Properties of the Induced Solubility Graph of Finite Non-Soluble Groups

Ziyou Hu

July 2023

1 Introduction

I explored the properties of induced solubility graphs of finite non-soluble groups. In the following discussion, G is assumed to be a finite non-soluble group.

The **commutator** of elements a and b is $[a, b] = a^{-1}b^{-1}ab$.
Define $G^n = [G^{n-1}, G^{n-1}]$ where $G^1 = [G, G] = \{\ [g_1, g_2] \mid g_1, g_2 \in G\ \}$.
A group G is **soluble** if there exists an n such that $G^n = 1$. Based on the context, 1 denotes the identity element in a group or a trivial group that just contains the identity element.

It is not difficult to see that if a group H is abelian, then [H, H] = 1 because the commutator of any 2 elements $a^{-1}b^{-1}ab = 1$. Therefore, all abelian groups are soluble.

The **solubility graph** of a group G is denoted as $\Gamma_s G$. The vertices are all the elements in G. There is an edge between vertices u and v if and only if $\langle u, v\rangle$ is a soluble subgroup of G.
John Thompson states that a finite group G is soluble if and only if the group generated by any 2 elements in the group is soluble [3]. Therefore, G is soluble if and only if $\Gamma_s G$ is a complete graph [1].

$Sol_G(g)$ is the set of **solublizers** of an element $g \in G$. $Sol_G(g) = \{$h $\in G \mid \langle h, g\rangle$ is soluble$\}$.
Then, I use $Sol_G(g)^c$ to denote the non-solubilizers of $g \in G$. $Sol_G(g)^c = \{$h $\in G \mid \langle h, g\rangle$ is non-soluble$\}$.

R(G) denotes the **radical of G**, which is the largest soluble normal subgroup of a group G. R(G) is also the set of universal vertices in $\Gamma_s G$. In other words, R(G) are elements that can generate a soluble subgroup with any one element in G.

Feit-Thompson Theorem: If a finite group has odd order, then the group is soluble [4].

The **induced solubility graph** of a group G is denoted as $\Delta_s G$. $\Delta_s G$ is the graph after removing R(G) in $\Gamma_s G$.

$\Delta_s G$ is always connected [1].

1.1 Lemma 1:

$|G \backslash R(G)| \geq 5$.

1.1.1 Proof for lemma 1:

According to Bhowal et al., the smallest clique number is 4 [2]. Therefore, there are at least 4 vertices in $G \backslash R(G)$. However, if there are only 4 vertices in $G \backslash R(G)$, then $\Delta_s G$ is a complete graph. This means that $\Gamma_s G$ is a complete graph, by Thompson's Theorem. Thus, G is soluble. $G \backslash R(G) = \emptyset$ because G is soluble. This is a contradiction. $\blacksquare$.

2 Index

- No Complete Multipartite Graph
- Failure of the Condition in Dirac's Theorem
- Failure of the Condition in Ore's Theorem
- Product of Induced Solubility Graph

3 No Complete Multipartite Graph

Bhowal et al. did not provide an explicit proof for the non-existence of complete k-partite graph. In this project, I proved the non-existence of complete k-partite graph for non-soluble groups through a different direction that Bhowal et al. suggested.

Theorem 1: The induced solubility graph cannot be complete multipartite for any non-soluble group G.

Proof:

Assume for contradiction that there exists a non-soluble group G such that $\Delta_s(G)$ is complete multipartite.

The vertices can be partitioned into k sets $P_1, P_2, ..., P_k$.

As we have proved earlier, there are at least 2 elements in $G\backslash R(G)$, otherwise G will be soluble.
Consider any 2 vertices $x, y \in G\backslash R(G)$. x, y can be in the same or different set P. I will show that under our assumption that the graph is k-partite, any vertices x and y are adjacent to each other.
For any two vertices, there are two possibilities:

1) $x^2, y^2 \in R(G)$

Then, $x^2 \in R(G)$ and $y^2 \in R(G)$, which means $x^2R(G) = R(G)$ and $y^2R(G) = R(G)$. Because $x \notin R(G)$ and $y \notin R(G)$, $xR(G) \neq R(G)$ and $yR(G) \neq R(G)$. Thus, $|xR(G)| = 2$ and $|yR(G)| = 2$. Any two elements of order 2 generate a dihedral group. Then, $\langle xR(G), yR(G)\rangle$ is a dihedral group, which is soluble. Then, xR(G) and yR(G) are adjacent to each other in $\Delta_s(G/R(G))$. Thus, x is adjacent to y in $\Delta_s(G)$.

