APPLIED MATHEMATICS II

CHAPTER-1

REAL SEQUENCES AND SERIES

Chapter one contents

1. REAL SEQUENCES AND SERIES

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1. 1 REAL SEQUENCES

1.1 Definitions And Notions

OBSI JUNIOR
JIT

Definition 1.1.1

A sequence is a set of numbers a_1 , a_2 , a_3 , ... , an, . . . in a definite order of arrangment and formed according to a definite rule.

The number a_1 is called the first term, a_2 is the second term, and in general a_n is the $\boldsymbol{n^{th}}$ **term** of the sequence.

Cont'd

Definition 1.1. 2

A **real *sequence*** is defined to be a real-valued function whose domain is a set of positive integers.

Examples:1.1.1

a) The set of numbers 1,6,11,16,… is an infinite sequence with n^{th} term a_n=5n-4, n=1,2,3,….

b) The set of numbers $\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\ldots$ is an infinite sequence with n^{th} term $a_n=\frac{1}{2^n}$, n=1,2,3,…..

Cont'd

Notation

The sequence $\{a_{1,}a_{2,}\, a_{3,}\ldots\}$ is denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

Sequences can be described by one of the following methods

i) listing the terms.

ii) Writing a formula for the n^{th} term.

iii) drawing graphs.

iv) using recurrence relations.

cont'd

a) $\{\frac{n}{n+2}\}_{n=1}^{\infty}$ $\quad a_n = \frac{n}{n+2}$ $\quad \{\frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \ldots, \frac{n}{n+2}, \ldots\}$

b) $\{(-1)^n(\frac{n}{n^2-1})\}_{n=2}^{\infty}$ $\quad a_n = (-1)^n\left(\frac{n}{n^2-1}\right)$

$\{\frac{2}{3}, -\frac{3}{8}, \frac{4}{15}, \ldots, (-1)^n\left(\frac{n}{n^2-1}\right), \ldots\}$

c) $\{\sin(\frac{n\pi}{2})\}_{n=1}^{\infty}$ $\quad a_n = \sin(\frac{n\pi}{2})$

$\{1, 0, -1, \ldots, \sin(\frac{n\pi}{2}), \ldots\}$

d) $a_{n+1} = \sqrt{3a_n}, a_1 = 1$

Arithmetic Sequence

Definition 1.1.3

An **arithmetic sequence** is a sequence in which the difference between consecutive terms is a constant. This constant is called the **common difference**.

Example. $-3, -1, 1, 3, 5, 7, 9, \ldots$ *is arithmetic sequence with* $d = 2$

Formula for the n^{th} term of Arithmetic sequence

If $\{A_{n,}\}$ is an arithmetic sequence with common difference d, then the n^{th} term is given by

$$\boldsymbol{A_n = A_1 + (n-1)d}$$

Example 1.1.3

Given an arithmetic sequence with first term 5 and common difference 4. Find the twentieth term.

Solution

Given : $A_1 = 5$ and $d = 4$ and

$$A_n = A_1 + (n-1)d$$

Therefore

$$A_{20} = 5 + (20-1)4 = 5 + 76 = 81$$

Geometric sequence

Definition1.1.4 A sequence is **geometric** if

$$\frac{\text{each term}}{\text{previous term}} = r$$

where r is a constant called the **common ratio**

Example1.1.4 a) The sequence 2, 8, 32, 128, 512, . . .

is geometric with **common ratio**, $r = 4$.

b) 1, 2, 4, 8, 16 … is a geometric sequence with first

term 1 and common ratio 2.

Formula for the n^{th} term of geometric sequence

The $\boldsymbol{n^{th}}$ **term**of a geometric sequence has the form

$$a_n = a_1 r^{n-1}$$

where a_1 *is the first term and* r is the common ratio of consecutive terms of the sequence.

Example 1.1.5

1) Find the seventh term of a geometric sequence whose first term is 4 and whose fourth term is 108.

Cont'd

Solution: First, you need to find the common ratio r,

by using the formula $a_n = r^{n-1}a_1$

$\Longrightarrow \quad a_4 = r^3 a_1$ where $a_1 = 4$ and $a_4 = 108$

$$\Rightarrow 108 = 4r^3 \qquad \Leftrightarrow \quad r^3 = 27$$

This gives $r = 3$

Thus, $a_7 = (4)(3)^6 = 2,916$

Cont'd

2. Find a_5 and a_n for the geometric sequence

$$4, -12, 36, -108, \ldots$$

Solution Here $a_1 = 4$ and $r = 36/-12 = -3$.

Using $n = 5$ in the formula

$$a_n = a_1 r^{n-1}$$

$$a_5 = 4 \cdot (-3)^{5-1} = 4 \cdot (-3)^4 = 324$$

In general

$$a_n = a_1 r^{n-1} = 4 \cdot (-3)^{n-1}$$

Cont’d

Exercise1.1

a) Find the 9th term of the geometric sequence

7, 21, 63, . . .

b) A population of fruit flies grows in such a way that each generation is 1.5 times the previous generation. There were 100 insects in the first generation. How many are in the fourth generation?

