2: Integration

2.1: The Definite Integration and Indefinite integration

- The integration that produces a value not merely a formula is called a **definite integration**.
- An integration without a value is called **indefinite integration**.

2.2: Relationship of integration to differentiation

- According to the fundamental theorem of calculus:
  $\delta A \approx f(x)\delta x$, and $\dfrac{dA}{dx} = f(x)$ in the limit $\delta x \to 0$
- And the area is the **antiderivative** of the area, written as $A = \displaystyle\int f(t)dt$

2.3: Volume of Revolution

- A volume of revolution is formed by rotating a curve about a axis, which may be the x-axis or y-axis or other line.
- If the rotation is about the x-axis,each with radius given by $f(x)$ and thickness δx, the volume of one disc is $\delta V \approx \pi f(x)^2 \delta x$
- And the volume between $x = a$ and $x = b$ is given by:
  $\lim_{\delta x \to 0} \Sigma_{x=a}^{x=b} \pi f(x)^2 \delta x$
- So $V = \displaystyle\int_a^b \pi y^2 dx$

2.4: Arc-length

- To find the length of a curve expressed as $y = f(x)$.
- $\delta s^2 \approx \delta x^2 + \delta y^2$, so $\delta s \approx \sqrt{\delta x^2 + \delta y^2} = \sqrt{1 + (\dfrac{\delta y}{\delta x})} dx$
- Then we take a limit:

$$s = \lim_{\delta x \to 0} \sum \delta s = \lim_{\delta x \to 0} \sqrt{1 + (\frac{\delta y}{\delta x})^2} \delta x = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx$$

2.5: Surface area or revolution

- The volume of revolution has a curved surface, which may be regarded as a sum of thin circular bands.
- For each, the area is approximately the circumference multiplied by the length of the small piece of arc.

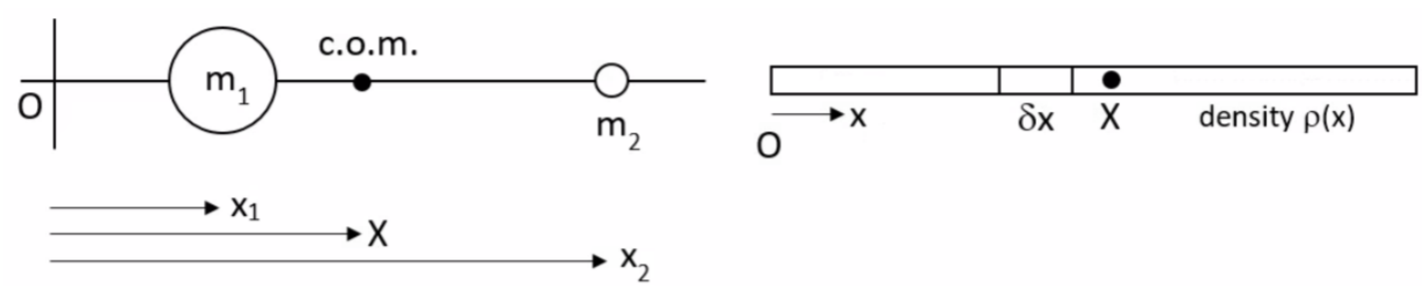
$$A_S \approx \sum 2\pi y \delta s = \sum 2\pi y \sqrt{1+(\frac{\delta y}{\delta x})^2}\delta x = \int_a^b 2\pi y \sqrt{1+(\frac{dy}{dx})^2} dx$$

2.6: Bending stiffness

- Initially, we will consider only a rectangular beam bending a small amount.Resistance to bending comes from stretching planes of material on the outside if the bend, and compressing in the inside bend, with: Resistance to bending $\pm$ amount of stretching $\times$ moment of about centre line.
- The strain (amount od stretch/compression) is proportional to the distance from the centre line, and the moment of that force is also proportional to the distance from the centre. Therefore we need to calculate the **second moment of area** about the centre line.
- For thickness δy and width w, we get second moment I $I \approx \sum (w\delta y)(y)(y)$,hence $I = \int_{-d/2}^{d/2} wy^2 dy$.
- For rectangular beam of width w and depth d, this gives $I = \dfrac{wd^3}{12}$.
- For a circular bar, it gives that $I = \pi r^4/4$, and hence for tube of inner radius a and outer radius b, we get $I = \pi(b^4 - a^4)/4$.
- Tubes and I beams have a large bending stiffness relative to their mass as much of the mass is away from the centre line (*neutral axis*).

