Robust Inventory Management: A Cycle-Based Approach

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outline

1. basic model(CB)
2. rolling-cycle implementation+ two solution approaches for CB: only extend to the backlogging model with a lead time
   - LP-based approach:general polyhedral uncertainty set
   - DP-based approach: box uncertainty set
   - implementation method
3. one-cycle look-ahead heuristic:
   1.extend to both the backlogging and lost-sales models with a lead time
   2.computationally tractable for large problem instances for two uncertainty sets

Basic model: Notation

parameters

- T: a finite discrete horizon of T periods
- K: fixed ordering cost
- c: the unit variable ordering cost
- h/s: the unit holding/shortage cost
- d_t: the stochastic demand in period t, taking values in a nonnegative bounded polyhedral uncertainty set

variables

- x_t: inventory available at the beginning of period t
- x_t^e: $x_t + u_t - d_t$ the inventory level at the end of period t
- u_t: the quantity ordered at the beginning of period t
- S_m: base stock levels
- M: circles
- w_m: m th cycle length
- $\overline{W}_m/\underline{W}_m$: the first/last period of the mth cycle

inventory cycles and cycle cost

- **inventory cycles**： a viable policy divides the planning horizon into a sequence of nonoverlapping inventory cycles, where an order is placed in the first period of a cycle and no orders are placed in the subsequent periods of the cycle.
- Consider a cycle starting in period r and lasting for w periods, let $y = x + u$ denote the inventory level right after the order u is received, then x_t^e has the following specification for t = r+1,...,r+w-1:

$$x_t^e = \begin{cases} y - \sum_{q=r}^{t} d_q, & \text{backlogging,} \\ \max\left\{y - \sum_{q=r}^{t-1} d_q, 0\right\} - d_t, & \text{lost sales.} \end{cases}$$

- let $C_{h/s}$ denote the inventory holding/shortage cost incurred over this cycle. $C_{h/s}(r, w, y, d_{r,r+w-1}) \triangleq \sum_{t=r}^{r+w-1} \max\{hx_t^e, -sx_t^e\}$

Proposition 1. The inventory holding/shortage cost $C_{h/s}(r, w, y, \mathbf{d}_{r+w-1})$ is a **piecewise-linear convex function** of $(y, \mathbf{d}_{r+w-1})$ on the set $\mathcal{B} \triangleq \{(y, \mathbf{d}_{r+w-1}) | \mathbf{d}_{r+w-1} \geq 0\}$ with w+1 linear pieces for both the backlogging and lost-sales models. The same conclusion holds for $C_{h/s} - cx_{r+w}$.

We partition $\mathcal{B} = \bigcup_{n=0}^{\omega} \mathcal{B}_n$, where

$$\mathcal{B}_n \triangleq \begin{cases} \{(y, d_{r,r+w-1}) \in \mathcal{B} \mid y \leq d_r\}, & n = 0, \\ \left\{(y, d_{r,r+w-1}) \in \mathcal{B} \mid \sum_{t=r}^{r+n-1} d_t \leq y \leq \sum_{t=r}^{r+n} d_t\right\}, & 1 \leq n \leq \omega - 1, \\ \left\{(y, d_{r,r+w-1}) \in \mathcal{B} \mid y \geq \sum_{t=r}^{r+\omega-1} d_t\right\}, & n = \omega. \end{cases}$$

- In the backlogging model, $C_{h/s}$ has the following representation on $\mathcal{B}_n$, for $0 \leq n \leq \omega$:

$$C_{h/s} = f_n \triangleq \sum_{t=r}^{r+n-1} h\left(y - \sum_{q=r}^{t} d_q\right) - \sum_{t=r+n}^{r+\omega-1} s\left(y - \sum_{q=r}^{t} d_q\right).$$

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it is straightforward to verify that $C_{h/s} = \max_{0\le n\le w}\{f_n\}$ on $\mathcal{B}$.

- cycle cost： $C_{h/s} - cx_{r+w}$

  $$C_{h/s} - cx_{r+w} = \tilde{f}_n \triangleq f_n - c\left(y - \sum_{q=r}^{r+w-1} d_q\right).$$

- In the lost-sales model $C_{h/s}$ has the following representation on B_n, for $0 \le n \le w$:

  $$C_{h/s} = g_n \triangleq \begin{cases} \sum_{t=r}^{r+n-1} h\left(y - \sum_{q=r}^{t} d_q\right) - s\left(y - \sum_{t=r}^{r+w-1} d_t\right), & 0 \le n \le w-1, \\ \sum_{t=r}^{r+w-1} h\left(y - \sum_{q=r}^{t} d_q\right), & n = w. \end{cases}$$

- ...

