form, *i.e.* the curvature of $\overline{\mathcal{P}^{\mathrm{amp}}}$. It is a semi-positive $(1,1)$-form. The following Proposition can also be deduced from R. de Jong's work [dJ16, (9.3)]:

Proposition 6.2. *We have the following identity of* $(1,1)$*-forms on* $\mathcal{M}_g$*:*

$$c_1(\overline{\mathscr{L}}_{\mathbb{C}}) = \beta_{\mathrm{GS}}.$$

In particular, $c_1(\overline{\mathscr{L}}_{\mathbb{C}})$ *is semi-positive.*

Proof. We use the notation in the proof of Theorem 6.1. For any $s \in \mathcal{M}_g(\mathbb{C})$, we obtain a neighborhood U such that (6.3) holds for any $s' \in U(\mathbb{C})$. So we get

$$\langle \Delta_{\mathrm{GS}}(\mathcal{C}_U), (t_1 \otimes t_2 \otimes t_3)^* \Delta_{\mathrm{GS}}(\mathcal{C}_U) \rangle_\infty = -\log \|\langle \ell_1, \ell_2, \ell_3 \rangle\|_\infty \quad \text{on } U.$$

By Proposition D.2, we have a section $\beta \in H^0(U, \mathcal{P}^{\mathrm{amp}})$ such that

$$\langle \Delta_{\mathrm{GS}}(\mathcal{C}_U), (t_1 \otimes t_2 \otimes t_3)^* \Delta_{\mathrm{GS}}(\mathcal{C}_U) \rangle_\infty = -\log \|\beta\| \quad \text{on } U.$$

So $\log \|\beta\| = \log \|\langle \ell_1, \ell_2, \ell_3 \rangle\|_\infty$ on U. Taking $\frac{\partial \bar{\partial}}{\pi i}$, we get that $c_1(\overline{\mathcal{P}^{\mathrm{amp}}}) = c_1(\overline{\mathscr{L}}_{\mathbb{C}})$ on U. Now we can conclude by letting s run over $\mathcal{M}_g(\mathbb{C})$. $\square$

6.3. **Bigness of the generic fiber of the adelic line bundle.** By [YZ21, Thm. 5.3.5] (which we reformulate as Theorem E.9), lower bounds of the height function $h_{\overline{\mathscr{L}}}$ correspond to bigness properties of $\overline{\mathscr{L}}$. In this paper, we prove the bigness of the *generic fiber* $\tilde{\mathscr{L}}$ of $\overline{\mathscr{L}}$ and deduce the desired height comparison from it. The definition of $\tilde{\mathscr{L}}$ is recalled in §E.5. Roughly speaking, there is a natural base change of adelic line bundles $\widehat{\mathrm{Pic}}(\mathcal{M}_g/\mathbb{Z})_{\mathbb{Q}} \to \widehat{\mathrm{Pic}}(\mathcal{M}_g/\mathbb{Q})_{\mathbb{Q}}$, and $\tilde{\mathscr{L}}$ is the image of $\overline{\mathscr{L}}$ under this base change.

The arithmetic volume of $\tilde{\mathscr{L}}$ is defined in (E.10). We will prove the following formula for $\widehat{\mathrm{vol}}(\tilde{\mathscr{L}})$. Notice that it does not follow from arithmetic Hilbert–Samuel because we have not proved the nefness of $\tilde{\mathscr{L}}$.

Theorem 6.3. *Let $S \subseteq \mathcal{M}_g$ be an irreducible subvariety, and let $\tilde{\mathscr{L}}_{\mathbb{C}}|_S$ be the image of $\tilde{\mathscr{L}}$ under the natural base change $\widehat{\mathrm{Pic}}(\mathcal{M}_g/\mathbb{Q})_{\mathbb{Q}} \to \widehat{\mathrm{Pic}}(S/\mathbb{C})_{\mathbb{Q}}$. Then we have*

$$\widehat{\mathrm{vol}}(\tilde{\mathscr{L}}_{\mathbb{C}}|_S) = \int_{S(\mathbb{C})} c_1(\overline{\mathscr{L}}_{\mathbb{C}})^{\wedge \dim S} = \int_{S(\mathbb{C})} \beta_{\mathrm{GS}}^{\wedge \dim S}. \tag{6.4}$$

The proof of this theorem shall be postponed to §6.4. An interesting corollary of this volume computation is as follows. For each subvariety S of $\mathcal{M}_g$, consider the Betti strata (1.2) with $\nu = \nu_{\mathrm{GS}}$.

Corollary 6.4. *Let S be a subvariety of $\mathcal{M}_g$ defined over a subfield k of $\mathbb{C}$. Then $S_{\mathscr{F}}(1)$ is also defined over k. In particular $\mathcal{M}_{g,\mathscr{F}}(1)$ is defined over $\mathbb{Q}$.*

Proof. Let S' be an irreducible component of $S_{\mathscr{F}}(1)$. Then $S' = S'_{\mathscr{F}}(1)$, and hence $\beta_{\mathrm{GS}}^{\wedge \dim S'} \equiv 0$ on S'; see (1.6) and Corollary D.7. So $\widehat{\mathrm{vol}}(\tilde{\mathscr{L}}_{\mathbb{C}}|_{S'}) = 0$ by Theorem 6.3. By Definition E.10, for any automorphism $\sigma \in \mathrm{Aut}(\mathbb{C}/k)$ we have

$$\widehat{\mathrm{vol}}(\tilde{\mathscr{L}}_{\mathbb{C}} \otimes_\sigma \mathbb{C}|_{S' \otimes_\sigma \mathbb{C}}) = \widehat{\mathrm{vol}}(\tilde{\mathscr{L}}_{\mathbb{C}}|_{S'}) = 0.$$

Let S'^σ denote $S' \otimes_\sigma \mathbb{C}$. Then S'^σ is also a closed subvariety of S. As $\tilde{\mathscr{L}}$ is defined over $\mathbb{Q}$, Proposition 6.3 implies

$$\int_{S'^\sigma(\mathbb{C})} c_1(\overline{\mathscr{L}}_{\mathbb{C}})^{\wedge \dim S'} = 0.$$

Thus $\beta_{\mathrm{GS}}^{\wedge \dim S'} \equiv 0$ on S'^σ by Proposition 6.2, and therefore

$$S'^\sigma(\mathbb{C}) = S'^\sigma_{\mathscr{F}}(1) \subseteq S_{\mathscr{F}}(1).$$

This shows that $S_{\mathscr{F}}(1) \otimes_\sigma \mathbb{C} = S_{\mathscr{F}}(1)$. Thus $S_{\mathscr{F}}(1)$ is defined over k. $\square$

Theorem 6.5. *Assume $g \geq 3$. Then the adelic line bundle $\tilde{\mathscr{L}}$ is big, i.e. $\widehat{\mathrm{vol}}(\tilde{\mathscr{L}}) > 0$.*

Proof. By the flatness of extension $\mathbb{Q} \subseteq \mathbb{C}$, we have $\widehat{\mathrm{vol}}(\tilde{\mathscr{L}}) = \widehat{\mathrm{vol}}(\tilde{\mathscr{L}}_{\mathbb{C}})$. So $\widehat{\mathrm{vol}}(\tilde{\mathscr{L}}) = \int_{\mathcal{M}_g(\mathbb{C})} \beta_{\mathrm{GS}}^{\wedge(3g-3)}$ by Theorem 6.3. Now $\beta_{\mathrm{GS}}^{\wedge(3g-3)} \geq 0$ since β_{GS} is semi-positive, and $\beta_{\mathrm{GS}}^{\wedge(3g-3)} \not\equiv 0$ by Corollary 5.3. So $\widehat{\mathrm{vol}}(\tilde{\mathscr{L}}) > 0$. $\square$

Now we are ready to prove our main theorem.

Proof of Theorem 1.1. It suffices to prove the claim for $\langle \mathrm{GS}(\xi), \mathrm{GS}(\xi) \rangle_{\mathrm{BB}}$ because, by work of the second-named author [Zha10], the Ceresa cycles and Gross–Schoen have the same height up to some positive multiple.

Next by [Zha10, Cor. 2.5.2], it suffices to prove for $\xi = \xi_s$. We start by showing: there exist a Zariski open dense subset U of $\mathcal{M}_g$ defined over $\mathbb{Q}$ and positive numbers ϵ and c such that

$$\langle \mathrm{GS}(\mathcal{C}_s), \mathrm{GS}(\mathcal{C}_s)\rangle_{\mathrm{BB}} \geq \epsilon h_{\mathrm{Fal}}(s) - c \tag{6.5}$$

for all $s \in U(\overline{\mathbb{Q}})$. The conclusion follows immediately from Theorem 6.5 and [YZ21, Thm. 5.3.5.(iii)] (which we reformulated as Theorem E.9). Notice that one can either take the adelic line bundle on $\mathcal{M}_g$, which defines the Faltings height (it exists by [YZ21, §2.6.2]), or one can take any ample line bundle on a suitable compactification of $\mathcal{M}_g$ and then use the comparison of the logarithmic Weil height and the Faltings height.

We can do better to get an explicit U. Suppose U_1, U_2 are open subsets of $\mathcal{M}_g$ over $\mathbb{Q}$ such that (6.5) holds true for all $s \in U_1(\overline{\mathbb{Q}})$ and $s \in U_2(\overline{\mathbb{Q}})$, then (6.5) holds true for all $s \in (U_1 \cup U_2)(\overline{\mathbb{Q}})$. Therefore there exists a maximal Zariski open dense subset U' of $\mathcal{M}_g$ on which (6.5) holds true. Let $S := \mathcal{M}_g \setminus U'$. Then S is proper Zariski closed in $\mathcal{M}_g$, and the height inequality (6.5) cannot hold for any non-empty open subvariety V of S. By Theorem E.9, $\tilde{\mathscr{L}}$ cannot be big on any irreducible component Z of S, and hence $c_1(\overline{\mathscr{L}}|_{Z(\mathbb{C})})^{\wedge \dim Z} \equiv 0$ by Proposition 6.3. Thus $Z_{\mathbb{C}} = Z_{\mathscr{F}}(1) \subseteq \mathcal{M}_{g,\mathscr{F}}(1)$. So $S_{\mathbb{C}} \subseteq \mathcal{M}_{g,\mathscr{F}}(1)$. So $S \subseteq \mathcal{M}_{g,\mathscr{F}}(1)$ by Corollary 6.4. Thus we can take $\mathcal{M}_g^{\mathrm{amp}}$ to be $\mathcal{M}_g \setminus \mathcal{M}_{g,\mathscr{F}}(1)$, which is Zariski open dense and is defined over $\mathbb{Q}$, in Theorem 1.1. ☐

6.4. **Proof of Theorem 6.3.** The second equality is immediate by Proposition 6.2. We will prove the first equality. We start to work on $\mathcal{M}_g$. To ease notation, denote by $\mathcal{M} = \mathcal{M}_g$ and $\mathcal{J} = \mathcal{J}_g$.

By [YZ21, Theorem 5.2.1], it suffices to construct a sequence of model (Hermitian) line bundles $(\mathscr{M}_i, \overline{\mathscr{L}}_i)$ which converges to $(\mathcal{M}, \overline{\mathscr{L}})$ (see the beginning of §E.1 for notation) such that

$$\lim_{i\to\infty} \mathrm{vol}(\mathscr{L}_{i,\mathbb{C}}|_{S_i}) = \int_{S(\mathbb{C})} c_1(\overline{\mathscr{L}}_{\mathbb{C}})^{\wedge \dim S}, \tag{6.6}$$

where S_i is the Zariski closure of $S \subseteq \mathcal{M} \subseteq \mathscr{M}_{i,\mathbb{C}}$.

We start the construction. First, the proof of [YZ21, Thm. 6.1.1] yields

- a sequence of projective integral models $\pi_i \colon \mathscr{J}_i \to \mathscr{M}$ of $\mathcal{J} \to \mathcal{M}$,
- an ample Hermitian line bundle $\overline{\mathscr{N}}$ on $\mathscr{M}$,
- a sequence of model line bundles $(\mathscr{J}_i, \overline{\mathscr{P}}_i^{\Delta})$ converging to $(\mathcal{J}, \overline{\mathscr{P}}^{\Delta})$,

such that $\overline{\mathscr{P}}_i^{\Delta} + 4^{-i}\pi_i^*\overline{\mathscr{N}}$ is nef. Denote by ℓ_i the connection morphism $\mathscr{P}^{\Delta} \xrightarrow{\sim} \mathscr{P}^{\Delta}_{i,\mathcal{J}} := \mathscr{P}_i^{\Delta} \times_{\mathscr{J}_i} \mathcal{J}$.

For each $n \in \mathbb{N}$, we have an action of $\mathcal{J}[n]$ on $\mathcal{J}$ by translating torsion points:

$$m_n \colon \mathcal{J}[n] \times_S \mathcal{J} \longrightarrow \mathcal{J}, \qquad (t, x) \mapsto x + t.$$

By Theorem of the square, the norm line bundle $\mathrm{N}_p m_n^* \mathscr{P}^{\Delta}$ is $(\mathscr{P}^{\Delta})^{\otimes n^{2g}}$, where p is the natural projection $\mathcal{J}[n] \times_S \mathcal{J} \to \mathcal{J}$. Thus, we obtain an adelic line bundle

$$\overline{\mathscr{P}}^{\Delta}_{i,n} := n^{-2g} \mathrm{N}_p m_n^* \overline{\mathscr{P}}^{\Delta}_i$$

which is realized on some model $\mathscr{J}_{i,n}$ of $\mathcal{J}$ with connection morphism $\ell_{i,n} : \mathscr{P}^{\Delta} \xrightarrow{\sim} \mathscr{P}^{\Delta}_{i,n,\mathcal{J}}$. Moreover $\mathrm{div}(\ell_{i,n})$ is in fact bounded by $\mathrm{div}(\ell_i)$. Over $\mathbb{C}$, the curvature form $c_1(\overline{\mathscr{P}}^{\Delta}_{i,n,\mathbb{C}})$ is obtained from the curvature form $c_1(\overline{\mathscr{P}}^{\Delta}_{i,\mathbb{C}})$ by taking over average over n-torsion points. It follows that these forms converge to $c_1(\overline{\mathscr{P}}^{\Delta})$ uniformly in any compact subset of $\mathcal{J}(\mathbb{C})$. These bundles also induce a double sequence of model line bundles $(\mathscr{M}_i, \overline{\mathscr{L}}_{i,n})$ of $(\mathcal{M}, \overline{\mathscr{L}})$ so that they convergent to $(\mathcal{M}, \overline{\mathscr{L}})$ as $i \to \infty$, and that the metric $c_1(\overline{\mathscr{L}}_{i,n,\mathbb{C}})$ convergent uniformly to $c_1(\overline{\mathscr{L}}_{\mathbb{C}})$ as $n \to \infty$.

More precisely, we let Ω_i be an increasing sequence of relatively compact open subsets of $\mathcal{M}(\mathbb{C})$ with $\mathcal{M}(\mathbb{C}) = \cup\Omega_i$ and ϵ_i be a decreasing sequence of positive numbers convergent to 0 such that

$$c_1(\overline{\mathscr{L}}_{\mathbb{C}}) \leq \epsilon_i^{-1} c_1(\overline{\mathscr{N}}_{\mathbb{C}}) \quad \text{on } \Omega_i. \tag{6.7}$$

Then for each i, choose n_i such that

$$-\epsilon_i^d c_1(\overline{\mathscr{N}}_{\mathbb{C}}) \leq c_1(\overline{\mathscr{L}}_{i,n_i,\mathbb{C}}) - c_1(\overline{\mathscr{L}}_{\mathbb{C}}) \leq \epsilon_i^d c_1(\overline{\mathscr{N}}_{\mathbb{C}}) \tag{6.8}$$

as Hermitian forms on the tangent bundle of Ω_i.

For each $i \geq 1$, set $\overline{\mathscr{L}}_i := \overline{\mathscr{L}}_{i,n_i}$. We will show that the sequence $(\mathscr{M}_i, \overline{\mathscr{L}}_i)$ is the desired sequence of model line bundles such that (6.6) holds true.

Using the reference curvature $c_1(\overline{\mathscr{N}}_{\mathbb{C}})$, we may talk about eigenvalues of $c_1(\overline{\mathscr{L}}_{\mathbb{C}})$ and $c_1(\overline{\mathscr{L}}_{i,\mathbb{C}})$. Recall that S_i is the closure of $S \subseteq \mathcal{M} \subseteq \mathscr{M}_{i,\mathbb{C}}$ for each i.

From now on, to ease notation we use $\mathscr{L}_{i,\mathbb{C}}$ to denote $\mathscr{L}_{i,\mathbb{C}}|_{S_i}$.

Let us apply Demailly's Morse inequality [Dem91] to the bundle $\overline{\mathscr{L}}_{i,\mathbb{C}}$ on $S_i(\mathbb{C})$. For each $q \in \mathbb{N}$, let $S_{i,q}$ denote the subset of $S_i(\mathbb{C})$ of points where $c_1(\overline{\mathscr{L}}_{i,\mathbb{C}})$ has q negative eigenvalues and $n-q$ positive eigenvalues. Then by [Dem91, (1.3), (1.5)], we have the following estimates as $k \to \infty$, with $d := \dim S$

$$h^q(k\mathscr{L}_{i,\mathbb{C}}) \leq \frac{k^d}{d!} \left| \int_{S_{i,q}} c_1(\overline{\mathscr{L}}_{i,\mathbb{C}})^{\wedge d} \right| + o(k^d),$$

$$\sum_q (-1)^q h^q(k\mathscr{L}_{i,\mathbb{C}}) = \frac{k^d}{d!} \int_{S(\mathbb{C})} c_1(\overline{\mathscr{L}}_{i,\mathbb{C}})^{\wedge d} + o(k^d).$$

It follows, combined with the definition of $\mathrm{vol}(\mathscr{L}_{i,\mathbb{C}})$, that

$$\left| \mathrm{vol}(\mathscr{L}_{i,\mathbb{C}}) - \int_{S(\mathbb{C})} c_1(\overline{\mathscr{L}}_{i,\mathbb{C}})^{\wedge d} \right| \leq \sum_{q>0} \left| \int_{S_{i,q}} c_1(\overline{\mathscr{L}}_{i,\mathbb{C}})^{\wedge d} \right|.$$

By [YZ21, Thm. 5.4.4],

$$\lim_{i\to\infty} \int_{S(\mathbb{C})} c_1(\overline{\mathscr{L}}_{i,\mathbb{C}})^{\wedge d} = \int_{S(\mathbb{C})} c_1(\overline{\mathscr{L}}_{\mathbb{C}})^{\wedge d}.$$

Thus to prove (6.6), it remains to prove

$$\lim_{i\to\infty} \int_{S_{i,q}} c_1(\overline{\mathscr{L}}_{i,\mathbb{C}})^{\wedge d} = 0 \quad \text{for each } q > 0. \tag{6.9}$$

We will prove (6.9) on $\Omega_i \cap S_{i,q}$ and on its complement $S_{i,q} \setminus \Omega_i$ respectively.

