A Concise Course in Arakelov Geometry

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Preface

The rudiment of Arakelov's theory comes from discussing the volume of lattices of number fields, using Minkowski's theory. By some non-geometric technologies, we can prove the Serre duality theorem and the Riemann-Roch theorem of a ring of algebraic integers (geometrically speaking, this is a 1-dimensional curve), but these formulas come from algebraic geometry. This implies that our algebraic number theory should have a geometric interpretation, and its high-dimensional version should contain more abundant arithmetic information. This geometry was later called Arakelov geometry.

The serious Arakelov theory of surfaces was established by Arakelov [Arak] in 1974. It is a kind of intersection theory connecting algebraic geometry and complex geometry. Then, in 1984, Faltings gave an arithmetic Riemann-Roch formula using Arakelov's theory [Falt], this showed that Arakelov geometry is powerful.

In 1990, Gillet and Soulé developed the Arakelov intersection theory on general arithmetic varieties [GiS1], and in 1992, they extended the arithmetic Riemann-Roch formula by using this theory [GiS2]. In 2008, their subsequent work with Rössler [GRS], proved the formula in the case of higher degrees.

We assume that the readers are familiar with algebraic geometry, differential geometry and algebraic number theory. Although this note does not presuppose knowledge of complex geometry, it is better if you master it.

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Surfaces and Arakelov Theory

In this chapter, we introduce the Arakelov theory of surfaces developed by Arakelov and Faltings.

2.1 Riemann Surfaces

Let X be a compact Riemann surface.

Definition 2.1.1 (Weil Functions)**.** Let $D = \{(U, f)\}$ be a Cartier divisor on X. A **Weil function** associated with D is a map

$$\lambda_D : X \setminus \mathrm{Supp}(D) \to \mathbb{R},$$

such that for every $P \in U \setminus \mathrm{Supp}(D)$, $\lambda_D(P) = -\log|f(P)| + \alpha(P)$ for some smooth function $\alpha : U \to \mathbb{R}$.

The function α here will be viewed as a metric. Note that $\lambda_D(P) = \infty$ is not well-defined when $P \in \mathrm{Supp}(D)$.

Definition 2.1.2 (Néron Functions)**.** Let $D = \{(U, f)\}$ be a Cartier divisor on X, consider the triple (U, f, α) where $\alpha : U \to \mathbb{R}$ is smooth. We say two triples (U, f, α) and (V, g, β) are compatible, if

- $(U, f), (V, g) \in D$. This implies $f/g \in \mathscr{O}_X(U \cap V)^\times$.
- $-\log|f/g| = \beta - \alpha$ holds on $U \cap V$.

A maximal family of compatible triples is called a **Néron divisor**, denoted by $D = \{(U, f, \alpha)\}$. All Néron divisors form an abelian group via

$$(U, f, \alpha) \cdot (V, g, \beta) := (U \cap V, (fg)|_{U \cap V}, (\alpha + \beta)|_{U \cap V}).$$

For a Néron divisor $D = \{(U, f, \alpha)\}$, define the Weil function associated with D to be

$$\lambda_D(P) := -\log|f(P)| + \alpha(P), \quad P \in U \setminus \mathrm{Supp}(D).$$

Néron divisors can be viewed as "metrized" Cartier divisors.

Exercise 2.1.3. *λ_D is independent of the choice of triple.*

Recall that there is a natural way to identify line bundles (or invertable sheaves) on X with Cartier (or Weil) divisors on X. Our goal is to make this correspondence metrically. Let us first review some geometric operations.

Let $\mathscr{L}$ be a line bundle on X, i.e. $X = \bigcup_i U_i$ such that for each i, $\phi_i : \mathscr{L}|_{U_i} \xrightarrow{\sim} \mathscr{O}_X|_{U_i}$ is an isomorphism of $\mathscr{O}_X|_{U_i}$-module, and satisfies

$$\begin{tikzcd}
\mathscr{L}|_{U_i\cap U_j} \arrow[r, "\mathrm{id}"] \arrow[d, "\phi_j"'] & \mathscr{L}|_{U_i\cap U_j} \arrow[d, "\phi_i"] \\
\mathscr{O}_X|_{U_i\cap U_j} \arrow[r, "\sim", "\phi_{ij}:=\phi_i\circ\phi_j^{-1}\in\mathscr{O}_X(U_i\cap U_j)^\times"'] & \mathscr{O}_X|_{U_i\cap U_j}
\end{tikzcd}$$

These functions ϕ_i are called **trivialization functions**. In particular, if $D = \{(U_i, f_i)\}$ is a Cartier divisor, then the line bundle associated with D

$$\mathscr{O}_X(D)(U) := \{f \in \mathscr{M}(U) : \mathrm{div}(f) + D \geq 0\}, \quad U \subseteq X$$

has a well-known trivialization

$$\big(U_i, f_i \times (\cdot) : \mathscr{O}_X(D)|_{U_i} \xrightarrow{\sim} \mathscr{O}_X|_{U_i}\big).$$

Definition 2.1.4 (Metrics)**.** Let $\mathscr{L}$ be a line bundle on X and has trivialization $\mathscr{L} = \{(U,\phi)\}$. Let $h : U \to \mathbb{R}_{>0}$ be smooth functions, consider the triples (U,ϕ,h). We say two triples (U,ϕ,h) and (V,ψ,m) are compatible, if

$$h(P) = |\phi \circ \psi^{-1}(P)|^2 \cdot m(P), \quad P \in U \cap V.$$

A maximal family of compatible triples is called a **metric** on $\mathscr{L}$, denoted by $(\mathscr{L},h)$. We also call it a **metrized line bundle**.

Remark 2.1.5. Let $\mathscr{L} = \{(U,\phi)\}$ be a line bundle on X. Let $s \in \Gamma(U,\mathscr{L})$ be a section, for $P \in U$, define a norm (hence induces a Hermitian inner product) on the one dimensional $\mathbb{C}$-linear space $\mathscr{L}_P$ (i.e. the fiber of $\mathscr{L}$ at P) to be

$$\|s(P)\|_h := \frac{|\phi_P(s(P))|}{\sqrt{h(P)}}.$$

This number does not depend on the choice of trivialization. It is easy to see that h and $\|\cdot\|_h$ are determine each other, so we will abuse them.

Remark 2.1.6. There are many ways to construct new metrized line bundles.

- Let $(\mathscr{L},h) = \{(U,\phi,h)\}$ be a metrized line bundle on X, define the **dual bundle** $(\mathscr{L}^{-1},h^{-1}) := \{(U,\phi^{-1},h^{-1})\}$.
- Let $(\mathscr{L},h) = \{(U,\phi,h)\}$, $(\mathscr{M},m) = \{(U,\psi,m)\}$ be metrized line bundles on X, define the **tensor bundle** $(\mathscr{L}\otimes\mathscr{M}, h\cdot m) := \{(U,\phi\cdot\psi,h\cdot m)\}$.

- Let $f: X \to Y$ be a morphism and $(\mathscr{L}, h) = \{(U, \phi, h)\}$ be a metrized line bundle on Y, then the **pull-back bundle** $f^*\mathscr{L}$ has a metric defined by $(f^*\mathscr{L}, h \circ f) = \{(f^{-1}U, f^\sharp \circ \phi, h \circ f)\}$, where $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$ and the trivialization $f^*\mathscr{L} \to \mathscr{O}_X$ comes from the adjoint pair (f^*, f_*).

Proposition 2.1.7. *Let $D = \{(U, f)\}$ be a Cartier divisor on X, then there is a one-to-one correspondence:*

$$\{\textit{metrics on } \mathscr{O}_X(D)\} \longleftrightarrow \{\textit{Weil functions associated with } D\},$$

given by $h \longmapsto (-\log|f| + \frac{1}{2}\log h)$; $(\lambda_D = -\log|f| + \alpha) \longmapsto e^{2\alpha}$.

Proof. Let $s = (U, s|_U \in \mathscr{M}(U))$ be a meromorphic global section of $\mathscr{O}_X(D)$ such that $\operatorname{div}(s) = D$, we already know there is a natural trivialization

$$f \times (\cdot) : \mathscr{O}_X(D)|_U \xrightarrow{\sim} \mathscr{O}_X|_U, \quad s|_U \mapsto f \cdot s|_U.$$

For a metric h on $\mathscr{O}_X(D)$, define a function associated with s by

$$\lambda_{h,s}(P) := -\log\|s(P)\|_h = -\log\frac{|f(P)\cdot s|_U(P)|}{\sqrt{h(P)}} = -\log|f(P)\cdot s|_U(P)| + \frac{1}{2}\log h(P).$$

Now take $s = 1_D$ and suppose $(\operatorname{Supp}(D)^c, 1) \in D$, then the log terms vanish, one can verify the bijection easily. □

In fact, the metric on a vector bundle reflects some geometrical and topological information of this bundle. We now introduce the Chern form of a metric, which can be viewed as an important characteristic class in the cohomology group. This class can be obtained from curvature in differential geometry.

But first, let us recall some notations. Let $z = x + iy$ be a local complex coordinate. Define the differential operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right); \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

For a smooth function f, define $\partial f := \frac{\partial f}{\partial z}dz \in \mathscr{A}^{1,0}$, $\overline{\partial} f := \frac{\partial f}{\partial \bar{z}}d\bar{z} \in \mathscr{A}^{0,1}$ and $d := \partial + \overline{\partial} \in \mathscr{A}^1$, $d^c := \frac{1}{4\pi i}(\partial - \overline{\partial}) \in \mathscr{A}^1$.

Exercise 2.1.8. *Prove:*

$$\partial\overline{\partial} = -\overline{\partial}\partial = -2\pi i dd^c = -\frac{i}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)dx \wedge dy \in \mathscr{A}^{1,1}.$$

Remark 2.1.9. This is a warning. Let (X, g) be a n dimensional projective complex manifold with Kähler metric g and its volume form $\mathrm{vol}_g \in \mathscr{A}^{2n}$. It is unreasonable to use Exercise 2.1.8 to define the Laplacian on X, since $\sum_{i=1}^{2n}\frac{\partial^2}{\partial x_i^2}$ can not carry metric information and may not glue into a global operator. But from linear algebra, g induces a Hermitian inner product $\widetilde{g}$ on $\bigwedge^k T^*X$, the space of k-forms, $0 \le k \le 2n$. Now define the L^2-scalar product

$$\langle\cdot,\cdot\rangle_g : \mathscr{A}^k \times \mathscr{A}^k \to \mathbb{C} \sqcup \{\infty\}, \quad \langle\omega,\eta\rangle_g \mapsto \int_X \widetilde{g}(\omega,\eta)\cdot \mathrm{vol}_g.$$

If we write the right adjoint of d for $\langle\cdot,\cdot\rangle_g$ as d^*, one can define the Laplacian $\Delta_{\mathrm{dR}} := dd^* + d^*d$, called the **Laplace-de Rham operator**. In Euclidean plane, this Laplacian coincides with the ordinary one (up to a sign).

Definition 2.1.10 (Chern Forms)**.** Let $(\mathscr{L},h) = \{(U,\phi,h)\}$ be a metrized line bundle on X, let s be a holomorphic section on U. Define the **Chern form** of $(\mathscr{L},h)$ to be

$$c_1(\mathscr{L},h) := dd^c \log h(z) = -dd^c \log \|s(z)\|_h^2, \quad z \in U \setminus \mathrm{Supp}(\mathrm{div}(s)).$$

Its cohomology class in the de Rham cohomology group $H^2(X,\mathbb{R})$ is called **Chern class**, also denoted by $c_1(\mathscr{L},h)$.

