Chapter 3

Preparation on analysis for the proof of Arithmetic Hilbert–Samuel

The goal of this chapter is to prove the Arithmetic Hilbert–Samuel Theorem. We follow the approach of Abbès–Bouche.

3.1 Distortion function

3.1.1 Fubini–Study metric

Let X be a connected complex manifold of dimension n, endowed with a smooth Hermitian metric (*i.e.* a J-invariant positive-definite Hermitian inner product $h(\cdot,\cdot)$ on T_X where J is the complex structure on X). This Hermitian metric induces a positive $(1,1)$-form $\omega = -\mathrm{Im}h$ on X, and hence a volume form $\mathrm{d}V := \omega^n/n!$ on X. Notice that h can be recovered from ω and J via the formula $h(u,v) = \omega(u, Jv) - \sqrt{-1}\omega(u,v)$.

Definition 3.1.1. *Such a complex manifold X is called a* **Kähler manifold** *if ω is closed.*

If X is a Kähler manifold, we usually call ω its *Kähler form.*

Example 3.1.2. *For $X = \mathbb{P}^n$, the* **Fubini–Study metric** *is defined as follows. We have the standard projection $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ by viewing $\mathbb{P}^n$ as the space consisting of all complex lines in $\mathbb{C}^{n+1}$. The standard Hermitian metric on $\mathbb{C}^{n+1}$ defines the following $(1,1)$-form on $\mathbb{P}^n$*

$$\omega_{\mathrm{FS}} = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(|z_0|^2 + \cdots + |z_n|^2)$$

with $(z_0, \ldots, z_n)$ the standard coordinate of $\mathbb{C}^{n+1}$. To see this, consider any open subset $U \subseteq \mathbb{P}^n$ such that natural projection admits a lifting $Z\colon U \to \mathbb{C}^{n+1} \setminus \{0\}$. Thenany other lifting Z' differs from Z by a non-zero holomorphic function f, and hence $\partial\overline{\partial}\log|Z'|^2 = \partial\overline{\partial}\log|fZ|^2 = \partial\overline{\partial}\log|Z|^2 + \partial\overline{\partial}\log(f\overline{f}) = \partial\overline{\partial}\log|Z|^2$. Thus the local $(1,1)$-forms $\partial\overline{\partial}\log|Z|^2$, with U varying, patch together to a global $(1,1)$-form, which is exactly $(2\pi/\sqrt{-1})\omega_{\mathrm{FS}}$.

Notice that $\mathrm{d}\omega_{\mathrm{FS}} = 0$, i.e. ω_{FS} is closed.

To see that ω_{FS} is a positive $(1,1)$-form, it suffices to prove that it is positive at one point since ω is invariant under the group action of $U(n+1)$ on $\mathbb{P}^n$ (which is transitive). Use $\{w_1, \ldots, w_n\}$ to denote the standard coordinate on the open subset $U_0 := \{z_0 \neq 0\} \subseteq \mathbb{P}^n$, i.e. $w_j = z_j/z_0$. Then

$$\omega_{\mathrm{FS}}|_{U_0} = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(1 + \sum w_j\overline{w}_j) = \frac{\sqrt{-1}}{2\pi}\left(\frac{\sum \mathrm{d}w_j \wedge \mathrm{d}\overline{w}_j}{1 + \sum w_j\overline{w}_j} - \frac{(\sum \overline{w}_j \mathrm{d}w_j) \wedge (\sum w_j \wedge \mathrm{d}\overline{w}_j)}{(1 + \sum w_j\overline{w}_j)^2}\right),$$

which is $\frac{\sqrt{-1}}{2\pi}\sum \mathrm{d}w_j \wedge \mathrm{d}\overline{w}_j$ *at* $[1:0:\cdots:0]$. *Thus* ω_{FS} *is positive.*

By the discussion above, ω_{FS} *defines a Hermitian metric on* $\mathbb{P}^n$*, which is called the Fubini–Study metric.*

By Example 3.1.2, the analytification of any smooth quasi-projective variety is a Kähler manifold.

3.1.2 Distortion function

Let X be a *compact* Kähler manifold. Let L be a line bundle on X, endowed with a smooth Hermitian metric $\|\cdot\|$ which is positive, *i.e.* $c_1(L,\|\cdot\|)$ is a positive $(1,1)$-form on X. By the Kodaira embedding theorem, L is an ample line bundle on X (and hence X is projective). Now for each $k \geq 0$, denote by $kL := L^{\otimes k}$, $V_k := H^0(X, kL)$ the space of holomorphic sections of kL on X, and

$$\Phi_k \colon X \to \mathbb{P}(V_k^{\vee}), \qquad x \mapsto H_x = \{\sigma \in V_k : \sigma(x) = 0\}. \tag{3.1.1}$$

Then Φ_k is a closed immersion with $\Phi_k^*\mathcal{O}_{\mathbb{P}(V_k^{\vee})}(1) \simeq kL$ for all $k \gg 1$.