On $\Omega_i \cap S_{i,q}$, by (6.7) and (6.8) we have

$$-\epsilon_i^d c_1(\overline{\mathscr{N}}_{\mathbb{C}}) \leq c_1(\overline{\mathscr{L}}_{i,\mathbb{C}}) \leq (\epsilon_i^{-1} + \epsilon_i^d) c_1(\overline{\mathscr{N}}_{\mathbb{C}}).$$

Thus on $\Omega_i \cap S_{i,q}$, $c_1(\overline{\mathscr{L}}_{i,\mathbb{C}})$ has all eigenvalue $\leq \epsilon_i^{-1} + \epsilon_i^d$ and one negative eigenvalue with absolute value bounded by $\leq \epsilon_i^d$. It follows that $|c_1(\mathscr{L}_{i,\mathbb{C}})^{\wedge d}|$ is bounded by

$$\epsilon_i^d(\epsilon_i^{-1} + \epsilon_i^d)^{d-1} c_1(\overline{\mathscr{N}}_{\mathbb{C}})^{\wedge d} = \epsilon_i (1 + \epsilon_i^{d+1})^{d-1} c_1(\overline{\mathscr{N}}_{\mathbb{C}})^d.$$

It follows that

$$\int_{S_{i,q}\cap\Omega_i} |c_1(\overline{\mathscr{L}}_i)^{\wedge d}| = O(\epsilon_i). \tag{6.10}$$

On $S_{i,q} \setminus \Omega_i$, using

$$\mathscr{P} = \frac{1}{2}(m^*\mathscr{P}^\Delta - p_1^*\mathscr{P}^\Delta - p_2^*\mathscr{P}^\Delta),$$

we may write $\overline{\mathscr{L}}_i = \overline{\mathscr{E}}_i - \overline{\mathscr{F}}_i$, where $\overline{\mathscr{E}}_i$ and $\overline{\mathscr{F}}_i$ are two sequences of new bundles convergent to $\overline{\mathscr{E}}$ and $\overline{\mathscr{F}}$ with smooth metrics respectively. Then we have

$$\int_{S_{i,q}\setminus\Omega_i} |c_1(\overline{\mathscr{L}}_{i,\mathbb{C}})^{\wedge d}| \leq \int_{S(\mathbb{C})\setminus\Omega_i} c_1(\overline{\mathscr{E}}_{i,\mathbb{C}} + \overline{\mathscr{F}}_{i,\mathbb{C}})^{\wedge d}. \tag{6.11}$$

Now let $\Omega_i' \subseteq \Omega_i$ be another increasing sequence of relatively compact open subsets so that $\cup\Omega_i' = S(\mathbb{C})$. Then we can construct an increasing sequence of continuous functions f_i so that $f_i \equiv 1$ on $S(\mathbb{C}) \setminus \Omega_i$ and $f_i \equiv 0$ on Ω_i'. Then for any $i \geq j$ we have

$$\int_{S(\mathbb{C})\setminus\Omega_i} c_1(\overline{\mathscr{E}}_{i,\mathbb{C}} + \overline{\mathscr{F}}_{i,\mathbb{C}})^{\wedge d} \leq \int_{S(\mathbb{C})} f_i \cdot c_1(\overline{\mathscr{E}}_{i,\mathbb{C}} + \overline{\mathscr{F}}_{i,\mathbb{C}})^{\wedge d} \leq \int_{S(\mathbb{C})} f_j \cdot c_1(\overline{\mathscr{E}}_{i,\mathbb{C}} + \overline{\mathscr{F}}_{i,\mathbb{C}})^{\wedge d}.$$

Fix j and take $i \longrightarrow \infty$, we get

$$\limsup_{i\to\infty} \int_{S(\mathbb{C})\setminus\Omega_i} c_1(\overline{\mathscr{E}}_{i,\mathbb{C}} + \overline{\mathscr{F}}_{i,\mathbb{C}})^{\wedge d} \leq \int_{S(\mathbb{C})} f_j \cdot c_1(\overline{\mathscr{E}} + \overline{\mathscr{F}})^{\wedge d} \leq \int_{S(\mathbb{C})\setminus\Omega_j'} c_1(\overline{\mathscr{E}} + \overline{\mathscr{F}})^{\wedge d}.$$

Letting $j \to \infty$, we get

$$\lim_{i\to\infty} \int_{S(\mathbb{C})\setminus\Omega_i} c_1(\overline{\mathscr{E}}_{i,\mathbb{C}} + \overline{\mathscr{F}}_{i,\mathbb{C}})^{\wedge d} = 0. \tag{6.12}$$

Now (6.9) follows immediately from (6.10), (6.11) and (6.12). We are done. □

Appendix A. Variation of mixed Hodge structures

A.1. **Definitions.** Let R be a subring of $\mathbb{R}$.

Definition A.1. *Let M be a free R-module of finite rank.*

(i) *An R-pure Hodge structure on M of weight n is a decreasing filtration $F^\bullet$ on $M_\mathbb{C}$ (the* Hodge filtration*) such that $M_\mathbb{C} = F^pM_\mathbb{C} \oplus \overline{F^{n+1-p}M_\mathbb{C}}$ for all $p \in \mathbb{Z}$.*

(ii) *An R-mixed Hodge structure on M is a triple $(M, W_\bullet, F^\bullet)$ consisting of two filtrations, an increasing filtration $W_\bullet$ on $M_\mathbb{Q}$ (the* weight filtration*) and a decreasing filtration $F^\bullet$ on $M_\mathbb{C}$ (the* Hodge filtration*), such that for each $k \in \mathbb{Z}$, $\mathrm{Gr}_k^W M_\mathbb{Q} = W_k/W_{k-1}$ is a $\mathbb{Q}$-pure Hodge structure of weight k for the filtration on $\mathrm{Gr}_k^W(M_\mathbb{C})$ deduced from $F^\bullet$.*

Pure Hodge structures of weight n can be defined in terms of bigradings. Indeed, set $M^{p,n-p} := F^p M_{\mathbb{C}} \cap \overline{F^{p+1} M_{\mathbb{C}}}$, then $M_{\mathbb{C}} = \oplus_p M^{p,n-p}$ (the *Hodge decomposition*) and $\overline{M^{n-p,p}} = M^{p,n-p}$. We have $F^p = \oplus_{p' \geq p} M^{p',n-p'}$.

For a mixed Hodge structure $(M, W_\bullet, F^\bullet)$, the numbers $k \in \mathbb{Z}$ such that $\mathrm{Gr}_k^W M_{\mathbb{Q}} \neq 0$ are called its *weights*, and the numbers $h^{p,q}(M) = \dim_{\mathbb{C}} F^p \mathrm{Gr}_{p+q}^W(M_{\mathbb{C}})/F^{p+1}\mathrm{Gr}_{p+q}^W(M_{\mathbb{C}})$ are called its *Hodge numbers* .

For each $n \in \mathbb{Z}$, define $R(n)$ to be the pure Hodge structure on R of weight $-2n$ such that $R(n)^{-n,-n} = \mathbb{C}$ and $R(n)^{p,q} = 0$ for all $(p,q) \neq (-n,-n)$.

A *polarization* on a pure Hodge structure V of weight n is a morphism of Hodge structures

$$Q\colon V_{\mathbb{Q}} \otimes V_{\mathbb{Q}} \longrightarrow \mathbb{Q}(-n)$$

such that the Hermitian form on $V_{\mathbb{C}}$ given by $Q(Cu, \bar{v})$ is positive-definite where C is the Weil operator ($C|_{H^{p,q}} = i^{p-q}$ for all p, q).

A.2. **Mumford–Tate group.** Now, let us turn to a more group theoretical point of view on mixed Hodge structures. Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{\mathrm{m},\mathbb{C}}$ be the Deligne torus, *i.e.* the real algebraic group such that $\mathbb{S}(\mathbb{R}) = \mathbb{C}^*$ and $\mathbb{S}(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^*$, and that the complex conjugation on $\mathbb{S}(\mathbb{C})$ sends $(z_1, z_2) \mapsto (\bar{z}_2, \bar{z}_1)$.

As for pure Hodge structures, mixed Hodge structures can also be equivalently defined in terms of *bigradings* by Deligne [Del71, 1.2.8]. Given a $\mathbb{Q}$-vector space M of finite dimension, a bigrading $M_{\mathbb{C}} = \oplus_{p,q\in\mathbb{Z}} I^{p,q}$ is equivalent to a homomorphism $h\colon \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(M_{\mathbb{C}})$. In particular, any mixed Hodge structure on M defines a unique homomorphism $h\colon \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(M_{\mathbb{C}})$, and we use (M, h) to denote this mixed Hodge structure.

Definition A.2. *For any mixed Hodge structure* (M, h)*, its* Mumford–Tate group *is the smallest* $\mathbb{Q}$*-subgroup* $\mathbf{G}$ *of* $\mathrm{GL}(M_{\mathbb{Q}})$ *such that* $h(\mathbb{S}_{\mathbb{C}}) \subseteq \mathbf{G}(\mathbb{C})$*.*

Now we assume that M has weight 0 and -1 with $M_{-1} := \mathrm{Gr}_{-1}^W M$ and $M_0 := \mathrm{Gr}_0^W M$. Then, we have an exact sequence

$$0 \longrightarrow M_{-1} \longrightarrow M \longrightarrow M_0 \longrightarrow 0.$$

It is clear that $h(\mathbb{S})$ stabilizes this exact sequence and induced identity on N. Let $\mathbf{G}_0$ denote the Mumford–Tate group of V. Then, we have an exact sequence of reductive groups:

$$0 \longrightarrow \mathbf{G}_{-1} \longrightarrow \mathbf{G} \longrightarrow \mathbf{G}_0 \longrightarrow 0,$$

where $\mathbf{G}_{-1}$ is a vector group included into $\mathrm{Hom}(M_0, V)$. If V is polarized, then $\mathbf{G}_0$ is reductive by Deligne. Thus, $\mathbf{G}_{-1}$ is the unipotent radical of $\mathbf{G}$.

The base change to $\mathbb{R}$ has a natural splitting $M_{\mathbb{R}} = V_{\mathbb{R}} \oplus M_{0,\mathbb{R}}$ of Hodge structures given by the inverse of the isomorphism $M_{\mathbb{R}} \cap F^0 M_{\mathbb{C}} \xrightarrow{\sim} M_{0,\mathbb{R}}$. This induces splittings

$$\mathbf{G}_{\mathbb{R}} = \mathbf{G}_{-1,\mathbb{R}} \rtimes \mathbf{G}_{0,\mathbb{R}}.$$

Moreover $h(\mathbb{S})$ is included into $G_{0,\mathbb{R}}$. Thus, we have proved the following:

Lemma A.3. *Assume* (M, h) *has weight* -1 *and* 0*. The following holds.*

(i) *The* h *is defined over* $\mathbb{R}$*.*
(ii) *The unipotent radical of* $\mathbf{G}$ *is a vector group.*

A.3. **Variation of mixed Hodge structures and admissibility.**

Definition A.4. *Let S be a connected complex manifold. A* variation of mixed Hodge structures (VMHS) *on S is a triple* $(\mathbb{M}_{\mathbb{Z}}, W_\bullet, \mathcal{F}^\bullet)$ *consisting of*

- *a local system $\mathbb{M}_{\mathbb{Z}}$ of free $\mathbb{Z}$-modules of finite rank on S ,*
- *a finite increasing filtration (*weight filtration*) $W_\bullet$ of the local system $\mathbb{M} := \mathbb{M}_{\mathbb{Z}} \otimes_{\mathbb{Z}_S} \mathbb{Q}_S$ by local subsystems,*
- *a finite decreasing filtration (*Hodge filtration*) $\mathcal{F}^\bullet$ of the holomorphic vector bundle $\mathcal{M} := \mathbb{M}_{\mathbb{Z}} \otimes_{\mathbb{Z}_S} \mathcal{O}_S$ by holomorphic subbundles*

satisfying the following properties

(i) *for each $s \in S$, the triple $(\mathbb{M}_s, W_\bullet, \mathcal{F}^\bullet_s)$ defines a mixed Hodge structure on $\mathbb{M}_s$,*
(ii) *the connection $\nabla\colon \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_S} \Omega^1_S$ whose sheaf of horizontal sections is $\mathbb{M}_{\mathbb{C}} := \mathbb{M}_{\mathbb{Z}} \otimes_{\mathbb{Z}_S} \mathbb{C}_S$ satisfies the* Griffiths' transversality *condition*

$$\nabla(\mathcal{F}^p) \subseteq \mathcal{F}^{p-1} \otimes \Omega^1_S.$$

The weights and Hodge numbers of $(\mathbb{M}_s, W_\bullet, \mathcal{F}^\bullet_s)$ are the same for all $s \in S$. We call them the *weights* and the *Hodge numbers* of the VMHS $(\mathbb{M}_{\mathbb{Z}}, W_\bullet, \mathcal{F}^\bullet)$.

If there is only one $n \in \mathbb{Z}$ such that $\mathrm{Gr}^W_n \mathbb{M} \neq 0$, then each fiber of this VMHS is a pure Hodge structure of weight n. In this case, the VMHS is said to be *pure*. More precisely, we have the following definition.

Definition A.5. *A* variation of Hodge structures (VHS) of weight n *on S is a pair $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet)$ such that $(\mathbb{V}_{\mathbb{Z}}, W_\bullet, \mathcal{F}^\bullet)$ is a VMHS, where $W_\bullet$ is the increasing filtration on $\mathbb{V} := \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}_S} \mathbb{Q}_S$ defined by $W_{n-1} = 0$ and $W_n = \mathbb{V}$.*

To each VMHS $(\mathbb{M}_{\mathbb{Z}}, W_\bullet, \mathcal{F}^\bullet)$ on S, we can associate variations of pure Hodge structures obtained from $\mathrm{Gr}^W_k \mathbb{M}$.

We shall use the following convention: For each $n \in \mathbb{Z}$ and any VHS $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet) \to S$, define $\mathbb{V}_{\mathbb{Z}}(n)$ to be the VHS $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^{\bullet - n})$. In particular $\mathbb{Z}(n)_S$ be the VHS on S of weight $-2n$ such that $(\mathbb{Z}(n)_S)_s = \mathbb{Z}(n)$ for each $s \in S$.

Definition A.6. *A* polarization *of VHS $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet)$ on S of weight n is a morphism of VHS $\mathbb{V} \otimes \mathbb{V} \to \mathbb{Q}(-n)_S$ inducing on each fiber a polarization.*

We say that a VMHS $(\mathbb{M}_{\mathbb{Z}}, W_\bullet, \mathcal{F}^\bullet)$ is graded-polarizable *if $\mathrm{Gr}^W_k \mathbb{M}$ has a polarization for each $k \in \mathbb{Z}$.*

Example A.7. Let $f\colon X \to S$ be a projective smooth morphism of algebraic varieties over $\mathbb{C}$ with irreducible fibers of dimension d. For each $s \in S(\mathbb{C})$, the cohomology group $H^n(X_s, \mathbb{Z})$ is endowed with a natural Hodge structure of weight n by the de Rham–Betti comparison, and this makes $R^n f_* \mathbb{Z}_X$ into a VHS on S^{an} of weight n. It is polarizable by the hard Lefschetz theorem and the Lefschetz decomposition.

We close this subsection with a discussion on the admissibility of VMHS. Let S be a smooth, complex quasi-projective variety. Given a VMHS on S^{an}, we often need to extend it to suitable compactifications of S^{an} and hence study the asymptotic behavior of the VMHS near the boundary. This leads to the definition of *admissible* VMHS, which are the graded-polarized VMHSs with good asymptotic properties. This concept was introduced by Steenbrick–Zucker [SZ85, Prop. 3.13] on a curve and Kashiwara [Kas86, 1.8 and 1.9] in general, and the property for a VMHS on S^{an} to be admissible does not depend on the choice of the compactification. We shall not recall the precise definition here (see, for example, [PS08, Defn. 14.49]), but point out the following:

(1) Any VMHS arising from geometry is admissible ([SZ85] and [Kas86]).
(2) Any VHS is admissible.

Appendix B. Classfying space and Mumford–Tate domain

We relate the Betti foliation on intermediate Jacobians to the fibered structure of certain Mumford–Tate domain parametrizing $\mathbb{Q}$-mixed Hodge structure of weight -1 and 0. In this appendix, we recall and prove some results about such classifying spaces and Mumford–Tate domains. The main result is Proposition B.3, which explains and compares the semi-algebraic structure and the complex structure of such Mumford–Tate domains. This comparison is used to study the Betti foliation of intermediate Jacobians.

B.1. **Classifying space.** Let V be a finite-dimensional $\mathbb{Q}$-vector space. Let $E = V \oplus \mathbb{Q}$.

B.1.1. *Pure Hodge structures.* Consider the polarized Hodge data on V: a non-degenerate skew pairing $Q_{-1}\colon V \otimes V \to \mathbb{Q}(1)$, and a partition $\{h_V^{p,q}\}_{p,q\in\mathbb{Z}}$ of $\dim V_{\mathbb{C}}$ into non-negative integers with $p+q=-1$ such that $h_V^{p,q} = h_V^{q,p}$. Then there exists a *classifying space* $\mathcal{M}_0$ parametrizing $\mathbb{Q}$-Hodge structures on V of weight -1 with a polarization by Q_{-1} such that the (p,q)-constituent of $V_{\mathbb{C}}$ has complex dimension $h^{p,q}$. Moreover, the $\mathbb{Q}$-group $\mathbf{G}_0^{\mathcal{M}} := \mathrm{Aut}(V, Q_{-1})$, the associated real Lie group $\mathbf{G}_0^{\mathcal{M}}(\mathbb{R})^+$ acts transitively on $\mathcal{M}_0$, *i.e.*

$$\mathcal{M}_0 = \mathbf{G}_0(\mathbb{R})^+ x_0$$

for any point $x_0 \in \mathcal{M}_0$. This makes $\mathcal{M}_0$ into a semi-algebraic open subset of a flag variety $\mathcal{M}_0^\vee$, which is a suitable $\mathbf{G}_0^{\mathcal{M}}(\mathbb{C})$-orbit, and hence endows $\mathcal{M}_0$ with a complex structure.

