III: Principal stresses and Failure Criteria

1: Principal Stresses

1.1: The Principal stresses: Max and Min

- The principal stresses σ_1 and σ_2 acts on the principal planes, which the **shear stress** is zero.
- They are orthogonal, σ_1 is oriented 90 degrees from σ_2.
- The third principal stress acts in the out-of-plane direction which is zero in **2D**.
- Usually,σ_1 is algebraically the max normal stress while σ_2 is the least.
- However the σ_2 maybe have large magnitudes when it is compressive (negative).
- Principal planes are mutually inclined at 90 degrees.

1.2: The formula of the max and min

- $\sigma_1 = (\frac{\sigma_{xx} + \sigma_{yy}}{2}) + \sqrt{(\frac{\sigma_{xx} - \sigma_{yy}}{2})^2 + \tau_{xy}^2}$
- $\sigma_2 = (\frac{\sigma_{xx} + \sigma_{yy}}{2}) - \sqrt{(\frac{\sigma_{xx} - \sigma_{yy}}{2})^2 + \tau_{xy}^2}$
- The direction θ can be defined as: $\theta = \frac{1}{2} \tan^{-1} \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}$

2: The third principal stress

- In 3-D situation, there is a σ_3 which is perpendicular to the σ_1 and σ_2.
- In 2-D the σ_3=0.

3: Maximum shear stress

- $\tau_{max} = \sqrt{(\frac{\sigma_{xx} - \sigma_{yy}}{2})^2 + \tau_{xy}^2}$
- When the maximum shear stress applied, thw normal stress can be defined as $\sigma_s = \frac{\sigma_{xx} + \sigma_{yy}}{2}$.
- The direction of the shear stress can be expressed as: (from the horizontal)

$$\theta = \frac{1}{2}\tan^{-1}(-(\frac{\sigma_{xx} - \sigma_{yy}}{2\tau_{xy}}))$$

- And the positive sign show the A/C while negative sign show the CW.

4: Relationship between principal stress and max shear stress

- $\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$
- The degree between the principal stress plane and max shear stress plane is 45 degrees.

5: The Failure Criteria

5.1: Introduction

- **Yield criteria** for ductile material (some of the metal) and **Fracture criteria** for brittle material (stone and ceramic) and for other failure material like buckling it need other criteria.

5.2: Failure criteria for ductile material (yield)

5.2.1: Tresca criterion

- Also known as the max shear stress criterion.
- $\tau_{max} = \frac{\sigma_Y}{2}$
- For σ_1 and σ_2 have different sign, taking $\sigma_1 > \sigma_2$:

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$$

- For σ_1 and σ_2 have same sign, taking $\sigma_1 > \sigma_2$:

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_1}{2}$$

- For **Tresca criterion**, we use the most severly stressed point to find the **factor of safety**:

$$n = \frac{\sigma_Y/2}{\tau_{max}}$$

5.2.2: **Von Mises Criterion

- Also known as distortion strain energy criterion。
- The Von Mises stress can be expressed as:

$$\sqrt{\frac{1}{2}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]}=\sigma_Y$$

- And the factor of safety can be expressed as σ_Y/von mises stress

6: Failure criteria for brittle materials (fracture)

- According to the Rankine criterion, the fracture occurs when: $\sigma_{max}=\sigma_{ult}$, where the $\sigma_{max}=max(|\sigma_1|,|\sigma_2|,|\sigma_3|)$.
- In this criterion:

$$n=\frac{\sigma_{ult}}{\sigma_{max}}$$