# A Concise Course in Arakelov Geometry

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# Contents

# Chapter 1

# Curves and Number Theory

In this chapter, we introduce the so-called "geometry of numbers", which studies fractional ideals of number fields by embedding them into the field of complex numbers. One of the most important theorems in this theory is the Riemann-Roch formula for arithmetic varieties.

## 1.1 Fractional Ideals and Invertible Sheaves

Let $K$ be a number field, write $X := \mathrm{Spec}(\mathscr{O}_K)$. There are three ways to study the primes in $\mathscr{O}_K$:

- Number theoretically. A **fractional ideal** of $K$ is a projective $\mathscr{O}_K$-submodule of $K$ of rank 1, a **principal fractional ideal** of $K$ is a fractional ideal has form $x\mathscr{O}_K$ for some $x \in K^\times$. Define the **ideal class group** of $K$ to be

  $$\mathrm{Cl}(\mathscr{O}_K) := \{\text{fractional ideals of } K\}/\{\text{principal fractional ideals of } K\},$$

  with the usual multiplication of ideals. The inverse of a fractional ideal $\mathfrak{a}$ in this group is $\mathfrak{a}^{-1} := \{x \in K : x\mathfrak{a} \subseteq \mathscr{O}_K\}$.

- Geometrically. Consider the scheme $X$, by a **invertible sheaf** (or a **line bundle**) we mean a rank 1 locally free $\mathscr{O}_X$-module on $X$. For a scheme $X$ we can define a **Picard group**

  $$\mathrm{Pic}(X) := \text{isomorphism classes of invertible sheaves on } X,$$

  with the multiplication given by tensor product. The inverse of a invertible sheaf is obtained by dualizing.

- Geometrically. Consider the scheme $X$. A **divisor** on $X$ is a codimension 1 subscheme of $X$. Since $X$ has Krull dimension 1, a divisor must be a finite formal sum of some closed points in $X$. A **principal divisor** on $X$ is a divisor has form $\sum_{\mathfrak{p}} \mathrm{ord}_{\mathfrak{p}}(x)\,\mathfrak{p}$ for some $x \in K^\times$. Define the **divisor class group**

  $$\mathrm{CH}^1(X) := \{\text{divisors on } X\}/\{\text{principal divisors on } X\},$$

with the addition given by the formal sum. The inverse of a divisor is added a minus sign.

Algebraic geometry tells us the second and third methods are essentially the same. That is, studying the codimension 1 closed subschemes is equivalent to studying the line bundles. Indeed, the above three methods are all equivalent.

**Proposition 1.1.1.** *There are isomorphisms*

$$\mathrm{Cl}(\mathscr{O}_K) \cong \mathrm{Pic}(X) \cong \mathrm{CH}^1(X) \cong K^\times \Big\backslash \mathbb{A}_K^\times \Big/ \widehat{\mathscr{O}_K^\times} K_\infty^\times,$$

*where $\mathbb{A}_K^\times$ is the group of units in the ring of adeles of $K$, and*

$$\widehat{\mathscr{O}_K^\times} := \prod_{v \text{ finite}} \mathscr{O}_{K_v}^\times \times \prod_{v \text{ infinite}} \{1\}, \quad K_\infty^\times := \prod_{v \text{ finite}} \{1\} \times \prod_{v \text{ infinite}} K_v^\times.$$

*Proof.* We only give the definitions of these maps.

- Since $X$ is affine, we have $\mathrm{Cl}(\mathscr{O}_K) \cong \mathrm{Pic}(X)$.
- $\mathrm{Cl}(\mathscr{O}_K) \cong \mathrm{CH}^1(X)$ is given by $\prod_{\mathfrak{p}} \mathfrak{p}^{-n_{\mathfrak{p}}} \leftrightarrow \sum_{\mathfrak{p}} n_{\mathfrak{p}} \mathfrak{p}$.
- The map $K^\times \Big\backslash \mathbb{A}_K^\times \Big/ \widehat{\mathscr{O}_K^\times} K_\infty^\times \cong \mathrm{Cl}(\mathscr{O}_K)$ is given by $(x_v)_v \leftrightarrow \prod_{v \text{ finite}} \mathfrak{p}_v^{n_v}$, where $\mathfrak{p}_v$ is the prime ideal corresponding to $v$ and $n_v$ is the $v$-adic valuation of $x_v$.

The remaining work is for you. □

**Remark 1.1.2.** The map from the Picard group $\mathrm{Pic}(X)$ to the divisor class group $\mathrm{CH}^1(X)$ is called the **first Chern class**, this map is given by taking "zeros minus poles" of some rational global section of a invertible sheaf. Its inverse $D \mapsto \mathscr{O}_X(D)$ can be defined as we will do in Chapter 2.1.

Recall we have the product formula

$$\prod_{v \text{ finite}} |x|_v = \prod_{v \text{ infinite}} |x|_v^{-1}$$

for all $x \in K^\times$. It implies that, in the premise of satisfying this formula, in order to obtain a "compactification" of $X$ in a certain sense, some information at infinite places can be added to the divisors defined previously. This is the original idea of Arakelov's theory.

**Definition 1.1.3.** Let us make some definitions in parallel.

- Number theoretically. We use the adelic version of $\mathrm{Cl}(\mathscr{O}_K)$ to generalize the definition. That is, we define the **Arakelov class group** to be the locally compact group
$$\widehat{\mathrm{Cl}}(\mathscr{O}_K) := K^\times \Big\backslash \mathbb{A}_K^\times \Big/ \widehat{\mathscr{O}_K^\times}\, \mathscr{O}_\infty^\times,$$
where $\mathscr{O}_\infty^\times$ is the maximal compact subgroup of $K_\infty^\times$.

- Geometrically. Consider the scheme $X$. A **metrized invertible sheaf** $(\mathscr{L}, \|\cdot\|_\tau)$ (or a **metrized line bundle**) is a invertible sheaf $\mathscr{L}$ together with a collection of non-trivial norms (hence induce Hermitian inner products) $\|\cdot\|_\tau$ on 1 dimensional complex linear spaces $\mathscr{L}\otimes_\tau \mathbb{C}$ for each embedding $\tau \in \mathrm{Hom}(K,\mathbb{C})$, invariant under the action of complex conjugation. Define the **arithmetic Picard group** of $X$ to be

$$\widehat{\mathrm{Pic}}(X) := \text{isometry classes of metrized line bundles on } X.$$

- Geometrically. Consider the scheme $X$. An **Arakelov divisor** on $X$ is an element in the group

$$\widehat{Z}^1(X) := \{\text{divisors on } X\} \oplus \left( \bigoplus_{\tau\in\mathrm{Hom}(K,\mathbb{C})} \mathbb{R} \right)^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})},$$

where $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts by $\tau \to \overline{\tau}$. Inside this group, we form a subgroup

$$\widehat{R}^1(X) := \left\{ \widehat{\mathrm{div}}(x) := \Big( \sum_{\mathfrak{p}} \mathrm{ord}_{\mathfrak{p}}(x)\,\mathfrak{p}, (-\log|\tau(x)|^2)_\tau \Big) : x \in K^\times \right\}.$$

The quotient group $\widehat{\mathrm{CH}}^1(X) := \widehat{Z}^1(X)/\widehat{R}^1(X)$ is called the **arithmetic Chow group**.

For the Arakelov case, we have a conclusion similar to Proposition 1.1.1.

**Proposition 1.1.4.** *There are isomorphisms*

$$\widehat{\mathrm{Cl}}(\mathscr{O}_K) \cong \widehat{\mathrm{Pic}}(X) \cong \widehat{\mathrm{CH}}^1(X).$$

*Proof.* We still only give the definitions of the maps.

- The isomorphism $\widehat{\mathrm{Pic}}(X) \xrightarrow{\sim} \widehat{\mathrm{CH}}^1(X)$ is given by

$$(\mathscr{L}, \|\cdot\|_\tau) \mapsto \widehat{\mathrm{div}}(s) := \left( \sum_{\mathfrak{p}} n_{\mathfrak{p}}\,\mathfrak{p}, (-\log\|s_\tau\|_\tau^2)_\tau \right)$$

for some rational global section $s$ of $\mathscr{L}$, where $s_\tau \in \mathscr{L}\otimes_\tau \mathbb{C}$ is the pull-back of $s$ by $\tau$, and $n_{\mathfrak{p}}$ is the order of vanishing of $s$ at $\mathfrak{p}$.

- To construct an isomorphism from $\widehat{\mathrm{Pic}}(X)$ to $\widehat{\mathrm{Cl}}(\mathscr{O}_K)$, suppose we have a metrized line bundle $(\mathscr{L}, \|\cdot\|_\tau)$ on $X$. Choose a rational section $s$ of $\mathscr{L}$, we associate $s$ an idele

$$\Big( (\varpi_v^{n_v})_{v \text{ finite}}, (\|s_\tau\|_\tau)_{\tau \text{ infinite}} \Big),$$

where $\varpi_v$ is a uniformizer of $K_v$ and $n_v$ is the order of vanishing of $s$ at $v$.

The remaining work is for you. $\square$

The map from $\widehat{\mathrm{Pic}}(X)$ to $\widehat{\mathrm{CH}}^1(X)$ is called the **first arithmetic Chern class**.

If we take $K=\mathbb{Q}$ in Definition 1.1.3, then $\widehat{\mathrm{Cl}}(\mathbb{Z})$ is actually a **Shimura variety** relative to the Shimura datum $(\mathrm{GL}_1, \{\mathrm{pt}\})$ with arithmetic subgroup $\{\pm 1\}$. Moreover, there is a natural isomorphism

$$\mathrm{Log} : \widehat{\mathrm{Cl}}(\mathbb{Z}) = \mathbb{Q}^\times \Big\backslash \mathbb{A}_{\mathbb{Q}}^\times \Big/ \left(\widehat{\mathbb{Z}}^\times \times \{\pm 1\}\right) \xrightarrow{\sim} \mathbb{R}, \quad (x_v)_v \longmapsto \sum_{p<\infty} \mathrm{ord}_p(x_p) \log p - \log |x_\infty|.$$

This map induces maps $\widehat{\mathrm{CH}}^1(\mathrm{Spec}(\mathbb{Z})) \to \mathbb{R}$ and $\widehat{\mathrm{Pic}}(\mathrm{Spec}(\mathbb{Z})) \to \mathbb{R}$ by Proposition 1.1.4. Based on this, we can define the most important invariant of metrized line bundles: the Arakelov degree map (here just only provide the definition in the case of curves, and of course, it can be generalized to higher dimensions using push-forward and the arithmetic Riemann-Roch formula).

**Definition 1.1.5** (Arithmetic Degree)**.** For a metrized line bundle $(\mathscr{L}, \|\cdot\|_\tau)$ on $X$, then $\mathscr{L}$ is a fractional ideal of $K$. Take $0 \neq s \in \mathscr{L}$, define

$$\widehat{\deg}(\mathscr{L}, \|\cdot\|_\tau) := \log \#(\mathscr{L}/s\cdot\mathscr{O}_K) - \sum_{\tau \in \mathrm{Hom}(K,\mathbb{C})} \log \|s_\tau\|_\tau \in \mathbb{R},$$

where $s_\tau \in \mathscr{L} \otimes_\tau \mathbb{C}$ is the pull-back of $s$ by $\tau$. By product formula, this definition is independent of the choice of $s$.

**Exercise 1.1.6.** *Verify the arithmetic degree* $\widehat{\deg} : \widehat{\mathrm{CH}}^1(\mathrm{Spec}(\mathbb{Z})) \to \mathbb{R}$ *is given by*

$$\left(\sum_p n_p[p], n_\infty\right) \mapsto \sum_p n_p \log p + \frac{n_\infty}{2}.$$

## 1.2 Riemann-Roch Theorem

In this section we will show the proof of the arithmetic Riemann-Roch formula for curves.

Let $K$ be a number field and $\mathscr{O}_K$ be its ring of integers, write $X := \mathrm{Spec}(\mathscr{O}_K)$, define $K_\mathbb{R} := \left(\prod_{\tau \in \mathrm{Hom}(K,\mathbb{C})} \mathbb{C}\right)^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} \cong K \otimes_\mathbb{Q} \mathbb{R}$.

Fix an Arakelov divisor $\left(\sum_\mathfrak{p} n_\mathfrak{p}\, \mathfrak{p}, (r_\tau)_\tau\right)$. By Proposition 1.1.4, it corresponds to a metrized line bundle

$$\mathscr{L} = \left(\prod_\mathfrak{p} \mathfrak{p}^{-n_\mathfrak{p}}, \left(\|1_\tau\|_\tau = e^{-\frac{1}{2}r_\tau}\right)_\tau\right).$$

Put $\mathscr{L}_{\mathrm{fin}} := \left(\prod_\mathfrak{p} \mathfrak{p}^{-n_\mathfrak{p}}, (1)_\tau\right)$ and $\mathscr{L}_{\mathrm{inf}} := \left((1), (e^{-r_\tau/2})_\tau\right)$. Define a map

$$j : K \to K_\mathbb{R}, \quad x \mapsto (\tau(x))_\tau,$$

then $j(\mathscr{L}_{\mathrm{fin}})$ is a lattice in $K_\mathbb{R}$. On the other hand, $\mathscr{L}_{\mathrm{inf}}$ induces a linear map

$$\rho_\mathscr{L} : K_\mathbb{R} \to K_\mathbb{R}, \quad (s_\tau)_\tau \mapsto \left(e^{-r_\tau/2} s_\tau\right)_\tau$$

of $\mathbb{R}$-linear spaces.

**Definition 1.2.1** (Characteristic)**.** Define a map

$$\widehat{\chi} : \widehat{\mathrm{Pic}}(X) \to \mathbb{R}, \quad \mathscr{L} \mapsto -\log\big(\mathrm{vol}\big(K_{\mathbb{R}}/(\rho_{\mathscr{L}} \circ j(\mathscr{L}_{\mathrm{fin}}))\big)\big),$$

call it the **arithmetic Euler characteristic**.

**Exercise 1.2.2.** *Verify this definition is well-defined, i.e. $\widehat{\chi}$ depends only on the class in $\widehat{Z}^1(X)$.*

Let $\mathscr{O}_X = \big((1),(1)_\tau\big)$ be the trivial invertible sheaf with the standard metric. The corresponding Arakelov divisor is $\big((0),(0)_\tau\big)$.

**Lemma 1.2.3.** $\widehat{\chi}(\mathscr{O}_X) = -\log\sqrt{|\mathrm{disc}(K/\mathbb{Q})|}$.

*Proof.* Let $e_1,\cdots,e_n$ be an integral basis, then the lattice $j(\mathscr{O}_{X,\mathrm{fin}})$ is spanned by the vectors $(\tau_1(e_j),\cdots,\tau_n(e_j))$ for $j=1,\cdots,n$, where $\tau_i \in \mathrm{Hom}(K,\mathbb{C})$ are embeddings. One can compute the volume is $\mathrm{vol}(K_{\mathbb{R}}/j(\mathscr{O}_{X,\mathrm{fin}})) = |\det(\tau_i(e_j))_{ij}| = \sqrt{|\mathrm{disc}(K/\mathbb{Q})|}$. □

Now we obtain the most important formula in this section:

**Theorem 1.2.4** (Arithmetic Riemann-Roch)**.** *For any $\mathscr{L} \in \widehat{\mathrm{Pic}}(X)$, we have*

$$\widehat{\chi}(\mathscr{L}) - \widehat{\chi}(\mathscr{O}_X) = \widehat{\deg}(\mathscr{L}).$$

*Proof.* Suppose $\mathscr{L} = \big(\prod_{\mathfrak{p}} \mathfrak{p}^{-n_{\mathfrak{p}}}, (e^{-r_\tau/2})_\tau\big)$, then

$$\widehat{\deg}(\mathscr{L}) = \sum_{\mathfrak{p}} n_{\mathfrak{p}} \log \# k(\mathfrak{p}) + \sum_{\tau} \frac{r_\tau}{2}.$$

Consider the sublattice $j(\mathscr{L}_{\mathrm{fin}})$ of $j(\mathscr{O}_{X,\mathrm{fin}})$, it defines a linear endomorphism $\Theta : K_{\mathbb{R}} \to K_{\mathbb{R}}$ such that $\Theta \circ j(\mathscr{O}_{X,\mathrm{fin}}) = j(\mathscr{L}_{\mathrm{fin}})$. Thus

$$|\det(\Theta)| = [j(\mathscr{O}_{X,\mathrm{fin}}) : j(\mathscr{L}_{\mathrm{fin}})] = \Big[\mathscr{O}_K : \prod_{\mathfrak{p}} \mathfrak{p}^{-n_{\mathfrak{p}}}\Big] = \prod_{\mathfrak{p}} \# k(\mathfrak{p})^{-n_{\mathfrak{p}}}.$$

Using this, we compute

$$\begin{aligned}
\widehat{\chi}(\mathscr{L}) &= -\log\big(\mathrm{vol}\big(K_{\mathbb{R}}/(\rho_{\mathscr{L}} \circ j(\mathscr{L}_{\mathrm{fin}}))\big)\big)\\
&= -\log\big(\mathrm{vol}\big(K_{\mathbb{R}}/(\rho_{\mathscr{L}} \circ \Theta \circ j(\mathscr{O}_{X,\mathrm{fin}}))\big)\big)\\
&= -\log\big(\det(\rho_{\mathscr{L}}) \cdot \det(\Theta) \cdot \mathrm{vol}(K_{\mathbb{R}}/j(\mathscr{O}_{X,\mathrm{fin}}))\big)\\
&= \sum_{\tau} \frac{r_\tau}{2} + \sum_{\mathfrak{p}} n_{\mathfrak{p}} \log \# k(\mathfrak{p}) + \widehat{\chi}(\mathscr{O}_X),
\end{aligned}$$

as desired. □

**Remark 1.2.5.** The arithmetic Riemann-Roch formula also studies the behavior of the push-forward of a metrized line bundle, or equivalently speaking, compute how push-forward affects the arithmetic degree. The so-called push-forward is the map

$$\pi_* : \widehat{\mathrm{Pic}}(X) \to \widehat{\mathrm{Pic}}(\mathrm{Spec}(\mathbb{Z})) \text{ or } \pi_* : \widehat{\mathrm{CH}}^1(X) \to \widehat{\mathrm{CH}}^1(\mathrm{Spec}(\mathbb{Z}))$$

induced by $\pi : X \to \mathrm{Spec}(\mathbb{Z})$, which is coincide with the norm map in algebraic number theory.

Indeed, for a metrized line bundle $\mathscr{L}$ on $\mathrm{Spec}(\mathscr{O}_K)$, there is an equality

$$\widehat{\deg}(\pi_* \mathscr{L}) = \widehat{\deg}(\mathscr{L}) - \log \sqrt{|\mathrm{disc}(K/\mathbb{Q})|}.$$

So there is no reason why $\widehat{\deg}$ and $\pi_*$ should commute, the commutativity issue is a well-known problem in algebraic geometry: Grothendieck-Riemann-Roch Theorem.

## 1.3 Counting Points on Metrized Lattices

To study bundles there are so many invariants. In this section, we only introduce one of these important invariants of metrized vector bundles on $X := \mathrm{Spec}(\mathbb{Z})$. It may be viewed as "Minkowski cohomology v.s. Tate cohomology". Recall we have defined metrized line bundles, hence in general, a **metrized vector bundle** $(\mathscr{E}, \|\cdot\|)$ on $X$ is just a finite rank locally free $\mathbb{Z}$-module together with a non-trivial norm on the real linear space $\mathscr{E}_{\mathbb{R}} := \mathscr{E} \otimes_{\mathbb{Z}} \mathbb{R}$. This pair can be viewed as a metrized lattice.

**Definition 1.3.1** (Cohomologies)**.** Let $(\mathscr{E}, \|\cdot\|)$ be a metrized vector bundle, view $\mathscr{E}$ as a subset of $\mathscr{E}_{\mathbb{R}}$ via the natural embedding. Define the **Minkowski cohomology** for $(\mathscr{E}, \|\cdot\|)$ to be a real number

$$h^0_M(X, (\mathscr{E}, \|\cdot\|)) := \log \#\{x \in \mathscr{E} : \|x\| \leq 1\};$$

Define the **Tate cohomology** for $(\mathscr{E}, \|\cdot\|)$ to be a real number

$$h^0_T(X, (\mathscr{E}, \|\cdot\|)) := \log \sum_{x \in \mathscr{E}} e^{-\pi \|x\|^2}.$$

Tate's thesis [Tate] provided a good Riemann-Roch formula for the Tate cohomology of adeles, which comes from the Poisson summation formula. In a more general perspective, the Tate cohomology is just the "partition function". To formulate this formula, one needs to use the Pontryagin duality. It connects the Fourier transform and the inverse Fourier transform.

Let $(\mathscr{E}, \|\cdot\|)$ be a metrized vector bundle, define its **dual bundle** as $(\mathscr{E}^\vee, \|\cdot\|^\vee)$, where $\mathscr{E}^\vee := \mathrm{Hom}_{\mathbb{Z}}(\mathscr{E}, \mathbb{Z})$ and

$$\|\cdot\|^\vee : \mathscr{E}_{\mathbb{R}}^\vee \to \mathbb{R}, \quad \xi \mapsto \|\xi\|^\vee := \max\{|\xi(x)| : x \in \mathscr{E}_{\mathbb{R}}, \|x\| \leq 1\}.$$

The **Fourier transform** for Schwartz functions is

$$\mathscr{F} : \mathscr{S}(\mathscr{E}_{\mathbb{R}}) \xrightarrow{\sim} \mathscr{S}(\widehat{\mathscr{E}_{\mathbb{R}}}), \quad f \longmapsto \left[\rho \mapsto \int_{\mathscr{E}_{\mathbb{R}}} f(x)\overline{\rho(x)} d_{\|\cdot\|} x\right],$$

where $d_{\|\cdot\|}x$ induced by $\|\cdot\|$ is a Haar measure on $\mathscr{E}_{\mathbb{R}}$, and

$$\widehat{\mathscr{E}_{\mathbb{R}}} := \{\rho : \mathscr{E}_{\mathbb{R}} \to S^1 \subseteq \mathbb{C} \text{ continuous homomorphism}\}$$

is the **Pontryagin dual group**. Use the identification

$$\mathscr{E}_{\mathbb{R}}^\vee \xrightarrow{\sim} \widehat{\mathscr{E}_{\mathbb{R}}}, \quad [\xi : x \mapsto \xi(x)] \longmapsto [\rho_\xi : x \mapsto e^{2\pi i \xi(x)}],$$

one can view

$$\mathscr{F}:\mathscr{S}(\mathscr{E}_{\mathbb{R}})\xrightarrow{\sim}\mathscr{S}(\mathscr{E}_{\mathbb{R}}^{\vee}),\quad f\longmapsto\Big[\xi\mapsto\int_{\mathscr{E}_{\mathbb{R}}}f(x)e^{-2\pi i\xi(x)}d_{\|\cdot\|}x\Big].$$

For $f\in\mathscr{S}(\mathscr{E}_{\mathbb{R}})$, recall the **Poisson summation formula** is

$$\sum_{v\in\mathscr{E}}f(x-v)=\frac{1}{\mathrm{vol}(\mathscr{E}_{\mathbb{R}}/\mathscr{E})}\sum_{\xi\in\mathscr{E}^{\vee}}\mathscr{F}(f)(\xi)\cdot e^{2\pi i\xi(x)}.$$

If we take $x=0$ and $f(x):=e^{-\pi\|x\|^2}\in\mathscr{S}(\mathscr{E}_{\mathbb{R}})$, then

$$\sum_{v\in\mathscr{E}}e^{-\pi\|v\|^2}=\frac{1}{\mathrm{vol}(\mathscr{E}_{\mathbb{R}}/\mathscr{E})}\sum_{\xi\in\mathscr{E}^{\vee}}e^{-\pi(\|\xi\|^{\vee})^2},$$

therefore

$$h_T^0(X,(\mathscr{E},\|\cdot\|))-h_T^0(X,(\mathscr{E}^{\vee},\|\cdot\|^{\vee}))=-\log(\mathrm{vol}(\mathscr{E}_{\mathbb{R}}/\mathscr{E})).$$

This is the explicit form of the **Tate-Riemann-Roch formula**. Note the right hand side is just the higher arithmetic Euler characteristic.

We have the following comparison theorem for these two cohomologies:

**Theorem 1.3.2.** *Let $(\mathscr{E},\|\cdot\|)$ be a rank $n$ metrized vector bundle, then*

$$-\pi\le h_T^0(X,(\mathscr{E},\|\cdot\|))-h_M^0(X,(\mathscr{E},\|\cdot\|))\le\frac{n}{2}\log n-\log\Big(1-\frac{1}{2\pi}\Big).$$

*Proof.* Write $Z_{(\mathscr{E},\|\cdot\|)}(t):=\log\sum_{v\in\mathscr{E}}e^{-\pi t\|v\|^2}$. It is easy to see that for any metrized vector bundle $(\mathscr{E},\|\cdot\|)$, $Z_{(\mathscr{E},\|\cdot\|)}(t)$ is a decreasing function when $t>0$. Since the Fourier transform of $f_t(x):=e^{-\pi t\|x\|^2}$ is $\mathscr{F}(f_t)(\xi)=t^{-\frac{n}{2}}e^{-\pi t^{-1}(\|\xi\|^{\vee})^2}$, so by Poisson summation formula, the function

$$\begin{aligned}W(t):=&Z_{(\mathscr{E},\|\cdot\|)}(t)+\frac{n}{2}\log t\\=&\Big(Z_{(\mathscr{E}^{\vee},\|\cdot\|^{\vee})}(t^{-1})-\frac{n}{2}\log t-\log(\mathrm{vol}(\mathscr{E}_{\mathbb{R}}/\mathscr{E}))\Big)+\frac{n}{2}\log t\\=&-\log(\mathrm{vol}(\mathscr{E}_{\mathbb{R}}/\mathscr{E}))+Z_{(\mathscr{E}^{\vee},\|\cdot\|^{\vee})}(t^{-1})\end{aligned}$$

is an increasing function when $t>0$. Now we show for $t>0$,

$$V(t):=\sum_{v\in\mathscr{E}}\|v\|^2e^{-\pi t\|v\|^2}-\frac{n}{2\pi t}\sum_{v\in\mathscr{E}}e^{-\pi t\|v\|^2}\le 0.$$

This is because

$$V(t)=-\frac{1}{\pi}\frac{d}{dt}\left(\sum_{v\in\mathscr{E}}e^{-\pi t\|v\|^2}\right)-\frac{n}{2\pi t}\sum_{v\in\mathscr{E}}e^{-\pi t\|v\|^2},$$

and $\frac{d}{dt}W(t)\ge 0$ implies $V(t)\le 0$. So for $t>0$ and $r>0$ we have

$$\sum_{v\in\mathscr{E},\|v\|\ge r}e^{-\pi t\|v\|^2}\le\frac{1}{r^2}\sum_{v\in\mathscr{E}}\|v\|^2e^{-\pi t\|v\|^2}\le\frac{n}{2\pi t r^2}\sum_{v\in\mathscr{E}}e^{-\pi t\|v\|^2}.$$

Let $r=1$. Choose a real number $t$ such that $t>\frac{n}{2\pi}$ and $t\geq 1$, then

$$\begin{aligned} h^0_M(X,(\mathscr{E},\|\cdot\|)) &= \log \sum_{v\in\mathscr{E},\|v\|\leq 1} 1 \\ &\geq \log \sum_{v\in\mathscr{E},\|v\|\leq 1} e^{-\pi t\|v\|^2} \\ &\geq \log\left(1-\frac{n}{2\pi t}\right) + \left(Z_{(\mathscr{E},\|\cdot\|)}(t) + \frac{n}{2}\log t\right) - \frac{n}{2}\log t \\ &\geq \log\left(1-\frac{n}{2\pi t}\right) - \frac{n}{2}\log t + W(1). \end{aligned}$$

So if we take $t=n$, then

$$h^0_M(X,(\mathscr{E},\|\cdot\|)) - h^0_T(X,(\mathscr{E},\|\cdot\|)) \geq \log\left(1-\frac{1}{2\pi}\right) - \frac{n}{2}\log n.$$

Furthermore,

$$h^0_T(X,(\mathscr{E},\|\cdot\|)) \geq \log \sum_{v\in\mathscr{E},\|v\|\leq 1} e^{-\pi\|v\|^2} \geq \log\left(e^{-\pi}\sum_{v\in\mathscr{E},\|v\|\leq 1} 1\right) = -\pi + h^0_M(X,(\mathscr{E},\|\cdot\|)),$$

so the theorem holds. □

**Remark 1.3.3.** Here we provide the Tate-Riemann-Roch formula for a line bundle (fractional ideal) $\mathfrak{a}$ of $K$ with the standard metric (i.e. view $\mathfrak{a}$ as a lattice in $\mathbb{C}$ via the natural embedding, $\|1\|=1$), where $K$ is an imaginary quadratic field. By arguments in [Tate], the Poisson summation formula expresses a Schwartz function of form $\sum_{a\in\mathfrak{a}} f(z+a)$ $(z\in\mathbb{C})$ as a Fourier expansion in terms of the basis elements from $\mathfrak{a}^\perp := \{\rho\in\widehat{\mathbb{C}} : \rho(\mathfrak{a})=1\}$. Tate also proved:

- $\mathbb{C}\cong\widehat{\mathbb{C}}$ as topological abelian groups, given by $s\mapsto\rho_s$, where $\rho_s : z\mapsto e^{-2\pi i(sz+\overline{sz})}$.
- Under the above isomorphism, one can identify $\mathfrak{a}^\perp\subseteq\widehat{\mathbb{C}}$ as $\mathfrak{a}^{-1}\mathscr{C}_{K/\mathbb{Q}}\subseteq\mathbb{C}$, where $\mathscr{C}_{K/\mathbb{Q}}$ is the codifferent of $K/\mathbb{Q}$ (see Example 2.4.4).

What we need to focus on is $f_\lambda(z) := e^{2\pi i|z|^2\lambda} = e^{2\pi x i|z|^2}e^{-2\pi y|z|^2}$, where $\lambda = x+iy$ with $y>0$ to make sure the series is uniformly convergent (Weierstrass M-test). Now, using the substitution $s=m+in, z=u+iv, d_{\|\cdot\|}z = 2dudv$, the Fourier transform of $f_\lambda$ is

$$\mathscr{F}(f_\lambda)(\rho_s) = \int_{\mathbb{C}} e^{2\pi x i|z|^2}e^{-2\pi y|z|^2}e^{2\pi i(sz+\overline{sz})}d_{\|\cdot\|}z = \frac{i}{\lambda}e^{-2\pi i\frac{|s|^2}{\lambda}} \quad \text{(check this)}.$$

Apply the Poisson summation formula, we obtain (where the coefficient is derived from the codifferent):

$$\sum_{a\in\mathfrak{a}} e^{2\pi i|z+a|^2\lambda} = \frac{1}{\sqrt{|\mathrm{disc}(K/\mathbb{Q})|}}\cdot\frac{1}{\#(\mathscr{O}_K/\mathfrak{a})}\cdot\frac{i}{\lambda}\sum_{b\in\mathfrak{a}^{-1}\mathscr{C}_{K/\mathbb{Q}}} e^{-2\pi i\frac{|b|^2}{\lambda}}e^{-2\pi i(bz+\overline{bz})}.$$

This formula will be used in Chapter 4.4. The readers can investigate what the Tate-Riemann-Roch formula looks like for metrized vector bundles on general $\mathrm{Spec}(\mathscr{O}_K)$.

# Chapter 2

# Surfaces and Arakelov Theory

In this chapter, we introduce the Arakelov theory of surfaces developed by Arakelov and Faltings.

## 2.1 Riemann Surfaces

Let $X$ be a compact Riemann surface.

**Definition 2.1.1** (Weil Functions)**.** Let $D = \{(U, f)\}$ be a Cartier divisor on $X$. A **Weil function** associated with $D$ is a map

$$\lambda_D : X \setminus \mathrm{Supp}(D) \to \mathbb{R},$$

such that for every $P \in U \setminus \mathrm{Supp}(D)$, $\lambda_D(P) = -\log|f(P)| + \alpha(P)$ for some smooth function $\alpha : U \to \mathbb{R}$.

The function $\alpha$ here will be viewed as a metric. Note that $\lambda_D(P) = \infty$ is not well-defined when $P \in \mathrm{Supp}(D)$.

**Definition 2.1.2** (Néron Functions)**.** Let $D = \{(U, f)\}$ be a Cartier divisor on $X$, consider the triple $(U, f, \alpha)$ where $\alpha : U \to \mathbb{R}$ is smooth. We say two triples $(U, f, \alpha)$ and $(V, g, \beta)$ are compatible, if

- $(U, f), (V, g) \in D$. This implies $f/g \in \mathscr{O}_X(U \cap V)^\times$.
- $-\log|f/g| = \beta - \alpha$ holds on $U \cap V$.

A maximal family of compatible triples is called a **Néron divisor**, denoted by $D = \{(U, f, \alpha)\}$. All Néron divisors form an abelian group via

$$(U, f, \alpha) \cdot (V, g, \beta) := (U \cap V, (fg)|_{U\cap V}, (\alpha + \beta)|_{U\cap V}).$$

For a Néron divisor $D = \{(U, f, \alpha)\}$, define the Weil function associated with $D$ to be

$$\lambda_D(P) := -\log|f(P)| + \alpha(P), \quad P \in U \setminus \mathrm{Supp}(D).$$

Néron divisors can be viewed as "metrized" Cartier divisors.

**Exercise 2.1.3.** *$\lambda_D$ is independent of the choice of triple.*

Recall that there is a natural way to identify line bundles (or invertible sheaves) on $X$ with Cartier (or Weil) divisors on $X$. Our goal is to make this correspondence metrically. Let us first review some geometric operations.

Let $\mathscr{L}$ be a line bundle on $X$, i.e. $X = \bigcup_i U_i$ such that for each $i$, $\phi_i : \mathscr{L}|_{U_i} \xrightarrow{\sim} \mathscr{O}_X|_{U_i}$ is an isomorphism of $\mathscr{O}_X|_{U_i}$-module, and satisfies

$$\begin{tikzcd}
\mathscr{L}|_{U_i\cap U_j} \arrow[r, "\mathrm{id}"] \arrow[d, "\phi_j"'] & \mathscr{L}|_{U_i\cap U_j} \arrow[d, "\phi_i"] \\
\mathscr{O}_X|_{U_i\cap U_j} \arrow[r, "\sim", "\phi_{ij}:=\phi_i\circ\phi_j^{-1}\in\mathscr{O}_X(U_i\cap U_j)^\times"'] & \mathscr{O}_X|_{U_i\cap U_j}
\end{tikzcd}$$

These functions $\phi_i$ are called **trivialization functions**. In particular, if $D = \{(U_i, f_i)\}$ is a Cartier divisor, then the line bundle associated with $D$

$$\mathscr{O}_X(D)(U) := \{f \in \mathscr{M}(U) : \mathrm{div}(f) + D \geq 0\}, \quad U \subseteq X$$

has a well-known trivialization

$$\big(U_i, f_i \times (\cdot) : \mathscr{O}_X(D)|_{U_i} \xrightarrow{\sim} \mathscr{O}_X|_{U_i}\big).$$

**Definition 2.1.4** (Metrics)**.** Let $\mathscr{L}$ be a line bundle on $X$ and has trivialization $\mathscr{L} = \{(U,\phi)\}$. Let $h : U \to \mathbb{R}_{>0}$ be smooth functions, consider the triples $(U,\phi,h)$. We say two triples $(U,\phi,h)$ and $(V,\psi,m)$ are compatible, if

$$h(P) = |\phi \circ \psi^{-1}(P)|^2 \cdot m(P), \quad P \in U \cap V.$$

A maximal family of compatible triples is called a **metric** on $\mathscr{L}$, denoted by $(\mathscr{L},h)$. We also call it a **metrized line bundle**.

**Remark 2.1.5.** Let $\mathscr{L} = \{(U,\phi)\}$ be a line bundle on $X$. Let $s \in \Gamma(U,\mathscr{L})$ be a section, for $P \in U$, define a norm (hence induces a Hermitian inner product) on the 1 dimensional $\mathbb{C}$-linear space $\mathscr{L}_P$ (i.e. the fiber of $\mathscr{L}$ at $P$) to be

$$\|s(P)\|_h := \frac{|\phi_P(s(P))|}{\sqrt{h(P)}}.$$

This number does not depend on the choice of trivialization. It is easy to see that $h$ and $\|\cdot\|_h$ are determine each other, so we will abuse them.

**Remark 2.1.6.** There are many ways to construct new metrized line bundles.

- Let $(\mathscr{L},h) = \{(U,\phi,h)\}$ be a metrized line bundle on $X$, define the **dual bundle** $(\mathscr{L}^{-1},h^{-1}) := \{(U,\phi^{-1},h^{-1})\}$. Sometimes we write $\mathscr{L}^\vee$ instead of $\mathscr{L}^{-1}$.

- Let $(\mathscr{L},h) = \{(U,\phi,h)\}$, $(\mathscr{M},m) = \{(U,\psi,m)\}$ be metrized line bundles on $X$, define the **tensor bundle** $(\mathscr{L}\otimes\mathscr{M},h\cdot m) := \{(U,\phi\cdot\psi,h\cdot m)\}$.

- Let $f : X \to Y$ be a morphism and $(\mathscr{L}, h) = \{(U, \phi, h)\}$ be a metrized line bundle on $Y$, then the **pull-back bundle** $f^*\mathscr{L}$ has a metric defined by $(f^*\mathscr{L}, h \circ f) = \{(f^{-1}U, f^\sharp \circ \phi, h \circ f)\}$, where $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$ and the trivialization $f^*\mathscr{L} \to \mathscr{O}_X$ comes from the adjoint pair $(f^*, f_*)$.

**Proposition 2.1.7.** *Let $D = \{(U, f)\}$ be a Cartier divisor on $X$, then there is a one-to-one correspondence:*

$$\{\textit{metrics on } \mathscr{O}_X(D)\} \longleftrightarrow \{\textit{Weil functions associated with } D\},$$

*given by $h \mapsto (-\log|f| + \frac{1}{2}\log h)$; $(\lambda_D = -\log|f| + \alpha) \mapsto e^{2\alpha}$.*

*Proof.* Let $s = (U, s|_U \in \mathscr{M}(U))$ be a meromorphic global section of $\mathscr{O}_X(D)$ such that $\operatorname{div}(s) = D$, we already know there is a natural trivialization

$$f \times (\cdot) : \mathscr{O}_X(D)|_U \xrightarrow{\sim} \mathscr{O}_X|_U, \quad s|_U \mapsto f \cdot s|_U.$$

For a metric $h$ on $\mathscr{O}_X(D)$, define a function associated with $s$ by

$$\lambda_{h,s}(P) := -\log\|s(P)\|_h = -\log\frac{|f(P)\cdot s|_U(P)|}{\sqrt{h(P)}} = -\log|f(P)\cdot s|_U(P)| + \frac{1}{2}\log h(P).$$

Now take $s = 1_D$ and suppose $(\operatorname{Supp}(D)^c, 1) \in D$, then $\log|f(P)\cdot s|_U(P)|$ vanishes, one can verify the bijection easily. $\square$

In fact, the metric on a vector bundle reflects some geometrical and topological information of this bundle. We now introduce the Chern form of a metric, which can be viewed as an important characteristic class in the cohomology group. This class can be obtained from curvature in differential geometry.

But first, let us recall some notations. Let $z = x + iy$ be a local complex coordinate. Define the differential operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right); \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

For a smooth function $f$, define $\partial f := \frac{\partial f}{\partial z}dz \in \mathscr{A}^{1,0}$, $\overline{\partial} f := \frac{\partial f}{\partial \bar{z}}d\bar{z} \in \mathscr{A}^{0,1}$ and $d := \partial + \overline{\partial} \in \mathscr{A}^1$, $d^c := \frac{1}{4\pi i}(\partial - \overline{\partial}) \in \mathscr{A}^1$.

**Exercise 2.1.8.** *Prove:*

$$\partial\overline{\partial} = -\overline{\partial}\partial = -2\pi i d d^c = -\frac{i}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)dx \wedge dy \in \mathscr{A}^{1,1}.$$

**Remark 2.1.9.** This is a warning. Let $(X, g)$ be a $n$ dimensional projective complex manifold with Kähler metric $g$ and its volume form $\mathrm{vol}_g \in \mathscr{A}^{2n}$. It is unreasonable to use Exercise 2.1.8 to define the Laplacian on $X$, since $\sum_{i=1}^{2n}\frac{\partial^2}{\partial x_i^2}$ can not carry metric information and may not glue into a global operator. But from linear algebra, $g$ induces a Hermitian inner product $\widetilde{g}$ on $\bigwedge^k T^*X$, the space of $k$-forms, $0 \le k \le 2n$. Now define the $L^2$-scalar product

$$\langle\cdot,\cdot\rangle_g : \mathscr{A}^k \times \mathscr{A}^k \to \mathbb{C}, \quad \langle\omega, \eta\rangle_g \mapsto \int_X \widetilde{g}(\omega, \eta)\cdot \mathrm{vol}_g.$$

If we write the right adjoint of $d$ for $\langle\cdot,\cdot\rangle_g$ as $d^*$, one can define the Laplacian $\Delta_{\mathrm{dR}} := dd^* + d^*d$, called the **Laplace-de Rham operator**. In Euclidean plane, this Laplacian coincides with the ordinary one (up to a sign).

**Definition 2.1.10** (Chern Forms)**.** Let $(\mathscr{L},h) = \{(U,\phi,h)\}$ be a metrized line bundle on $X$, let $s$ be a holomorphic section on $U$. Define the **Chern form** of $(\mathscr{L},h)$ to be

$$c_1(\mathscr{L},h) := dd^c \log h(z) = -dd^c \log \|s(z)\|_h^2, \quad z \in U \setminus \mathrm{Supp}(\mathrm{div}(s)).$$

Its cohomology class in the de Rham cohomology group $H^2_{\mathrm{dR}}(X)$ is called the **first Chern class**, also denoted by $c_1(\mathscr{L},h)$.

Since the transition functions are holomorphic non-zero, it follows that one can glue $c_1(\mathscr{L},h)$ into a global form in $\mathscr{A}^{1,1}$.

**Remark 2.1.11.** In complex geometry, let $\mathscr{E}$ be a Hermitian vector bundle on a complex manifold $X$. For each $P \in X$, the fiber $\mathscr{E}_P$ is a finite dimensional $\mathbb{C}$-linear space and has a Hermitian inner product

$$\langle\cdot,\cdot\rangle_P : \mathscr{E}_P \times \mathscr{E}_P \to \mathbb{C}.$$

Suppose $\mathscr{E}$ has a frame $\{e_i\}$ composed of global sections. There are some important matrices:

- The metric matrix $H := [\langle e_i, e_j\rangle] \in \mathrm{Mat}_n^{\text{0-form}}$. It is not hard to see $H = \overline{H}^T$.
- The connection matrix $W \in \mathrm{Mat}_n^{\text{1-form}}$. Let

  $$\nabla : \Gamma(U,\mathscr{E}) \to \Gamma(U, T^*X \otimes \mathscr{E}) \cong \Gamma(U, \mathrm{Hom}(TX,\mathscr{E}))$$

  be the connection induced by $H$, then $W := [w_{ij}]$ is defined by $\nabla e_j = \sum_{i=1}^n w_{ij} e_i$. One can show that $W = \overline{H}^{-1} \partial(\overline{H})$.
- The curvature matrix $\Omega = dW + W \wedge W = \overline{\partial}(\partial(H) \cdot H^{-1})^T \in \mathrm{Mat}_n^{\text{2-form}}$, by Bianchi identity.

