Chapter 5

Borel–Serre compactification

5.1 Borel–Serre compactification

Let $(\mathbf{G}, X)$ be a Shimura datum. By abuse of notation use X to denote a connected component.

5.1.1 Revision on the rational symmetric spaces and Siegel sets

Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}^{\mathrm{der}}$.

Recall the rational Langlands decomposition of $\mathbf{P}$ from (4.5.9)

$$P(\mathbb{R})^+ \simeq N_P(\mathbb{R}) \times A_{\mathbf{P}}(\mathbb{R})^+ \times M_{\mathbf{P}}(\mathbb{R})$$

and the induced rational horospherical decomposition (4.5.10)

$$h\colon X \simeq N_P(\mathbb{R}) \times A_{\mathbf{P}}(\mathbb{R})^+ \times X_{\mathbf{P}}$$

with the rational boundary symmetric space

$$X_{\mathbf{P}} = M_{\mathbf{P}}(\mathbb{R})^+/(M_{\mathbf{P}} \cap K_\infty).$$

Let $\Delta(A_{\mathbf{P}}, P) = \{\alpha_1, \ldots, \alpha_r\}$ be the subset of simple roots defined as in (4.5.8); they are characters of a maximal $\mathbb{Q}$-split torus $\mathbf{A}_{\mathbf{P}}$ contained in $\mathbf{P}$. Then we have an isomorphism

$$A_{\mathbf{P}}(\mathbb{R})^+ \xrightarrow{\sim} \mathbb{R}^r_{>0}, \qquad a \mapsto (\alpha_1(a)^{-1}, \ldots, \alpha_r(a)^{-1}). \tag{5.1.1}$$

A Siegel set in X associated with $\mathbf{P}$ is of the form

$$\Sigma_{\mathbf{P},U,t,V} := h^{-1}(U \times A_{\mathbf{P},t} \times V) \subseteq X$$

with $U \subseteq N_P(\mathbb{R})$ and $V \subseteq X_{\mathbf{P}}$ bounded and

$$A_{\mathbf{P},t} := \{a \in A_{\mathbf{P}}(\mathbb{R})^+ : \alpha(a)^{-1} \leq t,\ \forall \alpha \in \Delta(A_{\mathbf{P}}, P)\}.$$

If $\mathbf{P} < \mathbf{Q}$ are parabolic subgroups of $\mathbf{G}^{\mathrm{der}}$, then $A_{\mathbf{P}} > A_{\mathbf{Q}}$. Moreover, it can be shown that the followings are equivalent: (i) $X_{\mathbf{P}}$ is compact, (ii) $M_{\mathbf{P}}$ has $\mathbb{Q}$-rank 0, (iii) $\mathbf{P}$ is minimal parabolic in $\mathbf{G}^{\mathrm{der}}$. Furthermore, *reduction theory* asserts the following: Let $\mathbf{P}$ be a minimal parabolic subgroup of $\mathbf{G}^{\mathrm{der}}$, then there exist a Siegel set $\Sigma = \Sigma_{\mathbf{P},U,t,V}$ associated with $\mathbf{P}$ and a finite set $J \subseteq \mathbf{G}(\mathbb{Q})$ such that $J \cdot \Sigma$ is a fundamental set for the action of Γ on X.

5.1.2 Borel–Serre partial compactification: definition

For any parabolic subgroup $\mathbf{P}$ of $\mathbf{G}^{\mathrm{der}}$, define the boundary component

$$e(\mathbf{P}) := N_P(\mathbb{R}) \times X_{\mathbf{P}}. \tag{5.1.2}$$

Since N_P is a normal subgroup of P, the boundary component $e(\mathbf{P}) \simeq P(\mathbb{R})^+/A_{\mathbf{P}}(\mathbb{R})^+(M_{\mathbf{P}} \cap K_\infty)$ is then an $N_P(\mathbb{R})$-principle bundle over the rational boundary symmetric space $X_{\mathbf{P}} \simeq P(\mathbb{R})^+/N_P(\mathbb{R})A_{\mathbf{P}}(\mathbb{R})^+(M_{\mathbf{P}} \cap K_\infty)$.

