Homework Assignment 3
ENM 502 – Numerical Methods
Due Tuesday 3/23/2021

A. General Objective Statement

Consider the following non-linear boundary-value problem defined on the unit-square domain $D = (0 \le x \le 1) \cup (0 \le y \le 1)$

$$\nabla^2 u + \lambda u(1+u) = 0 \qquad (1)$$
$$u(x, y) = 0 \text{ on all boundaries,}$$

where λ is a known parameter.

(a) Discretize eq. (1) and boundary conditions using centered finite difference approximations and thereby generate a set of coupled, non-linear algebraic equations of the form $\mathbf{R}(\mathbf{u}) = \mathbf{0}$. This system of equations should be solved using Newton's Method for given values of the parameter λ.

(b) Track all (there are several—we will discuss this in class) non-trivial solution branches for $0 \le \lambda \le 60$ using arc-length continuation.

B. Numerical Method Considerations

(i) Use the intrinsic MATLAB function, '\' to numerically solve any linear system of equations that arise. Investigate the use of the sparse matrix capability in MATLAB to see if this offers computational advantages.

(ii) Use a 30x30 uniform finite difference grid for all calculations.

(iii) First use the solutions to the linearized problem to jump onto the various solution branches as discussed in class. Then use arc-length continuation to trace out the solutions as a function of λ.

(iv) Call your Newton's method routine from another M-file that contains the continuation code. The continuation code should consist of an outer loop that steps in the value of the continuation parameter.

(v) Notes describing arc-length continuation implementation are provided at the end of the assignment statement.

C. Additional Notes

(i) Your major result should be summarized on plots of $\|\mathbf{u}\|_2(\lambda)$ versus λ for all solution branches in the interval $0 \le \lambda \le 60$. Note that while the 2-norm is always positive, it is useful for clarity to assign 'negative' and 'positive' sub-branches for a given branch of the solution. For example, 'hill-type' solutions could be assigned as positive and 'bowl-like' solutions negative. Whatever convention you decide to use, you should make sure that you clearly define it.

(ii) Plot representative contour maps of the solutions found along the different solution branches so that it is clear how the solution changes along each branch.

(iii) Hand in printouts of your MATLAB code along with the rest of the report. It is in your best interest to provide as much documentation with your codes as possible.

SUPPLEMENTARY NOTES for ARC-LENGTH CONTINUATION

Use the following approach to begin your arc-length continuation (ALC) solution tracking:

1. Start with an initial guess for $\mathbf{u}_0$ *at* $\lambda = \lambda_0$
2. Using Newton's Method get the converged solution at $\mathbf{u}_0$ at $\lambda = \lambda_0$ (you select the initial value of λ).
3. Use one step of analytic continuation (AC) to obtain the converged solution at $\mathbf{u}_1$ *at* $\lambda_1 = \lambda_0 + \delta\lambda$ (you select the step size in λ).
4. Once you have the solutions at λ_0, λ_1, you can use ALC to obtain the solution (and the value of λ) at the next point '2' as described below. Note that you will need the two previous values to perform ALC as described in more detail next.

Arc-Length Continuation (ALC)

The initial guess for $\lambda_2, \mathbf{u}_2$ using ALC is given by:

$$\begin{aligned} \mathbf{u}_2^0 &= \mathbf{u}_1 + (\delta s)\left(\frac{\partial \mathbf{u}}{\partial s}\right)_1 \\ \lambda_2^0 &= \lambda_1 + (\delta s)\left(\frac{\partial \lambda}{\partial s}\right)_1 \end{aligned} \tag{1}$$

Where, here, $\delta s \equiv s_2 - s_1$, is the arc-length of a small segment of a curve defined by

$$\left(\delta S\right)^2 = \left(\delta\lambda\right)^2 + \left\|\delta\mathbf{u}\right\|_2^2 .$$