If $\mathscr{E}$ is a line bundle, then $\Omega = -\partial\overline{\partial}(\log H)$. This explains why we define Chern forms in a strange expression.

**Proposition 2.1.12.** *Let $(\mathscr{L},h)$ be a metrized line bundle on $X$, then*

$$\int_X c_1(\mathscr{L},h) = \deg(\mathscr{L}).$$

*Proof.* Let $s$ be a meromorphic section, so $c_1(\mathscr{L},h) = -dd^c \log \|s(z)\|_h^2$ outside the support of $\mathrm{div}(s)$. At each point $P$ where $s$ has a zero or pole, we pute a small circle $C(P,r)$ of radius $r$. Represent $\|s(z)\|_h^2 = f\overline{f}g$ where $f$ is meromorphic at $P$ and $g$ is smooth positive, apply Stokes' formula we have

$$\int_X c_1(\mathscr{L},h) = \lim_{r\to 0} \sum_P \int_{C(P,r),\curvearrowleft} d^c \log \|s(z)\|_h^2 = \lim_{r\to 0} \sum_P \int_{C(P,r),\curvearrowleft} \frac{\partial - \overline{\partial}}{4\pi i} (\log f + \log \overline{f} + \log g).$$

The $\log g$ term is bounded locally so the integral of this term tends to 0. Obviously, $\overline{\partial}\log f = \partial \log \overline{f} = 0$, so the integral becomes

$$\int_X c_1(\mathscr{L},h) = \frac{1}{4\pi i}\lim_{r\to 0}\sum_P 2i\cdot \operatorname{Im}\int_{C(P,r),\curvearrowleft}\frac{f'}{f}dz = \sum_P \operatorname{ord}_P(f) = \deg(\mathscr{L}),$$

as desired. $\square$

**Exercise 2.1.13.** *The* ***Fubini-Study metric*** *is defined by*

$$h:\mathbb{C}=\mathbb{P}^1(\mathbb{C})\setminus\{\infty\}\to\mathbb{R}_{>0},\quad z\mapsto 1+|z|^2.$$

*Find the curvature $c_1(\mathscr{L},h)$ and which line bundle where h lives in (recall the degree of holomorphic tangent bundle on $\mathbb{P}^1(\mathbb{C})$ is 2).*

## 2.2 Green Functions and Metrics

Let $X$ be a compact Riemann surface. In this section, we study a special case of Weil functions on $X$, which are Green functions.

If the genus $g$ of $X$ is not 0, define the **canonical volume form** on $X$ to be

$$\mu := \frac{i}{2g}\sum_{k=1}^{g}\omega_k\wedge\overline{\omega_k}\in\mathscr{A}^{1,1},$$

where $\omega_1,\cdots,\omega_g$ are orthonormal basis for the Hermitian inner product

$$\Gamma(X,\Omega^1_{X/\mathbb{C}})\times\Gamma(X,\Omega^1_{X/\mathbb{C}})\to\mathbb{C},\quad \langle\omega,\eta\rangle\mapsto\frac{i}{2}\int_X\omega\wedge\overline{\eta}.$$

One can check that $\int_X\mu = 1$.

**Definition 2.2.1** (Green Functions)**.** A **Green function (of logarithmic type) with respect to** $\mu$ is a function $g: X\times X\to\mathbb{R}$ smooth outside the diagonal $\Delta(X)\subseteq X\times X$, and satisfying the following conditions:

For any point $P\in X$,

- Any affine open neighbourhood $U$ of $P$ with local coordinate $z$, we have

$$g(P,z) = -\log|z-P|^2 + \text{real smooth function in } z,\quad z\in U\setminus\{P\}.$$

- For any $z\neq P$,

$$dd^c g(P,z)=\mu.$$

- $\displaystyle\int_X g(P,z)\mu = 0.$

One can prove that the Green function exists uniquely.

**Remark 2.2.2.** In the case of genus $g = 0$, i.e. $X = \mathbb{P}^1(\mathbb{C})$ a Riemann sphere, define a Green function on $\mathbb{P}^1(\mathbb{C})$ in terms of the affine coordinates $(z, w)$ by

$$g(z,w) := -\log \frac{|z-w|^2}{(1+z\bar{z})(1+w\overline{w})},$$

up to an appropriate additive constant. This function with respect to the form

$$\mu := \frac{i}{2\pi} \frac{dz \wedge d\bar{z}}{(1+z\bar{z})^2} \in \mathscr{A}^{1,1}.$$

There is an important formula:

**Proposition 2.2.3.** *Let $X$ be a compact Riemann surface. For all smooth real-valued functions $f$ on $X$,*

$$\int_X g(P,z) dd^c f + f(P) = \int_X f\mu.$$

**Exercise 2.2.4.** *Let $C(P,r)$ be a neighborhood of $P$ of radius $r$, show that:*

- *If $g \in \mathscr{C}^\infty(C(P,r))$ and $f = \gamma \log h + \mathscr{C}^\infty$-function for some constant $\gamma$, then*

$$\lim_{r\to 0} \int_{C(P,r),\curvearrowleft} f d^c g = 0.$$

- *If $g = \log h^2 + \mathscr{C}^\infty$-function and $f$ is continuous, then*

$$\lim_{r\to 0} \int_{C(P,r),\curvearrowleft} f d^c g = f(P).$$

*Proof.* (of Proposition). Write $g_P := g(P,\cdot)$, we calculate directly

$$\begin{aligned}
\int_X (g_P dd^c f - f\mu) &= \int_X (g_P dd^c f - f dd^c g_P) && (\mu = dd^c g_P)\\
&= \int_X d(g_P d^c f - f d^c g_P) && (df \wedge d^c g = dg \wedge d^c f)\\
&= \lim_{r\to 0} \int_{C(P,r),\curvearrowright} (g_P d^c f - f d^c g_P) && \text{(Stokes' formula)}\\
&= \lim_{r\to 0} \int_{C(P,r),\curvearrowleft} f d^c g_P - \lim_{r\to 0} \int_{C(P,r),\curvearrowleft} g_P d^c f && \text{(Exercise 2.2.4)}\\
&= -f(P),
\end{aligned}$$

as desired. $\square$

Let $\omega \in \mathscr{A}^{p,q}$ $(0 \le p,q \le 1)$, define some linear operators

$$[\omega] : \mathscr{A}^{1-p,1-q} \to \mathbb{R}, \quad \eta \mapsto \int_X \omega \wedge \eta$$

and

$$\delta_P : \mathscr{C}^\infty \to \mathbb{R}, \quad f \mapsto f(P).$$

Define $dd^c[\omega](\eta) := -[\omega](dd^c\eta)$, write $g_P := g(P,\cdot)$, then the previous proposition can be expressed as the equality of operators (afterwards, they will be called currents):

$$dd^c[-g_P] + \delta_P = [\mu].$$

This is the well-known **Poincaré-Lelong formula**, see Theorem 3.2.6 for the situation of currents.

**Exercise 2.2.5.** *Let $P_j$ ($j = 1,2$) be two different points, write $g_{P_j} := g(P_j,\cdot)$ are Green functions with respect to $\mu$. Show*

$$g_{P_1}(P_2) - g_{P_2}(P_1) = \int_X (g_{P_1} dd^c g_{P_2} - g_{P_2} dd^c g_{P_1}) = 0.$$

It reminds us to consider the compact complex manifold $X \times X$ and the diagonal divisor $\Delta(X)$. One can choose an appropriate metric $h$ on $\mathscr{O}_{X\times X}(\Delta(X))$ such that if $s$ is a section of $\mathscr{O}_{X\times X}(\Delta(X))$ with $\mathrm{div}(s) = \Delta(X)$, then $-\log\|s\|_h^2$ is the Green function with respect to $\mu$ on $X$. This means that in some neighborhood of $\Delta(X)$ in $X \times X$, one has the expansion

$$g(z,w) = -\log|z-w|^2 + \text{real analytic function in } (z,w).$$

So we can study the analytic properties of Green functions locally in $X \times X$.

**Remark 2.2.6.** The Green functions can be used to define metrics on line bundles on $X$, under the requirements of Proposition 2.1.12 since Green functions are special Weil functions.

- We first consider the case of degree one line bundle $\mathscr{O}_X(P)$ for some prime divisor $P \in X$. Let $1_P$ be a meromorphic section of $\mathscr{O}_X(P)$ which is constant outside $P$, due to Proposition 2.1.7 one can define

$$\|1_P(z)\| := \exp\left(-\frac{1}{2}g(P,z)\right), \quad z \neq P.$$

- For the case of general line bundle

$$\mathscr{O}_X(D) = \bigotimes_P \mathscr{O}_X(P)$$

  where $D = \sum_P P$. Let $1_D$ be a meromorphic section of $\mathscr{O}_X(D)$ which is constant outside $D$, due to Remark 2.1.6 one can define

$$\|1_D(z)\| := \exp\left(-\frac{1}{2}\sum_P g(P,z)\right), \quad z \notin \mathrm{Supp}(D).$$

  We usually write $\sum_P g(P,z)$ as $g(D,z)$.

These metrics are derived from the Green function and are the metrics used in Arakelov geometry. We will emphasize them in the following sections.

**Exercise 2.2.7.** *Under the assumption of Remark 2.2.6, verify Proposition 2.1.12.*

## 2.3 Arakelov Intersection Pairing

Let $K$ be a number field and $\mathcal{O}_K$ its ring of integers. An **arithmetic variety** $X$ is an integral, regular, projective, flat scheme over $\mathcal{O}_K$ of finite type with generic fiber $X_K = X \times_{\mathcal{O}_K} K$ is smooth over $K$. So

$$X(\mathbb{C}) := \bigsqcup_{\tau \in \mathrm{Hom}(K,\mathbb{C})} X_\tau(\mathbb{C}), \quad \text{where } X_\tau(\mathbb{C}) := \text{complex points of } X_K \times_\tau \mathbb{C},$$

is a family of compact Riemann surfaces.

An arithmetic variety with Krull dimension 2 will be called an **arithmetic surface**. The Arakelov theory of arithmetic surfaces is important because it can be calculated directly and used as important examples.

**Definition 2.3.1** (Arakelov Divisors)**.** Let $X$ be an arithmetic surface. Define the group of **Arakelov divisors** on $X$ is the group

$$\widehat{\mathrm{Div}}(X) := \mathrm{Div}(X) \oplus \left( \bigoplus_{\tau \in \mathrm{Hom}(K,\mathbb{C})} \mathbb{R} \cdot X_\tau(\mathbb{C}) \right)^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})},$$

where $\mathrm{Div}(X)$ denotes the group of Weil divisors on $X$ and the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on the infinite part by $\tau \mapsto \overline{\tau}$. Thus, an Arakelov divisor on $X$ is an expression of the type $D = D_{\mathrm{fin}} + D_{\mathrm{inf}}$.

**Definition 2.3.2** (Principal Arakelov Divisors)**.** Let $f \in k(X)^\times$. We associate an Arakelov divisor to $f$ in the following way

$$\widehat{\mathrm{div}}(f) := (f)_{\mathrm{fin}} + (f)_{\mathrm{inf}},$$

where $(f)_{\mathrm{fin}}$ is the principal Weil divisor $\mathrm{div}(f)$ associated with $f$ and

$$(f)_{\mathrm{inf}} := \sum_{\tau \in \mathrm{Hom}(K,\mathbb{C})} \left( -\frac{g_\tau(\mathrm{div}(f_\tau), z)}{2} - \log|f_\tau(z)| \right) \cdot X_\tau(\mathbb{C}),$$

where $g_\tau$ is the unique Green function on $X_\tau(\mathbb{C})$ with respect to the canonical volume form $\mu_\tau$ on $X_\tau(\mathbb{C})$ invariant under $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, and $f_\tau$ is the pull-back of $f$ by $X_\tau(\mathbb{C}) \to X$.

**Exercise 2.3.3.** *Check the coefficients in the sum above*

$$\gamma_\tau(f) := -\frac{g_\tau(\mathrm{div}(f_\tau), z)}{2} - \log|f_\tau(z)|$$

*are constant functions in z.*

These principal Arakelov divisors form a subgroup of $\widehat{\mathrm{Div}}(X)$, the quotient group is denoted by $\widehat{\mathrm{CH}}^1(X, \{\mu_\tau\})$ or simply $\widehat{\mathrm{CH}}^1(X)$, called the **arithmetic Chow group** of $X$.

Arakelov proved that there exists an intersection theory on an arithmetic surface:

**Theorem 2.3.4** (Arakelov)**.** *Let $X$ be an arithmetic surface defined over $\mathscr{O}_K$, the intersection number at a closed point $x \in X$ is denoted by $i_x(\cdot,\cdot)$. With these notations, there exists a unique symmetric bilinear pairing*

$$(\cdot,\cdot) : \widehat{\mathrm{Div}}(X) \times \widehat{\mathrm{Div}}(X) \to \mathbb{R},$$

*satisfying the following conditions:*

- (FINITE DIVISOR, FINITE DIVISOR): $($*D, vertical divisor E lies over a finite prime* $\mathfrak{p}) =$
$$\sum_{x|\mathfrak{p}} i_x(D,E)\log\#k(x).$$
- (FINITE DIVISOR, FINITE DIVISOR): $($*horizontal divisor D, horizontal divisor* $E) = (D,E)_{\mathrm{fin}} + (D,E)_{\mathrm{inf}}$, *where*
$$(D,E)_{\mathrm{fin}} = \sum_{\mathfrak{p}\in\mathrm{Spec}(\mathscr{O}_K)}\sum_{x|\mathfrak{p}} i_x(D,E)\log\#k(x); \quad (D,E)_{\mathrm{inf}} = \sum_{\tau\in\mathrm{Hom}(K,\mathbb{C})}\frac{1}{2}g_\tau(D_\tau,E_\tau).$$
- (FINITE DIVISOR, INFINITE DIVISOR): $($*horizontal divisor D,* $X_\tau(\mathbb{C})) = \deg(D)$.
- (FINITE DIVISOR, INFINITE DIVISOR): $($*vertical divisor D,* $X_\tau(\mathbb{C})) = 0$.
- (INFINITE DIVISOR, INFINITE DIVISOR): $(X_\tau(\mathbb{C}), X_\sigma(\mathbb{C})) = 0$.
- (PRINCIPAL DIVISOR, ANY DIVISOR): $($*principal Arakelov divisor,* $\cdot) = 0$. *Therefore the pair* $\widehat{\mathrm{Div}}(X) \times \widehat{\mathrm{Div}}(X) \to \mathbb{R}$ *defines a symmetric bilinear form on* $\widehat{\mathrm{CH}}^1(X)$.

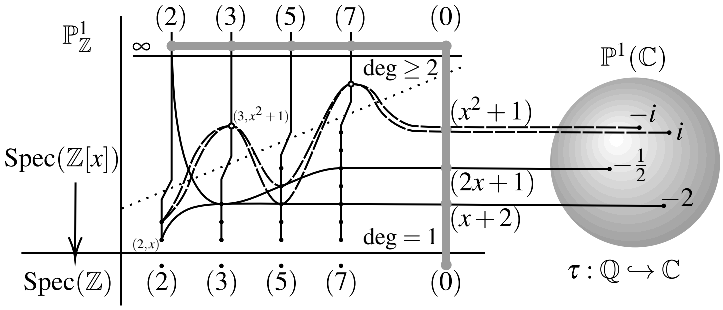
*Proof.* (only prove the last item). For example, given a horizontal divisor $D = D_{\mathrm{fin}}$, write $D_\tau = \sum_{i=1}^{\deg(D)} P_{i,\tau}$ such that each $P_{i,\tau}$ is prime. We have

$$\begin{aligned}
&(\widehat{\mathrm{div}}(f), D)\\
=&((f)_{\mathrm{fin}}, D) + ((f)_{\mathrm{inf}}, D)\\
=&((f)_{\mathrm{fin}}, D)_{\mathrm{fin}} + ((f)_{\mathrm{fin}}, D)_{\mathrm{inf}} + \sum_{\tau\in\mathrm{Hom}(K,\mathbb{C})}\gamma_\tau(f)\deg(D)\\
=&\sum_{\mathfrak{p}\in\mathrm{Spec}(\mathscr{O}_K)} -\log\left(\prod_{x|\mathfrak{p}}\left|f|_D\right|_x\right) + \sum_{\tau\in\mathrm{Hom}(K,\mathbb{C})}\frac{g_\tau(\mathrm{div}(f_\tau),D_\tau)}{2} + \sum_{\tau\in\mathrm{Hom}(K,\mathbb{C})}\gamma_\tau(f)\deg(D)\\
=&\sum_{\mathfrak{p}\in\mathrm{Spec}(\mathscr{O}_K)} -\log\left|\mathrm{Nm}_{k(D)/K}(f|_D)\right|_{\mathfrak{p}} + \sum_{\tau\in\mathrm{Hom}(K,\mathbb{C})}\left(\frac{g_\tau(\mathrm{div}(f_\tau),D_\tau)}{2} + \sum_{i=1}^{\deg(D)}\gamma_\tau(f)\right)\\
=&\sum_{\tau\in\mathrm{Hom}(K,\mathbb{C})}\left(\log\left|\tau\left(\mathrm{Nm}_{k(D)/K}(f|_D)\right)\right| + \sum_{i=1}^{\deg(D)}\left(\frac{g_\tau(\mathrm{div}(f_\tau),P_{i,\tau})}{2} + \gamma_\tau(f)\right)\right)\\
=&\sum_{\tau\in\mathrm{Hom}(K,\mathbb{C})}\left(\log\left|\tau\left(\prod_{\varphi\in\mathrm{Gal}(k(D)/K)}\varphi(f|_D)\right)\right| - \log\left|\tau\left(\prod_{i=1}^{\deg(D)} f_\tau(P_{i,\tau})\right)\right|\right).
\end{aligned}$$

The last term is 0 by Galois theory. $\square$

**Exercise 2.3.5.** *Let $K=\mathbb{Q},\mathscr{O}_K=\mathbb{Z}$, so there is a unique embedding $\tau:\mathbb{Q}\hookrightarrow\mathbb{C}$. Suppose $X=\mathbb{P}^1_{\mathbb{Z}}$, and the Green function $g$ on $X_\tau(\mathbb{C})$ is given by Remark 2.2.2, let $D=\widehat{\operatorname{div}}(x^2+1)$, $E=$ the closed subscheme defined by the prime ideal $(x+2)$, show that:*

- $(D_{\text{fin}},E)_{\text{fin}}=i_{(5,x+2)}(x^2+1,x+2)\log 5=\log 5.$
- $(D_{\text{fin}},E)_{\text{inf}}=\frac{1}{2}(g(i,-2)+g(-i,-2)-2g(\infty,-2))=\log\frac{2}{5}.$
- $\gamma_\tau(x^2+1)=-\log 2$. *Hence the intersection number $(D,E)=0$.*




We can also identify Arakelov divisors with some special metrized line bundles on an arithmetic surface, as in algebraic geometry. The **metrized line bundle** on an arithmetic variety $X$ is a rank one locally free $\mathscr{O}_X$-module $\mathscr{L}$ together with a collection of non-trivial metrized line bundles $\big([X_\tau(\mathbb{C})\to X]^*\mathscr{L},\|\cdot\|_\tau\big)$ on compact complex manifolds $X_\tau(\mathbb{C})$, $\tau\in\operatorname{Hom}(K,\mathbb{C})$, and invariant under $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$. Two metrized line bundles on $X$ are **isometric** if they are isomorphic on $X$ and the pull-back of this isomorphism is an isometry on $X_\tau(\mathbb{C})$.

**Exercise 2.3.6.** *This exercise introduces the height, which is an important invariant measures arithmetic complexity of rational points in Diophantine geometry.*

*Consider the arithmetic surface $X=\mathbb{P}^1_{\mathbb{Z}}$ defined over $\mathbb{Z}$, for $P\in\mathbb{P}^1(\overline{\mathbb{Q}})$, the set of algebraic points of the generic fiber of $X$, let $D_P$ be the Zariski closure of $P$ in $X$. Suppose $D_P$ has normalization $\widetilde{D_P}$. Show that:*

- $\widetilde{D_P}$ *has form* $\operatorname{Spec}(\mathscr{O}_K)$, *where $K$ is the function field of $D_P$.*
- $\widetilde{D_P}$ *is finite and flat over* $\operatorname{Spec}(\mathbb{Z})$.

*Now let $(\mathscr{L},\|\cdot\|)$ be a matrized line bundle on $X$, define the* ***Arakelov height (of algebraic numbers) with respect to*** *$(\mathscr{L},\|\cdot\|)$ as*

$$h_{(\mathscr{L},\|\cdot\|)}:\mathbb{P}^1(\overline{\mathbb{Q}})\to\mathbb{R},\quad P\mapsto\frac{1}{[K:\mathbb{Q}]}\widehat{\deg}\Big(\big(\widetilde{D_P}\to X\big)^*\mathscr{L},\|\cdot\||_{\widetilde{D_P}}\Big).$$

*Show that:*

- *For the metrized line bundle*

$$\overline{\mathscr{O}(1)}:=\big(\textit{Serre twisting sheaf }\mathscr{O}_X(1),\textit{Fubini-Study metric}\big)$$

*(see Exercise 2.1.13), try to compute the Arakelov height $h_{\overline{\mathscr{O}(1)}}$ explicitly.*

- *(Northcott Theorem). For all $A,B>0$, the set*

$$\left\{P\in\mathbb{P}^1(\overline{\mathbb{Q}}):h_{\overline{\mathscr{O}(1)}}(P)<A,\ [\mathbb{Q}(P):\mathbb{Q}]<B\right\}$$

  *is finite.*

Now let us go back to our previous discussion. Suppose

$$D=D_{\text{fin}}+\sum_{\tau\in\mathrm{Hom}(K,\mathbb{C})}r_\tau\cdot X_\tau(\mathbb{C})$$

is an Arakelov divisor on an arithmetic surface. Define a metrized line bundle associated with $D$ to be

$$\widehat{\mathscr{O}_X}(D):=\left(\mathscr{O}_X(D_{\text{fin}}),\left\{\left\|1_{D_{\text{fin}},\tau}(z)\right\|_\tau:=\exp\left(-\frac{1}{2}g_\tau(D_{\text{fin},\tau},z)-r_\tau\right)\right\}\right).$$

A metrized line bundle with this form is called **admissible**.

**Exercise 2.3.7.** *Show that two equivalent Arakelov divisors $D_1\sim D_2$ in $\widehat{\mathrm{CH}}^1(X)$ induce an isometry between $\widehat{\mathscr{O}_X}(D_1)$ and $\widehat{\mathscr{O}_X}(D_2)$.*

Because of the correspondence between arithmetic Chow group and isometry classes of admissible line bundles, one can transplant the intersection theory of divisors to the intersection theory of line bundles. Therefore, when discussing intersections on an arithmetic surface later, we will abuse divisors and admissible line bundles. For example,

**Proposition 2.3.8.** *Let $D=D_{\text{fin}}+D_{\text{inf}}$ be an Arakelov divisor on an arithmetic surface $X$ defined over $\mathscr{O}_K$, and let $E$ be a horizontal prime divisor has form $\mathrm{Spec}(\mathscr{O}_{k(E)})$ with function field $k(E)$. Then $(D,E)=\widehat{\deg}(\widehat{\mathscr{O}_X}(D)|_E)$. The right hand side can be viewed as $(\widehat{\mathscr{O}_X}(D),E)$.*

*Proof.* Write $D_{\text{fin}}=\{(U,f)\},D_{\text{inf}}=\sum_\tau r_\tau\cdot X_\tau(\mathbb{C})$, choose a special rational section $1_{D_{\text{fin}}}$, we can compute

$$\begin{aligned}
&\widehat{\deg}(\widehat{\mathscr{O}_X}(D)|_E)\\
=&\widehat{\deg}\left(\mathscr{O}_X(D_{\text{fin}})|_E,\left\{\left\|1_{D_{\text{fin}},\tau}(E_\tau)\right\|_\tau=\exp\left(-\frac{1}{2}g_\tau(D_{\text{fin},\tau},E_\tau)-r_\tau\deg(E)\right)\right\}\right)\\
=&\log\#\big(\mathscr{O}_X(D_{\text{fin}})|_E/\mathscr{O}_{k(E)}\big)-\sum_{\tau\in\mathrm{Hom}(k(E),\mathbb{C})}\log\|1_{D_{\text{fin}},\tau}(E_\tau)\|_\tau\\
=&\sum_{\mathfrak{p}\in E}\mathrm{ord}_{\mathfrak{p}}(f|_E)\log\#k(\mathfrak{p})+\sum_{\tau\in\mathrm{Hom}(k(E),\mathbb{C})}\left(\frac{1}{2}g_\tau(D_{\text{fin},\tau},E_\tau)+r_\tau\deg(E)\right)\\
=&(D_{\text{fin}},E)_{\text{fin}}+(D_{\text{fin}},E)_{\text{inf}}+(D_{\text{inf}},E).
\end{aligned}$$

The last term is $(D,E)$ by Theorem 2.3.4. □

## 2.4 Adjunction Formula

Recall that the classicial adjunction formula in algebraic geometry states that let $f : X \to Y, g : Y \to Z$ be quasi-projective local complete intersection (l.c.i, to abbreviate) morphisms, then we have a canonical isomorphism

$$\omega_{X/Z} \cong \omega_{X/Y} \otimes_{\mathscr{O}_X} f^* \omega_{Y/Z},$$

where $\omega$ are relative canonical sheaves. This formula can be understood by differential geometry.

Let $Y$ be a 2 dimensional compact complex manifold, and let $X$ be a 1 dimensional regular submanifold of $Y$. For each $P \in X$, there are two linear spaces $T_P X \subseteq T_P Y$. The complementary of $T_P X$ in $T_P Y$ means all normal vectors of $X$ at $P$ relative to $Y$. In the language of sheaf theory, there is an exact sequence of sheaves on $X$

$$0 \to (\Omega^1_{X/\mathbb{C}})^\vee \to (\Omega^1_{Y/\mathbb{C}}|_X)^\vee \to \mathscr{N}\mathrm{or}_{X/Y} \to 0,$$

where $\Omega^1_{(\cdot)/\mathbb{C}}$ means the sheaf of holomorphic 1-forms, i.e. the holomorphic cotangent sheaf of $(\cdot)$, it is a locally free $\mathscr{O}_{(\cdot)}$-module with rank equal to the dimension of $(\cdot)$. Take dual and take determinant of this sequence, we get

$$\omega_{X/Y} := \bigwedge^2 \Omega^1_{Y/\mathbb{C}}|_X \cong \Omega^1_{X/\mathbb{C}} \otimes \mathscr{N}\mathrm{or}^\vee_{X/Y} = \omega_{X/\mathbb{C}} \otimes \mathscr{N}\mathrm{or}^\vee_{X/Y}.$$

If one can show $\mathscr{N}\mathrm{or}_{X/Y} \cong \mathscr{O}_Y(X)|_X$, then the adjunction formula

$$\omega_{X/\mathbb{C}} \cong \omega_{X/Y} \otimes \mathscr{O}_Y(X)|_X$$

holds and can be generalized to general cases. In algebraic geometry, one can study these sheaves locally, just use commutative algebra on each affine open subset.

**Exercise 2.4.1.** *Let $X$ be a 1 dimensional regular submanifold of a 2 dimensional compact complex manifold $Y$. Define the conormal sheaf $\mathscr{N}\mathrm{or}^\vee_{X/Y}$ on $X$ of $i : X \hookrightarrow Y$ to be $i^*(\mathscr{I}/\mathscr{I}^2)$, where $\mathscr{I} := \mathscr{O}_Y(-X)$ (a line bundle). Show that:*

- *$i^*(\mathscr{I}/\mathscr{I}^2) \cong i^*\mathscr{I}$, and so $\mathscr{N}\mathrm{or}^\vee_{X/Y} \cong \mathscr{O}_Y(-X)|_X$. You can check this locally: given a commutative ring $A$ and an ideal $I$, there is an isomorphism $I/I^2 \otimes_A A/I \cong I \otimes_A A/I \cong I/I^2$.*
- *In particular, let $Y = X \times X$ and let $i : X \hookrightarrow X \times X$ be the diagonal embedding. Now $i^*(\mathscr{I}/\mathscr{I}^2) \cong \Omega^1_{X/\mathbb{C}}$. Locally, for a $\mathbb{C}$-algebra $A$, assume $I$ is the kernel of $A \otimes_{\mathbb{C}} A \to A, a_1 \otimes a_2 \mapsto a_1 a_2$ and set a $A$-module structure $a(a_1 \otimes a_2) := aa_1 \otimes a_2$ on it. Define*

$$\Omega^1_{A/\mathbb{C}} := \frac{\text{free } A\text{-module generated by the symbols } da, a \in A}{\langle d(a_1 + a_2) - da_1 - da_2, d(a_1 a_2) - a_1 da_2 - a_2 da_1 : a_i \in A\rangle}.$$

*Recall that $\Omega^1_{X/\mathbb{C}}|_U = \widetilde{\Omega^1_{\mathscr{O}_X(U)/\mathbb{C}}}$ for any affine open subset $U \subseteq X$, so the isomorphism we want is locally given by*

$$\Omega^1_{A/\mathbb{C}} \xrightarrow{\sim} I/I^2, \quad da \mapsto [a \otimes 1 - 1 \otimes a].$$

Let $X$ be an arithmetic surface defined over $\mathscr{O}_K$. We will show that there is an analogy in Arakelov geometry.

**Definition 2.4.2** (Dualizing Sheaves)**.** Let $\pi : X \to \mathscr{O}_K$ be a flat, projective, l.c.i. morphism. Let $i : X \hookrightarrow Y$ be an immersion into $Y$ and $Y$ is smooth over $\mathscr{O}_K$. Consider the following diagram

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "\pi"'] & Y \arrow[d, "p"] \\ & \mathscr{O}_K \end{tikzcd}$$

Analogous to the previous discussion, we define the **relative canonical sheaf** of $\pi$ to be

$$\omega_{X/\mathscr{O}_K} := \det(i^*\Omega^1_{Y/\mathscr{O}_K}) \otimes_{\mathscr{O}_X} \det(\mathscr{N}\mathrm{or}_{X/Y}),$$

where $\Omega^1_{Y/\mathscr{O}_K} := \Delta^*(\mathscr{I}/\mathscr{I}^2)$ and $\mathscr{I} := \mathscr{O}_{Y\times_{\mathscr{O}_K} Y}(-\Delta(Y))$, $\Delta : Y \to Y \times_{\mathscr{O}_K} Y$.

Sometimes we call $\omega_{X/\mathscr{O}_K}$ the **dualizing sheaf** with respect to $\pi$, and abbreviate it as $\omega_\pi$. It can be shown that dualizing sheaf is independent of the choice of the decomposition $X \hookrightarrow Y \to \mathscr{O}_K$ up to isomorphisms.

**Remark 2.4.3.** Let $\omega_\pi$ be a dualizing sheaf with respect to $\pi$, then one can find a trace morphism $\mathrm{tr}_\pi : H^{\cdot}(X, \omega_\pi) \to \mathscr{O}_K$, such that for all coherent sheaves $\mathscr{F}$ on $X$, the natural pairing

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \omega_\pi) \times H^{\cdot}(X, \mathscr{F}) \longrightarrow H^{\cdot}(X, \omega_\pi) \xrightarrow{\mathrm{tr}_\pi} \mathscr{O}_K$$

followed by $\mathrm{tr}_\pi$ gives an isomorphism $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \omega_\pi) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_K}(H^{\cdot}(X, \mathscr{F}), \mathscr{O}_K)$.

**Example 2.4.4.** The concept of dualizing sheaves in number theory is corresponds to the codifferents. Let $L/K$ be a finite extension of number fields and let $\theta : \mathrm{Spec}(\mathscr{O}_L) \to \mathrm{Spec}(\mathscr{O}_K)$ be the induced morphism. Now we have $\omega_\theta \cong \mathrm{Hom}_{\mathscr{O}_K}(\mathscr{O}_L, \mathscr{O}_K)$, since

$$\mathrm{Hom}_{\mathscr{O}_L}(\cdot, \mathrm{Hom}_{\mathscr{O}_K}(\mathscr{O}_L, \mathscr{O}_K)) \cong \mathrm{Hom}_{\mathscr{O}_K}(\cdot, \mathscr{O}_K) = \mathrm{Hom}_{\mathscr{O}_K}(H^0(\mathrm{Spec}(\mathscr{O}_L), \cdot), \mathscr{O}_K).$$

Recall that in algebraic number theory there is an isomorphism

$$\left\{y \in L : \mathrm{tr}_{L/K}(y\mathscr{O}_L) \subseteq \mathscr{O}_K\right\} \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_K}(\mathscr{O}_L, \mathscr{O}_K), \quad y \mapsto \mathrm{tr}_{L/K}(y\cdot).$$

So the dualizing sheaf $\omega_{\mathscr{O}_L/\mathscr{O}_K}$ in this case is just the codifferent $\mathscr{C}_{L/K}$ of a field extension.

Let us do Arakelov geometry now. Let $\tau \in \mathrm{Hom}(K, \mathbb{C})$, also write $\tau : X_\tau(\mathbb{C}) \to X$. Then by base-change, on $X_\tau(\mathbb{C})$ we have $\tau^*\omega_{X/\mathscr{O}_K} \cong \Omega^1_{X_\tau(\mathbb{C})/\mathbb{C}}$. One can equip $\omega_{X/\mathscr{O}_K}$ a suitable metric to make it become an admissible line bundle:

**Proposition 2.4.5.** *The Green function $g_\tau$ on $X_\tau(\mathbb{C}) \times X_\tau(\mathbb{C})$ induces a metric $\|\cdot\|_{\omega,\tau}$ on $\Omega^1_{X_\tau(\mathbb{C})/\mathbb{C}}$ for each $\tau \in \mathrm{Hom}(K, \mathbb{C})$. These metrics make $\widehat{\omega}_{X/\mathscr{O}_K} := (\omega_{X/\mathscr{O}_K}, \|\cdot\|_{\omega,\tau})$ an admissible metrized line bundle.*

*Proof.* Note that

$$\Omega^1_{X_\tau(\mathbb{C})/\mathbb{C}} = \Delta^* \mathscr{O}_{X_\tau(\mathbb{C})\times X_\tau(\mathbb{C})}(-\Delta(X_\tau(\mathbb{C}))),$$

so we should choose the Cartier divisor $(z-w)$ in some neighborhood of $\Delta(X_\tau(\mathbb{C})) \subseteq X_\tau(\mathbb{C}) \times X_\tau(\mathbb{C})$ as a section. Use the expansion

$$\frac{1}{2} g_\tau(z,w) = -\log|z-w| + \text{real analytic function } f_\tau \text{ in } (z,w)$$

and by Proposition 2.1.7 one gets a metric $e^{2f_\tau(z,z)}$ on $(\Omega^1_{X_\tau(\mathbb{C})/\mathbb{C}})^\vee$, hence it induces a metric $e^{-2f_\tau(z,z)}$ on $\Omega^1_{X_\tau(\mathbb{C})/\mathbb{C}}$. $\Big($Furthermore, one can show that

$$\int_{X_\tau(\mathbb{C})} dd^c \log\left(e^{-2f_\tau(z,z)}\right) = \deg\left(\Omega^1_{X_\tau(\mathbb{C})/\mathbb{C}}\right) = 2g(X_\tau(\mathbb{C})) - 2,$$

to prove this formula one needs some analytic techniques which are not introduced, however, you can check this for a special case: Remark 2.2.2$\Big)$. One can verify the Arakelov divisor corresponds to $\widehat{\omega}_{X/\mathscr{O}_K}$ is nothing but $K_{X/\mathscr{O}_K} + 0_{\text{inf}}$, where $K_{X/\mathscr{O}_K}$ is the canonical divisor on $X$. $\square$

**Proposition 2.4.6** (Arithmetic Adjunction Formula)**.** *Let $E$ be a normal horizontal prime divisor on an arithmetic surface $X$ defined over $\mathscr{O}_K$. Suppose $E$ has form* $\mathrm{Spec}(\mathscr{O}_{k(E)})$ *with function field $k(E)$, then*

$$\left(\widehat{\omega}_{X/\mathscr{O}_K} \otimes \widehat{\mathscr{O}_X}(E), E\right) = \log\#\left(\mathscr{C}_{k(E)/K}\big/\mathscr{O}_{k(E)}\right) - \frac{1}{2}\sum_{\tau\in\mathrm{Hom}(k(E),\mathbb{C})}\sum_{i\neq j} g_\tau(P_{i,\tau}, P_{j,\tau}),$$

*where $E_\tau = \sum_{j=1}^{\deg(E)} P_{j,\tau}$. In particular, if* $\deg(E) = 1$ *then* $\left(\widehat{\omega}_{X/\mathscr{O}_K} \otimes \widehat{\mathscr{O}_X}(E), E\right) = 0$.

*Proof.* We have $\omega_{E/\mathscr{O}_K} \cong \widetilde{\mathscr{C}_{k(E)/K}}$ by Example 2.4.4. On the other hand, note that $i : E \hookrightarrow X$ is a closed regular immersion, so $\omega_{E/\mathscr{O}_K} = i^*\omega_{X/\mathscr{O}_K} \otimes \mathscr{N}\mathrm{or}_{E/X}$ by classical adjunction formula. But $i^*\mathscr{O}_X(E) \cong \mathscr{N}\mathrm{or}_{E/X}$, therefore

$$\omega_{E/\mathscr{O}_K} \cong \left(\omega_{X/\mathscr{O}_K} \otimes \mathscr{O}_X(E)\right)\big|_E.$$

Now use Proposition 2.3.8 to compute

$$\left(\widehat{\omega}_{X/\mathscr{O}_K} \otimes \widehat{\mathscr{O}_X}(E), E\right) = \widehat{\deg}(\widehat{\omega}_{E/\mathscr{O}_K}) = \log\#\left(\mathscr{C}_{k(E)/K}\big/\mathscr{O}_{k(E)}\right) - \sum_{\tau\in\mathrm{Hom}(k(E),\mathbb{C})} \log\|1_\tau\|_{\omega,\tau}.$$

By Remark 2.2.6 and Proposition 2.4.5, the metric at $P_{j,\tau}$ in the last term above is

$$\|1_\tau(P_{j,\tau})\|_{\omega,\tau} = \exp\left(\frac{1}{2}\sum_{i\neq j} g_\tau(P_{i,\tau}, P_{j,\tau})\right).$$

It only needs to run out all $j$. $\square$

## 2.5 Faltings-Riemann-Roch Theorem

Recall that the Riemann-Roch formula for a line bundle $\mathscr{L}$ on a Riemann surface $X$ is

$$\chi(\mathscr{L}) := \dim H^0(X,\mathscr{L}) - \dim H^1(X,\mathscr{L}) = 1 - \text{genus}(X) + \deg(\mathscr{L}).$$

In this section, we will introduce the analogy of this formula in Arakelov geometry.

Let $V$ be a $\mathbb{C}$-linear space of dimension $n$, define $\det(V) := \bigwedge^n V$. For $\mathscr{L}$ a line bundle on a genus $g > 0$ Riemann surface $X$, let

$$\begin{aligned}\lambda(R\Gamma(X,\mathscr{L})) &:= \mathrm{Hom}_{\mathbb{C}}\big(\det(H^1(X,\mathscr{L})), \det(H^0(X,\mathscr{L}))\big) \\ &\cong \det(H^0(X,\mathscr{L})) \otimes \det(H^1(X,\mathscr{L}))^\vee.\end{aligned}$$

We call $\lambda(R\Gamma(X,\mathscr{L}))$ the **determinant of cohomology**.

The aim of this section is to discuss some suitable volume forms on the formal difference $H^0(X,\mathscr{L}) - H^1(X,\mathscr{L})$, i.e. some suitable Hermitian inner product on $\lambda(R\Gamma(X,\mathscr{L}))$, when $\mathscr{L}$ is an admissible metrized line bundle (we discard symbols $\|\cdot\|$ or $h$ for simplicity), and discuss how these volume forms (Hermitian inner product) give rise to an Euler characteristic $\chi(\mathscr{L})$ with desirable properties, e.g. for which one has a Riemann-Roch formula.

Let $D$ be a divisor on $X$, and let $P \in X$ be a point. There is an exact sequence

$$0 \to \mathscr{O}_X(D) \to \mathscr{O}_X(D+P) \to \mathbb{C}_P \to 0.$$

The metrics on $\mathscr{O}_X(D)$ and $\mathscr{O}_X(D+P)$ give rise to a metric on $\Gamma(X,\mathbb{C}_P)$, is simply the restriction of the metric on $\mathscr{O}_X(D+P)$ to the fiber at $P$. However, this metric is depend on $D$, so we write $\mathbb{C}_P$ as $\mathbb{C}_P(D)$ to emphasize this.

One has $H^1(X,\mathbb{C}_P(D)) = 0$, so there is a long exact sequence

$$\begin{aligned}0 \to \Gamma(X,\mathscr{O}_X(D)) \to \Gamma(X,\mathscr{O}_X(D+P)) \to \Gamma(X,\mathbb{C}_P(D)) \to \\ \to H^1(X,\mathscr{O}_X(D)) \to H^1(X,\mathscr{O}_X(D+P)) \to 0.\end{aligned}$$

**Exercise 2.5.1.** *Show that:*

- *Let*

  $$0 \to A_1 \to A_2 \to \cdots \to A_n \to 0$$

  *be a long exact sequence of finite dimensional $\mathbb{C}$-linear spaces, then*

  $$\left(\bigotimes_{i\geq 0} \det(A_{2i+1})\right) \otimes \left(\bigotimes_{i\geq 1} \det(A_{2i})^\vee\right) \cong \mathbb{C}.$$

- *The long exact sequence above gives an isomorphism*

  $$\lambda(R\Gamma(X,\mathscr{O}_X(D+P))) \cong \lambda(R\Gamma(X,\mathscr{O}_X(D))) \otimes \Gamma(X,\mathbb{C}_P(D)).$$

Faltings proved the following result in 1984 (here we omit the proof):

**Proposition 2.5.2** (Faltings)**.** *There is a unique way to assign to each admissible metrized line bundle $\mathscr{L}$ on $X$ a Hermitian inner product on $\lambda(R\Gamma(X,\mathscr{L}))$ such that the following properties hold:*

- *An isometry of metrized line bundles induces an isometry of the corresponding $\lambda(R\Gamma(X,\mathscr{L}))$.*
- *If the metric on $\mathscr{L}$ is changed by a factor $C>0$, then the metric on $\lambda(R\Gamma(X,\mathscr{L}))$ is changed by $C^{\chi(\mathscr{L})}$.*
- *The metrics on $\lambda(R\Gamma(X,\mathscr{L}))$ are compatible with the metrics on $\mathbb{C}_P(D)$, in the following sense: Suppose $D$ and $D+P$ are divisors on $X$, then the isomorphsim*

$$\lambda(R\Gamma(X,\mathscr{O}_X(D+P)))\cong\lambda(R\Gamma(X,\mathscr{O}_X(D)))\otimes\Gamma(X,\mathbb{C}_P(D))$$

  *is an isometry.*
- *The metric on $\lambda(R\Gamma(X,\Omega^1_{X/\mathbb{C}}))\cong\bigwedge^g(\Gamma(X,\Omega^1_{X/\mathbb{C}}))$ is the one determined by the Hermitian inner product $\Gamma(X,\Omega^1_{X/\mathbb{C}})\times\Gamma(X,\Omega^1_{X/\mathbb{C}})\to\mathbb{C}$, $\langle\omega,\eta\rangle\mapsto\frac{i}{2}\int_X\omega\wedge\overline{\eta}$.*

Let us do Arakelov geometry now.