On kL, we have the natural Hermitian metric $\|\cdot\|_k$, which is the metric of $(L,\|\cdot\|)^{\otimes k}$. This induces the L^2-norm on V_k defines by

$$\|s\|_{L^2}^2 = \int_X \|s(x)\|_k^2 \mathrm{d}V \qquad \text{for all } s \in V_k = H^0(X, kL).$$

Notice that this defines a metric on V_k. Notice that $H^0(\mathbb{P}(V_k^{\vee}), \mathcal{O}_{\mathbb{P}(V_k^{\vee})}(1))$ is naturally in bijection with the space of complex lines in V_k. Thus the construction of Example 3.1.2 gives a Hermitian metric on $\mathcal{O}_{\mathbb{P}(V_k^{\vee})}(1)$, which we also call the *Fubini–Study metric*. Thus its pullback via Φ_k defines a Hermitian metric on kL, which we call $\|\cdot\|_{k^{\mathrm{FS}}}$.

Thus on kL, we have two Hermitian metrics: $\|\cdot\|_k$ and $\|\cdot\|_{k^{\mathrm{FS}}}$.

Definition 3.1.3. *The k-th* **distortion function** *is*

$$b_k \colon X \to \mathbb{R}, \qquad x \mapsto \frac{\|\xi\|_k^2}{\|\xi\|_{k^{\mathrm{FS}}}^2}$$

for any $\xi \in L_x^k \setminus \{0\}$.

Here is a more explicit expression of the distortion function. Let $s_1, \ldots, s_N$ be an orthonormal basis of $V_k = H^0(X, kL)$ for the L^2-norm defined above. Then obviously $\|s_j(x)\|_{k^{\mathrm{FS}}}^2 = 1$ for each j, and hence $\|\xi\|_{k^{\mathrm{FS}}}^2 = (\|s_1(x)\|_k^2 + \cdots + \|s_N(x)\|_k^2)^{-1}\|\xi\|_k^2$. Thus

$$b_k(x) = \sum_{j=1}^{N} \|s_j(x)\|_k^2. \tag{3.1.2}$$

3.1.3 Main result on the distortion function

The main result about the distortion function is the following:

Theorem 3.1.4. *The function* $(b_k)^{1/k}$ *converges to* 1 *uniformly on* X*. Namely for any* $\epsilon > 0$*, there exists* k_0 *such that* $|b_k(x)^{1/k} - 1| < \epsilon$ *for all* $k \geq k_0$ *and all* $x \in X$.

In other terminology, the Fubini–Study metric on L flattens uniformly into the initial metric $\|\cdot\|$.

We shall prove a more precise version of this theorem. For the statement need to introduce the following notion. Locally on X we can find a suitable complex coordinate $(z_1, \ldots, z_n)$ of X such that: (i) $\omega = \frac{\sqrt{-1}}{2}\sum_{j=1}^n \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$ (in other words, $(\mathrm{d}z_1, \mathrm{d}\overline{z}_1 \ldots, \mathrm{d}z_n, \mathrm{d}\overline{z}_n)$ is an orthonormal frame of T_X^* with respect to the Hermitian metric), (ii) the $(1,1)$-form $c_1(L, \|\cdot\|)$ equals $\frac{\sqrt{-1}}{2}\sum_{j=1}^n \alpha_j(x)\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$ with $\alpha_j(x) > 0$.

Definition 3.1.5. *The functions $\alpha_1, \ldots, \alpha_n$ are called the* **eigenfunctions** *of $c_1(L, \|\cdot\|)$ with respect to ω (or with respect to the Hermitian metric on X). The* **determinant** *is defined to be the smooth function on X*

$$\det c_1(L, \|\cdot\|) := \alpha_1 \cdots \alpha_n.$$

Theorem 3.1.6. *When $k \to \infty$, the function*

$$\frac{b_k}{k^n \det c_1(L, \|\cdot\|)}$$

converges to 1 uniformly on X.

Theorem 3.1.6 implies Theorem 3.1.4 immediately.

3.2 Proof of the main theorem on the distortion function via heat kernel

Let X be a connected compact Kähler manifold of dimension n, and let $\mathrm{d}V$ be the volume form on X. Let L be a line bundle on X, endowed with a smooth Hermitian metric $\|\cdot\|$.

3.2.1 Kodaira Laplacian and Harmonic forms

For any $k \geq 1$, denote by $A^{0,q}(X, kL)$ the space of smooth global $(0,q)$-forms with values in $kL := L^{\otimes k}$ (*i.e.* global sections of $(T_X^{0,q})^* \otimes L^{\otimes k}$). If $q = 0$, notice that $A^{0,0}(X, kL)$ is preciselythe space of smooth (real) sections of kL over X.

The Hermitian metric on X and the Hermitian metric on L together induce a Hermitian metric on $(T_X^{0,q})^* \otimes L^{\otimes k}$ which we denote by $\|\cdot\|_{k,q}$. Then we can endow $A^{0,q}(X, kL)$ with norms, for example the L^2-norm

$$\|\sigma\|_{L^2} := \left(\int_X \|\sigma(x)\|_{k,q}^2 \mathrm{d}V\right)^{1/2}, \qquad \forall \sigma \in A^{0,q}(X, kL).$$

Each such norm defines a sesquilinear pairing $(\cdot,\cdot)_q$ on $A^{0,q}(X, kL)$.

Denote by $L_q^2(X, kL)$ the completion of $A^{0,q}(X, kL)$ with respect to the L^2-norm. It is a Hilbert space which contains $A^{0,q}(X, kL)$ as a dense subset.