We can be more explicit on the action of $\mathbf{G}_0^{\mathcal{M}}(\mathbb{R})^+$ on $\mathcal{M}_0$. For each $x_0 \in \mathcal{M}_0$, we have a Hodge decomposition and a Hodge filtration $F_{x_0}^\bullet$

$$V_{\mathbb{C}} = \bigoplus\nolimits_{p+q=-1} (V_{x_0})^{p,q}, \qquad F_{x_0}^p V_{\mathbb{C}} = \bigoplus\nolimits_{p'\geq p} (V_{x_0})^{p',q'} \tag{B.1}$$

with $(V_{x_0})^{q,p} = \overline{(V_{x_0})^{p,q}}$. The inclusion $\mathcal{M}_0 \subseteq \mathcal{M}_0^\vee$ is given by $x_0 \mapsto F_{x_0}^\bullet$.

For the Deligne torus $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{\mathrm{m},\mathbb{C}}$, the bi-grading decomposition above defines a morphism $h_{x_0}\colon \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$, with $(V_{x_0})^{p,q}$ the eigenspace of the character $z \mapsto z^{-p}\bar{z}^{-q}$ of $\mathbb{S}$. It is known that $h_{x_0}(\mathbb{S}) < \mathbf{G}_0^{\mathcal{M}}(\mathbb{R})$ for all $x_0 \in \mathcal{M}_0$. Hence we have a $\mathbf{G}_0^{\mathcal{M}}(\mathbb{R})^+$-equivariant map, which is known to be injective,

$$\mathcal{M}_0 \to \mathrm{Hom}(\mathbb{S}, \mathbf{G}_{0,\mathbb{R}}^{\mathcal{M}}), \quad x_0 \mapsto h_{x_0} \tag{B.2}$$

with the action of $\mathbf{G}_0^{\mathcal{M}}(\mathbb{R})^+$ on $\mathrm{Hom}(\mathbb{S}, \mathbf{G}_{0,\mathbb{R}}^{\mathcal{M}})$ given by conjugation. So we will view $\mathcal{M}_0$ as a subset of $\mathrm{Hom}(\mathbb{S}, \mathbf{G}_{0,\mathbb{R}}^{\mathcal{M}})$.

We close this subsubsection with the following remark on the Mumford–Tate group MT_{x_0} of the pure Hodge structure on V determined by x_0. We have that MT_{x_0} is a subgroup of $\mathbf{G}_0^{\mathcal{M}}$ for all $x_0 \in \mathcal{M}_0$ and equals $\mathbf{G}_0^{\mathcal{M}}$ for some $x_0 \in \mathcal{M}_0$. Moreover, MT_{x_0} is a reductive group for each $x_0 \in \mathcal{M}_0$.

B.1.2. *Weight* -1 *and* 0. Next, we turn to mixed Hodge structures of weight of -1 and 0.

Fix the following data on $E = V \oplus \mathbb{Q}$: the weight filtration

$$W_\bullet := (0 = W_{-2}E \subseteq W_{-1}E = V \subseteq W_0E = E);$$

the partition $\{h^{p,q}\}_{p,q\in\mathbb{Z}}$ of $\dim E_{\mathbb{C}}$ into non-negative integers, with $h^{p,q} = h_V^{p,q}$ for $p+q = -1$ and $h^{0,0} = 1$ and $h^{p,q} = 0$ otherwise.

There exists the *classifying space* $\mathcal{M}$ parametrizing $\mathbb{Q}$-mixed Hodge structures $(E, W_\bullet, F^\bullet)$ of weight -1 and 0 such that:

(a) the (p,q)-constituent $\mathrm{Gr}_F^p\mathrm{Gr}_{p+q}^W E_{\mathbb{C}}$ has complex dimension $h^{p,q}$;
(b) $\mathrm{Gr}_{-1}^W E = V$ is polarized by Q_{-1}.

See for example [Pea00, below (3.7) to the Remark below Lem. 3.9]. Notice that $\mathrm{Gr}_0^W E = \mathbb{Q}$ is polarized by $Q_0\colon \mathbb{Q}\otimes\mathbb{Q}\to\mathbb{Q}$ is $a\otimes b\mapsto ab$. So E is *graded-polarized.*

In our case, we need a better understanding of the structure of $\mathcal{M}$ than [Pea00]. The map $x\mapsto F_x^\bullet$ realizes $\mathcal{M}$ as a semi-algebraic open subset of a suitable flag variety $\mathcal{M}^\vee$, which is easily seen to be an orbit under $\mathbf{G}^{\mathcal{M}}(\mathbb{C})$ for the $\mathbb{Q}$-group

$$\mathbf{G}^{\mathcal{M}} := \mathrm{Aut}(E, Q_{-1}, W) = V \rtimes \mathrm{Aut}(V, Q_{-1}) = V \rtimes \mathbf{G}_0^{\mathcal{M}}. \tag{B.3}$$

Indeed, each $v\in V$ is consider as an element of $\mathrm{GL}(E)$ by sending $(w,a)\in V\oplus\mathbb{Q}$ to $(w+av, a)$. Moreover, since the morphism h_x is defined over $\mathbb{R}$ for each $x\in\mathcal{M}$ by Lemma A.3, we have (see for example [Pea00, last Remark of §3])

$$\mathcal{M} = \mathbf{G}^{\mathcal{M}}(\mathbb{R})^+ x \tag{B.4}$$

and a $\mathbf{G}^{\mathcal{M}}(\mathbb{R})^+$-equivariant map, which is known to be injective

$$\mathcal{M} \longrightarrow \mathrm{Hom}(\mathbb{S}, \mathbf{G}_{\mathbb{R}}^{\mathcal{M}}), \quad x\mapsto h_x. \tag{B.5}$$

As in the pure case, for the Mumford–Tate group MT_x for the mixed Hodge structure on E determined by x, we have that $\mathrm{MT}_x < \mathbf{G}^{\mathcal{M}}$ for all $x\in\mathcal{M}$, and $\mathrm{MT}_x = \mathbf{G}^{\mathcal{M}}$ for some $x\in\mathcal{M}$. Unlike the pure case, MT_x is not a reductive group in general.

B.1.3. For each $x\in\mathcal{M}$, we have an induced Hodge structure $p(x)$ on V and a unique vector $q(x)\in V_{\mathbb{R}}$ such that $(q(x),1)\in F_x^0E_{\mathbb{C}}$. Thus, we have a Hodge decomposition

$$E_{\mathbb{C}} = \bigoplus_{p+q=-1,0} E_x^{p,q}, \qquad E_x^{p,q} := F_x^pE_{\mathbb{C}}\cap\overline{F_x^qE_{\mathbb{C}}}.$$

The sum over $p+q=-1$ is the Hodge decomposition on V and $E_x^{0,0} = \mathbb{C}(q(x),1)$. This decomposition induces a bijection

$$(q,p)\colon \mathcal{M}\xrightarrow{\sim} V_{\mathbb{R}}\times\mathcal{M}_0, \qquad x\mapsto (q(x), p(x)) \tag{B.6}$$

where $p(x)$ is the induced Hodge structure on $V = W_{-1}E$. For the inverse map, for each $(v,x_0)\in V_{\mathbb{R}}\times\mathcal{M}_0$ we take $x\in\mathcal{M}$ be the unique Hodge structure h on E extending x_0 on $V_{\mathbb{C}}$ so that $F_x^0E_{\mathbb{C}} = F_{x_0}^0V_{\mathbb{C}} + \mathbb{C}(v,1)$. The real semi-algebraic structure on $\mathcal{M}$ is given by the product in the above isomorphism. The complex structure on $\mathcal{M}$ is given by making $\mathcal{M}$ into a vector bundle over $\mathcal{M}_0$ with fibers $V_{\mathbb{R}}\xrightarrow{\sim} V_{\mathbb{C}}/F_x^0V_{\mathbb{C}}$. Notice that for each $a\in V(\mathbb{R})$, the submanifold $(q,p)^{-1}(\{a\}\times\mathcal{M}_0)$ is a semi-algebraic and complex submanifold of $\mathcal{M}$. By abuse of notation, we also denote the group quotient still by $p\colon \mathbf{G}^{\mathcal{M}}\to\mathbf{G}_0^{\mathcal{M}} = \mathbf{G}^{\mathcal{M}}/V$.

For each $x \in \mathcal{M}$, we have a surjective morphism $\mathrm{MT}_x \longrightarrow \mathrm{MT}_{p(x)}$ with kernel $V_x := V \cap \mathrm{MT}_x$. Thus V_x is the unipotent radical of MT_x. This surjection has a section $s\colon \mathrm{MT}_{p(x)} \longrightarrow \mathrm{MT}_x$. Via (B.3), this section can be written as $s(g) = (\sigma(g), g)$ with σ a cocycle in $Z^1(\mathrm{MT}_x, V)$. But $H^1(\mathrm{MT}_{p(x)}, V) = 0$ since $\mathrm{MT}_{p(x)}$ is reductive. Thus σ is a coboundary, *i.e.* $\sigma(g) = v_0 - gv_0$ for some $v_0 \in V$ unique up to addition by invariant vectors of V under action by $\mathrm{MT}_{p(x)}$. Thus, we get a precise Levi decomposition

$$\mathrm{MT}_x = V_x \rtimes \mathrm{MT}_{x_0} = \mathrm{Ad}(v_0)(V_x \rtimes \mathrm{MT}_{p(x)}), \tag{B.7}$$

where $V_x = V \cap \mathrm{MT}_x$, $x_0 = (v_0, p(x))$ is a point of $\mathcal{M}$ via the isomorphism (B.6). With this isomorphism, we see that $x = (v_1 + v_0, p(x))$ with $v_1 \in V_{\mathbb{R}}$ which has Zariski closure V_x over $\mathbb{Q}$.

We will study the fibered structure given by p more carefully. We will do this in the finer setting of Mumford–Tate domains.

B.2. **Mumford–Tate domains.**

Definition B.1. *A subset $\mathcal{D}$ of the classifying space $\mathcal{M}$ is called a* (mixed) Mumford–Tate domain *if there exists an element $x \in \mathcal{D}$ such that $\mathcal{D} = \mathbf{G}(\mathbb{R})^+x$, where $\mathbf{G} = \mathrm{MT}(h_x)$.*

The group $\mathbf{G}$ in the definition above is called the *generic Mumford–Tate group of* $\mathcal{D}$ and is denoted by $\mathrm{MT}(\mathcal{D})$. It is known that $\mathbf{G} < \mathbf{G}^{\mathcal{M}}$.

Here are some basic properties of Mumford–Tate domains (for a reference see [GK24, §2.4]): $\mathcal{M}$ is a Mumford–Tate domain in itself with $\mathrm{MT}(\mathcal{M}) = \mathbf{G}^{\mathcal{M}}$, every Mumford–Tate domain is a complex analytic subspace of $\mathcal{M}$, and the collection of Mumford–Tate domains is stable under intersection.

Recall that $\mathcal{M}$ is a semi-algebraic open subset in some algebraic variety $\mathcal{M}^\vee$ over $\mathbb{C}$. Hence, $\mathcal{D}$ is a semi-algebraic open subset in some algebraic variety $\mathcal{D}^\vee$ over $\mathbb{C}$. This endows $\mathcal{D}$ with a semi-algebraic structure and a complex structure.

Definition B.2. *A subset of $\mathcal{D}$ is said to be* irreducible algebraic *if it is both complex analytic irreducible and semi-algebraic.*

In view of [KUY16, Lem. B.1 and its proof], a subset of $\mathcal{D}$ is irreducible algebraic if and only if it is a component of $U \cap \mathcal{D}$ with U an algebraic subvariety of $\mathcal{D}^\vee$.

Now, let us take a closer look at the semi-algebraic structure and a complex structure on $\mathcal{D}$.

The unipotent radical of $\mathbf{G} = \mathrm{MT}(\mathcal{D})$ equals $\mathbf{G}_{-1} := V \cap \mathbf{G}$ by reason of weight. Let $\mathbf{G}_0 := \mathbf{G}/\mathbf{G}_{-1}$ be the reductive part. Set $\mathcal{D}_0 := p(\mathcal{D}) \subseteq \mathcal{M}_0$ for the map p defined in (B.6). Then $\mathcal{D}_0$ is a $\mathbf{G}_0(\mathbb{R})^+$-orbit and is in fact a (pure) Mumford–Tate domain in the classifying space $\mathcal{M}_0$, and $\mathrm{MT}_{x_0} < \mathbf{G}_0$ for all $x_0 \in \mathcal{D}_0$.

By abuse of notation, we also use p to denote the natural projections

$$p\colon \mathbf{G} \to \mathbf{G}_0 = \mathbf{G}/\mathbf{G}_{-1} \quad \text{and} \quad p\colon \mathcal{D} \to \mathcal{D}_0. \tag{B.8}$$

Fix a Levi decomposition $\mathbf{G} = \mathbf{G}_{-1} \rtimes \mathbf{G}_0$, such as one given by B.7. Identify $\mathbf{G}_0$ with $\{0\} \times \mathbf{G}_0$.

Recall that each $x_0 \in \mathcal{M}_0$ endows V with a Hodge structure of weight -1. For $x_0 \in \mathcal{D}_0$, $\mathbf{G}_{-1}$ is a sub-Hodge structure because $\mathbf{G}_{-1}$ is a $\mathbf{G}_0$-submodule of V and that $\mathrm{MT}_{x_0} < \mathbf{G}_0$.

Finally, consider the constant bundle $\mathbf{G}_{-1}(\mathbb{C}) \times \mathcal{D}_0 \to \mathcal{D}_0$. Define the holomorphic subbundle $F^0(\mathbf{G}_{-1}(\mathbb{C}) \times \mathcal{D}_0)$ to be such that the fiber over each $x_0 \in \mathcal{D}_0$ is $F^0_{x_0}\mathbf{G}_{-1,\mathbb{C}}$.

Proposition B.3. *The following is true:*

(i) *The semi-algebraic structure on* $\mathcal{D} \subseteq \mathrm{Hom}(\mathbb{S}, \mathbf{G}_{\mathbb{R}})$ *is given by*

$$\mathbf{G}_{-1}(\mathbb{R}) \times \mathcal{D}_0 \xrightarrow{\sim} \mathcal{D}, \quad (v, x_0) \mapsto \mathrm{Int}(v) \circ h_{x_0}.$$

(ii) *The complex structure on* $\mathcal{D}$ *is given by* $\mathcal{D} \xrightarrow{\sim} (\mathbf{G}_{-1}(\mathbb{C}) \times \mathcal{D}_0)/F^0(\mathbf{G}_{-1}(\mathbb{C}) \times \mathcal{D}_0)$.

(iii) *These two structures are related by the natural bijection*

$$\mathbf{G}_{-1}(\mathbb{R}) \times \mathcal{D}_0 \subseteq \mathbf{G}_{-1}(\mathbb{C}) \times \mathcal{D}_0 \longrightarrow (\mathbf{G}_{-1}(\mathbb{C}) \times \mathcal{D}_0)/F^0(\mathbf{G}_{-1}(\mathbb{C}) \times \mathcal{D}_0). \tag{B.9}$$

Proof. Fix $x \in \mathcal{D}$ and let $h_x : \mathbb{S} \to \mathbf{G}_{\mathbb{R}}$ be the corresponding homomorphism.

The Levi decomposition $\mathbf{G} = \mathbf{G}_{-1} \rtimes \mathbf{G}_0$ induces a bijection $\mathbf{G}(\mathbb{R})^+x \simeq \mathbf{G}_{-1}(\mathbb{R}) \times \mathcal{D}_0$, which is semi-algebraic. This establishes (i).

Set $\mathcal{X} := \mathbf{G}(\mathbb{R})^+\mathbf{G}_{-1}(\mathbb{C})\cdot h_x \subseteq \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, \mathbf{G}_{\mathbb{C}})$, where the action is given by conjugation. The quotient $p: \mathbf{G}^{\mathcal{M}} \to \mathbf{G}_0^{\mathcal{M}}$ induces a natural surjective map $\mathcal{X} \to \mathcal{D}_0$, and by [Pin89, 1.8(a)] each fiber of this map is a $\mathbf{G}_{-1}(\mathbb{C})$-torsor.

The Levi decomposition $\mathbf{G} = \mathbf{G}_{-1} \rtimes \mathbf{G}_0$ induces a global section of $\mathcal{X} \to \mathcal{D}_0$, and hence an isomorphism $\mathcal{X} \simeq \mathbf{G}_{-1}(\mathbb{C}) \times \mathcal{D}_0$ over $\mathcal{D}_0$.

Consider the following surjective equivariant map

$$\varphi: \mathcal{X} \to \mathcal{D}, \qquad gh_xg^{-1} \mapsto g \cdot x. \tag{B.10}$$

By [Pin89, 1.8(b)], for each $x \in \mathcal{D}$, the fiber $\varphi^{-1}(x)$ is a principle homogeneous space under $F^0_{x_0}\mathbf{G}_{-1,\mathbb{C}}$ with $x_0 = p(x)$. Hence φ gives a bijection $(\mathbf{G}_{-1}(\mathbb{C}) \times \mathcal{D}_0)/F^0(\mathbf{G}_{-1}(\mathbb{C}) \times \mathcal{D}_0) \to \mathcal{D}$ over $\mathcal{D}_0$. Moreover, the complex structure of $\mathcal{D}$ is precisely given by this bijection; see [GK24, Proof of Prop. 2.6 in Appendix A] which is a consequence of [Pea00, Thm. 3.13]. This establishes (ii).

To see (iii), let $\mathcal{X}_{\mathbb{R}} := \mathbf{G}(\mathbb{R})^+ \cdot h_x \subseteq \mathcal{X}$. Then (B.9) is $\mathcal{X}_{\mathbb{R}} \subseteq \mathcal{X} \xrightarrow{\varphi} \mathcal{D}$ under the Levi decomposition $\mathbf{G} = \mathbf{G}_{-1} \rtimes \mathbf{G}_0$. Now we are done. ☐

Corollary B.4. *Let* $\tilde{Z}_0 \subseteq \mathcal{D}_0$ *be an irreducible algebraic subset. Then for any* $a \in \mathbf{G}_{-1}(\mathbb{R})$, *the subset* $\{a\} \times \tilde{Z}_0 \subseteq \mathbf{G}_{-1}(\mathbb{R}) \times \mathcal{D}_0 \simeq \mathcal{D}$ *is irreducible algebraic.*

Proof. $\{a\} \times \tilde{Z}_0$ is clearly semi-algebraic in $\mathcal{D}$. In view of (B.9), $\{a\} \times \tilde{Z}_0$ is also complex analytic in $\mathcal{D}$. Hence, we are done. ☐

Lemma B.5. *Let* $\mathcal{D}'$ *be a sub-Mumford–Tate domain of* $\mathcal{D}$. *Then under the identification* $\mathcal{D} = \mathbf{G}_{-1}(\mathbb{R}) \times \mathcal{D}_0$ *in Proposition B.3.(i), we have* $\mathcal{D}' = (\mathbf{G}'_{-1}(\mathbb{R}) + v_0) \times p(\mathcal{D}')$ *for some* $v_0 \in \mathbf{G}_{-1}(\mathbb{Q})$.