1.2 Bounded And Monotonic Sequences

1.2.1 Monotonic Sequences

Definition 1.2.1 A sequence $\{a_n\}$ is called

- **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$.
- **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$.
- **monotonic** if it is either increasing or decreasing.

Example1.2.1 The sequence $\left\{\frac{3}{n+5}\right\}$ is decreasing $\forall n \geq 1$

And the sequence $\left\{\frac{2^n-1}{2^n}\right\}$ is increasing $\forall n \geq 1$

cont'd

Example 1.2.2. Show that the sequence $a_n = \left(\frac{n}{n^2+1}\right)$ is decreasing.

Solution : The sequence is decreasing if

$$a_n > a_{n+1} \quad \Rightarrow \quad \frac{n}{n^2+1} > \frac{n+1}{(n+1)^2+1}$$

$$\Leftrightarrow \; n[(n+1)^2+1] > (n^2+1)(n+1)$$

$$\Leftrightarrow \; n^2+n > 1$$

Since $n \geq 1$, the inequality $n^2+n > 1$ is true.

Therefore, $a_n > a_{n+1}$

Hence , by definition $\{a_n\}$ is decreasing.

cont'd

Alternative method . Consider the function

$$f(x) = \frac{x}{x^2+1}$$

$$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0 \quad \text{if} \quad x^2 > 1$$

Thus, f is decreasing on $(1, \infty)$ by FDT .

so $f(n) = a_n > f(n+1) = a_{n+1}$ on $[1, \infty)$.

Therefore, $\{a_n\}$ is decreasing

Exercise 1.2

1. Determine whether the sequences are monotone or not.

a) $a_n = \frac{1}{5^n}$

b) $a_n = ne^{-n}$

c) $a_n = (-1)^2 n$

d) $a_n = \frac{\sqrt{n}}{n+1}$

e) $a_n = \left(1 + \frac{1}{n}\right)^{n+1}$

f) $a_{n+1} = \sqrt{3\, a_n}$, $a_1 = 1$

2. *Show that the sequence* $\{\frac{10^n}{n!}\}_{n=1}^{\infty}$ *is eventually decreasing.*

1.2.2 Bounded Sequences

Definition 1.2.2 A sequence $\{a_n\}$

- is ***bounded above*** if there is a number M such that

  $a_n \leq M$ for all $n \geq 1$.

- is ***bounded below*** if there is a number m such that

  $m \leq a_n$ for all $n \geq 1$.

- is **a *bounded*** *sequence* if it is bounded above and below.

cont'd

Example 1.2.3.

a) The sequence $a_n = n$ is bounded below $(a_n \geq 1)$ but not above.

b) The sequence $a_n = \frac{n}{n+1}$ is bounded because $0 < a_n < 1$ for all $n \geq 1$.

c) The sequence $a_n = (-1)^n$ is bounded because $-1 \leq a_n \leq 1$ for all $n \geq 1$.

cont'd

Exercise

1. *Prove that the sequence with* n^{th} *term* $a_n = \frac{2n-7}{3n+2}$

a) *Is monotonic increasing*

b) *Is bounded*

c) *has a limit*

2. *Prove that a convergent sequence is bounded.*

1.3 Convergence Of Sequences

Definition1.3.1 A sequence $\{a_n\}$ has the limit L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or } a_n \to L \quad \text{as} \quad n \to \infty$$

if for every ε there is a corresponding integer $N > 0$ such that if $n > N$,then $|a_n - L| < \varepsilon$.

If the limit L of the sequence exists, we say the sequence **converges** . Otherwise we say the sequence **diverges .**

Cont'd

Example 1.3.1

The sequence

a) $\{1, {}^1/_2, {}^1/_3, \ldots\}$ and $\{\frac{3n+1}{n}\}_{n=1}^{\infty}$ *are convergent.*

b) $\{1, 2, 3, 4, \ldots\}$ *and* $\{(-1)^n\}_{n=1}^{\infty}$ *are divergent.*

c) $\{-1, 1, -1, 1, -1, 1, -1, \ldots\}$ *is* divergent .

Cont'd

Example 1.3.2 Determine if the sequence $\{\frac{4n^2}{2n^2+1}\}$ is convergent or divergent.

Solution: The sequence converges if $\lim\limits_{n\to\infty} \frac{4n^2}{2n^2+1}$ exists.

$$\lim_{n\to\infty} \frac{4n^2}{2n^2+1} = \lim_{n\to\infty} \frac{4}{2+\frac{1}{n^2}} = 2$$

Therefore, by definition the given sequence is convergent.

Cont'd

Example1.3.3

Determine whether the sequence $\{n\sin\left(\frac{\pi}{n}\right)\}$ is convergent or divergent.

Solution: The sequence converges if $\lim\limits_{n\to\infty} n\sin\left(\frac{\pi}{n}\right)$ exists. Let $f(x) = \lim\limits_{x\to\infty} x\sin\left(\frac{\pi}{x}\right)$.

It can be written as

$$f(x) = \lim_{x\to\infty} x\sin\left(\frac{\pi}{x}\right) = \lim_{x\to\infty} \frac{\sin\left(\frac{\pi}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{-\frac{\pi}{x^2}\cos\left(\frac{\pi}{x}\right)}{-\frac{1}{x^2}} = \pi$$

Therefore, the given sequence converges to π.

