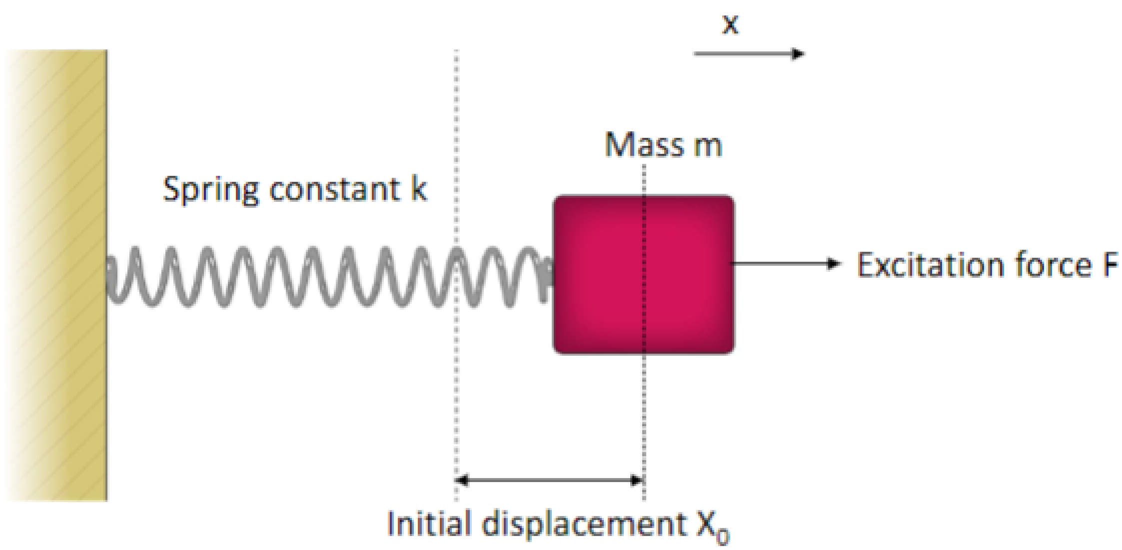
4: Forced undamped vibration



- The system is now acted on by a periodic force (**Excitation force**):

$$F = F_0 \cos(\omega t)$$

- If we consider the static condition firstly:

$$X_{static} = \frac{F_0}{k}$$

- To start with the EOM:

$$F_0 \cos(\omega t) - kx = m\ddot{x}(t)$$

- If we assume the equation of x:

$$x = X_0 \cos(\omega t_\phi)$$

- The substitution occurs as:

$$F_0 \cos(\omega t) - kX_0 \cos(\omega t + \phi) = -m\omega^2 \cos(\omega t + \phi)$$

- As the equation above is satisfied in all situations, so we assume $\omega t = \frac{\pi}{2}$, and we find that $\phi = 0$. So there is no phrase.
- Then we got:

$$\frac{X_0}{F_0} = \frac{1}{k - m\omega^2}$$

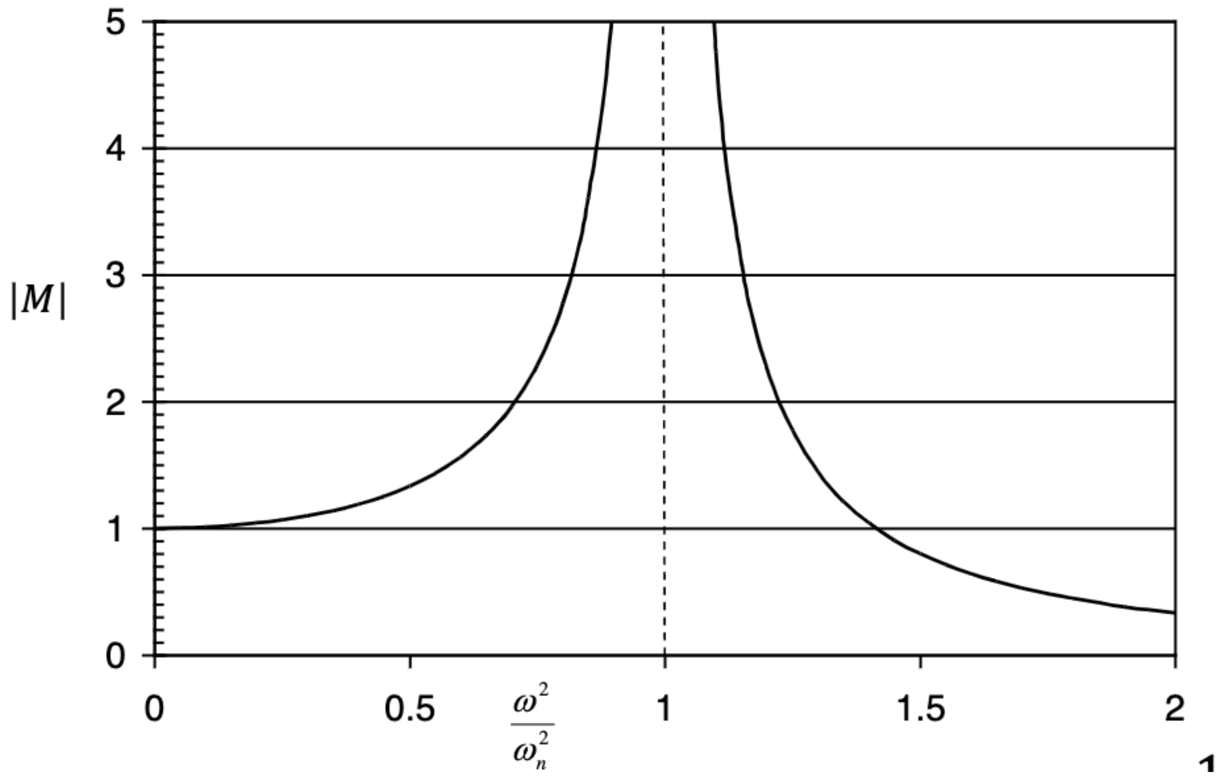
- If we use the conditions we got previously:

$$x_{static} = \frac{F_0}{k}$$
$$\omega_n = \frac{k}{m}$$

- We got the **Magnification factor**:

$$M = \frac{X_0}{x_{static}} = \frac{1}{1 - (\frac{\omega}{\omega_n})^2}$$

- When $\omega \to \omega_n$, we have the resonance.
- The amplitude (X_0) is maximized when ω is minimized or F_0 maximized.
- If we plot the result:



- Another way of looking at the effect of the forcing term is **transmissibility (T)**.
- **T** looks at how much of forcing term is transmitted to the system.
- Force transmitted is connected to the spring force:

$$F_s = kx = kX_0 \cos(\omega t)$$

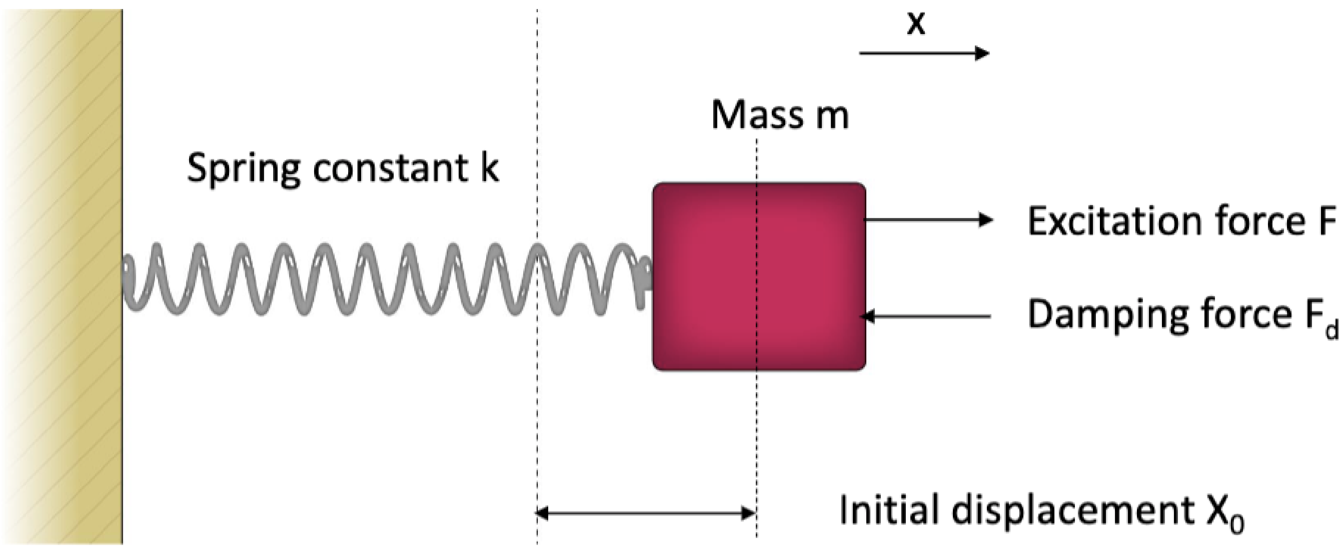
- Transmissibility is defined as the ratio of the amplitude of the force transmitted to he amplitude of the applied force:(F_0 as external force)