**Definition 2.5.3.** Let $M$ be a finitely generated $\mathbb{Z}$-module, suppose on $M\otimes_{\mathbb{Z}}\mathbb{R}$ we have a Haar measure. Define

$$\widehat{\chi}_{\mathbb{Z}}(M):=-\log\left(\mathrm{vol}(M\otimes_{\mathbb{Z}}\mathbb{R}/M)/\#M_{\mathrm{tor}}\right).$$

In the case of $\mathbb{Z}$-module $\mathscr{O}_K$ when $K$ is a number field, we choose the normalized Haar measure on $\mathscr{O}_K\otimes_{\mathbb{Z}}\mathbb{R}$ to be the one (indeed, it is just the usual Lebesgue measure) such that $\mathrm{vol}(\mathscr{O}_K\otimes_{\mathbb{Z}}\mathbb{R}/\mathscr{O}_K)=\sqrt{|\mathrm{disc}(K/\mathbb{Q})|}$.

The following definition generalizes Definition 2.5.3, since $\mathrm{disc}(\mathbb{Q}/\mathbb{Q})=1$.

**Definition 2.5.4.** Let $M$ be a finitely generated $\mathscr{O}_K$-module, define

$$\widehat{\chi}_K(M):=\widehat{\chi}_{\mathbb{Z}}(M)-\mathrm{rank}_{\mathscr{O}_K}(M)\cdot\widehat{\chi}_{\mathbb{Z}}(\mathscr{O}_K).$$

We need a lemma to summarize some properties of bundles on an arithmetic surface.

**Lemma 2.5.5.** *Let $X$ be an arithmetic surface defined over $\mathscr{O}_K$. For any coherent sheaf $\mathscr{F}$ on $X$, we have:*

- *$H^i(X,\mathscr{F})=0$ for $i\geq 2$.*
- *Denote $i:\mathscr{O}_K\hookrightarrow K$, then $H^{\cdot}(X,\mathscr{F})\otimes_{\mathscr{O}_K}K\cong H^{\cdot}(X_K,i^*\mathscr{F})$. Furthermore, if $\tau\in\mathrm{Hom}(K,\mathbb{C})$, then $H^{\cdot}(X,\mathscr{F})\otimes_\tau\mathbb{C}\cong H^{\cdot}(X_\tau(\mathbb{C}),\tau^*\mathscr{F})$.*

*Proof.* We compute the first one by using Čech cohomology. After localizing $\mathscr{O}_K$ at its primes, we are reduced to the case when $R$ is a discrete valuation ring. Since $X$ is projective over $R$, there exist homogeneous polynomials $f_1,\cdots,f_n$ with coefficients in $R$ such that

$$X\cap[f_1=0]\cap\cdots\cap[f_n=0]\text{ is empty.}$$

Now $X$ is covered by affine open subsets $f_j\neq 0$ for $1\leq j\leq n$, so $\check{H}^i(X,\mathscr{F})=0$ for $i\geq 2$. The second item is followed by flat base-change, since $K$ and $\mathbb{C}$ are flat over $\mathscr{O}_K$. □

Let $X$ be an arithmetic surface defined over $\mathscr{O}_K$. For an admissible metrized line bundle $\mathscr{L}$, we only need to consider $H^0(X,\mathscr{L})$ (resp. $H^0(X_\tau(\mathbb{C}),\tau^*\mathscr{L})$) and $H^1(X,\mathscr{L})$ (resp. $H^1(X_\tau(\mathbb{C}),\tau^*\mathscr{L})$) by Lemma 2.5.5.

For any embedding $\tau\in\mathrm{Hom}(K,\mathbb{C})$, we naturally have $\lambda(R\Gamma(X_\tau(\mathbb{C}),\tau^*\mathscr{L}))$, and by Proposition 2.5.2 it admits a Hermitian inner product, i.e. a volume form on the formal difference

$$H^0(X_\tau(\mathbb{C}),\tau^*\mathscr{L}))-H^1(X_\tau(\mathbb{C}),\tau^*\mathscr{L})).$$

By Lemma 2.5.5, this induces a Haar measure on

$$H^0(X,\mathscr{L})\otimes_\tau\mathbb{C}-H^1(X,\mathscr{L})\otimes_\tau\mathbb{C}.$$

This Haar measure is compatible with complex conjugation, so in fact there is a Haar measure on

$$H^0(X,\mathscr{L})\otimes_{\mathscr{O}_K}\mathbb{R}-H^1(X,\mathscr{L})\otimes_{\mathscr{O}_K}\mathbb{R}.$$

Combine these with Definition 2.5.3, we make the following definition.

**Definition 2.5.6.** Let $\mathscr{L}$ be an admissible metrized line bundle on an arithmetic surface $X$ defined over $\mathscr{O}_K$, define

$$\widehat{\chi}(\mathscr{L}):=\widehat{\chi}_K(H^0(X,\mathscr{L}))-\widehat{\chi}_K(H^1(X,\mathscr{L})).$$

The main theorem is:

**Theorem 2.5.7** (Faltings-Riemann-Roch)**.** *For $\mathscr{L}$ an admissible metrized line bundle on an arithmetic surface $X$ defined over $\mathscr{O}_K$, one has the following Riemann-Roch formula*

$$\widehat{\chi}(\mathscr{L})-\widehat{\chi}(\mathscr{O}_X)=\frac{1}{2}(\mathscr{L},\mathscr{L}\otimes\widehat{\omega}^\vee_{X/\mathscr{O}_K}),$$

*where $\mathscr{O}_X$ is equipped with the standard metric.*

*Proof.* (proof sketch). By induction, our goal is to prove the formula changes by the same amount both sides when add some divisor. Firstly, the statement is clearly true for $\mathscr{L}=\mathscr{O}_X$. Let $D$ be an Arakelov divisor on $X$ such that $\mathscr{L}$ is isometric to $\widehat{\mathscr{O}_X}(D)$. Now suppose we add to $D$ a divisor $n\cdot X_\tau(\mathbb{C})$, by definition and the ordinary Riemann-Roch formula one can compute

$$\begin{aligned}
&\frac{1}{2}\Big(D+nX_\tau(\mathbb{C}),D+nX_\tau(\mathbb{C})-\widehat{\omega}_{X/\mathscr{O}_K}\Big)-\frac{1}{2}(D,D-\widehat{\omega}_{X/\mathscr{O}_K})\\
=&\frac{1}{2}\Big(2(nX_\tau(\mathbb{C}),D)+(nX_\tau(\mathbb{C}),nX_\tau(\mathbb{C}))-(nX_\tau(\mathbb{C}),\widehat{\omega}_{X/\mathscr{O}_K})\Big)\\
=&n(X_\tau(\mathbb{C}),D_{\mathrm{fin}})-\frac{n}{2}(X_\tau(\mathbb{C}),\omega_{X/\mathscr{O}_K})\\
=&n(\deg(D_{\mathrm{fin}})+1-g)\\
=&n\big(\dim_{\mathbb{C}}H^0(X_\tau(\mathbb{C}),D_{\mathrm{fin},\tau})-\dim_{\mathbb{C}}H^1(X_\tau(\mathbb{C}),D_{\mathrm{fin},\tau})\big)\\
=&\widehat{\chi}(\widehat{\mathscr{O}_X}(D+nX_\tau(\mathbb{C})))-\widehat{\chi}(\widehat{\mathscr{O}_X}(D)).
\end{aligned}$$

So we may assume $D=D_{\mathrm{fin}}$. Now we should check the same thing when we add to $D$ a prime horizontal divisor $C$ on $X$. To do this, we can replace $K$ with a finite extension

$K'/K$ to make $C' := C \times_{\mathcal{O}_K} \operatorname{Spec}(\mathcal{O}_{K'})$ a section. Write $f : X' := X \times_{\mathcal{O}_K} \operatorname{Spec}(\mathcal{O}_{K'}) \to X$, the problem becomes proving the statement on $X'$ when add a section $C'$. But $X'$ may not regular, so we have to take a desingularization $g : X'' \to X'$, this induces $g : C'' \to C'$. Recall in the semistable case the cohomology groups will not change when pull-back by $g$, and Lemma 2.5.5 gives a description $H^{\cdot}(X', f^*(\cdot)) \cong H^{\cdot}(X, \cdot) \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$. Hence, for simplicity, we may assume $X'' = X$ and $C'' = C$, in this case $C$ is a section with function field $K$. Under these assumption, there is an exact sequence

$$\begin{aligned} 0 \to H^0(X,D) \to H^0(X,D+C) \to H^0\big(\operatorname{Spec}(\mathcal{O}_K), (D+C)|_{\operatorname{Spec}(\mathcal{O}_K)}\big) \to \\ \to H^1(X,D) \to H^1(X,D+C) \to 0. \end{aligned}$$

By properties of $\widehat{\chi}$ (Proposition 2.5.2), we have

$$\widehat{\chi}(D) + \widehat{\chi}_K\big(H^0\big(\operatorname{Spec}(\mathcal{O}_K), (D+C)|_{\operatorname{Spec}(\mathcal{O}_K)}\big)\big) = \widehat{\chi}(D+C).$$

But the middle term is

$$\widehat{\deg}\big((D+C)|_{\operatorname{Spec}(\mathcal{O}_K)}\big) = (D+C,C) = (D,C) - (C, \widehat{\omega}_{X/\mathcal{O}_K})$$

by Proposition 2.3.8 and Proposition 2.4.6. Similarly, one can easily compute

$$\frac{1}{2}\Big((D+C, D+C-\widehat{\omega}_{X/\mathcal{O}_K}) - (D, D-\widehat{\omega}_{X/\mathcal{O}_K})\Big) = (D,C) - (C, \widehat{\omega}_{X/\mathcal{O}_K}).$$

So the arithmetic Riemann-Roch formula holds. $\square$

# Chapter 3

# Higher Arakelov Geometry

In this chapter, we introduce the arithmetic intersection theory on arithmetic varieties developed by Gillet and Soulé.

## 3.1 Some Intersection Theory and K-Theory

There is an important model establishes the intersection theory locally:

**Remark 3.1.1.** Let $R$ be a noetherian regular local ring with residue field $k$. A finitely generated $R$-module has finite length if and only if it is supported at the closed point of $\mathrm{Spec}(R)$. By dévissage, the $K_0$ of the category of modules of finite length is isomorphic to the $K_0$ of the category of $k$-linear spaces, i.e. to $\mathbb{Z}$. Now let $M,N$ are finitely generated $R$-modules (hence have finite length), the supports of which intersect only at the closed point of $\mathrm{Spec}(R)$. Serre defines their **intersection multiplicity**

$$i(M,N) := \sum_{k\geq 0}(-1)^k \ell(\mathrm{Tor}_k^R(M,N)).$$

This formula will be served as the standard model for the general intersection theory.

Let $X$ be a noetherian, regular, separated scheme of dimension $d$ defined over a noetherian ring.

For any $p \in \mathbb{Z}_{\geq 0}$, denoted by $X^{(p)}$ the set of points of codimension $p$ in $X$. Let $Z^p(X)$ be the free abelian group generated by $X^{(p)}$, the elements in it are called $p$**-cycles**. Two $p$-cycles $Z_1, Z_2$ are called **rationally equivalent** if there exist finitely many functions $f_i \in k(y_i)^\times$, $y_i \in X^{(p-1)}$ such that

$$Z_2 - Z_1 = \sum_i \mathrm{div}(f_i),$$

where

$$\mathrm{div}(f_i) = \sum_{x\in X^{(p)}\cap\overline{\{y_i\}}} \mathrm{ord}_{\mathscr{O}_{\overline{\{y_i\}},x}}(f_i)\cdot\overline{\{x\}}.$$

**Definition 3.1.2** (Chow Groups)**.** The $p$-th **Chow group** $\mathrm{CH}^p(X)$ of $X$ is the quotient group

$$\mathrm{CH}^p(X) := Z^p(X)/\text{rational equivalence}.$$

For a closed subscheme $Y \subseteq X$ we define $Z^p_Y(X)$ as the group of cycles of codimension $p$ on $X$ supported in the closed subset attached to $Y$, then define

$$\mathrm{CH}^p_Y(X) := Z^p_Y(X)/\langle \mathrm{div}(f) : f \in k(y)^\times, y \in X^{(p-1)} \cap Y\rangle,$$

call it the Chow group of codimension $p$ of $X$ with supports in $Y$.

**Definition 3.1.3** (Intersections)**.** Two cycles $Y \in Z^p(X), Z \in Z^q(X)$ **intersect properly**, if $\mathrm{codim}_X(Y \cap Z) = p+q$. Assume $Y,Z$ intersect properly, define the **intersection multiplicity** $i_x(Y,Z)$ for $x \in Y \cap Z \cap X^{(p+q)}$ is the integer

$$i_x(Y,Z) := \sum_{k \geq 0} (-1)^k \ell_{\mathscr{O}_{X,x}}(\mathrm{Tor}_k^{\mathscr{O}_{X,x}}(\mathscr{O}_{Y,x}, \mathscr{O}_{Z,x})).$$

Write $(\cdot)_{\mathbb{Q}} := (\cdot) \otimes_{\mathbb{Z}} \mathbb{Q}$ (the reason for tensor $\mathbb{Q}$ is given by K-theory). The main conclusions of this section are as follows.

**Theorem 3.1.4.** *Let $Y,Z$ be closed subschemes of $X$, then there exists a bilinear pairing*

$$(\cdot,\cdot) : \mathrm{CH}^p_Y(X)_{\mathbb{Q}} \times \mathrm{CH}^q_Z(X)_{\mathbb{Q}} \to \mathrm{CH}^{p+q}_{Y \cap Z}(X)_{\mathbb{Q}}$$

*satisfying the following properties:*

- $\bigoplus_Y \bigoplus_p \mathrm{CH}^p_Y(X)_{\mathbb{Q}}$ *is a commutative ring with unit* $[X] \in \mathrm{CH}^0(X)$.
- *It is compatible with change of supports* $\mathrm{CH}^p_Y(X)_{\mathbb{Q}} \to \mathrm{CH}^p_{Y'}(X)_{\mathbb{Q}}$ *associated to inclusions* $Y \subseteq Y'$.
- *For* $[Y_1] \in \mathrm{CH}^p_Y(X), [Z_1] \in \mathrm{CH}^q_Z(X)$ *with* $Y_1, Z_1$ *intersect properly, we have*

$$([Y_1],[Z_1]) \mapsto \left[ \sum_{x \in Y_1 \cap Z_1 \cap X^{(p+q)}} i_x(Y_1,Z_1) \cdot \overline{\{x\}} \right].$$

*In particular, there exists a unique pairing*

$$\mathrm{CH}^p(X)_{\mathbb{Q}} \otimes \mathrm{CH}^q(X)_{\mathbb{Q}} \to \mathrm{CH}^{p+q}(X)_{\mathbb{Q}}$$

*such that for* $Y \in Z^p(X), Z \in Z^q(X)$ *interesting properly, we have*

$$([Y],[Z]) \mapsto \left[ \sum_{x \in Y \cap Z \cap X^{(p+q)}} i_x(Y,Z) \cdot \overline{\{x\}} \right].$$

The pairing above is given by tensor product of bundles in $K_0$ group. So if we want to prove this theorem, we need to introduce some K-theory.

**Definition 3.1.5** (Grothendieck Groups)**.** Let $Y$ be a closed subscheme of $X$. Define:

- $K_0(X)$ to be the **Grothendieck group** of coherent locally free $\mathscr{O}_X$-modules (i.e. finite dimensional vector bundles). More precisely,

$$K_0(X) := \frac{\text{the free abelian group generated by coherent locally free } \mathscr{O}_X\text{-modules}}{\langle \mathscr{F}' - \mathscr{F} + \mathscr{F}'' : 0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0 \text{ exact}\rangle}.$$

- $K_0'(X)$ to be the Grothendieck group of coherent $\mathscr{O}_X$-modules.
- $K_0^Y(X)$ to be the Grothendieck group of bounded complexes of locally free $\mathscr{O}_X$-modules acyclic outside $Y$ modulo quasi-isomorphisms and $(\mathfrak{F}.' - \mathfrak{F}. + \mathfrak{F}.'')$ if there is an exact sequence $0 \to \mathfrak{F}.' \to \mathfrak{F}. \to \mathfrak{F}.'' \to 0$.

Since our $X$ is regular, $K_0(X) \cong K_0'(X)$. The map from $K_0'(X)$ to $K_0(X)$ is given by the finite and locally free resolution

$$0 \to \mathscr{F}_n \to \cdots \to \mathscr{F}_1 \to \mathscr{F}_0 \to \mathscr{F} \to 0$$

of a coherent $\mathscr{O}_X$-module $\mathscr{F}$, for this we can send $[\mathscr{F}]$ to $\sum_{i=0}^n (-1)^i [\mathscr{F}_i]$.

We now review some facts in algebraic geometry, but omit the proof.

**Proposition 3.1.6.** *Let $Y, Z$ (not necessarily regular) be closed subschemes of $X$, their closed immersions to $X$ are denoted as $i$.*

- *(Excision Theorem). There is an exact sequence $K_0'(Y) \to K_0'(X) \to K_0'(X \setminus Y) \to 0$.*
- *There is a bilinear pairing*

$$K_0^Y(X) \times K_0^Z(X) \to K_0^{Y \cap Z}(X), \quad ([\mathfrak{F}.], [\mathfrak{G}.]) \mapsto [\mathrm{Tot}(\mathfrak{F}. \otimes \mathfrak{G}.)],$$

*where the total complex of a double complex $\mathfrak{F}. \otimes \mathfrak{G}.$, $\mathfrak{F}. = \{\mathscr{F}_i, d_i\}, \mathfrak{G}. = \{\mathscr{G}_j, \delta_j\}$, is defined by*

$$\mathrm{Tot}(\mathfrak{F}. \otimes \mathfrak{G}.) := \left\{ \left( \bigoplus_{i+j=n} \mathscr{F}_i \otimes \mathscr{G}_j \right)_n, \bigoplus_{i+j=n} \Big( d_i \otimes \mathrm{id} + (-1)^i \mathrm{id} \otimes \delta_j \Big) \right\}.$$

- *There is an isomorphism $K_0'(Y) \xrightarrow{\sim} K_0^Y(X), [\mathscr{F}] \mapsto [\mathfrak{F}.]$, where $\mathfrak{F}.$ is a finite free resolution of $i_* \mathscr{F}$.*
- *(Projection Formula). Let $f : X \to X'$ be a proper morphism. The homomorphism $f^* : K_0(X') \to K_0(X), [\mathscr{F}'] \mapsto [f^* \mathscr{F}']$ and the homomorphism $f_* : K_0'(X) \to K_0'(X'), [\mathscr{F}] \mapsto \sum_i (-1)^i [R^i f_* \mathscr{F}]$ satisfy the formula*

$$f_*(f^*[\mathscr{F}'] \otimes [\mathscr{F}]) = [\mathscr{F}'] \otimes f_*[\mathscr{F}], \quad \textit{for } [\mathscr{F}] \in K_0'(X), [\mathscr{F}'] \in K_0(X').$$

In order to state Theorem 3.1.4 using K-theory, we make the following definition.

**Definition 3.1.7.** On $K_0^Y(X)$ we define a decreasing filtration

$$K_0^Y(X) = F^0 K_0^Y(X) \supseteq F^1 K_0^Y(X) \supseteq \cdots \supseteq F^d K_0^Y(X) \supseteq F^{d+1} K_0^Y(X) = \{0\}$$

by

$$F^p K_0^Y(X) := \bigcup_{Z \subseteq Y, \mathrm{codim}_X Z \geq p} \mathrm{im}\Big( K_0^Z(X) \to K_0^Y(X) \Big).$$

Define $\mathrm{Gr}^p K_0^Y(X) := F^p K_0^Y(X) / F^{p+1} K_0^Y(X)$.

**Theorem 3.1.8.** *Using the terminologies above,*

- $F^pK_0^Y(X)_{\mathbb{Q}} \cdot F^qK_0^Z(X)_{\mathbb{Q}} \subseteq F^{p+q}K_0^{Y\cap Z}(X)_{\mathbb{Q}}$, *given by take the total complex of the tensor product double complex in Proposition 3.1.6.*

- *Let* $Z \in Z_Y^p(X)$ *be an irreducible cycle, then we can take a finite locally free resolution of* $i_*\mathscr{O}_Z$. *This induces an isomorphism* $\mathrm{CH}_Y^p(X)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Gr}^pK_0^Y(X)_{\mathbb{Q}}$.

We are not going to provide the proof of Theorem 3.1.8 here because it is very complicated and requires a lot of K-theory.

*Proof.* (of Theorem 3.1.4). It is not hard to see that Theorem 3.1.8 implies Theorem 3.1.4, since on $\mathrm{Gr}^{\cdot}K_0(X)_{\mathbb{Q}}$ we can do tensors naturally. The intersection number $i_x(\cdot,\cdot)$ comes from Proposition 3.1.6, because we can define a bilinear pairing

$$K_0'(Y)\times K_0'(Z)\to K_0'(Y\cap Z),\quad ([\mathscr{F}],[\mathscr{G}])\mapsto \sum_k(-1)^k[\mathscr{H}_k(\mathrm{Tot}(\mathfrak{P}.\otimes\mathfrak{Q}.))],$$

where $\mathfrak{P}.\to i_*\mathscr{F}\to 0$, $\mathfrak{Q}.\to i_*\mathscr{G}\to 0$ are free resolutions of $\mathscr{O}_X$-modules (here $i$ denotes the closed immersion to $X$). By homological algebra we have $\mathscr{H}_k(\mathrm{Tot}(\mathfrak{P}.\otimes\mathfrak{Q}.))\cong \mathscr{T}\mathrm{or}_k^{\mathscr{O}_X}(i_*\mathscr{F},i_*\mathscr{G})$, so the intersection number $i_x(Y,Z)$ will be defined to be the local information at $x\in Y\cap Z$ of the image of $([\mathscr{O}_Y],[\mathscr{O}_Z])$. Since $\mathscr{T}\mathrm{or}$ and $i_*$ commute with colimits, this is the alternating sum of the lengthes of the stalks at $x$ of Tor sheaves, due to Remark 3.1.1. $\square$

Note that the complex

$$\cdots\to 0\to 0\to i_*\mathscr{F}\otimes i_*\mathscr{G}\to 0\to 0\to\cdots$$

is isomorphic to $i_*\mathscr{F}\otimes\mathfrak{Q}.$ or $\mathfrak{P}.\otimes i_*\mathscr{G}$ in the derived category, so the right hand side of the map above is equal to $[i_*\mathscr{F}\otimes i_*\mathscr{G}]$ in $K_0'(X)$. Therefore the intersection bilinear form is "almost" the tensor product of $\mathscr{O}_X$-modules.

**Remark 3.1.9.** When $X$ is a smooth variety over a field, the Chow groups can also be defined by using sheaf cohomology. The **Bloch's formula** tells us that there is an isomorphism:

$$\mathrm{CH}^p(X)\cong H^p\Big(X,\big(U\mapsto \pi_{p+1}\mathscr{B}\mathscr{Q}\{\text{finitely generated projective }\mathscr{O}_X(U)\text{-module}\}\big)^{\dagger}\Big),$$

where $\dagger$ means sheafification, $\mathscr{Q}$ means add some arrows to the category, $\mathscr{B}$ means geometric realization (to make a category into a topological space) and $\pi_{p+1}$ means the $(p+1)$-th homotopy group. Some technical tools can be found in Quillen's higher K-theory and homotopy theory.

**Exercise 3.1.10.** *As a computable case, we compute the Chow groups of* $X=\mathbb{P}^1(\mathbb{C})$.

- *If on X there is*

$$\bigwedge^k\big(\mathscr{O}_X(l)^{\oplus n}\big)=\mathscr{O}_X(kl)^{\oplus\Phi(n,k)},$$

*find* $\Phi(n,k)$. *In particular,* $\bigwedge^n(\mathscr{O}_X(l)^{\oplus n})=\mathscr{O}_X(nl)$.

- *There is a split exact sequence*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} K_0'(X) \xrightarrow{\beta} \mathrm{Pic}(X) \longrightarrow 0,$$

*where* $\alpha : n \mapsto n[\mathscr{O}_X]$*;* $\alpha^{-1} = \mathrm{rank}$*;* $\beta = \det$*;* $\beta^{-1} : \mathscr{O}_X(P) \mapsto [\mathbb{C}_P] = [\mathscr{O}_X(P)] - [\mathscr{O}_X]$.

- *The exact sequence above induces an isomorphism of groups*

$$K_0'(X) \xrightarrow{\sim} \mathrm{Pic}(X) \oplus \mathbb{Z} \xrightarrow{\sim} \mathbb{Z}^2, \quad [\mathscr{F}] \mapsto (\det(\mathscr{F}), \mathrm{rank}(\mathscr{F})) \mapsto (\deg(\det(\mathscr{F})), \mathrm{rank}(\mathscr{F})).$$

*Moreover, if one makes* $\mathbb{Z}^2$ *into a ring by define* $(a,b)\cdot(c,d) := (ad+bc, bd)$*, so* $\mathbb{Z}^2 \cong \mathbb{Z}[x]/(x^2)$*. This makes* $K_0'(X)$ *into a ring, the multiplicative structure is given by the tensor product of* $\mathscr{O}_X$*-modules. (Indeed,* $\mathbb{Z}[x]/(x^2)$ *is the Chow ring* $\mathrm{CH}^{\cdot}(X)$ *of* $X$*, where* $x = [P]$ *corresponds to the skyscraper sheaf* $[\mathbb{C}_P]$ *in* $K_0'(X)$*).*

- *Verify* $\mathrm{Gr}^0 K_0^X(X) \cong \mathbb{Z}$*,* $\mathrm{Gr}^1 K_0^X(X) \cong \mathbb{Z}$*. Hence the Chow ring* $\mathrm{CH}^{\cdot}(X) = \mathbb{Z} \oplus \mathbb{Z}$ *by Theorem 3.1.8.*

- *Let* $P$ *be a closed point in* $X$*. Verify* $\mathrm{Gr}^0 K_0^P(X) = 0$*,* $\mathrm{Gr}^1 K_0^P(X) \cong \mathbb{Z}$*. Hence the Chow ring with supports in* $P$ *is* $\mathrm{CH}^{\cdot}_P(X) = \mathbb{Z}$ *by Theorem 3.1.8.*

## 3.2 Green Currents

In this section, we introduce some preliminaries of complex geometry. The arithmetic variety must have a smooth generic fiber, so the infinite part is a smooth projective complex variety. Now let $X$ be a smooth projective complex manifold, we will define some currents on it with respect to some closed irreducible subvarieties $Z \subseteq X$. But $Z$ may not be smooth! Therefore, in order to make a definition of integrating on $Z$, one may need the resolution of singularities.

Now we review some basic concepts of differential forms.

Let $X$ be a smooth projective complex equidimensional variety of complex dimension $d$, denote

$$\mathscr{A}^{p,q} := \text{the linear space of } \mathbb{C}\text{-valued smooth differential forms of type } (p,q).$$

More precisely, if $(z_1, \cdots, z_d)$ are local coordinates, then any element in $\mathscr{A}^{p,q}$ has form

$$\sum_{\substack{1\le i_1<\cdots<i_p\le d \\ 1\le j_1<\cdots<j_q\le d}}^{<\infty} f_{i_1,\cdots,i_p;j_1,\cdots,j_q}(z_1,\cdots,z_d;\overline{z_1},\cdots,\overline{z_d})dz_{i_1}\wedge\cdots\wedge dz_{i_p}\wedge d\overline{z_{j_1}}\wedge\cdots\wedge d\overline{z_{j_q}},$$

where $f_{i_1,\cdots,i_p;j_1,\cdots,j_q}$ are smooth functions. Denote by $\mathscr{A}^n := \bigoplus_{p+q=n} \mathscr{A}^{p,q}$ the space of differential forms of degree $n$, and denote by $\partial$, $\overline{\partial}$, $d$ the usual differentials.

**Definition 3.2.1** (Currents)**.** Define

$$\mathscr{D}_n := \left\{ F : \mathscr{A}^n \to \mathbb{C} \text{ linear} : \begin{array}{l} \text{for any compact } K, \text{ if } \{\omega_r\} \subseteq \mathscr{A}^n \text{ s.t. } \mathrm{Supp}(\omega_r) \subseteq K \\ \text{and all derivatives of all coefficients of } \omega_r \rightrightarrows 0 \text{ on } K \\ \text{if } r \to \infty, \text{ then } F(\omega_r) \to 0. \end{array} \right\},$$

then we obtain the decomposition $\mathscr{D}_n = \bigoplus_{p+q=n} \mathscr{D}_{p,q}$. Now define the space of $n$-**currents** (resp. $(p,q)$-currents) to be $\mathscr{D}^n := \mathscr{D}_{d-n}$ (resp. $\mathscr{D}^{p,q} := \mathscr{D}_{d-p,d-q}$).

The differentials $\partial, \overline{\partial}, d$ induce:

- $\partial : \mathscr{D}^{p,q} \longrightarrow \mathscr{D}^{p+1,q}$, given by
$$\big(T : \mathscr{A}^{d-p,d-q} \to \mathbb{C}, \omega \mapsto T(\omega)\big) \longmapsto \big(\partial T : \mathscr{A}^{d-p-1,d-q} \to \mathbb{C}, \omega \mapsto (-1)^{p+q+1} T(\partial\omega)\big).$$

- $\overline{\partial} : \mathscr{D}^{p,q} \longrightarrow \mathscr{D}^{p,q+1}$, given by
$$\big(T : \mathscr{A}^{d-p,d-q} \to \mathbb{C}, \omega \mapsto T(\omega)\big) \longmapsto \big(\overline{\partial} T : \mathscr{A}^{d-p,d-q-1} \to \mathbb{C}, \omega \mapsto (-1)^{p+q+1} T(\overline{\partial}\omega)\big).$$

- $d = \partial + \overline{\partial} : \mathscr{D}^{p,q} \longrightarrow \mathscr{D}^{p+q+1}$, given by
$$\big(T : \mathscr{A}^{d-p,d-q} \to \mathbb{C}, \omega \mapsto T(\omega)\big) \longmapsto \big(dT : \mathscr{A}^{d-p-q-1} \to \mathbb{C}, \omega \mapsto (-1)^{p+q+1} T(d\omega)\big).$$

Let us complete the definitions that first appeared in Section 2.2, which give several important examples.

- Let $\omega \in L^1 \otimes_{\mathscr{C}^\infty} \mathscr{A}^{p,q}$, then we can define a current $[\omega] \in \mathscr{D}^{p,q}$ induced by $\omega$ to be
$$[\omega](\eta) := \int_X \omega \wedge \eta, \quad \eta \in \mathscr{A}^{d-p,d-q}.$$

- Let $Y$ be a codimension $p$ irreducible **smooth** complex submanifold of $X$, then we get a **Dirac current** $\delta_Y \in \mathscr{D}^{p,p}$ defined by
$$\delta_Y(\eta) := \int_Y \eta, \quad \eta \in \mathscr{A}^{d-p,d-p}.$$

- More generally, for a codimension $p$ irreducible complex submanifold $Y$ (not necessarily smooth) of $X$ with embedding $i : Y \hookrightarrow X$, define a current $\delta_Y \in \mathscr{D}^{p,p}$ by
$$\delta_Y(\eta) := \int_{\text{non-singular locus of } Y} i^*\eta, \quad \eta \in \mathscr{A}^{d-p,d-p}.$$

  The well-definedness of this $\delta_Y$ is given by the following Hironaka's theorem on the resolution of singularities.

  **Theorem 3.2.2** (Hironaka)**.** *Given any $Z \subseteq Y$, where $Z$ contains the singular locus of $Y$, there exists a proper map $\pi : \widetilde{Y} \to Y$ such that:*

  - *$\widetilde{Y}$ is smooth.*
  - *$\pi^{-1}(Z)$ is a divisor with normal crossings.*
  - *$\pi : \widetilde{Y} \setminus \pi^{-1}(Z) \to Y \setminus Z$ is an isomorphism.*
  - $\delta_Y(\eta) = \int_{Y \setminus Z} i^*\eta = \int_{\widetilde{Y}} \pi^* i^* \eta.$

  By linearity we extend this definition to arbitrary codimension $p$ complex submanifolds.

One can check that:

**Exercise 3.2.3.** *There are some identities:*

- $dd^cT(\omega) = -T(dd^c\omega)$, *so* $dd^c[\omega](\eta) = -[\omega](dd^c\eta)$.
- $d[\omega] = [d\omega]$ *(use Stokes' formula). But* $d^c[\omega] \neq [d^c\omega]$, *so* $dd^c[\omega] \neq [dd^c\omega]$ *in general. However, if* $\omega$ *is smooth, then* $dd^c[\omega] = [dd^c\omega]$.

Recall Proposition 2.2.3 states that if $g_P$ is a Green function (of logarithmic type) with respect to $\mu$, then $dd^c[-g_P] + \delta_P = [\mu]$. Refer to this fact, we may define a class of special currents. First of all, let us review the definition of forms of logarithmic type, this is to get some moderate growing forms to make the integrals converge.

A smooth form $\omega$ on $X \setminus Y$ is said to be **of logarithmic type along** $Y$, if there exists a projective map $\pi : \widetilde{X} \to X$ such that $\pi^{-1}(Y)$ is a divisor with normal crossings, $\pi : \widetilde{X} \setminus \pi^{-1}(Y) \to X \setminus Y$ is smooth and $\omega$ is the direct image by $\pi$ of a form $\alpha$ on $\widetilde{X} \setminus \pi^{-1}(Y)$ with the following property: near each $x \in \widetilde{X}$, let $z_1 \cdots z_k = 0$ be a local equation of $\pi^{-1}(Y)$, then there exists $\partial$ and $\overline{\partial}$ closed smooth forms $\alpha_i$ and a smooth form $\gamma$ such that

$$\alpha = \sum_{i=1}^{k} \alpha_i \log|z_i|^2 + \gamma.$$

**Definition 3.2.4** (Green Currents)**.** A **Green current** for a codimension $p$ complex submanifold $Y$ (not necessarily irreducible, but does not contain any irreducible components of $X$), is a current $g_Y \in \mathscr{D}^{p-1,p-1}$ such that $-dd^c g_Y + \delta_Y = [\omega]$ for some $\omega \in \mathscr{A}^{p,p}$.

**Theorem 3.2.5.** *If $X$ is Kähler, then every $Y \subseteq X$ has a Green current. If $g_1, g_2$ are two Green currents for $Y$, then*

$$g_1 - g_2 = [\eta] + \partial S_1 + \overline{\partial} S_2$$

*for some* $\eta \in \mathscr{A}^{p-1,p-1}$, $S_1 \in \mathscr{D}^{p-2,p-1}$, $S_2 \in \mathscr{D}^{p-1,p-2}$. *In particular, there exists a smooth form $g_Y$ on $X \setminus Y$ of logarithmic type along $Y$ such that $[g_Y]$ is a Green current for $Y$, i.e. $dd^c[-g_Y] + \delta_Y = [\omega]$ for some smooth form $\omega$.*

*Proof.* First, let us show that if $T \in \mathscr{D}^{p,p}$ with $T = dS$ for some current $S$, then $T = dd^cU$ with $U \in \mathscr{D}^{p-1,p-1}$. This is the so-called $\partial\overline{\partial}$-lemma.

Recall the **Hodge decomposition** states that for a Kähler manifold $X$ we get adjoints $\partial^*, \overline{\partial}^*, d^*$ of $\partial, \overline{\partial}, d$ for the $L^2$-scalar product on forms (see Remark 2.1.9), and if $\mathbb{H}^{p,q} := \ker(\Delta_{\mathrm{dR}})$ denotes the space of **harmonic forms** then under the $L^2$-scalar product we have

$$\mathscr{A}^{p,q} = \mathbb{H}^{p,q} \oplus \mathrm{im}(d) \oplus \mathrm{im}(d^*) = \mathbb{H}^{p,q} \oplus \mathrm{im}(\partial) \oplus \mathrm{im}(\partial^*) = \mathbb{H}^{p,q} \oplus \mathrm{im}(\overline{\partial}) \oplus \mathrm{im}(\overline{\partial}^*),$$

(or there may be such a decomposition for $\mathscr{D}^{p,q}$). Hence, if write $T = \partial S + \overline{\partial} S$, by Hodge decomposition $S = h_1 + \partial x_1 + \partial^* y_1 = h_2 + \overline{\partial} x_2 + \overline{\partial}^* y_2$, so $\partial S = \partial\overline{\partial} x_2 + \partial\overline{\partial}^* y_2$ and $\overline{\partial} S = \overline{\partial}\partial x_1 + \overline{\partial}\partial^* y_1$. Thus

$$T = \overline{\partial}\partial x_1 + \partial\overline{\partial} x_2 + \partial\overline{\partial}^* y_2 + \overline{\partial}\partial^* y_1.$$

Now $dT = 0$ implies $\partial T = \overline{\partial} T = 0$, so $\partial\overline{\partial}\partial^* y_1 = 0$ and $\overline{\partial}\partial\overline{\partial}^* y_2 = 0$. Therefore

$$0 = \langle \partial\overline{\partial}\partial^* y_1, \overline{\partial} y_1 \rangle_{L^2} = -\langle \overline{\partial}\partial^* y_1, \overline{\partial}\partial^* y_1 \rangle_{L^2},$$

so $\overline{\partial}\partial^* y_1 = 0$, and similarly $\partial\overline{\partial}^* y_2 = 0$. Hence $T = \partial\overline{\partial}(x_2 - x_1)$, i.e. $U = 2\pi i(x_1 - x_2)$.

Here we only prove the existence of Green currents (we omit the proof of the existence of logarithmic type Green currents). By Stokes' formula we have $d\delta_Y = 0$, hence by Hodge decomposition we deduce $\delta_Y = [\omega] + dS$ for some $\omega \in \mathscr{A}^{p,p}$ and some current $S$. By $\partial\overline{\partial}$-lemma, we have $[\omega] - \delta_Y = -dS = -dd^c g$ for some $g \in \mathscr{D}^{p-1,p-1}$. $\square$

**Theorem 3.2.6** (Poincaré-Lelong Formula)**.** *Let $\mathscr{L}$ be a Hermitian line bundle on $X$ with metric $\|\cdot\|$, suppose $s$ is a meromorphic section of $\mathscr{L}$, then $-\log\|s\|^2 \in L^1$ hence induces $[-\log\|s\|^2] \in \mathscr{D}^{0,0}$. This is a Green current for* $\operatorname{div}(s)$*, in fact*

$$dd^c[\log\|s\|^2] + \delta_{\operatorname{div}(s)} = [c_1(\mathscr{L}, \|\cdot\|)].$$

*Proof.* Refer to the proof of Proposition 2.1.12. Note that $c_1(\mathscr{L}, \|\cdot\|) = -dd^c \log\|s\|^2$ for some meromorphic section $s$. After resolving the singularities, we may assume that in a local chart $U \xrightarrow{\sim} \mathbb{C}^d$, $\operatorname{div}(s)$ has equation $z_1 \cdots z_k = 0$. By linearity we are reduced to the case $s = z_1$. Since $-dd^c \log|z_1|^2 = 0$, apply Exercise 3.2.3, what we have to show is

$$\int_U \log|z_1|^2 dd^c\omega = \int_{z_1=0} \omega,$$

where $\omega \in \mathscr{A}^{d-1,d-1}$ with compact support in $U$. Indeed,

$$\begin{aligned}
&\int_U \log|z_1|^2 dd^c\omega \\
=&\lim_{r\to 0}\int_{|z_1|\geq r} \log|z_1|^2 \wedge dd^c\omega \\
=&\lim_{r\to 0}\int_{|z_1|=r,\curvearrowright} \log|z_1|^2 \wedge d^c\omega - \lim_{r\to 0}\int_{|z_1|\geq r} d\log|z_1|^2 \wedge d^c\omega && \text{(Stokes' formula)} \\
=&\lim_{r\to 0}\int_{|z_1|\geq r} d^c\log|z_1|^2 \wedge d\omega && \text{(Exercise 2.2.4)} \\
=&\lim_{r\to 0}\int_{|z_1|\geq r} dd^c\log|z_1|^2 \wedge \omega - \lim_{r\to 0}\int_{|z_1|=r,\curvearrowright} d^c\log|z_1|^2 \wedge \omega && \text{(Stokes' formula)} \\
=&\int_{z_1=0} \omega. && (dd^c\log|z_1|^2 = 0)
\end{aligned}$$

The last term is because $d^c\log|z_1|^2 = \frac{\partial-\overline{\partial}}{4\pi i}\log(z_1\overline{z_1}) = \frac{1}{2\pi}\operatorname{Im}(\frac{dz_1}{z_1}) = \frac{d\arg(z_1)}{2\pi}$. $\square$

As the end of this section, we introduce the $\star$-product of Green currents.

**Definition 3.2.7** ($\star$-Product)**.** Let $g_Y$ be a form of logarithmic type for $Y$ given by Theorem 3.2.5 such that $dd^c[-g_Y] + \delta_Y = [\omega_Y]$, let $g_Z$ be a Green current for $Z$. Define their **$\star$-product** to be

$$[g_Y] \star g_Z := [g_Y] \wedge \delta_Z + [\omega_Y] \wedge g_Z,$$

where $([g_Y] \wedge \delta_Z)(\eta) := \int_Z g_Y \wedge \eta$ (assume the singularities have been resolved) and $([\omega_Y] \wedge g_Z)(\eta) := g_Z(\omega_Y \wedge \eta)$.

This definition is well-defined, but not trivial. In contrast, the following fact is more important.

**Proposition 3.2.8.** *If $Y, Z$ intersect properly, then*

$$-dd^c([g_Y]\star g_Z) = [\omega_Y \wedge \omega_Z] - \sum_x i_x(Y,Z)\delta_x,$$

*where $x$ runs out of all irreducible components of $Y\cap Z$.*

*Proof.* We prove this formally. Indeed,

$$\begin{aligned} dd^c([g_Y]\star g_Z) &= dd^c[g_Y]\wedge\delta_Z + [\omega_Y]\wedge dd^c g_Z \\ &= (\delta_Y - [\omega_Y])\wedge\delta_Z + [\omega_Y]\wedge(\delta_Z - [\omega_Z]) \\ &= \delta_Y\wedge\delta_Z - [\omega_Y]\wedge[\omega_Z] \\ &= -([\omega_Y\wedge\omega_Z] - \delta_{Y\cap Z}), \end{aligned}$$

as desired. □

**Remark 3.2.9.** Let $Y\subseteq X$ be a closed irreducible submanifold and $g_Y$ a Green current for $Y$. By Theorem 3.2.5, there exists a Green form $\widetilde{g_Y}$ of logarithmic type for $Y$ such that

$$g_Y - [\widetilde{g_Y}] = [\eta] + \partial S_1 + \overline{\partial} S_2,$$

so every Green current for $Y$ may be represented by a Green form of logarithmic type along $Y$ modulo $\mathrm{im}(\partial)+\mathrm{im}(\overline{\partial})$, since $\eta$ is a smooth form. Hence, if $Y,Z\subseteq X$ are closed irreducible submanifolds such that $Z\nsubseteq Y$ and $g_Y$ (resp. $g_Z$) a Green current for $Y$ (resp. $Z$), then we can define the $\star$-product of $g_Y$ with $g_Z$ by

$$g_Y\star g_Z := [\widetilde{g_Y}]\star g_Z (\mathrm{mod}\ \mathrm{im}(\partial)+\mathrm{im}(\overline{\partial})).$$

One can show that this definition does not depend on the choice of $\widetilde{g_Y}$.