The differential operator $\overline{\partial}\colon (T_X^{0,q})^* \to (T_X^{0,q+1})^*$ induces a differential operator $\overline{\partial}_{k,q}\colon A^{0,q}(X, kL) \to A^{0,q+1}(X, kL)$. And $\overline{\partial}_{k,q}$ has an adjoint $\overline{\partial}_{k,q}^*\colon A^{0,q+1}(X, kL) \to A^{0,q}(X, kL)$ with respect to the given norms, determined by $(\overline{\partial}_{k,q}u, u')_{q+1} = (u, \overline{\partial}_{k,q}^* u')_q$.

Definition 3.2.1. *The* **anti-holomorphic Kodaira Laplacian** *is*

$$\Delta''_{k,q} := \overline{\partial}_{k,q}\overline{\partial}_{k,q}^* + \overline{\partial}_{k,q}^*\overline{\partial}_{k,q}.$$

A smooth $(0,q)$-form u is called a **harmonic form** *if $\Delta''_{k,q}u = 0$.*

In our case, we are interested in the operator

$$\overline{\square}_k^q := \frac{2}{k}\Delta''_{k,q}. \tag{3.2.1}$$

Notice that $\mathrm{Ker}\overline{\square}_k^q = \mathrm{Ker}\Delta''_{k,q}$.

The cohomology of the Dolbeault complex $\cdots \to A^{0,q}(X,kL) \xrightarrow{\overline{\partial}} A^{0,q+1}(X,kL) \to \cdots$ gives $H^{0,q}(X,kL) \simeq H^q(X,\Omega_X^0 \otimes L^{\otimes k}) = H^q(X,kL)$.

We state the following lemma without proof (the proof is not hard).

Lemma 3.2.2. *A $\overline{\partial}$-closed form $u \in A^{0,q}(X,kL)$ is of minimal norm in $u + \mathrm{Im}\overline{\partial}$ if and only if $\overline{\partial}^* u = 0$.*

This lemma (formally) implies that the Dolbeault cohomology group $H^{0,q}(X,kL)$ is represented exactly by solutions of two first-order equations

$$\overline{\partial}u = 0, \quad \overline{\partial}^* u = 0,$$

which can be replaced by the single second-order equation

$$\Delta''_{k,q}u = 0.$$

Thus we have

$$H^q(X,kL) \simeq \mathrm{Ker}\Delta''_{k,q} = \mathrm{Ker}\overline{\square}_k^q.$$

In particular if $q = 0$, then this realizes $H^0(X,kL)$ as the subspace $\mathrm{Ker}\overline{\square}_k^0$ of $A^{0,0}(X,kL)$.

In general, we have an L^2-orthogonal decomposition

$$A^{0,q}(X,kL) = \mathrm{Ker}\overline{\square}_k^q \oplus \mathrm{Im}\overline{\partial}_{k,q-1} \oplus \mathrm{Im}\overline{\partial}^*_{k,q+1}.$$

Recall that X is compact. We state the following (special case of a) theorem on the spectrum of any self-adjoint elliptic operator which is semi-bounded.

Theorem 3.2.3 (Spectral theorem)**.** *The operator $\overline{\square}_k^q$ has discrete spectrum (of eigenvalues)*

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots \to \infty$$

and there exists a corresponding orthonormal basis consisting of smooth eigenforms $\{\psi_m\}$, i.e. $\overline{\square}_k^q \psi_m = \lambda_m \psi_m$ for non-zero ψ_m.

In general, this theorem can be applied to any self-adjoint elliptic operator P which is semi-bounded (*i.e.* $(Pu,u)_{L^2} \geq -c\|u\|^2_{L^2}$ for some fixed $c \in \mathbb{R}$) and with 0 replaced by $-c$.

3.2.2 Heat kernel associated with the anti-holomorphic Kodaira Laplacian

We shall assume the following proposition which claims the existence of the heat kernel, which is our main tool to prove Theorem 3.1.6.

Proposition 3.2.4. *The operator $\overline{\square}_k^q$ admits a smooth (heat) kernel $e_k^q(t,x,y)$, uniquely determined by the following properties:*

(i) It is a smooth function on $\mathbb{R}_{>0} \times X \times X$ taking values in $\mathrm{End}((T_X^{0,q})^* \otimes L^{\otimes k})$.

(ii) $(\frac{\partial}{\partial t} + \overline{\square}_k)e_k^q = 0$ with $\overline{\square}_k^q$ acting on the x-variable.

(iii) $e_k^q(t,x,y) \to \delta_y$ *(Dirac function) when* $t \to 0^+$.

