2.4:Systems of ODE I

Example_1:

$$m_1 y_1'' = -k_1 y_1 + k_2 (y_2 - y_1)$$

$$m_2 y_2'' = -k_2 (y_2 - y_1)$$

$$\begin{pmatrix} \ddot{y_1} \\ \ddot{y_2} \end{pmatrix} = \begin{pmatrix} -\frac{(k_1+k_2)}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_1} & -\frac{k_2}{m_2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

- We can take the oscillation (for the second order) :
  - $y_1 = c\cos(\omega t - \alpha_1)$
  - $\ddot{y_1} = \omega^2 y_1$
  - Same for $y_2 = \omega^2 y_2$
  - Any oscillation will got same answer.
  - If we treat that 2×2 matrix as A:

$$\omega^2 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

- It is same as the $Ax = \lambda x$, and the answer of these equations is x, which is the eigenvectors.

2.5: Systems of ODESs II

$$\frac{dx}{dt} = -4x + y$$

$$\frac{dy}{dt} = -5x + 2y$$

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- Let $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{\lambda t}$

- x_0 and y_0 are constant.
- Then we get:

$$\lambda \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

- Use the way of E-value and E-vectors can find the solution.

2.6: Systems of ODEs III

$$\dot{x} = x + y - 2z$$

$$\dot{y} = -x + 2y + z$$

$$\dot{z} = -y - z$$

$$\frac{d}{dt} = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- Just find the E-vector, the equations can be solved.

3: Diagonalisation of matrices and decoupling of systems of equations

- Example:

$$\frac{dx}{dt} = -4x + y$$

$$\frac{dy}{dt} = -5x + 2y$$

- Then:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

- If we assumed $y = Pz$
- It is found that $\lambda_1 = 1$, $x_1 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ and $\lambda_2 = -3$, $x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
- So $P = \begin{pmatrix} 1 & 1 \\ 5 & 1 \end{pmatrix}$
- The diagonalisation $Z = P^{-1}AP^{-1}$
- $Z = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$
- $\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = Z\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$
- It is easy to find $z_1 = Ae^t$ and $z_2 = Be^{-3t}$.
- $y = Pz \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$
- Then the expression of x and y can be found.