Under the assumption of Remark 3.2.9, the $\star$-product satisfy some operational laws.

**Proposition 3.2.10.** *After modulo* $\mathrm{im}(\partial)+\mathrm{im}(\overline{\partial})$*, the $\star$-product is commutative and associative.*

*Proof.* We also compute formally with currents as if they are forms. If $Y,Z,W\subseteq X$ are closed irreducible submanifolds meeting properly with currents $g_Y, g_Z, g_W$, respectively, then

$$\begin{aligned} g_Y\star g_Z &= g_Y\wedge\delta_Z + \omega_Y\wedge g_Z && (-dd^c g_Y + \delta_Y = [\omega_Y]) \\ &= g_Y\wedge\delta_Z + \delta_Y\wedge g_Z - dd^c g_Y\wedge g_Z \\ &= g_Y\wedge\delta_Z + \delta_Y\wedge g_Z - g_Y\wedge dd^c g_Z && \text{(general case of Exercise 2.2.5)} \\ &= g_Z\star g_Y, \end{aligned}$$

and

$$g_Y\star(g_Z\star g_W) = g_Y\wedge\delta_Z\wedge\delta_W + \omega_Y\wedge g_Z\wedge\delta_W + \omega_Y\wedge\omega_Z\wedge g_W = (g_Y\star g_Z)\star g_W.$$

The strict proof uses the precise form of Hironaka's theorem on the resolution of singularities. □

We emphasize that it is not necessary to write down the proofs of these propositions strictly here. This is because these rules are abstractly proved for the so-called Green objects using Deligne-Beilinson cohomology in a general setting [BKK].

## 3.3 Gillet-Soulé Intersection Pairing

In this section, we develop Arakelov geometry in higher dimensions. Let $X$ be an integral regular projective flat scheme over $\mathbb{Z}$ with smooth generic fiber (i.e. an arithmetic variety), we will define the higher arithmetic Chow groups, and study the arithmetic intersection theory. The methods in which these theories are established are quite different from Chapter 2.3, but we will assert that these seemingly different geometries are essentially the same.

Let us introduce some notations. Assume $X$ is an arithmetic variety over $\mathbb{Z}$, denote the complex conjugation by $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$, it is a continuous involution of $X(\mathbb{C})$. Put

- $A^{p,p}(X) := \{\omega \in \mathscr{A}^{p,p}(X(\mathbb{C})) : \omega \text{ real}, F_\infty^*\omega = (-1)^p\omega\}$.
- $\widetilde{A}^{p,p}(X) := A^{p,p}(X)/(\mathrm{im}(\partial)+\mathrm{im}(\overline{\partial}))$.
- $Z^{p,p}(X) := \ker\big(d : A^{p,p}(X) \to \mathscr{A}^{2p+1}(X(\mathbb{C}))\big) \subseteq A^{p,p}(X)$.
- $H^{p,p}(X) := \ker\big(dd^c : A^{p,p}(X) \to A^{p+1,p+1}(X)\big)/(\mathrm{im}(\partial)+\mathrm{im}(\overline{\partial})) \subseteq \widetilde{A}^{p,p}(X)$.
- $D^{p,p}(X) := \{T \in \mathscr{D}^{p,p}(X(\mathbb{C})) : T \text{ real}, F_\infty^* T = (-1)^p T\}$.

The notations above fit into the following diagram:

$$\begin{tikzcd}
Z^{p,p}(X) \arrow[r, hook] \arrow[d, "\omega\mapsto\omega(\mathrm{mod\ im}(\partial)+\mathrm{im}(\overline{\partial}))"'] & A^{p,p}(X) \arrow[r, hook, "\omega\mapsto[\omega]"] \arrow[d] & D^{p,p}(X) \\
H^{p,p}(X) \arrow[r, hook] & \widetilde{A}^{p,p}(X) &
\end{tikzcd}$$

Now we give the definition of higher arithmetic Chow groups.

**Definition 3.3.1** (Arithmetic Chow Groups)**.** Let $X$ be an arithmetic variety over $\mathbb{Z}$. Define the group of $p$**-arithmetic cycles** to be

$$\widehat{Z}^p(X) := \big\{(Z, g_Z) : Z \in Z^p(X), g_Z \in \mathscr{D}^{p-1,p-1}(X(\mathbb{C})) \text{ a Green current for } Z(\mathbb{C})\big\},$$

with addition defined componentwise. Let $\widehat{R}^p(X) \subseteq \widehat{Z}^p(X)$ be the subgroup generated by pairs

$$\big(0, \partial(u)+\overline{\partial}(v)\big) \text{ and } \big(\mathrm{div}(f), -[\log|f_{\mathbb{C}}|^2]\big),$$

where $u$ (resp. $v$) is a current of type $(p-2,p-1)$ (resp. $(p-1,p-2)$), $f \in k(y)^\times$ for some $y \in X^{(p-1)}$, and $f_{\mathbb{C}}$ is the pull-back of $f$ by $X(\mathbb{C}) \to X$. The quotient group $\widehat{\mathrm{CH}}^p(X) := \widehat{Z}^p(X)/\widehat{R}^p(X)$ is called the $p$-th **arithmetic Chow group**.

Since $X$ is projective, one can choose a "canonical volume form" such that $X(\mathbb{C})$ has finite volume. Hence, in the definition above, after resolving the singularities if needed, $-\log|f_{\mathbb{C}}|^2$ is a Lebesgue integrable function on $y(\mathbb{C})$ (because there are as many zeros and poles) and induces a Green current

$$-[\log|f_{\mathbb{C}}|^2] \in \mathscr{D}^{p-1,p-1}(X(\mathbb{C})), \quad \omega \mapsto \int_{y(\mathbb{C})} -\log|f_{\mathbb{C}}|^2 \wedge (\omega|_{y(\mathbb{C})})$$

for $\mathrm{div}(f)(\mathbb{C})$ by Theorem 3.2.6.

**Theorem 3.3.2.** *There are two exact sequences:*

- $H^{p-1,p-1}(X) \xrightarrow{\alpha} \widehat{\mathrm{CH}}^p(X) \xrightarrow{(\phi,\psi)} \mathrm{CH}^p(X) \oplus Z^{p,p}(X)$.
- $\widetilde{A}^{p-1,p-1}(X) \xrightarrow{\alpha} \widehat{\mathrm{CH}}^p(X) \xrightarrow{\phi} \mathrm{CH}^p(X) \longrightarrow 0$.

*If we assume $-dd^c g_Z + \delta_Z = [\omega_Z]$ for $Z \in Z^p(X)$, then the maps are $\alpha : \omega \mapsto [(0,[\omega])]$; $\phi : [(Z,g_Z)] \mapsto [Z]$; $\psi : [(Z,g_Z)] \mapsto \omega_Z$.*

*Proof.* We only prove the first one, all maps are well-defined. Indeed, $\alpha$ is well-defined because $\omega \in H^{p-1,p-1}(X)$ always a Green current for the zero cycle (this also implies $(\phi,\psi)\circ\alpha = 0$); $\psi$ is well-defined because the Stokes' formula implies $d\delta_Z = 0$, so $\omega_Z$ is closed. To show $\ker(\phi,\psi) \subseteq \mathrm{im}(\alpha)$, we know $(\phi,\psi)[(Z,g_Z)] = 0$ if and only if $-dd^c g_Z + \delta_Z = 0$ for some $Z = \sum_y \mathrm{div}(f_y)$, where $f_y \in k(y)^\times$, $y \in X^{(p-1)}$. Thus

$$[(Z,g_Z)] = \Big[\Big(\sum_y \mathrm{div}(f_y), g_Z\Big)\Big] = \Big[\Big(0, g_Z + \sum_y [\log|f_{y,\mathbb{C}}|^2]\Big)\Big] =: [(0,G)].$$

Since $-dd^c G = -dd^c g_Z + \delta_Z = 0$, so $G$ and 0 are Green currents for the zero cycle. By Theorem 3.2.5 we have $G = [\eta] + \partial S_1 + \overline{\partial} S_2$ for some smooth form $\eta$, one can verify $\alpha(\eta) = [(Z,g_Z)]$. □

**Example 3.3.3.** In fact, some extra terms can be added to the left and the right of the exact sequences in Theorem 3.3.2. For instance, in the case of curves, let $X = \mathrm{Spec}(\mathscr{O}_K)$ for a number field $K$, then Theorem 3.3.2 becomes

$$\Big(0 \to \mu_K \to \mathscr{O}_K^\times \to\Big)\, \mathbb{R}^{r_1+\frac{r_2}{2}} \to \widehat{\mathrm{Pic}}(X) \to \mathrm{Cl}(\mathscr{O}_K)\Big(\to 0\Big),$$

where $r_1$ (resp. $r_2$) is the number of real (resp. complex) embeddings. All things have been defined in Chapter 1.1.

We now introduce Gillet-Soulé's arithmetic intersection theory.

**Theorem 3.3.4** (Gillet-Soulé)**.** *Let $X$ be an arithmetic variety over $\mathbb{Z}$, then there exists a bilinear pairing*

$$(\cdot,\cdot) : \widehat{\mathrm{CH}}^p(X) \times \widehat{\mathrm{CH}}^q(X) \to \widehat{\mathrm{CH}}^{p+q}(X)_{\mathbb{Q}},$$

*It turns $\bigoplus_p \widehat{\mathrm{CH}}^p(X)_{\mathbb{Q}}$ into a commutative graded $\mathbb{Q}$-algebra with unit $[(X,0)] \in \widehat{\mathrm{CH}}^0(X)$. Moreover,*

- $\phi\big([(Z,g_Z)],[(W,g_W)]\big) = \big(\phi[(Z,g_Z)],\phi[(W,g_W)]\big) = ([Z],[W])$.
- $\psi\big([(Z,g_Z)],[(W,g_W)]\big) = \psi[(Z,g_Z)] \wedge \psi[(W,g_W)]$.

*Proof.* (proof sketch). Suppose we have two arithmetic cycles $(Z,g_Z)$ and $(W,g_W)$, our aim is to define $[(Z\cap W, g_{Z\cap W})]$. Indeed, the cycle $Z \cap W$ is given by Theorem 3.1.4 algebraically. Now let

$$\mathrm{CH}^p_{\mathrm{fin}}(X) := \frac{\{Z \in Z^p(X) : \mathrm{Supp}(Z) \cap X_{\mathbb{Q}} = \emptyset\}}{\big\langle \mathrm{div}(f) : f \in k(y)^\times \text{ for some } y \in X^{(p-1)} \setminus X_{\mathbb{Q}}\big\rangle},$$

there is a canonical map

$$Z^p(X) \to \mathrm{CH}^p(X) \to \mathrm{CH}^p_{\mathrm{fin}}(X) \oplus Z^p(X_{\mathbb{Q}}), \quad Z \mapsto Z_{\mathrm{fin}} + Z_{\mathbb{Q}}.$$

To define $g_{Z\cap W}$, one just needs the generic part $Z^{\cdot}(X_{\mathbb{Q}})$, because the finite part $\mathrm{CH}^{\cdot}_{\mathrm{fin}}(X)$ does not produce cycles in $X(\mathbb{C})$. We can assume that $Z_{\mathbb{Q}}$ and $W_{\mathbb{Q}}$ intersect properly (use the $\mathbb{Q}$-moving lemma), then Theorem 3.1.4 implies there exists a well-defined intersection cycle $(Z_{\mathbb{Q}}, W_{\mathbb{Q}})$ in $X_{\mathbb{Q}}$. The current for $Z \cap W$ is now defined to be $g_{Z_{\mathbb{Q}}(\mathbb{C})} \star g_{W_{\mathbb{Q}}(\mathbb{C})}$, since Proposition 3.2.8 guarantees it is a Green current for $(Z\cap W)(\mathbb{C})$. □

In the proof of Theorem 3.3.4, we decompose the so-called intersection pairing into the finite part and the generic part, in order to correspond with the vertical divisors and horizontal divisors in Arakelov's intersection theory in Chapter 2.3. These two types of intersections should be treated differently. We use an exercise to summarize this.

**Exercise 3.3.5.** *Let $X$ be an arithmetic surface over $\mathbb{Z}$ and assume $\mu$ is a canonical volume form on $X(\mathbb{C})$. Show that there is an embedding*

$$\widehat{\mathrm{CH}}^1(X,\mu) \hookrightarrow \widehat{\mathrm{CH}}^1(X), \quad (Z,r) \mapsto \big(Z, [g_{Z(\mathbb{C})} + 2r]\big),$$

*where $g_{Z(\mathbb{C})}$ is the Green function of logarithmic type with respect to $\mu$. Try to reconstruct the Arakelov intersection theory (Theorem 2.3.4) by using the Gillet-Soulé intersection theory (Theorem 3.3.4). For the finite part, you should define a natural degree map from the algebraic Chow group to $\mathbb{R}$; for the infinite part, you should use $\star$-product to define intersect currents and compute their values at the constant function $1/2$.*

**Theorem 3.3.6.** *Let $X,Y$ be arithmetic varieties, and let $f: X \to Y$ be a flat $\mathbb{Z}$-morphism.*

- *There is a pull-back homomorphism $f^*: \widehat{\mathrm{CH}}^p(Y) \to \widehat{\mathrm{CH}}^p(X)_{\mathbb{Q}}$, it is multiplicative.*
- *If $f$ is proper, $f_{\mathbb{Q}}: X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is smooth and $X,Y$ are equidimensional, then there is a push-forward homomorphism $f_*: \widehat{\mathrm{CH}}^p(X) \to \widehat{\mathrm{CH}}^{p-(\dim(X)-\dim(Y))}(Y)$.*
- *The projection formula holds:*

$$f_*(f^*\alpha, \beta) = (\alpha, f_*\beta), \quad \text{for } \alpha \in \widehat{\mathrm{CH}}^p(Y), \beta \in \widehat{\mathrm{CH}}^q(X).$$

- *$(\cdot)^*$ is a contravariant functor, $(\cdot)_*$ is a covariant functor.*

*Proof.* (proof sketch). We only need to prove the first three items.

- Let $[(Z, g_Z)] \in \widehat{\mathrm{CH}}^p(Y)$ where $Z$ is irreducible and assume $\mathrm{codim}_{X_{\mathbb{Q}}}(f^{-1}(Z)_{\mathbb{Q}}) = p$ (any general case can be reduced to this case). By functoriality of K-theory or Chow theory (Theorem 3.1.4), we form a class $f^*[Z] \in \mathrm{CH}^p(X)_{\mathbb{Q}}$, its image under

$$\mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{CH}^p_{\mathrm{fin}}(X)_{\mathbb{Q}} \oplus Z^p(X_{\mathbb{Q}})_{\mathbb{Q}}$$

is also denoted by $f^*[Z]$. Furthermore, one can verify the pull-back $f^*_{\mathbb{C}}(g_Z)$ is also a Green current. So

$$f^*[(Z, g_Z)] := [(f^*[Z], f^*_{\mathbb{C}}(g_Z))] \in \widehat{\mathrm{CH}}^p(X)_{\mathbb{Q}}.$$

- Denote $q := \dim(X) - \dim(Y)$, we construct the map $\widehat{Z}^p(X) \to \widehat{Z}^{p-q}(Y)$, given by $(Z, g_Z) \mapsto (f_*(Z), f_{\mathbb{C},*}(g_Z))$, where

$$f_*(Z) := \begin{cases} [k(Z) : k(f(Z))] \cdot f(Z) & \dim(f(Z)) = \dim(Z) \\ 0 & \dim(f(Z)) < \dim(Z) \end{cases}.$$

To study $f_{\mathbb{C},*}(g_Z)$, observe that for a differential form $\eta$ on $Y(\mathbb{C})$ of appropriate degree, we have

$$\begin{aligned} & (f_{\mathbb{C},*}(\delta_Z))(\eta) \\ = & \int_{Z(\mathbb{C})} f_{\mathbb{C}}^*(\eta|_{f(Z(\mathbb{C}))}) \\ = & \begin{cases} [k(Z(\mathbb{C})) : k(f(Z(\mathbb{C})))] \cdot \int_{f(Z(\mathbb{C}))} \eta & \dim(f(Z(\mathbb{C}))) = \dim(Z(\mathbb{C})) \\ 0 & \dim(f(Z(\mathbb{C}))) < \dim(Z(\mathbb{C})) \end{cases}. \end{aligned}$$

Hence $f_{\mathbb{C},*}(\delta_Z) = \delta_{f_*(Z)}$, from which we deduce

$$-dd^c(f_{\mathbb{C},*}(g_Z)) = [f_{\mathbb{C},*}(\omega_Z)] - \delta_{f_*(Z)}.$$

So the map we have constructed is reasonable. It is easy to see this map sends $\widehat{R}^p(X)$ into $\widehat{R}^{p-q}(Y)$, because for a rational function $h \in k(W)^\times$ with $\dim(f(W)) = \dim(W)$ for some $W \in X^{(p-1)}$, one can check the image of $\big(\mathrm{div}(h), -[\log|h_{\mathbb{C}}|^2]\big)$ is

$$\Big(\mathrm{div}\big(\mathrm{Nm}_{k(W)/k(f(W))}(h)\big), -\big[\log\big|\mathrm{Nm}_{k(W(\mathbb{C}))/k(f(W(\mathbb{C})))}(h_{\mathbb{C}})\big|^2\big]\Big).$$

For the third item, we already have the projection formula for algebraic cycles, see Proposition 3.1.6. Therefore, we are left to prove it for Green currents. This can be calculated by the definition of the $\star$-product, we leave it as an exercise. □

## 3.4 A Story for Arithmetic Riemann-Roch Theorem

Some of the story in this section was shared by Dr. Bo LIU.

In algebraic geometry, the **Grothendieck-Riemann-Roch theorem** states that, for a flat and projective $S$-morphism $f : X \to Y$ between regular schemes which are quasi-projective and flat over some "nice" scheme $S$ (e.g. $S = \mathrm{Spec}(\mathbb{Z})$ or $S = \mathrm{Spec}(\mathbb{C})$), there is a commutative diagram

$$\begin{tikzcd} K_0(X) \arrow[r, "\mathrm{ch}(\cdot)\,\mathrm{td}(T_f)"] \arrow[d, "\sum(-1)^i R^i f_*(\cdot)"'] & \mathrm{CH}^{\cdot}(X) \arrow[d, "f_*"] \\ K_0(Y) \arrow[r, "\mathrm{ch}(\cdot)"] & \mathrm{CH}^{\cdot}(Y) \end{tikzcd}$$

where $T_f$ is the tangent vector bundle along the fibers of $f$. The characteristic classes $\mathrm{ch}(\cdot)$ and $\mathrm{td}(\cdot)$ here will be defined later. If we take $Y = \mathrm{Spec}(\mathbb{C})$ as a point, we get the **Hirzebruch-Riemann-Roch theorem**, which states for a rank $r$ holomorphic vector bundle $\mathscr{E}$ on a compact complex manifold $X$, we have

$$\sum_{i=0}^{r}(-1)^i \dim_{\mathbb{C}} H^i(X,\mathscr{E}) = \int_X \text{the degree } r \text{ part of } \operatorname{ch}(\mathscr{E})\wedge \operatorname{td}(TX).$$

Our ultimate goal is to introduce the arithmetic Riemann-Roch theorem in Arakelov geometry, where we need some materials.

**Remark 3.4.1.** In this long remark, we introduce the materials we need and explain the significance of their appearance.

- (**Chern and Todd characteristic classes**). Let us review Chern-Weil theory first. Let $\mathscr{E}$ be a rank $r$ Hermitian vector bundle on a complex manifold $X$ (we discard the symbols $\|\cdot\|$ or $h$ for simplicity), suppose its curvature matrix is $\Omega$, then the **total Chern form** of $\mathscr{E}$ is

$$\det\left(I+\frac{i}{2\pi}\Omega\right) = 1+c_1(\mathscr{E})+\cdots+c_r(\mathscr{E}),$$

  where $c_i(\mathscr{E})$ is the $i$-th Chern form and its cohomology class in $H^{2i}_{\mathrm{dR}}(X)$ is the $i$-th Chern class. These Chern classes do not depend on the Hermitian metrics on $\mathscr{E}$.

  – Consider the **Chern polynomial**

$$\operatorname{ch}(x_1,\cdots,x_r) := \sum_{j=1}^{r} e^{t_j} = r+\operatorname{ch}_1(x_1,\cdots,x_r)+\operatorname{ch}_2(x_1,\cdots,x_r)+\cdots,$$

  where

$$\operatorname{ch}_k(x_1,\cdots,x_r) := \sum_{i=1}^{r}\frac{t_i^k}{k!}$$

  and $x_1,\cdots,x_r$ are elementary symmetric polynomials in $t_1,\cdots,t_r$. Then the **Chern character** of $\mathscr{E}$ is defined to be a differential form

$$\operatorname{ch}(\mathscr{E}) := \operatorname{ch}(c_1(\mathscr{E}),\cdots,c_r(\mathscr{E})) \in \bigoplus_{p\geq 0}\mathscr{A}^{p,p}.$$

  – Consider the **Todd polynomial**

$$\operatorname{td}(x_1,\cdots,x_r) := \prod_{j=1}^{r}\frac{t_j}{1-e^{-t_j}} = 1+\operatorname{td}_1(x_1,\cdots,x_r)+\operatorname{td}_2(x_1,\cdots,x_r)+\cdots,$$

  where

$$\operatorname{td}_k(x_1,\cdots,x_r) := \text{degree } k \text{ part in } \prod_{j=1}^{r}\frac{t_j}{1-e^{-t_j}}; \qquad \frac{t}{1-e^{-t}} = 1+\frac{t}{2}+\frac{t^2}{12}-\cdots$$

  and $x_1,\cdots,x_r$ are elementary symmetric polynomials in $t_1,\cdots,t_r$. For example, one can compute $\operatorname{td}_1(x_1,\cdots,x_r) = \frac{x_1}{2}, \operatorname{td}_2(x_1,\cdots,x_r) = \frac{x_1^2+x_2}{12},\cdots$. Define the **Todd character** of $\mathscr{E}$ as a differential form

$$\operatorname{td}(\mathscr{E}) := \operatorname{td}(c_1(\mathscr{E}),\cdots,c_r(\mathscr{E})) \in \bigoplus_{p\geq 0}\mathscr{A}^{p,p}.$$

The cohomology classes of these two forms in $H^{\cdot}_{\mathrm{dR}}(X)$ are called the **characteristic classes**, they also do not depend on the Hermitian metrics on $\mathscr{E}$. It should emphasize that as classes, $\mathrm{ch}(\cdot)$ ($\mathrm{td}(\cdot)$, resp.) is additive (multiplicative, resp.) for any short exact sequence

$$\mathfrak{E}: \quad 0 \to \mathscr{E}_1 \to \mathscr{E}_2 \to \mathscr{E}_3 \to 0$$

of holomorphic vector bundles (with suitable Hermitian metrics if needed). However, as forms, the exact non-trivial differences $\mathrm{ch}(\mathscr{E}_1)+\mathrm{ch}(\mathscr{E}_3)-\mathrm{ch}(\mathscr{E}_2)$ and $\mathrm{td}(\mathscr{E}_1)\wedge \mathrm{td}(\mathscr{E}_3)-\mathrm{td}(\mathscr{E}_2)$ are described by the so-called Bott-Chern secondary forms. Here we only introduce the secondary Chern form

$$\left\{\begin{array}{l}\text{short exact sequences of Hermitian vector bundles} \\ \text{(metrics are independent from the exact sequence)}\end{array}\right\} \xrightarrow{\widetilde{\mathrm{ch}}} \frac{\bigoplus_{n\geq 0}\mathscr{A}^n}{\mathrm{im}(\partial)+\mathrm{im}(\overline{\partial})} \cap \bigoplus_{p\geq 0}\mathscr{A}^{p,p},$$

the secondary Todd form $\widetilde{\mathrm{td}}(\cdot)$ is similar.

**Theorem 3.4.2.** *There is a unique way to attach to every sequence of Hermitian vector bundles*

$$\mathfrak{E}: \quad 0 \to \mathscr{E}_1 \to \mathscr{E}_2 \to \mathscr{E}_3 \to 0$$

*a form $\widetilde{\mathrm{ch}}(\mathfrak{E})$, called the **(Bott-Chern) secondary Chern form** of $\mathfrak{E}$, satisfying the following properties:*

- $dd^c\,\widetilde{\mathrm{ch}}(\mathfrak{E}) = \mathrm{ch}(\mathscr{E}_1)-\mathrm{ch}(\mathscr{E}_2)+\mathrm{ch}(\mathscr{E}_3)$.
- *For every holomorphic map $f: Y\to X$, $f^*\,\widetilde{\mathrm{ch}}(\mathfrak{E}) = \widetilde{\mathrm{ch}}(f^*\mathfrak{E})$.*
- *If $\mathfrak{E}$ is split, then $\widetilde{\mathrm{ch}}(\mathfrak{E})=0$.*
- *If $\mathscr{F}$ is a Hermitian vector bundle, then $\widetilde{\mathrm{ch}}(\mathfrak{E}\otimes\mathscr{F}) = \widetilde{\mathrm{ch}}(\mathfrak{E})\wedge\mathrm{ch}(\mathscr{F})$.*
- *If $\mathfrak{E}_1$, $\mathfrak{E}_2$ are two exact sequences, then $\widetilde{\mathrm{ch}}(\mathfrak{E}_1\oplus\mathfrak{E}_2) = \widetilde{\mathrm{ch}}(\mathfrak{E}_1)+\widetilde{\mathrm{ch}}(\mathfrak{E}_2)$.*

*Proof.* (proof sketch). We can construct a suitable Hermitian vector bundle $(\widetilde{\mathscr{E}},\widetilde{h})$ on $X\times\mathbb{P}^1(\mathbb{C})$ which describes the process of deformation from $\mathscr{E}_2$ to $\mathscr{E}_1\oplus\mathscr{E}_3$ on $X$. Now one can check the following integral is what we need

$$\widetilde{\mathrm{ch}}(\mathfrak{E}) := -\int_{\mathbb{P}^1(\mathbb{C})} \log|z|^2 \wedge \mathrm{ch}(\widetilde{\mathscr{E}},\widetilde{h}).$$

This does not depend on the construction of $\widetilde{h}$ when modulo $\mathrm{im}(\partial)+\mathrm{im}(\overline{\partial})$. □

In some concrete cases, $\widetilde{\mathrm{ch}}(\cdot)$ and $\widetilde{\mathrm{td}}(\cdot)$ can be computed explicitly.

Chern classes and Chern characters also have arithmetic analogies in Arakelov geometry. More generally, for any characteristic class, it always has arithmetic analogues:

**Theorem 3.4.3.** *Let $X$ be an arithmetic variety over $\mathbb{Z}$, let $\xi\in\mathbb{Q}[[x_1,\cdots,x_r]]$ be the Chern polynomial or the Todd polynomial. Then we can uniquely assign $\widehat{\xi}(\mathscr{E},\|\cdot\|)\in\widehat{\mathrm{CH}}^{\cdot}(X)_{\mathbb{Q}}$ to any rank $r$ metrized vector bundle $(\mathscr{E},\|\cdot\|)$ on $X$ with the following properties:*

– *For every morphism* $f : Y \to X$, $f^*(\widehat{\xi}(\mathscr{E}, \|\cdot\|)) = \widehat{\xi}(f^*(\mathscr{E}, \|\cdot\|))$.

– $\psi(\widehat{\xi}(\mathscr{E}, \|\cdot\|)) = \xi(\mathscr{E}_{\mathbb{C}}, \|\cdot\|)$, *where* $\psi$ *is defined in Theorem 3.3.2.*

– *For any exact sequence of metrized vector bundles*

$$\mathfrak{E}: \quad 0 \to (\mathscr{E}_1, \|\cdot\|_1) \to (\mathscr{E}_2, \|\cdot\|_2) \to (\mathscr{E}_3, \|\cdot\|_3) \to 0,$$

*it holds that* $\widehat{\xi}\big((\mathscr{E}_1, \|\cdot\|_1) \oplus (\mathscr{E}_3, \|\cdot\|_3)\big) - \widehat{\xi}(\mathscr{E}_2, \|\cdot\|_2) = \alpha(\widetilde{\xi}(\mathfrak{E}_{\mathbb{C}}))$, *where* $\alpha$ *is defined in Theorem 3.3.2.*

– *If* $(\mathscr{E}, \|\cdot\|) = \bigoplus_{i=1}^r (\mathscr{L}_i, \|\cdot\|_i)$ *for metrized line bundles, then*

$$\widehat{\xi}(\mathscr{E}, \|\cdot\|) = \xi\big(\widehat{c}_1(\mathscr{L}_1, \|\cdot\|_1), \cdots, \widehat{c}_1(\mathscr{L}_r, \|\cdot\|_r)\big),$$

*where* $\widehat{c}_1(\mathscr{L}_i, \|\cdot\|_i) := \big[(\mathrm{div}(s_i), -[\log \|s_{i,\mathbb{C}}\|_i^2])\big]$ *and for each* $i$, $s_i$ *is a non-zero rational section of* $\mathscr{L}_i$.

We call $\widehat{\mathrm{ch}}(\cdot)$ ($\widehat{\mathrm{td}}(\cdot)$, resp.) the **arithmetic Chern (Todd, resp.) character**. For example, the arithmetic Chern character for a metrized line bundle $(\mathscr{L}, \|\cdot\|)$ is

$$\widehat{\mathrm{ch}}(\mathscr{L}, \|\cdot\|) = \exp(\widehat{c}_1(\mathscr{L}, \|\cdot\|)) \in \widehat{\mathrm{CH}}^{\cdot}(X)_{\mathbb{Q}}.$$

Let $X$ be an arithmetic variety over $\mathbb{Z}$. The **arithmetic Grothendieck group** $\widehat{K_0}(X)$ is the abelian group generated by

$$\{\text{metrized vector bundles on } X\}/\text{isometry} \quad \text{and} \quad \bigoplus_{p \geq 0} \widetilde{A}^{p,p}(X),$$

with relations:

– $(\mathscr{E}_1, \|\cdot\|_1) + (\mathscr{E}_3, \|\cdot\|_3) - (\mathscr{E}_2, \|\cdot\|_2) = \widetilde{\mathrm{ch}}(\mathfrak{E}_{\mathbb{C}})$, if there is an exact sequence of metrized vector bundles

$$\mathfrak{E}: \quad 0 \to (\mathscr{E}_1, \|\cdot\|_1) \to (\mathscr{E}_2, \|\cdot\|_2) \to (\mathscr{E}_3, \|\cdot\|_3) \to 0.$$

– $\omega = \omega_1 + \omega_2 \in \bigoplus_{p \geq 0} \widetilde{A}^{p,p}(X)$, if $[\omega] = [\omega_1] + [\omega_2]$.

The elements in $\widehat{K_0}(X)$ are in the form $(\mathscr{E}, \|\cdot\|) + \omega$. We can consider $\widehat{K_0}(X)$ to be a commutative ring with multiplication $((\mathscr{E}_1, \|\cdot\|_1) + \omega_1) \cdot ((\mathscr{E}_2, \|\cdot\|_2) + \omega_2)$ defined by

$$(\mathscr{E}_1, \|\cdot\|_1) \otimes (\mathscr{E}_2, \|\cdot\|_2) + \mathrm{ch}(\mathscr{E}_{1,\mathbb{C}}) \wedge \omega_2 + \omega_1 \wedge \mathrm{ch}(\mathscr{E}_{2,\mathbb{C}}) - (dd^c \omega_1) \wedge \omega_2.$$

Hence, $\widehat{\mathrm{ch}}(\cdot)$ induces a ring homomorphism from $\widehat{K_0}(X)$ to $\widehat{\mathrm{CH}}^{\cdot}(X)_{\mathbb{Q}}$ if we add that $\widehat{\mathrm{ch}}(\omega) := \alpha(\omega)$. So it is easy to verify $\psi \circ \widehat{\mathrm{ch}}(\omega) = \psi \circ \alpha(\omega) = -dd^c \omega$ by Exercise 3.2.3. Moreover, this map induces an isomorphism $\widehat{K_0}(X)_{\mathbb{Q}} \xrightarrow{\sim} \widehat{\mathrm{CH}}^{\cdot}(X)_{\mathbb{Q}}$.

- (**Analytic torsions, heat kernels and Quillen metrics**). In this part, we introduce some other materials, they are analytic torsions and Quillen metrics on the determinant of cohomologies, which appear in the formulation of the arithmetic Riemann-Roch theorem.

  Let $(X,g)$ be a $n$ dimensional projective complex manifold with Kähler metric $g$, and let $(\mathscr{E},h)$ be a rank $r$ Hermitian vector bundle on $X$. Then one can define the $L^2$-scalar product induced by $g$ and $h$ on

$$\mathscr{A}^{p,q}(\mathscr{E}) := \text{smooth global sections of } \left(\bigwedge^{p} T^{*(1,0)}X \otimes \bigwedge^{q} T^{*(0,1)}X\right) \otimes \mathscr{E},$$

  regard as a natural generalization of Remark 2.1.9, by

$$\langle\cdot,\cdot\rangle_{g,h} : \mathscr{A}^{p,q}(\mathscr{E}) \times \mathscr{A}^{p,q}(\mathscr{E}) \to \mathbb{C}, \quad \langle s,t\rangle_{g,h} := \int_X (\widetilde{g}\otimes h)(s(x),t(x)) \cdot \mathrm{vol}_g(x).$$

  In particular, there is a Hermitian inner product on $\mathscr{A}^{0,q}(\mathscr{E}), 0 \leq q \leq n$. If we define $\overline{\partial} : \mathscr{A}^{0,\cdot}(\mathscr{E}) \to \mathscr{A}^{0,\cdot+1}(\mathscr{E})$ in the usual way, we obtain the **Dolbeault complex**

$$\mathfrak{A}: \quad \mathscr{A}^{0,0}(\mathscr{E}) \xrightarrow{\overline{\partial}} \mathscr{A}^{0,1}(\mathscr{E}) \xrightarrow{\overline{\partial}} \mathscr{A}^{0,2}(\mathscr{E}) \xrightarrow{\overline{\partial}} \cdots,$$

  whose cohomology $H^q_{\mathrm{Dol}}(X,\mathscr{E})$ is called the $q$-th **Dolbeault cohomology group**. Moreover, $\overline{\partial}$ has a right adjoint $\overline{\partial}^* : \mathscr{A}^{0,\cdot+1}(\mathscr{E}) \to \mathscr{A}^{0,\cdot}(\mathscr{E})$ for $\langle\cdot,\cdot\rangle_{g,h}$. Hence, we get the **Laplace operator** $\Delta^q := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ on $\mathscr{A}^{0,q}(\mathscr{E})$, this is a self-adjoint second-order elliptic differential operator. Sometimes we write $\sqrt{\Delta} := \overline{\partial} + \overline{\partial}^*$, which acts on $\bigoplus_q \mathscr{A}^{0,q}(\mathscr{E})$, call it the **Dirac operator**. Since $X$ is compact, we have the following facts:

  **Theorem 3.4.4.** *With the notations above,*

  - *(Spectral Theorem). The spectrum (eigenvalues) of $\Delta^q$ denoted by $\{\lambda^{(q)}_{i\geq 1}\}$ is discrete, real and non-negative, each eigenspace is finite dimensional. Moreover, the eigensections of $\Delta^q$ form a complete basis of $\mathscr{A}^{0,q}(\mathscr{E})$ with the $L^2$-scalar product.*
  - *(Weyl's Law). The function*

$$\theta^q_\Delta(t) := \sum_{\lambda^{(q)}_i \neq 0} e^{-t\lambda^{(q)}_i} \quad \textit{(count with multiplicities)}$$

    *exists and is smooth on $(0,+\infty)$. When $t \to 0^+$, one has*

$$\theta^q_\Delta(t) = \frac{\mathrm{vol}_g(X) \cdot \mathrm{rank}\big(\bigwedge^q T^{*(0,1)}X \otimes \mathscr{E}\big)}{(4\pi)^n} \cdot t^{-n} + O(t^{-n+1}).$$

  Now we want to define the "determinant" of $\sqrt{\Delta}$, which will be called the analytic torsion of $\mathfrak{A}$. To achieve this, fix an integer $q$ with $0 \leq q \leq n$, we first formally and degree-wisely define

$$|\det(\sqrt{\Delta}|_q)| := \sqrt{|\det(\Delta^q)|} := \prod_{\lambda^{(q)}_i \neq 0} \sqrt{\lambda^{(q)}_i} = \exp\left(-\frac{1}{2}\frac{d\zeta^q_\Delta}{ds}\Big|_{s=0}\right) \in \mathbb{R},$$

where $\zeta_\Delta^q(s) := \sum_{\lambda_i^{(q)}\neq 0}(\lambda_i^{(q)})^{-s}$ (count with multiplicities) is called the **Selberg zeta function** for $\Delta^q$.

**Proposition 3.4.5.** *$\zeta_\Delta^q$ converges for* $\mathrm{Re}(s)$ *sufficiently large and has a meromorphic continuation to the whole complex plane (also denote by $\zeta_\Delta^q$), which is holomorphic at $s = 0$. Hence, $|\det(\sqrt{\Delta}|_q)| < \infty$ is well-defined.*

*Proof.* (proof sketch). It is easy to see $\theta_\Delta^q(t)$ converges uniformly and has an upper bound when $t$ is large enough. Furthermore, Theorem 3.4.4 implies $\theta_\Delta^q(t) \sim Ct^{-n}$ when $t \to 0^+$. Hence, for $\mathrm{Re}(s) > n$, the series $\sum_{\lambda_i^{(q)}\neq 0} e^{-t\lambda_i^{(q)}} t^{s-1}$ is uniformly and absolutely convergent on $(0, +\infty)$, thus

$$\zeta_\Delta^q(s) = \sum_{\lambda_i^{(q)}\neq 0} \frac{1}{\Gamma(s)} \int_0^{+\infty} e^{-t\lambda_i^{(q)}} t^{s-1} dt = \frac{1}{\Gamma(s)} \int_0^{+\infty} \theta_\Delta^q(t) t^{s-1} dt$$

exists and is holomorphic when $\mathrm{Re}(s) > n$. □

Now we need to combine these $|\det(\sqrt{\Delta}|_q)|$ by let

$$\zeta_\Delta(s) := \sum_{q=0}^{n} (-1)^{q-1} q \cdot \zeta_\Delta^q(s),$$

and define the **analytic torsion** of $\mathfrak{A}$ to be a real number

$$T_\Delta := |\det(\sqrt{\Delta})| := \exp\left(-\frac{1}{2}\frac{d\zeta_\Delta}{ds}\Big|_{s=0}\right).$$

**Remark 3.4.6.** Here we give some interpretations of analytic torsions.

– Torsion was first studied by Reidemeister, it was originally used to distinguish topological spaces that are homotopy equivalent but not homeomorphic. Ray and Singer generalized this concept to the form we discussed above. It is necessary to illustrate the geometric meaning of torsions, so we make a remark here. By Hodge theory, $\mathbb{H}^q := \ker(\Delta^q) \subseteq \mathscr{A}^{0,q}(\mathscr{E})$ is the space of $q$-th **Dolbeault harmonic forms**. For $\lambda_i^{(q)} > 0$, we denote $\mathscr{A}_i^q$ to be the eigenspace of $\Delta^q$ with respect to eigenvalue $\lambda_i^{(q)}$, which is finite dimensional by Theorem 3.4.4. Then one can verify

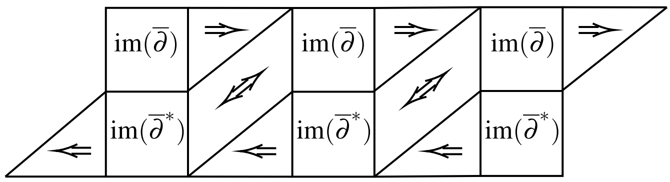
$$\mathfrak{A}_i: \quad 0 \to \mathscr{A}_i^0 \xrightarrow{\overline{\partial}} \mathscr{A}_i^1 \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathscr{A}_i^n \to 0$$

and

$$\mathfrak{A}_i^*: \quad 0 \leftarrow \mathscr{A}_i^0 \xleftarrow{\overline{\partial}^*} \mathscr{A}_i^1 \xleftarrow{\overline{\partial}^*} \cdots \xleftarrow{\overline{\partial}^*} \mathscr{A}_i^n \leftarrow 0$$

are exact. Recall that Hodge decomposition gives $\mathscr{A}_i^q = \overline{\partial}\mathscr{A}_i^{q-1} \oplus \overline{\partial}^*\mathscr{A}_i^{q+1}$, and $\overline{\partial} : \overline{\partial}^*\mathscr{A}_i^{q+1} \xrightarrow{\sim} \overline{\partial}\mathscr{A}_i^q$ is an isomorphism.

$$\underset{\overline{\partial}^*}{\overset{\overline{\partial}}{\rightleftarrows}} \mathscr{A}_i^{q-1} \underset{\overline{\partial}^*}{\overset{\overline{\partial}}{\rightleftarrows}} \mathscr{A}_i^{q} \underset{\overline{\partial}^*}{\overset{\overline{\partial}}{\rightleftarrows}} \mathscr{A}_i^{q+1} \underset{\overline{\partial}^*}{\overset{\overline{\partial}}{\rightleftarrows}}$$




We now pick $s_q \in \bigwedge^{\dim(\overline{\partial}\mathscr{A}_i^q)} \mathscr{A}_i^q$ s.t. $\overline{\partial} s_q \neq 0$ but $\overline{\partial}^* s_q = 0$ for $0 \leq q \leq n-1$, so $0 \neq \overline{\partial} s_q \wedge s_{q+1} \in \det(\mathscr{A}_i^{q+1}) \cong \mathbb{C}$. Consider the alternating tensor product

$$T_i := s_0^\vee \otimes (\overline{\partial} s_0 \wedge s_1) \otimes (\overline{\partial} s_1 \wedge s_2)^\vee \otimes \cdots \otimes (\overline{\partial} s_{n-1})^{(-1)^{n-1}} \in \bigotimes_{q=0}^{n} \det(\mathscr{A}_i^q)^{(-1)^{q-1}},$$

where $s^\vee$ is the element in the dual space such that $s^\vee(s) = 1$. One can show $T_i$ is independent of the choice of $\{s_i\}$, and the collection of all $T_i$ for $\lambda_i^{(q)} \neq 0$ essentially corresponds to a real number

$$\prod_{q=0}^{n} |\det(\sqrt{\Delta}|_q)|^{q(-1)^{q-1}} = \exp\left(-\frac{1}{2}\frac{d\zeta_\Delta}{ds}\Big|_{s=0}\right) = T_\Delta.$$

– Here we provide a heat equation perspective when $q = 0$. A **heat equation** is a PDE in form $(\frac{\partial}{\partial t} + \Delta)u(t,x) = 0$, where the Laplacian $\Delta = \Delta^0 : \mathscr{A}^{0,0}(\mathscr{E}) \to \mathscr{A}^{0,0}(\mathscr{E})$ only acts on the second component of $u$, and for each $t \in \mathbb{R}$, $u(t,\cdot) \in \mathscr{A}^{0,0}(\mathscr{E})$. Let $x, y \in X$ and $t > 0$, A **heat kernel** for $\Delta$ is a class of linear maps $p_t(x,y) : \mathscr{E}_y \to \mathscr{E}_x$, such that $(t,x,y) \mapsto p_t(x,y)$ is differentiable (resp. second-order differentiable) with respect to $t$ (resp. $x, y$) and the partial derivatives are continuous, satisfy the following two conditions:
  - $p_t(x,y)$ satisfies the heat equation: $(\frac{\partial}{\partial t} + \Delta_x)(p_t(x,y)v) = 0$ for all $v \in \mathscr{E}_y$.
  - $p_t(x,y)$ satisfies the initial condition: for any $\xi \in \mathscr{A}^{0,0}(\mathscr{E})$, the limit

    $$\lim_{t\to 0^+} \int_X p_t(x,y)\xi(y)\mathrm{vol}_g(y) = \xi(x)$$

    converges uniformly.