More concretely, (ii) and (iii) mean the following: For each $u_0(x)$, there exists a unique smooth solution $u = u(t,x)\colon \mathbb{R}_{\geq 0} \times X \to \mathrm{End}((T_X^{0,q})^* \otimes L^{\otimes k})$ to the heat equation

$$\begin{cases} (\frac{\partial}{\partial t} + \overline{\square}_k^q)u = 0 \\ u(0,x) = u_0(x), \end{cases}$$

which can be obtained as

$$u(t,x) = \int_X e_k^q(t,x,y)u_0(y)\mathrm{d}y. \tag{3.2.2}$$

We sometimes call $e_k^q(t,x,y)$ the *fundamental solution* of $(\frac{\partial}{\partial t} + \overline{\square}_k)u = 0$. It is known that under the eigenbasis given by Theorem 3.2.3, we have

$$e_k^q(t,x,y) = \sum_{m\geq 1} e^{-\lambda_m t}\psi_m(x) \otimes \psi_m^*(y).$$

We shall be interested in the *diagonal* of the heat kernel, which for simplicity we denote by

$$e_k^q(t,x) := e_k^q(t,x,x) = \sum_\lambda e^{-\lambda t}\|\psi_\lambda(x)\|_{k,q}^2 \tag{3.2.3}$$

for the L^2-orthonormal eigenbasis $(\lambda, \psi_\lambda)_\lambda$ given by Theorem 3.2.3.

The following theorem is the main theorem on heat kernel expansion and is of fundamental importance. We state the theorem without proof.

Let $\alpha_1, \ldots, \alpha_n$ be the eigenfunctions of $c_1(L, \|\cdot\|)$ with respect to the Hermitian metric on X. For any multi-index J, set $\overline{\alpha}_J := \sum_{j\notin J}\alpha_j - \sum_{j\in J}\alpha_j$. Define

$$e_\infty^q(t,x) := \alpha_1(x)\cdots\alpha_n(x)\frac{\sum_{|J|=q} e^{t\overline{\alpha}_J(x)}}{\prod_{j=1}^n (e^{t\alpha_j(x)} - e^{-t\alpha_j(x)})}. \tag{3.2.4}$$

Theorem 3.2.5. *There exists a real number* $\epsilon > 0$ *with the following property. When* $k \to \infty$, *the function* $k^{-n}e_k^q(t,x)$ *converges to* $e_\infty^q(t,x)$ *uniformly with respect to* $x \in X$ *and* $t \in (0, k^{2\epsilon})$.

3.2.3 Application to the proof of Theorem 3.1.6

Let us prove Theorem 3.1.6 by using the results on heat kernel above.

Let $(\lambda, \psi_\lambda)_\lambda$ be an L^2-orthonormal eigenbasis for the operator $\overline{\square}_k^0$ from Theorem 3.2.3. Recall that $H^0(X, kL)$ is precisely the subspace $\mathrm{Ker}\overline{\square}_k^0$ of $A^{0,0}(X, kL)$. Thus

$$e_k^0(t,x) = \sum_\lambda e^{-t\lambda}\|\psi_\lambda(x)\|_{k,0}^2 = b_k(x) + \sum_{\lambda>0} e^{-t\lambda}\|\psi_\lambda(x)\|_{k,0}^2 \tag{3.2.5}$$

where the second equality follows from (3.1.2).

We will study the asymptotic behavior of $e_k^0(t,x)$ and of $e_k^0(t,x) - b_k(x) = \sum_{\lambda>0} e^{-t\lambda}\|\psi_\lambda(x)\|_{k,0}^2$ separately.

By Theorem 3.2.5 with $q = 0$, we get

$$e_k^0(t,x) = \left(\alpha_1(x)\cdots\alpha_n(x)\prod_{j=1}^n \frac{1}{1 - e^{-2t\alpha_j(x)}}\right)k^n + o(k^n)$$

uniformly in $x \in X$ and in $t \in (0, k^{2\epsilon})$ for a fixed ϵ. Taking $t = k^{\epsilon} \to \infty$, we get

$$e_k^0(k^{\epsilon}, x) \sim \alpha_1(x) \cdots \alpha_n(x) k^n. \tag{3.2.6}$$

On the other hand for each $\lambda > 0$, we have $e^{-t\lambda/2}\|\psi_\lambda(x)\|_{k,0}^2 < e_k^0(t/2, x)$ by (3.2.5). Thus

$$\sum_{\lambda>0} e^{-t\lambda}\|\psi_\lambda(x)\|_{k,0}^2 < e_k^0(t/2, x) \sum_{\lambda>0} e^{-t\lambda/2} \tag{3.2.7}$$

Lemma 3.2.6. *Let $\lambda > 0$ be an eigenvalue of $\overline{\square}_k^0$. For any eigenfunction ψ_λ associated with λ, the $(0,1)$-form $\overline{\partial}\psi_\lambda$ is an eigenform for $\overline{\square}_k^1$ associated with λ.*

Sometimes we say that *the positive spectrum of $\overline{\square}_k^0$ injects into the positive spectrum of $\overline{\square}_k^1$*. Notice that this lemma immediately implies that $\overline{\partial}\psi_\lambda = \overline{\partial}\psi'_\lambda$ if and only if $\psi_\lambda = \psi'_\lambda$.

Proof. We have $\overline{\square}_k^0\psi_\lambda = \lambda\psi_\lambda$. Applying $\overline{\partial}$ to both sides, we get $\overline{\partial}\overline{\partial}^*\overline{\partial}\psi_\lambda = (k/2)\lambda\overline{\partial}\psi_\lambda$. Thus $\overline{\square}_k^1(\overline{\partial}\psi_\lambda) = \lambda\overline{\partial}\psi_\lambda$.