Since the transition functions are holomorphic non-zero, it follows that one can glue $c_1(\mathscr{L},h)$ into a global form in $\mathscr{A}^{1,1}$.

Remark 2.1.11. In complex geometry, let $\mathscr{E}$ be a Hermitian vector bundle on a complex manifold X. For each $P \in X$, the fiber $\mathscr{E}_P$ is a finite dimensional $\mathbb{C}$-linear space and has a Hermitian inner product

$$\langle\cdot,\cdot\rangle_P : \mathscr{E}_P \times \mathscr{E}_P \to \mathbb{C}.$$

Suppose $\mathscr{E}$ has a frame $\{e_i\}$ composed of global sections. There are some important matrices:

- The metric matrix $H := [\langle e_i, e_j\rangle] \in \mathrm{Mat}_n^{\text{0-form}}$. It is not hard to see $H = \overline{H}^T$.
- The connection matrix $W \in \mathrm{Mat}_n^{\text{1-form}}$. Let

  $$\nabla : \Gamma(U,\mathscr{E}) \to \Gamma(U, T^*X \otimes \mathscr{E}) \cong \Gamma(U, \mathrm{Hom}(TX,\mathscr{E}))$$

  be the connection induced by H, then $W := [w_{ij}]$ is defined by $\nabla e_j = \sum_{i=1}^n w_{ij} e_i$. One can show that $W = \overline{H}^{-1}\partial(\overline{H})$.
- The curvature matrix $\Omega = dW + W \wedge W = \overline{\partial}(\partial(H)\cdot H^{-1})^T \in \mathrm{Mat}_n^{\text{2-form}}$, by Bianchi identity.

If $\mathscr{E}$ is a line bundle, then $\Omega = -\partial\overline{\partial}(\log H)$. This explains why we define Chern forms in a strange expression.

Proposition 2.1.12. *Let $(\mathscr{L},h)$ be a metrized line bundle on X, then*

$$\int_X c_1(\mathscr{L},h) = \deg(\mathscr{L}).$$

Proof. Let s be a meromorphic section, so $c_1(\mathscr{L},h) = -dd^c \log\|s(z)\|_h^2$ outside the support of $\mathrm{div}(s)$. At each point P where s has a zero or pole, we pute a small circle $C(P,r)$ of radius r. Represent $\|s(z)\|_h^2 = f\overline{f}g$ where f is meromorphic at P and g is smooth positive, apply Stokes' formula we have

$$\int_X c_1(\mathscr{L},h) = \lim_{r\to 0}\sum_P \int_{C(P,r),\curvearrowright} d^c \log\|s(z)\|_h^2 = \lim_{r\to 0}\sum_P \int_{C(P,r),\curvearrowright} \frac{\partial-\overline{\partial}}{4\pi i}(\log f + \log\overline{f} + \log g).$$

the $\log g$ term is bounded locally so the integral of this term tends to 0. Obviously, $\overline{\partial}\log f = \partial \log \overline{f} = 0$, so the integral becomes

$$\int_X c_1(\mathscr{L},h) = \frac{1}{4\pi i}\lim_{r\to 0}\sum_P 2i\mathrm{Im}\int_{C(P,r),\curvearrowleft}\frac{f'}{f}dz = \sum_P \mathrm{ord}_P(f) = \deg(\mathscr{L}),$$

as desired. □

Exercise 2.1.13. *Let $[x:y]\in\mathbb{P}^1(\mathbb{C})$, define a metric*

$$h:\mathbb{C}^\times\to\mathbb{R}_{>0},\quad [x:y]\mapsto |x|^2+|y|^2.$$

- *If $x\neq 0$, then $z:=[x:y]=[1:\frac{y}{x}]\in\mathbb{C}$. Show the metric h in this coordinate is $z\mapsto 1+|z|^2$.*
- *Find the line bundle where h lives in.*
- *If we consider $dd^c\log h\in\mathscr{A}^{1,1}$ as a measure, this defines a Hermitian metric on $\mathbb{P}^1(\mathbb{C})$, called the* ***Fubini-Study metric****. Find the curvature of this metric and which line bundle where the metric lives in (recall the degree of holomorphic tangent bundle on $\mathbb{P}^1(\mathbb{C})$ is 2).*

2.2 Green Functions and Metrics

Let X be a compact Riemann surface. In this section, we study a special case of Weil functions on X, which are Green functions.

If the genus g of X is bigger than 0, define the **canonical volume form** on X to be

$$\mu := \frac{i}{2g}\sum_{k=1}^{g}\omega_k\wedge\overline{\omega_k}\in\mathscr{A}^{1,1},$$

where $\omega_1,\cdots,\omega_g$ are orthonormal basis for the Hermitian inner product

$$\Gamma(X,\Omega^1_{X/\mathbb{C}})\times\Gamma(X,\Omega^1_{X/\mathbb{C}})\to\mathbb{C},\quad \langle\omega,\eta\rangle\mapsto\frac{i}{2}\int_X\omega\wedge\overline{\eta}.$$

One can check that $\int_X\mu = 1$.

Definition 2.2.1 (Green Functions)**.** A **Green function (of logarithmic type) with respect to μ** is a function $g:X\times X\to\mathbb{R}$ smooth outside the diagonal $\Delta(X)\subseteq X\times X$, and satisfying the following conditions:

Fix a point $P\in X$,

- Any affine open neighbourhood U of P with local coordinate z, we have

$$g(P,z) = -\log|z-P|^2 + \text{real smooth function in } z,\quad z\in U\setminus\{P\}.$$

- For all points $z\neq P$,

$$\partial\overline{\partial}g(P,z) = -2\pi i\mu.$$

- $\int_X g(P,z)\mu = 0$.

One can prove that the Green function exists uniquely.

Remark 2.2.2. In the case of genus $g=0$, i.e. $X=\mathbb{P}^1(\mathbb{C})$ a Riemann sphere, define a Green function on $\mathbb{P}^1(\mathbb{C})$ in terms of the affine coordinates (z,w) by

$$g(z,w) := -\log \frac{|z-w|^2}{(1+z\bar{z})(1+w\overline{w})},$$

up to an appropriate additive constant. This function with respect to the Fubini-Study form

$$\mu := \frac{i}{2\pi}\frac{dz\wedge d\bar{z}}{(1+z\bar{z})^2} \in \mathscr{A}^{1,1}.$$

There is an important formula:

Proposition 2.2.3. *Let X be a compact Riemann surface. For all smooth real-valued functions f on X,*

$$\int_X g(P,z)dd^c f + f(P) = \int_X f\mu.$$

Exercise 2.2.4. *Let $C(P,r)$ be a neighborhood of P of radius r, show that:*

- *If $g\in\mathscr{C}^\infty(C(P,r))$ and $f=\gamma\log h+\mathscr{C}^\infty$-function for some constant γ, then*

$$\lim_{r\to 0}\int_{C(P,r),\curvearrowleft} f d^c g = 0.$$

- *If $g=\log h^2+\mathscr{C}^\infty$-function and f is continuous, then*

$$\lim_{r\to 0}\int_{C(P,r),\curvearrowleft} f d^c g = f(P).$$

Proof. (of Proposition). Write $g_P := g(P,\cdot)$, we calculate directly

$$\begin{aligned}
\int_X (g_P dd^c f - f\mu) &= \int_X (g_P dd^c f - f dd^c g_P) && (\mu = dd^c g_P)\\
&= \int_X d(g_P d^c f - f d^c g_P) && (d\alpha\wedge d^c\beta = d\beta\wedge d^c\alpha)\\
&= \lim_{r\to 0}\int_{C(P,r),\curvearrowright}(g_P d^c f - f d^c g_P) && \text{(Stokes' formula)}\\
&= \lim_{r\to 0}\int_{C(P,r),\curvearrowleft} f d^c g_P - \lim_{r\to 0}\int_{C(P,r),\curvearrowleft} g_p d^c f && \text{(Exercise 2.2.4)}\\
&= -f(P),
\end{aligned}$$

as desired. □

Let $\omega\in\mathscr{A}^{p,q}$ $(0\le p,q\le 1)$, define some linear operators

$$[\omega] : \mathscr{A}^{1-p,1-q}\to\mathbb{R},\quad \eta\mapsto\int_X \omega\wedge\eta$$

and

$$\delta_P : \mathscr{C}^\infty \to \mathbb{R}, \quad f \mapsto f(P).$$

Define $dd^c[\omega](\eta) := -[\omega](dd^c\eta)$, write $g_P := g(P,\cdot)$, then the previous proposition can be expressed as the equality of operators (afterwards, they will be called currents):

$$dd^c[-g_p] + \delta_P = [\mu].$$

Exercise 2.2.5. *Let P_j $(j=1,2)$ be two different points, write $g_{P_j} := g(P_j,\cdot)$ are Green functions with respect to μ. Show*

$$g_{P_1}(P_2) - g_{P_2}(P_1) = \int_X (g_{P_1} dd^c g_{P_2} - g_{P_2} dd^c g_{P_1}) = 0.$$

It reminds us to consider the compact complex manifold $X \times X$ and the diagonal divisor $\Delta(X)$. One can choose an appropriate metric h on $\mathscr{O}_{X\times X}(\Delta(X))$ such that if s is a section of $\mathscr{O}_{X\times X}(\Delta(X))$ with $\mathrm{div}(s) = \Delta(X)$, then $-\log\|s\|_h^2$ is the Green function with respect to μ on X. This means that in the neighborhood of $\Delta(X)$ in $X \times X$, on has the expansion

$$g(z,w) = -\log|z-w|^2 + \text{real analytic function in } (z,w).$$

So we can study the analytic properties of Green functions locally in $X \times X$.

Remark 2.2.6. The Green functions can be used to define metrics on line bundles on X, under the requirements of Proposition 2.1.12 since Green functions are special Weil functions.

- We first consider the case of degree one line bundle $\mathscr{O}_X(P)$ for some prime divisor $P \in X$. Let 1_P be a meromorphic section of $\mathscr{O}_X(P)$ which is constant outside P, due to Proposition 2.1.7 one can define

$$\|1_P(z)\| := \exp\left(-\frac{1}{2}g(P,z)\right), \quad z \neq P.$$

- For the case of general line bundle

$$\mathscr{O}_X(D) = \bigotimes_P \mathscr{O}_X(P)$$

where $D = \sum_P P$. Let 1_D be a meromorphic section of $\mathscr{O}_X(D)$ which is constant outside D, due to Remark 2.1.6 one can define

$$\|1_D(z)\| := \exp\left(-\frac{1}{2}\sum_P g(P,z)\right), \quad z \neq \mathrm{Supp}(D).$$

We usually write $\sum_P g(P,z)$ as $g(D,z)$.

These metrics are derived from the Green function and are the metrics used in Arakelov geometry. We will emphasize them in the following sections.

Exercise 2.2.7. *Under the assumption of Remark 2.2.6, verify Proposition 2.1.12.*

2.3 Arakelov Intersection Pairing

Let K be a number field and $\mathscr{O}_K$ its ring of integers. An **arithmetic variety** X is an integral, regular, projective, flat scheme over $\mathscr{O}_K$ with generic fiber $X_K = X \times_{\mathscr{O}_K} K$ is smooth (not necessarily connected) over K. So

$$X(\mathbb{C}) := \bigsqcup_{\tau \in \mathrm{Hom}(K,\mathbb{C})} X_\tau(\mathbb{C}), \quad \text{where } X_\tau(\mathbb{C}) := \text{complex points of } X_K \times_\tau \mathbb{C},$$

is a family of compact Riemann surfaces.