  It is well-known $\Delta$ always has a unique heat kernel which is smooth in $t, x, y$. For $t > 0$, define the **heat kernel operator**

  $$e^{-t\Delta} : \overline{\mathscr{A}^{0,0}(\mathscr{E})} \longrightarrow \mathscr{A}^{0,0}(\mathscr{E}), \quad \xi \longmapsto \left[e^{-t\Delta}(\xi) : x \mapsto \int_X p_t(x,y)\xi(y)\mathrm{vol}_g(y)\right].$$

  This is a compact self-adjoint operator, and there is a one-to-one correspondence

  $$\{\text{eigenvalues of } \Delta\} \longleftrightarrow \{\text{eigenvalues of } e^{-t\Delta}\},$$

  given by $\lambda \leftrightarrow Ce^{-t\lambda}$ for some constant $C$. This correspondence reverses the order. Now $\theta_\Delta^0(t)$ is the trace for $e^{-t\Delta}$, that is,

  $$\mathrm{tr}(e^{-t\Delta}) = \int_X \mathrm{tr}(p_t(x,x))\mathrm{vol}_g(x).$$

It has a very deep connection with the Selberg trace formula and the Weyl's law in Theorem 3.4.4. Moreover, one can use this to prove the general Atiyah-Singer index theorem.

So far, we only considered the positive-eigenvalue part of $\bigoplus_q \mathscr{A}^{0,q}(\mathscr{E})$. We also hope to take account of the zero-eigenvalue part, namely, the space of Dolbeault harmonic forms $\mathbb{H}^q$. We are trying to define a "norm" that take account for all eigenvalues, the idea of which is to multiply the norms from

$$\det(\mathbb{H}) := \bigotimes_{q=0}^{n} \det(\mathbb{H}^q)^{(-1)^{q-1}}$$

and

$$\lambda(R\Gamma(X,\mathscr{E})) := \bigotimes_{q=0}^{n} \det(H^q(X,\mathscr{E}))^{(-1)^{q-1}}.$$

We can regard the analytic torsion $T_\Delta$ to be the "norm" on $\lambda(R\Gamma(X,\mathscr{E}))$, and equip $\det(\mathbb{H})$ a natural norm (Hermitian inner product) $|\cdot|_{\lambda(\mathscr{E})}$ since $\mathbb{H}^q \cong H^q_{\mathrm{Dol}}(X,\mathscr{E}) \cong H^q(X,\mathscr{E}) \subseteq \mathscr{A}^{0,q}(\mathscr{E})$. Multiplying the two norms together, we give rise to the definition of **Quillen metric** $\|\cdot\|_{\mathrm{Qui}} := T_\Delta|\cdot|_{\lambda(\mathscr{E})}$ on $\lambda(R\Gamma(X,\mathscr{E}))$.

Here we just consider the complex case, that is $X \to \mathbb{C}$ a projective complex manifold. More generally, suppose $f : X \to Y$ is a flat proper morphism, and $Y$ is regular. For a coherent $\mathscr{O}_X$-module $\mathscr{E}$, define

$$\lambda(Rf_*\mathscr{E}) := \bigotimes_{q} \det(R^q f_*\mathscr{E})^{(-1)^{q-1}},$$

this is a bundle on $Y$. Let $X_y$ denote the fiber at $y \in Y$, the flatness of $f$ implies

$$\lambda(Rf_*\mathscr{E})_y = \bigotimes_{q} \det(H^q(X_y,\mathscr{E}|_{X_y}))^{(-1)^{q-1}} = \lambda(R\Gamma(X_y,\mathscr{E}|_{X_y})),$$

which was discussed before. Thus, we can equip $\lambda(Rf_*\mathscr{E})_y \otimes \mathbb{C}$ a Quillen metric naturally if $\mathscr{E}$ is metrized, and study $\lambda(Rf_*\mathscr{E})$ fiber-wisely.

- (**R-genus**). To study the arithmetic Riemann-Roch theorem for trivial line bundles on $\mathbb{P}^n$, Gillet and Soulé defined a formal power series

$$\mathscr{R}(x) := \sum_{\text{odd } k\geq 1} \left( \sum_{j=1}^{k} \frac{1}{j} + 2\frac{\zeta'(-k)}{\zeta(-k)} \right) \zeta(-k)\frac{x^k}{k!},$$

where $\zeta(s)$ is the Riemann zeta function. For a Hermitian line bundle $(\mathscr{L},\|\cdot\|)$ on some complex manifold, we define its **R-genus** to be the form $\mathscr{R}(\mathscr{L},\|\cdot\|) := \mathscr{R}(c_1(\mathscr{L},\|\cdot\|))$. In general, for a rank $r$ Hermitian vector bundle $(\mathscr{E},\|\cdot\|)$ with curvature matrix $\Omega$, define its R-genus to be the form $\mathscr{R}(\mathscr{E},\|\cdot\|) := \mathrm{tr}\big(\mathscr{R}(\frac{i}{2\pi}\Omega)\big)$.

**Remark 3.4.7.** Let $f : X \to Y$ be a flat projective morphism of arithmetic varieties, let $\overline{\mathscr{E}}$ be a metrized vector bundle on $X$. The original arithmetic Riemann-Roch formula which was established by Gillet and Soulé states in $\widehat{\mathrm{CH}}^{\cdot}(Y)_{\mathbb{Q}}$ the following equality holds

$$\widehat{c}_1(\overline{\lambda(Rf_*\mathscr{E})}) = f_*(\widehat{\mathrm{ch}}(\overline{\mathscr{E}})\widehat{\mathrm{td}}(\overline{T_f}))^{(1)} - \alpha\Big(f_*\big(\mathrm{ch}(\overline{\mathscr{E}_{\mathbb{C}}})\,\mathrm{td}(\overline{T_{f,\mathbb{C}}})\mathscr{R}(\overline{T_{f,\mathbb{C}}})\big)^{(1)}\Big),$$

where the metric on $\lambda(Rf_*\mathscr{E})_{\mathbb{C}}$ is the "Quillen metric fiber-wisely". We omit the proof of this formula, but provide an idea for a more general case [GRS] in Theorem 3.4.8.

Now we state the main theorem of this section:

**Theorem 3.4.8** (Rössler-Gillet-Soulé-Riemann-Roch)**.** *Let $f : X \to Y$ be a flat projective $\mathbb{Z}$-morphism of arithmetic varieties, which is smooth over $\mathbb{Q}$. We abbreviate the metric as a bar on the head. Now we have:*

- *$\widehat{K_0}(X)$ can be generated by $f$-acyclic (that is, the $i$-th higher direct image by $f$ vanishes for $i > 0$) metrized vector bundles on $X$ and elements of $\bigoplus_{p\geq 0}\widetilde{A}^{p,p}(X)$.*
- *Fix a conjugation invariant Kähler metric $g_X$ on $X(\mathbb{C})$, then the following diagram commutes*

$$\begin{tikzcd}
K_0(X) \arrow[rr, "\mathrm{ch}(\cdot)\,\mathrm{td}(T_f)"] \arrow[dd, "\sum(-1)^i R^i f_*(\cdot)"'] & & \mathrm{CH}^{\cdot}(X)_{\mathbb{Q}} \arrow[dd, "f_*"] & \\
& \widehat{K_0}(X) \arrow[ul, two heads] \arrow[rr, "\widehat{\mathrm{ch}}(\cdot)\widehat{\mathrm{td}}(\overline{T_f})(1-\alpha(\mathscr{R}(\overline{T_{f,\mathbb{C}}})))"] \arrow[dd, "\widehat{f}_*"] & & \widehat{\mathrm{CH}}^{\cdot}(X)_{\mathbb{Q}} \arrow[ul, two heads] \arrow[dd, "f_*"] \\
K_0(Y) \arrow[rr, "\mathrm{ch}(\cdot)"] & & \mathrm{CH}^{\cdot}(Y)_{\mathbb{Q}} & \\
& \widehat{K_0}(Y) \arrow[ul, two heads] \arrow[rr, "\widehat{\mathrm{ch}}(\cdot)"] & & \widehat{\mathrm{CH}}^{\cdot}(Y)_{\mathbb{Q}} \arrow[ul, two heads]
\end{tikzcd}$$

*where*

- *$\overline{T_f}$ is the metrized relative tangent bundle on $X$ whose metric induced by $g_X$.*
- *For a $f$-acyclic $\overline{\mathscr{E}}$, $\widehat{f}_* : \widehat{K_0}(X) \to \widehat{K_0}(Y)$ maps $\overline{\mathscr{E}} + \omega$ to $\overline{R^0 f_*\mathscr{E}} + f_{\overline{\mathscr{E}}} + f_\omega$. Here,*
  - *$R^0 f_*\mathscr{E} = f_*\mathscr{E}$ is locally free (by semi-continuity theorem) and moreover, the natural map $(f_{\mathbb{C},*}\mathscr{E}_{\mathbb{C}})_y \xrightarrow{\sim} H^0\big(X(\mathbb{C})_y, \mathscr{E}_{\mathbb{C}}|_{X(\mathbb{C})_y}\big)$ is an isomorphism for every point $y \in Y(\mathbb{C})$. Thus, one can endow $R^0 f_*\mathscr{E}$ a metric fiber-wisely by the $L^2$-scalar product on $H^0\big(X(\mathbb{C})_y, \mathscr{E}_{\mathbb{C}}|_{X(\mathbb{C})_y}\big)$, which depends on $f, g_X, \overline{\mathscr{E}}$.*
  - *$f_{\overline{\mathscr{E}}} \in \bigoplus_{p\geq 0}\widetilde{A}^{p,p}(Y)$ such that the equation*

$$dd^c f_{\overline{\mathscr{E}}} = \int_{X(\mathbb{C})/Y(\mathbb{C})} \mathrm{ch}(\overline{\mathscr{E}_{\mathbb{C}}})\,\mathrm{td}(\overline{T_{f,\mathbb{C}}}) - \mathrm{ch}(\overline{(R^0 f_*\mathscr{E})_{\mathbb{C}}})$$

    *holds more precisely than the Hirzebruch-Riemann-Roch theorem. The existence of $f_{\overline{\mathscr{E}}}$ was proved by Bismut and Köhler. We call it the **higher torsion form**, this is because $f_{\overline{\mathscr{E}}} = \log(T_\Delta) \in \mathbb{R}$ if $Y$ is a point.*
  - *$f_\omega := \int_{X(\mathbb{C})/Y(\mathbb{C})} \mathrm{td}(\overline{T_{f,\mathbb{C}}})\omega$.*

– $\widehat{\mathrm{td}}(\overline{T_f})$ *satisfies the following property. Let*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & \mathbb{P}^n_Y \arrow[d, "p"] \\ & Y \end{tikzcd}$$

*be a factorization of $f$ into a closed immersion $i$ and a projective smooth morphism $p$, consider the exact sequence*

$$\mathfrak{N}: \quad 0 \to T_f \to i^* T_p \to \mathscr{N}\mathrm{or}_{X/\mathbb{P}^n_Y} \to 0,$$

*then in $\widehat{\mathrm{CH}}^{\cdot}(X)_{\mathbb{Q}}$ we have*

$$\widehat{\mathrm{td}}(\overline{T_f}) \wedge \widehat{\mathrm{td}}(\overline{\mathscr{N}\mathrm{or}_{X/\mathbb{P}^n_Y}}) - \widehat{\mathrm{td}}(i^*\overline{T_p}) = \alpha(\widetilde{\mathrm{td}}(\mathfrak{N}_{\mathbb{C}})).$$

*Proof.* (proof sketch). To prove the commutation of the diagram, we only need to prove that for $\Theta \in \widehat{K_0}(X)$,

$$\delta(\Theta, f, g_X) := \widehat{\mathrm{ch}}(\widehat{f_*}\Theta) - f_*\left(\widehat{\mathrm{ch}}(\Theta)\widehat{\mathrm{td}}(\overline{T_f})(1-\alpha(\mathscr{R}(\overline{T_{f,\mathbb{C}}})))\right) = 0.$$

There are five steps:

(1). Show that $\delta(\cdot, f, g_X)$ is linear. This is straightforward from the definition.

(2). Show that if $\Theta \in \bigoplus_{p\geq 0} A^{p,p}(X)$, then $\delta(\Theta, f, g_X) = 0$. Moreover, $\delta(\overline{\mathscr{E}}, f, g_X)$ does not depend on the metric of $\overline{\mathscr{E}}$, so we drop the bar. This step is also easy.

(3). $\delta(\Theta, f, g_X)$ does not depend on the choice of Kähler metric $g_X$ on $X(\mathbb{C})$. This was proved by Bismut and Köhler in 1992. We thus denote $\delta(\Theta, f) := \delta(\Theta, f, g_X)$ from now on.

(4). Suppose we have a factorization $X \xrightarrow{i} \mathbb{P}^n_Y \xrightarrow{p} Y$ of $f$. Let $\mathscr{E}$ be a vector bundle on $X$, then $i_*\mathscr{E}$ is a coherent sheaf on $\mathbb{P}^n_Y$. Consider a locally free resolution

$$0 \to \mathscr{F}_m \to \mathscr{F}_{m-1} \to \cdots \to \mathscr{F}_0 \to i_*\mathscr{E} \to 0,$$

what we need is $\sum_{i=0}^m (-1)^i \delta(\mathscr{F}_i, p) = \delta(\mathscr{E}, f)$.

(5). For any vector bundle $\mathscr{F}$ on $\mathbb{P}^n_Y$, $\delta(\mathscr{F}, p) = 0$.

Combined with the previous steps, one concludes that $\delta(\Theta, f, g_X) = 0$ always holds, finishing the proof of the arithmetic Riemann-Roch theorem. Of these, step (4) plays a key role in the proof, and is not easy at all. We now take a closer look at this step.

For simplicity, let us assume that all of the bundles here are $f$-acyclic or $p$-acyclic, and admit suitable metrics. After a simple calculation we get

$$\begin{aligned}
\sum_{i=0}^{m}(-1)^i\delta(\overline{\mathscr{F}_i},p)-\delta(\overline{\mathscr{E}},f)=&\left(\sum_{i=0}^{m}(-1)^i\widehat{\mathrm{ch}}(p_*\overline{\mathscr{F}_i})-\widehat{\mathrm{ch}}(f_*\overline{\mathscr{E}})\right) && \text{(I)}\\
&+p_*\left(-\widehat{\mathrm{td}}(\overline{T_p})\sum_{i=0}^{m}(-1)^i\widehat{\mathrm{ch}}(\overline{\mathscr{F}_i})\right) && \text{(II)}\\
&+p_*\left(\widehat{\mathrm{td}}(\overline{T_p})\alpha(\mathscr{R}(\overline{T_{p,\mathbb{C}}}))\sum_{i=0}^{m}(-1)^i\widehat{\mathrm{ch}}(\overline{\mathscr{F}_i})\right) && \text{(III)}\\
&+\left(\widehat{\mathrm{ch}}\left(\sum_{i=0}^{m}(-1)^i p_{\overline{\mathscr{F}_i}}\right)-\widehat{\mathrm{ch}}(f_{\overline{\mathscr{E}}})\right) && \text{(IV)}\\
&+f_*\left(\widehat{\mathrm{ch}}(\overline{\mathscr{E}})\widehat{\mathrm{td}}(\overline{T_f})(1-\alpha(\mathscr{R}(\overline{T_{f,\mathbb{C}}})))\right). && \text{(V)}
\end{aligned}$$

We have:

- By Theorem 3.4.3, (I) $=\alpha\left((-1)^m\widetilde{\mathrm{ch}}(\mathfrak{H}_{\mathbb{C}})\right)$, where $\mathfrak{H}_{\mathbb{C}}$ is the exact sequence

$$\begin{aligned}
&0\to H^0(\mathbb{P}^n_Y(\mathbb{C})/Y(\mathbb{C}),\mathscr{F}_{m,\mathbb{C}})\to\cdots\\
&\qquad\cdots\to H^0(\mathbb{P}^n_Y(\mathbb{C})/Y(\mathbb{C}),\mathscr{F}_{0,\mathbb{C}})\to H^0(X(\mathbb{C})/Y(\mathbb{C}),\mathscr{E}_{\mathbb{C}})\to 0.
\end{aligned}$$

- To calculate (II), Bismut, Gillet and Soulé showed that for $[(Z,g_Z)]\in\widehat{\mathrm{CH}}^{\cdot}(\mathbb{P}^n_Y)_{\mathbb{Q}}$, there exists a current $D_{\overline{\mathscr{E}}}$ satisfy

$$dd^cD_{\overline{\mathscr{E}}}=i_*\left(\mathrm{td}^{-1}(\overline{(\mathscr{N}\mathrm{or}_{X/\mathbb{P}^n_Y})_{\mathbb{C}}})\,\mathrm{ch}(\overline{\mathscr{E}_{\mathbb{C}}})\right)-\sum_i(-1)^i\,\mathrm{ch}(\overline{\mathscr{F}_{i,\mathbb{C}}}),$$

  such that

$$\begin{aligned}
p_*\left([(Z,g_Z)],\sum_i(-1)^i\widehat{\mathrm{ch}}(\overline{\mathscr{F}_i})\right)=&f_*\left(i^*([(Z,g_Z)])\widehat{\mathrm{td}}^{-1}(\overline{\mathscr{N}\mathrm{or}_{X/\mathbb{P}^n_Y}})\widehat{\mathrm{ch}}(\overline{\mathscr{E}})\right)\\
&-\alpha\left(\int_{\mathbb{P}^n_Y(\mathbb{C})/Y(\mathbb{C})}\psi([(Z,g_Z)])\wedge D_{\overline{\mathscr{E}}}\right).
\end{aligned}$$

  In consequence,

$$\text{(II)}=-f_*\left(\widehat{\mathrm{td}}(i^*\overline{T_p})\widehat{\mathrm{td}}^{-1}(\overline{\mathscr{N}\mathrm{or}_{X/\mathbb{P}^n_Y}})\widehat{\mathrm{ch}}(\overline{\mathscr{E}})\right)+\alpha\left(\int_{\mathbb{P}^n_Y(\mathbb{C})/Y(\mathbb{C})}\mathrm{td}(\overline{T_{p,\mathbb{C}}})\wedge D_{\overline{\mathscr{E}}}\right).$$

- By Theorem 3.3.6 and the formula $\alpha(\psi([(Z,g_Z)])\wedge\omega)=([(Z,g_Z)],\alpha(\omega))$,

$$\text{(III)}=\alpha\left(\int_{\mathbb{P}^n_Y(\mathbb{C})/Y(\mathbb{C})}\mathrm{td}(\overline{T_{p,\mathbb{C}}})\mathscr{R}(\overline{T_{p,\mathbb{C}}})\sum_i(-1)^i\,\mathrm{ch}(\overline{\mathscr{F}_{i,\mathbb{C}}})\right).$$

- (IV) $=\alpha\left(\sum_i(-1)^ip_{\overline{\mathscr{F}_i}}-f_{\overline{\mathscr{E}}}\right)$.

- One can compute formally

$$
\begin{aligned}
(\mathrm{V}) =& f_*\left(\left(\widehat{\mathrm{td}}(i^*\overline{T_p})\widehat{\mathrm{td}}^{-1}(\overline{\mathscr{N}\mathrm{or}_{X/\mathbb{P}^n_Y}})+\alpha(\widetilde{\mathrm{td}}(\mathfrak{N}_{\mathbb{C}}))\widehat{\mathrm{td}}^{-1}(\overline{\mathscr{N}\mathrm{or}_{X/\mathbb{P}^n_Y}})\right)\widehat{\mathrm{ch}}(\overline{\mathscr{E}})(1-\alpha(\mathscr{R}(\overline{T_{f,\mathbb{C}}})))\right)\\
=& f_*\left(\widehat{\mathrm{td}}(i^*\overline{T_p})\widehat{\mathrm{td}}^{-1}(\overline{\mathscr{N}\mathrm{or}_{X/\mathbb{P}^n_Y}})\widehat{\mathrm{ch}}(\overline{\mathscr{E}})\right)\\
&+\alpha\left(\int_{X(\mathbb{C})/Y(\mathbb{C})}\widetilde{\mathrm{td}}(\mathfrak{N}_{\mathbb{C}})\,\mathrm{td}^{-1}(\overline{(\mathscr{N}\mathrm{or}_{X/\mathbb{P}^n_Y})_{\mathbb{C}}})\,\mathrm{ch}(\overline{\mathscr{E}_{\mathbb{C}}})\right)\\
&-\alpha\left(\int_{X(\mathbb{C})/Y(\mathbb{C})}\mathrm{td}(i^*_{\mathbb{C}}\overline{T_{p,\mathbb{C}}})\,\mathrm{td}^{-1}(\overline{(\mathscr{N}\mathrm{or}_{X/\mathbb{P}^n_Y})_{\mathbb{C}}})\,\mathrm{ch}(\overline{\mathscr{E}_{\mathbb{C}}})\mathscr{R}(\overline{T_{f,\mathbb{C}}})\right)\\
&+\alpha\left(\int_{X(\mathbb{C})/Y(\mathbb{C})}dd^c\widetilde{\mathrm{td}}(\mathfrak{N}_{\mathbb{C}})\,\mathrm{td}^{-1}(\overline{(\mathscr{N}\mathrm{or}_{X/\mathbb{P}^n_Y})_{\mathbb{C}}})\,\mathrm{ch}(\overline{\mathscr{E}_{\mathbb{C}}})\mathscr{R}(\overline{T_{f,\mathbb{C}}})\right).
\end{aligned}
$$

Thus, it is time for the grand finale, where we accomplish the great summation. Note that

$$
\begin{aligned}
&(-1)^m\,\widetilde{\mathrm{ch}}(\mathfrak{H}_{\mathbb{C}})-\sum_i(-1)^i p_{\overline{\mathscr{F}_i}}+f_{\overline{\mathscr{E}}}\\
=&\int_{X(\mathbb{C})/Y(\mathbb{C})}\left(\widetilde{\mathrm{td}}(\mathfrak{N}_{\mathbb{C}})\,\mathrm{td}^{-1}(\overline{(\mathscr{N}\mathrm{or}_{X/\mathbb{P}^n_Y})_{\mathbb{C}}})+\mathrm{td}(\overline{T_{f,\mathbb{C}}})\mathscr{R}(\overline{T_{f,\mathbb{C}}})\right)\mathrm{ch}(\overline{\mathscr{E}_{\mathbb{C}}})\\
&-\int_{\mathbb{P}^n_Y(\mathbb{C})/Y(\mathbb{C})}\mathrm{td}(\overline{T_{p,\mathbb{C}}})\left(D_{\overline{\mathscr{E}}}+\mathscr{R}(\overline{T_{p,\mathbb{C}}})\sum_i(-1)^i\,\mathrm{ch}(\overline{\mathscr{F}_{i,\mathbb{C}}})\right),
\end{aligned}
$$

which was proved by Bismut using the Quillen metric. Hence,

$$
\sum_{i=0}^m(-1)^i\delta(\overline{\mathscr{F}}_i,p)-\delta(\overline{\mathscr{E}},f)=(\mathrm{I})+(\mathrm{II})+(\mathrm{III})+(\mathrm{IV})+(\mathrm{V})=0
$$

vanishes. □

# Chapter 4

# The Gross-Zagier Formula

In this chapter, we aim to provide a concise introduction to the Gross-Zagier formula. We will endeavor to present details, but some overly complex proofs will be omitted.

We will use the same notation for elements and the classes they represent, except for the ideals. We use lower-case Gothic letters for ideals and upper-case Gothic letters for their classes. The reader should keep in mind whether our notations refer to elements or to their classes in context.

This chapter was co-authored with Mingqiang FENG.

## 4.1 Elliptic Curves and Moduli

**Definition 4.1.1** (Elliptic Curves)**.** Let $L$ be a field. An **elliptic curve** defined over $L$ is a smooth, projective genus 1 algebraic curve $E$ defined over $L$, on which there is a specified $L$-point $O$. We write $(E/L,O)$ for such a curve. If $L=\mathbb{C}$, all the elliptic curves defined over $\mathbb{C}$ can be described by $(\mathbb{C}/\Lambda,0)$, where the discrete subgroup $\Lambda\subseteq\mathbb{C}$ is of rank 2 as a $\mathbb{Z}$-module. The Mordell-Weil theorem states that the set of $L$-points $E(L)$ has a finitely generated abelian group structure with $O$ the zero element.

The map between two elliptic curves $\phi:(E/L,O)\to(E'/L,O')$ is called an **isogeny** over $L$, if $\phi$ is a surjective $L$-morphism of curves that induces a group homomorphism from $E(\overline{L})$ to $E'(\overline{L})$. Two elliptic curves $(E/L,O)$ and $(E'/L,O')$ are **isomorphic** if their exist isogenies $\phi:(E/L,O)\to(E'/L,O')$ and $\psi:(E'/L,O')\to(E/L,O)$ whose composition is the identity.

Suppose $(E/\mathbb{C},O)$ is a complex elliptic curve, let $\mathscr{O}_K$ be the ring of integers of some imaginary quadratic field $K$. We say $(E/\mathbb{C},O)$ **has complex multiplication by** $\mathscr{O}_K$, if there exists a ring isomorphism $\mathscr{O}_K\to\mathrm{End}(E/\mathbb{C},O):=\{\text{complex isogenies }(E/\mathbb{C},O)\to(E/\mathbb{C},O)\}$.

Sometimes we abbreviate $(E/L,O)$ to $E$ for convenience.

**Remark 4.1.2.** Here we review the moduli problem for complex elliptic curves. Obviously, there are one-to-one correspondences:

$$Y(1)(\mathbb{C}):=\frac{\{\text{elliptic curves }E/\mathbb{C}\}}{\mathbb{C}\text{-isomorphism}}\xleftrightarrow{\sim}\frac{\{\text{genus 1 Riemann surfaces}\}}{\text{biholomorphy}}\xleftrightarrow{\sim}\frac{\{\text{lattices in }\mathbb{C}\}}{\text{homothety}}.$$

The set $Y(1)(\mathbb{C})$ has some extra structure.

- There is a bijection $\mathrm{SL}_2(\mathbb{Z})\backslash\mathscr{H} \xrightarrow{\sim} Y(1)(\mathbb{C})$, where $\mathscr{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ is the upper half-plane and the action of $\mathrm{SL}_2(\mathbb{Z})$ is given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z) := \frac{az+b}{cz+d}$. Hence, $Y(1)(\mathbb{C})$ is a non-compact Riemann surface. Indeed, as Riemann surfaces, we have $j : Y(1)(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}$, where the map $j$ is the $j$-invariant
$$j(E) := 1728\frac{g_4(\Lambda_E)^3}{g_4(\Lambda_E)^3 - g_6(\Lambda_E)^2} = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots,$$
here $q := e^{2\pi i z_E}$, $g_k(\Lambda) := (2\zeta(k))^{-1}\sum_{\lambda\in\Lambda\setminus\{0\}}\lambda^{-k}$, and $z_E \in \mathscr{H}, \Lambda_E \in \{\text{lattices in } \mathbb{C}\}$ are the elements correspond to $E$.
- Adding the cusps (i.e. $\mathrm{SL}_2(\mathbb{Z})$-orbits of $\mathbb{P}^1(\mathbb{Q})$, indeed this action is transitive) to $\mathrm{SL}_2(\mathbb{Z})\backslash\mathscr{H}$, is the same as compactify the complex plane $\mathbb{C}$ by adding the point $\infty$. This gives a compact Riemann surface $\mathrm{SL}_2(\mathbb{Z})\backslash\big(\mathscr{H} \sqcup \mathbb{P}^1(\mathbb{Q})\big) \cong \mathbb{P}^1(\mathbb{C})$, call it the **modular curve** $X(1)(\mathbb{C})$.
- The Riemann-Roch theorem for divisors $kO$ on $E(\mathbb{C})$ implies a bijection $Y(1)(\mathbb{C}) \xrightarrow{\sim} \big\{\text{smooth curves in } \mathbb{P}^2_{\mathbb{C}} \text{ defined by } y^2z = 4x^3 + axz^2 + bz^3, a, b \in \mathbb{C}\big\}$. Hence, $Y(1)(\mathbb{C})$ is the parameter space of complex algebraic elliptic curves.

Elliptic curves with complex multiplication form a special subset of $Y(1)(\mathbb{C})$. Let $K$ be an imaginary quadratic field, write
$$\mathrm{Ell}(\mathscr{O}_K) := \big\{\text{elliptic curves } E/\mathbb{C} \text{ have complex multiplication by } \mathscr{O}_K\big\}\big/\mathbb{C}\text{-isomorphsim}.$$
By Remark 4.1.2, $\mathrm{Ell}(\mathscr{O}_K) = \big\{\text{lattices } \Lambda \text{ with } \{z \in \mathbb{C} : z\Lambda \subseteq \Lambda\} \cong \mathscr{O}_K\big\}\big/\text{homothety}$. We now list some important facts about the set $\mathrm{Ell}(\mathscr{O}_K)$ without proof.

**Proposition 4.1.3.** *Let $K$ be an imaginary quadratic field with integers $\mathscr{O}_K$.*

- *The action of* $\mathrm{Cl}(\mathscr{O}_K)$ *on* $\mathrm{Ell}(\mathscr{O}_K)$ *given by* $\mathfrak{A} : \Lambda \mapsto \mathfrak{a}^{-1}\Lambda := \{a_1\lambda_1 + \cdots + a_k\lambda_k : a_i \in \mathfrak{a}^{-1}, \lambda_i \in \Lambda\}$, *where $\mathfrak{a}$ is some fractional ideal in $\mathfrak{A}$, is well-defined and simply-transitive. We usually denote this action by $\mathfrak{A} * (\cdot)$. Hence,* $\#\mathrm{Cl}(\mathscr{O}_K) = \#\mathrm{Ell}(\mathscr{O}_K)$.
- *The action of* $\mathrm{Gal}(\overline{K}/K)$ *on* $\mathrm{Ell}(\mathscr{O}_K)$ *given by $\sigma : E \mapsto \sigma(E)$ is well-defined, this induces a well-defined homomorphism* $F : \mathrm{Gal}(\overline{K}/K) \to \mathrm{Cl}(\mathscr{O}_K)$ *characterized by the property $F(\sigma) * E = \sigma(E)$, for any $E/\mathbb{C}$ has complex multiplication by $\mathscr{O}_K$. The kernel is* $\ker(F) = \mathrm{Gal}(\overline{K}/H_K)$, *where $H_K$ is the* ***Hilbert class field*** *of $K$, i.e. the maximal unramified abelian extension of $K$. Hence,* $[H_K : K] = \#\mathrm{Gal}(H_K/K) = \#\mathrm{Cl}(\mathscr{O}_K)$.
- *Let $E/\mathbb{C}$ be an elliptic curve with complex multiplication by $\mathscr{O}_K$, then the $j$-invariant $j(E)$ is an algebraic integer, and $H_K = K(j(E))$.*
- *The class field theory gives a group isomorphism, called the* ***Artin reciprocity map*** $\mathrm{Art}_K : \mathrm{Cl}(\mathscr{O}_K) \xrightarrow{\sim} \mathrm{Gal}(H_K/K), \mathfrak{A} \mapsto (\mathrm{Frob}_{\mathfrak{p}_1} \circ \cdots \circ \mathrm{Frob}_{\mathfrak{p}_k})$, *where $\mathfrak{a} = \prod_{i=1}^{k}\mathfrak{p}_i$ is a prime decomposition for some $\mathfrak{a} \in \mathfrak{A}$. This map commutes with $j$ in the sense* $\mathrm{Art}_K(\mathfrak{A})(j(E)) = j(\mathfrak{A} * E)$, *for any $E/\mathbb{C}$ has complex multiplication by $\mathscr{O}_K$.*

**Exercise 4.1.4.** *Use Proposition 4.1.3, prove* $\mathrm{Art}_K(\mathfrak{A})(E) = F(\mathrm{Art}_K(\mathfrak{A})) * E = \mathfrak{A} * E$.

We now introduce the **modular curve** $X_0(N)$ as an analogy of Remark 4.1.2, which plays a central role in the following research. However, even if the properties of modular curves $X(N)$ and $X_1(N)$ are known [Grad], we will not study them in this book. Let

$$\Gamma_0(N) := \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\} \subseteq \mathrm{SL}_2(\mathbb{Z})$$

denote the congruence subgroup of level $N$. Define a compact Riemann surface

$$X_0(N)(\mathbb{C}) := \Gamma_0(N) \backslash \big(\mathscr{H} \sqcup \mathbb{P}^1(\mathbb{Q})\big),$$

which is the compactification of $Y_0(N)(\mathbb{C}) := \Gamma_0(N)\backslash\mathscr{H}$ by adding the cusps. In fact, these cusps are described by

$$\Gamma_0(N)\backslash\mathbb{P}^1(\mathbb{Q}) \overset{\sim}{\longleftrightarrow} \bigsqcup_{d|N,d>0} \big(\mathbb{Z}/(d,N/d)\,\mathbb{Z}\big)^\times,$$

whose cardinal number is $\sum_{d|N,d>0} \Phi((d,N/d))$, where $\Phi$ is the Euler function (see [DiSh], note that the cusps $\infty$ and 0 correspond to $d = N$ and $d = 1$ respectively).

**Exercise 4.1.5.** $X_0(1)(\mathbb{C}) = X(1)(\mathbb{C}) = \mathbb{P}^1(\mathbb{C})$. *The degree of the natural ramified covering map* $X_0(N)(\mathbb{C}) \to X_0(1)(\mathbb{C}) = \mathbb{P}^1(\mathbb{C})$ *is* $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \cdot \prod_{p|N}(1+p^{-1})$.

For $X_0(N)$, we have:

**Theorem 4.1.6.** *Let $N > 0$ be an integer.*

- *(Moduli Interpretation). There is a bijection*

$$Y_0(N)(\mathbb{C}) \overset{\sim}{\longleftrightarrow} \frac{\{\textit{isogenies } E \to E' \textit{ over } \mathbb{C} \textit{ with kernel isomorphic to } \mathbb{Z}/N\mathbb{Z}\}}{\mathbb{C}\textit{-isomorphism}},$$

  *given by* $z \longmapsto (\mathbb{C}/\langle z,1\rangle \to \mathbb{C}/\langle z,1/N\rangle)$.

- *(Rational Model). The function field of the Riemann surface $X_0(N)(\mathbb{C})$ (as a projective $\mathbb{C}$-scheme) is $\mathbb{C}(j(z),j(Nz)) \cong \mathbb{C}[x,y]/\langle\Phi_N(x,y)\rangle$, where $\Phi_N(x,y) \in \mathbb{Q}[x,y]$. Thus, $X_0(N)/\mathbb{C}$ (or $X_0(N)(\mathbb{C})$) has a $\mathbb{Q}$-model, denotes as $X_0(N)/\mathbb{Q}$, which is a projective non-singular curve (after desingularization).*

- *(Representable). The functor induced by the moduli interpretation above is representable by a modular stack. If we take the compactified coarse moduli space of this functor we get a finite type $\mathbb{Z}$-scheme $X_0(N)/\mathbb{Z}$. Outside $p|N$ it is an arithmetic surface.*

- *(Special Fiber). For the scheme $X_0(N)/\mathbb{Z}$ above, $X_0(N) \times_{\mathbb{Z}} \mathbb{Z}[1/N]$ is smooth and proper over* $\mathrm{Spec}(\mathbb{Z}[1/N])$. *More precisely, let $X_0(N)_{\mathbb{F}_p} := X_0(N) \times_{\mathbb{Z}} \mathbb{F}_p$, for a prime number $p$, we have:*
  - *For $p \nmid N$, the scheme $X_0(N)_{\mathbb{F}_p}$ over $\mathbb{F}_p$ is a smooth and projective curve.*
  - *For $p|N$, write $N = p^kM$ with $(p,M) = 1$. The scheme $X_0(N)_{\mathbb{F}_p}$ is reducible and has $k+1$ irreducible components $\mathscr{F}_{a,b}$, indexed by $\{(a,b) : a,b \in \mathbb{Z}_{\geq 0}, a+b = k\}$, satisfy:*

  * *Each $\mathscr{F}_{a,b}$ is isomorphic to $X_0(M)_{\mathbb{F}_p}$ as $\mathbb{F}_p$-scheme and occurs with multiplicity $\Phi(p^{\min\{a,b\}})$ in $X_0(N)_{\mathbb{F}_p}$.*
  * *All of the $\mathscr{F}_{a,b}$ intersect at each supersingular point, these are the points correspond to $\mathbb{F}_p$-isogenies $E \to E'$ with supersingular elliptic curves $E$ and $E'$.*
  * *Any non-supersingular point of $\mathscr{F}_{a,b}$ corresponds to a $\mathbb{F}_p$-isogeny of elliptic curves or Néron polygons (we call them **generalized elliptic curves**, indeed the isogenies of Néron polygons are located exactly on the divisor of cusps), whose kernel is isomorphic locally to $\mu_{p^a} \times \mathbb{Z}/p^b M\mathbb{Z}$ as group schemes.*

  *For a special case $X_0(p^2)$, we provide a picture in Chapter 4.3.*

- *(Modularity Theorem).*
  - *Let $E/\mathbb{C}$ be an elliptic curve with $j(E) \in \mathbb{Q}$. Then for some positive integer $N$ there exists a surjective holomorphic function of compact Riemann surfaces $X_0(N)(\mathbb{C}) \to E(\mathbb{C})$.*
  - *Let $E/\mathbb{Q}$ be an elliptic curve. Then for some positive integer $N$ there exists a surjective $\mathbb{Q}$-morphism $X_0(N)/\mathbb{Q} \to E/\mathbb{Q}$.*

The modularity theorem was conjectured by Shimura-Taniyama-Weil, and proved by Breuil-Conrad-Diamond-Taylor-Wiles. It was the key ingredient in the proof of Fermat's Last Theorem.

**Definition 4.1.7** (Modular Forms with Nebentypus)**.** A **modular form** of level $\Gamma_0(N)$ and weight $k$ is a holomorphic function $f : \mathscr{H} \to \mathbb{C}$ satisfying the following two conditions:

- For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, we have $[f|_k\gamma](z) := (cz+d)^{-k} f(\gamma z) = f(z)$.
- $f$ is bounded near all cusps of $\Gamma_0(N)$.

The $\mathbb{C}$-linear space of all modular forms of level $\Gamma_0(N)$ and weight $k$ is denoted by $M_k(\Gamma_0(N))$. In addition, a modular form is called a **cusp form** if it is vanish at all cusps of $\Gamma_0(N)$. Cusp forms of level $\Gamma_0(N)$ and weight $k$ form a $\mathbb{C}$-linear subspace $S_k(\Gamma_0(N))$ of $M_k(\Gamma_0(N))$. On $S_k(\Gamma_0(N))$ we can define the **Petersson inner product**, by let

$$(\cdot,\cdot)_{\mathrm{Pet}} : S_k(\Gamma_0(N)) \times S_k(\Gamma_0(N)) \to \mathbb{C}, \quad (f,g)_{\mathrm{Pet}} := \int_{\Gamma_0(N)\backslash\mathscr{H}} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2},$$

where $z = x+iy \in \mathscr{H}$ (the well-definedness is guaranteed by Exercise 4.1.8). The space of **new forms** $S_k^{\mathrm{new}}(\Gamma_0(N))$ is defined as the orthogonal complement of the space of **old forms** that spanned by the set

$$\bigcup_{\text{proper divisors } M \text{ of } N} \left( \bigcup_{\text{positive divisors } d \text{ of } N/M} \{f(dz) : f \in S_k(\Gamma_0(M))\} \right) \subseteq S_k(\Gamma_0(N)),$$

with respect to the Petersson inner product.

Let $\varepsilon$ be a Dirichlet character modulo $N$, define the $\varepsilon$-eigenspace

$$M_k(\Gamma_0(N),\varepsilon) := \left\{ f \in M_k(\Gamma_0(N)) : [f|_k\gamma] = \varepsilon(d)\cdot f \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \right\}.$$

Elements in this space are called **modular forms with Nebentypus $\varepsilon$**.

Let $f \in S_k(\Gamma_0(N))$. Note that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_0(N)$, this implies $f$ is a periodic function, so we have the Fourier expansion $f(z) = \sum_{n>0} a(n)e^{2\pi inz}$. The **Hecke $L$-function** of $f$ is defined to be the series $L(f,s) := \sum_{n>0} \frac{a(n)}{n^s}$. This series converges absolutely for $\mathrm{Re}(s)$ sufficiently large and has a meromorphic continuation to the whole complex plane (see [DiSh]).

**Exercise 4.1.8.** *Let $f,g \in S_k(\Gamma_0(N))$, show that $f(z)\overline{g(z)}\mathrm{Im}(z)^k$ is $\Gamma_0(N)$-invariant.*

## 4.2 Automorphic Green Functions and Intersections

Fix an imaginary quadratic field $K$, its Hilbert class field is denoted as $H_K$. Let $N > 1$, consider the scheme $X_0(N)/H_K := X_0(N) \times_{\mathbb{Q}} H_K$, suppose $\mathscr{X}_0(N)/\mathscr{O}_{H_K}$ is a proper regular model of $X_0(N)/H_K$.

One side of the Gross-Zagier formula deals with canonical Néron-Tate height, which can be defined in some different ways. In this section, we follow the ideas in Chapter 2.3 and study the arithmetic intersection theory of rational points on $X_0(N)/H_K$ $\big($indeed, this is the so-called **Néron pairing** on the group of degree zero divisors $\mathrm{Div}^0(X_0(N)(H_K))\big)$ as a way to define heights. However, the statement of intersection pairing in Theorem 2.3.4 needs to be revised because of the singularity at cusps. In the next section we will study the intersections associated with Heegner points.

Our aim is to establish a pairing

$$\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{\mathrm{fin}} + \langle\cdot,\cdot\rangle_{\mathrm{inf}} : \mathrm{Div}^0(X_0(N)(H_K)) \times \mathrm{Div}^0(X_0(N)(H_K)) \to \mathbb{R}.$$

Establishing $\langle\cdot,\cdot\rangle_{\mathrm{fin}}$ is easy. Let $D \in \mathrm{Div}^0(X_0(N)(H_K))$ be a divisor on $X_0(N)/H_K$, this is a linear combination of $H_K$-points. For each $\mathfrak{P} \in \mathrm{Spec}(\mathscr{O}_{H_K})$, the scheme $\mathscr{X}_0(N)_{\mathfrak{P}} := \mathscr{X}_0(N) \times_{\mathscr{O}_{H_K}} \mathscr{O}_{H_K,\mathfrak{P}}$ is regular, and one can extend $D$ to a horizontal (rational) divisor $D_{\mathfrak{P}}$ on $\mathscr{X}_0(N)_{\mathfrak{P}}$ such that

$$i_{\mathfrak{P}}(D_{\mathfrak{P}}, F_{\mathfrak{P},t}) := \sum_{x|\mathfrak{P}} i_x(D_{\mathfrak{P}}, F_{\mathfrak{P},t})[k(x):k(\mathfrak{P})] = 0$$

for all $t$, where $\{F_{\mathfrak{P},t}\}$ is the set of (reduced) irreducible components of the special fiber of $\mathscr{X}_0(N)_{\mathfrak{P}}$. Such $D_{\mathfrak{P}}$ can be computed more precisely: let $\widetilde{D}_{\mathfrak{P}}$ be the Zariski closure of $D$ in $\mathscr{X}_0(N)_{\mathfrak{P}}$, we can find a vertical (rational) divisor $V_{\mathfrak{P}}$ so that $i_{\mathfrak{P}}(V_{\mathfrak{P}}, F_{\mathfrak{P},t}) = i_{\mathfrak{P}}(\widetilde{D}_{\mathfrak{P}}, F_{\mathfrak{P},t})$ for all $t$, then $D_{\mathfrak{P}} = \widetilde{D}_{\mathfrak{P}} - V_{\mathfrak{P}}$ is our desired extension.

