X: Maclaurin Series

1: The Exponential, Sine and Cosine Series

1.1: An infinite power Series

1.1.1: Introduction

- Consider the infinite Series:

$$S = 1 + x + \frac{x^2}{2!} + \frac{x^2}{3!} + \frac{x^4}{4!} + ...$$

- If we differentiate both sides, we can get:

$$\frac{dS}{dx} = S$$

- Noting that $\frac{d}{dx}e^x$, so we see that this is the **exponential Series**:

$$e^x = exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

1.1.2: Properties

- This is a **infinite power series** about $x = 0$, which means it works at $x = 0$ and nearby.
- The series for e^x, converges for all values of x, and x can also be complex number.
- We can say it has an **infinite radius of convergence**.
- It works best for small values of $|x|$ or $|z|$.
- A **Maclaurin Series** in a variable x is just a power series about $x = 0$.
- The series of cosine, sine and exponential have same derivation way. (Differential)

1.2: Using the exponential series

- $e^{10.1} = e^{10}e^{0.1} = e^{10}[1 + 0.1 + \frac{0.01}{2} + ...]$. If we get the value of e^{10}, then we can choose various adaptations of power series like this to get a desired level of accuracy.
- $e^{1+x} = e^1 e^x = e[1 + x + \frac{x^2}{2} + ...]$

- Using a complex argument is permitted, and the choice of $i\theta$ results in two more useful series: (θ in radian)

$$e^{i\theta} = \cos\theta + i\sin\theta = [1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ....] + i[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ...]$$

- This the power series of $\cos(\theta)$ and $\sin(\theta)$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ....$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ...$$

2: The Logarithmic Series

- If there is a series like this: $ln(x) = a_0 + a_1x + a_2x^2 + ...$
- If we change another form of this series:

$$ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$$

- For more usual form, we get **Taylor Series**:

$$ln(y) = (y-1) - \frac{(y-1)^2}{2} + \frac{(y-1)^3}{3} - ....$$

- **Taylor Series** is valid near $y = 1$, and converges if $0 < y \leq 2$. This is a Taylor Series about $y = 1$.
- If we differentiate the series, we get: $\frac{1}{1+x} = 1 - x + x^2 - x^3 + ...$, which is a *geometric series* with first term $a = 1$, ratio $r = -x$.
- The geometric series converges if $|x| < 1$.

3: The Binomial Series

3.1: The Binomial Expansion

$$(a+b)^n = \sum_{r=0}^{n} C_n^r a^{n-r} b^r$$

- This series is **finite** if n is a positive integer.

3.2: The Binomial Series

- If n is negative or a fraction, and we use the same series as above, th series will be **infinite**.
- Firstly, $(a+b)^n = a^n[1+(\frac{b}{a})]^n$, then we use the binomial expansion:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + ...$$

- If the power p is a positive integer, the series terminated as before and always convergent.
- If p is negative integer or a fraction, the series is infinite, and converges only if $|x| < 1$.

4: Applications and Combining series

4.1: Combining series

- If we have a function composed of two or more standard series, we can combined, if we correctly identify the overall domain of convergence for the final series.
- For example:

$$\begin{aligned} f(x) &= \frac{ln(1-2x)}{\sqrt{1-3x}} \\ &= ln(1-2x)(1-3x)^{-1/2} \\ &= [-2x - 2x^2 - \frac{8}{3}x^3 - ...] \times [1 + \frac{3}{2}x + \frac{27}{8}x^2 + ...] \\ &= (-2)x + (3-2)x^2 + (-\frac{27}{4} + 3 - \frac{8}{3})x^3 + ... \end{aligned}$$

- The combined series will converge if both series converge, so we must satisfy both convergence conditions:

$$-1 < -2x \neq 1 \qquad -1 < -3x < 1$$

- So $|x| < \frac{1}{3}$ works for both conditions.

4.2: Applications for Infinite Series

- Infinite Series can be used to find e^x and $\sin(x)$ in computers.

- Series can be used to evaluate the integral, such as binomial expansion can be used to estimate $\displaystyle\int_0^1 \frac{1}{(1+x^2)^3}\mathrm{d}x$.
- To solve more complicated differential equations.