2) At least one element, say $y^2 \notin R(G)$. This case covers the possibility that x^2 and y^2 are both in $G\backslash R(G)$.

Suppose for contradiction that $\langle x, y\rangle$ is non-soluble.
$y^2 \neq 1$, otherwise $y^2 \in R(G)$. So, $|y| \geq 3$. Thus, $y^{-1} \neq y$ and $y^{-1} = y^m$ for some m ≥ 2. We know that $\langle g, y\rangle$ is non-soluble for some $g \in G$. Because $\langle g, y\rangle = \langle g, y^{-1}\rangle$, $\langle g, y^{-1}\rangle$ is also non-soluble. Thus, $y^{-1} \in G\backslash R(G)$.

$\langle x, y\rangle$ is non-soluble, so $\langle x, y^{-1}\rangle$ is also non-soluble. Because $\langle x, y\rangle$ is non-soluble and $\Delta_s(G)$ is complete k-partite graph, $x, y \in P_i$ for some i between 1 and k. By the same reasoning, $y^{-1} \in P_i$. $\langle y, y^k\rangle = \langle y\rangle$, which is a cyclic group. All cyclic groups are abelian, so $\langle y, y^k\rangle$ is soluble. Then, y and y^k are adjacent. However, y, $y^k \in P_i$.

This contradicts the definition of k-partite graphs. The contradiction is caused by our assumption that $\langle x, y\rangle$ is non-soluble. Hence, $\langle x, y\rangle$ is soluble, which means $x \sim y$.

Therefore, I have shown that any 2 vertices in $\Delta_s(G)$ are adjacent to each other.

Therefore, if we assume the graph to be complete k-partite, then $\Delta_s(G)$ is a complete graph. Then, for some element a $\in$ G. $\forall g \in R(G) \cup G\backslash R(G) = G$, $\langle a, g\rangle$ is soluble. Then, $a \in R(G)$ by definition of R(G). However, $a \in G\backslash R(G)$. This is a contradiction. Therefore, $\Delta_s(G)$ is not a complete k-partite graph for any group G. ■

4 Failure of the Condition in Dirac's Theorem

Theorem 2: The induced solubility graph for all non-soluble groups G does not satisfy the condition in Dirac's Theorem.

Proof:
Dirac's condition in the context of induced solubility graph is that $|Sol_G(x)| \geq \frac{|G|+R(G)+2}{2}$ $\forall x \in G \backslash R(G)$.

Assume for contradiction that for some non-soluble group G $|Sol_G(x)| \geq \frac{|G|+R(G)+2}{2}$ $\forall x \in G \backslash R(G)$. $Sol_G(x) = G$ $\forall x \in R(G)$. So, $|Sol_G(x)| > \frac{|G|+R(G)+2}{2}$ $\forall x \in R(G)$.

$|\{(u,v) \in G \times G : \langle u,v \rangle \textit{ is soluble}\}| = |\{(x,y) : x \in G \textit{ and } y \in Sol_G(x)\}| = \Sigma_{x \in G}|Sol_G(x)|$. Then, $\Sigma_{x \in G}|Sol_G(x)| \geq |G| * \frac{|G|+|R(G)|+2}{2}$.

Then, $P_s(G) = \frac{\Sigma_{x \in G}|Sol_G(x)|}{|G|^2} \geq \frac{|G| * \frac{|G|+|R(G)|+2}{2}}{|G|^2} = \frac{|G|+|R(G)|+2}{2|G|} = \frac{|G|}{2|G|} + \frac{|R(G)|+2}{2|G|} = \frac{1}{2} + \frac{|R(G)|+2}{2|G|}$. Because $\frac{|R(G)|+2}{2|G|} > 0$, $P_s(G) > \frac{1}{2}$. This is a contradiction.