Now define

$$\langle D,E\rangle_{\mathrm{fin}} := \sum_{\mathfrak{P}\in\mathrm{Spec}(\mathscr{O}_{H_K})} i_{\mathfrak{P}}(D_{\mathfrak{P}}, \widetilde{E}_{\mathfrak{P}}) \log\#k(\mathfrak{P}),$$

where we need for each $\mathfrak{P} \in \mathrm{Spec}(\mathscr{O}_{H_K})$, $D_{\mathfrak{P}}$, $\widetilde{E}_{\mathfrak{P}}$ (Zariski closure of $E$) has no common support.

To define $\langle\cdot,\cdot\rangle_{\inf} := \sum_{\tau\in\mathrm{Hom}(H_K,\mathbb{C})}\langle\cdot,\cdot\rangle_\tau$, one needs the so-called automorphic Green functions $\mathbf{g}_{N,\tau}$.

We refer to [Iwan]. Recall the distance function on the hyperbolic Riemann manifold

$$\left(\mathscr{H}, ds^2 = \frac{dx^2+dy^2}{y^2}, \mathrm{vol} = \frac{dxdy}{y^2}\right)$$

is given exactly by

$$d(z,w) = \cosh^{-1}(1+2u(z,w)), \quad \text{where } u(z,w) = \frac{|z-w|^2}{4\mathrm{Im}(z)\mathrm{Im}(w)}.$$

Let

$$\Delta_{\mathrm{Bt}} := y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) = -(z-\bar{z})^2\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}$$

be the **hyperbolic Laplace-Beltrami opeartor**, which is invariant under the action of $\mathrm{SL}_2(\mathbb{R})$. A great deal of Fourier analysis, or Harmonic analysis on $\mathscr{H}$ concerns decomposition of functions $f:\mathscr{H}\to\mathbb{C}$ into eigenspaces of $\Delta_{\mathrm{Bt}}$.

The Cartan decomposition $\mathrm{SL}_2(\mathbb{R}) = KAK$, where

$$K := \left\{k(\varphi) = \begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix} : 0\le\varphi<\pi\right\}, A := \left\{a(d) = \begin{pmatrix}e^{-d/2} & 0\\ 0 & e^{d/2}\end{pmatrix} : d\ge 0\right\},$$

provides the geodesic polar coordinate

$$k(\varphi)a(d)i = \Big(\frac{\sinh d\sin(2\varphi)}{\cosh d+\sinh d\cos(2\varphi)}\Big) + i\Big(\frac{1}{\cosh d+\sinh d\cos(2\varphi)}\Big)\in\mathscr{H},$$

where $d = d(i, e^{-d}i)$. If we write $\cosh d = 1+2u$ for $u = u(i,e^{-d}i)$, then in geodesic polar coordinates the operator $\Delta_{\mathrm{Bt}}$ takes the form

$$\Delta_{\mathrm{Bt}} = u(u+1)\frac{\partial^2}{\partial u^2} + (2u+1)\frac{\partial}{\partial u} + \frac{1}{16u(u+1)}\frac{\partial^2}{\partial\varphi^2}.$$

Since we expect the Green function to be independent of direction, we remove the term containing $\varphi$ by separation of variables. At this point, the spectrum of $\Delta_{\mathrm{Bt}}$ is divided into discrete and continuous spectra, corresponding to two important eigenfunctions: automorphic Green functions and Eisenstein series.

**Definition 4.2.1** (Automorphic Green Functions)**.** Given $\lambda = s(1-s)\in\mathbb{C}$ such that $\mathrm{Re}(s)>0$. A **pre-automorphic Green function** is a solution $G_s(u)$ of the ordinary differential equation

$$(\Delta_{\mathrm{Bt}}+\lambda)G_s(u) = \left(u(u+1)\frac{d^2}{du^2} + (2u+1)\frac{d}{du} + s(1-s)\right)G_s(u) = 0$$

that exactly given by

$$G_s(u) := \frac{1}{4\pi}\int_0^1\frac{(t(1-t))^{s-1}}{(t+u)^s}dt.$$

Taking averages, the series

$$g_{N,s}(z,w) := \sum_{\gamma\in\Gamma_0(N)} G_s(u(z,\gamma w)), \quad \text{if } z \not\equiv w (\operatorname{mod} \Gamma_0(N)),$$

is called the **automorphic Green function**.

Since $\Delta_{\mathrm{Bt}}$ is an elliptic operator with real analytic coefficients, it forces its eigenfunction $G_s(u)$ to be real analytic if $s \in \mathbb{R}_{>0}$. Using the language of hypergeometric function $F(\cdot,\cdot;\cdot;\cdot)$, the integral $G_s(u)$ with parameter $u$ can be written as

$$G_s(u) = \frac{1}{4\pi}\frac{\Gamma(s)^2}{\Gamma(2s)}u^{-s}F(s,s;2s;u^{-1}).$$

In addition, we list some more necessary properties about the automorphic green functions here.

**Proposition 4.2.2.** *$G_s(u)$ and $g_{N,s}(z,w)$ satisfy the following properties:*

- *The integral $G_s(u)$ converges absolutely for $\operatorname{Re}(s) > 0$, and*
  - $G_s(u) = -\frac{1}{4\pi}\log u + O(1)$, $u \to 0$.
  - $G_s(u) = o\big(u^{-\operatorname{Re}(s)}\big)$, $u \to +\infty$.
  - *$G_s(u(z,w))$ is singular on the diagonal, that is*

  $$G_s(u(z,w)) = -\frac{1}{4\pi}\log|z-w|^2 + \textit{smooth function in } (z,w).$$

- *The series $g_{N,s}(z,w)$ converges absolutely for $\operatorname{Re}(s) > 1$, and it can be extended to a meromorphic function in $s$ with residue*

  $$\kappa_N := \operatorname{vol}(Y_0(N)(\mathbb{C}))^{-1} = \big(\operatorname{vol}(Y_0(1)(\mathbb{C}))[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]\big)^{-1} = \frac{3}{N\pi}\prod_{p|N}\frac{p}{1+p}$$

  *(independent of $z,w$) at the simple pole $s = 1$. Furthermore, for $z \in \mathscr{H}$ we have*

  $$g_{N,s}(z,w) = -\frac{1}{4\pi}\cdot\#(\operatorname{Stab}_{\Gamma_0(N)}(z))\cdot\log|z-w|^2 + O(1), \quad \textit{as } w \to z.$$

- *$g_{N,s}(z,w)$ is $\Gamma_0(N)$-invariant in each variable, that is*

  $$g_{N,s}(\gamma z,\gamma' w) = g_{N,s}(z,w) = g_{N,s}(w,z), \quad \textit{for } \gamma,\gamma' \in \Gamma_0(N), z \not\equiv w (\operatorname{mod} \Gamma_0(N)).$$

  *Therefore, $g_{N,s}(z,w)$ induces a Green function on $Y_0(N)(\mathbb{C})$.*

- *(Poincaré-Lelong Formula). If $f$ is smooth and bounded on $\mathscr{H}$, then*

  $$\int_{\mathscr{H}} G_s(u(z,w))(\Delta_{\mathrm{Bt}}+\lambda)f(w)\operatorname{vol}(w) + f(z) = 0$$

  *holds for $\operatorname{Re}(s) > 1$. Moreover, if $f(z)$ is smooth on $Y_0(N)(\mathbb{C})$ and it is a high-order infinitesimal of $\operatorname{Im}(z)^{\delta}$ near any cusp, then*

  $$\int_{\Gamma_0(N)\backslash\mathscr{H}} g_{N,s}(z,w)(\Delta_{\mathrm{Bt}}+\lambda)f(w)\operatorname{vol}(w) + f(z) = 0$$

  *holds for $\operatorname{Re}(s) > 1+\delta$.*

The Green function $g_{N,s}(z,w)$ here is not currently harmonic. In order to obtain a real analytic harmonic function, we need to modify it with Eisenstein series.

Recall $\Gamma_0(N)\backslash\mathbb{P}^1(\mathbb{Q})$ is the set of cusps of $\Gamma_0(N)$. If $\mathfrak{c}$ is a cusp, there exists $\xi_{\mathfrak{c}} \in \mathrm{SL}_2(\mathbb{R})$ such that $\xi_{\mathfrak{c}}\infty = \mathfrak{c}$ and $\xi_{\mathfrak{c}}^{-1}\gamma_{\mathfrak{c}}\xi_{\mathfrak{c}} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, where $\langle\gamma_{\mathfrak{c}}\rangle = \mathrm{Stab}_{\Gamma_0(N)}(\mathfrak{c})$.

**Definition 4.2.3** (Eisenstein Series)**.** Let $z \in \mathscr{H}$ and $\mathrm{Re}(s) > 1$. Define the **Eisenstein series** for the cusp $\mathfrak{c}$ of $\Gamma_0(N)$ as

$$E_{N,\mathfrak{c}}(z,s) := \sum_{\gamma \in \mathrm{Stab}_{\Gamma_0(N)}(\mathfrak{c})\backslash\Gamma_0(N)} \mathrm{Im}(\xi_{\mathfrak{c}}^{-1}\gamma z)^s.$$

For $\mathfrak{c} = \infty$ we have $\xi_{\mathfrak{c}} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, one can write this series more explicitly. Note that the summation indices are taken from the set

$$\begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix}\Big\backslash\Gamma_0(N) \overset{\sim}{\longleftrightarrow} \frac{\{(c,d) \in \mathbb{Z}^2 : (c,d) = 1, N|c\}}{(c,d) \sim (c',d') \Leftrightarrow cd' = c'd}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (c,d),$$

therefore,

$$E_{N,\infty}(z,s) = \frac{1}{2}\sum_{N|c,(c,d)=1} \frac{\mathrm{Im}(z)^s}{|cz+d|^{2s}}.$$

**Proposition 4.2.4.** *$E_{N,\mathfrak{c}}(z,s)$ satisfy the following properties:*

- *For* $\mathrm{Re}(s) > 1$*, $E_{N,\mathfrak{c}}(z,s)$ is an eigenfunction of $\Delta_{\mathrm{Bt}}$ with eigenvalue $-\lambda = -s(1-s)$, i.e. $(\Delta_{\mathrm{Bt}} + \lambda)E_{N,\mathfrak{c}}(z,s) = 0$.*
- *(Fourier Expansion). Let $\mathfrak{c}, \mathfrak{e}$ be two cusps. For $s$ on the line* $\mathrm{Re}(s) = \sigma > 1$ *we have*

$$E_{N,\mathfrak{c}}(\xi_{\mathfrak{e}}z,s) = \delta_{\mathfrak{c}\mathfrak{e}}y^s + \rho_{\mathfrak{c}\mathfrak{e}}(s)y^{1-s} + O((1+y^{-\sigma})e^{-2\pi y})$$

*uniformly in $z = x+iy \in \mathscr{H}$, where $\delta_{\mathfrak{c}\mathfrak{e}}$ is the Kronecker delta and*

$$\rho_{\mathfrak{c}\mathfrak{e}}(s) := \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)}\sum_c c^{-2s}\cdot\#\left\{d (\mathrm{mod}\ c) : \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \xi_{\mathfrak{c}}^{-1}\Gamma_0(N)\xi_{\mathfrak{e}}\right\}.$$

*In particular, if $N = 1$ and $\mathfrak{c} = \mathfrak{e} = \infty$, one has*

$$E_{1,\infty}(z,s) = y^s + \rho_{\infty\infty}(s)y^{1-s} + O((1+y^{-\sigma})e^{-2\pi y}),$$

*where*

$$\rho_{\infty\infty}(s) = \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)} = \frac{\frac{3}{\pi}}{s-1} - \frac{3}{\pi}\left(2\Gamma'(1) + 2\log 2 + 2\frac{\zeta'(2)}{\zeta(2)}\right) + O(s-1).$$

- *$E_{N,\mathfrak{c}}(z,s)$ can be extended to a meromorphic function in $s$ with residue at the simple pole $s = 1$ is*

$$\mathrm{vol}(Y_0(N)(\mathbb{C}))^{-1} = \kappa_N.$$

- *Let the Möbius function be* $\mu(n) := \begin{cases} 1 & n=1 \\ (-1)^k & \text{square free } n = p_1\cdots p_k \\ 0 & \text{otherwise} \end{cases}$*, then*

$$E_{N,\infty}(z,s) = N^{-s}\prod_{p|N}(1-p^{-2s})^{-1}\cdot\sum_{d|N}\frac{\mu(d)}{d^s}E_{1,\infty}\left(\frac{N}{d}z,s\right).$$

- *The following functional equation holds*

$$g_{N,s}(z,w) - g_{N,1-s}(z,w) = -\frac{1}{2s-1}\sum_{\mathfrak{c}\ cusp} E_{N,\mathfrak{c}}(z,1-s)E_{N,\mathfrak{c}}(w,s).$$

- *Let* $\mathfrak{c},\mathfrak{e}$ *be cusps of* $\Gamma_0(N)$*, let* $\mathrm{Re}(s)>1$*. Then for* $z = x+iy, w = x'+iy' \in \mathscr{H}$ *with* $y$ *sufficiently large, we have*

$$g_{N,s}(\xi_{\mathfrak{c}}z,\xi_{\mathfrak{e}}w) = \frac{1}{2s-1}E_{N,\mathfrak{c}}(\xi_{\mathfrak{e}}w,s)y^{1-s} - \frac{\delta_{\mathfrak{c}\mathfrak{e}}}{\pi}\log|1-e^{2\pi i(z-w)}| + O(e^{-2\pi(y-y')}).$$

Proofs of Proposition 4.2.2 and Proposition 4.2.4 can be found in [Iwan] (some typos are corrected here), except for the so-called Möbius inversion formula. We provide a proof of this formula here.

*Proof.* For $N = p$ a prime, one has

$$\begin{aligned} E_{1,\infty}(z,s) &= \frac{1}{2}\sum_{p|c,(c,d)=1}\frac{\mathrm{Im}(z)^s}{|cz+d|^{2s}} + \frac{1}{2}\sum_{(c,pd)=1}\frac{\mathrm{Im}(z)^s}{|cz+d|^{2s}} \\ &= E_{p,\infty}(z,s) + p^s\cdot\frac{1}{2}\sum_{p|t,(c,t)=1}\frac{\mathrm{Im}(pz)^s}{|c(pz)+t|^{2s}} \\ &= E_{p,\infty}(z,s) + p^s\big(E_{1,\infty}(pz,s) - p^sE_{p,\infty}(z,s)\big). \end{aligned}$$

Hence, $E_{p,\infty}(z,s) = p^{-s}(1-p^{-2s})^{-1}\big(E_{1,\infty}(pz,s) - p^{-s}E_{1,\infty}(z,s)\big)$. The next step involves the induction on the number of prime factors, which is equivalent to using the principle of inclusion-exclusion. $\square$

**Theorem 4.2.5.** *Let* $N>1$*, the limit*

$$\mathbf{g}_N(z,w) := \lim_{s\to1}\left(g_{N,s}(z,w) - E_{N,\infty}\left(-\frac{1}{Nz},s\right) - E_{N,\infty}(w,s) + \frac{\kappa_N}{s-1}\right) + C_N$$

*exists, where* $C_N$ *is a constant such that* $\lim_{z\to\infty,w\to w_0}\mathbf{g}_N(z,w) = 0$ *for any fixed* $\infty \neq w_0 \in X_0(N)(\mathbb{C})$*. Moreover,* $\mathbf{g}_N(z,w)$ *satisfy the conditions:*

- $\mathbf{g}_N(w,z) = \mathbf{g}_N(-\frac{1}{Nz},-\frac{1}{Nw})$*, i.e.* $\mathbf{g}_N$ *is not symmetric (unlike* $g_{N,s}$*).*
- $\mathbf{g}_N(\gamma z,\gamma' w) = \mathbf{g}_N(z,w)$ *for all* $\gamma,\gamma' \in \Gamma_0(N), z,w \in \mathscr{H}$*.*
- $\mathbf{g}_N(z,w) = -\frac{1}{4\pi}\#(\mathrm{Stab}_{\Gamma_0(N)}(z))\cdot\log|z-w|^2 + O(1)$*, as* $w\to z$*.*

- $\mathbf{g}_N(z,w)$ *is continuous and harmonic for* $z \not\equiv w (\mathrm{mod}\ \Gamma_0(N))$.
- *For* $z \in \mathscr{H}$ *fixed, we have* $\mathbf{g}_N(z,w) = -\mathrm{Im}(w) + O(1)$ *as* $w \to \infty$ *and* $\mathbf{g}_N(z,w) = O(1)$ *as* $w$ *tends to cusps of* $\Gamma_0(N)$ *other than* $\infty$*; for* $w \in \mathscr{H}$ *fixed, we have* $\mathbf{g}_N(z,w) = -\frac{\mathrm{Im}(z)}{N|z|^2} + O(1)$ *as* $z \to 0$ *and* $\mathbf{g}_N(z,w) = O(1)$ *as* $z$ *tends to cusps other than* $0$.

*Proof.* We are going to keep using Proposition 4.2.2 and Proposition 4.2.4. The first three items are directly derived from the definitions and properties of $g_{N,s}$ and $E_{N,\infty}$. Note that $s = 1$ is not a pole of the function in the limit of $\mathbf{g}_N(z,w)$ because its residue at $s = 1$ is equal to 0. So $\mathbf{g}_N(z,w)$ is well-defined and continuous. To verify the harmonicity, we just calculate (for the variable $z$, say)

$$\Delta_{\mathrm{Bt}}\mathbf{g}_N(z,w) = \lim_{s\to 1}\left(-s(1-s)g_{N,s}(z,w) + s(1-s)E_{N,\infty}\left(-\frac{1}{Nz}, s\right)\right) = 0,$$

since $g_{N,s}$ and $E_{N,\infty}$ are both have residue $\kappa_N$ at $s = 1$. For the last item, for example, we fix $w$ and study the behavior of the limit. By Proposition 4.2.4 we may consider

$$\begin{aligned} F(s) :=& g_{N,s}(\xi_{\mathfrak{c}}z,w) - E_{N,\infty}(\xi_0\xi_{\mathfrak{c}}z,s) - E_{N,\infty}(w,s) + \frac{\kappa_N}{s-1} \\ =& -\delta_{\infty\mathfrak{e}}y^s + \left(\frac{1}{2s-1}E_{N,\mathfrak{c}}(w,s) - \rho_{\infty\mathfrak{e}}(s)\right)y^{1-s} + \left(\frac{\kappa_N}{s-1} - E_{N,\infty}(w,s)\right) + O((1+y^{-\sigma})e^{-2\pi y}) \end{aligned}$$

as $y = \mathrm{Im}(z) \to \infty$, where $\xi_0 = \begin{pmatrix} 0 & 1/\sqrt{N} \\ -\sqrt{N} & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $\mathfrak{e}$ is a cusp such that $\xi_0\xi_{\mathfrak{c}} = \xi_{\mathfrak{e}}$. For $\mathfrak{c} \neq 0$ we have $\mathfrak{e} \neq \infty$, by Proposition 4.2.4 the terms in brackets above are both have residues 0 at $s = 1$, hence $\lim_{s\to 1} F(s)$ is bounded as a function in $y$ (this can be obtained by taking Laurent expansions at $s = 1$), which means $\mathbf{g}_N(z,w) = O(1)$ as $z$ tends to cusps other than 0. For $\mathfrak{c} = 0$ we have $\mathfrak{e} = \infty$, the same argument points out $\lim_{s\to 1} F(s) = -y + O(1)$, which means $\mathbf{g}_N(z,w) = -\frac{\mathrm{Im}(z)}{N|z|^2} + O(1)$ as $z \to 0$ (here replace $\xi_0 z$ with $z$). The assertions for fixed $z$ can be proved similarly. □

**Remark 4.2.6.** The constant $C_N$ in Theorem 4.2.5 is

$$C_N = \kappa_N\left(2 - \log N - \left(2\Gamma'(1) + 2\log 2 + 2\frac{\zeta'(2)}{\zeta(2)}\right) + 2\sum_{p|N}\frac{p\log p}{p^2-1}\right).$$

This is obtained by computing the Laurent expansion for each term appears in the limit of Proposition 4.2.5 at $s = 1$. The reader can find $C_N$ by analogy with the proof of Theorem 4.3.11.

We recommend that readers compare Theorem 4.2.5 with Definition 2.2.1, these two kinds of settings are essentially compatible. Hence, one can define in parallel that, for $D, E \in \mathrm{Div}^0(X_0(N)(H_K))$, $\tau \in \mathrm{Hom}(H_K, \mathbb{C})$,

$$\langle D, E\rangle_\tau := 2\pi\sum_{P,Q} m_P n_Q \mathbf{g}_{N,\tau}(P_\tau, Q_\tau),$$

where $D = \sum_P m_P P$ and $E = \sum_Q n_Q Q$. We have

- $\langle\cdot,\cdot\rangle_\tau$ is bilinear, but **NOT** symmetric.

- $\langle P-\infty, Q-0\rangle_\tau = 2\pi \mathbf{g}_{N,\tau}(P_\tau, Q_\tau)$, hence $\langle D,E\rangle_\tau = \sum_{P,Q} m_P n_Q \langle P-\infty, Q-0\rangle_\tau$.
- $\langle D, \sum_Q n_Q Q\rangle_\tau$ is continuous outside $\mathrm{Supp}(D)$ with respect to each variable $Q$.
- $\langle \sum_P m_P P, \mathrm{div}(f)\rangle_\tau = -\sum_P m_P \log|f_\tau(P_\tau)|$, where $\mathrm{div}(f)$ is a principle divisor.

This pairing is also unique in some sense. Put all the embeddings together we get the final definition

$$\langle \cdot,\cdot\rangle_{\mathrm{inf}} := \sum_{\tau\in \mathrm{Hom}(H_K,\mathbb{C})} \langle \cdot,\cdot\rangle_\tau : \mathrm{Div}^0(X_0(N)(H_K)) \times \mathrm{Div}^0(X_0(N)(H_K)) \to \mathbb{R}.$$

All of the above is the definition of $\langle \cdot,\cdot\rangle = \langle \cdot,\cdot\rangle_{\mathrm{fin}} + \langle \cdot,\cdot\rangle_{\mathrm{inf}}$. For $D \in \mathrm{Div}^0(X_0(N)(H_K))$, one can define its **canonical Néron-Tate height** as the self-intersection $\widehat{h}(D) := \langle D,D\rangle$. It should be noted that there are some equivalent definitions for this height, for instance, using the theta divisor, or taking limits of some iterations, see [HiSi].

## 4.3 Heegner Points and Hecke Actions

We begin this section with the following well-known exercise.

**Exercise 4.3.1.** *Let $t \neq 0,1$ be a square free integer, and $D := \mathrm{disc}(\mathbb{Q}[\sqrt{t}]/\mathbb{Q})$. Show that*

- $\begin{cases} D = t, \mathscr{O}_{\mathbb{Q}[\sqrt{t}]} = \mathbb{Z}\left[\frac{1+\sqrt{t}}{2}\right] & t \equiv 1 (\mathrm{mod}\ 4) \\ D = 4t, \mathscr{O}_{\mathbb{Q}[\sqrt{t}]} = \mathbb{Z}[\sqrt{t}] & t \equiv 2,3 (\mathrm{mod}\ 4) \end{cases}.$
- $\begin{cases} (p) \text{ ramified} & & p \mid D \\ (p) \text{ unramified} \begin{cases} (2) \text{ splits} \Leftrightarrow t \equiv 1 (\mathrm{mod}\ 8) & p=2 \\ (p) \text{ splits} \Leftrightarrow t \text{ is a square in } \mathbb{F}_p & p \geq 3 \end{cases} & & p \nmid D \end{cases}.$

Let $N>0$ be an integer, and let $K$ be an imaginary quadratic field with discriminant $D_K := \mathrm{disc}(K/\mathbb{Q}) < 0$, such that $(D_K, N) = 1$. We always assume $D_K$ is odd, square free and congruent to $1 (\mathrm{mod}\ 4)$. Suppose $\mathfrak{n}$ is an ideal of $\mathscr{O}_K$ such that $\mathscr{O}_K/\mathfrak{n} \cong \mathbb{Z}/N\mathbb{Z}$.

**Proposition 4.3.2.** *Such $\mathfrak{n}$ exists if and only if $D_K$ is a square in $\mathbb{Z}/4N\mathbb{Z}$, and there are exactly $2^s$ many such $\mathfrak{n}$, where $s$ is the number of distinct prime factors in $N$.*

*Proof.* Suppose $N = \prod_i p_i^{r_i}$ and $\mathfrak{n} = \prod_j \mathfrak{p}_j^{t_j}$. By the Chinese Remainder Theorem, we have

$$\prod_j \mathscr{O}_K/\mathfrak{p}_j^{t_j} \cong \mathscr{O}_K/\mathfrak{n} \cong \mathbb{Z}/N\mathbb{Z} \cong \prod_i \mathbb{Z}/p_i^{r_i}\mathbb{Z},$$

so there must hold $(p_i)$ splits and corresponds to a unique $\mathfrak{p}_i | p_i$ in $\mathscr{O}_K$. Now, the existence of $\mathfrak{n}$ is equivalent to that $(p_i)$ are all split in $\mathscr{O}_K$, which implies $D_K$ is a square in $\mathbb{Z}/4N\mathbb{Z}$. Obviously, there are $2^s$ choices of the decomposition of $\mathfrak{n}$. □

Points in $X_0(N)(\mathbb{C})$ corresponding to isogenies whose both sides have the same complex multiplication are the distinguished points we now study.

**Definition 4.3.3** (Heegner Points)**.** Let $K$ be an imaginary quadratic field as above. Elements in $\bigsqcup_{\{\mathfrak{n} \text{ ideal}:\mathscr{O}_K/\mathfrak{n}\cong\mathbb{Z}/N\mathbb{Z}\}} \mathrm{im}(\Theta_\mathfrak{n})$ are called **Heegner points** of discriminant $D_K$, where

$$\Theta_\mathfrak{n} : \mathrm{Cl}(\mathscr{O}_K) \longrightarrow \frac{\left\{\mathbb{C}\text{-isogeny } E \xrightarrow{\phi} E' : E, E' \in \mathrm{Ell}(\mathscr{O}_K), \ker(\phi) \cong \mathbb{Z}/N\mathbb{Z}\right\}}{\mathbb{C}\text{-isomorphism}},$$

given by $\mathfrak{a} \longmapsto (\mathbb{C}/\mathfrak{a} \to \mathbb{C}/\mathfrak{n}^{-1}\mathfrak{a})$, where $\mathfrak{a}$ is a representative element in some ideal class. Indeed, $\mathrm{im}(\Theta_\mathfrak{n}) \subseteq X_0(N)(H_K)$ by Proposition 4.1.3. Hence the abelian group $(\mathbb{Z}/2\mathbb{Z})^s \times \mathrm{Gal}(H_K/K)$ acts simply-transitively on the set of Heegner points of discriminant $D_K$.

**Exercise 4.3.4.** *Let $\mathfrak{B} \in \mathrm{Cl}(\mathscr{O}_K)$, we have $\mathrm{Art}_K(\mathfrak{B}^{-1})(\Theta_\mathfrak{n}(\mathfrak{A})) = \Theta_\mathfrak{n}(\mathfrak{B}\mathfrak{A})$.*

Any Heegner point can be written as a root in $\mathscr{H} \cap K$ of some quadratic integral equation with discriminant $D_K$ explicitly. Indeed, there is an injection

$$\begin{aligned}\{\text{Heegner points of discriminant } D_K\} &\xrightarrow{\sim} \{\beta \in \mathbb{Z}/2N\mathbb{Z} : \beta^2 \equiv D_K (\mathrm{mod}\ 4N)\} \times \mathrm{Cl}(\mathscr{O}_K)\\ &\hookrightarrow \Gamma_0(N)\backslash\left\{\tau \in \mathscr{H}\cap K : a\tau^2 + b\tau + c = 0, (a,b,c) = 1, a > 0, N|a, b^2 - 4ac = D_K\right\}.\end{aligned}$$

Suppose $\mathfrak{n}$ has form $\mathbb{Z}N + \mathbb{Z}\frac{-\beta+\sqrt{D_K}}{2}$, then the maps are given by $\Theta_\mathfrak{n}(\mathfrak{a}) \longmapsto (\beta, \mathfrak{a}) \longmapsto \frac{-b+\sqrt{D_K}}{2a}$, where $a = \#(\mathscr{O}_K/\mathfrak{a}), b \equiv \beta (\mathrm{mod}\ 2N)$. Here $\mathfrak{a}$ has form $\mathbb{Z}a + \mathbb{Z}\frac{-b+\sqrt{D_K}}{2}$.

Recall that the dual isogeny of $\phi : E \to E'$ is $\phi^\vee : E' \to E$ such that $\phi^\vee \circ \phi = [\deg(\phi)]$ on $E$ and $\phi \circ \phi^\vee = [\deg(\phi)]$ on $E'$, where $[m]$ means adding an element to itself $m-1$ times.

**Definition 4.3.5** (Atkin-Lehner Involution)**.** Let $d > 1$ be a factor of $N$ such that $(d, N/d) = 1$, define

$$w_d : Y_0(N)(\mathbb{C}) \longrightarrow Y_0(N)(\mathbb{C}), \quad (\phi : E \to E') \longmapsto (E/D \to E'/D'),$$

where $D, D'$ are the unique subgroups of $\ker(\phi)$ and $\ker(\phi^\vee)$ of order $d$.

**Proposition 4.3.6.** *Under the assumptions above,*

- *$w_d$ can be written as a unique determinant $d$ matrix in the set $\Gamma_0(N)\backslash\left\{\left(\begin{smallmatrix} d\mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & d\mathbb{Z}\end{smallmatrix}\right)\right\}$. In particular, if $d = N$, then $w_N(z) = -\frac{1}{Nz}$, which maps isogeny to its dual isogeny.*
- *Suppose $(d, N/d) = (d', N/d') = 1$. Then $w_d \circ w_{d'} = w_{dd'/(d,d')^2}$. Hence, the set of Atkin-Lehner involutions form an abelian group, which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^s$, if we define $w_1 :=$ id. Moreover, $w_d$ maps Heegner points to Heegner points.*

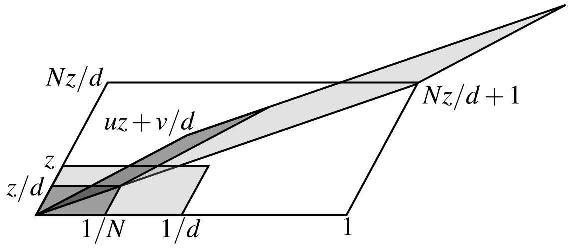
*Proof.* We only need to prove the first item, and leave the unproven part for exercises. The uniqueness is trivial: indeed,

$$\begin{pmatrix} x'wd - y'zN/d & xy' - yx' \\ N(z'w - w'z) & xw'd - yz'N/d \end{pmatrix}\begin{pmatrix} dx & y \\ Nz & dw \end{pmatrix} = \begin{pmatrix} dx' & y' \\ Nz' & dw' \end{pmatrix}.$$

Without loss of generality we may assume $E/\mathbb{C} \cong \mathbb{C}/\langle z, 1\rangle$ and $E'/\mathbb{C} \cong \mathbb{C}/\langle z, 1/N\rangle$, with $\ker(\phi) \cong \frac{\langle z,1/N\rangle}{\langle z,1\rangle}$ and $\ker(\phi^\vee) \cong \frac{\langle z/N,1/N\rangle}{\langle z,1/N\rangle}$. Suppose $D \cong \frac{\langle z,1/d\rangle}{\langle z,1\rangle}$ and $D' \cong \frac{\langle z/d,1/N\rangle}{\langle z,1/N\rangle}$. Now $w_d(\phi)$ can be written as

$$\left[\mathbb{C}/\langle z, 1/d\rangle \cong E/D \to E'/D' \cong \mathbb{C}/\langle z/d, 1/N\rangle\right] \in Y_0(N)(\mathbb{C}).$$

Since $(d, N/d) = 1$, one can replace the generators with those marked in the figure below, for some $u, v \in \mathbb{Z}$:



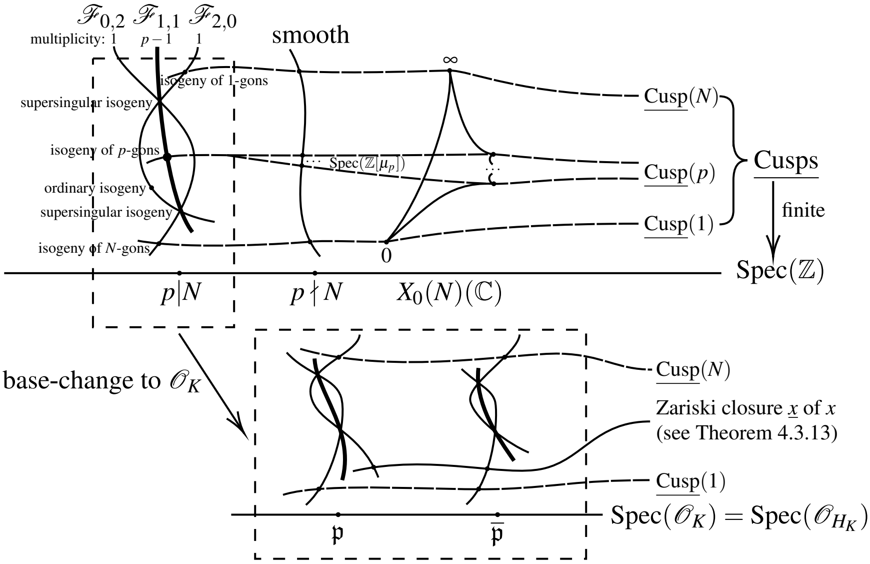

By the definition of $Y_0(N)(\mathbb{C})$, we have $w_d(\phi) = \frac{uz+v/d}{Nz/d+1} = \begin{pmatrix} du & v \\ N & d \end{pmatrix}(z)$. □

We will focus on the intersection number $\langle x - \infty, T_m \circ \sigma(x - 0)\rangle$ (see Chapter 4.2), where $x = \Theta_{\mathfrak{n}}(\mathfrak{A}) \in X_0(N)(H_K)$ is a Heegner point , $\infty$ and 0 are cusps (they are extended to horizontal divisors $\underline{x}$, $\underline{\infty}$ and $\underline{0}$ on $\mathscr{X}_0(N)_{\mathfrak{P}}$ for each $\mathfrak{P}$), $\sigma \in \mathrm{Gal}(H_K/K)$ and $T_m$ is a Hecke operator.

Here is a picture of the case $N = p^2$, suppose our $K$ has trivial class group and $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ is split in $\mathscr{O}_K$. There are two Heegner points (divisors), we draw one of them, the Zariski closure of $x = \Theta_{\mathfrak{p}^2}(\mathscr{O}_K)$, and $\Theta_{\overline{\mathfrak{p}}^2}(\mathscr{O}_K)$ is similar. The cusps

$$\mathrm{Cusps} = \bigsqcup_{d|N, d>0} \mathrm{Cusp}(d) = \mathrm{Cusp}(1) \sqcup \mathrm{Cusp}(p) \sqcup \mathrm{Cusp}(N)$$

are extended to isogenies of Néron polygons, where $\underline{0} = \underline{\mathrm{Cusp}}(1)$ and $\underline{\infty} = \underline{\mathrm{Cusp}}(N)$ are degree one sections, but $\underline{\mathrm{Cusp}}(p)$ is irreducible of degree $p-1$ and isomorphic to $\mathrm{Spec}(\mathbb{Z}[\mu_p])$.




We must emphasize that some points correspond to supersingular isogenies may still occur when $p \nmid N$.

Let us introduce the Hecke operators appear in the intersection.

**Definition 4.3.7** (Hecke Operators)**.** Let $m,N \geq 1$ be integers, the **Hecke operator** $T_m$ on $X_0(N)/\mathbb{C}$ is defined linearly by

$$T_m : \mathrm{Div}(X_0(N)) \longrightarrow \mathrm{Div}(X_0(N)), \quad (\phi : E \to E') \longmapsto \sum_C (E/C \to E'/\phi(C)),$$

where the sum is taken over all subgroups $C$ of order $m$ in $E$ with $C \cap \ker(\phi) = \{0\}$. The commutative $\mathbb{Z}$-algebra $\mathfrak{T} := \mathbb{Z}[T_1, T_2, \cdots]$ is called the **Hecke algebra** of $\Gamma_0(N)$.

Through the language of lattices, for a homomorphism $\phi : \Lambda \to \Lambda'$, we can express this operator as

$$T_m(\Lambda \to \Lambda') = \sum_{\Lambda_0 \supseteq \Lambda, [\Lambda_0:\Lambda]=m, \Lambda_0 \cap \Lambda' = \Lambda} (\Lambda_0 \to \Lambda_0 + \Lambda').$$

Thus the operator $T_m$ can act on modular forms, since modular forms are lattice functions.

**Exercise 4.3.8.** *$\Lambda_0 \nsubseteq \Lambda'$ if $m > 1$ (note all isogenies have the same kernel $\mathbb{Z}/N\mathbb{Z}$).*

If $(m,N) = 1$, then $C$ has no non-trivial intersection with $\ker(\phi)$, like Proposition 4.3.6 our $T_m$ can be written as

$$T_m(z) = \sum_{\gamma \in \Gamma_0(N) \backslash \left\{ \left( \begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} \end{smallmatrix} \right) \right\}, \det(\gamma)=m} \gamma z \quad \in \mathrm{Div}(X_0(N)),$$

and in particular, $T_m$ acts on a cusp $\mathfrak{c}$ by $T_m(\mathfrak{c}) = \sigma_1(m)\mathfrak{c}$, where $\sigma_r(m) := \sum_{d|m, d>0} d^r$. Indeed, $\sigma_1(m) = \#\left\{ \left( \begin{smallmatrix} m/d & b \\ 0 & d \end{smallmatrix} \right) : d|m, 0 \leq b < d \right\}$ is the number of summation indices.

**Proposition 4.3.9.** *Suppose $m \geq 1, N > 1$ and $(m,N) = 1$. Let $x \in X_0(N)(H_K)$ be a Heegner point and $\sigma \in \mathrm{Gal}(H_K/K)$. Then $x - \infty$ and $T_m \circ \sigma(x - 0)$ have no common support if and only if $r_\sigma(m) = 0$, where*

$$r_\sigma(m) := \#\left\{ \mathfrak{s} \in \mathrm{Art}_K^{-1}(\sigma) : \mathfrak{s} \subseteq \mathscr{O}_K, \#(\mathscr{O}_K/\mathfrak{s}) = m \right\}.$$

*Proof.* By linearity, one needs to compute the intersections:

$$x \text{ and } T_m(\sigma(x)); \quad x \text{ and } 0; \quad \infty \text{ and } T_m(\sigma(x)); \quad \infty \text{ and } 0.$$

Obviously, ordinary points do not intersect with cusp points. The same is true for $\infty$ and 0 by the discussion above ($\infty \neq 0$ since $N > 1$). So we only need to study the first intersection. Without loss of generality, let us assume $x : \mathbb{C}/\mathfrak{a} \to \mathbb{C}/\mathfrak{n}^{-1}\mathfrak{a}$, then $\sigma(x) : \mathbb{C}/\mathfrak{s}^{-1}\mathfrak{a} \to \mathbb{C}/\mathfrak{s}^{-1}\mathfrak{n}^{-1}\mathfrak{a}$ for some $\mathfrak{s} \in \mathfrak{S} := \mathrm{Art}_K^{-1}(\sigma)$ by Exercise 4.3.4. Thus

$$T_m(\sigma(x)) = \sum_{\Lambda_0 \supseteq \mathfrak{s}^{-1}\mathfrak{a}, [\Lambda_0 : \mathfrak{s}^{-1}\mathfrak{a}]=m} \left( \mathbb{C}/\Lambda_0 \to \mathbb{C}/(\Lambda_0 + \mathfrak{s}^{-1}\mathfrak{n}^{-1}\mathfrak{a}) \right).$$

Hence, the multiplicity of $x$ in $T_m(\sigma(x))$ is the number of diagrams

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{a} \arrow[r, "x"] \arrow[d, "\sim"'] & \mathfrak{n}^{-1}\mathfrak{a} \arrow[r] \arrow[d, dashed, "\sim"'] & \mathfrak{n}^{-1}\mathfrak{a}/\mathfrak{a} \arrow[r] \arrow[d, "\sim"'] & 0 \\
0 \arrow[r] & \Lambda_0 \arrow[r] & \Lambda_0 + \mathfrak{s}^{-1}\mathfrak{n}^{-1}\mathfrak{a} \arrow[r, "(m,N)=1"] & \mathfrak{s}^{-1}\mathfrak{n}^{-1}\mathfrak{a}/\mathfrak{s}^{-1}\mathfrak{a} \arrow[r] & 0
\end{tikzcd}$$

with property $[\mathscr{O}_K : \mathfrak{s}^{-1}\mathfrak{a}\Lambda_0^{-1}] = m$. Since $\mathfrak{a}\Lambda_0^{-1}$ is principal, one concludes that $\mathfrak{s}^{-1}\mathfrak{a}\Lambda_0^{-1} \in \mathfrak{S}^{-1} = \overline{\mathfrak{S}}$ and there are $r_\sigma(m)$ many such $\Lambda_0$. □

Suppose $m \geq 1, N > 1$ and $(m,N) = 1$. Let $x = \Theta_\mathfrak{n}(\mathfrak{A}) \in X_0(N)(H_K)$ be a Heegner point. We are going to compute the intersection number

$$\langle x-\infty, T_m \circ \sigma(x-0)\rangle = \langle x-\infty, T_m \circ \sigma(x-0)\rangle_{\text{fin}} + \langle x-\infty, T_m \circ \sigma(x-0)\rangle_{\text{inf}}.$$

These two parts will be given in Corollary 4.3.12 and Corollary 4.3.18 respectively.