Before we can use eq. (1) to provide intelligent initial guesses for λ_2 , $\mathbf{u}_2$ we need to evaluate

$$\left(\frac{\partial\mathbf{u}}{\partial s}\right)_1, \left(\frac{\partial\lambda}{\partial s}\right)_1 .$$

The preceding quantities can be evaluated from the following equation obtained by expanding the augmented residual, $\hat{\mathbf{R}} = \begin{pmatrix}\mathbf{R}\\ \eta\end{pmatrix}$, in a Taylor Series about point '1' to give the following:

$$\hat{\mathbf{J}}\Big|_1 \begin{pmatrix}\dfrac{\partial\mathbf{u}}{\partial s}\\ \dfrac{\partial\lambda}{\partial s}\end{pmatrix}_1 = -\left(\frac{\partial\hat{\mathbf{R}}}{\partial s}\right)_1, \tag{2}$$

where

$$\hat{\mathbf{J}}\Big|_1 = \begin{pmatrix}\dfrac{\partial\mathbf{R}}{\partial\mathbf{u}}\Big|_1 & \dfrac{\partial\mathbf{R}}{\partial\lambda}\Big|_1\\ \dfrac{\partial\eta}{\partial\mathbf{u}}\Big|_1 & \dfrac{\partial\eta}{\partial\lambda}\Big|_1\end{pmatrix}, \tag{3}$$

and

$$\eta(s,\lambda,\mathbf{u})\Big|_1 = \left|s_1 - s_0\right|^2 - \left\|\mathbf{u}(s_1) - \mathbf{u}(s_0)\right\|^2 - \left|\lambda(s_1) - \lambda(s_0)\right|^2 .$$

The augmented residual derivative w.r.t *s* is given explicitly by

$$\left(\frac{\partial\hat{\mathbf{R}}}{\partial s}\right)_1 = \begin{pmatrix}\dfrac{\partial\mathbf{R}}{\partial s}\\ \dfrac{\partial\eta}{\partial s}\end{pmatrix}_1 = \begin{pmatrix}0\\ \dfrac{\partial\eta}{\partial s}\end{pmatrix}_1, \tag{4}$$

where

$$\begin{aligned}\frac{\partial\eta}{\partial\lambda}\Big|_1 &= -2\left(\lambda(s_1) - \lambda(s_0)\right)\\ \frac{\partial\eta}{\partial u_i}\Big|_1 &= -2\left(u_i(s_1) - u_i(s_0)\right) \quad \forall i = 1,2,.......n\\ \frac{\partial\eta}{\partial s}\Big|_1 &= 2\left(s_1 - s_0\right)\end{aligned} \tag{5}$$

We can now compute

$$\left.\left(\frac{\partial \mathbf{u}}{\partial s}\right)\right|_1, \left.\left(\frac{\partial \lambda}{\partial s}\right)\right|_1.$$

The latter expressions can then be substituted back into eq. (1), to obtain the initial guess for $\underline{u}_2, \lambda_2$. This completes a full cycle of the arc-length continuation scheme and further stepping can be achieved by repeating the above sequence of steps. Finally, for reference, the equations for the Full Newton Method are provided below.

Full Newton's Method with an Augmented Residual/Jacobian:

Once we obtain the initial guess at point $s = s_2, \mathbf{u} = \mathbf{u}_2, \lambda = \lambda_2$, we need to iterate using Newton's Method until both $\|\delta \mathbf{u}\|, |\delta \lambda\,|$ are below the suggested tolerance values.

For the Augmented system the equation for Full Newton's Method looks like:

$$\left.\left(\hat{\mathbf{J}}^k\right)\right|_{s2} \left.\begin{pmatrix} \delta \mathbf{u}^k \\ \\ \delta \lambda^{\,k} \end{pmatrix}\right|_{s2} = -\left.\left(\hat{\mathbf{R}}^k\right)\right|_{s2}$$

$$\lambda^{k+1} = \lambda^k + \delta \lambda^{\,k}$$

$$\mathbf{u}^{k+1} = \mathbf{u}^k + \delta \mathbf{u}^k$$