An arithmetic variety with Krull dimension 2 will be called an **arithmetic surface**. The Arakelov theory of arithmetic surfaces is important because it can be calculated directly and used as important examples.

Definition 2.3.1 (Arakelov Divisors)**.** Let X be an arithmetic surface. Define the group of **Arakelov divisors** on X is the group

$$\widehat{\mathrm{Div}}(X) := \mathrm{Div}(X) \oplus \left(\bigoplus_{\tau \in \mathrm{Hom}(K,\mathbb{C})} \mathbb{R} \cdot X_\tau(\mathbb{C}) \right)^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})},$$

where $\mathrm{Div}(X)$ denotes the group of Weil divisors on X and the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on the infinite part by $\tau \mapsto \overline{\tau}$. Thus, an Arakelov divisor on X is an expression of the type $D = D_{\mathrm{fin}} + D_{\mathrm{inf}}$.

Definition 2.3.2 (Principal Arakelov Divisors)**.** Let $f \in k(X)^\times$. We associate an Arakelov divisor to f in the following way

$$\widehat{\mathrm{div}}(f) := (f)_{\mathrm{fin}} + (f)_{\mathrm{inf}},$$

where $(f)_{\mathrm{fin}}$ is the principal Weil divisor $\mathrm{div}(f)$ associated with f and

$$(f)_{\mathrm{inf}} := \sum_{\tau \in \mathrm{Hom}(K,\mathbb{C})} \left(-\frac{g_\tau(\mathrm{div}(f_\tau), z)}{2} - \log|f_\tau(z)| \right) \cdot X_\tau(\mathbb{C}),$$

where g_τ is the unique Green function on $X_\tau(\mathbb{C})$ with respect to the canonical volume form μ_τ on $X_\tau(\mathbb{C})$ invariant under $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, and f_τ is the pull-back of f by $X_\tau(\mathbb{C}) \to X$.

Exercise 2.3.3. *Check the coefficients in the sum above*

$$\gamma_\tau(f) := -\frac{g_\tau(\mathrm{div}(f_\tau), z)}{2} - \log|f_\tau(z)|$$

are constant functions in z.

These principal Arakelov divisors form a subgroup of $\widehat{\mathrm{Div}}(X)$, the quotient group is denoted by $\widehat{\mathrm{CH}}^1(X, \{\mu_\tau\})$ or simply $\widehat{\mathrm{CH}}^1(X)$, called the **arithmetic Chow group** of X.

Arakelov proved that there exists an intersection theory on an arithmetic surface:

Theorem 2.3.4 (Arakelov). *Let X be an arithmetic surface defined over $\mathscr{O}_K$, the intersection number at a closed point $x \in X$ is denoted by $i_x(\cdot,\cdot)$. With these notations, there exists a unique symmetric bilinear pairing*

$$(\cdot,\cdot) : \widehat{\mathrm{Div}}(X) \times \widehat{\mathrm{Div}}(X) \to \mathbb{R},$$

satisfying the following conditions:

- (FINITE DIVISOR, FINITE DIVISOR): $($*D, vertical divisor E lies over a finite prime* $\mathfrak{p}) =$
$$\sum_{x|\mathfrak{p}} i_x(D,E)\log \# k(x).$$
- (FINITE DIVISOR, FINITE DIVISOR): $($*horizontal divisor D, horizontal divisor* $E) = (D,E)_{\mathrm{fin}} + (D,E)_{\mathrm{inf}}$, *where*
$$(D,E)_{\mathrm{fin}} = \sum_{\mathfrak{p}\in \mathrm{Spec}(\mathscr{O}_K)} \sum_{x|\mathfrak{p}} i_x(D,E)\log \# k(x); \quad (D,E)_{\mathrm{inf}} = \sum_{\tau \in \mathrm{Hom}(K,\mathbb{C})} \frac{1}{2} g_\tau(D_\tau, E_\tau).$$
- (FINITE DIVISOR, INFINITE DIVISOR): $($*horizontal divisor D,* $X_\tau(\mathbb{C})) = \deg(D)$.
- (FINITE DIVISOR, INFINITE DIVISOR): $($*vertical divisor D,* $X_\tau(\mathbb{C})) = 0$.
- (INFINITE DIVISOR, INFINITE DIVISOR): $(X_\tau(\mathbb{C}), X_\sigma(\mathbb{C})) = 0$.
- (PRINCIPAL DIVISOR, ANY DIVISOR): $($*principal Arakelov divisor,* $\cdot) = 0$. *Therefore the pair $\widehat{\mathrm{Div}}(X) \times \widehat{\mathrm{Div}}(X) \to \mathbb{R}$ defines a symmetric bilinear form on $\widehat{\mathrm{CH}}^1(X)$.*

Proof. (only prove the last item). For example, given a horizontal divisor $D = D_{\mathrm{fin}}$, write $D_\tau = \sum_{i=1}^{\deg(D)} P_{i,\tau}$ such that each $P_{i,\tau}$ is prime. We have

$$\begin{aligned}
&(\widehat{\mathrm{div}}(f), D)\\
=&((f)_{\mathrm{fin}}, D) + ((f)_{\mathrm{inf}}, D)\\
=&((f)_{\mathrm{fin}}, D)_{\mathrm{fin}} + ((f)_{\mathrm{fin}}, D)_{\mathrm{inf}} + \sum_{\tau\in\mathrm{Hom}(K,\mathbb{C})} \gamma_\tau(f)\deg(D)\\
=&\sum_{\mathfrak{p}\in\mathrm{Spec}(\mathscr{O}_K)} -\log\left(\prod_{x|\mathfrak{p}} \left|f|_D\right|_x\right) + \sum_{\tau\in\mathrm{Hom}(K,\mathbb{C})} \frac{g_\tau(\mathrm{div}(f_\tau), D_\tau)}{2} + \sum_{\tau\in\mathrm{Hom}(K,\mathbb{C})} \gamma_\tau(f)\deg(D)\\
=&\sum_{\mathfrak{p}\in\mathrm{Spec}(\mathscr{O}_K)} -\log\left|\mathrm{Nm}_{k(D)/K}(f|_D)\right|_{\mathfrak{p}} + \sum_{\tau\in\mathrm{Hom}(K,\mathbb{C})}\left(\frac{g_\tau(\mathrm{div}(f_\tau), D_\tau)}{2} + \sum_{i=1}^{\deg(D)} \gamma_\tau(f)\right)\\
=&\sum_{\tau\in\mathrm{Hom}(K,\mathbb{C})}\left(\log\left|\tau\left(\mathrm{Nm}_{k(D)/K}(f|_D)\right)\right| + \sum_{i=1}^{\deg(D)}\left(\frac{g_\tau(\mathrm{div}(f_\tau), P_{i,\tau})}{2} + \gamma_\tau(f)\right)\right)\\
=&\sum_{\tau\in\mathrm{Hom}(K,\mathbb{C})}\left(\log\left|\tau\left(\prod_{\varphi\in\mathrm{Gal}(k(D)/K)} \varphi(f|_D)\right)\right| - \log\left|\tau\left(\prod_{i=1}^{\deg(D)} f_\tau(P_{i,\tau})\right)\right|\right).
\end{aligned}$$

The last term is 0 by Galois theory. $\square$

Exercise 2.3.5. *Let $K=\mathbb{Q}, \mathscr{O}_K=\mathbb{Z}$, so there is a unique embedding $\tau:\mathbb{Q}\hookrightarrow\mathbb{C}$. Suppose $X=\mathbb{P}^1_{\mathbb{Z}}$, and the Green function g on $X_\tau(\mathbb{C})$ is given by Remark 2.2.2, let $D=\widehat{\operatorname{div}}(x^2+1)$, $E=$ the closed subscheme defined by a prime ideal $(x+2)$, show that:*

- $(D_{\text{fin}},E)_{\text{fin}}=i_{(5,x+2)}(x^2+1,x+2)\log 5=\log 5.$
- $(D_{\text{fin}},E)_{\text{inf}}=\frac{1}{2}(g(i,-2)+g(-i,-2)-2g(\infty,-2))=\log\frac{2}{5}.$
- $\gamma_\tau(x^2+1)=-\log 2$. *Hence the intersection number $(D,E)=0$.*

We can also identify Arakelov divisors with admissible metrized line bundles on an arithmetic surface, as in algebraic geometry. The **metrized line bundle** on an arithmetic variety X is a rank one locally free $\mathscr{O}_X$-module $\mathscr{L}$ together with a collection of non-trivial metrized line bundles $\left([X_\tau(\mathbb{C})\to X]^*\mathscr{L},\|\cdot\|_\tau\right)$ on compact complex manifolds $X_\tau(\mathbb{C})$, $\tau\in\operatorname{Hom}(K,\mathbb{C})$, and invariant under $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$.

Now let

$$D=D_{\text{fin}}+\sum_{\tau\in\operatorname{Hom}(K,\mathbb{C})} r_\tau\cdot X_\tau(\mathbb{C})$$

be an Arakelov divisor on an arithmetic surface. Define a metrized line bundle associated with D to be

$$\widehat{\mathscr{O}_X}(D):=\left(\mathscr{O}_X(D_{\text{fin}}),\left\{\left\|1_{D_{\text{fin}},\tau}(z)\right\|_\tau:=\exp\left(-\frac{1}{2}g_\tau(D_{\text{fin},\tau},z)-r_\tau\right)\right\}\right).$$

The most important invariant of metrized line bundles is the Arakelov degree map (here just only defined on the arithmetic curve, and of course, it can be generalized to higher dimensions using push-forward).

Definition 2.3.6 (Arithmetic Degree)**.** For a metrized line bundle $(\mathscr{L},\|\cdot\|_\tau)$ on $\operatorname{Spec}(\mathscr{O}_K)$, then $\mathscr{L}$ is a fractional ideal of $\mathscr{O}_K$. Take $0\neq s\in\mathscr{L}$, define

$$\widehat{\deg}(\mathscr{L},\|\cdot\|_\tau):=\log\#(\mathscr{L}/s\cdot\mathscr{O}_K)-\sum_{\tau\in\operatorname{Hom}(K,\mathbb{C})}\log\|s_\tau\|_\tau\in\mathbb{R},$$

where $s_\tau\in\mathscr{L}\otimes_\tau\mathbb{C}$ is the pull-back of s by τ. By product formula, this definition is independent of the choice of s.