Cont’d

Exercise 1.3.1

1. Show that a sequence $a_n=\sqrt[n]{n}$ *is convergent.*

2. By using the definition of convergence prove that

a) $\lim\limits_{n\to\infty} \dfrac{n}{2n-3} = \frac{1}{2}$ **b**) $\lim\limits_{n\to\infty}\left(\dfrac{4n^2+8}{2n^2+n}\right) = 2$

3. Prove that a convergent sequence has a unique limit.

Definition 1.3.2 $\lim\limits_{n\to\infty} a_n = \infty$ means that for every positivenumber M there is an integer N such that $a_n > M$ whenever $n > N$.

Limit laws for sequences

If $\{a_n\}$ and $\{b_n\}$are convergent sequences and c is a constant, then

i. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm$

ii. $\lim_{n\to\infty}(a_n\, b_n) = \lim_{n\to\infty} a_n \,.\, \lim_{n\to\infty} b_n$

iii. $\lim_{n\to\infty}(c\, a_n) = c \lim_{n\to\infty} a_n$

iv. $\lim_{n\to\infty}(\frac{a_n}{b_n}) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, provided $\lim_{n\to\infty} b_n \neq 0$

v. $\lim_{n\to\infty} a_n{}^{\boldsymbol{p}} = [\lim_{n\to\infty} a_n]^{\boldsymbol{p}}$ if $p > 0$ and $a_n > 0$

The Squeeze Theorem

If $a_n \le b_n \le c_n$ for $n > N$ and

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$,then $\lim_{n\to\infty} b_n = L.$

Example 1.3.4 *Show that* $\lim_{n\to\infty} \frac{sin^2 n}{n} = 0$

solution. $0 \le \sin^2 n \le 1,$ *so for* $n > 0$

$$0 \le \frac{\sin^2 n}{n} \le \frac{1}{n} \quad \Rightarrow 0 \le \lim_{n\to\infty}\left(\frac{\sin^2 n}{n}\right) \le 0$$

Hence, By squeesing theorem $\lim_{n\to\infty}\left(\frac{\sin^2 n}{n}\right) = 0$

cont'd

Theorem 1.2

If $\lim_{n\to\infty}|a_n| = 0$,then $\lim_{n\to\infty} a_n = 0.$

Example 1.3.5 Evaluate $\lim_{n\to\infty}\frac{(-1)^n}{n}$ if it exists.

Solution $\lim_{n\to\infty}\left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty}\frac{1}{n} = 0$

Therefore, by the above theorem

$$\lim_{n\to\infty}\frac{(-1)^n}{n} = 0$$

Monotonic Sequence Theorem

Theorem1. 3 : Every bounded, monotonic sequence is convergent. proof(exercise)

Example 1.3.6

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence defined by the recurrence relation

$$a_1 = 1 \quad , \quad a_{n+1} = \frac{1}{2}\left(5 - \frac{2}{a_n}\right) \quad , \forall\, n \geq 1$$

i. Show that $\{a_n\}_{n=1}^{\infty}$ is convergent.

ii. Find $\lim\limits_{n\to\infty} a_n$

cont'd

Solution: We begin by computing the first few terms:

$$a_1 = 1, a_2 = \tfrac{1}{2}(5 - \tfrac{2}{a_1}) = \frac{3}{2}, a_3 = \tfrac{1}{2}(5 - \tfrac{2}{a_2}) = \frac{11}{6}, \ldots$$

These initial terms suggest that the sequence is increasing and the terms are approaching 2.

To confirm that the sequence is increasing

($a_n < a_{n+1} \forall n \geq 1$) we use mathematical induction.

It is true for $n = 1$ because $a_2 = \frac{3}{2} > a_1 = 1$.

cont'd

If we assume that it is true for $n = k$, then we have

$$a_k < a_{k+1}$$

$$-\frac{2}{a_k} < -\frac{2}{a_{k+1}}$$

$$\Leftrightarrow \quad 5 - \frac{2}{a_k} < 5 - \frac{2}{a_{k+1}}$$

$$\Leftrightarrow \frac{1}{2}(5 - \frac{2}{a_k}) < \frac{1}{2}(5 - \frac{2}{a_{k+1}})$$

Thus, $a_{k+1} < a_{k+2}$

cont'd

We have deduced that $a_n < a_{n+1}$ is true for $n = k + 1$.

Therefore the inequality is true for all n by induction.

Hence , the sequence is increasing.

Next, we verify that $\{a_n\}$ is bounded by showing that

$a_n < 2 \ \forall n$. Since the sequence is increasing, we already

know that it has a lower bound: $a_n \geq a_1 = 1$

We know that $a_1 < 2$, so the assertion is true for $n = 1$.

cont'd

Suppose it is true for $n = \text{k}$.

Then

$$a_k < 2$$

$$\Rightarrow -\frac{2}{a_k} < -1$$

$$\Leftrightarrow\ 5 - \frac{2}{a_k} < 5 - 1$$

$$\Leftrightarrow\ \frac{1}{2}\left(5 - \frac{2}{a_k}\right) < \frac{1}{2}(4) = 2$$

Thus $a_{k+1} < 2$

This shows, by mathematical induction, that $a_n < 2$ for all n. $\Rightarrow \{a_n\}$ is bounded

cont'd

Therefore, by monotonic sequence theorem it is convergent and it has a limit.