$$T = \frac{kX_0}{F_0}$$

- As we defined $M = \frac{X_0}{F_0/k}$, so:

$$T = \frac{1}{|1 - \frac{\omega^2}{\omega_n^2}|}$$

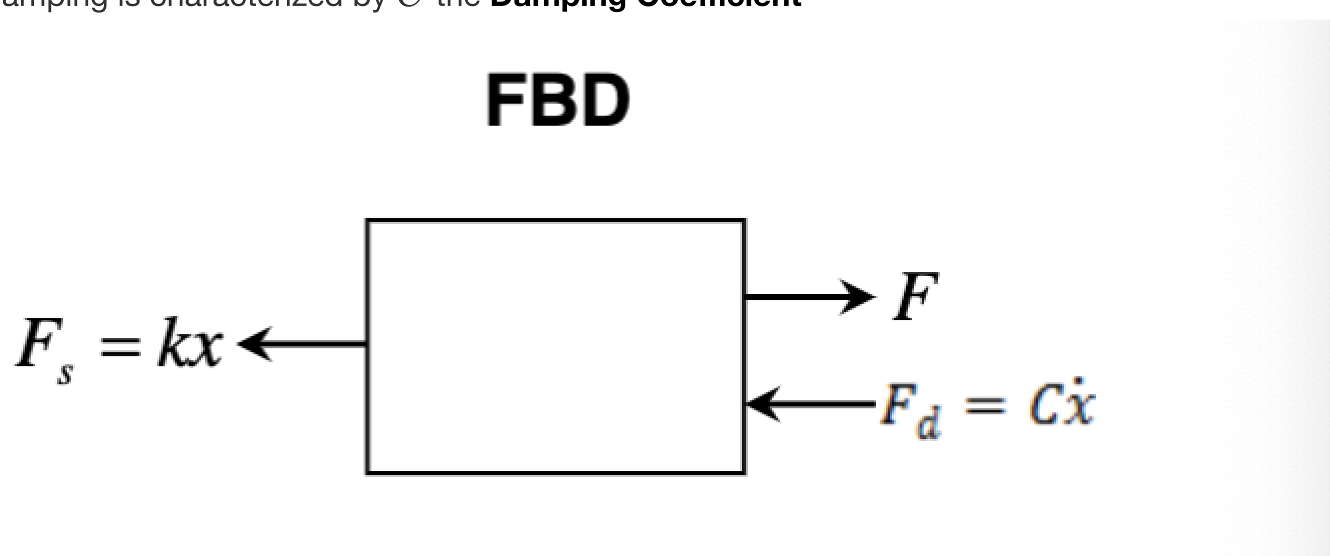
5: Forced Damped Vibration



- The system is acted on a periodic force (**excitation force**) $F = F_0 \cos \omega t$ and there is a damping force which opposes motion.
- We assume that the damping is proportional to velocity (linear damping):

$$F_d = -C\dot{x}$$

- Damping is characterized by C the **Damping Coefficient**



- Firstly, the equation of motion:

$$F - kx - C\ddot{x} = m\ddot{x}$$

- Taking the steady sate form of the excitation force and system response:

$$F = F_0 \cos(\omega t)$$
$$x = X_0 \cos(\omega t + \phi)$$

- The do the substitution:

$$-mX_0\omega^2 \cos(\omega t + \phi) - CX_0\omega \sin(\omega t + \phi) + kX_0 \cos(\omega t + \phi) = F_0 \cos(\omega t)$$

- If we set $\omega t = \frac{\pi}{2}$ to find the ϕ:

$$-(-m\omega^2 + k)\sin(\phi) - C\omega \cos(\phi) = 0$$
$$\tan(\phi) = \frac{-C\omega}{(-m\omega^2 + k)}$$