Because of the correspondence between Arakelov divisors and "admissible" line bundles (admissible means the line bundle has form $\widehat{\mathscr{O}_X}(D)$ when D is an Arakelov divisor), one can transplant the intersection theory of divisors to the intersection theory of line bundles. Therefore, when discussing intersections on an arithmetic surface later, we will abuse divisors and admissible line bundles. For example,

Proposition 2.3.7. *Let $D=D_{\text{fin}}+D_{\text{inf}}$ be an Arakelov divisor on an arithmetic surface X defined over $\mathscr{O}_K$, and let E be a horizontal prime divisor has form* $\operatorname{Spec}(\mathscr{O}_{k(E)})$ *with residue field $k(E)$. Then $(D,E)=\widehat{\deg}(\widehat{\mathscr{O}_X}(D)|_E)$. The right hand side can be viewed as $(\widehat{\mathscr{O}_X}(D),E)$.*

Proof. Write $D_{\text{fin}} = \{(U,f)\}, D_{\text{inf}} = \sum_\tau r_\tau \cdot X_\tau(\mathbb{C})$, choose a special rational section $1_{D_{\text{fin}}}$, we can compute

$$\begin{aligned}
&\widehat{\deg}(\widehat{\mathscr{O}_X}(D)|_E) \\
=&\widehat{\deg}\left(\mathscr{O}_X(D_{\text{fin}})|_E, \left\{\left\|1_{D_{\text{fin}},\tau}(E_\tau)\right\|_\tau = \exp\left(-\frac{1}{2}g_\tau(D_{\text{fin},\tau}, E_\tau) - r_\tau \deg(E)\right)\right\}\right) \\
=&\log\#\big(\mathscr{O}_X(D_{\text{fin}})|_E \big/ \mathscr{O}_{k(E)}\big) - \sum_{\tau\in\mathrm{Hom}(k(E),\mathbb{C})} \log\|1_{D_{\text{fin}},\tau}(E_\tau)\|_\tau \\
=&\sum_{\mathfrak{p}\in E} \mathrm{ord}_{\mathfrak{p}}(f|_E)\log\#k(\mathfrak{p}) + \sum_{\tau\in\mathrm{Hom}(k(E),\mathbb{C})} \left(\frac{1}{2}g_\tau(D_{\text{fin},\tau}, E_\tau) + r_\tau \deg(E)\right) \\
=&(D_{\text{fin}}, E)_{\text{fin}} + (D_{\text{fin}}, E)_{\text{inf}} + (D_{\text{inf}}, E).
\end{aligned}$$

The last term is (D,E) by Theorem 2.3.4. $\square$

2.4 Adjunction Formula*

Recall that the classicial adjunction formula in algebraic geometry states that let $f: X \to Y, g: Y \to Z$ be quasi-projective local complete intersection (l.c.i, to abbreviate) morphisms, then we have a canonical isomorphism

$$\omega_{X/Z} \cong \omega_{X/Y} \otimes_{\mathscr{O}_X} f^*\omega_{Y/Z},$$

where ω are relative canonical sheaves. This formula can be understood by differential geometry.

Let Y be a 2 dimensional compact complex manifold, and let X be a 1 dimensional regular submanifold of Y. For each $P \in X$, there are two linear spaces $T_PX \subseteq T_PY$. The complementary of T_PX in T_PY means all normal vectors of X at P relative to Y. In the language of sheaf theory, there is an exact sequence of sheaves on X

$$0 \to (\Omega^1_{X/\mathbb{C}})^\vee \to (\Omega^1_{Y/\mathbb{C}}|_X)^\vee \to \mathscr{N}\mathrm{or}_{X/Y} \to 0,$$

where $\Omega^1_{(\cdot)/\mathbb{C}}$ means the sheaf of holomorphic 1-forms, i.e. the holomorphic cotangent sheaf of $(\cdot)$, it is a locally free $\mathscr{O}_{(\cdot)}$-module with rank equal to the dimension of $(\cdot)$. Take dual and take determinant of this sequence, we get

$$\omega_{X/Y} := \bigwedge^2 \Omega^1_{Y/\mathbb{C}}|_X \cong \Omega^1_{X/\mathbb{C}} \otimes \mathscr{N}\mathrm{or}^\vee_{X/Y} = \omega_{X/\mathbb{C}} \otimes \mathscr{N}\mathrm{or}^\vee_{X/Y}.$$

If one can show $\mathscr{N}\mathrm{or}_{X/Y} \cong \mathscr{O}_Y(X)|_X$, then the adjunction formula

$$\omega_{X/\mathbb{C}} \cong \omega_{X/Y} \otimes \mathscr{O}_Y(X)|_X$$

holds and can be generalized to general cases. In algebraic geometry, one can study these sheaves locally, just use commutative algebra on each affine open subset.

Exercise 2.4.1. *Let X be a 1 dimensional regular submanifold of a 2 dimensional compact complex manifold Y. Define the conormal sheaf $\mathscr{N}\mathrm{or}^\vee_{X/Y}$ on X of $i: X \hookrightarrow Y$ to be $i^*(\mathscr{I}/\mathscr{I}^2)$, where $\mathscr{I} := \mathscr{O}_Y(-X)$ (a line bundle). Show that:*

- *$i^*(\mathscr{I}/\mathscr{I}^2) \cong i^*\mathscr{I}$, and so $\mathscr{N}\mathrm{or}^{\vee}_{X/Y} \cong \mathscr{O}_Y(-X)|_X$. You can check this locally: given a commutative ring A and an ideal I, there is an isomorphism $I/I^2 \otimes_A A/I \cong I \otimes_A A/I \cong I/I^2$.*

- *In particular, let $Y = X \times X$ and let $i : X \hookrightarrow X \times X$ be the diagonal embedding. Now $i^*(\mathscr{I}/\mathscr{I}^2) \cong \Omega^1_{X/\mathbb{C}}$. Locally, for a $\mathbb{C}$-algebra A, assume I is the kernel of $A \otimes_{\mathbb{C}} A \to A, a_1 \otimes a_2 \mapsto a_1 a_2$ and set a A-module structure $a(a_1 \otimes a_2) := aa_1 \otimes a_2$ on it. Define*

$$\Omega^1_{A/\mathbb{C}} := \frac{\text{free } A\text{-module generated by the symbols } da, a \in A}{\langle d(a_1+a_2) - da_1 - da_2, d(a_1a_2) - a_1da_2 - a_2da_1 : a_i \in A\rangle}.$$

Recall that $\Omega^1_{X/\mathbb{C}}|_U = \widetilde{\Omega^1_{\mathscr{O}_X(U)/\mathbb{C}}}$ for any affine open subset $U \subseteq X$, so the isomorphism we want is locally given by

$$\Omega^1_{A/\mathbb{C}} \xrightarrow{\sim} I/I^2, \quad da \mapsto [a \otimes 1 - 1 \otimes a].$$

Let X be an arithmetic surface defined over $\mathscr{O}_K$. We will show that there is an analogy in Arakelov geometry.

Definition 2.4.2 (Dualizing Sheaves)**.** Let $\pi : X \to \mathscr{O}_K$ be a flat, projective, l.c.i. morphism. Let $i : X \hookrightarrow Y$ be an immersion into Y and Y is smooth over $\mathscr{O}_K$. Consider the following diagram

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "\pi"'] & Y \arrow[d, "p"] \\ & \mathscr{O}_K \end{tikzcd}$$

Analogous to the previous discussion, we define the **relative canonical sheaf** of π to be

$$\omega_{X/\mathscr{O}_K} := \det(i^*\Omega^1_{Y/\mathscr{O}_K}) \otimes_{\mathscr{O}_X} \det(\mathscr{N}\mathrm{or}_{X/Y}),$$

where $\Omega^1_{Y/\mathscr{O}_K} := \Delta^*(\mathscr{I}/\mathscr{I}^2)$ and $\mathscr{I} := \mathscr{O}_{Y \times_{\mathscr{O}_K} Y}(-\Delta(Y)), \Delta : Y \to Y \times_{\mathscr{O}_K} Y$.

Sometimes we call $\omega_{X/\mathscr{O}_K}$ the **dualizing sheaf** with respect to π, and abbreviate it as ω_π. It can be shown that dualizing sheaf is independent of the choice of the decomposition $X \hookrightarrow Y \to \mathscr{O}_K$ up to isomorphisms.

Remark 2.4.3. Let ω_π be a dualizing sheaf with respect to π, then one can find a trace morphism $\mathrm{tr}_\pi : H^\cdot(X, \omega_\pi) \to \mathscr{O}_K$, such that for all coherent sheaves $\mathscr{F}$ on X, the natural pairing

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \omega_\pi) \times H^\cdot(X, \mathscr{F}) \longrightarrow H^\cdot(X, \omega_\pi) \xrightarrow{\mathrm{tr}_\pi} \mathscr{O}_K$$

followed by tr_π gives an isomorphism $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \omega_\pi) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_K}(H^\cdot(X, \mathscr{F}), \mathscr{O}_K)$.

Example 2.4.4. The concept of dualizing sheaves in number theory is corresponds to the codifferents. Let L/K be a finite extension of number fields and let $\theta : \mathrm{Spec}(\mathscr{O}_L) \to \mathrm{Spec}(\mathscr{O}_K)$ be the induced morphism. Now we have $\omega_\theta \cong \mathrm{Hom}_{\mathscr{O}_K}(\mathscr{O}_L, \mathscr{O}_K)$, since

$$\mathrm{Hom}_{\mathscr{O}_L}(\cdot, \mathrm{Hom}_{\mathscr{O}_K}(\mathscr{O}_L, \mathscr{O}_K)) \cong \mathrm{Hom}_{\mathscr{O}_K}(\cdot, \mathscr{O}_K) = \mathrm{Hom}_{\mathscr{O}_K}(H^0(\mathrm{Spec}(\mathscr{O}_L), \cdot), \mathscr{O}_K).$$

Recall that in algebraic number theory there is an isomorphism

$$\{y \in L : \mathrm{tr}_{L/K}(y\mathscr{O}_L) \subseteq \mathscr{O}_K\} \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_K}(\mathscr{O}_L, \mathscr{O}_K), \quad y \mapsto \mathrm{tr}_{L/K}(y\cdot).$$

So the dualizing sheaf $\omega_{\mathscr{O}_L/\mathscr{O}_K}$ in this case is just the codifferent $\mathscr{C}_{L/K}$ of a field extension.

Let us do Arakelov geometry now. Let $\tau \in \mathrm{Hom}(K,\mathbb{C})$, also write $\tau : X_\tau(\mathbb{C}) \to X$. Then by base-change, on $X_\tau(\mathbb{C})$ we have $\tau^*\omega_{X/\mathscr{O}_K} \cong \Omega^1_{X_\tau(\mathbb{C})/\mathbb{C}}$. So there is an admissible metric on $\Omega^1_{X_\tau(\mathbb{C})/\mathbb{C}} = \Delta^*\mathscr{O}_{X_\tau(\mathbb{C})\times X_\tau(\mathbb{C})}(-\Delta(X_\tau(\mathbb{C})))$, since we have the Green function on the right hand side.

Proposition 2.4.5 (Arithmetic Adjunction Formula)**.** *Let E be a horizontal prime divisor on an arithmetic surface X defined over $\mathscr{O}_K$. Suppose E has form* $\mathrm{Spec}(\mathscr{O}_{k(E)})$*, then*

$$\left(\widehat{\omega_{X/\mathscr{O}_K}} \otimes \widehat{\mathscr{O}_X}(E), E\right) = \log\#\left(\mathscr{C}_{k(E)/K}\big/\mathscr{O}_{k(E)}\right) + \frac{1}{2}\sum_{\tau\in\mathrm{Hom}(k(E),\mathbb{C})}\sum_{i\neq j} g_\tau(P_{i,\tau}, P_{j,\tau}),$$

where $E_\tau = \sum_{j=1}^{\deg(E)} P_{j,\tau}$.