Basic model: objective function

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$$\min_{M,w,S} \quad \max_{d_{1,T}\in\Omega} \quad C_{total}(M, w, S, d_{1,T})$$

In the backlogging model:

$C_{\text{total}} =$

$MK + c\left(S_M - x_1 + \sum_{t=1}^{\overline{W}_{M-1}} d_t\right) + \sum_{m=1}^{M} C_{\mathrm{h/s}}\left(\underline{W}_m, w_m, S_m, d_{\underline{W}_m,\overline{W}_m}\right)$

In the lost-sales model:

$C_{total} = MK + c(S_1 - x_1) + \sum_{m=2}^{M} c(S_m - x_{\underline{W}_m}) + \sum_{m=1}^{M} C_{h/s}^{m}$

$= MK + c\left(\sum_{m=1}^{M} S_m - x_1\right) + C_{h/s}^{M} + \sum_{m=1}^{M-1}\left(C_{h/s}^{m} - cx_{\underline{W}_{m+1}}\right)$

$$
\begin{aligned}
\text{(CB)} \quad \min_{M,w,S} \quad \max_{d_{1,T}\in\Omega} \quad & C_{\text{total}} \\
\text{s.t.} \quad & 1 \le M \le T, \quad M \in \mathbb{Z}, \\
& \sum_{m=1}^{M} w_m = T, \\
& w_m \ge 1, \quad w_m \in \mathbb{Z}, \\
& S \in \mathcal{S}_{M,w},
\end{aligned}
$$

backlogging: $S \in S_{M,w} \triangleq \left\{ \begin{array}{l} S_1 > x_1, S_m > S_{m-1} \\ \\ -\min_{d_{1,T}\in\Omega} \sum_{t=W_{m-1}}^{\overline{W}_{m-1}} d_t \text{ for } m = 2, ..., M \end{array} \right\}.$

lost-sales: $S \in S_{M,w} \triangleq \left\{ \begin{array}{l} S_1 > x_1, S_m > S_{m-1} \\ \\ -\min_{d_{1,T}\in\Omega} \sum_{t=W_{m-1}}^{\overline{W}_{m-1}} d_t \textit{ and } S_m > 0 \text{ for } m = 2, .. \end{array} \right.$

- When Ω is a general polyhedral uncertainty set, we solve (CB) by first solving it with respect to S and then enumerating all feasible values for (M, w).
- For a fixed value of (M, w), we compute the constants in $S_{M,w}$ by solving LPs.
- Consider the inner maximization problem, and get a mixed-integer quadratic program by replacing the objective function with additional adversarial binary decision variables $z_{i,k}$, and linearize the quadratic term

$$\begin{aligned}
\underset{\zeta\in\mathbb{R}^m}{\text{maximize}} \quad & \sum_{i=1}^{N}\left\{\max_k \mathbf{c}_{i,k}^T\zeta + d_{i,k}\right\} && \text{(3a)}\\
\text{subject to} \quad & \mathbf{A}\zeta \leqslant \mathbf{b}, && \text{(3b)}\\
& \|\zeta\|_\infty \leqslant 1, && \text{(3c)}\\
& \|\zeta\|_1 \leqslant \Gamma. && \text{(3d)}
\end{aligned}$$

$$\max_{\{\mathbf{z}\in\{0,1\}^{N\times K} \mid \sum_{k=1}^{K} z_{i,k}=1, \forall i\}} \sum_{i=1}^{N}\sum_{k=1}^{K} z_{i,k}(\mathbf{c}_{i,k}^T\zeta + d_{i,k}),$$

Rolling-Cycle Implementation

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- reoptimization is carried out at the beginning of each cycle
- at the beginning of the first cycle, we solve a modified version of (CB) to compute (M, w, S), but only implement w_1 and S_1
- at the beginning of a cycle that starts in period r, we formulate (CB_R) for the horizon consisting of periods r, r + 1, ... , T, compute the lengths and base-stock levels for all cycles, but again only implement the decisions for the current cycle

One-Cycle Look-Ahead Heuristic

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- robust cycle–based policy + its rolling-cycle implementation :
  1. computationally tractable for large problem instances only when the uncertain demand belongs to a box uncertainty set;
  2.only be tractably extended to the backlogging model with a positive lead time
- hence,propose a one-cycle lookahead heuristic + rolling-cycle implementation
- one-cycle lookahead heuristic
  1.solves an optimization problem at the beginning of each cycle to compute the cycle length w and base-stock level S
  2.Instead of adopting the total cost incurred over periods r, r + 1,..., T as the cost function, the heuristic considers C_{cycle} and a term cx_{r+w} that approximates the impact of (w, S) on the cost incurred over the future cycles

$$\mathbb{C}_{\text{cycle}} = \begin{cases} K1(S-x>0)+c(S-x)+\mathbb{C}_{h/s}(r,w,S,\mathbf{d}_{r,r+w-1})-\mathrm{c}x_{r+w}, \\ \quad r+w-1<T, \\ K1(S-x>0)+c(S-x)+\mathbb{C}_{h/s}(r,w,S,\mathbf{d}_{r,r+w-1}), \\ \quad r+w-1=T. \end{cases}$$