3: Centre of Mass and Multiple Integrals

3.1: Centre of Mass in 1D



- For two objects, we calculate the *centre of mass* by looking at total mass and the total moment.Take the moment about O, with X as the coordinate of the c.o.m:

$$m_1x_1 + m_2x_2 = (m_1 + m_2)X, X = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{totalmoment}{totalmass}$$

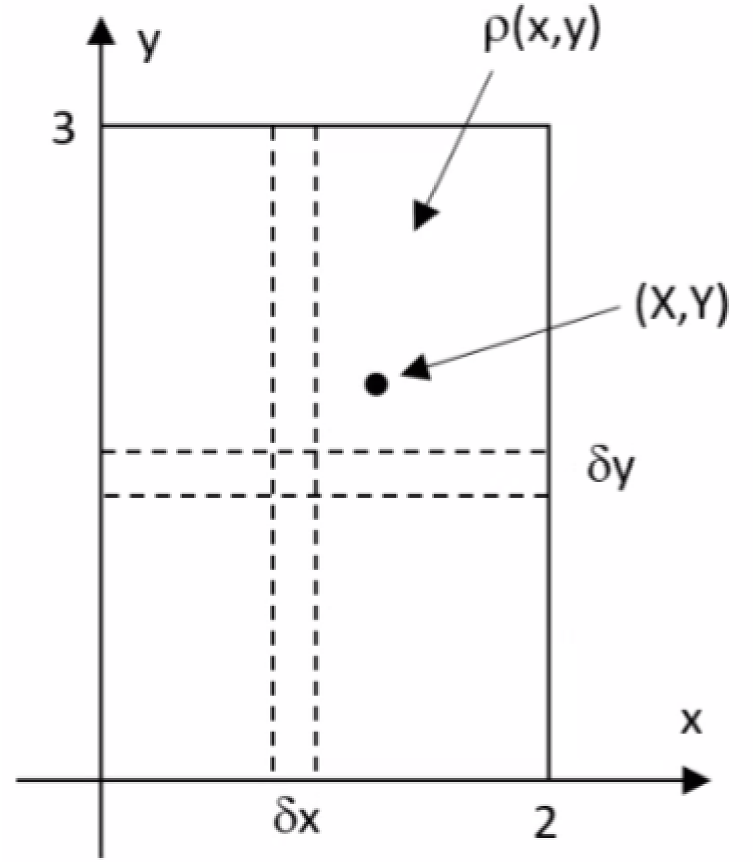
- A collection of discrete objects but all on the same line:

$$X = \frac{\sum m_ix_i}{m_i}$$

- For a continuous bar in 1D, but with varying density $\rho(x)kg.m^{-1}$, we use the same starting with one small piece of the rod:
  - For the mass: $\delta m \approx \rho\delta x, m = \int_0^L \rho(x)dx$
  - For the moment about the O: $\delta M \approx \rho x\delta x, M = \int_0^L \rho(x)xdx$
  - Then the position of X is given by the same ratio, $X = \dfrac{M}{m}$

3.2: Multiple integrals

- Integration in 2D or 3D uses the same concept - the limit of a sum of small contributions.**For example**, to find the mass of a rectangular plate with density $\rho = 3 + x + 2y$, extending from $x = 0 \to 2$ and $y = 0 \to 3$



- For one small piece, $\delta m \approx \rho x\delta y$. For a strip parallel to the x-axis, mass $\approx [\sum \rho\delta x]\delta y$.
- Overall,$m \approx \sum[\sum \rho\delta x]\delta y$, and in the limit $\delta x \to 0$, $\delta y \to 0$.

- $m = \int_{y=0}^{3} [\int_{x=0}^{2} \rho(x,y)dx]dy = \int_{0}^{3}\int_{0}^{2} \rho(x,y)dxdy$
- Note that the inner integral has the end-points 0,2 and is treated as a function of x, with y held constant, before complete the outer integral of y.(Order doesn't matter)
- To find the c.o.m with coordinates(X,Y), we need moments about two axes: $M_X = \iint x\rho dxdy$ (About the y-axis), $M_Y = \iint y\rho dxdy$ (About the x-axis)
- Then $X = \frac{M_X}{m}$,$Y = \frac{M_Y}{m}$
- It is possible to adapt this approach to find the c.o.m, of non-rectangular regions.
- Other applications includes the case of a 2D or 3D body with internal heat sources or heat sinks. If there is equilibrium, then the heat crossing the boundary of the region must equal to the total heat source or sink within it.
- Thus we can equate the **volume integral** of heat produced with the **surface integral** of heat crossing a boundary.