Proof. We have $\omega_{E/\mathscr{O}_K} \cong \widetilde{\mathscr{C}_{k(E)/K}}$ by Example 2.4.4. On the other hand, note that $i : E \hookrightarrow X$ is a closed regular immersion, so $\omega_{E/\mathscr{O}_K} = i^*\omega_{X/\mathscr{O}_K} \otimes \mathscr{N}\mathrm{or}_{E/X}$ by classical adjunction formula. But $i^*\mathscr{O}_X(E) \cong \mathscr{N}\mathrm{or}_{E/X}$, therefore

$$\omega_{E/\mathscr{O}_K} \cong \left(\omega_{X/\mathscr{O}_K} \otimes \mathscr{O}_X(E)\right)\big|_E.$$

Now use Proposition 2.3.7 to compute

$$\left(\widehat{\omega_{X/\mathscr{O}_K}} \otimes \widehat{\mathscr{O}_X}(E), E\right) = \widehat{\deg}(\widehat{\omega_{E/\mathscr{O}_K}}) = \log\#\left(\mathscr{C}_{k(E)/K}\big/\mathscr{O}_{k(E)}\right) - \sum_{\tau\in\mathrm{Hom}(k(E),\mathbb{C})} \log\|1_\tau\|_\tau.$$

By Remark 2.2.6, the metric on $\Omega^1_{X_\tau(\mathbb{C})/\mathbb{C}} \otimes \mathscr{O}_{X_\tau(\mathbb{C})}(E_\tau)$ at $P_{j,\tau}$ is

$$\|1(P_{j,\tau})\|_\tau = \exp\left(-\frac{1}{2}\sum_{i\neq j} g_\tau(P_{i,\tau}, P_{j,\tau})\right).$$

It only needs to run out all j. □

2.5 Faltings-Riemann-Roch Theorem*

Recall that the Riemann-Roch formula for a line bundle $\mathscr{L}$ on a Riemann surface X is

$$\chi(\mathscr{L}) := \dim H^0(X,\mathscr{L}) - \dim H^1(X,\mathscr{L}) = 1 - \mathrm{genus}(X) + \deg(\mathscr{L}).$$

In this section, we will introduce the analogy of this formula in Arakelov geometry.

Let V be a $\mathbb{C}$-linear space of dimension n, define $\det(V) := \bigwedge^n V$. For $\mathscr{L}$ a line bundle on a genus $g > 0$ Riemann surface X, let

$$\begin{aligned}\lambda(R\Gamma(X,\mathscr{L})) :=& \mathrm{Hom}_{\mathbb{C}}\left(\det(H^1(X,\mathscr{L})), \det(H^0(X,\mathscr{L}))\right)\\ \cong& \det(H^0(X,\mathscr{L})) \otimes \det(H^1(X,\mathscr{L}))^\vee.\end{aligned}$$

The object of this section will be to discuss volume forms on the formal difference $H^0(X,\mathscr{L}) - H^1(X,\mathscr{L})$, i.e. hermitian inner product on $\lambda(R\Gamma(X,\mathscr{L}))$, when $\mathscr{L}$ is a metrized line bundle (we discard symbols $\|\cdot\|$ or h for simplicity) discussed in Remark 2.2.6, and discuss how these volume forms (hermitian inner product) give rise to an Euler characteristic $\chi(\mathscr{L})$ with desirable properties, e.g. for which one has a Riemann-Roch formula.

Let D be a divisor on X, and let $P \in X$ be a point. There is an exact sequence

$$0 \to \mathscr{O}_X(D) \to \mathscr{O}_X(D+P) \to \mathbb{C}_P \to 0.$$

The metrics on $\mathscr{O}_X(D)$ and $\mathscr{O}_X(D+P)$ give rise to a metric on $\Gamma(X,\mathbb{C}_P)$, is simply the restriction of the metric on $\mathscr{O}_X(D+P)$ to the fiber at P. However, this metric is depend on D, so we write $\mathbb{C}_P$ as $\mathbb{C}_P(D)$ to emphasize this.

One has $H^1(X,\mathbb{C}_P(D)) = 0$, so there is a long exact sequence

$$\begin{aligned} 0 \to \Gamma(X,\mathscr{O}_X(D)) \to \Gamma(X,\mathscr{O}_X(D+P)) \to \Gamma(X,\mathbb{C}_P(D)) \to \\ \to H^1(X,\mathscr{O}_X(D)) \to H^1(X,\mathscr{O}_X(D+P)) \to 0. \end{aligned}$$

Exercise 2.5.1. *Show that:*

- *Let*

$$0 \to A_1 \to A_2 \to \cdots \to A_n \to 0$$

  be a long exact sequence of finite dimensional $\mathbb{C}$-linear spaces, then

$$\left(\bigotimes_{i\geq 0} \det(A_{2i+1})\right) \otimes \left(\bigotimes_{i\geq 1} \det(A_{2i})^\vee\right) \cong \mathbb{C}.$$

- *The long exact sequence above gives an isomorphism*

$$\lambda(R\Gamma(X,\mathscr{O}_X(D+P))) \cong \lambda(R\Gamma(X,\mathscr{O}_X(D))) \otimes \Gamma(X,\mathbb{C}_P(D)).$$

Faltings proved the following result in 1984 (here we omit the proof):

Proposition 2.5.2 (Faltings)**.** *There is a unique way to assign to each admissible metrized line bundle $\mathscr{L}$ (admissible means the metric on $\mathscr{L}$ comes from Remark 2.2.6) on X a hermitian inner product on $\lambda(R\Gamma(X,\mathscr{L}))$ such that the following properties hold:*

- *An isometry of metrized line bundles induces an isometry of the corresponding $\lambda(R\Gamma(X,\mathscr{L}))$.*
- *If the metric on $\mathscr{L}$ is changed by a factor $C > 0$, then the metric on $\lambda(R\Gamma(X,\mathscr{L}))$ is changed by $C^{\chi(\mathscr{L})}$.*
- *The metrics on $\lambda(R\Gamma(X,\mathscr{L}))$ are compatible with the metrics on $\mathbb{C}_P(D)$, in the following sense: Suppose D and $D+P$ are divisors on X, then the isomorphsim*

$$\lambda(R\Gamma(X,\mathscr{O}_X(D+P))) \cong \lambda(R\Gamma(X,\mathscr{O}_X(D))) \otimes \Gamma(X,\mathbb{C}_P(D))$$

  is an isometry.

- *The metric on* $\lambda(R\Gamma(X,\Omega^1_{X/\mathbb{C}})) \cong \bigwedge^g(\Gamma(X,\Omega^1_{X/\mathbb{C}}))$ *is the one determined by the Hermitian inner product* $\Gamma(X,\Omega^1_{X/\mathbb{C}}) \times \Gamma(X,\Omega^1_{X/\mathbb{C}}) \to \mathbb{C}$, $\langle \omega, \eta \rangle \mapsto \frac{i}{2}\int_X \omega \wedge \overline{\eta}$.

Let us do Arakelov geometry now.

Definition 2.5.3. Let M be a finitely generated $\mathbb{Z}$-module, suppose on $M \otimes_{\mathbb{Z}} \mathbb{R}$ we have a Haar measure. Define

$$\chi_{\mathbb{Z}}(M) := -\log\left(\mathrm{vol}(M \otimes_{\mathbb{Z}} \mathbb{R}/M)\big/\#M_{\mathrm{tor}}\right).$$

In the case of $\mathbb{Z}$-module $\mathscr{O}_K$ when K is a number field, we choose the normalized Haar measure on $\mathscr{O}_K \otimes_{\mathbb{Z}} \mathbb{R}$ to be the one such that $\mathrm{vol}(\mathscr{O}_K \otimes_{\mathbb{Z}} \mathbb{R}/\mathscr{O}_K) = \sqrt{|\mathrm{disc}(K/\mathbb{Q})|}$.

The following definition generalizes Definition 2.5.3, since $\mathrm{disc}(\mathbb{Q}/\mathbb{Q}) = 1$.

Definition 2.5.4. Let M be a finitely generated $\mathscr{O}_K$-module, define

$$\chi_K(M) := \chi_{\mathbb{Z}}(M) - \mathrm{rank}_{\mathscr{O}_K}(M) \cdot \chi_{\mathbb{Z}}(\mathscr{O}_K).$$

We need a lemma to summarize some properties of bundles on an arithmetic surface.

Lemma 2.5.5. *Let X be an arithmetic surface defined over $\mathscr{O}_K$. For any coherent sheaf $\mathscr{F}$ on X, we have:*

- $H^i(X,\mathscr{F}) = 0$ *for* $i \geq 2$.
- *Denote* $i : \mathscr{O}_K \hookrightarrow K$, *then* $H^i(X,\mathscr{F}) \otimes_{\mathscr{O}_K} K \cong H^i(X_K, i^*\mathscr{F})$. *Furthermore, if* $\tau \in \mathrm{Hom}(K,\mathbb{C})$, *then* $H^i(X,\mathscr{F}) \otimes_\tau \mathbb{C} \cong H^i(X_\tau(\mathbb{C}), \tau^*\mathscr{F})$.

Proof. We compute the first one by using Čech cohomology. After localizing $\mathscr{O}_K$ at its primes, we are reduced to the case when R is a discrete valuation ring. Since X is projective over R, there exist homogeneous polynomials $f_1, \cdots, f_n$ with coefficients in R such that

$$X \cap [f_1 = 0] \cap \cdots \cap [f_n = 0] \text{ is empty.}$$

Now X is covered by affine open subsets $f_j \neq 0$ for $1 \leq j \leq n$, so $\check{H}^i(X,\mathscr{F}) = 0$ for $i \geq 2$. The second item is followed by flat base-change, since K and $\mathbb{C}$ are flat over $\mathscr{O}_K$. □

Let X be an arithmetic surface defined over $\mathscr{O}_K$. For an admissible metrized line bundle $\mathscr{L}$, we only need to consider $H^0(X,\mathscr{L})$ (resp. $H^0(X_\tau(\mathbb{C}),\tau^*\mathscr{L})$) and $H^1(X,\mathscr{L})$ (resp. $H^1(X_\tau(\mathbb{C}),\tau^*\mathscr{L})$) by Lemma 2.5.5.

For any embedding $\tau \in \mathrm{Hom}(K,\mathbb{C})$, we naturally have $\lambda(R\Gamma(X_\tau(\mathbb{C}),\tau^*\mathscr{L}))$, and by Proposition 2.5.2 it admits a hermitian inner product, i.e. a volume form on the formal difference

$$H^0(X_\tau(\mathbb{C}),\tau^*\mathscr{L})) - H^1(X_\tau(\mathbb{C}),\tau^*\mathscr{L})).$$

By Lemma 2.5.5, this induces a Haar measure on

$$H^0(X,\mathscr{L}) \otimes_\tau \mathbb{C} - H^1(X,\mathscr{L}) \otimes_\tau \mathbb{C}.$$

This Haar measure is compatible with complex conjugation, so in fact there is a Haar measure on

$$H^0(X,\mathscr{L}) \otimes_{\mathscr{O}_K} \mathbb{R} - H^1(X,\mathscr{L}) \otimes_{\mathscr{O}_K} \mathbb{R}.$$

Combine these with Definition 2.5.3, we make the following definition.