Proof. It suffices to apply (B.7) to both $\mathrm{MT}(\mathcal{D})$ and $\mathrm{MT}(\mathcal{D}')$. ☐

B.3. **Quotient by a normal subgroup and weak Mumford–Tate domains.** Let $\mathcal{D}$ be a Mumford–Tate domain in $\mathcal{M}$ with $\mathrm{MT}(\mathcal{D}) = \mathbf{G}$. Let $N \lhd \mathrm{MT}(\mathcal{D})$. By [GK24, Prop. 5.1], we have a quotient in the category of complex varieties

$$p_N : \mathcal{D} \to \mathcal{D}/N$$

with the following properties: (i) $\mathcal{D}/N$ is a Mumford–Tate domain in some classifying space of mixed Hodge structures (which must be of weight -1 and 0) and $\mathrm{MT}(\mathcal{D}/N) =$

$\mathbf{G}/N$; (ii) each fiber of p_N is an $N(\mathbb{R})^+$-orbit. It is clearly true that p_N is semi-algebraic. More precisely, let V/N be the maximal quotient of V on which N acts trivially, and let $M/N = V/N \oplus \mathbb{Q}$. For each $x \in \mathcal{D}$, the composition

$$\bar{h}_x : \mathbb{S} \xrightarrow{h_x} \mathrm{GL}(M_{\mathbb{R}}) \longrightarrow \mathrm{GL}((M/N)_{\mathbb{R}})$$

defines some Hodge structure on M/N with weight -1 on V/N and weight 0 on the quotient $\mathbb{Q}$. The Hodge numbers on M/N defined by h_x do not depend on the choice of x. Let $\mathcal{M}'$ denote the corresponding classification space. Then $\bar{h}_x$ defined a point $\bar{x}$ in $\mathcal{M}'$. The images of such $\bar{x}$ form a Mumford–Tate domain denoted by $\mathcal{D}/N$. Thus, we have the desired quotient map $p_N : \mathcal{D} \longrightarrow \mathcal{D}/N$ which is both complex analytic and real semi-algebraic.

Assume Γ is an arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$. Then, the quotient $\Gamma\backslash\mathcal{D}$ is an orbifold. Denote by $\Gamma_{/N}$ the image of Γ under the quotient $\mathbf{G} \to \mathbf{G}/N$. Then the quotient p_N induces

$$[p_N] \colon \Gamma\backslash\mathcal{D} \to \Gamma_{/N}\backslash(\mathcal{D}/N).$$

We will pay special attention to the fibers of p_N (and $[p_N]$). More generally, we define

Definition B.6. *A subset $\mathcal{D}_N$ of $\mathcal{D}$ is called a* weak Mumford–Tate domain *if there exist $x \in \mathcal{D}$ and a normal subgroup N of* $\mathrm{MT}(x)$ *such that $\mathcal{D}_N = N(\mathbb{R})^+x$.*

In this definition, if x is taken to be a Hodge generic point, *i.e.* $\mathrm{MT}(x) = \mathbf{G}$, then the weak Mumford–Tate domain thus obtained is a fiber of p_N.

Appendix C. Intermediate Jacobians and normal functions

Let $n \in \mathbb{Z}$. Let $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet) \to S$ be a polarized VHS of weight $2n-1$ over a complex manifold. As we shall see at the end of §C.1, the essential case is when $n = 0$.

Write $\mathcal{V} = \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_S$ for the holomorphic vector bundle obtained from $\mathbb{V}_{\mathbb{Z}}$.

C.1. Definition and basic property of Intermediate Jacobians.

Definition C.1. *The quotient $\mathcal{V}/(\mathcal{F}^n + \mathbb{V}_{\mathbb{Z}})$ is called the* (relative) intermediate Jacobian *of $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet) \to S$, and is denoted by $\mathcal{J}(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet)$ (or $\mathcal{J}(\mathbb{V}_{\mathbb{Z}})$ when the Hodge filtration is clear in the context).*

Lemma C.2. *$\pi \colon \mathcal{J}(\mathbb{V}_{\mathbb{Z}}) \to S$ is torus fibration, i.e. a holomorphic family of compact complex torus.*

Proof. For each $s \in S$, we have $\mathbb{V}_{\mathbb{C},s} = \mathcal{F}^n_s \oplus \overline{\mathcal{F}^n_s}$ because $\mathbb{V}_{\mathbb{Z},s}$ has weight $2n-1$. In particular, $\dim \mathcal{F}^n_s = \frac{1}{2}\dim \mathcal{V}_s$ and $\mathbb{V}_{\mathbb{Z},s} \cap \mathcal{F}^n_s = \{0\}$. Hence, each fiber $\mathcal{J}(\mathbb{V}_{\mathbb{Z}})_s = \mathcal{V}_s/(\mathcal{F}^n_s + \mathbb{V}_{\mathbb{Z},s})$ is a compact complex torus, and this yields the claim. □

Example C.3. Let us look at the following example from geometry. Let $f \colon X \to S$ be a smooth projective morphism of relative dimension d over a complex quasi-projective variety such that each fiber is irreducible. For each $n \geq 1$, the relative intermediate Jacobian of the (polarizable) VHS $R^{2n-1}f_*\mathbb{Z}(n)_X$ is called the *n-th intermediate Jacobian* of $X \to S$ and is denoted by $\mathcal{J}^n(X/S)$. If S is a point, then we simply write $\mathcal{J}^n(X)$.

For each $s \in S$ we have $H_1(\mathcal{J}(\mathbb{V}_\mathbb{Z})_s, \mathbb{Z}) = \mathbb{V}_{\mathbb{Z},s}$ as $\mathbb{Z}$-modules. So the local system $(R^1\pi_*\mathbb{Z}_{\mathcal{J}(\mathbb{V}_\mathbb{Z})})^\vee$ is $\mathbb{V}_\mathbb{Z}$. In particular, $\mathcal{V}_s = H_1(\mathcal{J}(\mathbb{V}_\mathbb{Z}, \mathbb{C}))$.

The dual of $H^{1,0}(\mathcal{J}(\mathbb{V}_\mathbb{Z})_s) \subseteq H^1_{\mathrm{dR}}(\mathcal{J}(\mathbb{V}_\mathbb{Z})_s)$ gives $H^1_{\mathrm{dR}}(\mathcal{J}(\mathbb{V}_\mathbb{Z})_s)^\vee \to \mathrm{Lie}(\mathcal{J}(\mathbb{V}_\mathbb{Z})_s)$. Thus the de Rham–Betti comparison $H^1_{\mathrm{dR}}(\mathcal{J}(\mathbb{V}_\mathbb{Z})_s) \simeq H^1(\mathcal{J}(\mathbb{V}_\mathbb{Z})_s, \mathbb{C})$ gives rise to a Hodge filtration $\mathcal{F}^\bullet_\mathcal{J}$ on $\mathcal{V}$ defined by $\mathcal{F}^1_\mathcal{J} = 0$, $\mathcal{F}^{-1}_\mathcal{J} = \mathcal{V}$ and

$$(\mathcal{F}^0_\mathcal{J})_s = \ker(\mathcal{V}_s = H_1(\mathcal{J}(\mathbb{V}_\mathbb{Z})_s, \mathbb{C}) \to \mathrm{Lie}(\mathcal{J}(\mathbb{V}_\mathbb{Z})_s)) \quad \text{for all } s \in S.$$

Lemma C.4. *$\mathcal{F}^0_\mathcal{J} = \mathcal{F}^n$.*

Proof. We computed in the proof of Lemma C.2 that $\mathcal{J}(\mathbb{V}_\mathbb{Z})_s = \mathbb{V}_{\mathbb{Z},s}\backslash\mathcal{V}_s/\mathcal{F}^n_s$. Hence $\mathrm{Lie}(\mathcal{J}(\mathbb{V}_\mathbb{Z})_s) = \mathcal{V}_s/\mathcal{F}^n_s$. So $(\mathcal{F}^0_\mathcal{J})_s = \mathcal{F}^n_s$ by defintion. We are done. □

By definition, $\mathcal{J}(\mathbb{V}_\mathbb{Z}) = \mathcal{J}(\mathbb{V}_\mathbb{Z}(n))$ and $\mathbb{V}_\mathbb{Z}(n)$ is of weight -1. Hence *in the rest of the paper, without loss of generality, we assume* $(\mathbb{V}_\mathbb{Z}, \mathcal{F}^\bullet)$ *to have weight* -1 *for the discussion on intermediate Jacobians.*

C.2. **Betti foliation.** Next, we discuss the Betti foliation on $\mathcal{J}(\mathbb{V}_\mathbb{Z})$. Write $\pi\colon \mathcal{J}(\mathbb{V}_\mathbb{Z}) \to S$ for the natural projection and d for the relative dimension.

For each open subset $\Delta \subseteq S$, We have the following exact sequence.

$$0 \longrightarrow \mathcal{F}^0_\mathcal{J}\mathcal{V}_\Delta \longrightarrow \mathcal{V}_\Delta \longrightarrow \mathrm{Lie}(\pi^{-1}(\Delta)/\Delta) \to 0.$$

Locally on S, the local system $\mathbb{V}_\mathbb{Z}|_\Delta$ is trivial and so is $(\mathcal{V}, \nabla)|_\Delta$. Thus $\mathbb{V}_\mathbb{Z}|_\Delta \subseteq \mathcal{V}_\Delta$ becomes $\mathbb{Z}^{2d} \times \Delta \subseteq \mathbb{C}^{2d} \times \Delta$, which extends to $\mathbb{R}^{2d} \times \Delta \subseteq \mathbb{C}^{2d} \times \Delta$. Notice that $(\mathbb{R}^{2d} \times \Delta) \cap \mathcal{F}^0_\mathcal{J}\mathcal{V}_\Delta$ is 0 on each fiber by weight reasons. Hence

$$\mathbb{R}^{2d} \times \Delta \subseteq \mathbb{C}^{2d} \times \Delta \to \mathrm{Lie}(\pi^{-1}(\Delta)/\Delta)$$

is a real analytic diffeomorphism, and over each $s \in \Delta$ it becomes a group homomorphism. Moreover, the image of $\{r\} \times \Delta$ is easily seen to be complex analytic for any $r \in \mathbb{R}^{2d}$.

Taking the quotient of $\mathbb{V}_\mathbb{Z}$ on both sides, we obtain a real analytic diffeomorphism $(\mathbb{R}^{2d}/\mathbb{Z}^{2d}) \times \Delta \xrightarrow{\sim} \pi^{-1}(\Delta)$. Let $b_\Delta\colon \pi^{-1}(\Delta) \to \mathbb{R}^{2d}/\mathbb{Z}^{2d}$ be the composite of its inverse with the projection to the first factor $(\mathbb{R}^{2d}/\mathbb{Z}^{2d}) \times \Delta \to \mathbb{R}^{2d}/\mathbb{Z}^{2d}$. Then, each fiber of b_Δ is complex analytic by the last sentence of the last paragraph.

The construction above patches to a real analytic homeomorphism

$$\mathcal{J}(\mathbb{V}_\mathbb{Z}) \xrightarrow{\sim} \mathbb{V}_\mathbb{R}/\mathbb{V}_\mathbb{Z}. \tag{C.1}$$

This gives a foliation $\mathscr{F}_{\mathrm{Betti}}$ on $\mathcal{J}(\mathbb{V}_\mathbb{Z})$ which we call the *Betti foliation.* More concretely $\mathscr{F}_{\mathrm{Betti}}$ is defined as follows: for each $x \in \pi^{-1}(\Delta)$, the local leaf through x is the fiber $b_\Delta^{-1}(b_\Delta(x))$. Each leaf is holomorphic to the discussion above. In fact, the Betti foliation is the unique foliation on $\mathcal{J}(\mathbb{V}_\mathbb{Z})$ which is everywhere transverse to the fibers of π and whose set of leaves contains all torsion multisections.

For each $x \in \mathcal{J}(\mathbb{V}_\mathbb{Z})$, the Betti foliation induces a decomposition

$$T_x\mathcal{J}(\mathbb{V}_\mathbb{Z}) = T_x\mathscr{F}_{\mathrm{Betti}} \oplus T_x\mathcal{J}(\mathbb{V}_\mathbb{Z})_{\pi(x)},$$

and $\pi\colon \mathcal{J}(\mathbb{V}_\mathbb{Z}) \to S$ induces a natural isomorphism $T_x\mathscr{F}_{\mathrm{Betti}} \simeq T_{\pi(x)}S$. The translation on the torus $\mathcal{J}(\mathbb{V}_{\mathbb{Z},\pi(x)}) = \mathcal{J}(\mathbb{V}_\mathbb{Z})_{\pi(x)}$ yields a canonical isomorphism $T_x\mathcal{J}(\mathbb{V}_{\mathbb{Z},\pi(x)}) = T_0\mathcal{J}(\mathbb{V}_{\mathbb{Z},\pi(x)})$. Hence, we have a linear map

$$q_x\colon T_x\mathcal{J}(\mathbb{V}_\mathbb{Z}) = T_x\mathscr{F}_{\mathrm{Betti}} \oplus T_x\mathcal{J}(\mathbb{V}_{\mathbb{Z},\pi(x)}) \to T_x\mathcal{J}(\mathbb{V}_{\mathbb{Z},\pi(x)}) = T_0\mathcal{J}(\mathbb{V}_{\mathbb{Z},\pi(x)}), \tag{C.2}$$

whose kernel is $T_x\mathscr{F}_{\mathrm{Betti}}$.

We close this subsection with the following discussion. Take a holomorphic section $\nu\colon S \to \mathfrak{J}(\mathbb{V}_{\mathbb{Z}})$ of the intermediate Jacobian. Then we have a linear map, at each $s \in S$,

$$\nu_{\mathrm{Betti},s}\colon T_sS \xrightarrow{\mathrm{d}\nu} T_{\nu(s)}\mathfrak{J}(\mathbb{V}_{\mathbb{Z}}) \xrightarrow{q_{\nu(s)}} T_0\mathfrak{J}(\mathbb{V}_{\mathbb{Z},s}). \tag{C.3}$$

Notice that, since π induces canonically $T_{\nu(s)}\mathscr{F}_{\mathrm{Betti}} = T_sS$, the map $\mathrm{d}\nu$ is exactly $(1, \nu_{\mathrm{Betti},s})\colon T_sS \to T_sS \oplus T_0\mathfrak{J}(\mathbb{V}_{\mathbb{Z},s}) = T_sS \oplus T_{\nu(s)}\mathfrak{J}(\mathbb{V}_{\mathbb{Z},s}) = T_{\nu(s)}\mathfrak{J}(\mathbb{V}_{\mathbb{Z}})$.

C.3. **Sections of intermediate Jacobians.** In this subsection, we explain how any holomorphic section $\nu\colon S \to \mathfrak{J}(\mathbb{V}_{\mathbb{Z}})$ defines a family of mixed Hodge structures on S which varies holomorphically (Definition A.4 without the Griffiths' transversality) of weight -1 and 0.

First we define a local system $\mathbb{E}_\nu$ on S associated with ν. Write $\mathfrak{J}$ for the sheaf of holomorphic sections of $\mathfrak{J}(\mathbb{V}_{\mathbb{Z}}) \to S$. Then we have the following exact sequence

$$0 \to \mathbb{V}_{\mathbb{Z}} \to \mathcal{V}/\mathcal{F}^0\mathcal{V} \to \mathfrak{J} \to 0.$$

Taking cohomology yields the boundary map.

$$c\colon H^0(S, \mathfrak{J}) \to H^1(S, \mathbb{V}_{\mathbb{Z}}).$$

On the other hand, it is known that $H^1(S, \mathbb{V}_{\mathbb{Z}})$ can be canonically identified with $\mathrm{Ext}_{\mathrm{loc.sys}}(\mathbb{Z}_S, \mathbb{V}_{\mathbb{Z}})$, the isomorphism classes of the extensions of local systems ($\mathbb{Z}_S$ by $\mathbb{V}_{\mathbb{Z}}$) on S, so $c(\nu) \in H^1(S, \mathbb{V}_{\mathbb{Z}})$ defines a local system $\mathbb{E}_\nu$ on S fitting into the short exact sequence $0 \to \mathbb{V}_{\mathbb{Z}} \to \mathbb{E}_\nu \to \mathbb{Z}_S \to 0$. Notice that this defines a weight filtration $W_\bullet$ on $\mathbb{E}_\nu$ of weight -1 and 0, by letting $W_{-2}\mathbb{E}_\nu = 0$ and $W_{-1}\mathbb{E}_\nu = \mathbb{V}_{\mathbb{Z}}$ and $W_0\mathbb{E}_\nu = \mathbb{E}_\nu$. Notice that in the category of local systems, this exact sequence is split after $\otimes\mathbb{Q}$ if S is simply connected.

Carlson [Car85] proved that $\mathfrak{J}(\mathbb{V}_{\mathbb{Z}})_s = \mathfrak{J}(\mathbb{V}_{\mathbb{Z},s})$ is canonically isomorphic to $\mathrm{Ext}_{\mathrm{MHS}}(\mathbb{Z}(0), \mathbb{V}_{\mathbb{Z},s})$, the set of congruence classes of extensions of $\mathbb{Z}(0)$ by $\mathbb{V}_{\mathbb{Z},s}$ in the category of mixed Hodge structures; see for example [BZ14, Thm. 8.4.2]. In our context, this says the following. For each $s \in S(\mathbb{C})$, $\nu(s)$ defines a mixed Hodge structure on $\mathbb{E}_{\nu,s}$, and hence a Hodge filtration $\mathcal{F}^\bullet_{\mathbb{E}}$ on the fiber $\mathbb{E}_{\nu,s} \otimes \mathbb{C}$. Since ν is holomorphic, the fiberwise Hodge filtrations give rise to a Hodge filtration on $\mathbb{E}_\nu$. So $(\mathbb{E}_\nu, W_\bullet, \mathcal{F}^\bullet_{\mathbb{E}})$ is a family of mixed Hodge structures on S which varies holomorphically and has weight -1 and 0. We are done.

Moreover, if $\mathbb{V}_{\mathbb{Z}}$ is polarized, then $\mathbb{E}_\nu$ is graded-polarized with this polarization on $\mathbb{V}_{\mathbb{Z}}$ and the canonical polarization on $\mathbb{Z}(0)$ given by $\mathbb{Q} \otimes \mathbb{Q} \to \mathbb{Q}$, $a \otimes b \mapsto ab$.

C.4. **Normal functions.** Each holomorphic section $\nu\colon S \to \mathfrak{J}(\mathbb{V}_{\mathbb{Z}})$ defines a family of mixed Hodge structures $(\mathbb{E}_\nu, W_\bullet, \mathcal{F}^\bullet_{\mathbb{E}})$ on S of weight -1 and 0 which varies holomorphically and is graded-polarized; see §C.3.