It remains to show that $\overline{\partial}\psi_\lambda \neq 0$. Suppose $\overline{\partial}\psi_\lambda = 0$. Then ψ_λ is a holomorphic function on X, and hence is constant since X is compact. But then $\overline{\square}_k^0\psi_\lambda = 0$, so $\psi_\lambda = 0$, which is a contradiction. □

These $(0,1)$-forms $\overline{\partial}\psi_\lambda$ are still orthogonal to each other, but they do not necessary have L^2-norm 1 (and hence should be normalized).

By Lemma 3.2.6 and (3.2.3), we have

$$\sum_{\lambda>0} e^{-t\lambda}\frac{\|\overline{\partial}\psi_\lambda(x)\|_{k,1}^2}{\|\overline{\partial}\psi_\lambda\|_{L^2}^2} < e_k^1(t, x).$$

Integrating on X and by the definition of the L^2-norm, we get

$$\sum_{\lambda>0} e^{-t\lambda} < \int_X e_k^1(t, x)\mathrm{d}V. \tag{3.2.8}$$

Now (3.2.7) and (3.2.8) together yield

$$\sum_{\lambda>0} e^{-t\lambda}\|\psi_\lambda(x)\|_{k,0}^2 < e_k^0(t/2, x)\int_X e_k^1(t/2, x)\mathrm{d}V. \tag{3.2.9}$$

By Theorem 3.2.5 with $q = 1$, we get

$$e_k^1(t, x) = \left(\alpha_1(x)\cdots\alpha_n(x)\sum_{j=1}^n \frac{1}{e^{2t\alpha_j(x)} - 1}\prod_{j'\neq j}\frac{1}{1 - e^{-2t\alpha_{j'}(x)}}\right) k^n + o(k^n)$$

uniformly in $x \in X$ and in $t \in (0, k^{2\epsilon})$. Set $\alpha_0 := 2\inf_{j,x}\alpha_j(x) > 0$. Then $e_k^1(t, x)$ is uniformly bounded above by $Ce^{-\alpha_0 t}k^n$ for some real number $C > 0$. Letting $t = k^{\epsilon}$, we get

$$e_k^0(k^{\epsilon}/2, x)\int_X e_k^1(k^{\epsilon}/2, x)\mathrm{d}V \le C'e^{-\alpha_0 k^{\epsilon}}k^{2n}$$

which converges to 0 uniformly in $x \in X$ when $k \to \infty$. Thus by (3.2.9) we have

$$\sum_{\lambda>0} e^{-k^{\epsilon}\lambda}\|\psi_\lambda(x)\|_{k,0}^2 \to 0 \quad \text{when } k \to \infty \tag{3.2.10}$$

uniformly in $x \in X$.

Now let $t = k^{\epsilon}$ in (3.2.5). Then Theorem 3.1.6 immediately follows from (3.2.6) and (3.2.10).

3.2.4 Application to a lower bound of the smallest non-zero eigenvalue

Lemma 3.2.7. *Let μ_k be the smallest non-zero eigenvalue of $\overline{\square}_k^0$ on X. Then*

$$\liminf_k \mu_k \geq \alpha_0$$

where $\alpha_0 := 2\inf_{j,x}\alpha_j(x) > 0$ for the eigenfunctions $\alpha_1,\dots,\alpha_n$ of $c_1(L,\|\cdot\|)$ with respect to the Hermitian metric on X.

Proof. By (3.2.8), we have $e^{-t\mu_k} < \int_X e_k^1(t,x)\mathrm{d}V$. By Theorem 3.2.5 with $q=1$, we get that $e_k^1(k^\epsilon,x)$ is uniformly bounded above in $x\in X$ by $Ce^{-\alpha_0 t}k^n$ for some real numbers $C>0$ and $\epsilon>0$ by the argument as above. Thus $e^{t\mu_k} < Ce^{-\alpha_0 k^\epsilon}k^n$. Taking the log of both sides and letting $k\to\infty$, we can conclude. □

3.3 L^2-existence

Let X be a connected Kähler manifold of dimension n with Kähler form ω, and let $\mathrm{d}V_\omega = \omega^n/n!$ be the volume form on X.

Let L be a line bundle on X, endowed with a smooth Hermitian metric $\|\cdot\|$.

3.3.1 Setup

Denote by $A_c^{p,q}(X,L)$ the space of compactly supported smooth global (p,q)-forms with values in L (*i.e.* global sections of $(T_X^{p,q})^*\otimes L$ which are compactly supported). The Hermitian metric on X and the Hermitian metric on L together induce a Hermitian metric on $(T_X^{p,q})^*\otimes L$ which we denote by $|\cdot|_{p,q}$. Then we can endow $A_c^{p,q}(X,L)$ with the L^2-norm

$$\|\sigma\|_{L^2} := \left(\int_X |\sigma(x)|_{p,q}^2\mathrm{d}V_\omega\right)^{1/2}, \qquad \forall\sigma\in A_c^{p,q}(X,L).$$

This norm defines a sesquilinear pairing $\langle\cdot,\cdot\rangle_{L^2}$ on $A^{p,q}(X,L)$.