Thus, there must be an x in $G \backslash R(G)$ such that $|Sol_G(x)| < \frac{|G|+|R(G)|+2}{2}$. ■

5 Failure of the Condition in Ore's Theorem

5.1 Ore's Theorem

Ore's Theorem provides another condition for Hamiltonian Cycle. The theorem states that, in a graph, if the sum of the degrees of any 2 nonadjacent vertices is greater than or equal to the vertex set, then the graph is Hamiltonian.

5.1.1 Ore's Condition in this Context

In this context, Ore's theorem suggests that there is a Hamiltonian cycle in $\Delta_s G$ if for any non-adjacent vertices x and y in $\Delta_s G$

$$\begin{aligned} |Sol(x)| + |Sol(y)| - 2|R(G)| - 2 &\geq |G| - |R(G)| \\ |Sol(x)| + |Sol(y)| &\geq |G| + |R(G)| + 2 \end{aligned}$$

.

Lemma 2: There is a bijection from $G \backslash R(G)$ to $G \backslash R(G)$ such that f(x) $\in Sol_G(x)^c$ if and only if for all subsets $\{a_1, ..., a_n\} \in G \backslash R(G)$, $|\cup_{i=1}^{n} Sol_G(a_i)^c| \geq n$.

Proof for the Lemma:

First Direction:
Suppose there is a bijection from $G \backslash R(G)$ to $G \backslash R(G)$.

Now, suppose for contradiction that there exists n elements $a_1, ..., a_n$ such that $|\cup_{i=1}^{n} Sol_G(a_i)^c| < n$.

Let $\{a_1, ..., a_n\}$ map to the set A. Because there is a bijection, then the mapping from $\{a_1, ..., a_n\}$ to the set A is also a bijection. Then, $|A|$ = n.

However, $\{a_1, ..., a_n\}$ can only map to their non-solubilizers $|\cup_{i=1}^{n} Sol_G(a_i)^c|$ $< n$. This means that $|A| < n$. This is a contradiction.

Therefore, if there is a bijection from $G\backslash R(G)$ to $G\backslash R(G)$, then for all sets of elements $\{a_1, ..., a_n\}$ in $G\backslash R(G)$ $|\cup_{i=1}^{n} Sol_G(a_i)^c| \geq n$.

Second Direction:

Suppose for every set of elements $\{a_1, ..., a_n\}$ $|\cup_{i=1}^{n} Sol_G(a_i)^c| \geq n$.

Induction:

Consider a set S of size m. Each element is associated with a nonempty subset of S that does not include the element itself. Call the subset that a_i associates with as B_i. Let P(m) be the statement that if for every subset $\{a_1, ..., a_n\}$ in S, $|\cup_{i=1}^{n} B_i| \geq n$, there is a bijection f from S to S such that $f(s) \in B_i$.

Base Case) Show P(2) is true.

There are 2 elements in S. Call them x and y. Because they are are both associated with an nonempty set that does not contain themselves. Then, x must be associated with $\{y\}$ and y must be associated with $\{x\}$. Thus, map x to y and y to x. Now, we have a bijection.

Inductive hypothesis: Assume the statement is true for all $m \leq k$.

Inductive case) Show P(k+1) is true.

Let us remove an element x_{k+1} . so that no element maps to x_{k+1} and x_{k+1} does not exist in the set S. Call this new set S^*.

Then, by our inductive hypothesis, there is a bijection from S^* to S^*.

Now, let us create a bijection from S to S using the current mapping.

In the current mapping, x_{k+1} does not map to any element and no element maps to x_{k+1}.

Choose an element in B_{k+1} and map x_{k+1} to that element.. Then, there are two elements that map to the same element. Call the element that clashes with x_{k+1} as x_1. Consider x_1 and x_{k+1}. Because $|B_{k+1} \cup B_1| \geq 2$. We must be able to map exactly one element to another element to resolve the clash.

Reassign one of them to another element. Now, we create a new clashing pair. Call the new element that clashes as x_2. We repeat the process while keeping a running set of all the elements that have clashed. Each time we assign a value, we create exactly one clashing pair. By the way we assign new values to resolve clashes, the new repeating value is distinct from all previously repeated values. Finally, stop the process when we assign an element to x_{k+1}

because no more clashing will be created. Now, we have our bijection.

