Homework 17

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[A] Calculate E[X1X2].

```r
M <- matrix(c(0.2,0.1,0.3,0.3,0.05,0.05), nrow = 2, byrow = TRUE)
X1 <- c(1, 2)
X2 <- c(1, 2, 3)
EX1X2 <- sum(X1 %*% t(X2) * M)
EX1X2
```

```
[1] 2.4
```

[A] Calculate the conditional distributions of X1 | X2 and X2 | X1

```r
# conditional distribution of X1 given X2
PX1_given_X2 <- t(t(M) / colSums(M))
PX1_given_X2
```

```
     [,1]      [,2]      [,3]
[1,]  0.4 0.6666667 0.8571429
[2,]  0.6 0.3333333 0.1428571
```

```r
# conditional distribution of X2 given X1
PX2_given_X1 <- M / rowSums(M)
PX2_given_X1
```

```
          [,1]      [,2]  [,3]
[1,] 0.3333333 0.1666667 0.500
[2,] 0.7500000 0.1250000 0.125
```

```
cat("\n")
```

```
#conditional distribution of X1 given X2
for (i in 1:2) {
  for (j in 1:3) {
    cat(paste0("Pr(X1 = ", i, " | X2 = ", j, ") = ", M[i, j], " / ", colSums(M)[j], " = ", PX1_
  }
}
```

```
Pr(X1 = 1 | X2 = 1) = 0.2 / 0.5 = 0.4
Pr(X1 = 1 | X2 = 2) = 0.1 / 0.15 = 0.666666666666667
Pr(X1 = 1 | X2 = 3) = 0.3 / 0.35 = 0.857142857142857
Pr(X1 = 2 | X2 = 1) = 0.3 / 0.5 = 0.6
Pr(X1 = 2 | X2 = 2) = 0.05 / 0.15 = 0.333333333333333
Pr(X1 = 2 | X2 = 3) = 0.05 / 0.35 = 0.142857142857143
```

```
cat("\n")
```

```
#conditional distribution of X2 given X1
for (i in 1:2) {
  for (j in 1:3) {
    cat(paste0("Pr(X2 = ", j, " | X1 = ", i, ") = ", M[i, j], " / ", rowSums(M)[i], " = ", PX2_
  }
}
```

```
Pr(X2 = 1 | X1 = 1) = 0.2 / 0.6 = 0.333333333333333
Pr(X2 = 2 | X1 = 1) = 0.1 / 0.6 = 0.166666666666667
Pr(X2 = 3 | X1 = 1) = 0.3 / 0.6 = 0.5
Pr(X2 = 1 | X1 = 2) = 0.3 / 0.4 = 0.75
Pr(X2 = 2 | X1 = 2) = 0.05 / 0.4 = 0.125
Pr(X2 = 3 | X1 = 2) = 0.05 / 0.4 = 0.125
```

[C] Use Gibbs sampling to simulate a Markov Chain of length m = 10000 that starts at X1 = 1, X2 = 1 and evolves according to the conditional distributions of X1 | X2 and X2 | X1. Use set.seed(440) in your simulation.

```r
set.seed(440)
m = 10000
X1 = 1
X2 = 1

# initialize matrix to store Markov chain
chain <- matrix(0, nrow = m, ncol = 2)
chain[1, ] <- c(X1, X2)
#Gibbs sampling with conditional distribution
for (i in 2:m) {
  # sample X1 given X2
  X1 <- sample(1:2, size = 1, prob = PX1_given_X2[,X2])

  # sample X2 given X1
  X2 <- sample(1:3, size = 1, prob = PX2_given_X1[X1,])

  # store current values in chain
  chain[i, ] <- c(X1, X2)
}
```

[C] Estimate Pr(X1 = 2, X2 = 1) using your simulated Gibbs samples from the previous part, and then compare it with the theoretical value.

```r
# estimate Pr(X1 = 2, X2 = 1)
count <- sum(chain[, 1] == 2 & chain[, 2] == 1)
est_prob <- count / m
cat(paste0("Estimated Pr(X1 = 2, X2 = 1) = ", est_prob, "\n"))
```

```
Estimated Pr(X1 = 2, X2 = 1) = 0.2971
```

```r
cat(paste0("Theoretical Pr(X1 = 2, X2 = 1) = ", M[2,1], "\n"))
```

```
Theoretical Pr(X1 = 2, X2 = 1) = 0.3
```

[C] Estimate E[X1X2] using your simulated Gibbs samples from the previous part, and then compare it with the theoretical value.

```r
# estimate E[X1X2]
est_EX1X2 <- mean(chain[, 1] * chain[, 2])
cat(paste0("Estimated E[X1X2] = ", est_EX1X2, "\n"))
```

```
Estimated E[X1X2] = 2.4132
```

```r
cat(paste0("Theoretical E[X1X2] = ", EX1X2, "\n"))
```

```
Theoretical E[X1X2] = 2.4
```