For the infinite part, according to Proposition 4.3.9, we may first assume that $r_\sigma(m) = 0$. By Theorem 4.2.5, we have

$$\langle x-\infty, T_m \circ \sigma(x-0)\rangle_{\text{inf}} = \sum_{\text{complex places } \{\tau,\overline{\tau}\} \text{ of } H_K} 4\pi \mathbf{g}_{N,\tau}(x_\tau, T_m \circ \sigma(x_\tau)).$$

We start with the most important case where $\tau = \mathrm{id}$. The operator $T_m$ multiplies the constant term by $\sigma_1(m)$ and $E_{N,\infty}(z,s)$ by $m^s\sigma_{1-2s}(m)$. For the latter case, since

$$\begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix} \Big\backslash \left\{ \det \begin{pmatrix} * & * \\ N* & * \end{pmatrix} = m \right\} \overset{\sim}{\longleftrightarrow} \bigcup_{l|m, l>0} \left\{ \begin{pmatrix} \frac{m}{l}v - k\frac{c}{l} & -\frac{m}{l}u - k\frac{d}{l} \\ c & d \end{pmatrix} : \begin{array}{l} c,d \geq 0, N|c, \\ (c,d) = l, \\ 0 \leq k < l. \end{array} \right\},$$

where $u,v$ are integers such that $cu + dv = l$, we have

$$E_{N,\infty}(T_m(z),s) = \sum_\gamma \mathrm{Im}(\gamma z)^s = \sum_{l|m,l>0} \left( \frac{l}{2} \sum_{N|c,(c,d)=l} \frac{\mathrm{Im}(z)^s}{|cz+d|^{2s}} \right) = \sum_{l|m,l>0} l^{1-2s} E_{N,\infty}(z,s),$$

where $\gamma$ runs through the set mentioned above. Therefore, $\mathbf{g}_N(x, T_m \circ \sigma(x))$ is equal to

$$\lim_{s\to 1} \left( g_{N,s,m}(x,\sigma(x)) - \sigma_1(m) E_{N,\infty}(w_N x, s) - m^s \sigma_{1-2s}(m) E_{N,\infty}(\sigma(x),s) + \frac{\sigma_1(m)\kappa_N}{s-1} \right) + \sigma_1(m) C_N,$$

where

$$g_{N,s,m}(z,w) = \sum_{\gamma = \left(\begin{smallmatrix} * & * \\ N* & * \end{smallmatrix}\right), \det(\gamma) = m} G_s(u(z,\gamma w)).$$

Write $\mathfrak{S} = \mathrm{Art}_K^{-1}(\sigma)$. For general $\tau$, note the complex places of $H_K$ are permuted simply-transitively by $\mathrm{Gal}(H_K/K) \cong \mathrm{Cl}(\mathscr{O}_K)$, so

$$\langle x-\infty, T_m \circ \sigma(x-0)\rangle_{\text{inf}} = \sum_{\mathfrak{B} = \mathfrak{S}^{-1}\mathfrak{A}} 2\langle \Theta_\mathfrak{n}(\mathfrak{A}) - \infty, T_m(\Theta_\mathfrak{n}(\mathfrak{B}) - 0)\rangle_{\text{id}},$$

and the intersection pairs in the summation have already been discussed. Hence,

$$\begin{aligned} \langle x-\infty, T_m \circ \sigma(x-0)\rangle_{\text{inf}} = 4\pi \lim_{s\to 1} \Bigg( & H_{N,s,m}(\mathfrak{S}) - \sigma_1(m) \sum_{\mathfrak{A}} E_{N,\infty}(w_N \Theta_\mathfrak{n}(\mathfrak{A}), s) \\ & - m^s \sigma_{1-2s}(m) \sum_{\mathfrak{A}} E_{N,\infty}(\Theta_\mathfrak{n}(\mathfrak{S}^{-1}\mathfrak{A}), s) + \frac{\#\mathrm{Cl}(\mathscr{O}_K)\sigma_1(m)\kappa_N}{s-1} \Bigg) \\ & + 4\pi \cdot \#\mathrm{Cl}(\mathscr{O}_K) \cdot \sigma_1(m) \cdot C_N, \end{aligned}$$

where $H_{N,s,m}(\mathfrak{S}) = \sum_{\mathfrak{B}=\mathfrak{S}^{-1}\mathfrak{A}} g_{N,s,m}(\Theta_{\mathfrak{n}}(\mathfrak{A}), \Theta_{\mathfrak{n}}(\mathfrak{B}))$.

By Proposition 4.2.4, in the Möbius inversion formula

$$\sum_{\mathfrak{A}} E_{N,\infty}(\Theta_{\mathfrak{n}}(\mathfrak{A}), s) = N^{-s} \prod_{p|N} (1-p^{-2s})^{-1} \cdot \sum_{d|N} \left( \frac{\mu(d)}{d^s} \sum_{\mathfrak{A}} E_{1,\infty}\left( \frac{N}{d} \Theta_{\mathfrak{n}}(\mathfrak{A}), s \right) \right),$$

the point $\frac{N}{d}\Theta_{\mathfrak{n}}(\mathfrak{A})$ is actually the Heegner point $\Theta_{\mathscr{O}_K}(\mathfrak{A}')$ (not necessarily $\mathfrak{A}$, but $\mathfrak{A} \mapsto \mathfrak{A}'$ is one-to-one) of discriminant $D_K$ in $X_0(1)(H_K)$. So if $d|N$ is fixed, one can replace $E_{1,\infty}\big(\frac{N}{d}\Theta_{\mathfrak{n}}(\mathfrak{A}), s\big)$ with $E_{1,\infty}(\Theta_{\mathscr{O}_K}(\mathfrak{A}), s)$. To compute this, we need:

**Remark 4.3.10.** We still assume $D_K \equiv 1 (\mathrm{mod}\ 4)$. Let $\mathfrak{A} \in \mathrm{Cl}(\mathscr{O}_K)$ be an ideal class with a representative element in form $\mathfrak{a}_0 = \mathbb{Z}a + \mathbb{Z}\frac{-b+\sqrt{D_K}}{2}$, where $a = \#(\mathscr{O}_K/\mathfrak{a}_0)$, $b$ is odd and $D_K = b^2 - 4ac$. Then there is a one-to-one correspondence

$$\big\{\mathfrak{a} \in \mathfrak{A} : \mathfrak{a} \subseteq \mathscr{O}_K, \#(\mathscr{O}_K/\mathfrak{a}) = m\big\} \xrightarrow{\sim} \big\{(x,y) \in \mathbb{Z}^2 : ax^2 + bxy + cy^2 = m\big\} \big/ \mathscr{O}_K^\times,$$

given by $\alpha\mathfrak{a}_0 \longmapsto (r,s)$, where $\alpha \in K^\times$ such that $\alpha\mathfrak{a}_0 \subseteq \mathscr{O}_K = \mathbb{Z}\big[\frac{b+\sqrt{D_K}}{2}\big]$ and a direct calculation shows that $\alpha = r + \frac{b+\sqrt{D_K}}{2a}s$ for $r,s \in \mathbb{Z}$. Using this correspondence, one can express the **partial zeta function** of $\mathfrak{A}$ as a **Epstein zeta function**:

$$\zeta_{\mathfrak{A}}(s) := \sum_{\mathfrak{a}\in\mathfrak{A}, \mathfrak{a}\subseteq\mathscr{O}_K} \frac{1}{\#(\mathscr{O}_K/\mathfrak{a})^s} = \sum_{n=1}^{\infty} \left( \sum_{\mathfrak{a}\in\mathfrak{A}, \mathfrak{a}\subseteq\mathscr{O}_K, \#(\mathscr{O}_K/\mathfrak{a})=n} \frac{1}{n^s} \right) = \sum_{(x,y)\in\mathbb{Z}^2} \frac{\#(\mathscr{O}_K^\times)^{-1}}{a^s \left| x - \frac{-b+\sqrt{D_K}}{2a} y \right|^{2s}}.$$

Thus, recall that $\Theta_{\mathscr{O}_K}(\mathfrak{A})$ corresponds to $\frac{-b+\sqrt{D_K}}{2a} \in \mathscr{H} \cap K$, we have

$$\begin{aligned} E_{1,\infty}(\Theta_{\mathscr{O}_K}(\mathfrak{A}), s) =& \frac{1}{2} \sum_{(c,d)=1} \frac{|D_K|^{s/2}(2a)^{-s}}{\left| c\frac{-b+\sqrt{D_K}}{2a} + d \right|^{2s}} \\ =& 2^{-s-1} a^{-s} |D_K|^{s/2} \zeta(2s)^{-1} \sum_{(c,d)\in\mathbb{Z}^2} \frac{1}{\left| c\frac{-b+\sqrt{D_K}}{2a} + d \right|^{2s}} \\ =& 2^{-s-1} \cdot \#(\mathscr{O}_K^\times) \cdot |D_K|^{s/2} \cdot \zeta(2s)^{-1} \cdot \zeta_{\mathfrak{A}}(s). \end{aligned}$$

Summarizing these calculations together, we obtain

**Theorem 4.3.11.** *Suppose $m \geq 1, N > 1$ and $(m,N) = 1$. Let $x = \Theta_{\mathfrak{n}}(\mathfrak{A}) \in X_0(N)(H_K)$ be a Heegner point, let $\sigma \in \mathrm{Gal}(H_K/K)$ and $\mathfrak{S} = \mathrm{Art}_K^{-1}(\sigma)$. Assume $r_\sigma(m) = 0$, then $\langle x - \infty, T_m \circ \sigma(x-0)\rangle_{\mathrm{inf}}$ is equal to*

$$4\pi \lim_{s\to 1} \left( H_{N,s,m}(\mathfrak{S}) - \frac{\#\mathrm{Cl}(\mathscr{O}_K)\sigma_1(m)\kappa_N}{s-1} \right) + 4\pi \cdot \#\mathrm{Cl}(\mathscr{O}_K)\kappa_N \left( \sigma_1(m) \left( 2 + \log\frac{N}{|D_K|} \right.\right.$$
$$\left.\left. + 2\sum_{p|N} \frac{\log p}{p^2-1} - 2\frac{L'(1,\varepsilon_K)}{L(1,\varepsilon_K)} + 2\frac{\zeta'(2)}{\zeta(2)} \right) + \sum_{d|m} d\log\frac{m}{d^2} \right),$$

*where $\zeta_K(s) = \zeta(s)L(s,\varepsilon_K) = \sum_{\mathfrak{A}} \zeta_{\mathfrak{A}}(s)$ is the Dedekind zeta function of $K$, $\varepsilon_K := \big(\frac{D_K}{\cdot}\big)$ is the Kronecker symbol, $H_{N,s,m}(\mathfrak{S}) = \#(\mathscr{O}_K^\times)^2 \cdot \sum_{n\geq 1} \delta(n) R_{\mathfrak{S}\mathfrak{n}}(n) r_\sigma(nN + m|D_K|) G_s\big(\frac{nN}{m|D_K|}\big)$, $\delta(n) := \prod_{p|(n,D_K)} 2$, and $R_{\mathfrak{A}}(n) := \#\big\{\mathfrak{a} \in [\mathfrak{A}] : [\mathfrak{A}] \in \mathrm{Cl}(\mathscr{O}_K)/2\mathrm{Cl}(\mathscr{O}_K), \#(\mathscr{O}_K/\mathfrak{a}) = n\big\}$.*

*Proof.* (proof sketch). Note that $w_N\Theta_{\mathfrak{n}}(\mathfrak{A}) = \Theta_{\overline{\mathfrak{n}}}(\mathfrak{A})$ and $\sum_{d|N}\mu(d)d^{-s} = \prod_{p|N}(1-p^{-s})$, so we need to compute

$$\lim_{s\to 1}\left(H_{N,s,m}(\mathfrak{S}) - \big(\sigma_1(m)+m^s\sigma_{1-2s}(m)\big)\frac{\#(\mathscr{O}_K^\times)|D_K|^{s/2}\zeta_K(s)}{2^{s+1}N^s\zeta(2s)\prod_{p|N}(1+p^{-s})} + \frac{\#\mathrm{Cl}(\mathscr{O}_K)\sigma_1(m)\kappa_N}{s-1}\right).$$

We now perform the Laurent expansion for each term at $s=1$.

- By definition, $\sigma_1(m)+m^s\sigma_{1-2s}(m) = 2\sigma_1(m)+\sum_{d|m} d\log\frac{d^2}{m}(s-1)+O(s-1)^2$.
- $\zeta(2s) = \zeta(2)+2\zeta'(2)(s-1)+O(s-1)^2$.
- $\zeta_K(s) = \left(\frac{1}{s-1}-\Gamma'(1)+O(s-1)\right)\left(L(1,\varepsilon_K)+L'(1,\varepsilon_K)(s-1)+O(s-1)^2\right) = \frac{L(1,\varepsilon_K)}{s-1} + \left(L'(1,\varepsilon_K)-\Gamma'(1)L(1,\varepsilon_K)\right)+O(s-1)$. Where, $L(1,\varepsilon_K) = \frac{2\pi\cdot\#\mathrm{Cl}(\mathscr{O}_K)}{\#(\mathscr{O}_K^\times)\sqrt{|D_K|}}$, by the class number formula.
- Since $\prod_{p|N}(1+p^{-1}) = \frac{3}{N\pi\kappa_N}$ (Proposition 4.2.2), we have $\prod_{p|N}(1+p^{-s}) = \frac{3}{N\pi\kappa_N} - \frac{3}{N\pi\kappa_N}\sum_{p|N}\frac{\log p}{p+1}(s-1)+O(s-1)^2$.

Hence, after a complicated but tedious calculation, the expansion of the middle term in the above limit is

$$-\frac{2\#\mathrm{Cl}(\mathscr{O}_K)\sigma_1(m)\kappa_N}{s-1} - \#\mathrm{Cl}(\mathscr{O}_K)\kappa_N\left(\sigma_1(m)\left(\log|D_K|+2\frac{L'(1,\varepsilon_K)}{L(1,\varepsilon_K)}-2\Gamma'(1)\right.\right.$$
$$\left.\left.-2\log 2N-4\frac{\zeta'(2)}{\zeta(2)}+2\sum_{p|N}\frac{\log p}{p+1}\right)+\sum_{d|m}d\log\frac{d^2}{m}\right)+O(s-1).$$

Here, the constant term combines with $\#\mathrm{Cl}(\mathscr{O}_K)\sigma_1(m)C_N$ to form

$$\#\mathrm{Cl}(\mathscr{O}_K)\kappa_N\left(\sigma_1(m)\left(2+\log\frac{N}{|D_K|}+2\sum_{p|N}\frac{\log p}{p^2-1}-2\frac{L'(1,\varepsilon_K)}{L(1,\varepsilon_K)}+2\frac{\zeta'(2)}{\zeta(2)}\right)+\sum_{d|m}d\log\frac{m}{d^2}\right).$$

Now compute the remaining $H_{N,s,m}(\mathfrak{S}) = \sum_{\mathfrak{B}=\mathfrak{S}^{-1}\mathfrak{A}}\ \sum_{\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right),N|c,\det(\gamma)=m} G_s\big(u(\Theta_{\mathfrak{n}}(\mathfrak{A}),\gamma\Theta_{\mathfrak{n}}(\mathfrak{B}))\big)$.

Suppose $\Theta_{\mathfrak{n}}(\mathfrak{A})$ and $\Theta_{\mathfrak{n}}(\mathfrak{B})$ correspond to $z=\frac{-\beta+\sqrt{D_K}}{2\alpha}$ and $w=\frac{-\beta+\sqrt{D_K}}{2\alpha'}$ in $\mathscr{H}\cap K$, respectively. A direct calculation shows that $u(z,\gamma w) = \frac{|czw+dz-aw-b|^2}{4m\mathrm{Im}(z)\mathrm{Im}(w)} = \frac{nN}{m|D_K|}$, where $n=\frac{\alpha\alpha'}{N}|czw+dz-aw-b|^2$. So $g_{N,s,m}(z,w) = \sum_{n\geq 1}\rho_m(n)G_s\left(\frac{nN}{m|D_K|}\right)$, where $\rho_m(n)$ is the cardinal number of

$$\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix} : N|c, ad-bc=m, \frac{\alpha\alpha'}{N}|czw+dz-aw-b|^2=n\right\}.$$

In fact, one can show there is a one-to-one correspondence from the above set to

$$\left\{(u,v)\in\mathfrak{a}^{-1}\overline{\mathfrak{b}}^{-1}\times\overline{\mathfrak{a}}^{-1}\overline{\mathfrak{b}}^{-1}\overline{\mathfrak{n}} : |u|^2=\frac{nN+m|D_K|}{\alpha\alpha'}, |v|^2=\frac{nN}{\alpha\alpha'}, \alpha\alpha'(u-v)\in(\sqrt{D_K})\right\},$$

where $\mathfrak{a} = \mathbb{Z}\alpha + \mathbb{Z}\frac{-\beta+\sqrt{D_K}}{2}$, $\mathfrak{b} = \mathbb{Z}\alpha' + \mathbb{Z}\frac{-\beta+\sqrt{D_K}}{2}$, $\mathfrak{n} = \mathbb{Z}N + \mathbb{Z}\frac{-\beta+\sqrt{D_K}}{2}$, given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (c\bar{z}w + d\bar{z} - aw - b, czw + dz - aw - b)$. Now, what we need is the sum $\sum_{\mathfrak{B}=\mathfrak{S}^{-1}\mathfrak{A}} \rho_m(n)$, which is equal to

$$\sum_{\mathfrak{C}} \left( \sum_{\mathfrak{A}^{-1}\mathfrak{B}=\overline{\mathfrak{S}}, \mathfrak{A}\mathfrak{B}\overline{\mathfrak{N}}=\mathfrak{C}} \rho_m(n) \right) = \sum_{\mathfrak{C}} \#(\mathscr{O}_K^\times)^2 \cdot \delta(n) r_\sigma(nN + m|D_K|) \cdot \#\{\mathfrak{c} \in \mathfrak{C} : \#(\mathscr{O}_K/\mathfrak{c}) = n\},$$

because $\mathfrak{a}^{-1}\overline{\mathfrak{b}}^{-1} \in \mathfrak{A}^{-1}\mathfrak{B} = \overline{\mathfrak{S}}, \overline{\mathfrak{a}}^{-1}\overline{\mathfrak{b}}^{-1}\overline{\mathfrak{n}} \in \mathfrak{A}\mathfrak{B}\overline{\mathfrak{N}}$. By genus theory (see [Cox] for details), the term in the summation is non-zero only when $[\mathfrak{C}\mathfrak{N}] = [\mathfrak{S}^{-1}] \in \mathrm{Cl}(\mathscr{O}_K)/2\mathrm{Cl}(\mathscr{O}_K)$, i.e. $[\mathfrak{S}\mathfrak{n}] = [\overline{\mathfrak{C}}]$. So the conclusion holds. $\square$

Remove the condition $r_\sigma(m) = 0$, we get the final formula for the infinite part. The proof is omitted here.

**Corollary 4.3.12.** *Under the conditions of Theorem 4.3.11, assume $r_\sigma(m) \neq 0$. Then $H_{N,s,m}(\mathfrak{S})$ is equal to the one in Theorem 4.3.11 plus a correction term, which is*

$$+\#\mathrm{Cl}(\mathscr{O}_K) \cdot \#(\mathscr{O}_K^\times) \cdot r_\sigma(m) \left( \frac{\Gamma'(s)}{\Gamma(s)} - \log 2\pi + \frac{L'(1,\varepsilon_K)}{L(1,\varepsilon_K)} + \frac{1}{2}\log|D_K| \right).$$

Now we compute the finite part $\langle x - \infty, T_m \circ \sigma(x-0) \rangle_{\mathrm{fin}}$. Recall that

$$\langle \cdot, \cdot \rangle_{\mathrm{fin}} = \sum_{\mathfrak{P} \in \mathrm{Spec}(\mathscr{O}_{H_K})} \langle \cdot, \cdot \rangle_{\mathfrak{P}},$$

where $\langle \cdot, \cdot \rangle_{\mathfrak{P}} = i_{\mathfrak{P}}\big((\cdot)_{\mathfrak{P}}, (\cdot)_{\mathfrak{P}}\big) \log \#k(\mathfrak{P})$ if one of these two points has zero intersection with every component of the special fiber of $\mathscr{X}_0(N)_{\mathfrak{P}}$. For this reason (and in order to use the theory of quaternion algebras over local fields), it is useful to consider the local field $H_{K,\mathfrak{P}}$ over $\mathbb{Q}_p$ with integers $\mathfrak{O}_{\mathfrak{P}} := \varprojlim_{n\geq 1} \mathscr{O}_{H_K}/\mathfrak{P}^n$. Note that replacing $\mathscr{X}_0(N)_{\mathfrak{P}}$ with $\widehat{\mathscr{X}_0(N)}_{\mathfrak{P}} := \mathscr{X}_0(N) \times_{\mathscr{O}_{H_K}} \mathfrak{O}_{\mathfrak{P}}$ does not change the intersection number, so the points we study can be extended to horizontal divisors on it.

Obviously, the sections $\underline{x}$ and $\sigma(\underline{x})$ do not intersect the divisor $\underline{\mathrm{Cusps}}$ on the special fiber of $\widehat{\mathscr{X}_0(N)}_{\mathfrak{P}}$. The theorem of Deuring states that the reduction of $\underline{x}$ in $\widehat{\mathscr{X}_0(N)}_{\mathfrak{P}}$ is ordinary for those $\mathfrak{P}|p$ where $(p)$ split in $\mathscr{O}_K$; and is supersingular for those $\mathfrak{P}|p$ where $(p)$ inert or ramified in $\mathscr{O}_K$.

**Theorem 4.3.13.** *Let $p|N$ and suppose $N = p^k M$, let $x = \Theta_{\mathfrak{n}}(\mathfrak{A})$ be a Heegner point. Then the sections $\underline{x}$ and $\sigma(\underline{x})$ are both reduced to ordinary points in the component*

- *$\mathscr{F}_{0,k}$, if $\mathfrak{P}|\overline{\mathfrak{n}}$.*
- *$\mathscr{F}_{k,0}$, if $\mathfrak{P}|\mathfrak{n}$.*

*Hence, one of the divisors $\underline{x} - \underline{\infty}$ and $\sigma(\underline{x}) - \underline{0}$ intersects every component in the special fiber trivially. Moreover, if $(m,N) = 1$ and $r_\sigma(m) = 0$ (this ensures that the divisors have no common support by Proposition 4.3.9) we have the formula*

$$\langle x - \infty, T_m \circ \sigma(x-0) \rangle_{\mathfrak{P}} = i_{\mathfrak{P}}(\underline{x}, T_m(\sigma(\underline{x}))) \log \#k(\mathfrak{P}).$$

*The right hand side is an intersection in $\widehat{\mathscr{X}_0(N)}_{\mathfrak{P}}$.*

*Proof.* If $\mathfrak{P}|\overline{\mathfrak{n}}$, the group scheme $\ker(\underline{x})$ (or $\ker(\sigma(\underline{x}))$) is étale over $\mathfrak{O}_{\mathfrak{P}}$ (i.e. the kernel of Verschiebung), so it is isomorphic to $\mathbb{Z}/N\mathbb{Z}$ over the residue field. Hence the reduction lies in $\mathscr{F}_{0,k}$ (recall $\mu_{p^a}$ is not étale) by Theorem 4.1.6, the other assertion is similar. Note that $\underline{0}$ also reduces to a point in $\mathscr{F}_{0,k}$, so $\sigma(\underline{x})-\underline{0}$ has zero intersection with each fibral component. Therefore,

$$\langle x-\infty, T_m\circ\sigma(x-0)\rangle_{\mathfrak{P}} = i_{\mathfrak{P}}(\underline{x}-\underline{\infty}, T_m(\sigma(\underline{x})-\underline{0}))\log\#k(\mathfrak{P}).$$

When $(m,N)=1$ and $r_\sigma(m)=0$, one can compute

$$\begin{aligned}
&i_{\mathfrak{P}}(\underline{x}-\underline{\infty}, T_m(\sigma(\underline{x})-\underline{0}))\\
=&i_{\mathfrak{P}}(\underline{x}, T_m(\sigma(\underline{x})))-i_{\mathfrak{P}}(\underline{x}, T_m(\underline{0}))-i_{\mathfrak{P}}(\underline{\infty}, T_m(\sigma(\underline{x})))+i_{\mathfrak{P}}(\underline{\infty}, T_m(\underline{0}))\\
=&i_{\mathfrak{P}}(\underline{x}, T_m(\sigma(\underline{x})))-0-0+0,
\end{aligned}$$

this proves the last conclusion. □

For $p\nmid N$, obviously the divisors $\underline{x}-\underline{\infty}$ and $\sigma(\underline{x})-\underline{0}$ are both have zero intersection with the irreducible smooth special fiber. Hence the formula in Theorem 4.3.13 still holds.

It remains to compute $i_{\mathfrak{P}}(\underline{x}, T_m(\sigma(\underline{x})))$. In order to study the reduction on the special fiber, we hope that the residue field is an algebraically closed field. To this end, let $W_{\mathfrak{P}}$ denote the completion of the maximal unramified extension of $\mathfrak{O}_{\mathfrak{P}}$, whose residue field is an algebraic closure of the residue field of $\mathfrak{O}_{\mathfrak{P}}$.

**Theorem 4.3.14.** *Assume $(m,N)=1$ and $r_\sigma(m)=0$. Then*

$$i_{\mathfrak{P}}(\underline{x}, T_m(\sigma(\underline{x}))) = \frac{1}{2}\sum_{n\geq 1}\#\Big(\mathrm{Hom}_{W_{\mathfrak{P}}/\varpi^n}(\sigma(\underline{x}),\underline{x})_{\deg=m}\Big),$$

*where $\varpi$ is a uniformizer of $\mathfrak{O}_{\mathfrak{P}}$ or $W_{\mathfrak{P}}$.*

*Proof.* The proof needs Serre and Tate's deformation theory, which is beyond the framework of this book. See [DaZh]. □

Let us compute $\mathrm{Hom}_{W_{\mathfrak{P}}/\varpi^n}(\sigma(\underline{x}),\underline{x})$ now. There are two situations: $(p)$ splits in $\mathscr{O}_K$, or does not split.

Note that if $n=0$, one can show there are isomorphisms

$$\mathrm{End}_{W_{\mathfrak{P}}}(\underline{x}) = \mathrm{End}_{W_{\mathfrak{P}}}(\sigma(\underline{x})) = \mathscr{O}_K;$$

and for any (integral) ideal $\mathfrak{s}\in\mathrm{Art}_K^{-1}(\sigma)$,

$$\mathrm{Hom}_{W_{\mathfrak{P}}}(\sigma(\underline{x}),\underline{x}) \xrightarrow{\sim} \mathfrak{s} \quad \text{(as left } \mathscr{O}_K\text{-modules)},$$

which maps a diagram (two vertical arrows are $W_{\mathfrak{P}}$-isogenies)

$$\begin{tikzcd}
\mathfrak{s}*E \arrow[r, "\sigma(\underline{x})"] \arrow[d] & \mathfrak{s}*E' \arrow[d] \\
E \arrow[r, "\underline{x}"] & E'
\end{tikzcd}$$

to an inclusion $\mathfrak{s}^{-1} \hookrightarrow \mathscr{O}_K$ naturally (this inclusion corresponds to an element in $\mathfrak{s}$, denote as $\phi \mapsto \phi^\flat \in \mathfrak{s}$). The degree of such a diagram $\phi$ (here means the degree of two vertical arrows) is equal to $\frac{|\phi^\flat|^2}{\#(\mathscr{O}_K/\mathfrak{s})}$.

This is enough to get to the first case:

**Proposition 4.3.15.** *If $(p)$ splits in $\mathscr{O}_K$ and $r_\sigma(m) = 0$, then $i_{\mathfrak{P}}(\underline{x}, T_m(\sigma(\underline{x}))) = 0$.*

*Proof.* When $(p)$ splits, it holds $\mathrm{Hom}_{W_{\mathfrak{P}}}(\sigma(\underline{x}), \underline{x}) \cong \mathrm{Hom}_{W_{\mathfrak{P}}/\varpi^n}(\sigma(\underline{x}), \underline{x})$ for all $n \geq 1$ by Deuring's theory [Deur]. Since $r_\sigma(m) = 0$, this group contains no degree $m$ elements. □

However, when $(p)$ does not split, we need to study the remaining $n > 0$ case, which leads to the following analogy. The reductions are supersingular now (by Deuring's theorem), so the endomorphism rings are orders in quaternion division algebras.

**Lemma 4.3.16.** *Suppose $(p) = \mathfrak{p}^e$ ($e = 1$ or $2$) is not split in $\mathscr{O}_K$ (of course, $p \nmid N$). The ring $R := \mathrm{End}_{W_{\mathfrak{P}}/\varpi}(\underline{x})$ is an (Eichler) order in a quaternion division algebra $B$ over $\mathbb{Q}$ with reduced discriminant $Np$, which is ramified at exactly $p$ and $\infty$. More specically,*

- *$B \otimes_{\mathbb{Q}} \mathbb{R}$, $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ are division algebras, $R \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a maximal order in $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$.*
- *For primes $\ell \neq p$, we have $B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong \mathrm{Mat}_2(\mathbb{Q}_\ell)$ and $R \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ is conjugate to the order*
$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}_\ell) : c \equiv 0 (\mathrm{mod}\ N) \right\} \subseteq \mathrm{Mat}_2(\mathbb{Q}_\ell).$$

*The embedding $\mathscr{O}_K = \mathrm{End}_{W_{\mathfrak{P}}}(\underline{x}) \hookrightarrow \mathrm{End}_{W_{\mathfrak{P}}/\varpi}(\underline{x}) = R$ extends to a $\mathbb{Q}$-linear map $K \hookrightarrow B$.*

*By Skolem-Noether theorem, there exists $j \in B$ such that $jaj^{-1} = \overline{a}$ (the conjugation in $\mathrm{Gal}(K/\mathbb{Q})$) for all $a \in K$, so we get a decomposition $B = K \oplus Kj, b = b^+ + b^-$. Recall the reduced norm on $B$ is defined by $\mathrm{Nrd}(b) := b\overline{b}$ (the conjugation on $B$), then one has*

$$\mathrm{Nrd}(b) = \mathrm{Nrd}(b^+) + \mathrm{Nrd}(b^-).$$

*With these notations, for $n \geq 1$,*

- $\mathrm{End}_{W_{\mathfrak{P}}/\varpi^n}(\underline{x}) = \left\{ b \in R : D_K \cdot \mathrm{Nrd}(b^-) \equiv 0 (\mathrm{mod}\ p \cdot \#(\mathscr{O}_K/\mathfrak{p})^{n-1}) \right\}$.
- $\mathrm{Hom}_{W_{\mathfrak{P}}/\varpi^n}(\sigma(\underline{x}), \underline{x}) \xrightarrow{\sim} \mathrm{End}_{W_{\mathfrak{P}}/\varpi^n}(\underline{x}) \cdot \mathfrak{s} \subseteq B$*, where $\mathfrak{s}$ is any (integral) ideal in $\mathfrak{S}$. Moreover, if we denote this map by $\phi \mapsto \phi^\flat \in B$, then the degree of $\phi$ is $\frac{\mathrm{Nrd}(\phi^\flat)}{\#(\mathscr{O}_K/\mathfrak{s})}$.*

This proof can also be found in [DaZh]. We omit it here. Consequently, we have:

**Proposition 4.3.17.** *Assume $r_\sigma(m) = 0$. Suppose $(p) = \mathfrak{p}^e$ ($e = 1$ or $2$) is not split in $\mathscr{O}_K$. Let $R := \mathrm{End}_{W_{\mathfrak{P}}/\varpi}(\underline{x})$ and $\mathfrak{s} \in \mathrm{Art}_K^{-1}(\sigma)$ be an (integral) ideal,*

- *If $(p)$ is inert and $\mathfrak{P}|p$, then*
$$i_{\mathfrak{P}}(\underline{x}, T_m(\sigma(\underline{x}))) = \sum_{b \in R\mathfrak{s}/\{\pm 1\}, \mathrm{Nrd}(b) = m \cdot \#(\mathscr{O}_K/\mathfrak{s})} \frac{1}{2}\Big(1 + \mathrm{ord}_p(\mathrm{Nrd}(b^-))\Big).$$

- *If $(p)$ is ramified and $\mathfrak{P}|p$, then*

$$i_{\mathfrak{P}}(\underline{x}, T_m(\sigma(\underline{x}))) = \sum_{b\in R\mathfrak{s}/\{\pm 1\}, \mathrm{Nrd}(b)=m\cdot\#(\mathscr{O}_K/\mathfrak{s})} \mathrm{ord}_p(D_K\cdot \mathrm{Nrd}(b^-)).$$

*Proof.* Combining Theorem 4.3.14 and Lemma 4.3.16 yields

$$i_{\mathfrak{P}}(\underline{x}, T_m(\sigma(\underline{x}))) = \frac{1}{2}\sum_{n\geq 1}\#\left\{b\in R\mathfrak{s} : m = \frac{\mathrm{Nrd}(b)}{\#(\mathscr{O}_K/\mathfrak{s})}, D_K\cdot\mathrm{Nrd}(b^-)\equiv 0 (\mathrm{mod}\ p\cdot\#(\mathscr{O}_K/\mathfrak{p})^{n-1})\right\}.$$

If $(p)$ is inert, then $p\nmid D_K$ and therefore $\#(\mathscr{O}_K/\mathfrak{p}) = p^2$. If $(p)$ is ramified, then $p|D_K$, so $\#(\mathscr{O}_K/\mathfrak{p}) = p$. □

As a corollary, remove the condition $r_\sigma(m) = 0$ by the moving argument, a correction term that depends on $r_\sigma(m)$ appears. Although the corollary is directly the sum of all the fiberal intersections, its proof is not easy, so we only provide a sketch of the proof.

**Corollary 4.3.18.** *Suppose $(m,N) = 1$, let $x\in X_0(N)(H_K)$ be a Heegner point. Then $\langle x-\infty, T_m\circ\sigma(x-0)\rangle_{\mathrm{fin}}$ is equal to*

$$-\frac{\#(\mathscr{O}_K^\times)^2}{4}\sum_{0<n\leq\frac{m|D_K|}{N}}\widetilde{\sigma}'(n) r_\sigma(m|D_K|-nN) + \frac{\#(\mathscr{O}_K^\times)}{2}\#\mathrm{Cl}(\mathscr{O}_K) r_\sigma(m)\log\frac{N}{m},$$

*where $\widetilde{\sigma}'(n)$ is described in Theorem 4.4.1 and Remark 4.4.2.*

*Proof.* (proof sketch). Let $p$ be a prime number, the sum $\sum_{\mathfrak{P}|p}\langle x-\infty, T_m\circ\sigma(x-0)\rangle_{\mathfrak{P}}$ is equal to (we write $\mathfrak{S} = \mathrm{Art}_K^{-1}(\sigma)$):

- $-\frac{\#(\mathscr{O}_K^\times)}{2}\#\mathrm{Cl}(\mathscr{O}_K) r_\sigma(m)\mathrm{ord}_p\left(\frac{m}{N}\right)\log p$, if $(p)$ is split in $\mathscr{O}_K$.
- $-\frac{\#(\mathscr{O}_K^\times)}{2}\#\mathrm{Cl}(\mathscr{O}_K) r_\sigma(m)\mathrm{ord}_p(m)\log p - \frac{\#(\mathscr{O}_K^\times)^2}{4}\log p\sum_{0<n<\frac{m|D_K|}{N}, p|n}\Big(\mathrm{ord}_p(pn) r_\sigma(m|D_K| - nN)\delta(n) R_{\mathfrak{S}\mathfrak{n}\mathfrak{q}}\left(\frac{n}{p}\right)\Big)$, if $(p)$ is inert in $\mathscr{O}_K$. Here, $q$ is a fixed auxiliary prime with $\left(\frac{q}{l}\right) = \left(\frac{-p}{l}\right)$ for all primes $l|D_K$, and $(q) = \mathfrak{q}\overline{\mathfrak{q}}$. The explicit quaternion algebra in this case is $B = \frac{(D_K, -pq)}{\mathbb{Q}}$.
- $-\frac{\#(\mathscr{O}_K^\times)}{2}\#\mathrm{Cl}(\mathscr{O}_K) r_\sigma(m)\mathrm{ord}_p(m)\log p - \frac{\#(\mathscr{O}_K^\times)^2}{4}\log p\sum_{0<n<\frac{m|D_K|}{N}, p|n}\Big(\mathrm{ord}_p(n) r_\sigma(m|D_K| - nN)\delta(n) R_{\mathfrak{S}\mathfrak{n}\mathfrak{p}\mathfrak{q}}\left(\frac{n}{p}\right)\Big)$, if $(p)$ is ramified in $\mathscr{O}_K$. Here, $q$ is a fixed auxiliary prime with $\left(\frac{q}{p'}\right) = \left(\frac{-1}{p'}\right)$ for all $p'\neq p$ which divide $D_K$, and $\left(\frac{-q}{p}\right) = -1$, then $(q) = \mathfrak{q}\overline{\mathfrak{q}}$. The explicit quaternion algebra in this case is $B = \frac{(D_K, -q)}{\mathbb{Q}}$.

For example, suppose $(p)$ is inert in $\mathscr{O}_K$. For concrete calculations, we choose an Eichler order

$$S = \left\{u + vj : u\in(\sqrt{D_K})^{-1}, v\in(\sqrt{D_K})^{-1}\mathfrak{q}^{-1}\mathfrak{n}, u\equiv v(\mathrm{mod}\ \mathscr{O}_{K,\mathfrak{f}}) \text{ for primes } \mathfrak{f}\supseteq(\sqrt{D_K})\right\}$$

with reduced discriminant $Np$ in $B = \frac{(D_K,-pq)}{\mathbb{Q}} = K \oplus Kj$, $j^2 = -pq$. The Eichler order $R$ in Lemma 4.3.16 must have form $R = \mathfrak{a}S\mathfrak{a}^{-1}$ for some ideal $\mathfrak{a} \subseteq \mathscr{O}_K$ (we obtain all the Eichler orders by ranging over all ideals). For an ideal $\mathfrak{s} \in \mathfrak{S}$, we have

$$R\mathfrak{s} = \left\{ u + vj : u \in (\sqrt{D_K})^{-1}\mathfrak{s}, v \in (\sqrt{D_K})^{-1}\mathfrak{q}^{-1}\mathfrak{n}\bar{\mathfrak{a}}\mathfrak{a}^{-1}\bar{\mathfrak{s}}, u \equiv (-1)^{\mathrm{ord}_{\mathfrak{f}}(\mathfrak{a})}v (\mathrm{mod}\ \mathscr{O}_{K,\mathfrak{f}}), \mathfrak{f} \supseteq (\sqrt{D_K}) \right\}.$$

The class $\mathfrak{A}$ of $\mathfrak{a}$ depends on the place divides $p$, and the different ideal classes which arise are permitted simply-transitively by $\mathrm{Gal}(H_K/K)$. Therefore, note that

$$\mathrm{Nrd}(u + vj) = |u|^2 + pq|v|^2 = m \cdot \#(\mathscr{O}_K/\mathfrak{s})$$

implies

$$\#\left(\frac{\mathscr{O}_K}{(u)(\sqrt{D_K})\mathfrak{s}^{-1}}\right) + pq \cdot \#\left(\frac{\mathscr{O}_K}{(v)(\sqrt{D_K})\bar{\mathfrak{a}}^{-1}\mathfrak{a}\bar{\mathfrak{s}}^{-1}}\right) = m|D_K|,$$

summing over $\mathfrak{P}|p$ (which means summing over all $(R,b)$ where $R$ is an Eichler order of reduced discriminant $Np$ and $b \in R\mathfrak{s}/\{\pm 1\}$ with $\mathrm{Nrd}(b) = m \cdot \#(\mathscr{O}_K/\mathfrak{s})$) yields the final result. See [GrZa] for details. □

## 4.4 Special Values of the $L$-function

For $\sigma \in \mathrm{Gal}(H_K/K)$, let $\mathfrak{s} \in \mathrm{Art}_K^{-1}(\sigma)$ be any integral ideal. Define a series

$$\theta_\sigma(z) := \frac{1}{\#(\mathscr{O}_K^\times)} + \sum_{n>0} r_\sigma(n)e^{2\pi inz} = \frac{1}{\#(\mathscr{O}_K^\times)} \sum_{\lambda \in \mathfrak{s}} \exp\left(\frac{2\pi i|\lambda|^2 z}{\#(\mathscr{O}_K/\mathfrak{s})}\right).$$

This is a modular form in $M_1(\Gamma_0(|D_K|), \varepsilon_K)$ (see [Heck], in this paper $r_\sigma(n)$ is the number of solutions of quadratic form, but by Remark 4.3.10, this definition coincides with the one in Proposition 4.3.9). If there is an additional modular form $f(z) = \sum_{n>0} a(n)e^{2\pi inz} \in S_{2k}^{\mathrm{new}}(\Gamma_0(N))$ (we still need $(D_K, N) = 1$), one can associate them ($\sigma$ and $f$) a $L$-function

$$L_\sigma(f,s) := \sum_{(n,N)=1} \frac{\varepsilon_K(n)}{n^{2s-2k+1}} \cdot \sum_{n=1}^{\infty} \frac{a(n)r_\sigma(n)}{n^s},$$

this is a product of a Dirichlet series and a Hecke $L$-function of the convolution $f * \theta_\sigma$.

Let us state the main result of this section, the Gross-Zagier formula.

**Theorem 4.4.1** (Gross-Zagier). *Let $\Lambda_\sigma(f,s) := (2\pi)^{-2s}N^s|D_K|^s\Gamma(s)^2L_\sigma(f,s)$ with $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(N))$, then the following functional equation holds:*

$$\Lambda_\sigma(f,s) = -\varepsilon_K(N)\Lambda_\sigma(f, 2k-s).$$

*Write $z = x + iy$, at the symmetric point of the functional equation,*

- *If $\varepsilon_K(N) = -1$, then $L_\sigma(f,k) = \frac{2^{2k+1}\pi^{k+1}}{(k-1)!\sqrt{|D_K|}}(f, \widetilde{\Psi}^-)_{\mathrm{Pet}}$, where $\widetilde{\Psi}^-(z)$ is equal to*

$$\sum_{m=0}^{\infty}\left(\sum_{0<n\leq\frac{m|D_K|}{N}} -\widetilde{\sigma}(n)r_\sigma(m|D_K|-nN)\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{\left(-\frac{4\pi nNy}{|D_K|}\right)^j}{j!} + \frac{\#\mathrm{Cl}(\mathscr{O}_K)}{\#(\mathscr{O}_K^\times)}r_\sigma(m)\right)y^{1-k}e^{2\pi imz}$$

*with $\widetilde{\sigma}(n) := \sum_{d|n,d>0}\varepsilon_\sigma(n,d)$ (see Proposition 4.4.6 for the definition of $\varepsilon_\sigma(n,d)$).*

- *If $\varepsilon_K(N)=1$, then $L'_\sigma(f,k)=\frac{2^{2k+1}\pi^{k+1}}{(k-1)!\sqrt{|D_K|}}(f,\widetilde{\Psi}^+)_{\mathrm{Pet}}$, where $\widetilde{\Psi}^+(z)$ is equal to*

$$\sum_{m=-\infty}^{\infty}\Bigg(-\sum_{0<n\le\frac{m|D_K|}{N}}\widetilde{\sigma}'(n)r_\sigma(m|D_K|-nN)\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{\left(-\frac{4\pi nNy}{|D_K|}\right)^j}{j!}$$
$$+2\frac{\#\mathrm{Cl}(\mathscr{O}_K)}{\#(\mathscr{O}_K^\times)}r_\sigma(m)\left(\log\left(\frac{N|D_K|y}{\pi}\right)+\frac{\Gamma'(k)}{\Gamma(k)}+2\frac{L'(1,\varepsilon_K)}{L(1,\varepsilon_K)}\right)$$
$$-\sum_{n=1}^{\infty}\widetilde{\sigma}(n)r_\sigma(m|D_K|+nN)\int_1^\infty\frac{(x-1)^{k-1}}{x^k}e^{-\frac{4\pi nNxy}{|D_K|}}dx\Bigg)y^{1-k}e^{2\pi imz}$$

  *with $\widetilde{\sigma}'(n):=\sum_{d|n,d>0}\varepsilon_\sigma(n,d)\log\frac{n}{d^2}$.*

*The functions $\widetilde{\Psi}^-$ and $\widetilde{\Psi}^+$ both satisfy that for any $\gamma\in\Gamma_0(N)$, $[\widetilde{\Psi}^\pm|_{2k}\gamma]=\widetilde{\Psi}^\pm$ and have at most polynomial growth at the cusps of $\Gamma_0(N)$ (they are not holomorphic modular forms in general).*

In Theorem 4.4.1, the functions $\widetilde{\Psi}^\pm(z)$ originate from

$$\widetilde{\Psi}^-(z):=\frac{N^{k-1}\sqrt{|D_K|}}{2\pi}\widetilde{\Phi}_{1-k}(z)\quad\text{and}\quad\widetilde{\Psi}^+(z):=\frac{N^{k-1}\sqrt{|D_K|}}{2\pi}\frac{\partial\widetilde{\Phi}_s(z)}{\partial s}\Big|_{s=1-k},$$

where $\widetilde{\Phi}_s(z)$ will be discussed in Proposition 4.4.5. Moreover, $\widetilde{\sigma}(n)$ and $\widetilde{\sigma}'(n)$ can be described in another way, whose proofs can be found in [GrZa]:

**Remark 4.4.2.** Let $n$ be an integer.

- Suppose $\varepsilon_K(N)n<0$ and $D_K|nN+\#(\mathscr{O}_K/\mathfrak{s})$ for some $\mathfrak{s}\in\mathfrak{S}:=\mathrm{Art}_K^{-1}(\sigma)$. Then $\widetilde{\sigma}(n)=\delta(n)R_{\mathfrak{S}\mathfrak{n}}(|n|)$.
- Suppose $\varepsilon_K(N)=1$ and $n>0$, then $\widetilde{\sigma}'(n)=\sum_{p|n}a_p(n)\log p$ with

$$a_p(n):=\begin{cases}0 & \varepsilon_K(p)=1\\ (\mathrm{ord}_p(n)+1)\delta(n)R_{\mathfrak{S}\mathfrak{n}\mathfrak{a}}(n/p) & \varepsilon_K(p)=-1\\ \mathrm{ord}_p(n)\delta(n)R_{\mathfrak{S}\mathfrak{n}\mathfrak{a}}(n/p) & \varepsilon_K(p)=0\end{cases},$$

  where $\mathfrak{a}$ is some integral ideal that satisfies $D_K|p+\#(\mathscr{O}_K/\mathfrak{a})$.