Thus, P(k+1) is true and the induction holds. ■

We can consider $G\backslash R(G)$ as S. Thus, suppose for every set of elements $\{a_1, ..., a_n\}$ $|\cup_{i=1}^{n} Sol_G(a_i)^c| \geq n$, then there is a bijection from $G\backslash R(G)$ to $G\backslash R(G)$.

Theorem 3:

If there is a bijection f: $G\backslash R(G) \to G\backslash R(G)$ such that f(x) $\in Sol_g(x)^c$, then $\Delta_s(G)$ does not satisfy the condition in Ore's Theorem.

Proof:

Assume for contradiction that there exists some group G such that and $|Sol(x)| + |Sol(y)| \geq |G| + |R(G)| + 2$ for elements x and y that generate non-soluble subgroups.

Then, we have a bijection f: $G\backslash R(G)$ to $G\backslash R(G)$ such that f(x) $\in Sol_G(x)^c$.

$\Sigma_{x\in G\backslash R(G)}(|Sol(x)| + |Sol(f(x))|) \geq \Sigma_{x\in G\backslash R(G)}(|G| + |R(G)| + 2)$.

The right hand side is equal to $(|G| - |R(G)|)(|G| + |R(G)| + 2)$.

Thus, $\Sigma_{x\in G\backslash R(G)}(|Sol(x)| + |Sol(f(x))|) \geq (|G| - |R(G)|)(|G| + |R(G)| + 2)$.

Left Hand Side of the Inequality

Because f is a bijection, $\Sigma_{x\in G\backslash R(G)}|Sol(f(x))| = \Sigma_{x\in G\backslash R(G)}|Sol(x)|$.

$$\begin{aligned}
LHS &= \Sigma_{x\in G\backslash R(G)}|Sol(x)| + \Sigma_{x\in G\backslash R(G)}|Sol(x)| \\
&= 2\Sigma_{x\in G\backslash R(G)}|Sol(x)| \\
&= 2(\Sigma_{x\in G\backslash R(G)}|Sol(x)| + \Sigma_{x\in R(G)}|Sol(x)| - \Sigma_{x\in R(G)}|Sol(x)|) \\
&= 2(\Sigma_{x\in G\backslash R(G)}|Sol(x)| + \Sigma_{x\in R(G)}|Sol(x)|) - 2(\Sigma_{x\in R(G)}|Sol(x)|) \\
&= 2\Sigma_{x\in G}|Sol(x)| - 2|R||G|
\end{aligned}$$

Right Hand Side of the Inequality

$$\begin{aligned}
RHS &= |G|^2 + |G||R(G)| + 2|G| - |R(G)||G| - |R(G)|^2 - 2|R(G)| \\
&= |G|^2 + 2|G| - |R(G)|^2 - 2|R(G)|
\end{aligned}$$

Rewrite of the inequality

$2\Sigma_{x\in G}|Sol(x)| - 2|R||G| \geq |G|^2 + 2|G| - |R(G)|^2 - 2|R(G)|$
Divide both sides by $|G|^2$, we get:

$$
\begin{aligned}
2P_s(G) - \frac{2|R(G)|}{|G|} &\geq 1 + \frac{2}{|G|} - \frac{|R(G)|^2 + 2|R(G)|}{|G|^2} \\
2P_s(G) &\geq 1 + \frac{2 + 2|R(G)|}{|G|} - \frac{|R(G)|^2 + 2|R(G)|}{|G|^2} \\
2P_s(G) &\geq 1 + \frac{2|G| + 2|R(G)||G| - |R(G)|^2 - 2|R(G)|}{|G|^2} \\
2P_s(G) &\geq 1 + \frac{2(|G| - |R(G)|) + |R(G)|(2|G| - |R(G)|)}{|G|^2}
\end{aligned}
$$

Divide both sides by 2, we get:
$P_s(G) \geq \frac{1}{2} + \frac{2(|G|-|R(G)|)+|R(G)|(2|G|-|R(G)|)}{2|G|^2}$.
Because G is a non-soluble group, $|G| > |R(G)|$. Then, $|G| - |R(G)| > 0$ and $2|G| - |R(G)| > 0$.
Hence, $P_s(G) > \frac{1}{2}$. However, for all non-soluble groups, $P_s(G) \leq \frac{11}{30}$. This is a contradiction. ■

6 Product of the Induced Solubility Graph

Let G_1 and G_2 be 2 finite non-solvable groups.
Consider the **Product of the Induced Solubility Graph** $\Delta_s(G_1) \times \Delta_s(G_2)$.
The vertex set is $G_1 \backslash R(G_1) \times G_2 \backslash R(G_2)$.
Two vertices (g_1, g_2) and (h_1, h_2) are adjacent if and only if $g_1 = h_1$ and g_2 and h_2 are adjacent in G_2, or $g_2 = h_2$ and g_1 is adjacent to h_1 in G_1. Also, (g_1, g_2) is not connected to itself.

