Homework 21

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Consider the following covariance matrix:

$$\Sigma = \begin{bmatrix} 6 & -3 \\ -3 & 4 \end{bmatrix}$$

(A) Find the SVD of Σ by hand and show all steps of your derivation.

```r
# Define the covariance matrix
Sigma <- matrix(c(6, -3, -3, 4), nrow = 2)
Sigma
```

```
     [,1] [,2]
[1,]    6   -3
[2,]   -3    4
```

```r
# Calculate Sigma * Sigma^T
Sigma %*% t(Sigma)
```

```
     [,1] [,2]
[1,]   45  -30
[2,]  -30   25
```

```r
# Compute the eigenvalues of Sigma
lambda1 <- 5*(7 + 2 * sqrt(10))
lambda2 <- 5*(7 - 2 * sqrt(10))
```

```r
# Print the eigenvalues
cat("The eigenvalues of Sigma are", lambda1, "and", lambda2, "\n")
```

```
The eigenvalues of Sigma are 66.62278 and 3.377223
```

```r
# Compute the eigenvectors corresponding to lambda1
x11 <- 1
x12 <- (45 - lambda1) /30
v1_norm = sqrt(x11^2 + x12^2)
v11 <- x11/v1_norm
v12 <- x12/v1_norm

# Print the eigenvector corresponding to lambda1
cat("The eigenvector corresponding to lambda1 is [", v11, v12, "]\n")
```

```
The eigenvector corresponding to lambda1 is [ 0.8112422 -0.5847103 ]
```

```r
# Compute the eigenvectors corresponding to lambda2
x21 <- 1
x22 <- (45 - lambda2) /30
v2_norm = sqrt(x21^2 + x22^2)
v21 <- x21/v2_norm
v22 <- x22/v2_norm
# Print the eigenvector corresponding to lambda2
cat("The eigenvector corresponding to lambda2 is [", v21, v22, "]\n")
```

```
The eigenvector corresponding to lambda2 is [ 0.5847103 0.8112422 ]
```

```r
# Construct the matrix V using the eigenvectors
V <- matrix(c(v11, v21, v12, v22), nrow = 2, byrow = TRUE)
V
```

```
           [,1]      [,2]
[1,]  0.8112422 0.5847103
[2,] -0.5847103 0.8112422
```

```r
# Calculate singular values
d <- matrix(c(sqrt(lambda1),0, 0, sqrt(lambda2)),nrow = 2 )
d
```

```
         [,1]     [,2]
[1,] 8.162278 0.000000
[2,] 0.000000 1.837722
```

```r
# Calculate U
u <- Sigma %*% V %*% solve(d)
u
```

```
           [,1]      [,2]
[1,]  0.8112422 0.5847103
[2,] -0.5847103 0.8112422
```

```r
# check
u %*% d %*% t(V)
```

```
     [,1] [,2]
[1,]    6   -3
[2,]   -3    4
```

(C) Write R codes to verify the SVD of Σ.

```r
# Verify the SVD of Sigma using the svd function
Sigma_svd <- svd(Sigma)
Sigma_svd
```

```
$d
[1] 8.162278 1.837722

$u
           [,1]      [,2]
[1,] -0.8112422 0.5847103
[2,]  0.5847103 0.8112422
```

```
$v
           [,1]      [,2]
[1,] -0.8112422 0.5847103
[2,]  0.5847103 0.8112422
```

```
U <- Sigma_svd$u
D <- diag(Sigma_svd$d)
V <- Sigma_svd$v
U %*% D %*% t(V)
```

```
     [,1] [,2]
[1,]    6   -3
[2,]   -3    4
```

(A) Use the SVD of Σ to find Σ^{-1} by hand and show all steps of your derivation.

```
solve(d)
```

```
          [,1]      [,2]
[1,] 0.1225148 0.0000000
[2,] 0.0000000 0.5441518
```

```
t(U)
```

```
           [,1]      [,2]
[1,] -0.8112422 0.5847103
[2,]  0.5847103 0.8112422
```

```
V %*% solve(d) %*% t(U)
```

```
          [,1] [,2]
[1,] 0.2666667  0.2
[2,] 0.2000000  0.4
```

(C) Write R codes to verify Σ^{-1}.

```r
solve(Sigma)
```

```
          [,1] [,2]
[1,] 0.2666667  0.2
[2,] 0.2000000  0.4
```