$\Longrightarrow \lim_{n\to\infty} a_n$ exists and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \mathrm{L}$

$$\Longrightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(5 - \frac{2}{a_n}\right) = \frac{1}{2}\left(5 - \frac{2}{\lim_{n\to\infty} a_n}\right)$$

$$\Leftrightarrow \frac{1}{2}\left(5 - \frac{2}{L}\right) = L$$

$\Leftrightarrow 2L^2 - 5L + 2 = 0 \quad \Leftrightarrow L = 2$ or $L = \frac{1}{2}$

Therefore, $\lim_{n\to\infty} a_n = 2$, as predicted.

Exercise 1.3.2

Determine whether the sequence converges or diverges. If it converges, find the limit.

a. $a_n = \frac{(-1)^{n-1}n}{n^2+1}$

b. $a_n = \left\{\frac{\ln n}{\ln 2n}\right\}$

c. $a_n = \frac{\sqrt{n}}{\sqrt{n}+1}$

d) $a_n = \frac{cos^2 n}{2^n}$

e) $a_n = (1 + \frac{2}{n})^{\frac{1}{n}}$

f. $a_1 = 2, a_{n+1} = \frac{1}{3-a_n}$

1.4 Real Series

Definition 1.4.1

- **A series** is the sum of the terms of a sequence.

i.e The sum $S = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$

is an infinite series

- The sum of the first n terms

$S_n = \sum_{n=1}^{n} a_n = a_1 + a_2 + a_3 + \cdots + a_n$ of a sequence

$\{a_n\}$ is called the ***n^{th}* partial sum.**

Geometric Series

A series of the form

$a + ar + ar^2 + ar^3 + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$ is called **geometric series** , where a and r are constants.

If a G.S has **first term *a_1*** and **common ratio *r***, then the sum of the first *n* terms is given by

$$S_n = \frac{a_1(1-r^n)}{1-r} \quad \text{where} \quad r \neq 1 \quad \text{and}$$

$$S_\infty = \frac{a_1}{1-r} \qquad \text{where } -1 < \mathrm{r} < 1$$

Cont'd

Example 1. Evaluate

$$\sum_{n=1}^{5} 3(2)^n$$

Solution For our series

$$3(2) + 3(2)^2 + 3(2)^3 + 3(2)^4 + 3(2)^5$$

$a_1 = 6, \; r = 2$ and $n = 5$ $\quad using \; S_n = \dfrac{a(r^n - 1)}{r - 1}$

$$S_5 = \frac{6(2^5 - 1)}{2 - 1} = \frac{6(31)}{1} = 186$$

2. $Find \; \displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n$

Solution: Here $a_1 = \frac{3}{5} = \mathrm{r}$ so

$$\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n = \frac{a_1}{1 - r} = \frac{\frac{3}{5}}{1 - \frac{3}{5}} = \frac{3}{2}$$

Harmonic Series

A series of the form

$$\frac{1}{m}+\frac{1}{m+d}+\frac{1}{m+2d}+\ldots.$$

Where m , d are numbers such that the denominators are never zero is called **harmonic Series.**

Example. The series

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n}$$

Is a harmonic series.

Note: Harmonic series is never an integer.

P-series

A series of the form

$$1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^p}$$

Is called **p-series** where $p\ any\ real\ number.$

Example . $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^2}$

is a p- series with p= 2.

Note : A series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^p}$ is alternating p-series

Telescoping series

A series $\sum_{n=1}^{\infty} a_n$ with a_n of the form $a_n = b_n - b_{n+1}$ is called a telescoping Series.

For a telescoping series

$S_n = \sum_{n=1}^{n} a_n = \sum_{n=1}^{n}(b_n - b_{n+1}) = b_1 - b_{n+1}$

Hence a telescoping series converges if

$\lim_{n \longrightarrow \infty} S_n$ exists.

Convergence of series

Let $S_n = \sum_{n=1}^{n} a_n = a_1 + a_2 + a_3 + \cdots + a_n$

be the $\boldsymbol{n^{th}}$ **partial sum** of a sequence $\{a_n\}$.

Then the series $\sum_{n=1}^{\infty} a_n$ **converges** if $\lim_{n\to\infty} S_n$ exists,

and $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$.

Otherwise we say the series **diverges**.

Cont'd

Example1.4.1. Show that the geometric series $\sum_{n=0}^{\infty} r^n$ converges to $\frac{1}{1-r}$ if $|r| < 1$ and diverge if $|r| \geq 1$

Solution. $S_n = 1 + r + r^2 + r^3 + \cdots + r^{n-1} + r^n$ …(i)

$rS_n = r + r^2 + r^3 + \cdots + r^{n-1} + r^n + r^{n+1}$ …(ii)

$$\Longrightarrow S_n(1-r) = 1 - r^{n+1}$$

$$\Longrightarrow S_n = \frac{1}{1-r} - \frac{r^{n+1}}{1-r}, r \neq 1 \Longrightarrow \lim_{n\to\infty} S_n = \frac{1}{1-r}$$

Therefore, the series $\sum_{n=0}^{\infty} r^n$ converges to $\frac{1}{1-r}$ for $|r| < 1$.