Definition 2.5.6. Let $\mathscr{L}$ be an admissible metrized line bundle on an arithmetic surface X defined over $\mathscr{O}_K$, define

$$\chi(\mathscr{L}) := \chi_K(H^0(X,\mathscr{L})) - \chi_K(H^1(X,\mathscr{L})).$$

The main theorem is:

Theorem 2.5.7 (Faltings)**.** *For $\mathscr{L}$ an admissible metrized line bundle on an arithmetic surface X defined over $\mathscr{O}_K$, one has the following Riemann-Roch formula*

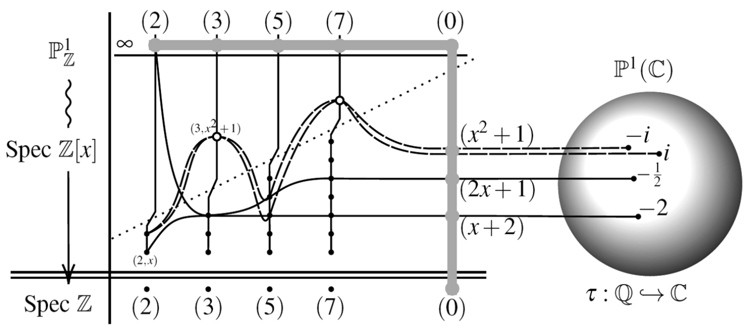
$$\chi(\mathscr{L}) - \chi(\mathscr{O}_X) = \frac{1}{2}(\mathscr{L}, \mathscr{L} - \omega_{X/\mathscr{O}_K}).$$

Proof. (proof sketch). Let D be an Arakelov divisor on X such that $\mathscr{L}$ is isometric to $\widehat{\mathscr{O}_X}(D)$. Let K'/K be a finite extension, write $\phi : X' := X \times_{\mathscr{O}_K} \mathrm{Spec}(\mathscr{O}_{K'}) \to X$. We wish to show that both sides of the Riemann-Roch formula are multiplied by $[K' : K]$ if X, $\mathscr{O}_K$ and D are replaced by X', $\mathscr{O}_{K'}$ and $\phi^*(D)$. So the Riemann-Roch formula holds under base-change.

When D is the trivial divisor, there is nothing to say. By passing to a suitable extension K', we are reduced to checking that both sides of the formula change by the same amount when we add to D a divisor of the following kinds:

- A real multiple of an infinite fibre of X.
- An irreducible component of a fibre of $X \to \mathscr{O}_K$ at a closed point of $\mathrm{Spec}(\mathscr{O}_K)$.
- The image of a section $s : \mathrm{Spec}(\mathscr{O}_K) \to X$.

Some of them are deduced from the arithmetic adjunction formula. $\square$



Chapter 3

Higher Arakelov Geometry

In this chapter, we introduce the arithmetic intersection theory on arithmetic varieties developed by Gillet and Soulé.

3.1 Some Intersection Theory and K-Theory

There is an important model establishes the intersection theory locally:

Remark 3.1.1. Let R be a noetherian regular local ring with residue field k. An R-module has finite length if and only if it is supported at the closed point of $\mathrm{Spec}(R)$. By dévissage, the K_0 of the category of modules of finite length is isomorphic to the K_0 of the category of k-linear spaces, i.e. to $\mathbb{Z}$. Now let M,N are finitely generated R-modules (hence have finite length), the supports of which intersect only at the closed point of $\mathrm{Spec}(R)$. Serre defines their **intersection multiplicity**

$$i(M,N) := \sum_{k\geq 0}(-1)^k \ell(\mathrm{Tor}_k^R(M,N)).$$

This formula will be served as the standard model for the general intersection theory.

Let X be a noetherian, regular, separated scheme of dimension d defined over a noetherian ring.

For any $p \in \mathbb{Z}_{\geq 0}$, denoted by $X^{(p)}$ the set of points of codimension p in X. Let $Z^p(X)$ be the free abelian group generated by $X^{(p)}$, the elements in it are called **p-cycles**. Two p-cycles Z_1, Z_2 are called **rationally equivalent** if there exist finitely many functions $f_i \in k(y_i)^\times$, $y_i \in X^{(p-1)}$ such that

$$Z_2 - Z_1 = \sum_i \mathrm{div}(f_i),$$

where

$$\mathrm{div}(f_i) = \sum_{x\in X^{(p)}\cap\overline{\{y_i\}}} \mathrm{ord}_{\mathscr{O}_{\overline{\{y_i\}},x}}(f_i)\cdot\overline{\{x\}}.$$

Definition 3.1.2 (Chow Groups)**.** The p-th **Chow group** $\mathrm{CH}^p(X)$ of X is the quotient group

$$\mathrm{CH}^p(X) := Z^p(X)/\text{rational equivalence}.$$

For a closed subscheme $Y \subseteq X$ we define $Z_Y^p(X)$ as the group of cycles of codimension p on X supported in the closed subset attached to Y, then define

$$\mathrm{CH}_Y^p(X) := Z_Y^p(X)/\langle \mathrm{div}(f) : f \in k(y)^\times, y \in X^{(p-1)} \cap Y \rangle,$$

call it the Chow group of codimension p of X with supports in Y.

Definition 3.1.3 (Intersections)**.** Two cycles $Y \in Z^p(X), Z \in Z^q(X)$ **intersect properly**, if $\mathrm{codim}_X(Y \cap Z) = p+q$. Assume Y, Z intersect properly, define the **intersection multiplicity** $i_x(Y,Z)$ for $x \in Y \cap Z \cap X^{(p+q)}$ is the integer

$$i_x(Y,Z) := \sum_{k \geq 0} (-1)^k \ell_{\mathscr{O}_{X,x}}(\mathrm{Tor}_k^{\mathscr{O}_{X,x}}(\mathscr{O}_{Y,x}, \mathscr{O}_{Z,x})).$$

Write $(\cdot)_{\mathbb{Q}} := (\cdot) \otimes_{\mathbb{Z}} \mathbb{Q}$ (the reason for tensor $\mathbb{Q}$ is given by K-theory). The main conclusions of this section are as follows.

Theorem 3.1.4. *Let Y, Z be closed subschemes of X, then there exists a bilinear pairing*

$$\mathrm{CH}_Y^p(X)_{\mathbb{Q}} \times \mathrm{CH}_Z^q(X)_{\mathbb{Q}} \to \mathrm{CH}_{Y \cap Z}^{p+q}(X)_{\mathbb{Q}}$$

satisfying the following properties:

- $\bigoplus_Y \bigoplus_p \mathrm{CH}_Y^p(X)_{\mathbb{Q}}$ *is a commutative ring with unit* $[X] \in \mathrm{CH}^0(X)$.
- *It is compatible with change of supports* $\mathrm{CH}_Y^p(X)_{\mathbb{Q}} \to \mathrm{CH}_{Y'}^p(X)_{\mathbb{Q}}$ *associated to inclusions* $Y \subseteq Y'$.
- *For* $[Y_1] \in \mathrm{CH}_Y^p(X), [Z_1] \in \mathrm{CH}_Z^p(X)$ *with* Y_1, Z_1 *intersect properly, we have*

$$([Y_1],[Z_1]) \mapsto \left[\sum_{x \in Y_1 \cap Z_1 \cap X^{(p+q)}} i_x(Y_1, Z_1) \cdot \overline{\{x\}} \right].$$

In particular, there exists a unique pairing

$$\mathrm{CH}^p(X)_{\mathbb{Q}} \otimes \mathrm{CH}^q(X)_{\mathbb{Q}} \to \mathrm{CH}^{p+q}(X)_{\mathbb{Q}}$$

such that for $Y \in Z^p(X), Z \in Z^q(X)$ *interesting properly, we have*

$$([Y],[Z]) \mapsto \left[\sum_{x \in Y \cap Z \cap X^{(p+q)}} i_x(Y,Z) \cdot \overline{\{x\}} \right].$$

The pairing above is given by tensor product of bundles in K_0 group. So if we want to prove this theorem, we need to introduce some K-theory.

Definition 3.1.5 (Grothendieck Groups)**.** Let Y be a closed subscheme of X. Define:

- $K_0(X)$ to be the **Grothendieck group** of coherent locally free $\mathscr{O}_X$-modules (i.e. finite dimensional vector bundles). More precisely,

$$K_0(X) := \frac{\text{the free abelian group generated by coherent locally free } \mathscr{O}_X\text{-modules}}{\langle \mathscr{F}' - \mathscr{F} + \mathscr{F}'' : 0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0 \text{ exact} \rangle}.$$

- $K_0'(X)$ to be the Grothendieck group of coherent $\mathscr{O}_X$-modules.
- $K_0^Y(X)$ to be the Grothendieck group of bounded complexes of locally free $\mathscr{O}_X$-modules acyclic outside Y modulo quasi-isomorphisms and $(\mathfrak{F}.' - \mathfrak{F} + \mathfrak{F}.'')$ if there is an exact sequence $0 \to \mathfrak{F}.' \to \mathfrak{F}. \to \mathfrak{F}.'' \to 0$.

Since our X is regular, $K_0(X) \cong K_0'(X)$. The map from $K_0'(X)$ to $K_0(X)$ is given by the finite and locally free resolution

$$0 \to \mathscr{F}_n \to \cdots \to \mathscr{F}_1 \to \mathscr{F}_0 \to \mathscr{F} \to 0$$

of a coherent $\mathscr{O}_X$-module $\mathscr{F}$, for this we can send $[\mathscr{F}]$ to $\sum_{i=0}^n (-1)^i [\mathscr{F}_i]$.

We now review some facts in algebraic geometry, but omit the proof.

Proposition 3.1.6. *Let Y, Z (not necessarily regular) be closed subschemes of X, their closed immersions to X are denoted as i.*

- *(Excision Theorem). There is an exact sequence $K_0'(Y) \to K_0'(X) \to K_0'(X \setminus Y) \to 0$.*
- *There is a bilinear pairing*

$$K_0^Y(X) \times K_0^Z(X) \to K_0^{Y \cap Z}(X), \quad ([\mathfrak{F}.], [\mathfrak{G}.]) \mapsto [\mathrm{Tot}(\mathfrak{F}. \otimes \mathfrak{G}.)],$$

where the total complex of a double complex $\mathfrak{F}. \otimes \mathfrak{G}.$ where $\mathfrak{F}. = \{\mathscr{F}_i, d_i\}, \mathfrak{G}. = \{\mathscr{G}_j, \delta_j\}$ is defined by

$$\mathrm{Tot}(\mathfrak{F}. \otimes \mathfrak{G}.) := \left\{ \left(\bigoplus_{i+j=n} \mathscr{F}_i \otimes \mathscr{G}_j \right)_n, \bigoplus_{i+j=n} \left(d_i \otimes \mathrm{id} + (-1)^i \mathrm{id} \otimes \delta_j \right) \right\}.$$

- *There is an isomorphism $K_0'(Y) \xrightarrow{\sim} K_0^Y(X), [\mathscr{F}] \mapsto [\mathfrak{F}.]$, where $\mathfrak{F}.$ is a finite free resolution of $i_* \mathscr{F}$.*
- *(Projection Formula). Let $f : X \to X'$ be a proper morphism. The homomorphism $f^* : K_0(X') \to K_0(X), [\mathscr{F}'] \mapsto [f^* \mathscr{F}']$ and the homomorphism $f_* : K_0'(X) \to K_0'(X'), [\mathscr{F}] \mapsto \sum_i (-1)^i [R^i f_* \mathscr{F}]$ satisfy the formula*

$$f_*(f^*[\mathscr{F}'] \otimes [\mathscr{F}]) = [\mathscr{F}'] \otimes f_*[\mathscr{F}], \quad \textit{for } [\mathscr{F}] \in K_0'(X), [\mathscr{F}'] \in K_0(X').$$

In order to state Theorem 3.1.4 using K-theory, we make the following definition.