Denote by $L^2_{p,q}(X,L)$ the completion of $A^{p,q}(X,L)$ with respect to the L^2-norm. It is a Hilbert space which contains $A_c^{p,q}(X,L)$ as a dense subset.

Let $\Lambda := \Lambda_\omega$ be the adjoint of the operator $\omega\wedge\colon A_c^{p,q}(X,L)\to A_c^{p+1,q+1}(X,L)$ with respect to the L^2-norm. Then we have a differential operator

$$A_\omega := [2\pi c_1(L,\|\cdot\|)\wedge,\Lambda] = 2\pi c_1(L,\|\cdot\|)\wedge\circ\Lambda - \Lambda\circ 2\pi c_1(L,\|\cdot\|)\wedge \tag{3.3.1}$$

on $A_c^{p,q}(X,L)$ for all $p,q\geq 1$.

Example 3.3.1. *Consider $X=\mathbb{C}^n$ with the standard metric, and $L=\mathcal{O}_X$ with the trivial metric (i.e. $(\mathcal{O}_X,\|\cdot\|)$ is the trivial Hermitian line bundle on $\mathbb{C}^n$). Then $\omega = 2\pi c_1(\mathcal{O}_X,\|\cdot\|) = \frac{\sqrt{-1}}{2}\sum_{j=1}^n \mathrm{d}z_j\wedge\mathrm{d}\overline{z}_j$. For each j, denote by $e_j\colon A_c^{p,q}(\mathbb{C}^n)\to A_c^{p+1,q}(\mathbb{C}^n)$ the operator $\mathrm{d}z_j\wedge$ (resp. $\overline{e}_j\colon A_c^{p,q}(\mathbb{C}^n)\to A_c^{p,q+1}(\mathbb{C}^n)$ the operator $\mathrm{d}\overline{z}_j\wedge$). Then their adjoints satisfy $e_j^*(\mathrm{d}z_J\wedge\mathrm{d}\overline{z}_{J'})=0$ if $j\notin J$ and $e_j^*(\mathrm{d}z_j\wedge\mathrm{d}z_J\wedge\mathrm{d}\overline{z}_{J'}) = 2\mathrm{d}z_J\wedge\mathrm{d}\overline{z}_{J'}$ (since the length of $\mathrm{d}z_j$ is 2), and $\overline{e}_j^*(\mathrm{d}z_J\wedge\mathrm{d}\overline{z}_{J'}) = 0$ if $j\notin J'$ and $\overline{e}_j^*(\mathrm{d}\overline{z}_j\wedge\mathrm{d}z_J\wedge\mathrm{d}\overline{z}_{J'}) = 2\mathrm{d}z_J\wedge\mathrm{d}\overline{z}_{J'}$. In this case, $\omega\wedge = \frac{\sqrt{-1}}{2}\sum e_j\overline{e}_j$ and $\Lambda = -\frac{\sqrt{-1}}{2}\sum\overline{e}_j^*e_j^*$. Thus $A_\omega = \frac{1}{4}\sum(\overline{e}_j\overline{e}_j^* - e_j^*e_j)$.*

Also we have $\overline{\partial} = \sum\overline{\partial}_j\overline{e}_j = \sum\overline{e}_j\overline{\partial}_j$, where $\overline{\partial}_j(\sum f_{JJ'}\mathrm{d}z_J\wedge\mathrm{d}\overline{z}_{J'}) = \sum\frac{\partial f_{JJ'}}{\partial\overline{z}_j}\mathrm{d}z_J\wedge\mathrm{d}\overline{z}_{J'}$. Then $\overline{\partial}^* = -\sum\partial_j\overline{e}_j^*$.

We need to extend the differential operators $\overline{\partial}$ and A_ω to $L^2_{p,q}(X,L)$. First, notice that A_ω extends to an operator on the whole $L^2_{p,q}(X,L)$ because both $2\pi c_1(L,\|\cdot\|)\wedge$ and Λ_ω do. Next, the differential operator $\overline{\partial}\colon A^{p,q}_{\mathrm{c}}(X,L)\to A^{p,q+1}_{\mathrm{c}}(X,L)$ then has an adjoint $\overline{\partial}^*\colon A^{p,q+1}_{\mathrm{c}}(X,L)\to A^{p,q}_{\mathrm{c}}(X,L)$ with respect to the L^2-norm. Let $\mathrm{dom}\overline{\partial}\subseteq L^2_{p,q}(X,L)$ consist of those u for which $\overline{\partial}u$, computed in the sense of distribution (*i.e.* using $\langle\overline{\partial}u,v\rangle_{L^2}:=\langle u,\overline{\partial}^*v\rangle_{L^2}$ for all $v\in A^{p,q+1}_{\mathrm{c}}(X,L)$), is in $L^2_{p,q+1}(X,L)$.

3.3.2 Classical L^2-existence

Theorem 3.3.2 (Classical L^2-existence)**.** *Assume X is geodesic complete for the Riemannian metric determined by ω.*

Assume that the operator A_ω is positive definite everywhere in $L^2_{p,q}(X,L)$. Assume $p\geq 0$, $q\geq 1$ and $u\in L^2_{p,q}(X,L)$ satisfies $\overline{\partial}u=0$ (in the sense of distributions) and $\langle A_\omega^{-1}u,u\rangle_{L^2}<\infty$.