Cont'd

Example1.4.2

1. Show that the repeating decimal geometric series

$$0.0808\overline{08}=\frac{8}{10^2}+\frac{8}{10^4}+\frac{8}{10^6}+\frac{8}{10^8}+\ldots \textit{converges}$$

$$a=\frac{8}{10^2}\ \textit{and}\ \ r=\frac{1}{10^2} \qquad \sum_{n=1}^{\infty}ar^{n-1}=\sum_{n=1}^{\infty}\frac{8}{10^2}\left(\frac{1}{10^2}\right)^{n-1}$$

$$\sum_{n=1}^{\infty}ar^{n-1}=\frac{a}{1-r}=\frac{\frac{8}{10^2}}{1-\frac{1}{10^2}}=\frac{8}{99}$$

The repeating decimal is converges to 8/99.

Cont'd

2. Show that $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\cdots$ converges.

Solution. The n^{th} term of the sequence is $\frac{1}{n(n+1)}$

Using partial fraction $a_n=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{(n+1)}$

Hence , the series can be written as $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=$

$$(1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+\ldots.+(\frac{1}{m-1}-\frac{1}{m})+(\frac{1}{m}-\frac{1}{(m+1)})+\ldots$$

$$\Longrightarrow S_m=1-\frac{1}{(m+1)} \qquad \Longrightarrow \lim_{m\to\infty} S_m=1$$

Therefore, by definition the telescoping series $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ converges to 1.

Cont'd

Exercise

1. Determine if the following series converges or diverges. If it converges find its value.

a) $\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3}$ b) $\sum_{n=1}^{\infty} \frac{n}{n+1}$

2. Prove that $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} < 1$

Properties of convergent infinite series

Theorem 1.4

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series and k is a real number, then

i. $\sum_{n=1}^{\infty} ka_n = k\sum_{n=1}^{\infty} a_n$

ii. $\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

Example1.4.3. Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{2}{n(n+1)} + \frac{1}{2^n}\right))$

Solution:

$$\sum_{n=1}^{\infty}\left(\frac{2}{n(n+1)} + \frac{2}{2^n}\right) = 2\sum_{n=1}^{\infty}\left(\frac{1}{n(n+1)}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right) = 2 + 1 = 3$$

Limit of n^{th} Term of a Convergent Series

Theorem1. 5 If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then

$$\lim_{n\to\infty} a_n = 0.$$

Proof. Let $\sum_{n=1}^{\infty} a_n$ converges to L

Claim: $\lim_{n\to\infty} a_n = 0$

Since $\sum_{n=1}^{\infty} a_n$ converges to L, $\lim_{n\to\infty} S_m = L$

$where\ \ S_m = \sum_{n=1}^{m} a_n$

As m$\rightarrow \infty, m-1 \rightarrow \infty$ so $\lim_{m\to\infty} S_{m-1} = L$

But $S_m = a_1 + a_2 + a_3 + \cdots + a_{m-1} + a_m$

$S_{m-1} = a_1 + a_2 + a_3 + \cdots + a_{m-1}$

$\Longrightarrow S_m - S_{m-1} = a_n \quad \Longrightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_m - S_{m-1}) = 0$

Cont'd

The converse of Theorem 1.5 is not true in general.

If $\lim_{n\to\infty} a_n = 0$, we cannot conclude that $\sum_{n=1}^{\infty} a_n$ is convergent.

Observe that for the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ we have $a_n = \frac{1}{n} \to 0 \; as \; n \to \infty$, but the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$ (or does not exist) , then $\sum_{n=1}^{\infty} a_n$ diverges.

Example 1.4.4 Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

Solution. Let $a_n = \frac{n^2}{5n^2+4}$. Then $\lim_{n\to\infty} a_n = \frac{1}{5} \neq 0$

Therefore , by the divergence test theorem the series

$\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ is divergent .

Exercise 1.4.1

1. Determine whether the series converges or diverges.

a) $3 + 2 + \frac{4}{3} + \frac{8}{9} + \ldots$ b) $\sum_{n=1}^{\infty} \frac{3^n + 2^n}{6^n}$

2. If the n^{th} partial sum of a series $\sum_{n=1}^{\infty} a_n$ is

$S_n = 3 - \frac{n}{2^n}$, find a_n and $\sum_{n=1}^{\infty} a_n$

3. Evaluate the sum $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \ldots + \frac{1}{99.100}$

4. Show that *the arithmetic series* $\sum_{k=1}^{\infty} (ak + b)$

converges $\Leftrightarrow$ *a* = *b* = 0.

The integral Test

If f is **a continuous, positive, decreasing** function on$[\mathbf{1},\infty)$ and a_n= f(n), then $\sum_{n=1}^{\infty} a_n$ and $\int_{\mathbf{1}}^{\infty} f(x)dx$ either both converge or both diverge.

In other words:

(i) If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Note : The function f is called testing function.

