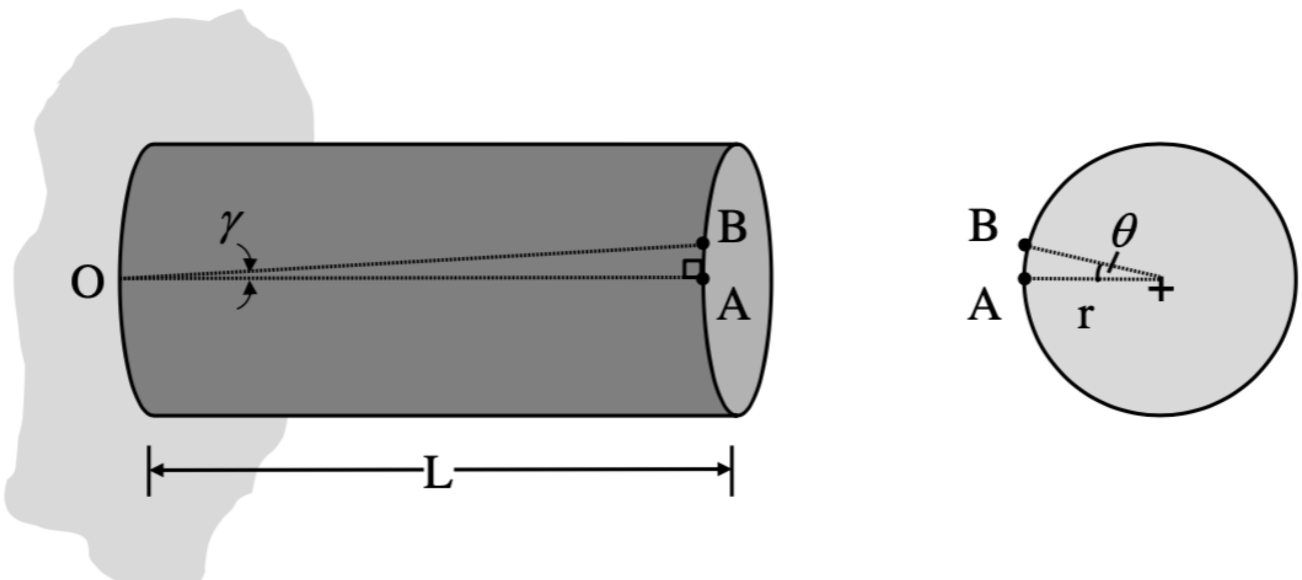
VI: Torsion of shafts

1: Introduction

- Torsion refers to the twisting of a straight shaft or bar when ist is subjected to a torque(s) which results in rotation about the longitudinal axis and induces shear stress.

2: Basic theory of torsion

2.1: Basic concepts

- Consider a solid shaft fixed at one end and twisted at the other end due to a torque T.
- 

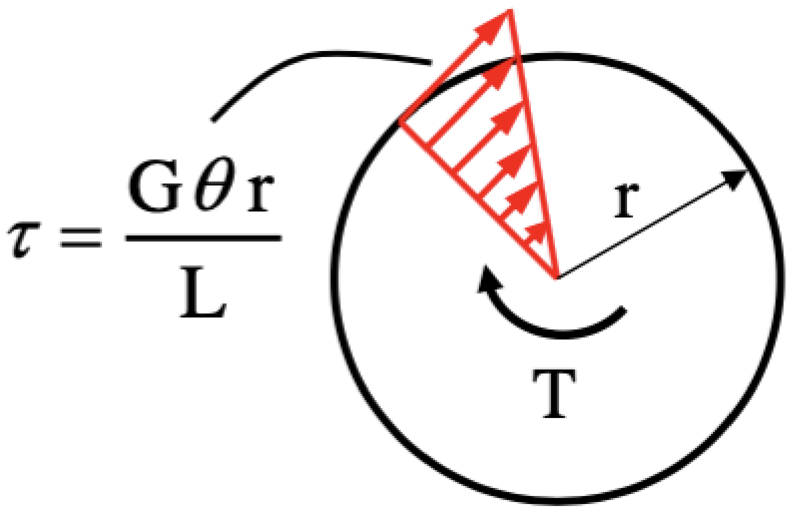
- The original shaft line OA rotates through a angle γ and form a new triangle OAB.
- $\tan\gamma = \dfrac{l_{AB}}{L}$
- Since the **angle γ is very small**, we can use $\tan\gamma = \gamma$.
- So $l_{AB} = L\gamma$
- At the cross section area, $l_{AB} = r\theta$
- Then $L\gamma = r\theta$
- $\gamma = \dfrac{r\theta}{L}$
- Recall the relationship between shear stress and shear strain:

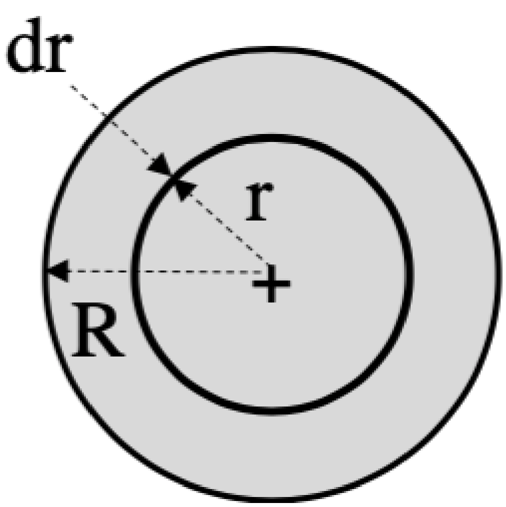
$$G=\frac{\tau}{\gamma} \Rrightarrow \gamma=\frac{\tau}{G}$$

- Finally, we can find:

$$\tau=\frac{G\theta r}{L}$$

2.2: Basic concepts_2

- As the equation show, the shear stress is zero at the centre and reach to the max at the outer of the shaft.
- 

- Consider a ring of material with the thickness dr and a radius r and R.
- 

- $dA=2\pi r \times dr$
- Thus, the force acting on the ring of material:

$$dF = \tau \times dA = \tau \times 2\pi r dr$$

- $dT = dF \times r = \tau \times 2\pi r dr \times r$
- It can be simplified to :

$$dT = \frac{G\theta}{L} 2\pi r^3 dr$$

- The torque for the entire cross-section can be determined as:

$$T = \frac{G\theta}{L} 2\pi \int_0^R r^3 dr$$

- In this equation, the term J is call **polar second moment of area**.

$$J = 2\pi \int_0^R r^3 dr$$

$$T = \frac{G\theta}{L} J$$

or

$$\frac{\tau}{r} = \frac{G\theta}{L} = \frac{T}{J}$$

2.3: The restriction of the formula

- The shaft is straight and of uniform cross-area over the length.
- The torque is constant.
- The cross-section remains circular after the torque applied.
- Radial line remains radial.
- Plane remain normal to the longitudinal axis of the shaft.

3: Torsional stiffness

- The torsional stiffness, k_t is defined as:

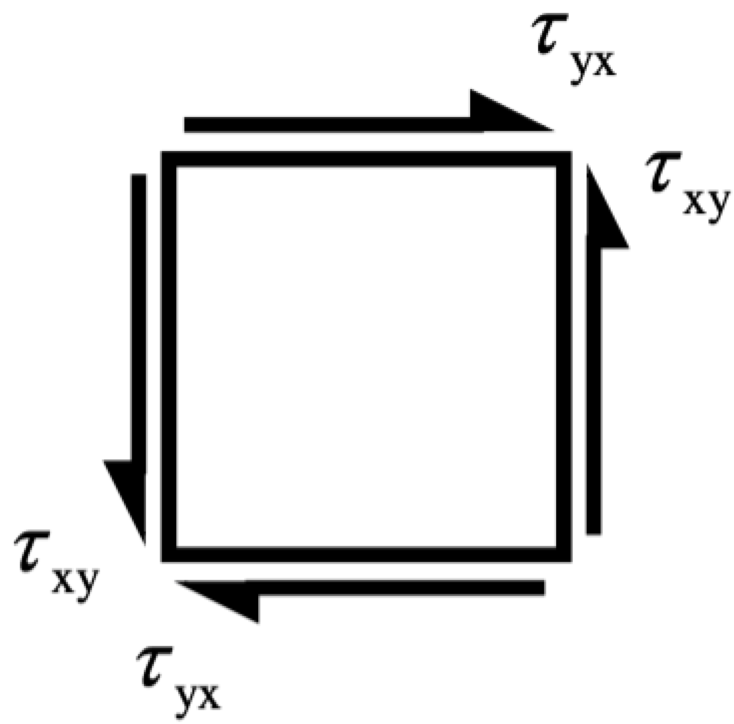
$$k_t = \frac{T}{\theta} = \frac{GJ}{L}$$

4: Polar second moment of area

- For a hollow circular shaft with internal diameter D_i and external diameter D_o:

$$J = \frac{\pi(D_o^4 - D_i^4)}{32}$$

5: Principal stresses and max shear stress

- 

- For the pure shear stress case ($\sigma_{xx} = \sigma_{yy} = 0$ and $\tau_{xy} \neq 0$), if we use the stress transformation equations, we obtained:

$$\sigma_\theta = \tau_{xy} \sin 2\theta$$

$$\tau_\theta = \tau_{xy} \cos 2\theta$$

- For max shear stress, τ_{max} occurs when $\theta = 0$ (vertical)
- For principal stresses:
  - Max occurs when $\theta = 45$
  - Min occurs when $\theta = -45$
- Since the principal stresses acts at 45 and -45 to the maximum shear stress, strain gauges are usually mounted at 45 to a shaft's longitudinal axis so that the principal strains can be measured

directly, which can be used to determine the principal stresses.

6: Torsion failure modes

- In uniaxial tension case the ductile materials fail on a plane of maximum shear stress (45 from the vertical), whereas the brittle materials fail on planes of maximum normal stress.(0 from vertical)
- When subjected to torsion, ductile materials will fails on max shear stress, however the max shear stress acts on 0 degrees from the vertical (perpendicular to shaft's longitudinal axis).
- Subjected to torsion, the brittle materials will fail on planes of max normal stress which is 45 degrees from the vertical.