Then there exists $f\in L^2_{p,q-1}(X,L)$ such that $\overline{\partial}f=u$ and $\|f\|^2_{L^2}\leq\langle A_\omega^{-1}u,u\rangle_{L^2}$.

We shall assume the following lemma, which is an easy application of the Bochner–Kodaira–Nakano identity (which itself is an easy computation via the Hodge identities).

Lemma 3.3.3. *For any $v\in A^{p,q}_{\mathrm{c}}(L)$ with $q\geq 1$, we have*

$$\|\overline{\partial}v\|^2_{L^2}+\|\overline{\partial}^*v\|^2_{L^2}\geq\langle A_\omega v,v\rangle_{L^2}.$$

Proof of Theorem 3.3.2. Both $\mathrm{Ker}\overline{\partial}$ and $\mathrm{Im}\overline{\partial}^*$ are closed subspaces of $L^2_{p,q}(X,L)$. General theory of Hilbert spaces gives the orthogonal decomposition $L^2_{p,q}(X,L)=\mathrm{Ker}\overline{\partial}\oplus\mathrm{Im}\overline{\partial}^*$.

Denote for simplicity by $C:=\langle A_\omega^{-1}u,u\rangle_{L^2}<\infty$. Consider the linear functional

$$\mathrm{Im}\overline{\partial}^*\subseteq L^2_{p,q}(X,L)\to\mathbb{C},\qquad \overline{\partial}^*v\mapsto\langle v,u\rangle_{L^2}. \tag{3.3.2}$$

We shall prove that the norm of this linear functional is bounded by $\sqrt{C}$, *i.e.*

$$\frac{|\langle v,u\rangle_{L^2}|^2}{\|\overline{\partial}^*v\|^2_{L^2}}\leq C\qquad\text{for all } v\in\mathrm{dom}\overline{\partial}^*. \tag{3.3.3}$$

We start with $v\in A^{p,q+1}_{\mathrm{c}}(X,L)$, and write $v=v_1+v_2$ according to the decomposition $L^2_{p,q}(X,L)=\mathrm{Ker}\overline{\partial}\oplus\mathrm{Im}\overline{\partial}^*$. Then Lemma 3.3.3 applied to v_1 implies

$$\|\overline{\partial}^*v\|^2_{L^2}=\|\overline{\partial}^*v_1\|^2_{L^2}\geq\langle A_\omega v_1,v_1\rangle_{L^2}.$$

On the other hand, Cauchy–Schwarz yields

$$|\langle v,u\rangle_{L^2}|^2=|\langle v_1,u\rangle_{L^2}|^2\leq\langle A_\omega v_1,v_1\rangle_{L^2}\langle A_\omega^{-1}u,u\rangle_{L^2}.$$

Thus (3.3.3) holds true for all $v\in A^{p,q+1}_{\mathrm{c}}(X,L)$.

To claim (3.3.3) for all $v\in\mathrm{dom}\overline{\partial}^*$, we need to use the geodesic completeness of ω. Indeed, under this assumption, the Andreotti–Vesentini lemma says that $A^{p,q+1}_{\mathrm{c}}(X,L)$ is dense in the graph norm of $\overline{\partial}^*$, and hence we can conclude for (3.3.3).

Thus we can apply the Riesz representation theorem to the *continuous* linear functional (3.3.2) to conclude that (3.3.2) is represented by an element $f\in L^2_{p,q-1}(X,L)$ of L^2-norm $\leq\sqrt{C}$, *i.e.* $\langle v,u\rangle_{L^2}=\langle\overline{\partial}^*v,f\rangle_{L^2}$ for all $v\in\mathrm{dom}\overline{\partial}^*$. Therefore $\overline{\partial}f=u$ as distributions. We are done. □

3.3.3 Hörmander's L^2-existence theorem

Theorem 3.3.4. *Assume X carries a Kähler form $\widehat{\omega}$ such that X is geodesic complete for the Riemannian metric determined by $\widehat{\omega}$.*

Assume $c_1(L, \|\cdot\|) > 0$. Assume $q \geq 1$ and $u \in L^2_{n,q}(X, L)$ satisfies $\overline{\partial} u = 0$ (in the sense of distributions) and $\langle A_\omega^{-1} u, u\rangle_{L^2} < \infty$.

Then there exists $f \in L^2_{n,q-1}(X, L)$ such that $\overline{\partial} f = u$ and $\|f\|^2_{L^2} \leq \langle A_\omega^{-1} u, u\rangle_{L^2}$.

Remark 3.3.5. *(i) A particularly important case for which X carries such a complete Kähler form $\widehat{\omega}$ is as follows: $X = X' \setminus Z$ where X' is a compact Kähler manifold and Z be an analytic subvariety.*

(ii) Since $c_1(L, \|\cdot\|) > 0$, locally on X we can find a suitable complex coordinate $(z_1, \ldots, z_n)$ of X such that: (i) $\omega = \frac{\sqrt{-1}}{2}\sum_{j=1}^n \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$, (ii) the $(1,1)$-form $c_1(L, \|\cdot\|)$ equals $\frac{\sqrt{-1}}{2}\sum_{j=1}^n \alpha_j(x)\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$ with $\alpha_j(x) > 0$. By the computation from Example 3.3.1, we have then $A_\omega = \frac{\pi}{2}\sum_j \alpha_j(\overline{e}_j\overline{e}_j^* - e_j^* e_j)$, which simplifies to $\frac{\pi}{2}\sum_j \alpha_j \overline{e}_j\overline{e}_j^*$ for (n,q)-forms. Thus A_ω is positive definite.