6.1 Degree of a Vertex

Consider the vertex (g_1, g_2) in the graph $\Delta_s(G_1) \times \Delta_s(G_2)$.
There are two types of vertices that (g_1, g_2) is adjacent to.

6.1.1 Type 1

All vertices in the form such us (g_1, h_2) where g_2 and h_2 are adjacent in G_2 and $h_2 \neq g_2$. There are $|Sol_{G_2} g_2| - |R(G_2)| - 1$ such vertices.

6.1.2 Type 2

All vertices in the form such us (h_1, g_2) where g_1 and h_1 are adjacent in G_1 and $h_1 \neq g_1$. There are $|Sol_{G_1}(g_1)| - |R(G_1)| - 1$ such vertices.

Therefore, in total, $deg(g_1, g_2) = |Sol_{G_2}(g_2)| - |R(G_2)| - 1 + |Sol_{G_1}(g_1)| - |R(G_1)| - 1$.

Notice that $|Sol_{G_2}(g_2)| - |R(G_1)| - 1 = deg_{G_2} g_2$ and $|Sol_{G_1}(g_1)| - |R(G_1)| - 1 = deg_{G_1} g_1$. Thus, $deg(g_1, g_2) = deg_{G_2} g_2 + deg_{G_1} g_1$.

Property 1: The product of induced solubility graph is connected.

Proof:

Let (g_1, g_2) and (h_1, h_2) be two arbitrary vertices in $\Delta_s(G_1) \times \Delta_s(G_2)$. Because G_2 is connected, there is a path from g_2 to h_2. Let the path be: $g_2, a_2, b_2, ..., h_2$. Then, by definition of $\Delta_s(G_1) \times \Delta_s(G_2)$, there is a path in $\Delta_s(G_1) \times \Delta_s(G_2)$: $(g_1, g_2), (g_1, a_2), (g_1, b_2), ..., (g_1, h_2)$.

We can repeat this process for g_1 and h_1. Because G_1 is connected, there is a path from g_1 to h_1. Let the path be: $g_1, a_1, b_1, ..., h_1$. Then, by definition of $\Delta_s(G_1) \times \Delta_s(G_2)$, there is a path in $\Delta_s(G_1) \times \Delta_s(G_2)$: $(g_1, h_2), (a_1, h_2), (b_1, h_2), ..., (h_1, h_2)$.

Hence, there is a path from (g_1, g_2) to (h_1, h_2) in $\Delta_s(G_1) \times \Delta_s(G_2)$:

$(g_1, g_2), (g_1, a_2), (g_1, b_2), ..., (g_1, h_2), (a_1, h_2), (b_1, h_2), ..., (h_1, h_2)$.

Thus, $\Delta_s(G_1) \times \Delta_s(G_2)$ is a connected graph. ■

Property 2: The product of induced solubility graph can never be a tree.

Proof:

Because $\Delta_s(G_2)$ is a connected graph that is not tree, there must be at least a cycle of at least length 3 in $\Delta_s(G_2)$. Let the cycle be b_1, b_2, ..., b_m, b_1.

Let a be a vertex in G_1. Then, the following is a cycle of length m in $\Delta_s(G_1) \times \Delta_s(G_2)$:

(a, b_1), (a, b_2), ..., (a, b_m), (a, b_1).

Therefore, $\Delta_s(G_1) \times \Delta_s(G_2)$ can never be a tree. ■

Property 3:
Question: When is the graph $\Delta_s(G_1) \times \Delta_s(G_2)$ Eulerian?