(CB_H)

$$\min_{w\in\{1,2,\ldots,T-r+1\},u\geq 0} \max_{d_{r,r}\in\Omega_r} \frac{C_{\text{cycle}}}{w}.$$

We solve (CB_H) by enumerating over w, and solving

$$\min_{u\geq 0} \max_{d_{r,r}\in\Omega_r} \frac{C_{\text{cycle}}}{w},$$

for each w. Proposition 2 shows that this can be solved extremely efficiently by exploiting the simple structure of C_{cycle}.

proposition 2 : the above problem can be solved using two LPs in both the backlogging and lost-sales models.

$$\min\left\{\min_{u\geq 0}\max_{\mathbf{d}_{r,T}\in\Omega_r}\frac{K+cu+C_{\mathrm{h/s}}(r,w,x+u,\mathbf{d}_{r,r+w-1})}{w},\ \max_{\mathbf{d}_{r,T}\in\Omega_r}\frac{C_{\mathrm{h/s}}(r,w,x,\mathbf{d}_{r,r+w-1})}{w}\right\}. \tag{A2}$$

$C_{h/s} = \max_{0\leq n\leq w}\{f_n\}$, where f_n's are linear functions of $(u, d_{r,r+w-1})$.

$$\begin{aligned} \min_{u\geq 0}\quad & z \\ \text{s.t.}\quad & z\geq \alpha_n u+\gamma_n+\max_{\mathbf{d}_{r,T}\in\Omega_r}\boldsymbol{\beta}_n^\top\mathbf{d}_{r,T}, \qquad \text{for } n=0,\ldots,w. \end{aligned} \tag{A4}$$

Since Ω_r is a bounded polyhedron, strong duality implies that $\max_{\mathbf{d}_{r,T}\in\Omega_r}\boldsymbol{\beta}_n^\top\mathbf{d}_{r,T} = \min_{\Delta_r^\top\mathbf{v}_n=\boldsymbol{\beta}_n,\,\mathbf{v}_n\geq 0}\delta_r^\top\mathbf{v}_n$. Replacing $\max_{\mathbf{d}_{r,T}\in\Omega_r}\boldsymbol{\beta}_n^\top\mathbf{d}_{r,T}$ with its dual

$$\begin{aligned} \min_{u\geq 0}\quad & z \\ \text{s.t.}\quad & z\geq \alpha_n u+\gamma_n+\boldsymbol{\delta}_r^\top\mathbf{v}_n,\ \boldsymbol{\Delta}_r^\top\mathbf{v}_n=\boldsymbol{\beta}_n,\ \mathbf{v}_n\geq 0, \qquad \text{for } n=0,\ldots,w. \end{aligned} \tag{A5}$$

Extension with a Positive Lead Time

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- The main difference introduced by L is that the decisions w and u now determine the inventory dynamics in periods r + L, ... , r + w - 1 + L instead of periods r, ... , r + w - 1, and hence should be associated with the inventory holding/shortage cost incurred in periods r + L, ..., r + w - 1 + L
- the order u is received in period r + L and no order is received in the subsequent periods r + L + 1, ... , r + w - 1 + L, this cost can be represented by $C_{h/s}(r+L, w, x_{r+L}, d_{r+L,r+L+w-1})$
- a heuristic term cx_{r+w+L} to approximate the impact on the cost incurred over the future cycles

Extension with a Positive Lead Time

$$C^L_{\text{cycle}} =$$
$$\begin{cases} K\mathbb{1}(u>0) + cu + C_{\text{h/s}}(r+L, w, x_{r+L}+u, d_{r+L,r+w-1+L}) - cx_{r+w+L}, r+w- \\ K\mathbb{1}(u>0) + cu + C_{\text{h/s}}(r+L, w, x_{r+L}+u, d_{r+L,r+w-1+L}), r+w-1 = T. \end{cases}$$

Here, $x_{r+w+L} = x_{r+L} + u - \sum_{q=r+L}^{r+w-1+L} dq$ in the backlogging model, and $x_{r+w+L} = \max\{x_{r+L} + u - \sum_{q=r+L}^{r+w-1+L} dq, 0\}$ in the lost-sales model.
We propose the following optimization problem for determining w and u:

$$(CB^L_H) \qquad \min_{w\in\{1,2,\ldots,T-r+1\}, u\geq 0} \; \max_{d_{r,T+L}\in\Omega^L_r} \frac{C^L_{\text{cycle}}}{w}.$$

Here $\Omega^L_r = \{d_{r,T+L} \in \mathbb{R}^{T-r+1+L} | \Delta^L_r d_{r,T+L} \leq \delta^L_r\}$ is a non-negative bounded polyhedral uncertainty set.

We solve (CB_H^L) by first solving the optimization problem

$$\min_{u \geq 0} \max_{d_{r,T+L} \in \Omega_r^L} \frac{C_{\text{cycle}}^L}{w},$$

for fixed w and then using a simple enumeration to compute the optimal w and its associated u.

Proposition 5. This problem can be solved using two LPs in both the backlogging and lost-sales models.

References I

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