With this observation, we shall reduce Theorem 3.3.4 to Theorem 3.3.2 using the following monotonicity result.

Proposition 3.3.6 (Monotonicity)**.** *Assume X has two Kähler metrics ω, ω' such that $\omega' \geq \omega$ pointwise. Then for any positive $(1,1)$-form β, we have*

$$|u|^2_{\omega'}\mathrm{d}V_{\omega'} \leq |u|^2_\omega \mathrm{d}V_\omega, \quad \langle[\beta, \Lambda_{w'}]u, u\rangle_{L^2,\omega'}\mathrm{d}V_{\omega'} \geq \langle[\beta, \Lambda_w]u, u\rangle_{L^2,\omega}\mathrm{d}V_\omega$$

for all (n,q)-form u.

Here is a brief explanation on the proof of Proposition 3.3.6. The conclusion can be checked locally, and hence it suffices to check for $X = \mathbb{C}^n$, ω the standard Kähler form, and $\omega' = (\sqrt{-1}/2)\sum \gamma_j \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$ for $\gamma_j \geq 1$. The proof is then a direct computation.

Proof of Theorem 3.3.4. For every $\epsilon > 0$, set $\omega_\epsilon := \omega + \epsilon\widehat{\omega}$. Since $c_1(L, \|\cdot\|) \geq 0$, we can apply Proposition 3.3.6 to $\beta = 2\pi c_1(L, \|\cdot\|)$ and to ω and $\omega' = \omega_\epsilon$ to get that u is L^2 with respect to ω_ϵ and

$$\langle A_{\omega_\epsilon}^{-1} u, u\rangle_{L^2,\omega_\epsilon} \leq \langle A_\omega^{-1} u, u\rangle_{L^2,\omega}.$$

It is known that ω_ϵ is complete. The argument of Remark 3.3.5.(ii) shows that A_{ω_ϵ} is positive definite. Thus we can apply Theorem 3.3.2 to the Kähler manifold (X, ω_ϵ). So we obtain an $f_\epsilon \in L^2_{n,q-1}(X, L)$ (with L^2 with respect to ω_ϵ) satisfying $\overline{\partial} f_\epsilon = u$ and $\|f_\epsilon\|^2_{L^2,\omega_\epsilon} \leq \langle A_{\omega_\epsilon}^{-1} u, u\rangle_{L^2,\omega_\epsilon}$. In particular, the family (f_ϵ) is locally bounded in the L^2-norm, and hence we can extract a weal limit f in L^2_{loc} (locally L^2-coefficients), which is the required f. □

3.3.4 Weighted L^2-existence

To prove the L^2-extension theorem in the next section, we need a fancier version of Hörmander's L^2-existence theorem by introducing weights on the operator A_ω. Let us explain this.

Let $\eta, \lambda \colon X \to \mathbb{R}_{>0}$ be smooth functions. Define

$$B_{\eta,\lambda,\omega} := [\big(\eta 2\pi c_1(L, \|\cdot\|) - \sqrt{-1}\partial\overline{\partial}\eta - \sqrt{-1}\lambda^{-1}\partial\eta \wedge \overline{\partial}\eta\big) \wedge,\ \Lambda_\omega]. \tag{3.3.4}$$

Theorem 3.3.7. *Assume X carries a Kähler form $\widehat{\omega}$ such that X is geodesic complete for the Riemannian metric determined by $\widehat{\omega}$.*

Assume that the $(1,1)$-form $\eta 2\pi c_1(L, \|\cdot\|) - \sqrt{-1}\partial\overline{\partial}\eta - \sqrt{-1}\lambda^{-1}\partial\eta \wedge \overline{\partial}\eta$ is positive.

Assume $q \geq 1$ and $u \in L^2_{n,q}(X, L)$ satisfies $\overline{\partial}u = 0$ (in the sense of distributions) and $\langle B^{-1}_{\eta,\lambda,\omega} u, u\rangle_{L^2} < \infty$.

Then there exists $f \in L^2_{n,q-1}(X, L)$ such that $\overline{\partial}f = u$ and

$$\left\| \frac{f}{\sqrt{\eta + \lambda}} \right\|^2_{L^2} \leq 2 \left\| \frac{f}{\eta^{1/2} + \lambda^{1/2}} \right\|^2_{L^2} \leq 2\langle B^{-1}_{\eta,\lambda,\omega} u, u\rangle_{L^2}.$$

The proof follows the same line as Theorem 3.3.4. The extra information needed is the following estimate: For all (n, q)-forms u, we have

$$\langle B^{-1}_{\eta,\lambda,\omega} u, u\rangle_{L^2} \leq \|(\eta^{1/2} + \lambda^{1/2})\overline{\partial}^* u\|^2_{L^2} \|\eta^{1/2}\overline{\partial}u\|^2_{L^2}.$$

3.4 L^2-extension