Definition C.5. *A holomorphic section $\nu\colon S \to \mathfrak{J}(\mathbb{V}_{\mathbb{Z}})$ is called an* admissible normal function *if $(\mathbb{E}_\nu, W_\bullet, \mathcal{F}^\bullet_{\mathbb{E}})$ is an admissible VMHS.*

C.5. **Normal functions arising from families of algebraic cycles.** Let X be a smooth irreducible projective variety over $\mathbb{C}$ with $\dim X = d$. For each n, the Chow group of n-cocycles $\mathrm{Ch}^n(X)$ is the group of algebraic cycles of codimension n on X modulo rational equivalence. Denote by $\mathrm{Ch}^n(X)_{\mathrm{hom}}$ the kernel of the cycle map $\mathrm{Ch}^n(X) \to H^{2n}(X, \mathbb{Z})$. An n-cocycle Z is said to be *homologically trivial* if its Chow class lies in

$\mathrm{Ch}^n(X)_{\mathrm{hom}}$. By abuse of notation, we will denote the Chow class of a (co)cycle Z also by Z.

Let $\mathcal{J}^n(X)$ be the intermediate Jacobian corresponding to $H^{2n-1}(X,\mathbb{Z})$ from Example C.3.

The *Abel–Jacobi map*

$$\mathrm{AJ}\colon \mathrm{Ch}^n(X)_{\mathrm{hom}} \longrightarrow \mathcal{J}^n(X) \tag{C.4}$$

is constructed by Griffiths and Carlson in two equivalent ways (up to sign). Let us sketch Griffiths' construction [Gri69, §11]. Take Z as a homologically trivial n-cocycle. Then Z equals the boundary $\partial\Gamma_Z$ of a $(2d-2n+1)$-chain Γ_Z in X, and any two such chains differ from an element of $H_{2d-2n+1}(X,\mathbb{Z})$. Then Z induces a functional $\omega \mapsto \int_{\Gamma_Z}\omega$ on $H^{2d-2n+1}(X,\mathbb{C})$. One can check that this functional lies in $(F^{d-n+1}H^{2d-2n+1}(X))^\vee$. So we obtain an element $[\int_{\Gamma_Z}]$ in $\frac{(F^{d-n+1}H^{2d-2n+1}(X))^\vee}{H_{2d-2n+1}(X,\mathbb{Z})}$. Finally $\mathcal{J}^n(X) = \frac{(F^{d-n+1}H^{2d-2n+1}(X))^\vee}{H_{2d-2n+1}(X,\mathbb{Z})}$ by Poincaré duality. The map AJ is defined by sending the class of Z to $[\int_{\Gamma_Z}]$.

Now, we turn to the family version and define the corresponding normal function. Let $f\colon X\to S$ be a smooth projective morphism of algebraic varieties with irreducible fibers of dimension d. Let Z be a *family of homologically trivial n-cocycle in X/S*, *i.e.* a formal sum of integral subschemes of X which are flat and dominant over S such that each fiber Z_s is a homologically trivial n-cocycle of X_s.

Theorem C.6 ([EZ86])**.** *The holomorphic section*

$$\nu_Z\colon S\to \mathcal{J}^n(X/S), \qquad s\mapsto \mathrm{AJ}(Z_s)$$

is an admissible normal function.

The following proposition is a simple application of the Abel–Jacobi map and the Betti foliation $\mathscr{F}_{\mathrm{Betti}}$ on intermediate Jacobians introduced in §C.2.

Proposition C.7. *Set $S^\circ := S\setminus S_{\mathscr{F}}(1)$.*

Assume X/S and Z are defined over $\overline{\mathbb{Q}}$. Then $[Z_s]\in \mathrm{Ch}^n(X_s)$ is non-torsion for any $s\in S^\circ(\mathbb{C})\setminus S^\circ(\overline{\mathbb{Q}})$.

Proof. Take $s\in S(\mathbb{C})$, and set $\bar{s}$ to be the $\overline{\mathbb{Q}}$-Zariski closure of s in S.

Assume $[Z_s]$ is a torsion point of $\mathrm{Ch}^n(X_s)$. Then $[Z_t]$ is a torsion point of $\mathrm{Ch}^n(X_t)$ for each $t\in\bar{s}(\overline{\mathbb{Q}})$, and $t\mapsto \mathrm{AJ}(Z_t)$ is a torsion section of $\mathcal{J}^n(X/S)\times_S\bar{s}\to\bar{s}$. So $\nu_Z(\bar{s})\subseteq\mathscr{F}_{\mathrm{Betti}}$. If $s\notin S(\overline{\mathbb{Q}})$, then $\dim\bar{s}\geq 1$, and hence $\bar{s}\subseteq S_{\mathscr{F}}(1)$ by definition. Now we are done. $\square$

C.6. **Normal functions associated with Gross–Schoen cycles and Ceresa cycles.** Let C be an irreducible smooth projective curve defined over a field k of genus $g\geq 3$. Let $e\in C(k)$. We have:

- For each subset $T\subseteq\{1,2,3\}$, the modified diagonal $\Delta_T(C)=\{(x_1,x_2,x_3): x_i = e \text{ for } i\notin T,\ x_j=x_{j'} \text{ for all } j,j'\in T\}\in\mathrm{Ch}^2(C^3)$.
- The (classical) Abel–Jacobi map $i_e\colon C\to\mathrm{Jac}(C)$ sending $x\mapsto[x-e]$.

Both $\Delta_T(C)$ and i_e can be extended to any $e=\sum_i n_ie_i\in\mathrm{Div}^1(C)(k)=\mathrm{Pic}^1(C)(k)$. This is classical and direct for i_e, and for $\Delta_T(C)$ one can define $\Delta_{123}(C)=\{(c,c,c): c\in C\}$, $\Delta_{12}(C)=\sum n_i\{(c,c,e_i):c\in C\}$, $\Delta_1(C)=\sum_{i,j}n_in_j\{(c,e_i,e_j):c\in C\}$, *etc.*

For any $e \in \mathrm{Pic}^1(C)$, define the *Gross–Schoen cycle based at* e (resp. *the Ceresa cycle based at* e) to be:

- (Gross–Schoen cycle based at e) $\Delta_{\mathrm{GS},e}(C) := \Delta_{123}(C) - \Delta_{12}(C) - \Delta_{13}(C) - \Delta_{23}(C) + \Delta_1(C) + \Delta_2(C) + \Delta_3(C) \in \mathrm{Ch}^2(C^3)$.
- (Ceresa cycle based at e) $\mathrm{Ce}_e(C) := i_e(C) - [-1]_* i_e(C) \in \mathrm{Ch}^{g-1}(\mathrm{Jac}(C))$.

$\mathrm{Ce}_e(C)$ is clearly homologically trivial because $[-1]$ acts trivially on even-degree cohomology groups. It is not hard to check that $\Delta_{\mathrm{GS},e}(C)$ is also homologically trivial; this follows for example from [GS95, Prop. 3.1] and because the map $\mathrm{Pic}^1(C) \to \mathrm{Ch}^2(C^3)$, $e \mapsto \Delta_{\mathrm{GS},e}(C)$ is a group homomorphism.

Let ω_C be the canonical divisor on C and $[\omega_C]$ its divisor class.

Definition C.8. *Let* $\xi \in \mathrm{Pic}^1(C)(k)$ *such that* $(2g-2)\xi = \omega_C$. *Define the* (canonical) Gross–Schoen cycle *and the* (canonical) Ceresa cycle *of* C *to be:*

- *(Gross–Schoen cycle)* $\Delta_{\mathrm{GS}}(C) := \Delta_{\mathrm{GS},\xi}(C) \in \mathrm{Ch}^2(C^3)_{\mathrm{hom}}$.
- *(Ceresa cycle)* $\mathrm{Ce}(C) := \mathrm{Ce}_\xi(C) \in \mathrm{Ch}^{g-1}(\mathrm{Jac}(C))_{\mathrm{hom}}$.

There are finitely many ξ's with $(2g-2)\xi = [\omega_C]$ and each two differ from a $(2g-2)$-torsion. Hence, the Gross–Schoen cycle and the Ceresa cycle of C are well-defined up to $(2g-2)$-torsions.

We will associate normal functions with the Gross–Schoen cycles and Ceresa cycles. First we have $\mathrm{AJ}(\Delta_{\mathrm{GS}}(C)) \in \mathfrak{J}^2(C^3) = \mathfrak{J}(H^3(C^3,\mathbb{Z})(2))$ and $\mathrm{AJ}(\mathrm{Ce}(C)) \in \mathfrak{J}^{g-1}(\mathrm{Jac}(C)) = \mathfrak{J}(H^{2g-3}(\mathrm{Jac}(C),\mathbb{Z})(g-1))$. Using the Poincaré duality on $\mathrm{Jac}(C)$ and on C, we get $H^{2g-3}(\mathrm{Jac}(C),\mathbb{Z})(g-1) = H_3(\mathrm{Jac}(C),\mathbb{Z})(-1) = \bigwedge^3 H_1(C,\mathbb{Z})(-1) = \bigwedge^3 H^1(C,\mathbb{Z})(2)$. So $\mathrm{AJ}(\mathrm{Ce}(C)) \in \mathfrak{J}(\bigwedge^3 H^1(C,\mathbb{Z})(2))$.

We have $\bigwedge^3 H^1(C,\mathbb{Z})(2) \subseteq H^3(C^3,\mathbb{Z})(2)$ in the following way. First, the Künneth formula gives a decomposition $H^3(C^3,\mathbb{Z}) = H^1(C,\mathbb{Z})^{\otimes 3} \bigoplus H^1(C,\mathbb{Z})(-1)^{\oplus 6}$. Next, $H^1(C,\mathbb{Z})^{\otimes 3}$, as a subspace of $H^3(C^3,\mathbb{Z})$, has a basis consisting of $\alpha_1 \wedge \alpha_2 \wedge \alpha_3$, with α_j the pullback of an element in $H^1(C,\mathbb{Z})$ under the j-th projection $C^3 \to C$. The symmetric group S_3 acts naturally on C^3, and this induces an action of S_3 on $H^1(C,\mathbb{Z})^{\otimes 3}$ with

$$\sigma(\alpha_1 \wedge \alpha_2 \wedge \alpha_3) = \mathrm{sgn}(\sigma)\alpha_{\sigma(1)} \wedge \alpha_{\sigma(2)} \wedge \alpha_{\sigma(3)}$$

for each $\sigma \in S_3$. Then $(H^1(C,\mathbb{Z})^{\otimes 3})^{S_3} = \bigwedge^3 H^1(C,\mathbb{Z})$ for this action, with each member having a basis consisting of $\sum_{\sigma \in S_3} \mathrm{sgn}(\sigma)\alpha_{\sigma(1)} \wedge \alpha_{\sigma(2)} \wedge \alpha_{\sigma(3)}$.

It is easy to check that the pushforward of $\Delta_{\mathrm{GS}}(C)$ to any two factors of C^3 is trivial. Hence $\mathrm{AJ}(\Delta_{\mathrm{GS}}(C)) \in \mathfrak{J}(H^1(C,\mathbb{Z})^{\otimes 3}(2))$. Moreover, the modified diagonal is easily seen to be invariant under the action of S_3 on C^3. Thus $\mathrm{AJ}(\Delta_{\mathrm{GS}}(C)) \in \mathfrak{J}(\bigwedge^3 H^1(C,\mathbb{Z})(2))$.

Lemma C.9. $\mathrm{AJ}(\Delta_{\mathrm{GS}}(C)) = 3\mathrm{AJ}(\mathrm{Ce}(C))$.

Proof. Consider $\iota_3 \colon C^3 \to \mathrm{Jac}(C)$, $(c_1,c_2,c_3) \mapsto i_\xi(c_1) + i_\xi(c_2) + i_\xi(c_3)$. The difference $(\iota_3)_*\Delta_{\mathrm{GS}}(C) - 3\mathrm{Ce}(C)$ was computed by the second-named author [Zha10, Thm. 1.5.5]. More precisely, the Fourier–Mukai transformation yields a spectrum decomposition $C = \sum_{j=0}^{g-1} C_j$ in $\mathrm{Ch}^{g-1}(\mathrm{Jac}(C))$, with $[n]_* C_j = n^{2+j} C_j$ for all $n \in \mathbb{Z}$ and $j \in \{0,\ldots,g-1\}$, and [Zha10, Thm. 1.5.5 and its proof] implies that

$$(\iota_3)_*\Delta_{\mathrm{GS}}(C) - 3\mathrm{Ce}(C) = \sum\nolimits_{j\geq 2} a_j C_j$$

For appropriate numbers a_j. The multiplication $[n]\colon \mathrm{Jac}(C) \to \mathrm{Jac}(C)$ induces

$$\bigwedge^3 H^1(C,\mathbb{Z}) \to \bigwedge^3 H^1(C,\mathbb{Z}), \quad x \mapsto n^3 x.$$

Hence $n^{2+j}\mathrm{AJ}(C_j) = n^3\mathrm{AJ}(C_j)$ for all $n \in \mathbb{Z}$, and so $\mathrm{AJ}(C_j) = 0$ for all $j \geq 2$.

Hence the conclusion follows because the induced map of ι_3 is the natural projection. $\square$

Now, we turn to defining and studying the normal functions. Let $\mathcal{M}_g$ be the moduli space of curves of genus g, and let $f\colon \mathcal{C}_g \to \mathcal{M}_g$ be the universal curve. We have a VHS $\bigwedge^3 R^1 f_*\mathbb{Z}_{\mathcal{C}_g} \to \mathcal{M}_g$. The fiberwise polarization

$$q_H\colon H^1(C,\mathbb{Z}) \otimes H^1(C,\mathbb{Z}) \xrightarrow{\cup} H^2(C,\mathbb{Z}) \simeq \mathbb{Z}(-1),$$

with $\cup$ the cup product (the dual of the intersection pairing) on $H^1(C,\mathbb{Z})$, induces a polarization on $\bigwedge^3 R^1 f_*\mathbb{Z}_{\mathcal{C}_g}$.

Now we have two normal functions

$$\begin{aligned} \nu_{\mathrm{GS}}\colon \mathbb{M}_g &\longrightarrow \mathcal{J}(\bigwedge^3 R^1 f_*\mathbb{Z}_{\mathcal{C}_g}), \qquad s \mapsto \mathrm{AJ}(\Delta_{\mathrm{GS}}(\mathcal{C}_s)), \\ \nu_{\mathrm{Ce}}\colon \mathbb{M}_g &\longrightarrow \mathcal{J}(\bigwedge^3 R^1 f_*\mathbb{Z}_{\mathcal{C}_g}), \qquad s \mapsto \mathrm{AJ}(\mathrm{Ce}(\mathcal{C}_s)). \end{aligned} \tag{C.5}$$

We can do better.

Proposition C.10. *Let $\mathbb{V}_{\mathbb{Z}}$ be the kernel of the morphism of VHS (called the* contractor*)*

$$c\colon \bigwedge^3 R^1 f_*\mathbb{Z}_{\mathcal{C}_g}(2) \longrightarrow R^1 f_*\mathbb{Z}_{\mathcal{C}_g}(1) \tag{C.6}$$

fiberwise defined by $x \wedge y \wedge z \mapsto q_H(y,z)x + q_H(z,x)y + q_H(x,y)z$. Then

(i) *both ν_{GS} and ν_{Ce} have images in $\mathcal{J}(\mathbb{V}_{\mathbb{Z}})$;*
(ii) *neither ν_{GS} nor ν_{Ce} is a torsion section;*
(iii) *$\mathbb{V}_{\mathbb{Z}}$ is an irreducible VHS on $\mathcal{M}_g$, i.e. the only sub-VHSs of $\mathbb{V}_{\mathbb{Z}}$ are 0 and itself.*

Proof. For each $s \in S(\mathbb{C})$, $\nu_{\mathrm{GS}}(s) \in \mathcal{J}(\mathbb{V}_{\mathbb{Z},s})$ by [Zha10, Lem. 5.1.5], and hence $\nu_{\mathrm{Ce}}(s) \in \mathcal{J}(\mathbb{V}_{\mathbb{Z},s})$ by Lemma C.9. So (i) holds.

For (ii), it suffices to find a curve C of genus g such that $\mathrm{AJ}(\mathrm{Ce}(C))$ is not torsion. There are many examples of such curves in existing literature. Alternatively, the result for ν_{Ce} can be already deduced from Ceresa's original argument in [Cer83].

For (iii), it suffices to prove that $V := \mathbb{V}_{\mathbb{Z},s} \otimes \mathbb{Q}$ is a simple Sp_{2g}-module for one (and hence all) $s \in \mathcal{M}_g(\mathbb{C})$, or equivalently V is an irreducible representation of Sp_{2g}. This is a standard result of the representation theory for $\mathfrak{sp}_{2g}$, see for example [FH13, Thm. 17.5]. $\square$

Appendix D. Metrized Poincaré bundle and local height pairing

Let S be a smooth irreducible quasi-projective variety.

D.1. **Height pairing at Archimedean places.** Let $f\colon X \to S$ be a smooth projective morphism of algebraic varieties with irreducible fibers of dimension d. Let p, q be non-negative integers such that $p + q = d + 1$.

Let Z, resp. W, be a family of homologically trivial p-cocycle in X/S, resp. a family of homologically trivial q-cocycle in X/S. Assume Z and W have disjoint supports over the generic fiber, then up to replacing S by a Zariski open dense subset, we may assume that Z_s and W_s have disjoint supports for all $s \in S(\mathbb{C})$.

In Arakelov's theory, the local height pairing at archimedean places in this context is defined by

$$\langle Z_s, W_s\rangle_\infty = \int_{Z_s} \eta_{W_s}. \tag{D.1}$$

where η_{W_s} is a Green's current for W_s. This pairing is known to be symmetric.

The two families of homologically trivial cocycles give rise to two admissible normal functions

$$\nu_Z\colon S \to \mathcal{J}^p(X/S) \quad \text{and} \quad \nu_W\colon S \to \mathcal{J}^q(X/S).$$

A particularly important case is when $p = q = \frac{d+1}{2}$ and that Z and W are rationally equivalent. In this case, we obtain the height pairings $\langle Z_s, Z_s\rangle_\infty$, or simply the *archimedean local height of* Z_s. More precisely, let Z be a family of homologically trivial p-cocycle in X/S with $p = \frac{d+1}{2}$, and let η be the generic point of S. By Moving Lemma, there exists a p-cocycle W_η of X_η whose class in $\mathrm{Ch}^p(X_\eta)$ is the same as the class of Z_η such that W_η and Z_η have disjoint supports. Then W_η extends to a family of homologically trivial p-cocycle W in X/S, and W_s and Z_s have disjoint supports for s in a Zariski open dense subset of S. Then we set $\langle Z_s, Z_s\rangle_\infty = \langle Z_s, W_s\rangle_\infty$. Notice that in this case $\nu_Z = \nu_W$.