Cont'd

Example 1.4.5

1. Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

Solution: The function $f(x) = \frac{1}{x^2+1}$ is continuous, positive, and decreasing on $[1, \infty)$. so we use the Integral Test:

$$\int_1^{\infty} \frac{1}{x^2+1} dx$$

$$= \lim_{n\to\infty} \int_1^{n} \frac{1}{x^2+1} dx = \lim_{n\to\infty} tan^{-1}x \Big|_{x=1}^{n} = \frac{\pi}{4}$$

Thus $\int_1^{\infty} \frac{1}{x^2+1} dx$ is a convergent integral and so, by the Integral Test, the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent.

Convergence of the p- series

2. Prove that the **p-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if p>1 and divergent if p$\leq$ 1.

Solution.

i. If $p = 1, the\ series\ becmes$ $\sum_{n=1}^{\infty} \frac{1}{n}$ which is harmonic series and is divergent.

ii. If $p > 1$, $\int_1^{\infty} \frac{1}{x^p} dx = \lim_{t \to \infty} \int_1^{t} \frac{1}{x^p} dx = \frac{-1}{1-p}$

Therefore, by integral test the series

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges

iii. If $p < 1$ the improper integral $\int_1^{\infty} \frac{1}{x^p} dx$ diverges .

(i),(ii) &(iii) $\Rightarrow$ the p-series converges if $p > 1$ and diverges if $p \leq 1$.

Note: We should ***not*** infer from the Integral Test that the sum of the series is equal to the value of the integral.

Example. 1 4.6 $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ whereas $\int_1^{\infty} \frac{1}{x^2} dx = 1$

Therefore, in general, $\sum_{n=1}^{\infty} a_n \neq \int_1^{\infty} f(x)dx$

Exercise 1.4.2

1. Use the Integral Test to determine whether the series is convergent or divergent .

$a)$ $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

$b)$ $\sum_{n=1}^{\infty} \frac{lnn}{n}$

c) $\sum_{n=1}^{\infty} ne^{-n}$

d) $\sum_{n=1}^{\infty} \frac{n+2}{n+1}$

2. Find the values of p for which the series $\sum_{n=1}^{\infty} n(1+n^2)^p$ is convergent .

Direct Comparison Test

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

(i) If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

(ii) If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \geq b_n$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} a_n$ is also divergent

PROOF (i) Let $S_n = \sum_{j=1}^{n} a_n$, $t_n = \sum_{j=1}^{n} b_n$ $t = \sum_{j=1}^{\infty} b_n$

Since both series have positive terms, the sequences $\{S_n\}$ and $\{t_n\}$ are increasing. Also $t_n \to t$, so $t_n \leq t$ for all n .

Since $a_i \leq b_i$ we have $S_n \leq t_n$. Thus $S_n \leq t$ for all n . This means that $\{S_n\}$ is increasing and bounded above and therefore Converges by the Monotonic Sequence Theorem.

Thus $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\sum_{n=1}^{\infty} b_n$ is divergent, then $t_n \to \infty$ (since $\{t_n\}$ is increasing). But $a_i \geq b_i$ so $S_n \geq t_n$. Thus $s_n \to \infty$. Therefore $\sum_{n=1}^{\infty} a_n$ diverges.

Cont'd

Example 1.4.7 Determine whether the series $\sum_{n=1}^{\infty} \frac{3}{2n^2+4n+5}$ converges or diverges.

Solution: Let $a_n = \frac{3}{2n^2+4n+5}$ and $b_n = \frac{3}{2n^2}$.

Observe that $a_n < b_n$. But we know that

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{3}{2n^2} = \frac{3}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent series

because it's a constant times a p-series with $p = 2 > 1$.

Therefore $\sum_{n=1}^{\infty} \frac{3}{2n^2+4n+5}$ is convergent by part (i) of the direct Comparison Test.

Cont'd

Note: The terms of the series being tested must be smaller than those of a convergent series or larger than those of a divergent series. If the terms are larger than the terms of a convergent series or smaller than those of a divergent series, then the Comparison Test doesn't apply.

Exercise 1.4.3

Determine whether the series converges or diverges

a) $\sum_{n=1}^{\infty} \frac{3sin^2 n}{n!}$ b) $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}-1}$ c) $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$

Limit Comparison Test

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms . If

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$$

where L is a finite number and positive , then either both series converge or both diverges. **Proof** (ex)

Example 1.4.8

Test the convergence or divergence of the following series.

a) $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{8n^2-5n}}$

Cont'd

Solution: a) We use the Limit Comparison Test with

$a_n = \frac{1}{2^n - 1}$ and $b_n = \frac{1}{2^n}$ and obtain

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{2^n}{2^n - 1}\right) = 1 > 0$$

Since this limit exists and $\sum_{n=1}^{\infty}\frac{1}{2^n}$ is a convergent geometric series, the given series converges by the Limit Comparison Test.

Cont'd

b) Let $a_n = \frac{1}{\sqrt[3]{8n^2-5n}}$ and $b_n = \frac{1}{\sqrt[3]{8n^2}}$.Then

$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\sqrt[3]{\frac{8n^2}{8n^2-5n}}\right) = 1 > 0$ and $\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{8n^2}} =$

$\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^{2/3}}$ is divergent because it is a constant times a

p- series with $p = \frac{2}{3}$<1.

Hence, by limit comparison test the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{8n^2-5n}}$

diverges.