The *Borel–Serre partial compactification* $\overline{X}^{\mathrm{BS}}$ is defined, as a set, to be

$$\overline{X}^{\mathrm{BS}} := X \cup \bigsqcup_{\mathbf{P}} e(\mathbf{P}). \tag{5.1.3}$$

To define the topology on $\overline{X}^{\mathrm{BS}}$, we only need to define the neighborhoods of the boundary points. For this purpose, we need to analyze boundary components $e(\mathbf{Q})$ and $e(\mathbf{P})$ for two parabolic subgroups $\mathbf{P} < \mathbf{Q}$ of $\mathbf{G}^{\mathrm{der}}$.

For the reductive subgroup $\mathbf{M_Q}$ of $\mathbf{Q}$, set $\mathbf{P}' := \mathbf{P} \cap \mathbf{M_Q}$. Then $\mathbf{P}'$ is a parabolic subgroup of $\mathbf{M_Q}$ such that, by looking at the root system construction,

$$\mathbf{M}_{\mathbf{P}'} = \mathbf{M_P},\ \mathbf{A_P} = \mathbf{A_Q} \times \mathbf{A}_{\mathbf{P}'},\ N_P = N_Q \rtimes N_{P'}. \tag{5.1.4}$$

Thus the horospherical decomposition of $X_{\mathbf{Q}}$ associated with $\mathbf{P}'$ is

$$X_{\mathbf{Q}} \simeq N_{P'}(\mathbb{R}) \times A_{\mathbf{P}'}(\mathbb{R})^+ \times X_{\mathbf{P}'} = N_{P'}(\mathbb{R}) \times A_{\mathbf{P}'}(\mathbb{R})^+ \times X_{\mathbf{P}}. \tag{5.1.5}$$

Next, we find another $\mathbb{Q}$-split torus $\mathbf{A_{P,Q}}$ of $\mathbf{A_P}$ which is isomorphic to $\mathbf{A}_{\mathbf{P}'}$. We start with the case where $\mathbf{P}$ is a standard parabolic subgroup. Namely, we fix a basis ${}_{\mathbb{Q}}\Delta$ of the relative root system ${}_{\mathbb{Q}}\Phi := \Phi(\mathbf{A}, \mathbf{G}^{\mathrm{der}})$ for some maximal $\mathbb{Q}$-split torus $\mathbf{A}$ in $\mathbf{G}^{\mathrm{der}}$, and then we obtain a minimal parabolic subgroup $\mathbf{P}_0$ of $\mathbf{G}^{\mathrm{der}}$ as below (4.4.2), and assume $\mathbf{P} = \mathbf{P}_I$ for some subset $I \subseteq {}_{\mathbb{Q}}\Delta$ as in Theorem 4.4.6. Since $\mathbf{Q} > \mathbf{P}$, we have $\mathbf{Q} > \mathbf{P}_0$ and hence $\mathbf{Q} = \mathbf{P}_J$ for some $J \subseteq {}_{\mathbb{Q}}\Delta$ by Theorem 4.4.6, and it is clear that $I \subseteq J$. By Lemma 4.4.7, we have then $\mathbf{A}_I > \mathbf{A}_J$. Better, using definitions of $\mathbf{A}_I$ and $\mathbf{A}_J$ we get that $\mathbf{A}_I = \mathbf{A}_{I,J} \times \mathbf{A}_J$, with $\mathbf{A}_{I,J} := \left(\bigcap_{\alpha' \in J \setminus I} \mathrm{Ker}\alpha'\right)^\circ$. Notice that in this case, $\Delta(A_{\mathbf{P}}, P) = {}_{\mathbb{Q}}\Delta \setminus I$, and hence $J \setminus I \subseteq \Delta(A_{\mathbf{P}}, P)$. In general, $\mathbf{P}$ is conjugate to a unique $\mathbf{P}_I$, and then the conjugation of $\mathbf{Q}$ by the same element in $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})$ is standard (*i.e.* contains $\mathbf{P}_0$), and hence $\mathbf{Q} = \mathbf{P}_J$ for some $J \subseteq {}_{\mathbb{Q}}\Delta$. Let $\mathbf{A_{P,Q}} < \mathbf{P}$ be the suitable conjugation of $\mathbf{A}_{I,J}$, and let $I_{\mathbf{P,Q}} \subseteq \Delta(A_{\mathbf{P}}, P)$ be the suitable conjugation of $J \setminus I$. Then we have $\mathbf{A_P} = \mathbf{A_Q} \times \mathbf{A_{P,Q}}$. Thus $\mathbf{A}_{\mathbf{P}'} \simeq \mathbf{A_{P,Q}}$ by the second equality in (5.1.4). So (5.1.5) becomes