Definition 3.1.7. On $K_0^Y(X)$ we define a decreasing filtration

$$K_0^Y(X) = F^0 K_0^Y(X) \supseteq F^1 K_0^Y(X) \supseteq \cdots \supseteq F^d K_0^Y(X) \supseteq F^{d+1} K_0^Y(X) = \{0\}.$$

by

$$F^p K_0^Y(X) := \bigcup_{Z \subseteq Y, \mathrm{codim}_X Z \geq p} \mathrm{im}\Big(K_0^Z(X) \to K_0^Y(X) \Big).$$

Define $\mathrm{Gr}^p K_0^Y(X) := F^p K_0^Y(X) / F^{p+1} K_0^Y(X)$.

Theorem 3.1.8. *Using the terminologies above,*

- $F^pK_0^Y(X)_{\mathbb{Q}} \cdot F^qK_0^Z(X)_{\mathbb{Q}} \subseteq F^{p+q}K_0^{Y\cap Z}(X)_{\mathbb{Q}}$, *given by take the total complex of the tensor product double complex in Proposition 3.1.6.*
- *Let* $Z \in Z_Y^p(X)$ *be an irreducible cycle, then we can take a finite locally free resolution of* $i_*\mathscr{O}_Z$. *This induces an isomorphism* $\mathrm{CH}_Y^p(X)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Gr}^pK_0^Y(X)_{\mathbb{Q}}$.

Proof. (of Theorem 3.1.4). It is not hard to see that Theorem 3.1.8 implies Theorem 3.1.4, since on $\mathrm{Gr}^{\cdot}K_0(X)_{\mathbb{Q}}$ we can do tensors naturally. The intersection number $i_x(\cdot,\cdot)$ comes from Proposition 3.1.6, because we can define a bilinear pairing

$$K_0'(Y)\times K_0'(Z)\to K_0'(Y\cap Z),\quad ([\mathscr{F}],[\mathscr{G}])\mapsto \sum_k(-1)^k[\mathscr{H}_k(\mathrm{Tot}(\mathfrak{P}.\otimes\mathfrak{Q}.))],$$

where $\mathfrak{P}.\to i_*\mathscr{F}\to 0$, $\mathfrak{Q}.\to i_*\mathscr{G}\to 0$ are free resolutions of $\mathscr{O}_X$-modules (here i denotes the closed immersion to X). By homological algebra we have $\mathscr{H}_k(\mathrm{Tot}(\mathfrak{P}.\otimes\mathfrak{Q}.))\cong \mathscr{T}\mathrm{or}_k^{\mathscr{O}_X}(i_*\mathscr{F},i_*\mathscr{G})$, so the intersection number $i_x(Y,Z)$ will be defined to be the local information at $x\in Y\cap Z$ of the image of $([\mathscr{O}_Y],[\mathscr{O}_Z])$. Since $\mathscr{T}\mathrm{or}$ and i_* commute with colimits, this is the alternating sum of the lengthes of the stalks at x of Tor sheaves, due to Remark 3.1.1. □

Note that the complex

$$\cdots\to 0\to 0\to i_*\mathscr{F}\otimes i_*\mathscr{G}\to 0\to 0\to\cdots$$

is isomorphic to $i_*\mathscr{F}\otimes\mathfrak{Q}.$ or $\mathfrak{P}.\otimes i_*\mathscr{G}$ in the derived category, so the right hand side of the map above is equal to $[i_*\mathscr{F}\otimes i_*\mathscr{G}]$ in $K_0'(X)$. Therefore the intersection bilinear form is "almost" the tensor product of $\mathscr{O}_X$-modules.

Remark 3.1.9. Chow groups can also be defined by using sheaf cohomology. The **Bloch's formula** tells us that there is an isomorphism:

$$\mathrm{CH}^p(X)\cong H^p\Big(X,\big(U\mapsto \pi_{p+1}\mathscr{B}\mathscr{Q}\{\text{finitely generated projective }\mathscr{O}_X(U)\text{-module}\}\big)^{\dagger}\Big),$$

where † means sheafification, $\mathscr{Q}$ means add some arrows to the category, $\mathscr{B}$ means geometric realization (to make a category into a topological space) and π_{p+1} means the $(p+1)$-th homotopy group. Some technical tools can be found in Quillen's higher K-theory and homotopy theory.

We will give a proof sketch of Theorem 3.1.8. Some preparation is needed.

Definition 3.1.10 (λ-Rings)**.** A **λ-ring** is a unitary ring R with operations $\lambda^k: R\to R$ ($k\geq 0$), satisfying

- $\lambda^0=1$; $\lambda^1=\mathrm{id}$; $\lambda^k(1)=0(\forall k>1)$.
- $\lambda^k(x+y)=\sum_{i=0}^k\lambda^i(x)\cdot\lambda^{k-i}(y)$.
- $\lambda^k(xy)=P_k(\lambda^1(x),\cdots,\lambda^k(x);\lambda^1(y),\cdots,\lambda^k(y))$, where P_k is a integral coefficient polynomial in $2k$ variables s.t. $P_k(e_1,\cdots,e_k;f_1,\cdots,f_k)$ is the coefficient of t^k in the expression $\prod_{i=1}^k\prod_{j=1}^k(1+tx_iy_j)$, where $e_1,\cdots,e_k$ (resp. $f_1,\cdots,f_k$) are elementary symmetric polynomials in $x_1,\cdots,x_k$ (resp. $y_1,\cdots,y_k$).

- $\lambda^k(\lambda^l(x)) = Q_{k,l}(\lambda^1(x), \cdots, \lambda^{kl}(x))$, where $Q_{k,l}$ is a integral coefficient polynomial in kl variables s.t. $Q_{k,l}(e_1, \cdots, e_{kl})$ is the coefficient of t^k in the expression $\prod_{1\le i_1<i_2<\cdots<i_k\le kl}(1+tx_{i_1}x_{i_2}\cdots x_{i_k})$, where $e_1, \cdots, e_{kl}$ are elementary symmetric polynomials in $x_1, \cdots, x_{kl}$.

Note that P_k, Q_{kl} are not depend on R.

Exercise 3.1.11. *$P_1(x;y) = xy$; $P_2(x,y;z,w) = x^2w + z^2y - 2yw$.*

The concept of λ-rings comes from the analogy of operations on vector bundles. Indeed, we can roughly view $(K_0, +, \otimes, \wedge)$ as a λ-ring with unit is the trivial bundle $[\mathscr{O}]$.

Exercise 3.1.12. *If on $X = \mathbb{P}^1(\mathbb{C})$ there is*

$$\bigwedge^k \left(\mathscr{O}_X(l)^{\oplus n}\right) = \mathscr{O}_X(kl)^{\oplus \Phi(n,k)},$$

find $\Phi(n,k)$. In particular, $\bigwedge^n(\mathscr{O}_X(l)^{\oplus n}) = \mathscr{O}_X(nl)$.

Definition 3.1.13 (Adams Operators)**.** Write $\lambda_t(x) := \sum_{k\ge 0} \lambda^k(x)t^k$. Put

$$\psi_{-t}(x) := -\frac{t}{\lambda_t(x)} \cdot \frac{d\lambda_t(x)}{dt},$$

and $\psi_t(x) := \sum_{k\ge 1} \psi^k(x)t^k$. The operators $\psi^k : R \to R$ are called the **Adams operators** on the λ-ring R.

There is an important principle in algebraic topology, called the **splitting principle**. That is, to check universal relations among operations on λ-rings, it is sufficient to check these on elements of the form $x = x_1 + \cdots + x_n$ with $\lambda^k(x_i) = 0$ for all $k > 1, i = 1, \cdots, n$. This is because for a vector bundle E on X, there is a **tautological exact sequence** on $\mathbb{P}E$

$$0 \to \mathscr{O}_{\mathbb{P}E}(-1) \to \pi^*E \to \mathrm{Quot} \to 0,$$

where $\pi : \mathbb{P}E \to X$, whose fiber at a point $x \in X$ is the usual projective space of lines in the fiber E_x. The fact is that $[\pi^*E]$ completely determines $[E]$ on X and $\mathrm{rank}(\mathrm{Quot}) < \mathrm{rank}(E)$, so one can continue this process by induction. Finally we get some line bundles from E. Since in the K_0 group we modulo exact sequences, the splitting principle is reasonable.

Proposition 3.1.14. *Let ψ^k $(k \ge 1)$ be Adams operators on a λ-ring R.*

- *ψ^k are ring endomorphisms.*
- *$\psi^k \circ \psi^l = \psi^{kl} = \psi^l \circ \psi^k$.*
- *$\psi^k = \mathrm{New}_k(\lambda^1, \cdots, \lambda^k)$, where New_k is the k-th Newton polynomial.*

Proof. We only prove the first one. Obviously, $\lambda_t(x+y) = \lambda_t(x)\cdot\lambda_t(y)$, so ψ^k preserves addition. To check ψ^k preserves multiplication, we use splitting principle. Let $x, y \in R$ with $\lambda^k(x) = \lambda^k(y) = 0$ for all $k > 1$, hence $\lambda^k(xy) = 0$ for all $k > 1$. Then $\lambda_t(xy) = 1 + txy$ and therefore $\psi_{-t}(xy) = \frac{-txy}{1+txy}$. This implies $\psi_t(xy) = \sum_{k\ge 1}(txy)^k$, so $\psi^k(xy) = (xy)^k = \psi^k(x)\psi^k(y)$. □

Exercise 3.1.15. *There is a unique λ-ring structure on $\mathbb{Z}$, given by*

$$\lambda^k : \mathbb{Z} \to \mathbb{Z}, \quad n \mapsto \binom{n}{k} = \dim\left(\bigwedge^k \mathbb{Q}^{\oplus n}\right).$$

The Adams operators on this λ-ring are $\psi^k = \mathrm{id}$ for all $k \geq 1$. Check this.

Proof. (proof sketch of Theorem 3.1.8). We take the following steps.

- (Fact i). There exists a functorial λ-ring structure on $\bigoplus_{Y \subseteq X} K_0^Y(X)$ such that λ^k maps $K_0^Y(X)$ to itself for every $k \geq 0$. Note that this does not mean that $K_0^Y(X)$ has an induced λ-ring structure, in fact it may not have a unit. In particular, if $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(R/(a))$, then

$$\psi^k : K_0^Y(X) \to K_0^Y(X), \quad [\Sigma_a] := [\cdots \to 0 \to R \xrightarrow{a} R \to 0 \to \cdots] \mapsto k \cdot [\Sigma_a].$$

- (Fact ii). Let A be a commutative ring, write $K.(A)$ instead of $K.(\mathrm{Spec}(A))$. In particular, if $A = F$ is a field, then

$$\begin{aligned} &K_0(F) \cong \mathbb{Z}, \\ &K_1(F) \cong F^\times, \\ &K_2(F) \cong F^\times \otimes_{\mathbb{Z}} F^\times \big/ \langle x \otimes (1-x) : x \in F^\times \setminus \{1\} \rangle. \end{aligned}$$

  The Adams operators ψ^k act on $K_0(F)$ by id, on $K_1(F)$ by multiplication by k.