Cont’d

Exercise 1.4.4

Determine whether the series converges or diverges.

$a)\ \sum_{n=1}^{\infty} \frac{4n-3}{n^3-5n-7}$

b) b) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

c)$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms.

1. The series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$.

2. The series $\sum_{n=1}^{\infty} a_n$ **diverges** if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$

   Or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$

3. The ratio test is inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$

Proof (exercise)

Cont'd

Example 1.4.9

Test the convergence of the following series

a) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

b) b) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$

d) $\sum_{n=2}^{\infty} \frac{n^2}{(2n-1)!}$

e) $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$

Cont'd

Solution. a) Let $a_n = \frac{2^n}{n!}$ then $a_{n+1} = \frac{2^{n+1}}{(n+1)!}$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2}{n+1}\right| = 0 < 1$$

Thus, by ratio test the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges.

b) Let $a_n = \frac{n!}{n^n}$ then $a_{n+1} = \frac{n!}{(n+1)^n}$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\left(\frac{n}{n+1}\right)^n\right| = \frac{1}{e} < 1$$

Thus, by the Ratio Test, the given series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is absolutely convergent and therefore convergent.

Cont'd

$$c)\ a_n = \frac{(-1)^n}{n^2+1}$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(n+1)^2+1}\,\frac{n^2+1}{(-1)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n^2+1}{(n+1)^2+1}\right| = 1$$

Therefore, the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+1}$ might converge or it might diverge. In this case the Ratio Test fails .

Cont'd

d) $a_n = \frac{n^2}{(2n-1)!}$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{(2n+1)!}\frac{(2n-1)!}{n^2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^2}{(2n+1)(2n)(2n-1)!}\frac{(2n-1)!}{n^2}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)(2n)(n^2)} = 0 < 1$$

Therefore, by ratio test this series is convergent

Cont'd

e) $a_n = \frac{2^n+5}{3^n}$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{(n+1)}+5}{3^{(n+1)}}\,\frac{3^n}{2^n+5}\right|$$

$$= \lim_{n\to\infty}\frac{2(2^n)+5}{3(3^n)}\,\frac{3^n}{2^n+5} = \lim_{n\to\infty}\frac{(2+\frac{5}{2^n})}{3(1+\frac{5}{2^n})}$$

$$= 2/3 < 1$$

Therefore, by ratio test this series is convergent

Exercise 1.4.5

Test the convergence of the following series

a) $\sum_{n=1}^{\infty}\frac{2^n}{n^2}$ b) $\sum_{n=1}^{\infty}\frac{(-1)^n n^3}{3^n}$ c) $\sum_{n=1}^{\infty} n^3 e^{-n^2}$

Root Test

i) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** (and therefore convergent).

(ii) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is **divergent**.

(iii) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

Cont'd

Example 1.4.10

Test the convergence of the series

$a)\ \sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n$ **b)** $\sum_{n=1}^{\infty}\left(\frac{\ln(n)}{1000}\right)^n$ **c)** $\sum_{n=1}^{\infty}\frac{(n+2)^n}{(n+5)^n}$

Solution a) $a_n = \left(\frac{2n+3}{3n+2}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{2n+3}{3n+2} = \frac{2}{3} < 1$$

Thus the given series converges by the Root Test.

b) $a_n = \left(\frac{\ln(n)}{1000}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{\ln(n)}{1000} = \infty$$

Thus the given series diverges by the Root Test.

Cont'd

c) $a_n = \frac{(n+2)^n}{(n+5)^n}$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(n+2)^n}{(n+5)^n}\right|} = \lim_{n\to\infty} \frac{n+2}{n+5}$$

$$= \lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{(n+2)^n}{(n+5)^n}$ might converge or it might diverge.

Cont'd

Exercise 1.4.6

Determine whether the series is absolutely convergent or divergent.

a) $\sum_{n=1}^{\infty}\left(\frac{n}{2^n}\right)$ e) $\sum_{n=1}^{\infty}\frac{2^n}{n^2}$

b) $\sum_{n=1}^{\infty}\left(\frac{3^{2n}}{n^n}\right)$ f) $\sum_{n=1}^{\infty}\frac{e^{3n}}{n^n}$

c) $\sum_{n=2}^{\infty}\left(\frac{-2}{n+1}\right)^{5n}$

d) $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$

Alternating series

Definition 1.4.2 **Alternating series** is a series whose terms are alternatively positive and negative.

Example 1.4.11 $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n} n}{n+1}$ arealternating series.

Alternating series Test (Leibniz Test)

Let $a_n > 0$. The alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ and $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converges if the following two conditions are met.

1. $\lim_{n\to\infty} a_n = 0$ 2. $a_{n+1} \le a_n, \forall n$

Cont'd

Example 1.4.12 Determine whether the series converges or diverges.

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ b) $\sum_{n=1}^{\infty} \frac{(-1)^{n} 3n}{4n-1}$

Solution . a) $a_n = \frac{1}{n}$ is decreasing and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

Hence, by alternating series test the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent.

Cont'd

b) $a_n = \frac{3n}{4n-1}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n}{4n-1} = \frac{3}{4} \neq 0$

Therefore ,by alternating series test the series

$\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ diverges.