We can extend this definition to $p \le \frac{d+1}{2}$ as follows. Fix a relatively ample line bundle L on X/S. Then intersection induces a family of maps $\cdot L_s^{d+1-2p}\colon \mathrm{Ch}^p(X_s)_{\mathrm{hom}} \to \mathrm{Ch}^q(X_s)_{\mathrm{hom}}$ for each $s \in S(\mathbb{C})$. Therefore we can define $\langle Z_s, Z_s\rangle_\infty$ to be $\langle Z_s, L_s^{d+1-2p}(Z_s)\rangle_\infty$; here one may also need to apply Moving Lemma.

In some cases, we can also extend this definition to $p > \frac{d+1}{2}$. In this case $q = d+1-p \le \frac{d+1}{2}$, and we have $\cdot L_s^{d+1-2q}\colon \mathrm{Ch}^q(X_s)_{\mathrm{hom}} \to \mathrm{Ch}^p(X_s)_{\mathrm{hom}}$. Now if L_s^{d+1-2q} is an isomorphism, then we can define $\langle Z_s, Z_s\rangle_\infty$ to be $\langle Z_s, (L_s^{d+1-2q})^{-1}(Z_s)\rangle_\infty$. This happens for example when X_s is an abelian variety.

Hain [Hai90, §3.3] related the local height pairing at archimedean places (D.1) to the metrized Poincaré bundle. We will review this in the next subsection.

D.2. **Metrized Poincaré bundle.** Let $(\mathbb{V}_\mathbb{Z}, \mathcal{F}^\bullet)$ be a polarized VHS on S of weight -1. The intermediate Jacobian $\mathcal{J}(\mathbb{V}_\mathbb{Z})$ is a torus fibration by Lemma C.2. The dual torus fibration $\mathrm{Pic}^0(\mathcal{J}(\mathbb{V}_\mathbb{Z}))$ can be described as follows. Set $\mathbb{V}_\mathbb{Z}^\vee := \mathrm{Hom}_{\mathrm{VHS}}(\mathbb{V}_\mathbb{Z}, \mathbb{Z}(1)_S)$ with the natural Hodge filtration. Then $\mathbb{V}_\mathbb{Z}^\vee$ is a VHS on S of weight -1, and there is a canonical isomorphism $\mathcal{J}(\mathbb{V}_\mathbb{Z}^\vee) = \mathrm{Pic}^0(\mathcal{J}(\mathbb{V}_\mathbb{Z}))$.

The general theory of biextension says that there exists a unique line bundle $\mathcal{P} \to \mathcal{J}(\mathbb{V}_\mathbb{Z}) \times_S \mathcal{J}(\mathbb{V}_\mathbb{Z}^\vee)$ satisfying the following properties:

(i) Over each $s \in S(\mathbb{C})$, we have $\mathcal{P}|_{\{0\}\times \mathcal{J}(\mathbb{V}_\mathbb{Z}^\vee)_s} \simeq \mathcal{O}_{\mathcal{J}(\mathbb{V}_\mathbb{Z}^\vee)_s}$,

(ii) over each $s \in S(\mathbb{C})$, $\mathcal{P}|_{\mathcal{J}(\mathbb{V}_\mathbb{Z})_s \times \{\lambda\}}$ represents $\lambda \in \mathrm{Pic}^0(\mathcal{J}(\mathbb{V}_\mathbb{Z})_s) = \mathcal{J}(\mathbb{V}_\mathbb{Z})_s^\vee$,

(iii) $\epsilon^*\mathcal{P} \simeq \mathcal{O}_S$ for the zero section ϵ of $\mathcal{J}(\mathbb{V}_\mathbb{Z}) \times_S \mathcal{J}(\mathbb{V}_\mathbb{Z})^\vee \to S$.

Moreover, $\mathcal{P}$ can be endowed a canonical Hermitian metric $\|\cdot\|_{\mathrm{can}}$ uniquely determined by the following properties: (i) the curvature of $\overline{\mathcal{P}}$ is translation invariant on each fiber $\mathcal{J}(\mathbb{V}_{\mathbb{Z}})_s \times \mathcal{J}(\mathbb{V}_{\mathbb{Z}}^{\vee})_s$; (ii) $\epsilon^*\overline{\mathcal{P}} \simeq (\mathcal{O}_S, \|\cdot\|_{\mathrm{triv}})$ for the trivial metric on $\mathcal{O}_S$.

Definition D.1. *The metrized line bundle* $\overline{\mathcal{P}} := (\mathcal{P}, \|\cdot\|_{\mathrm{can}})$ *is called the* metrized Poincaré bundle.

To relate it to the local height pairing (D.1), we need the following Hodge theoretic construction of $\overline{\mathcal{P}}$ by Hain [Hai90, §3.2]. Denote by $\mathcal{P}^*$ the associated $\mathbb{G}_{\mathrm{m}}$-torsor, *i.e.* $\mathcal{P}$ with the zero section removed. Over each $s \in S$, $\mathcal{P}^*_s$ equals $\mathscr{B}(\mathbb{V}_{\mathbb{Z},s})$, which is the set of mixed Hodge structures M of weight $0, -1, -2$ such that $\mathrm{Gr}^W_0 M = \mathbb{Z}(0)$, $\mathrm{Gr}^W_{-1} M = \mathbb{V}_{\mathbb{Z},s}$ and $\mathrm{Gr}^W_{-2} M = \mathbb{Z}(1)$. The projection $\mathcal{P}^* \to \mathcal{J}(\mathbb{V}_{\mathbb{Z}}) \times_S \mathcal{J}(\mathbb{V}_{\mathbb{Z}}^{\vee})$ is fiberwise given by sending $M \mapsto (M/\mathrm{Gr}^W_{-2}M, W_{-1}M)$.

Going from $\mathbb{Z}$ to $\mathbb{R}$-coefficients, one has the set $\mathscr{B}(\mathbb{V}_{\mathbb{R},s})$ of mixed $\mathbb{R}$-Hodge structures of weight $0, -1, -2$ whose weight graded pieces are $\mathbb{R}(0)$, $\mathbb{V}_{\mathbb{R},s}$ and $\mathbb{R}(1)$. One can check that $\mathscr{B}(\mathbb{V}_{\mathbb{R},s})$ is canonically isomorphic to $\mathbb{R}$. Then on $\mathcal{P}^*_s = \mathscr{B}(\mathbb{V}_{\mathbb{Z},s})$,

$$\|p\|_{\mathrm{can}} = e^{\mathrm{f}_s(p)}$$

where f_s is the forgetful map $\mathscr{B}(\mathbb{V}_{\mathbb{Z},s}) \to \mathscr{B}(\mathbb{V}_{\mathbb{R},s}) = \mathbb{R}$.

Now, let us go back to the setting of §D.1. Recall the section $(\nu_Z, \nu_W)\colon S \to \mathcal{J}^p(X/S) \times_S \mathcal{J}^q(X/S)$ obtained from the two families of homologically trivial cocycles Z and W.

By Poincaré duality, $\mathcal{J}^p(X/S)$ and $\mathcal{J}^q(X/S)$ are dual to each other. Hence we have the metrized Poincaré bundle $\overline{\mathcal{P}}$ on $\mathcal{J}^p(X/S) \times_S \mathcal{J}^q(X/S)$. Thus we obtain a metrized line bundle $(\nu_Z, \nu_W)^*\overline{\mathcal{P}}$ on S. Use $\|\cdot\|$ to denote this induced metric on $(\nu_Z, \nu_W)^*\mathcal{P}$.

Hain [Hai90, Prop. 3.3.2] constructed a section $\beta_{Z,W}$ of the line bundle $(\nu_Z, \nu_W)^*\mathcal{P} \to S$ in view of the Hodge-theoretic construction of $\mathcal{P}$ explained below Definition D.1, and proved [Hai90, Prop. 3.3.12] $\log \|\beta_{Z,W}(s)\| = -\int_{Z_s} \eta_{W_s}$ for all $s \in S(\mathbb{C})$. To summarize, we have

Proposition D.2 (Hain)**.** *We have* $-\log \|\beta_{Z,W}(s)\| = \langle Z_s, W_s \rangle_\infty$ *for all* $s \in S(\mathbb{C})$.

D.3. **Metrized tautological bundle.** As explained at the end of §D.1, we are often more interested in the case where $\mathcal{J}^p(X/S)$ is self-dual and $\nu_Z = \nu_W$. In this case, $(\nu_Z, \nu_W)^*\overline{\mathcal{P}} = \nu_Z^*\Delta^*\overline{\mathcal{P}}$ for the diagonal $\Delta\colon \mathcal{J}^p(X/S) \to \mathcal{J}^p(X/S) \times_S \mathcal{J}^p(X/S)$. We discuss this case in this and the next subsections in a more general setting.

Let $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet)$ be a VHS on S of weight -1, with a polarization $\mathcal{Q}\colon \mathbb{V}_{\mathbb{Z}} \otimes \mathbb{V}_{\mathbb{Z}} \to \mathbb{Z}(1)_S$. Then $\mathcal{Q}$ induces a morphism of VHS $\mathbb{V}_{\mathbb{Z}} \to \mathbb{V}_{\mathbb{Z}}^{\vee}$, and hence a morphism between the intermediate Jacobians $i_{\mathcal{Q}}\colon \mathcal{J}(\mathbb{V}_{\mathbb{Z}}) \longrightarrow \mathcal{J}(\mathbb{V}_{\mathbb{Z}})^{\vee}$. We thus have a morphism

$$\Delta_{\mathcal{Q}} = (1, i_{\mathcal{Q}})\colon \mathcal{J}(\mathbb{V}_{\mathbb{Z}}) \to \mathcal{J}(\mathbb{V}_{\mathbb{Z}}) \times_S \mathcal{J}(\mathbb{V}_{\mathbb{Z}})^{\vee}.$$

This is the case, for example, for Poincaré duality, where the cup product gives the polarization.

Definition D.3. *The metrized line bundle* $\overline{\mathcal{P}^{\Delta_{\mathcal{Q}}}} := \Delta_{\mathcal{Q}}^*\overline{\mathcal{P}}$ *is called the* metrized tautological bundle *on* $\mathcal{J}(\mathbb{V}_{\mathbb{Z}})$. *When the polarization* $\mathcal{Q}$ *is clear in context, we simply denote it by* $\overline{\mathcal{P}^{\Delta}}$.

By [HR04, Prop. 7.1 and 7.3], the curvature form $c_1(\overline{\mathcal{P}^{\Delta}})$ is a closed 2-form, uniquely determined by the following properties.

Proposition D.4. *Recall the Betti foliation $\mathscr{F}_{\mathrm{Betti}}$ on $\mathcal{J}(\mathbb{V}_{\mathbb{Z}})$ defined in §C.2.*

(i) *Restricted to each fiber $\mathcal{J}(\mathbb{V}_{\mathbb{Z}})_s$, $c_1(\overline{\mathcal{P}^{\Delta}})$ is the unique translation invariant 2-form ω_s given by $2\mathcal{Q}_s$,*
(ii) *$c_1(\overline{\mathcal{P}^{\Delta}})$ vanishes along each leaf of the Betti foliation.*

Here is a more explicit formula for $c_1(\overline{\mathcal{P}^{\Delta}})$. Write $\pi\colon \mathcal{J}(\mathbb{V}_{\mathbb{Z}}) \to S$ for the natural projection. Recall the linear map (C.2)

$$q_x\colon T_x\mathcal{J}(\mathbb{V}_{\mathbb{Z}}) \to T_0\mathcal{J}(\mathbb{V}_{\mathbb{Z},\pi(x)}).$$

with kernel $T_x\mathscr{F}_{\mathrm{Betti}}$. Now for any $v_1, v_2 \in T_x\mathcal{J}(\mathbb{V}_{\mathbb{Z}})$, we have

$$c_1(\overline{\mathcal{P}^{\Delta}})(v_1, v_2) = 2\mathcal{Q}_{\pi(x)}(q_x(v_1), q_x(v_2)). \tag{D.2}$$

D.4. **Pullback by admissible normal functions.** Retain the setup in §D.3. Now we turn to admissible normal functions $\nu\colon S \to \mathcal{J}(\mathbb{V}_{\mathbb{Z}})$.

Hain proved [Hai13, Thm. 13.1] that $\nu^*c_1(\overline{\mathcal{P}^{\Delta}})$ is a semi-positive $(1,1)$-form.

Definition D.5. *The semi-positive $(1,1)$-form $\nu^*c_1(\overline{\mathcal{P}^{\Delta}})$ is called the* Betti form associated with ν*, which we denote by β_ν.*

The Betti foliation $\mathscr{F}_{\mathrm{Betti}}$ on $\mathcal{J}(\mathbb{V}_{\mathbb{Z}})$ induces a linear map (C.3) at each $s \in S(\mathbb{C})$

$$\nu_{\mathrm{Betti},s}\colon T_sS \xrightarrow{\mathrm{d}\nu} T_{\nu(s)}\mathcal{J}(\mathbb{V}_{\mathbb{Z}}) \xrightarrow{q_{\nu(s)}} T_0\mathcal{J}(\mathbb{V}_{\mathbb{Z},s}).$$

By (D.2), $\beta_\nu(u,\bar{u}) = 2\mathcal{Q}_s(\nu_{\mathrm{Betti},s}(u), \overline{\nu_{\mathrm{Betti},s}(u)})$ for any $u \in T_sS$. By *Griffiths' transversality*, $\nu_{\mathrm{Betti},s}(u) \in \mathbb{V}_s^{-1,0}$. So

$$\beta_\nu(u,\bar{u}) \geq 0 \text{ for all } u \in T_sS, \text{ with equality if and only if } \nu_{\mathrm{Betti},s}(u) = 0. \tag{D.3}$$

Notice that this also explains the semi-positivity of β_ν. Now we use (D.3) to prove the following proposition.

Proposition D.6. *For any $s \in S(\mathbb{C})$, the following are equivalent:*

(i) $(\beta_\nu^{\wedge \dim S})_s \neq 0$;
(ii) $\dim \nu_{\mathrm{Betti},s}(T_sS) = \dim S$.

This proposition has the following immediate corollary.

Corollary D.7. $\beta_\nu^{\wedge \dim S} \not\equiv 0$ *if and only if* $\max_{s\in S(\mathbb{C})} \dim \nu_{\mathrm{Betti},s}(T_sS) = \dim S$.

We will call $\max_{s\in S(\mathbb{C})} \dim \nu_{\mathrm{Betti},s}(T_sS)$ the *Betti rank* of ν and denote it by $r(\nu)$.

Proof of Proposition D.6. Assume (i) is false, *i.e.* $(\beta_\nu^{\wedge \dim S})_s = 0$. By (D.3) there exists $0 \neq u \in T_sS$ with $\nu_{\mathrm{Betti},s}(u) = 0$. Thus $\ker \nu_{\mathrm{Betti},s} \neq 0$, and therefore $\dim \nu_{\mathrm{Betti},s}(T_sS) < \dim S$. So (ii) is also false.

Assume (ii) is false. Then there exists $0 \neq u \in \ker \nu_{\mathrm{Betti},s}$. By (D.3), $\beta_\nu(u,\bar{u}) = 0$. Thus u is an eigenvector of the Hermitian matrix defining β_ν with eigenvalue 0. Hence the determinant of this matrix is 0, so $\beta_\nu^{\wedge \dim S} = 0$ at s. So (i) is also false. $\square$

Appendix E. Adelic line bundles over quasi-projective varieties

The goal of this appendix is to review some basic notions and facts about adelic line bundles over quasi-projective varieties developed by the second-named author and Yuan [YZ21], generarlizing [Zha95]. See also [Yua21, §2.1] for another quick summary.

Let K be a number field and let $\mathscr{O}_K$ be its ring of integers. Through the whole appendix, X is a flat and quasi-projective variety defined over K. Via the natural inclusions $\mathbb{Z} \subseteq \mathbb{Q} \subseteq K$, we can see X as a scheme over $\mathrm{Spec}\mathbb{Z}$.

By an *arithmetic variety*, we mean an integral scheme which is flat, separated and of finite type over $\mathrm{Spec}\mathbb{Z}$.

E.1. **Adelic line bundles on X as limits of model Hermitian line bundles.** An *adelic line bundle $\overline{L}$ on X* is, roughly speaking, the limit of a sequence of *model (Hermitian) line bundles* $(\mathcal{X}_i, \overline{\mathcal{L}}_i)$, where $\mathcal{X}_i$ is a projective arithmetic variety such that X is Zariski open in $\mathcal{X}_{i,\mathbb{Q}}$. We say that the sequence $(\mathcal{X}_i, \overline{\mathcal{L}}_i)$ converges to $(X, \overline{L})$.

For this limit/convergence to make sense, we will construct *the group of isomorphism classes of adelic line bundles on X*, denoted by $\widehat{\mathrm{Pic}}(X/\mathbb{Z})$, in the following steps.

(i) Consider all quasi-projective models $\mathcal{U}$ of X, *i.e.* $\mathcal{U}$ is an integral scheme which is quasi-projective over $\mathrm{Spec}\mathscr{O}_K$, such that X is open in the generic fiber of $\mathcal{U} \to \mathrm{Spec}\mathscr{O}_K$. These quasi-projective models of X form an inverse system.

(ii) Define for each quasi-projective model $\mathcal{U}$ of X the *groups of isomorphism classes adelic line bundles* $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$ and of *adelic divisors* $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$ on $\mathcal{U}$. They are defined as suitable completions of the groups of isomorphism classes of *model adelic line bundles* $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$ and of *model adelic divisors* $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$.

(iii) Define $\widehat{\mathrm{Pic}}(X/\mathbb{Z})$ to be $\varinjlim_{\mathcal{U}} \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$ where the limits are taken on the inverse system of quasi-projective models of X. Similarly define the group of *adelic divisors* on X to be $\widehat{\mathrm{Div}}(X/\mathbb{Z}) := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$.

Steps (i) and (iii) are formal. We will explain step (ii) in two different points of views: in §E.3 by constructing $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$ and $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$ via a limit process, and in §E.4 by explaining how to see adelic line bundles on $\mathcal{U}$ as metrized line bundles on the (Berkovich) analytification $\mathcal{U}^{\mathrm{an}}$ of $\mathcal{U}$ and to see adelic divisors on $\mathcal{U}$ as arithmetic divisors on $\mathcal{U}^{\mathrm{an}}$. Before doing so, we need to recall some basic definitions about Hermitian line bundles and arithmetic divisors on projective arithmetic varieties in §E.2.