$$X_{\mathbf{Q}} \simeq N_{P'}(\mathbb{R}) \times A_{\mathbf{P,Q}}(\mathbb{R})^+ \times X_{\mathbf{P}}. \tag{5.1.6}$$

Therefore by (5.1.4) and (5.1.6), we have

$$e(\mathbf{Q}) = N_Q \times X_{\mathbf{Q}} \simeq N_P(\mathbb{R}) \times A_{\mathbf{P,Q}}(\mathbb{R})^+ \times X_{\mathbf{P}}. \tag{5.1.7}$$

Definition 5.1.1. *The topology on $\overline{X}^{\mathrm{BS}}$ is defined as follows: (i) on X it is the natural one, (ii) for each parabolic subgroup $\mathbf{P}$ of $\mathbf{G}^{\mathrm{der}}$, the neighborhoods of a point $(n, z) \in e(\mathbf{P}) = N_P(\mathbb{R}) \times X_{\mathbf{P}}$ is $\bigsqcup_{\mathbf{Q} > \mathbf{P}} U \times A_{\mathbf{P,Q},t} \times V$ for all neighborhoods U of n in $N_P(\mathbb{R})$, all neighborhoods V of z in $X_{\mathbf{P}}$, and all $t > 0$, with*

$$A_{\mathbf{P,Q},t} := \{a \in A_{\mathbf{P,Q}}(\mathbb{R})^+ : \alpha(a)^{-1} < t,\ \forall \alpha \in I_{\mathbf{P,Q}}\}.$$

5.1.3 Borel–Serre partial compactification: corners and Hausdorff property

Recall the isomorphism $A_{\mathbf{P}}(\mathbb{R})^+ \simeq \mathbb{R}^r_{>0}$ from (5.1.1). Use $\overline{A_{\mathbf{P}}}$ to denote the closure of $A_{\mathbf{P}}(\mathbb{R})^+$ in $\mathbb{R}^r$ under the natural inclusion $\mathbb{R}^r_{>0} \subseteq \mathbb{R}^r$. The discussion on the topology of $\overline{X}^{\mathrm{BS}}$ in the previous subsection yields easily the following results.

Lemma 5.1.2. *We have a disjoint decomposition*

$$\overline{A_{\mathbf{P}}} = A_{\mathbf{P}}(\mathbb{R})^+ \cup \bigsqcup_{\mathbf{Q}>\mathbf{P}} (A_{\mathbf{P},\mathbf{Q}}(\mathbb{R})^+ \times 0_{\mathbf{Q}})$$

where $0_{\mathbf{Q}}$ is the origin of the real vector space $\mathbb{R}^{r'}$ arising from $A_{\mathbf{Q}}(\mathbb{R})^+ \simeq \mathbb{R}^{r'}_{>0} \subseteq \mathbb{R}^{r'}$.

Proposition 5.1.3. *The embedding $N_P(\mathbb{R}) \times A_{\mathbf{P}}(\mathbb{R})^+ \times X_{\mathbf{P}} \simeq X \subseteq \overline{X}^{\mathrm{BS}}$ can be naturally extended to an open embedding $N_P(\mathbb{R}) \times \overline{A_{\mathbf{P}}} \times X_{\mathbf{P}} \hookrightarrow \overline{X}^{\mathrm{BS}}$. Moreover, the image of $N_P(\mathbb{R}) \times \overline{A_{\mathbf{P}}} \times X_{\mathbf{P}}$ in $\overline{X}^{\mathrm{BS}}$ is equal to the subset*

$$X \cup \textstyle\bigsqcup_{\mathbf{Q}>\mathbf{P}} e(\mathbf{Q}) \tag{5.1.8}$$

in $\overline{X}^{\mathrm{BS}}$.