Exercise 1.4.7

Test the series for convergence or divergence

a) $\frac{-1}{3} + \frac{2}{4} - \frac{3}{5} + \frac{4}{6} - \frac{5}{7} + \cdots$

b) $\sum_{n=1}^{\infty} \left(\frac{-n}{5}\right)^n$

c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$

Absolute convergence

Definition 1.4.2

A series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Example 1.4.13

The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$$

is absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

is a convergent -series $(p = 2)$

Conditional convergence

DEFINITION 1.4.3

A series $\sum_{n=1}^{\infty} a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

In other word:

A series $\sum_{n=1}^{\infty} a_n$ is called **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Cont'd

Example 1.4.14 We know that the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent , but it is not absolutely convergent because the corresponding series of absolute values $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n}\right| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ which is the harmonic series (P –series with p=1) and is therefore divergent.

Cont'd

THEOREM 1.7 If a series is absolutely convergent, then it is convergent.

Proof Observe that the inequality $0 \le a_n + |a_n| \le 2|a_n|$

is true because $|a_n|$ is either a_n or $-a_n$.

If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} |a_n|$ is convergent, so $\sum_{n=1}^{\infty} 2|a_n|$ is convergent. Therefore, by the Comparison Test, $\sum_{n=1}^{\infty}(a_n + |a_n|)$ is convergent.

Then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(a_n + |a_n|) - \sum_{n=1}^{\infty}|a_n|$ is the difference of two convergent series and is therefore convergent.

Cont'd

Example 1.4.15

1. Show that the series $\sum_{n=1}^{\infty} \frac{sin(n)}{n^3}$ converges.

Solution. $\left|\frac{sin(n)}{n^3}\right| \leq \frac{1}{n^3}, \forall n \geq 1.$

Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges the series $\sum_{n=1}^{\infty} \left|\frac{sin(n)}{n^3}\right|$ converges by direct comparison test.

Since $\sum_{n=1}^{\infty} \left|\frac{sin(n)}{n^3}\right|$ converges, the series $\sum_{n=1}^{\infty} \frac{sin(n)}{n^3}$ converges absolutely.

Therefore the series converges by the above theorem.

Cont'd

2. Is the given series absolutely convergent or conditionally convergent or divergent?

$$a)\ \sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)/2}}{3^n} \qquad c)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$$

$$b)\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n} \qquad d)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

Cont'd

$$a)\ \sum_{n=1}^{\infty}\frac{(-1)^{n(n+1)/2}}{3^n}=-\frac{1}{3}-\frac{1}{9}+\frac{1}{27}+\frac{1}{81}-\ldots$$

This is not an alternating series, but since

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n(n+1)/2}}{3^n}\right|=\sum_{n=1}^{\infty}\frac{1}{3^n}$$

Is a convergent geometric series, then the given Series is <u>absolutely convergent</u>.

Cont'd

$$b)\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n} = \frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} +$$

By the nth term test for divergence, the series the series Diverges.

Cont'd

$$d)\ \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}} = -\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}$$

Converges by the alternating series test.

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{\sqrt{n}}\right| = \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}$$

Diverges since it is a p-series with p <1. The Given series is <u>conditionally convergent.</u>

Cont'd

$$c)\ \sum_{n=1}^{\infty}\frac{(-1)^n}{\ln(n+1)} = -\frac{1}{\ln 2}+\frac{1}{\ln 3}-\frac{1}{\ln 4}+\ldots\ldots\frac{n!}{r!(n-r)!}$$

Converges by the Alternating series test.

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{\ln(n+1)}\right| = \frac{1}{\ln 2}+\frac{1}{\ln 3}+\frac{1}{\ln 4}+\ldots\ldots$$

Diverges with direct comparison with the harmonic Series. The given series is <u>conditionally convergent</u>.

Cont'd

Exercise 1.4.8

1. Show that $\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$converges absolutely for $|x| < 1$, converges conditionally for $x = -1$ and diverges for $x = 1$ and for $|x| > 1$.
2. Given any series $\sum_{n=1}^{\infty} a_n$, we define a series $\sum_{n=1}^{\infty} {a_n}^{+}$ whose terms are all the positive terms of $\sum_{n=1}^{\infty} a_n$ and a series $\sum_{n=1}^{\infty} {a_n}^{-}$ whose terms are all the negative terms of $\sum_{n=1}^{\infty} a_n$.

Cont'd

To be specific, we let $a_n^{\,+} = \frac{a_n+|a_n|}{2}$ and $a_n^{\,-} = \frac{a_n-|a_n|}{2}$

Notice that if $a_n > 0$, then $a_n = a_n^{\,+}$ and $a_n^{\,-} = 0$,

whereas if $a_n < 0$, then $a_n = a_n^{\,-}$ and $a_n^{\,+} = 0$

(a) If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, show that both of the series $\sum_{n=1}^{\infty} a_n^{\,+}$ and $\sum_{n=1}^{\infty} a_n^{\,-}$ are convergent.

(b) If $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, show that both of the series $\sum_{n=1}^{\infty} a_n^{\,+}$ and $\sum_{n=1}^{\infty} a_n^{\,-}$ are divergent.