E.2. **Hermitian line bundles and arithmetic divisors.** Let $\mathcal{X}$ be a projective arithmetic variety.

A *Hermitian line bundle* $\overline{\mathcal{L}}$ on $\mathcal{X}$ is a pair $(\mathcal{L}, \|\cdot\|)$ where $\mathcal{L}$ is a line bundle on $\mathcal{X}$ and $\|\cdot\|$ is a Hermitian metric of $\mathcal{L}(\mathbb{C})$ on $\mathcal{X}(\mathbb{C})$ which is invariant under the complex conjugation on $\mathcal{X}(\mathbb{C})$. More precisely, for each local section s of $\mathcal{L}(\mathbb{C})$ on any open subset U of $\mathcal{X}(\mathbb{C})$, the function $x \mapsto \|s(x)\|^2$ is continuous in $x \in U$ and $\|s(x)\| = \|s(\overline{x})\|$ for all $x \in U$. Two Hermitian line bundles $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ and $\overline{\mathcal{L}}' = (\mathcal{L}', \|\cdot\|')$ are said to be isomorphic if there exists an isomorphism $\mathcal{L} \to \mathcal{L}'$ of line bundles on $\mathcal{X}$ such that the pullback of $\|\cdot\|'$ is $\|\cdot\|$. Denote by

$$\widehat{\mathrm{Pic}}(\mathcal{X}) := \{\text{Hermitian line bundles on } \mathcal{X}\}/\simeq .$$

Then $\widehat{\mathrm{Pic}}(\mathcal{X})$ has a natural group structure, with the identity element being $(\mathscr{O}_X, \|\cdot\|_{\mathrm{triv}})$ and the group law being the tensor product.

An *arithmetic divisor* $\overline{D}$ on $\mathcal{X}$ is a pair (D, g) where D is a Cartier divisor on $\mathcal{X}$ and g is a continuous Green's function of $|D|(\mathbb{C})$ on $\mathcal{X}(\mathbb{C})$ which is invariant under the complex conjugation. Here a *Green's function* is a continuous function $g\colon \mathcal{X}(\mathbb{C})\setminus|D|(\mathbb{C}) \to \mathbb{R}$ such that for any meromorphic function f on an open subset U of $\mathcal{X}$ with $\mathrm{div}(f) = D|_U$, the function $g + \log|f|$ extends to a continuous function on U. Next, a *principle arithmetic divisor* is defined to be an arithmetic divisor of the form $(\mathrm{div}(f), -\log|f|)$ for some non-zero rational function f on $\mathcal{X}$. Denote by

$$\widehat{\mathrm{Div}}(\mathcal{X}) := \{\text{the group of arithmetic divisors on } \mathcal{X}\}$$

and $\widehat{\mathrm{Prin}}(\mathcal{X}) := \{\text{the group of principal arithmetic divisors on } \mathcal{X}\}$.

There exists a canonical isomorphism

$$\widehat{\mathrm{Div}}(\mathcal{X})/\widehat{\mathrm{Prin}}(\mathcal{X}) \xrightarrow{\sim} \widehat{\mathrm{Pic}}(\mathcal{X}), \tag{E.1}$$

defined by sending $\overline{D} = (D, g) \mapsto (\mathscr{O}(D), \|\cdot\|_g)$ where $\|\cdot\|$ is determined by $\|1_D\|_g = e^{-g}$ for the canonical section 1_D of $\mathscr{O}(D)$ on $\mathcal{X}(\mathbb{C})$. The inverse is defined by $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|) \mapsto \widehat{\mathrm{div}}(s) := (\mathrm{div}(s), -\log\|s\|)$ for any non-zero rational section s of $\mathcal{L}$ on $\mathcal{X}$.

A Hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|) \in \widehat{\mathrm{Pic}}(\mathcal{X})$ is said to be *nef* if $\|\cdot\|$ is semi-positive and $\overline{\mathcal{L}}$ has non-negative arithmetic degree on any closed integral 1-dimensional subscheme of $\mathcal{X}$.

An arithmetic divisor $\overline{D} = (D, g) \in \widehat{\mathrm{Div}}(\mathcal{X})$ is called *effective* if $D \geq 0$ and $g \geq 0$ on $\mathcal{X}(\mathbb{C}) \setminus |D|(\mathbb{C})$. It is called *strictly effective* if furthermore $g > 0$ on $\mathcal{X}(\mathbb{C}) \setminus D(\mathbb{C})$.

E.3. **Adelic line bundles on $\mathcal{U}$ via limit process.** Let $\mathcal{U}$ be a quasi-projective arithmetic variety. In this subsection, we construct the group of isomorphism classes of adelic line bundles $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$ via a limit process on the projective models of $\mathcal{U}$, where a *projective model* $\mathcal{X}$ of $\mathcal{U}$ is a projective arithmetic variety $\mathcal{X}$ which contains $\mathcal{U}$ as an open subscheme.

A *model adelic line bundle on* $\mathcal{U}$ is a pair $(\mathcal{X}, \overline{\mathcal{L}})$ where $\mathcal{X}$ is a projective model of $\mathcal{U}$ and $\overline{\mathcal{L}} \in \widehat{\mathrm{Pic}}(\mathcal{X})_{\mathbb{Q}}$ such that $\mathcal{L}|_{\mathcal{U}} \in \mathrm{Pic}(\mathcal{U})$. The group of isomorphism classes of model adelic line bundles on $\mathcal{U}$ is denoted by $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$.

An *adelic line bundle on* $\mathcal{U}$ is a pair $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ where $\mathcal{L} \in \mathrm{Pic}(\mathcal{U})$, $(\mathcal{X}_i, \overline{\mathcal{L}}_i)$ is a model adelic line bundle for each $i \geq 1$ and $\ell_i\colon \mathcal{L} \to \mathcal{L}_i|_{\mathcal{U}}$ is an isomorphism in $\mathrm{Pic}(\mathcal{U})_{\mathbb{Q}}$ for each $i \geq 1$, such that the sequence $(\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1}$ satisfies a suitable *Cauchy condition*.

It is more convenient to define this Cauchy condition on model adelic divisors. So now let us construct the group of adelic divisors $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$ in the following steps.

(a) Consider all projective models $\mathcal{X}$ of $\mathcal{U}$. These projective models of $\mathcal{U}$ form an inverse system.

(b) For each projective model $\mathcal{X}$ of $\mathcal{U}$, define $\widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$ to be the fiber product $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}} \times_{\mathrm{Div}(\mathcal{X})_{\mathbb{Q}}} \mathrm{Div}(\mathcal{U})$, where the fiber product is taken with respect to the forgetful map $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}} \to \mathrm{Div}(\mathcal{X})_{\mathbb{Q}}$ and the natural map $\mathrm{Div}(\mathcal{U}) \to \mathrm{Div}(\mathcal{X}) \to \mathrm{Div}(\mathcal{X})_{\mathbb{Q}}$. Roughly speaking, $\widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$ consists of $\mathbb{Q}$-arithmetic divisors $\overline{D} = (D, g)$ on $\mathcal{X}$ (with $D \in \mathrm{Div}(\mathcal{X})_{\mathbb{Q}}$) such that $D|_{\mathcal{U}}$ lies in in $\mathrm{Div}(\mathcal{U})$.

(c) Define the group of *model adelic divisors on* $\mathcal{U}$ to be $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}} := \varinjlim_{\mathcal{X}} \widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$.

(d) Finally, define $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$ to be the equivalence classes of Cauchy sequences in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$ with respect to the *boundary topology*.

We need to explain the boundary topology on $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$ in the last step. First, notice that there is a partial order $\leq$ on $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$: Let $\overline{D}_1, \overline{D}_2 \in \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$, then $\overline{D}_1, \overline{D}_2 \in \widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$ for some projective model $\mathcal{X}$, and we say that $\overline{D}_1 \leq \overline{D}_2$ if the images of $\overline{D}_2 - \overline{D}_1 \in \widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$ in $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$ and in $\mathrm{Div}(\mathcal{U})$ are both effective. Next, fix a pair $(\mathcal{X}_0, \overline{\mathcal{E}}_0)$ with $\mathcal{X}_0$ a projective model of $\mathcal{U}$ and $\overline{\mathcal{E}}_0$ a strictly effective arithmetic divisor on $\mathcal{X}_0$ supporting at $\mathcal{X}_0 \setminus \mathcal{U}$. It defines the following *boundary norm*

$$\| \cdot \|_{\overline{\mathcal{E}}_0} \colon \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}} \to [0, \infty]$$

by sending $\overline{D} \mapsto \inf\{\epsilon \in \mathbb{Q}_{>0} : -\epsilon\overline{\mathcal{E}}_0 \leq \overline{D} \leq \epsilon\overline{\mathcal{E}}_0\}$. This boundary norm induces a topology on $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$, which we call the *boundary topology*. We remark that the boundary topology does not depend on the choice of the pair $(\mathcal{X}_0, \overline{\mathcal{E}}_0)$.

Now we are ready to state the Cauchy condition in the definition of adelic line bundles on $\mathcal{U}$. For a pair $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1}) \in \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$, for each $i \geq 1$ we obtain an isomorphism $\ell_i\ell_1^{-1} \colon \mathcal{L}_1|_{\mathcal{U}} \to \mathcal{L}_i|_{\mathcal{U}}$ of $\mathbb{Q}$-line bundles on $\mathcal{U}$, and hence a rational map $\ell_i\ell_1^{-1} \colon \overline{\mathcal{L}}_1 \dashrightarrow \overline{\mathcal{L}}_i$, and therefore a model adelic divisor $\widehat{\mathrm{div}}(\ell_i\ell_1^{-1}) \in \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$. Then the *Cauchy condition* on $(\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1}$ is that the sequence $\{\widehat{\mathrm{div}}(\ell_i\ell_1^{-1})\}_{i\geq 1}$ is a Cauchy sequence in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$ under the boundary topology.

E.4. **Adelic line bundles on $\mathcal{U}$ as metrized line bundles.** Let $\mathcal{U}$ be a quasi-projective arithmetic variety. A more intrinsic way to understand adelic line bundles is that they are still line bundles with a suitable metric, but now on the (Berkovich) analytification $\mathcal{U}^{\mathrm{an}}$ of $\mathcal{U}$.

We start by constructing $\mathcal{U}^{\mathrm{an}}$. If $\mathcal{U} = \mathrm{Spec} A$, then $\mathcal{U}^{\mathrm{an}}$ is defined to be $\mathscr{M}(A)$ of multiplicative semi-norms $|\cdot|$ on A whose restriction on $\mathbb{Z}$ is an absolute value, endowed with the weakest topology such that $\mathscr{M}(A) \to \mathbb{R}$, $|\cdot| \mapsto |f|$ is continuous for all $f \in A$. Denote by $|\cdot|_x$ the semi-norm on A corresponding to $x \in \mathscr{M}(A)$. The *contraction* is the map

$$\kappa \colon (\mathrm{Spec} A)^{\mathrm{an}} \to \mathrm{Spec} A$$

defined by sending each $x \in \mathscr{M}(A)$ to the prime ideal $\bar{x} := \ker(|\cdot|_x)$. For each $x \in (\mathrm{Spec} A)^{\mathrm{an}}$, the *residue field* H_x of x is defined to be the completion of $\mathrm{Frac}(A/\ker(|\cdot|_x))$. Now $|\cdot|_x$ factors through

$$|\cdot|_x \colon A \to H_x \xrightarrow{|\cdot|} \mathbb{R}.$$

In general $\mathcal{U}$ is covered by an affine open cover $\{\mathrm{Spec} A_i\}_i$, and $\mathcal{U}^{\mathrm{an}}$ is defined to be the union of $\mathscr{M}(A_i)$, glued in the canonical way, such that each $\mathscr{M}(A_i)$ is an open subspace of $\mathcal{U}^{\mathrm{an}}$. Then we can define the *contraction*

$$\kappa \colon \mathcal{U}^{\mathrm{an}} \to \mathcal{U}, \qquad x \mapsto \bar{x}$$

and the residue field H_x of each $x \in \mathcal{U}^{\mathrm{an}}$ since the definitions are local. We again have a semi-norm $|\cdot| \colon H_x \to \mathbb{R}$ induced by $|\cdot|_x$ for each $x \in \mathcal{U}^{\mathrm{an}}$.

Notice that in the constructions above, we did not use the flatness of $\mathcal{U} \to \mathrm{Spec}\mathbb{Z}$. For example, the construction works with any quasi-projective variety X defined over a number field K, viewed as a scheme over $\mathrm{Spec}\mathbb{Z}$ via $\mathbb{Z} \subseteq \mathbb{Q} \subseteq K$.

Example E.1. *Let us look at* $(\mathrm{Spec}\mathbb{Z})^{\mathrm{an}}$. *It is the union of the closed line segments*

$$[0,1]_\infty := \{|\cdot|_\infty^t : 0 \le t \le 1\}$$

and the closed line segments

$$[0,\infty]_p := \{|\cdot|_p^t : 0 \le t \le \infty\}$$

for all finite prime numbers $p > 0$, by identifying the endpoints $|\cdot|_\infty^0$ and $|\cdot|_p^0$ for all p with the trivial norm $|\cdot|_0$ on $\mathbb{Z}$. In particular, $(\mathrm{Spec}\mathbb{Z})^{\mathrm{an}}$ *is compact and path-connected.*

The contraction map leaves $|\cdot|_p^\infty$ stable and sends all other points to $|\cdot|_0$.

For each finite prime p, the residue field of $|\cdot|_p^t$ is $\mathbb{Q}_p$ when $t \in (0,\infty)$ and is $\mathbb{F}_p$ when $t = \infty$. The residue field of $|\cdot|_\infty^t$ is $\mathbb{R}$; this includes the case of $|\cdot|_0$.

And $(\mathrm{Spec}\mathbb{Q})^{\mathrm{an}}$ *is* $(\mathrm{Spec}\mathbb{Z})^{\mathrm{an}}$ *with $|\cdot|_p^\infty$ removed for all $p > 0$.*

In general, the structural morphism $\mathcal{U} \to \mathrm{Spec}\mathbb{Z}$ gives a structural map $\mathcal{U}^{\mathrm{an}} \to (\mathrm{Spec}\mathbb{Z})^{\mathrm{an}}$. This divides $\mathcal{U}^{\mathrm{an}}$ into

$$\mathcal{U}^{\mathrm{an}} = \mathcal{U}^{\mathrm{an}}[f] \coprod \mathcal{U}^{\mathrm{an}}[\infty], \tag{E.2}$$

where $\mathcal{U}^{\mathrm{an}}[f]$ is the inverse image of $(0,\infty]_p$ and $\mathcal{U}^{\mathrm{an}}[\infty]$ is the inverse image of $[0,1]_\infty$. Denote by $\mathcal{U}^{\mathrm{an}}_{\mathrm{triv}}$ the fiber over the point $|\cdot|_0 \in (\mathrm{Spec}\mathbb{Z})^{\mathrm{an}}$ corresponding to the trivial norm on $\mathbb{Z}$.

Now we are ready to define metrized line bundles and arithmetic divisors on $\mathcal{U}^{\mathrm{an}}$.

A *metrized line bundle* $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ on $\mathcal{U}^{\mathrm{an}}$ is a pair with $\mathcal{L} \in \mathrm{Pic}(\mathcal{U})$ and a continuous metric $\|\cdot\|$ of $\mathcal{L}$ on $\mathcal{U}^{\mathrm{an}}$. Here a continuous metric of $\mathcal{L}$ on $\mathcal{U}^{\mathrm{an}}$ is defined to be a continuous metric on $\coprod_{x \in \mathcal{U}^{\mathrm{an}}} \mathcal{L}^{\mathrm{an}}(x)$ (with $\mathcal{L}^{\mathrm{an}}(x) := \mathcal{L}(\bar{x}) \otimes H_x$) which is compatible with the semi-norms on $\mathscr{O}_{\mathcal{U}}$, *i.e.* for each $x \in \mathcal{U}^{\mathrm{an}}$, assign a norm $\|\cdot\|_x$ on $\mathcal{L}^{\mathrm{an}}(x)$ such that $\|f\ell\|_x = |f|_x \|\ell\|_x$ for all $f \in H_x$ and $\ell \in \mathcal{L}^{\mathrm{an}}(x)$, and that for any local section ℓ of $\mathcal{L}$ on $\mathcal{U}$ the function $\|\ell(x)\|_x$ is continuous in $x \in \mathcal{U}^{\mathrm{an}}$. The group of isomorphism classes of metrized line bundles on $\mathcal{U}^{\mathrm{an}}$ is denoted by $\widehat{\mathrm{Pic}}(\mathcal{U}^{\mathrm{an}})$.

Similarly, an *arithmetic divisor* on $\mathcal{U}^{\mathrm{an}}$ is a pair $\overline{D} = (D, g)$ where D is a Cartier divisor on $\mathcal{U}$ and g is a continuous Green's function of $|D|^{\mathrm{an}}$ on $\mathcal{U}^{\mathrm{an}}$, *i.e.* a continuous function $g\colon \mathcal{U}^{\mathrm{an}} \backslash |D|^{\mathrm{an}} \to \mathbb{R}$ such that for any meromorphic function f on an open subset $\mathcal{V}$ of $\mathcal{U}$ with $\mathrm{div}(f) = D|_{\mathcal{V}}$, the function $g + \log|f|$ extends to a continuous function on $\mathcal{V}^{\mathrm{an}}$. The group of arithmetic divisors on $\mathcal{U}^{\mathrm{an}}$ is denoted by $\widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})$. We also have the group of *principal arithmetic divisors* on $\mathcal{U}^{\mathrm{an}}$, denoted by $\widehat{\mathrm{Prin}}(\mathcal{U}^{\mathrm{an}})$, consisting of arithmetic divisors of the form $\widehat{\mathrm{div}}(f) := (\mathrm{div}(f), -\log|f|)$ for some non-zero rational function f on $\mathcal{U}$.

As in the case of Hermitian line bundles and arithmetic divisors on projective arithmetic varieties, we also have a canonical isomorphism

$$\widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})/\widehat{\mathrm{Prin}}(\mathcal{U}^{\mathrm{an}}) \xrightarrow{\sim} \widehat{\mathrm{Pic}}(\mathcal{U}^{\mathrm{an}}) \tag{E.3}$$

with a similar construction as (E.1).