Case 1) $\Delta_s(G_1)$ and $\Delta_s(G_2)$ are both Eulerian. In other words, $|Sol_G(x)|$ is even and $|R(G)|$ is odd for both groups (This is the condition for $\Delta_s G$ to be Eulerian found by the rest of my research group).

The degree of every vertex is even. The sum of even number is always even. Then, $deg(g_1, g_2) = deg_{G_2} g_2 + deg_{G_1} g_1$ is always even for every vertex in the graph $\Delta_s(G_1) \times \Delta_s(G_2)$. Thus, $\Delta_s(G_1) \times \Delta_s(G_2)$ is Eulerian.

Case 2) One of the two graphs is not Eulerian and the other graph is Eulerian.

Without loss of generality, let $\Delta_s G_1$ be not Eulerian and $\Delta_s G_2$ be Eulerian.

Then, there exists a vertex x in $\Delta_s G_1$ such that $deg_{G_1} x$ is odd. The degree of every vertex in $\Delta_s G_2$ is even. Then, for any vertex in the form of (x, g_2), deg(x, g_2) = $deg_{G_2} g_2 + deg_x g_1$ is odd. Thus, $\Delta_s G_1 \times \Delta_s G_2$ is not Eulerian.

Case 3) Both graphs are not Eulerian.

Claim: If both graphs are not Eulerian, $\Delta_s(G_1) \times \Delta_s(G_2)$ is Eulerian if and only if all vertices are odd in both graphs.

Proof:

There must exist at least one odd vertex in each graph.

First, suppose every vertex in $\Delta_s(G_1)$ and $\Delta_s(G_2)$ are odd.

$deg(g_1, g_2) = deg_{G_2} g_2 + deg_{G_1} g_1$ is always even. Thus, $\Delta_s(G_1) \times \Delta_s(G_2)$ is Eulerian.

Second, let us show that if $\Delta_s(G_1) \times \Delta_s(G_2)$ is Eulerian, then every vertex in $\Delta_s(G_1)$ and $\Delta_s(G_2)$ are odd.

Suppose not every vertex in $\Delta_s(G_1)$ and $\Delta_s(G_2)$ are odd. Let g_2 be an even vertex in G_2 and g_1 be an odd vertex in G_1.

Then, $deg(g_1, g_2) = deg_{G_2} g_2 + deg_{G_1} g_1$ is odd.

If both graphs are not Eulerian, $\Delta_s(G_1) \times \Delta_s(G_2)$ is Eulerian if and only if all vertices are odd in both graphs.

Let us rewrite the condition that "all vertices are odd in both graphs".

Remember that deg(x) = $|Sol(x)|$ - 1 - $|R(G)|$.

There are two ways for deg(x) to be odd. Either $|Sol(x)|$ and $|R(G)|$ are both even or $|Sol(x)|$ and $|R(G)|$ are both odd.

Suppose $|Sol(x)|$ are odd for all vertices in both graphs and R(G) are odd for both graphs.

We can follow a similar line of argument as the previous Eulerian proof completed by the other members of the research group. $|G/R(G)|$ is even otherwise G is solvable by Extension Theorem. Then, by Cauchy's Theorem, there exists and element xR(G) of order 2. This means that $|x|$ is even. Also, we know that $|x|$ devides $|Sol(x)|$, but $|Sol(x)|$ is always odd. This is a contradiction.

Therefore, $|Sol(x)|$ and $|R(G)|$ must be always even. Hence, if both graphs are not Eulerian, $\Delta_s(G_1) \times \Delta_s(G_2)$ is Eulerian if and only if $|Sol(x)|$ and $|R(G)|$ are both even. ■

References

[1] Bhagyesh Akbari, Mark Lewis, J. Mirzajani, and Alireza Moghaddamfar. The solubility graph associated with a finite group. *arXiv*, 03 2020.

[2] Parthajit Bhowal, Deiborlang Nongsiang, and Rajat Kanti Nath. Solvable graphs of finite groups. *Hacettepe Journal of Mathematics and Statistics*, 49(6):1955–1964, 2020.

[3] John G. Thompson. Nonsolvable finite groups all of whose local subgroups are solvable. *Bulletin of the American Mathematical Society*, 74(3):383–437, 1968.

[4] John G. Thompson and Walter Feit. *Solvability of Groups of Odd Order*. Pacific Journal of Mathematics, 1963.