We aim to modify the functions obtained in Theorem 4.4.1 into holomorphic modular forms and interpret their Fourier coefficients as arithmetic intersection numbers. To this end, we need:

**Remark 4.4.3.** Let $k>1$. For the function $\widetilde{\Psi}^\pm:=\widetilde{\Psi}^-$ or $\widetilde{\Psi}^+$ in Theorem 4.4.1, it will produce a holomorphic cusp form $\Psi^\pm\in S_{2k}(\Gamma_0(N))$ such that $(\widetilde{\Psi}^\pm,f)_{\mathrm{Pet}}=(\Psi^\pm,f)_{\mathrm{Pet}}$ for all $f\in S_{2k}(\Gamma_0(N))$. Indeed, for any $m\ge1$, consider the series

$$P_m(z)=\sum_{\gamma\in\left(\begin{smallmatrix}\pm1&*\\0&\pm1\end{smallmatrix}\right)\backslash\Gamma_0(N)}[e^{2\pi imz}|_{2k}\gamma]=\frac{1}{2}\sum_{N|c,(c,d)=1}(cz+d)^{-2k}e^{2\pi im\frac{az+b}{cz+d}}\in S_{2k}(\Gamma_0(N))$$

which is absolutely convergent (bounded by $O(y^{1-2k}), y \to \infty$) because $k > 1$. Note that $\widetilde{\Psi}^{\pm}(z) = \sum_{m\in\mathbb{Z}} a(m,y)e^{2\pi imz}$ is small at all cusps (that is, $[\widetilde{\Psi}^{\pm}|_{2k}\gamma](x+iy) = O(y^{-\varepsilon})$ as $y \to \infty$ for some $\varepsilon > 0$ and every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$), this makes

$$\begin{aligned}(\widetilde{\Psi}^{\pm}, P_m)_{\mathrm{Pet}} &= \sum_{\gamma\in\left(\begin{smallmatrix}\pm1 & *\\ 0 & \pm1\end{smallmatrix}\right)\backslash\Gamma_0(N)} \int_{\Gamma_0(N)\backslash\mathscr{H}} \widetilde{\Psi}^{\pm}(z)\overline{[e^{2\pi imz}|_{2k}\gamma]}y^{2k}\frac{dxdy}{y^2}\\
&= \int_{\left(\begin{smallmatrix}\pm1 & *\\ 0 & \pm1\end{smallmatrix}\right)\backslash\mathscr{H}} \widetilde{\Psi}^{\pm}(z)\overline{e^{2\pi imz}}y^{2k}\frac{dxdy}{y^2} && \text{(substitute variables)}\\
&= \int_0^{\infty}\left(\int_0^1 \widetilde{\Psi}^{\pm}(z)\overline{e^{2\pi imz}}dx\right)y^{2k-2}dy\\
&= \int_0^{\infty} a(m,y)e^{-4\pi my}y^{2k-2}dy && \text{(Parseval identity)}\end{aligned}$$

converges absolutely and hence the operator $(\widetilde{\Psi}^{\pm}, \cdot)_{\mathrm{Pet}}$ can be represented by $(\Psi^{\pm}, \cdot)_{\mathrm{Pet}}$ for some $\Psi^{\pm} \in S_{2k}(\Gamma_0(N))$. The Fourier coefficients of $\Psi^{\pm}$ (they are do not depend on $y$) are obtained by computing $(\Psi^{\pm}, P_m)_{\mathrm{Pet}} = (\widetilde{\Psi}^{\pm}, P_m)_{\mathrm{Pet}}$, $m \geq 1$:

$$\Psi^{\pm}(z) = \sum_{m=1}^{\infty} \frac{(4\pi m)^{2k-1}}{(2k-2)!}\left(\int_0^{\infty} a(m,y)e^{-4\pi my}y^{2k-2}dy\right)e^{2\pi imz}.$$

Now suppose $k = 1$ (which is the case we need). If $\varepsilon_K(N) = -1$, then $\widetilde{\Psi}^{-}$ is already holomorphic (but not a cusp form) because it is $\mathbb{C}$-linearly dependent with $\theta_\sigma$, so there is nothing to do. If $\varepsilon_K(N) = 1$, $\widetilde{\Psi}^{+}$ can still produce a holomorphic cusp form $\Psi^{+}$, but the argument becomes much more intricate due to the non-convergence of Eisenstein series. Therefore, we need to improve the above method. We only state the final results here: if $(m,N) = 1$, the $m$-th Fourier coefficient of $\Psi^{+}(z)$ is

$$\begin{aligned}&-\sum_{0<n\leq\frac{m|D_K|}{N}} \widetilde{\sigma}'(n)r_\sigma(m|D_K|-nN) + 2\frac{\#\mathrm{Cl}(\mathscr{O}_K)}{\#(\mathscr{O}_K^{\times})}r_\sigma(m)\left(\log\left(\frac{N|D_K|}{4\pi^2m}\right) + 2\frac{\Gamma'(1)}{\Gamma(1)} + 2\frac{L'(1,\varepsilon_K)}{L(1,\varepsilon_K)}\right)\\
&+16\pi\lim_{s\to1}\left(-\sum_{n=1}^{\infty}\widetilde{\sigma}(n)r_\sigma(m|D_K|+nN)G_s\left(\frac{nN}{m|D_K|}\right) - \frac{1}{\#(\mathscr{O}_K^{\times})^2}\frac{\#\mathrm{Cl}(\mathscr{O}_K)\sigma_1(m)\kappa_N}{s-1}\right)\\
&+16\pi\frac{\#\mathrm{Cl}(\mathscr{O}_K)\kappa_N}{\#(\mathscr{O}_K^{\times})^2}\left(\sigma_1(m)\left(2+\log\frac{N}{|D_K|} + \sum_{p|N}\frac{2\log p}{p^2-1} - 2\frac{L'(1,\varepsilon_K)}{L(1,\varepsilon_K)} + 2\frac{\zeta'(2)}{\zeta(2)}\right) + \sum_{d|m} d\log\frac{m}{d^2}\right).\end{aligned}$$

**Corollary 4.4.4** (Gross-Zagier)**.** *Let $x \in X_0(N)(H_K)$ be a Heegner point. For $\sigma \in \mathrm{Gal}(H_K/K)$, we have*

$$g_\sigma := \sum_{m\geq1}\langle x-\infty, T_m\circ\sigma(x-0)\rangle e^{2\pi imz} \in S_2(\Gamma_0(N)),$$

*which is independent of the choice of Heegner point, and*

$$L'_\sigma(f,1) = \frac{32\pi^2}{\#(\mathscr{O}_K^{\times})^2\sqrt{|D_K|}}(f,g_\sigma)_{\mathrm{Pet}}$$

*holds for $f \in S_2^{\mathrm{new}}(\Gamma_0(N))$.*

*Proof.* Let $\mathfrak{T}$ be the Hecke algebra of $\Gamma_0(N)$. It is not hard to verify that, the bilinear map

$$(\mathfrak{T}\otimes_{\mathbb{Z}}\mathbb{C})\times S_2(\Gamma_0(N))\longrightarrow\mathbb{C},\quad \Big(T,f(z)=\sum_{m\geq 1}a_f(m)e^{2\pi imz}\Big)\longmapsto a_{T(f)}(1)$$

is a perfect pairing, so any $\mathbb{C}$-linear map $\alpha:\mathfrak{T}\otimes_{\mathbb{Z}}\mathbb{C}\to\mathbb{C}$ can be represented as the image of $(\cdot,f_\alpha)$ for some $f_\alpha\in S_2(\Gamma_0(N))$ which has expansion $f_\alpha(z)=\sum_{m\geq 1}\alpha(T_m)e^{2\pi imz}$ since $\alpha(T_m)=a_{T_m(f_\alpha)}(1)=a_{f_\alpha}(m)$. Apply this fact to the $\mathbb{C}$-linear map $T\mapsto\langle x-\infty,T\circ\sigma(x-0)\rangle$, we obtain the form $g_\sigma=\sum_{m\geq 1}\langle x-\infty,T_m\circ\sigma(x-0)\rangle e^{2\pi imz}$. If $(m,N)=1$, the intersection number $\langle x-\infty,T_m\circ\sigma(x-0)\rangle=\langle x-\infty,T_m\circ\sigma(x-0)\rangle_{\mathrm{inf}}+\langle x-\infty,T_m\circ\sigma(x-0)\rangle_{\mathrm{fin}}$ is specified by Theorem 4.3.11 $\big($use the property $\widetilde{\sigma}(-n)=-\widetilde{\sigma}(n)$ to simplify the term $\delta(n)R_{\mathfrak{S}\mathfrak{n}}(n)$ in $H_{N,s,m}(\mathfrak{S})\big)$, Corollary 4.3.12 and Corollary 4.3.18. Compare this result with the last case in Remark 4.4.3, our $g_\sigma$ is equal to $\frac{\#(\mathscr{O}_K^\times)^2}{4}\Psi^+$ up to an old form, so the desired formula holds. $\square$

Now we return to the definition of $\widetilde{\Phi}_s$.

**Proposition 4.4.5.** *Let $K$ be an imaginary quadratic field with discriminate $D_K$, $N\geq 1$ prime to $D_K$. Define the trace map $\mathrm{tr}_N^M(f):=\sum_{\gamma\in\Gamma_0(M)\backslash\Gamma_0(N)}[f|_{2k}\gamma]$, then*

$$\widetilde{\Phi}_s(z):=\mathrm{tr}_N^{N|D_K|}\Big(\theta_\sigma(z)E_s^{(1)}(Nz)\Big)$$

*satisfies that for any $\gamma\in\Gamma_0(N)$, $[\widetilde{\Phi}_s|_{2k}\gamma]=\widetilde{\Phi}_s$ and has at most polynomial growth at the cusps of $\Gamma_0(N)$. Moreover, the formula*

$$(4\pi)^{-s-2k+1}N^s\Gamma(s+2k-1)L_\sigma(f,s+2k-1)=(f,\widetilde{\Phi}_{\bar{s}})_{\mathrm{Pet}}$$

*holds for any $f\in S_{2k}^{\mathrm{new}}(\Gamma_0(N))$. Here*

$$E_s^{(1)}(z):=\frac{1}{2}\sum_{(c,d)\in\mathbb{Z}^2,D_K|c}\frac{\varepsilon_K(d)}{(cz+d)^{2k-1}}\frac{y^s}{|cz+d|^{2s}}.$$

*Proof.* Note that the series $\sum_{n=1}^\infty a(n)r_\sigma(n)n^{-(s+2k-1)}$ converges absolutely, we have

$$\begin{aligned}
&\frac{\Gamma(s+2k-1)}{(4\pi)^{s+2k-1}}\sum_{n=1}^\infty\frac{a(n)r_\sigma(n)}{n^{s+2k-1}}\\
=&\int_0^\infty\left(\sum_{n=1}^\infty a(n)r_\sigma(n)e^{-4\pi ny}\right)y^{s+2k-2}dy && \Big(\Gamma(s)=\int_0^\infty y^{s-1}e^{-y}dy\Big)\\
=&\int_0^\infty\left(\int_0^1 f(x+iy)\overline{\theta_\sigma(x+iy)}dx\right)y^{s+2k-2}dy && \text{(Parseval identity)}\\
=&\int_{\left(\begin{smallmatrix}\pm1&*\\0&\pm1\end{smallmatrix}\right)\backslash\mathscr{H}}f(z)\overline{\theta_\sigma(z)}y^{s+2k}\frac{dxdy}{y^2}. && (z=x+iy)
\end{aligned}$$

Recall $f\in S_{2k}^{\mathrm{new}}(\Gamma_0(N))$ and $\theta_\sigma\in M_1(\Gamma_0(|D_K|),\varepsilon_K)$, one can formulate a decomposition

$$\left(\begin{smallmatrix}\pm1&*\\0&\pm1\end{smallmatrix}\right)\backslash\mathscr{H}=\bigsqcup_{\gamma\in\left(\begin{smallmatrix}\pm1&*\\0&\pm1\end{smallmatrix}\right)\backslash\Gamma_0(N|D_K|)}\gamma\cdot(\Gamma_0(N|D_K|)\backslash\mathscr{H}),$$

so the above integral becomes

$$\sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \left(\begin{smallmatrix} \pm 1 & * \\ 0 & \pm 1 \end{smallmatrix}\right) \backslash \Gamma_0(N|D_K|)} \int_{\Gamma_0(N|D_K|)\backslash \mathscr{H}} f(z)\overline{\theta_\sigma(z)} \frac{\varepsilon_K(d) y^{s+2k}}{(c\bar{z}+d)^{2k-1}|cz+d|^{2s}} \frac{dxdy}{y^2}.$$

Multiplying by $\sum_{(n,N)=1} \varepsilon_K(n) n^{-(2(s+2k-1)-2k+1)}$, we obtain $N^{-s}$ times the left-hand side of the desired formula, which is equal to

$$\int_{\Gamma_0(N|D_K|)\backslash \mathscr{H}} f(z)\overline{\theta_\sigma(z)} \left( \sum_{(n,ND_K)=1} \frac{\varepsilon_K(n)}{n^{2s+2k-1}} \cdot \sum_{ND_K|c,(c,d)=1} \frac{\varepsilon_K(d) y^s}{2(c\bar{z}+d)^{2k-1}|cz+d|^{2s}} \right) y^{2k} \frac{dxdy}{y^2}.$$

Since for integers $c,d,M(=ND_K)$, one can check that $[M|c] \wedge [(d,M)=1]$ is equivalent to $[Mn|c] \wedge [(n,M)=1]$ with $n=(c,d)$, this means the product of two summations in the integral above is

$$\sum_{ND_K|c,(c,d)=1,(n,ND_K)=1} \frac{\varepsilon_K(nd) y^s}{2(nc\bar{z}+nd)^{2k-1}|ncz+nd|^{2s}} = \sum_{ND_K|c,(d,ND_K)=1} \frac{\varepsilon_K(d) y^s}{2(c\bar{z}+d)^{2k-1}|cz+d|^{2s}}.$$

Thus, we derive the Petersson inner product we need under the setting of level $\Gamma_0(N|D_K|)$, and to view it within $\Gamma_0(N)$, we simply take the trace. Note that the above summation can be rewritten using the Möbius function

$$\sum_{t|(N,d)} \mu(t) \sum_{ND_K|c} \frac{\varepsilon_K(d) y^s}{2(c\bar{z}+d)^{2k-1}|cz+d|^{2s}} = \sum_{t|N} \frac{\mu(t)\varepsilon_K(t)}{t^{2s+2k-1}} (N/t)^{-s} E_s^{(1)}(N\bar{z}/t),$$

since $\varepsilon_K(d)$ vanishes if $(d,D_K)>1$, $\sum_{t|(N,d)} \mu(t)$ vanishes if $(N,d)>1$. So we can drop the $t>1$ part in the trace because they contribute terms of level $N/t<N$ which are orthogonal to $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(N))$ with respect to the Petersson inner product. □

It should be noted that the function $\widetilde{\Phi}_s$ in Proposition 4.4.5 is not necessarily a modular form, as $E_s^{(1)}$ is not holomorphic in $z$.

To study the Fourier development of $\widetilde{\Phi}_s$, we need the expansions of $\theta_\sigma$ and $E_s^{(1)}$ at the various cusps of $\Gamma_0(|D_K|)$. According to the correspondence in Chapter 4.1, let us suppose $|D_K| = \delta_1\delta_2$ (note that $D_K$ is square free), where $\delta_i = |D_i|$ with $D_i$ are two discriminates of quadratic fields (one real and one imaginary) and $D_K = D_1 D_2 < 0$, these give two Kronecker symbols $\varepsilon_i = \left(\frac{D_i}{\cdot}\right)$. We also have a character

$$\chi_{D_1D_2} : \mathrm{Cl}(\mathscr{O}_K) \to \mathrm{Cl}(\mathscr{O}_K)/2\mathrm{Cl}(\mathscr{O}_K) \to \{\pm 1\}, \quad \mathfrak{a} \mapsto \left(\frac{D_1}{\#(\mathscr{O}_K/\mathfrak{a})}\right) = \left(\frac{D_2}{\#(\mathscr{O}_K/\mathfrak{a})}\right)$$

for (integral) ideals $\mathfrak{a} \subseteq \mathscr{O}_K$ prime to $D_K$.

**Proposition 4.4.6.** *Under the assumptions above, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $(c,D_K)=\delta_2$, let $c^*$ be an inverse of $c (\mathrm{mod}\, D_1)$, we have*

- $[\theta_\sigma|_1\gamma](z) = \hbar(D_1)\varepsilon_1(c/\delta_2)\varepsilon_2(d)\delta_1^{-1/2}\chi_{D_1D_2}(\mathrm{Art}_K^{-1}(\sigma))\theta_{\sigma(D_1)}\left(\frac{z+c^*d}{\delta_1}\right)$, *where $\hbar(D_1)$ denotes* 1 *or* $-i$ *according as $D_1>0$ and $D_1<0$, $\sigma(D_1) := \sigma \circ \mathrm{Art}_K\big((D_1, \sqrt{D_K})\big)$.*

- $[E_s^{(1)}|_{2k-1}\gamma](z) = \varepsilon_1(c)\varepsilon_2(d\delta_1)\delta_1^{-s-2k+1}E_s^{(D_1)}\left(\frac{z+c^*d}{\delta_1}\right)$, *where*

$$E_s^{(D_1)}(z) = \frac{1}{2}\sum_{(m,n)\in\mathbb{Z}^2, D_2|m} \frac{\varepsilon_1(m)\varepsilon_2(n)}{(mz+n)^{2k-1}}\frac{y^s}{|mz+n|^{2s}}$$

*satisfies that for any* $\gamma\in\Gamma_0(|D_K|)$, $[E_s^{(D_1)}|_{2k-1}\gamma] = \varepsilon_K(d)\cdot E_s^{(D_1)}$ *and has at most polynomial growth at the cusps of* $\Gamma_0(|D_K|)$.

*Moreover, the function* $\widetilde{\Phi}_s$ *in Proposition 4.4.5 has the following representation*

$$\begin{aligned}\widetilde{\Phi}_s(z) =& \mathbf{U}_{|D_K|}\left(\theta_\sigma(z)\cdot\sum_{D_K=D_1D_2}\frac{\varepsilon_1(N)\chi_{D_1D_2}(\mathrm{Art}_K^{-1}(\sigma))}{\hbar(D_1)|D_1|^{s+2k-\frac{3}{2}}}E_s^{(D_1)}(|D_2|Nz)\right)\\ =& \sum_{n\in\mathbb{Z}, l\geq 0, D_K|nN+l} e_s\left(n,\frac{Ny}{|D_K|}\right)r_\sigma(l)\exp\left(\frac{2\pi i}{|D_K|}\left((nN+l)x+ily\right)\right),\end{aligned}$$

*where the operator* $\mathbf{U}_n$ *acts on the set of periodic functions of period* 1 *on* $\mathscr{H}$ *by*

$$f(z)\mapsto\frac{1}{n}\sum_{m\in\mathbb{Z}/n\mathbb{Z}} f\left(\frac{z+m}{n}\right) \quad \text{or} \quad \sum_{m\in\mathbb{Z}} a(m,y)e^{2\pi imx}\mapsto\sum_{m\in\mathbb{Z}} a(mn,y/n)e^{2\pi imx} \quad (z=x+iy),$$

*and*

$$e_s(n,y) = \begin{cases} L(2s+2k-1,\varepsilon_K)(|D_K|y)^s + \frac{\varepsilon_K(N)}{\sqrt{D_K}}V_s(0)L(2s+2k-2,\varepsilon_K)(|D_K|y)^{-s-2k+2} & n=0\\ \frac{\varepsilon_K(N)}{\sqrt{D_K}}(|D_K|y)^{-s-2k+2}V_s(ny)\sum_{d|n,d>0}\frac{\varepsilon_\sigma(n,d)}{d^{2s+2k-2}} & n\neq 0\end{cases},$$

*where*

$$V_s(t) := \int_{\mathbb{R}}\frac{e^{-2\pi ixt}}{(x+i)^{2k-1}(x^2+1)^s}dx, \quad t\in\mathbb{R}, \mathrm{Re}(s)>1-k,$$

*and*

$$\varepsilon_\sigma(n,d) := \delta_{(d,n/d,D_K),1}\cdot\varepsilon_1(d)\varepsilon_2(-nN/d)\chi_{D_1D_2}(\mathrm{Art}_K^{-1}(\sigma)), \quad \delta_2 = (d,D_K).$$

**Remark 4.4.7.** The function $V_s(t)$ in Proposition 4.4.6 has the following properties (see [GrZa]):

- $V_s(0) = 2^{-2s-2k+3}\pi i(-1)^k\dfrac{\Gamma(2s+2k-2)}{\Gamma(s)\Gamma(s+2k-1)}$.
- For $t\neq 0$, the function $V_s(t)$ can be extended to a holomorphic function in $s\in\mathbb{C}$ and $V_s(t) = |t|^{O(1)}e^{-2\pi|t|}$, $|t|\to\infty$.
- For $t\neq 0$, let $V_s^*(t) := (\pi|t|)^{-s-2k+1}\Gamma(s+2k-1)V_s(t)$, then $V_s^*(t)$ is holomorphic in $s\in\mathbb{C}$ and $V_s^*(t) = \mathrm{sgn}(t)V_{2-2k-s}^*(t)$.
- Let $0\leq r\leq k-1$ be an integer, then $V_{-r}(t) = \begin{cases}0 & t<0\\ 2\pi i(-1)^{k-r}p_{k,r}(4\pi t)e^{-2\pi t} & t>0\end{cases}$, where $p_{k,r}(t) := \left(\frac{t}{2}\right)^{2k-2r-2}\sum_{j=0}^{r}\binom{r}{j}\frac{(-t)^j}{(2k-2r-2+j)!}$ is a polynomial.

- For $t<0$, we have $\displaystyle \frac{\partial V_s(t)}{\partial s}\Big|_{s=1-k} = -2\pi i e^{-2\pi t}\int_1^\infty \frac{(x-1)^{k-1}}{x^k}e^{-4\pi x|t|}dx.$

The proof of Proposition 4.4.6 is an application of the Poisson summation formula.

*Proof.* (proof sketch of Proposition 4.4.6). We only compute $[\theta_\sigma|_1\gamma](z)$, and $[E_s^{(1)}|_{2k-1}\gamma](z)$ is similar. This can be reduced to the case $c=\delta_2$, because any matrix $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{Z})$ with $(c,D_K)=\delta_2$ can be transformed into the form $\gamma\left(\begin{smallmatrix} x & *\\ -\delta_2 & *\end{smallmatrix}\right)=\left(\begin{smallmatrix} ax-b\delta_2 & *\\ cx-d\delta_2 & *\end{smallmatrix}\right)\in\Gamma_0(|D_K|)$ with $(x,\delta_2)=1$, $cx\equiv d\delta_2 (\mathrm{mod}\ \delta_1)$ for some $x\in\mathbb{Z}$, and (apply the special case of $c=\delta_2$)

$$\begin{aligned}[\theta_\sigma|_1\gamma](z) =&\varepsilon_K(ax-b\delta_2)\hbar(D_1)\varepsilon_2(x)\delta_1^{-1/2}\chi_{D_1D_2}(\mathrm{Art}_K^{-1}(\sigma))\theta_{\sigma(D_1)}\left(\frac{z+\delta_2^*x}{\delta_1}\right)\\ =&\varepsilon_1(c)\varepsilon_1(\delta_2)\varepsilon_2(a)\varepsilon_2(x)\varepsilon_2(x)\hbar(D_1)\delta_1^{-1/2}\chi_{D_1D_2}(\mathrm{Art}_K^{-1}(\sigma))\theta_{\sigma(D_1)}\left(\frac{z+\delta_2^*x}{\delta_1}\right)\end{aligned}$$

since $\theta_\sigma\in M_1(\Gamma_0(|D_K|),\varepsilon_K)$. So from now on, we assume $c=\delta_2$ and write $\zeta:=-\frac{1}{c(cz+d)}$, then

$$\theta_\sigma(\gamma z)=\theta_\sigma\left(\frac{a}{c}+\zeta\right)=\frac{1}{\#(\mathscr{O}_K^\times)}\sum_{\lambda\in\mathfrak{s}}\exp\left(2\pi i\frac{|\lambda|^2}{\#(\mathscr{O}_K/\mathfrak{s})}\frac{a}{\delta_2}\right)\exp\left(2\pi i\frac{|\lambda|^2}{\#(\mathscr{O}_K/\mathfrak{s})}\zeta\right),$$

where $\mathfrak{s}\in\mathrm{Art}_K^{-1}(\sigma)$. Note that the integer $\frac{|\lambda|^2}{\#(\mathscr{O}_K/\mathfrak{s})}(\mathrm{mod}\ \delta_2)$ only depends on the class of $\lambda$ in $\mathscr{O}_K/\mathfrak{s}(D_2,\sqrt{D_K})$ $\Big($because for $s\in\mathfrak{s}$ and $t\in(D_2,\sqrt{D_K})$ we have $\frac{|\lambda|^2}{\#(\mathscr{O}_K/\mathfrak{s})}=\frac{|\lambda+st|^2}{\#(\mathscr{O}_K/\mathfrak{s})}-\frac{st\overline{\lambda}+\overline{st}\lambda}{\#(\mathscr{O}_K/\mathfrak{s})}$, but $s\overline{\lambda}\in\mathfrak{s}\overline{\mathfrak{s}}$ is principal, hence $s\overline{\lambda}=\alpha\cdot\#(\mathscr{O}_K/\mathfrak{s})$ for some $\alpha\in\mathscr{O}_K$, so $\frac{st\overline{\lambda}+\overline{st}\lambda}{\#(\mathscr{O}_K/\mathfrak{s})}=2\mathrm{Re}(\alpha t)\in(\delta_2)\Big)$, we may formulate

$$\theta_\sigma(\gamma z)=\frac{1}{\#(\mathscr{O}_K^\times)}\sum_{\lambda\in\mathfrak{s}/\mathfrak{s}(D_2,\sqrt{D_K})}\exp\left(2\pi i\frac{|\lambda|^2}{\#(\mathscr{O}_K/\mathfrak{s})}\frac{a}{\delta_2}\right)\sum_{\mu\in\mathfrak{s}(D_2,\sqrt{D_K})}\exp\left(2\pi i\frac{|\lambda+\mu|^2}{\#(\mathscr{O}_K/\mathfrak{s})}\zeta\right).$$

By the Poisson summation formula (see Remark 1.3.3, here we take $\mathfrak{a}=\mathfrak{s}(D_2,\sqrt{D_K})$), the summation over $\mu$ above equals

$$\frac{i|D_K|^{-1/2}\zeta^{-1}}{\#(\mathscr{O}_K/(D_2,\sqrt{D_K}))}\sum_{\nu\in\mathfrak{s}^{-1}(D_2,\sqrt{D_K})^{-1}(\sqrt{D_K})^{-1}}\exp\left(-2\pi i\frac{\#(\mathscr{O}_K/\mathfrak{s})|\nu|^2}{\zeta}\right)\exp\left(-2\pi i(\nu\lambda+\overline{\nu\lambda})\right),$$

therefore

$$\theta_\sigma(\gamma z)=\frac{-i(cz+d)}{\#(\mathscr{O}_K^\times)\sqrt{|D_K|}}\sum_{\lambda,\nu}\exp\left(2\pi i\frac{|\lambda|^2}{\#(\mathscr{O}_K/\mathfrak{s})}\frac{a}{\delta_2}\right)e^{2\pi i\delta_2(cz+d)\cdot\#(\mathscr{O}_K/\mathfrak{s})|\nu|^2}e^{2\pi i(\nu\lambda+\overline{\nu\lambda})}.$$

Due to this, replacing $\nu$ with $\nu/c=\nu/\delta_2$ yields

$$[\theta_\sigma|_1\gamma](z)=\frac{-i}{\#(\mathscr{O}_K^\times)\sqrt{|D_K|}}\sum_{\nu\in\mathfrak{s}^{-1}(D_1,\sqrt{D_K})^{-1}}C(\nu)e^{2\pi i(z+d/c)\cdot\#(\mathscr{O}_K/\mathfrak{s})|\nu|^2},$$

where $C(v) = \sum_{\lambda\in\mathfrak{s}/\mathfrak{s}(D_2,\sqrt{D_K})} \exp\left(2\pi i\left(\frac{a|\lambda|^2}{\#(\mathscr{O}_K/\mathfrak{s})} + (v\lambda + \overline{v\lambda})\right)/\delta_2\right)$. To calculate this, let $\lambda_0 \in \mathfrak{s}$ with $(\lambda_0)\mathfrak{s}^{-1}$ is prime to $(D_2, \sqrt{D_K})$ such that when $\omega$ runs through $\mathscr{O}_K/(D_2,\sqrt{D_K})$, $\lambda = \lambda_0\omega$ runs through $\mathfrak{s}/\mathfrak{s}(D_2,\sqrt{D_K})$. Write $R := \frac{a|\lambda_0|^2}{\#(\mathscr{O}_K/\mathfrak{s})} \in \mathbb{Z}$, then $(R,\delta_2) = 1$ so we can choose $R^* > 0$ as an inverse of $R (\mathrm{mod}\ \delta_2)$ such that $\delta_1 | R^*$. Now,

$$\begin{aligned}
C(v) &= \sum_{\omega\in\mathscr{O}_K/(D_2,\sqrt{D_K})} \exp\left(2\pi i R\frac{|\omega|^2 + (R^* v\lambda_0\omega + \overline{R^* v\lambda_0\omega})}{\delta_2}\right)\\
&= \sum_{\omega\in\mathscr{O}_K/(D_2,\sqrt{D_K})} \exp\left(2\pi i R\frac{|\omega + R^* v\lambda_0|^2 - |R^* v\lambda_0|^2}{\delta_2}\right) && \text{(replace } \overline{\omega} \text{ with } \omega)\\
&= \exp\left(-2\pi i R^*\frac{|v\lambda_0|^2}{\delta_2}\right)\sum_{\omega\in\mathscr{O}_K/(D_2,\sqrt{D_K})} \exp\left(2\pi i R\frac{|\omega|^2}{\delta_2}\right) && \text{(replace } \omega + R^* v\lambda_0 \text{ with } \omega)\\
&= \exp\left(-2\pi i R^*\frac{|v\lambda_0|^2}{\delta_2}\right)\sum_{\omega\in\mathbb{Z}/\delta_2\mathbb{Z}} \exp\left(2\pi i R\frac{\omega^2}{\delta_2}\right)\\
&= e^{-2\pi i R^*|v\lambda_0|^2/\delta_2}\hbar(D_2)\delta_2^{1/2}\varepsilon_2(R). && \text{(Gauss sum)}
\end{aligned}$$

Since one can verify $\frac{R^*|\lambda_0|^2}{\#(\mathscr{O}_K/\mathfrak{s})} \equiv d (\mathrm{mod}\ \delta_2)$; $\frac{R^*|\lambda_0|^2}{\#(\mathscr{O}_K/\mathfrak{s})} \equiv 0 (\mathrm{mod}\ \delta_1)$, this implies

$$\begin{aligned}
e^{2\pi i(z+d/c)\cdot\#(\mathscr{O}_K/\mathfrak{s})|v|^2} e^{-2\pi i R^*|v\lambda_0|^2/\delta_2} &= e^{2\pi i\cdot\#(\mathscr{O}_K/\mathfrak{s})|v|^2\left(z + \frac{d - R^*|\lambda_0|^2/\#(\mathscr{O}_K/\mathfrak{s})}{\delta_2}\right)}\\
&= e^{2\pi i\cdot\#(\mathscr{O}_K/\mathfrak{s}(D_1,\sqrt{D_K}))|v|^2\frac{z+c^*d}{\delta_1}}.
\end{aligned}$$

Combining the above calculations, note that $\varepsilon_2(R) = \varepsilon_2(d)\chi_{D_1D_2}(\mathrm{Art}_K^{-1}(\sigma))$, we have

$$[\theta_\sigma|_1\gamma](z) = \frac{-i\hbar(D_2)\varepsilon_2(d)\chi_{D_1D_2}(\mathrm{Art}_K^{-1}(\sigma))}{\#(\mathscr{O}_K^\times)\delta_1^{1/2}}\sum_v \exp\left(2\pi i\cdot\#(\mathscr{O}_K/\mathfrak{s}(D_1,\sqrt{D_K}))|v|^2\frac{z+c^*d}{\delta_1}\right),$$

as desired.

With the results about $\theta_\sigma(z)$ and $E_s^{(1)}(Nz)$ in hand, we can directly compute $\widetilde{\Phi}_s(z)$ according to its definition. This calculation is not hard, the only thing that needs to be mentioned is that the trace from $\Gamma_0(N|D_K|)$ to $\Gamma_0(N)$ is given by summing over $\sum_{\delta_1|D_K}\delta_1$ representatives $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ of $\Gamma_0(N|D_K|)\backslash\Gamma_0(N)$, the representatives being characterized by the value $\delta_2 = (c, D_K)$ and by the residue class of $c^*d (\mathrm{mod}\ \delta_1)$.

Now we compute the Fourier coefficients of $\widetilde{\Phi}_s(z)$. Since $|D_1| > 1$ implies $\varepsilon_1(0) = 0$, we have

$$E_s^{(D_1)}(z) = \frac{\delta_{D_1,1}}{2}\sum_{n\in\mathbb{Z}}\frac{\varepsilon_K(n)y^s}{n^{2s+2k-1}} + \frac{1}{2}\sum_{n\in\mathbb{Z},m\in\mathbb{Z}\backslash\{0\},D_2|m}\frac{\varepsilon_1(m)\varepsilon_2(n)}{(mz+n)^{2k-1}}\frac{y^s}{|mz+n|^{2s}},$$

whose the first term is $\delta_{D_1,1}\cdot L(2s+2k-1,\varepsilon_K)y^s$, the second term is

$$\sum_{n\in\mathbb{Z},r\in\mathbb{Z}_{>0}}\frac{\varepsilon_1(r\delta_2)\varepsilon_2(n)}{(r\delta_2 z+n)^{2k-1}}\frac{y^s}{|r\delta_2 z+n|^{2s}} = \sum_{r>0}\sum_{j\in\mathbb{Z}/\delta_2\mathbb{Z}}\sum_{t\in\mathbb{Z}}\frac{\varepsilon_1(r\delta_2)\varepsilon_2(j)y^s}{(r\delta_2 z+t\delta_2+j)^{2k-1}|r\delta_2 z+t\delta_2+j|^{2s}}.$$

Apply the Poisson summation formula to $f_z(u) := \frac{1}{(z+u)^{2k-1}|z+u|^{2s}}$ $(z = x+iy)$ we get

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} f_z(n) &= \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \frac{e^{2\pi i n u}}{(z+u)^{2k-1}|z+u|^{2s}} du \\
&= \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \frac{e^{2\pi i n v y} e^{-2\pi i n x}}{(i+v)^{2k-1}|i+v|^{2s} y^{2s+2k-2}} dv \qquad\qquad (u = vy - x) \\
&= \sum_{n\in\mathbb{Z}} \frac{V_s(-ny)e^{-2\pi i n x}}{y^{2s+2k-2}},
\end{aligned}$$

so we have

$$\begin{aligned}
E_s^{(D_1)}(z) =& \delta_{D_1,1} \cdot L(2s+2k-1,\varepsilon_K)y^s + \sum_{r>0}\sum_{t\in\mathbb{Z}}\sum_{j\in\mathbb{Z}/\delta_2\mathbb{Z}} \frac{\varepsilon_1(r\delta_2)\varepsilon_2(j)y^s}{\delta_2^{2s+2k-1}} \frac{V_s(try)e^{2\pi i t(rx+j/\delta_2)}}{(ry)^{2s+2k-2}} \\
=& \delta_{D_1,1} \cdot L(2s+2k-1,\varepsilon_K)y^s + \sum_{r>0}\sum_{t\in\mathbb{Z}} \frac{\varepsilon_1(r\delta_2)y^s}{\delta_2^{2s+2k-1}} \frac{V_s(try)e^{2\pi i trx}}{(ry)^{2s+2k-2}} \sum_{j\in\mathbb{Z}/\delta_2\mathbb{Z}} \varepsilon_2(j)e^{2\pi i t j/\delta_2} \\
=& \delta_{D_1,1} \cdot L(2s+2k-1,\varepsilon_K)y^s + \sum_{r>0}\sum_{t\in\mathbb{Z}} \frac{\varepsilon_1(r\delta_2)y^s}{\delta_2^{2s+2k-1}} \frac{V_s(try)e^{2\pi i trx}}{(ry)^{2s+2k-2}} \varepsilon_2(t)\hbar(D_2)\sqrt{\delta_2} \\
=& \delta_{D_1,1} \cdot L(2s+2k-1,\varepsilon_K)y^s + \sum_{m\in\mathbb{Z}} \frac{\varepsilon_1(\delta_2)V_s(my)\hbar(D_2)}{\delta_2^{2s+2k-\frac{3}{2}} y^{s+2k-2}} \sum_{r|m,r>0} \frac{\varepsilon_1(r)\varepsilon_2(m/r)}{r^{2s+2k-2}} e^{2\pi i m x},
\end{aligned}$$

by the famous Gauss sum. According to the definitions of $\mathbf{U}_{|D_K|}$ and $\theta_\sigma$, substituting this result yields the expansion we want. □

*Proof.* (proof sketch of Theorem 4.4.1). According to Proposition 4.4.6, let $e_s^*(n,y) := \pi^{-s}|D_K|^s\Gamma(s+2k-1)e_s(n,y)$. We are going to show $e_s^*(n,y) = -\varepsilon_K(N)e^*_{2-2k-s}(n,y)$, for $n\in\mathbb{Z}$ satisfying $D_K|nN+l$ for some $l = \#(\mathscr{O}_K/\mathfrak{s})$, $\mathfrak{s}\in \mathrm{Art}_K^{-1}(\sigma)$. This implies the functional equation, because replace $s$ with $1-s$ yields the relationship between the Fourier coefficients of $\widetilde{\Phi}_{1-s}$ and $\widetilde{\Phi}_{s-2k+1}$:

$$e_{1-s}(n,y) = -\varepsilon_K(N)\pi^{2k-2s}|D_K|^{2s-2k}\frac{\Gamma(s)}{\Gamma(2k-s)}e_{s-2k+1}(n,y),$$

then by Proposition 4.4.5, take the Petersson inner product with $f$ leading to the conclusion we need. For $n=0$, use the facts $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$ we have

$$\begin{aligned}
e_s^*(0,y) =& \prod_{j=0}^{k-2}(s+k+j)\cdot\pi^{-s}|D_K|^s\Gamma(s+k)L(2s+2k-1,\varepsilon_K)(|D_K|y)^s \\
&+ \prod_{j=0}^{k-2}(-s-j)\cdot\varepsilon_K(N)\pi^{\frac{1}{2}-s}|D_K|^{s-\frac{1}{2}}\Gamma\Big(s+k-\frac{1}{2}\Big)L(2s+2k-2,\varepsilon_K)(|D_K|y)^{2-2k-s}
\end{aligned}$$

by Proposition 4.4.6 and Remark 4.4.7, and it is sufficient to substitute the functional equation of the Dirichlet series. For $n\neq 0$, the same argument yields

$$e_s^*(n,y) = i\varepsilon_K(N)\pi^{2k-1}|D_K|^{\frac{3}{2}-2k}|n|^k y\cdot V_s^*(ny)\sum_{d|n,d>0}\varepsilon_\sigma(n,d)\Big(\frac{|n|}{d^2}\Big)^{s+k-1},$$

then apply the functional equation of $V_s^*$ in Remark 4.4.7 will suffice.

As an example, we only calculate $L_\sigma(f,k)$ for $\varepsilon_K(N) = -1$. By Proposition 4.4.5 and Proposition 4.4.6, we set $s = 1-k$ to obtain

$$L_\sigma(f,k) = \frac{(4\pi)^k}{N^{1-k}(k-1)!}(f, \widetilde{\Phi}_{1-k})_{\mathrm{Pet}}$$

and

$$\widetilde{\Phi}_{1-k}(z) = \sum_{m,n\in\mathbb{Z}, 0\le n\le \frac{m|D_K|}{N}} e_{1-k}\left(n, \frac{Ny}{|D_K|}\right) r_\sigma(m|D_K| - nN)\exp\left(2\pi\left(\left(\frac{nN}{|D_K|} - m\right)y + imx\right)\right),$$

where

$$e_{1-k}\left(n, \frac{Ny}{|D_K|}\right) = \begin{cases} 0 & n<0 \\ \frac{2\pi}{\sqrt{|D_K|}}\frac{\#\mathrm{Cl}(\mathscr{O}_K)}{\#(\mathscr{O}_K^\times)}N^{1-k}y^{1-k} & n=0 \\ -\frac{2\pi}{\sqrt{|D_K|}}p_{k,k-1}\left(\frac{4\pi nNy}{|D_K|}\right)\widetilde{\sigma}(n)\exp\left(-\frac{2\pi nNy}{|D_K|}\right)N^{1-k}y^{1-k} & n>0 \end{cases}$$

by Remark 4.4.7 (note that when $n=0$ we have $r_\sigma(m|D_K|) = r_\sigma(m)$ since the different of $K/\mathbb{Q}$ is principal). Then, a simple calculation yields the final expansion of $\widetilde{\Phi}_{1-k}$. □

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