By [YZ21, Prop. 3.4.1], there exists a canonical injective map

$$\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z}) \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{U}^{\mathrm{an}}). \tag{E.4}$$

Moreover, any $\overline{\mathcal{L}} \in \widehat{\mathrm{Pic}}(\mathcal{U}^{\mathrm{an}})$ in the image of this map is *norm-equivariant*, *i.e.* for any rational section s of $\mathcal{L}$ on $\mathcal{U}$ and any points $x, x' \in \mathcal{U}^{\mathrm{an}} \setminus |\mathrm{div}(s)|^{\mathrm{an}}$ satisfying $|\cdot|_x = |\cdot|_{x'}^t$ (with $0 \leq t < \infty$) locally on $\mathscr{O}_{\mathcal{U}}$, we have $\|s\|_x = \|s\|_{x'}^t$. Similarly there exists a canonical injective map $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z}) \longrightarrow \widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})$ mapping $\widehat{\mathrm{Prin}}(\mathcal{U}/\mathbb{Z})$ into $\widehat{\mathrm{Prin}}(\mathcal{U}^{\mathrm{an}})$.

In this point of view, the completion process to get $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$ from $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$ can be taken as follows. Fix a pair $(\mathcal{X}_0, \overline{\mathcal{E}}_0)$ with $\mathcal{X}_0$ a projective model of $\mathcal{U}$ and $\overline{\mathcal{E}}_0 = (\mathcal{E}_0, g_0)$ an arithmetic divisor on $\mathcal{X}_0$ with $|\mathcal{E}_0| = \mathcal{X}_0 \setminus \mathcal{U}$ and $g_0 > 0$. Then $\overline{\mathcal{E}}_0$ induces an arithmetic divisor $(\mathcal{E}_0, \widetilde{g}_0)$ on $\mathcal{X}_0^{\mathrm{an}}$ which restricts to a continuous function $\mathcal{U}^{\mathrm{an}} \to \mathbb{R}_{\geq 0}$; see [YZ21, Lem. 3.3.2] for more details[2]. Thus one can define the *boundary topology* on $C(\mathcal{U}^{\mathrm{an}})$, the space of continuous functions on $\mathcal{U}^{\mathrm{an}}$, by the extended norm

$$\|f\|_{\overline{\mathcal{E}}_0} := \sup_{x \in \mathcal{U}^{\mathrm{an}},\ \widetilde{g}_0(x) > 0} \frac{|f(x)|}{\widetilde{g}_0(x)}.$$

Now, we say that a sequence $\{\overline{\mathcal{L}}_i = (\mathcal{L}_i, \|\cdot\|_i)\}_{i \geq 1}$ in $\widehat{\mathrm{Pic}}(\mathcal{U}^{\mathrm{an}})$ *converges* to $\overline{\mathcal{L}} := (\mathcal{L}, \|\cdot\|) \in \widehat{\mathrm{Pic}}(\mathcal{U}^{\mathrm{an}})$ if there exist isomorphisms $\tau_i \colon \mathcal{L} \to \mathcal{L}_i$ such that the sequence $\{-\log \frac{\tau_i^* \|\cdot\|_i}{\|\cdot\|}\}_{i \geq 1}$ converges to 0 in $C(\mathcal{U}^{\mathrm{an}})$ under the boundary topology above. Thus an adelic line bundle $\overline{\mathcal{L}} \in \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$ is a metrized line bundle in $\widehat{\mathrm{Pic}}(\mathcal{U}^{\mathrm{an}})$ which is isomorphic to the limit of a sequence in $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$.

E.5. **The generic fiber of an adelic line bundle.** For each adelic line bundle $\overline{\mathcal{L}} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$, we give two equivalent constructions of its *generic fiber* (or its *geometric part*) $\widetilde{\mathcal{L}}$.

The first way is to construct $\widehat{\mathrm{Pic}}(X/\mathbb{Q})$. The construction is similar to $\widehat{\mathrm{Pic}}(X/\mathbb{Z})$, but much easier because there are no arithmetic inputs now. First define $\widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathrm{mod}}$ to be the group of isomorphism classes of pairs (Y, L) where Y is a projective model of X (*i.e.* Y is a projective variety defined over $\mathbb{Q}$ which contains X as an open subset) and $L \in \mathrm{Pic}(Y)_{\mathbb{Q}}$ such that $L|_X \in \mathrm{Pic}(X)$. Next define $\widehat{\mathrm{Pic}}(X/\mathbb{Q})$ as the group of isomorphism classes of pairs $(L, (Y_i, L_i, \ell_i)_{i \geq 1})$ where $L \in \mathrm{Pic}(X)$, $(Y_i, L_i) \in \widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathrm{mod}}$ and $\ell_i \colon L \to L_i|_X$ is an isomorphism in $\mathrm{Pic}(X)_{\mathbb{Q}}$ such that the sequence $\{\mathrm{div}(\ell_i \ell_1^{-1})\}_{i \geq 1}$ satisfies the Cauchy condition defined using the boundary topology on $\mathrm{Div}(X/\mathbb{Q})_{\mathrm{mod}}$. Here the boundary topology is defined using any fixed projective model Y_0 of X such that $Y_0 \setminus X$ is a divisor, which defines a boundary norm on $\mathrm{Div}(X/\mathbb{Q})_{\mathrm{mod}}$ by sending $\widetilde{D} \mapsto \inf\{\epsilon \in \mathbb{Q}_{>0} : -\epsilon(Y_0 \setminus X) \leq \widetilde{D} \leq \epsilon(Y_0 \setminus X)\}$.

Remark E.2. *Let k be any field (notably a field finitely generated over $\mathbb{Q}$), and let X' be a variety defined over k. The construction above defines $\widehat{\mathrm{Pic}}(X'/k)$ by replacing $\mathbb{Q}$ by k and X by X'.*

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[2]On $\mathcal{X}_0^{\mathrm{an}}[f]$ it is defined using the local meromorphic function on $\mathcal{X}_0$. On $\mathcal{X}_0^{\mathrm{an}}[\infty]$ it is defined using the Green's function g_0 and then extended using norm-equivariance.

Now for any projective model $\mathcal{X}$ of X over $\mathbb{Z}$, the generic fiber $\mathcal{X}_{\mathbb{Q}}$ is a projective model of X, and hence we have a natural map $\widehat{\mathrm{Pic}}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_{\mathbb{Q}})$. This induces the desired map

$$\widehat{\mathrm{Pic}}(X/\mathbb{Z}) \longrightarrow \widehat{\mathrm{Pic}}(X/\mathbb{Q}), \qquad \overline{\mathcal{L}} \mapsto \widetilde{\mathcal{L}}, \tag{E.5}$$

with $\widetilde{\mathcal{L}}$ called the *generic fiber* of $\overline{\mathcal{L}}$.

The second way is to see $\widehat{\mathrm{Pic}}(X/\mathbb{Z})$ as metrized line bundles on the (Berkovich) analytification X^{an} of X; see the comment above Example E.1. Then the structural morphism $X \to \mathrm{Spec}K$ gives a structural map $X^{\mathrm{an}} \to (\mathrm{Spec}K)^{\mathrm{an}}$. Denote by $X^{\mathrm{an}}_{\mathrm{triv}}$ the fiber over the point $v_0 \in (\mathrm{Spec}K)^{\mathrm{an}}$ which corresponds to the trivial absolute value $|\cdot|_0$ on K, *i.e.* $|0|_0 = 0$ and $|a|_0 = 1$ for all $a \in K^*$. We have the pullback

$$\widehat{\mathrm{Pic}}(X^{\mathrm{an}}) \longrightarrow \widehat{\mathrm{Pic}}(X^{\mathrm{an}}_{\mathrm{triv}}) \tag{E.6}$$

of the groups of isomorphism classes of metrized line bundles. The canonical injective map (E.4) induces a canonical injective map $\widehat{\mathrm{Pic}}(X/\mathbb{Z}) \longrightarrow \widehat{\mathrm{Pic}}(X^{\mathrm{an}})$. Similarly there is a canonical injective map $\widehat{\mathrm{Pic}}(X/\mathbb{Q}) \longrightarrow \widehat{\mathrm{Pic}}(X^{\mathrm{an}}_{\mathrm{triv}})$. Thus we obtain a diagram

$$\begin{array}{ccc} \widehat{\mathrm{Pic}}(X/\mathbb{Z}) & \longrightarrow & \widehat{\mathrm{Pic}}(X/\mathbb{Q}) \\ \downarrow & & \downarrow \\ \widehat{\mathrm{Pic}}(X^{\mathrm{an}}) & \longrightarrow & \widehat{\mathrm{Pic}}(X^{\mathrm{an}}_{\mathrm{triv}}). \end{array} \tag{E.7}$$

The following proposition says that the generic fiber of an adelic line bundle is obtained by pulling back of the corresponding metrized line bundle to the fiber over the trivial norm.

Proposition E.3. *The diagram above commutes.*

E.6. **Nefness and integrability.**

Definition E.4. *Let $\mathcal{U}$ be a quasi-projective arithmetic variety, and let $\overline{\mathcal{L}} \in \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$.*

- $\overline{\mathcal{L}}$ *is called* strongly nef *if it is the limit of a sequence of model adelic line bundles $(\mathcal{X}_i, \overline{\mathcal{L}}_i)$ with $\overline{\mathcal{L}}_i$ nef as a Hermitian $\mathbb{Q}$-line bundle on $\mathcal{X}_i$;*
- $\overline{\mathcal{L}}$ *is called* nef *if there exists a strongly nef adelic line bundle $\overline{\mathcal{M}}$ on $\mathcal{U}$ such that $a\overline{\mathcal{L}} + \overline{\mathcal{M}}$ is strongly nef for all $a \in \mathbb{Z}_{>0}$;*
- $\overline{\mathcal{L}}$ *is called* integrable *if it is isomorphic to $\overline{\mathcal{L}}_1 - \overline{\mathcal{L}}_2$ for two strongly nef adelic line bundles $\overline{\mathcal{L}}_1$ and $\overline{\mathcal{L}}_2$ on $\mathcal{U}$.*

With this definition, we can write the following subgroups of $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$:

$$\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{snef}}, \quad \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{nef}}, \quad \text{and} \quad \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{int}}$$

of isomorphism classes of strongly nef, nef, and integrable adelic line bundles on $\mathcal{U}$.

Now back to our quasi-projective variety X defined over a number field K. Define the following subgroups of $\widehat{\mathrm{Pic}}(X/\mathbb{Z}) = \varinjlim_{\mathcal{U}} \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$:

$$\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{snef}} := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{snef}}$$
$$\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{nef}} := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{nef}}$$
$$\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{int}} := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})_{\mathrm{int}}.$$

The adelic line bundles on X in these subgroups are said to be *strongly nef*, *nef*, *integrable* respectively.

We can also define the similar notions for the generic fiber of an adelic line bundle on X. Recall the group $\widehat{\mathrm{Pic}}(X/\mathbb{Q})$ defined at the beginning of §E.5. A $\widetilde{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Q})$ is called *strongly nef* if it is the limit of $\{(Y_i, L_i)\}_{i\geq 1}$ in $\widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathrm{mod}}$ with each L_i a nef $\mathbb{Q}$-line bundle on Y_i. And $\widetilde{L}$ is called *nef* if there exists a strongly nef $\widetilde{M} \in \widehat{\mathrm{Pic}}(X/\mathbb{Q})$ such that $a\widetilde{L} + \widetilde{M}$ is strongly nef for all $a \in \mathbb{Z}_{>0}$, and $\widetilde{L}$ is called *integrable* if it is isomorphic to the difference of two strongly nef elements in $\widehat{\mathrm{Pic}}(X/\mathbb{Q})$. We also use $\widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathrm{snef}}$, $\widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathrm{nef}}$, and $\widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathrm{int}}$ to denote the relevant groups. Then the operation of taking the generic fiber (E.5) yields

$$\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathbf{P}} \longrightarrow \widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathbf{P}}, \qquad \overline{L} \mapsto \widetilde{L}$$

with $\mathbf{P}$ being snef, nef, or int.

E.7. **Height defined by adelic line bundles.** Extending Gillet–Soulé's arithmetic intersection theory, one has an arithmetic intersection pairing

$$\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{int}}^{\dim X+1} \longrightarrow \mathbb{R}, \qquad (\overline{L}_1, \ldots, \overline{L}_{\dim X+1}) \mapsto \langle \overline{L}_1, \ldots, \overline{L}_{\dim X+1} \rangle, \tag{E.8}$$

such that $\langle \overline{L}_1, \ldots, \overline{L}_{\dim X+1} \rangle \geq 0$ if $\overline{L}_i \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{nef}}$ for all $i \in \{1, \ldots, \dim X + 1\}$.

When $\dim X = 0$, then this intersection is called the *arithmetic degree* of the adelic line bundle and is denoted by

$$\widehat{\deg} \colon \widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{int}} \longrightarrow \mathbb{R}.$$

Definition E.5. *Let $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{int}}$. The* height function *defined by $\overline{L}$ is*

$$h_{\overline{L}} \colon X(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}, \qquad x \mapsto \frac{\widehat{\deg}(\overline{L}|_x)}{[K(x):K]}.$$

As a remark, we point out that $\overline{L}$ needs not be integrable to define the height function.

Denote by $(L, \|\cdot\|) \in \widehat{\mathrm{Pic}}(X^{\mathrm{an}})$ the image of $\overline{L}$ under the canonical injective map $\widehat{\mathrm{Pic}}(X/\mathbb{Z}) \to \widehat{\mathrm{Pic}}(X^{\mathrm{an}})$. Then the formula for the height function is:

$$h_{\overline{L}}(x) = -\frac{1}{[K(x):K]} \sum_{p \in M_{\mathbb{Q}}} \sum_{z \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)x \times_{\mathbb{Q}} \mathbb{Q}_p} \log \|s(z^{\mathrm{an}})\|^{\deg_{\mathbb{Q}_p} z} \tag{E.9}$$

for any non-zero rational section s of L on X with $x \notin |\mathrm{div}(s)|$; here p can be infinity.

E.8. **Volume, bigness, and height inequality.** Let $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$. Write $(L, \|\cdot\|) \in \widehat{\mathrm{Pic}}(X^{\mathrm{an}})$ for the image of $\overline{L}$ under the canonical injective map $\widehat{\mathrm{Pic}}(X/\mathbb{Z}) \to \widehat{\mathrm{Pic}}(X^{\mathrm{an}})$.

To get lower bounds on $h_{\overline{L}}$, it is important to find "small" sections, as suggested by (E.9). Define the sup-norm of a section $s \in H^0(X, L)$ to be

$$\|s\|_{\sup} := \sup_{x \in X^{\mathrm{an}}} \|s(x)\|.$$

Define the arithmetic H^0 to be

$$\widehat{H}^0(X, \overline{L}) := \{s \in H^0(X, L) : \|s\|_{\sup} \leq 1\}.$$

Now that $H^0(X, L)$ is a lattice in $H^0(X, L)_{\mathbb{R}}$ and $\widehat{H}^0(X, \overline{L})$ is contained in the unit ball, we have that $\widehat{H}^0(X, \overline{L})$ is a finite set. Define

$$\widehat{h}^0(X, \overline{L}) := \log \# \widehat{H}^0(X, \overline{L}).$$

Theorem-Definition E.6 ([YZ21, Thm. 5.2.1.(1)])**.** *The following limit exists*

$$\widehat{\mathrm{vol}}(X, \overline{L}) := \lim_{m \to \infty} \frac{(\dim X + 1)!}{m^{\dim X + 1}} \widehat{h}^0(X, m\overline{L}),$$

and is defined to be the (arithmetic) volume *of* $\overline{L}$*.*

Moreover, part (2) of [YZ21, Thm. 5.2.1] asserts that $\widehat{\mathrm{vol}}(X, \overline{L})$ can be computed as the limit of the arithmetic volumes of any sequence of model (Hermitian) line bundles converging to $\overline{L}$.

Definition E.7. *An adelic line bundle* $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$ *is said to be* big *if* $\widehat{\mathrm{vol}}(X, \overline{L}) > 0$.

Theorem E.8 ([YZ21, Thm. 5.2.2 and 5.2.9])**.** *We have:*

- *(Arithmetic Hilbert–Samuel) If* $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$ *is nef, then* $\widehat{\mathrm{vol}}(X, \overline{L}) = \overline{L}^{\cdot(\dim X + 1)}$.
- *(Arithmetic Siu) Let* $\overline{L}, \overline{M} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{nef}}$*. Then*

$$\widehat{\mathrm{vol}}(X, \overline{L} - \overline{M}) \geq \overline{L}^{\cdot(\dim X + 1)} - (\dim X + 1)\overline{L}^{\cdot \dim X} \cdot \overline{M}.$$

- *(continuity) Let* $\overline{L}, \overline{M} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})$*. Then* $\lim_{t \to 0} \widehat{\mathrm{vol}}(\overline{L} + t\overline{M}) = \widehat{\mathrm{vol}}(\overline{L})$*.*

The discussion above applies to $\widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathrm{int}}$. For each $\widetilde{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathrm{int}}$, define

$$\widehat{h}^0(X, \widetilde{L}) := \dim\{s \in H^0(X, L) : \sup_{x \in X^{\mathrm{an}}_{\mathrm{triv}}} \|s(x)\| \leq 1\}.$$

The limit

$$\widehat{\mathrm{vol}}(X, \widetilde{L}) := \lim_{m \to \infty} \frac{(\dim X)!}{m^{\dim X}} \widehat{h}^0(X, m\widetilde{L}) \tag{E.10}$$

exists and is defined to be the *volume* of $\widetilde{L}$. Then $\widetilde{L}$ is said to be *big* if $\widehat{\mathrm{vol}}(X, \widetilde{L}) > 0$. We also have arithmetic Hilbert–Samuel and Siu for elements in $\widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathrm{nef}}$, with $\dim X$ replaced by $\dim X - 1$ in the formulae.

With these preparation, we are ready to state the height inequality.

Theorem E.9 ([YZ21, Thm. 5.3.5]). *Let $\pi\colon X \to S$ be a morphism of quasi-projective varieties defined over a number field K. Let $\overline{L} \in \widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{int}}$ and $\overline{M} \in \widehat{\mathrm{Pic}}(M/\mathbb{Z})_{\mathrm{int}}$.*

If the generic fiber $\widetilde{L}$ of $\overline{L}$ is big, then there exist a Zariski open dense subset U of X and positive constants ϵ and c, such that

$$h_{\overline{L}}(x) \geq \epsilon h_{\overline{M}}(\pi(x)) - c, \qquad \textit{for all } x \in U(\overline{\mathbb{Q}}).$$

If moreover $\overline{L}$ is big, then one can remove the constant c from the inequality above.

When $\overline{L}$ is big, Arithmetic Siu and continuity together imply $\widehat{\mathrm{vol}}(\overline{L} - \epsilon\pi^*\overline{M}) > 0$ by definition of bigness. Thus the conclusion follows immediately from (E.9). The general case is reduced to the case where $\overline{L}$ is big by [YZ21, Lem. 5.2.10], which asserts that the bigness of $\widetilde{L}$ is not far from the bigness of the whole $\overline{L}$.

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