We will call (5.1.8) the *corner associated with* $\mathbf{P}$ and denote it by $X(\mathbf{P})$. Then we have

$$X(\mathbf{P}) \simeq X \times_{A_{\mathbf{P}}(\mathbb{R})^+} \overline{A_{\mathbf{P}}}, \quad e(\mathbf{P}) = N_P(\mathbb{R}) \times \{(0, \cdots, 0)\} \times X_{\mathbf{P}}, \quad X(\mathbf{P}) \simeq e(\mathbf{P}) \times [0, \infty)^r.$$

Another corollary of Lemma 5.1.2 is the following description of neighborhood bases of points in the boundaries.

Corollary 5.1.4. *For any point $(n, z) \in e(\mathbf{P}) = N_P(\mathbb{R}) \times X_{\mathbf{P}}$, a neighborhood basis in $\overline{X}^{\mathrm{BS}}$ is given by $U \times \overline{A_{\mathbf{P}}}_{,t} \times V \subseteq X(\mathbf{P})$, where $n \in U, z \in V$ are bases of neighborhoods of n and z respectively, and $t > 0$ with*

$$\overline{A_{\mathbf{P}}}_{,t} := \{a \in \overline{A_{\mathbf{P}}} : \alpha(a)^{-1} < t, \ \forall \alpha \in \Delta(A_{\mathbf{P}}, P)\}.$$

Finally, we close this subsection by the following proposition.

Proposition 5.1.5. *$\overline{X}^{\mathrm{BS}}$ is a Hausdorff space.*

Proof. Take two distinct points $y_1, y_2 \in \overline{X}^{\mathrm{BS}} \setminus X$, with $y_j \in e(\mathbf{P}_j)$.

If $\mathbf{P}_1 = \mathbf{P}_2$, then $e(\mathbf{P}_1) = e(\mathbf{P}_2)$ and clearly there are open neighborhoods of y_1 and y_2 which are disjoint.

From now on assume $\mathbf{P}_1 \neq \mathbf{P}_2$. Assume that y_1 and y_2 have open neighborhoods which are non-disjoint. By Corollary 5.1.4, we may assume that the neighborhoods are $U_1 \times \overline{A_{\mathbf{P}_1}}_{,t} \times V_1$ and $U_2 \times \overline{A_{\mathbf{P}_2}}_{,t} \times V_2$ with $t > 0$. We may furthermore assume that U_1, V_1, U_2, V_2 are bounded. Call the intersection W. Then W is open in $U_j \times \overline{A_{\mathbf{P}_j}}_{,t} \times V_j$.

Because $U_j \times A_{\mathbf{P}_j,t} \times V_j$ is open and dense in $U_j \times \overline{A_{\mathbf{P}_j}}_{,t} \times V_j$, we have that $W \cap (U_j \times A_{\mathbf{P}_j,t} \times V_j)$ is open and dense in W. So $W \cap (U_1 \times A_{\mathbf{P}_1,t} \times V_1) \cap (U_2 \times A_{\mathbf{P}_2,t} \times V_2)$ is non-empty.

But $\mathbf{P}_1 \neq \mathbf{P}_2$, so general theory of Siegel sets says that $(U_1 \times A_{\mathbf{P}_1,t} \times V_1) \cap (U_2 \times A_{\mathbf{P}_2,t} \times V_2) = \emptyset$ for $t \gg 1$ (say $t \geq t_0$ for some fixed $t_0 \in \mathbb{R}$). Therefore by the previous paragraph, $t < t_0$. Hence we find open neighborhoods $U_1 \times \overline{A_{\mathbf{P}_1}}_{,t_0} \times V_1$ of y_1 and $U_2 \times \overline{A_{\mathbf{P}_2}}_{,t_0} \times V_2$ of y_2 which are disjoint. We are done. $\square$