- (Claim iii). Let $Y \subseteq X$ be a closed subscheme with codimension p, then there exists an exact sequence

$$0 \longrightarrow F^{p+1}K_0^Y(X) \longrightarrow K_0^Y(X) \longrightarrow \bigoplus_{V \in Y \cap X^{(p)} \text{ closed}} K_0^V(\mathscr{O}_{X,V}) \longrightarrow 0.$$

  Proof. Since X is regular, by Proposition 3.1.6 we have for $Z \subseteq Y$ a closed subscheme, there is an exact sequence

$$0 \longrightarrow \mathrm{im}\big(K_0^Z(X) \to K_0^Y(X)\big) \longrightarrow K_0^Y(X) \longrightarrow K_0^{Y \setminus Z}(X \setminus Z) \longrightarrow 0.$$

  Take colimit over all closed subschemes $Z \subseteq Y$ with codimension $\geq p+1$, we get

$$0 \longrightarrow F^{p+1}K_0^Y(X) \longrightarrow K_0^Y(X) \longrightarrow \varinjlim_{Z \subseteq Y, \mathrm{codim}_X Z \geq p+1} K_0^{Y \setminus Z}(X \setminus Z) \longrightarrow 0.$$

  The colimit in the sequence above is $\displaystyle\bigoplus_{V \in Y \cap X^{(p)} \text{ closed}} K_0^V(\mathscr{O}_{X,V})$. □

- (Fact iv). Fix $k \geq 2$. For $i \geq 0$, denote $K_0^Y(X)^{[i]} := \{x \in K_0^Y(X)_{\mathbb{Q}} : \psi^k(x) = k^i x\}$ (it does not depend on the choice of k by Proposition 3.1.14, but this fact is not trivial). Then we have (i.e. the action of ψ^k is diagonalizable with eigenvalues in $\{k^0, \cdots, k^d\}$)

$$F^p K_0^Y(X)_{\mathbb{Q}} \cong \bigoplus_{i \geq p} K_0^Y(X)^{[i]}.$$

This was first proven by Grothendieck. The idea of proof is to use descending induction on $\operatorname{codim}_X(Y)$. To start with, let Y be a closed point, then $K_0^Y(X) \cong K_0'(Y) \cong K_0(k(Y)) \cong \mathbb{Z}$ (this is not a λ-ring, otherwise, it will contradict Exercise 3.1.15), so we only need to check the action of ψ^k on some element is multiplication by k^d. This is given by (Fact i), since one can compute $\psi^k\left(\left[\bigotimes_{i=1}^d \Sigma_{a_i}\right]\right) = k^d\left[\bigotimes_{i=1}^d \Sigma_{a_i}\right]$ for some a_i. To prove the general case, one should use (Claim iii).

- (Claim v). $F^pK_0^Y(X)_{\mathbb{Q}} \cdot F^qK_0^Z(X)_{\mathbb{Q}} \subseteq F^{p+q}K_0^{Y\cap Z}(X)_{\mathbb{Q}}$.

  Proof. Let $x \in F^pK_0^Y(X)_{\mathbb{Q}}$ and $y \in F^qK_0^Z(X)_{\mathbb{Q}}$, by (Fact iv) we have $x = \sum_{i\geq p} x_i$ with $x_i \in K_0^Y(X)^{[i]}$ and $y = \sum_{j\geq q} y_j$ with $y_j \in K_0^Z(X)^{[j]}$. Therefore $xy = \sum_{i,j} x_iy_j$ with $\psi^k(x_iy_j) = k^{i+j}x_iy_j$ for $i+j \geq p+q$. So $xy \in F^{p+q}K_0^{Y\cap Z}(X)_{\mathbb{Q}}$. $\square$

- (Fact vi). The higher K groups fit a spectral sequence, its E_1 page is

$$E_{1,Y}^{p,q}(X) = \begin{cases} \bigoplus_{x\in Y\cap X^{(p)}} K_{-p-q}(k(x)), & p\geq 0, p+q\leq 0, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, (Fact ii) implies $E_{1,Y}^{p-1,-p}(X) \cong \bigoplus_{y\in Y\cap X^{(p-1)}} k(y)^\times$ and $E_{1,Y}^{p,-p}(X) \cong \bigoplus_{x\in Y\cap X^{(p)}} \mathbb{Z} \cong Z_Y^p(X)$.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 & 0 \arrow[d] & 0 \arrow[d, "d_1'"] & 0 \arrow[d] & 0 & 0 \arrow[ddl] & 0 \arrow[ddl] \\
E_0^{0,-2} \arrow[r] & E_0^{0,-1} \arrow[r] & E_0^{0,0} & E_1^{0,-2} \arrow[d, "d_1'"] & E_1^{0,-1} \arrow[d, "d_1"] & E_1^{0,0} \arrow[d] & E_2^{0,-2} & E_2^{0,-1} \arrow[ddl] & E_2^{0,0} \arrow[ddl] \\
E_0^{1,-2} \arrow[r] & E_0^{1,-1} \arrow[r] & E_0^{1,0} & E_1^{1,-2} \arrow[d, "d_1"] & E_1^{1,-1} \arrow[d] & 0 \arrow[d] & E_2^{1,-2} & E_2^{1,-1} & 0 \\
E_0^{2,-2} \arrow[r] & E_0^{2,-1} \arrow[r] & E_0^{2,0} & E_1^{2,-2} & 0 & 0 & E_2^{2,-2} & 0 & 0
\end{tikzcd}$$

The maps d_1 in the E_1 page are

$$d_1 : \bigoplus_{y\in Y\cap X^{(p-1)}} k(y)^\times \to Z_Y^p(X), \quad (f_y) \mapsto \sum_y \operatorname{div}(f_y).$$

It is easy to see $Z_Y^p(X)/\operatorname{im}(d_1) \cong \mathrm{CH}_Y^p(X)$, i.e. $E_{2,Y}^{p,-p}(X) \cong \mathrm{CH}_Y^p(X)$. Similarly,

$$E_{2,Y}^{p-1,-p}(X) \cong \left\{ (f_y) \in \bigoplus_{y\in Y\cap X^{(p-1)}} k(y)^\times : \sum_y \operatorname{div}(f_y) = 0 \right\} \Big/ \operatorname{im}(d_1').$$

- (Fact vii). There are λ-ring structures $\lambda^k : E_{r,Y}^{p,q}(X) \to E_{r,Y}^{p,q}(X)$, and they converge to $\lambda^k : K_{-p-q}^Y(X) \to K_{-p-q}^Y(X)$. For the isomorphisms in (Fact vi)

$$\varepsilon : \bigoplus_{x\in Y\cap X^{(p)}} K_{-p-q}(k(x)) \xrightarrow{\sim} E_{1,Y}^{p,q}(X),$$

the Adams operators ψ^k satisfy $\psi^k \circ \varepsilon = k^p \cdot \varepsilon \circ \psi^k$.

This is proven by using Quillen's higher K-theory to compute cohomology groups.

- (Claim viii). $E_{2,Y}^{p,-p}(X)_{\mathbb{Q}} \cong \mathrm{Gr}^p K_0^Y(X)_{\mathbb{Q}}$.

  Proof. (Fact vii) implies that ψ^k acts on $E_{r,Y}^{p-1,-p}(X)$ by multiplication by k^p, on $E_{r,Y}^{p-1+r,-(p-1+r)}(X)$ by multiplication by k^{p-1+r}. Since the differentials

  $$d_r^{p-1,-p} : E_{r,Y}^{p-1,-p}(X) \to E_{r,Y}^{p-1+r,-(p-1+r)}$$

  commute with ψ^k, we have $k^p(k^{r-1}-1)d_r^{p-1,-p} = 0$. Hence, if $r \geq 2$, after tensoring with $\mathbb{Q}$, $d_r^{p-1,-p}$ vanishes, i.e. $E_{2,Y}^{p,-p}(X)_{\mathbb{Q}} \cong E_{\infty,Y}^{p,-p}(X)_{\mathbb{Q}}$. Now we can compute the E_∞ page by filtering, and finally get $E_{\infty,Y}^{p,-p}(X)_{\mathbb{Q}} = \mathrm{Gr}^p K_0^Y(X)_{\mathbb{Q}}$. □

To summarize, we obtain $\mathrm{CH}_Y^p(X)_{\mathbb{Q}} \cong \mathrm{Gr}^p K_0^Y(X)_{\mathbb{Q}}$. □

Exercise 3.1.16. *As a computable case, we compute the Chow groups of $X = \mathbb{P}^1(\mathbb{C})$.*

- *There is a split exact sequence*

  $$0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} K_0'(X) \xrightarrow{\beta} \mathrm{Pic}(X) \longrightarrow 0,$$

  where $\alpha : n \mapsto n[\mathscr{O}_X]$; $\alpha^{-1} = \mathrm{rank}$; $\beta = \det$; $\beta^{-1} : \mathscr{O}_X(P) \mapsto [\mathbb{C}_P] = [\mathscr{O}_X(P)] - [\mathscr{O}_X]$.

- *The exact sequence above induces an isomorphism of groups*

  $$K_0'(X) \xrightarrow{\sim} \mathrm{Pic}(X) \oplus \mathbb{Z} \xrightarrow{\sim} \mathbb{Z}^2, \quad [\mathscr{F}] \mapsto (\det(\mathscr{F}), \mathrm{rank}(\mathscr{F})) \mapsto (\deg(\det(\mathscr{F})), \mathrm{rank}(\mathscr{F})).$$

  Moreover, if one makes $\mathbb{Z}^2$ into a ring by define $(a,b)\cdot(c,d) := (ad+bc, bd)$, so $\mathbb{Z}^2 \cong \mathbb{Z}[x]/(x^2)$. This makes $K_0'(X)$ into a ring, the multiplicative structure is given by the tensor product of $\mathscr{O}_X$-modules. (Indeed, $\mathbb{Z}[x]/(x^2)$ is the Chow ring $\mathrm{CH}^{\cdot}(X)$ of X, where $x = [P]$ corresponds to the skyscraper sheaf $[\mathbb{C}_P]$ in $K_0'(X)$).

- *Verify $\mathrm{Gr}^0 K_0^X(X) \cong \mathbb{Z}$, $\mathrm{Gr}^1 K_0^X(X) \cong \mathbb{Z}$. Hence the Chow ring $\mathrm{CH}^{\cdot}(X) = \mathbb{Z} \oplus \mathbb{Z}$ by Theorem 3.1.8.*

- *Let P be a closed point in X. Verify $\mathrm{Gr}^0 K_0^P(X) = 0$, $\mathrm{Gr}^1 K_0^P(X) \cong \mathbb{Z}$. Hence the Chow ring with supports in P is $\mathrm{CH}_P^{\cdot}(X) = \mathbb{Z}$ by Theorem 3.1.8.*

- *Consider the natural λ-ring structure given by the wedge product on $K_0'(X) \cong \mathbb{Z}^2$. Verify the second component of λ^2 is $(m,n) \mapsto \frac{n(n-1)}{2}$, and the second component of ψ^2 is $(m,n) \mapsto n$.*

3.2 Currents

In this section, we introduce some preliminaries of complex geometry. The arithmetic variety must have a smooth generic fiber, so the infinite part is a smooth projective complex variety. Now let X be a smooth projective complex manifold, we will define some currents on it with respect to some closed irreducible subvarieties $Z \subseteq X$. But Z may not be smooth! Therefore, in order to make a definition of integrating on Z, one may need the resolution of singularities.

3.3 Gillet-Soulé Intersection Pairing

In contrast, the methods in which these theories are established are quite different, but we will assert that these seemingly different geometries are essentially the same.

3.4 Characteristic Classes

3.5 Extra Analytic Terms

3.6 Cohomology

3.7 Arithmetic Riemann-Roch Theorem

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