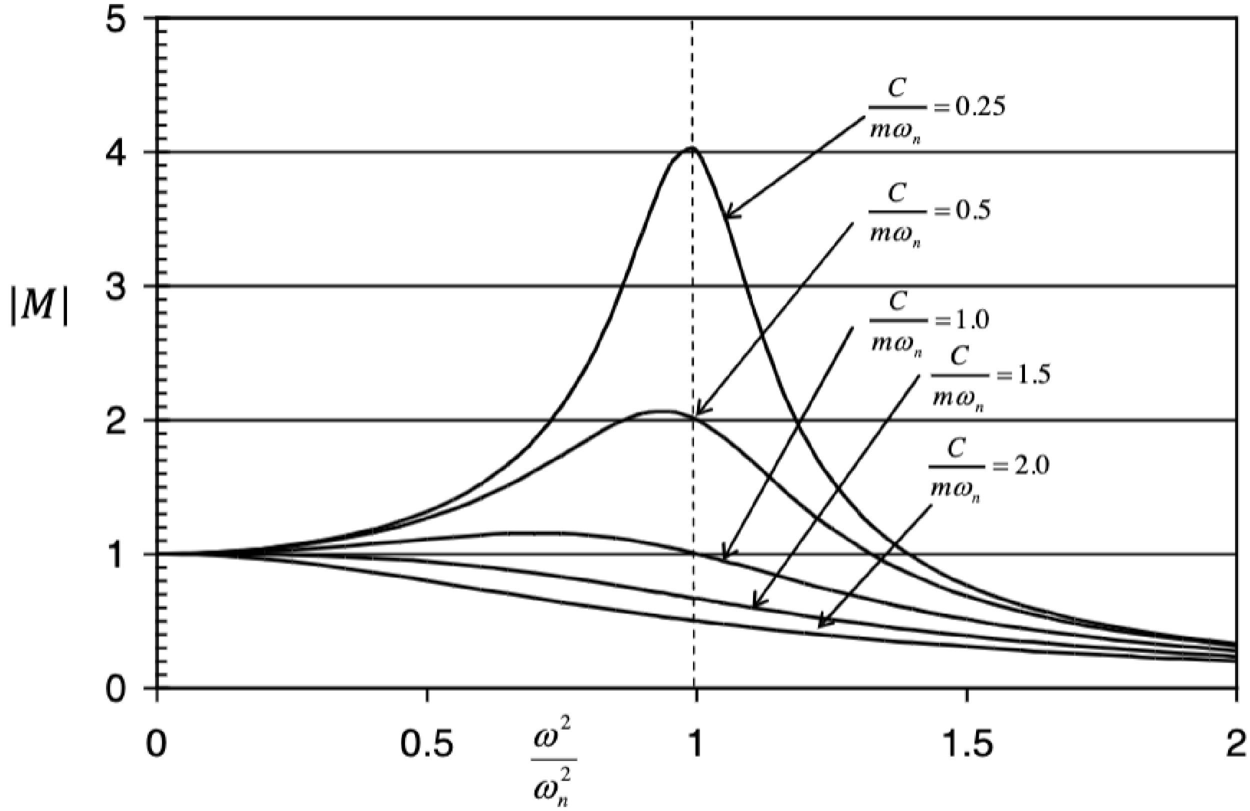
- By setting $\omega t = 0$:

$$\frac{X_0}{F_0} = \frac{1}{(-m\omega^2 + k)\cos(\phi)}$$

- Use the former result of tangent of ϕ:

$$\frac{X_0}{F_0/k} = \frac{1}{\sqrt{(1 - \frac{\omega^2}{\omega_n^2})^2 + (\frac{C}{m\omega_n})^2 \frac{\omega^2}{\omega_n^2}}}$$

- If we plot the result:



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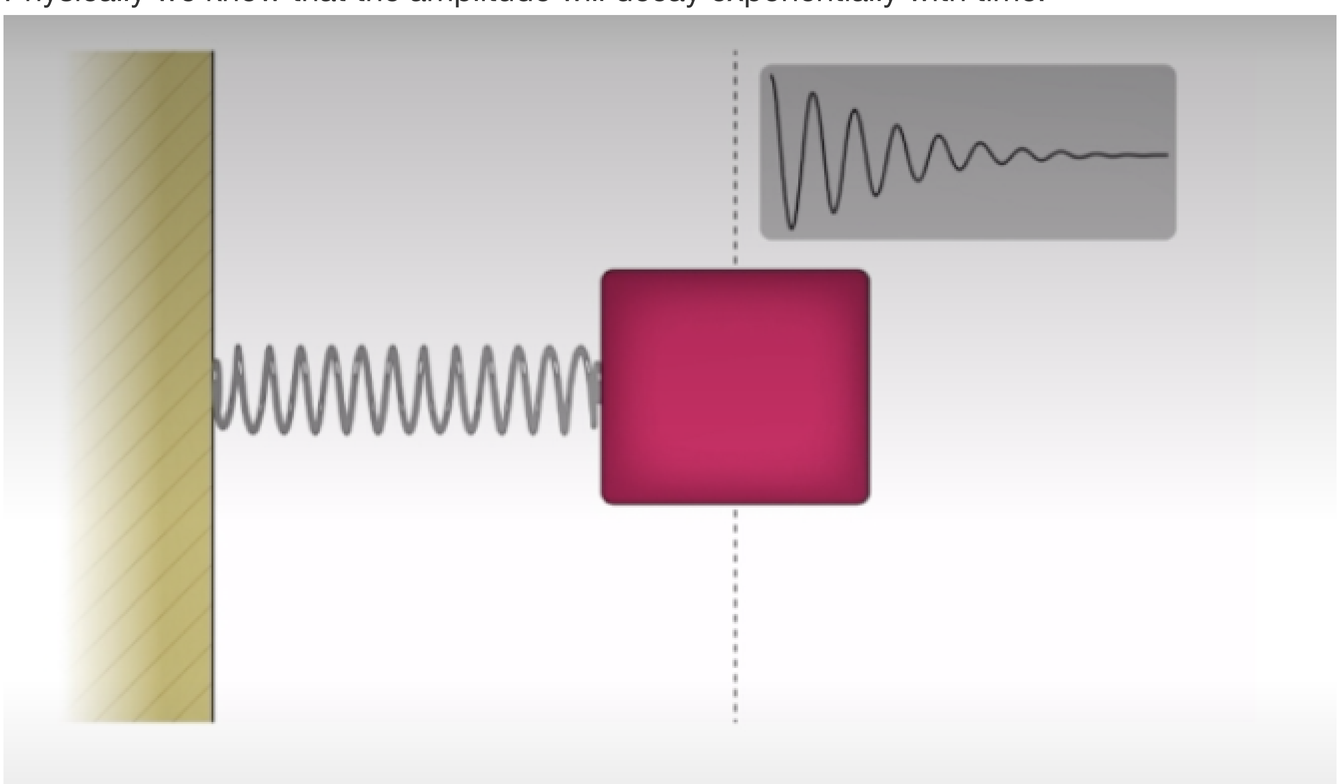
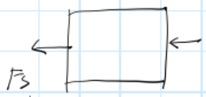
6: Unforced (Free) damped vibration

- Physically we know that the amplitude will decay exponentially with time:



- So $X \to Xe^{-nt}$, where n is an unknown constant.

Free damped vibration:



$F_d = c\dot{x}$

$F_s = kx$

As we treat this example as linear vibration, $F_d = c \cdot \dot{x}$ (damping $\propto v$)

Newton's 2nd Law:

$$-kx - c\dot{x} = m\ddot{x}$$

Assume the form of x with decaying amplitude:

$$x = X_0 e^{-nt} \cos(wt + \phi)$$

Substitude back:

$$m\left[(n^2 - w^2)\cos(wt+\phi) + 2nw\sin(wt+\phi)\right] + c\left[-n\cos(wt+\phi) - w\sin(wt+\phi)\right] + k\cos(wt+\phi) = 0$$

① if we treat $t = 0$, $x = X_0$, $\phi = 0$ for convenience, then set $wt = \frac{\pi}{2}$ to find n:

$$m\left[(2nw)\right] + c\left[-w\right] = 0 \Rightarrow r = \frac{c}{2m}$$

$$\therefore x = X_0 e^{-\frac{c}{2m}t} \cos(wt)$$

② To find w we set $wt = 0$:

$$w^2 = \frac{k}{m} - \frac{cn}{m} + n^2 = \frac{k}{m} - \frac{c^2}{4m^2}$$

$$\Rightarrow W_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$$ — The damping frequency of system

$$\Rightarrow W_d = w_n \cdot \sqrt{1 - \left(\frac{1}{2}\frac{c}{mw_n}\right)^2}$$

Define Critical damping: $c = C_c = 2mW_n$

$$\therefore W_d = w_n \sqrt{1 - \left(\frac{c}{C_c}\right)^2} \Rightarrow x = X_0 e^{-\frac{c}{2m}t} \cdot \cos\left[W_n t \sqrt{1 - \left(\frac{c}{C_c}\right)^2}\right]$$

7: Types of Damping Coefficient

7.1: Critically damped

- The system is at the limit of vi
- vibration:

$$\frac{C}{C_c} = 1$$

7.2: Over damped

- The system does not vibrate and return to equilibrium:

$$\frac{C}{C_c} > 1$$

7.3: Under damped

- The system vibrates with reducing amplitude at a reduced frequency:

$$\frac{C}{C_c} < 1$$