VI: Fourier Series

1: Periodic functions and Fourier Series representation

1.1 Periodic Functions

- The pattern that repeat themselves in the period of t or x.
- These functions can be interpreted as a sum of sine and/or cosine, i.e the sum of the frequencies.
- The mix of the frequencies makes the voice and the sound.
- The combining of sine and cosine components is called Fourier synthesis.
- Breaking down the function into components is called Fourier Analysis.

1.2 Representing a function using harmonic functions

- The most of the function we considered have the period of 2π.
- Let:

$$f(t) = C + a_1 \cos(t) + a_2 \cos(2t) + ... + b_1 sin(t) + b_2 \sin(2t)...$$

$$i.e: f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(nt) + b_n \sin(nt)]$$

- The term C represent the mean value of the function (DC component in oscillation)

1.3 Finding the coefficient

- To find the coefficient term, we calculated the mean value of $f(t)$, i.e.$C = \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt$
- The way to solve the components is to multiply $\cos(t)$ in both sides of the $f(t)$.
- Then integrate both sides in $[-\pi, \pi]$.
- Cause the integration of $\cos(t)$ in the range $[-\pi, \pi]$ =0.

- So we can find that:
  $a_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cos(t)dt$
- It is clear that we can find the related component by multiply the related cos or sin.
- The final solutions:

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt$$

$$b_n = \frac{1}{\pi}$$

1.4 Examples

1.4.1 The square wave

- $f(t) = -2$ if $-\pi \leq t < 0$
- $f(t) = 2$ if $0 \leq t \leq \pi$
- This a odd function, so we need sine only.($a_n = 0$)
- $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt$
- $b_n = \frac{8}{n\pi}$ if n is odd or =0 if n is even.
- $f(x) = \frac{8}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)}\sin(2m-1)t$
- We look at the partial sum and the discontinuity of this function.

1.4.2 $f(t) = t^2$ on the interval $[-\pi, \pi]$

- $f(t)$ is a even, so only cosines are needed, and $b_n = 0$
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$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 dt = \frac{\pi^2}{3}$$

- $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \cos(nt)dt = \frac{4}{n^2}(-1)^n$
- So the final form of the Fourier Series is:

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

2:Complex form of the Fourier Series

- We can combine the cosine and sine to a complex form:$c_n = a_n + ib_n$
- $f(x) = \sum_{-\infty}^{\infty} c_n e^{-inx}$
- $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx$

3: Other forms of Fourier Series

3.1 General interval

- If we use a general interval in $[a, b]$, the series takes this form:
  - $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{2\pi nt}{b-a} + b_n \sin(\frac{2\pi nt}{b-a})$

3.2: Half-range series

- $f(t) = \sum a_n \cos(nt/2)$
- $f(t) = \sum b_n \sin(nt/2)$

3.3 Discrete data

If the function we wish to analyse is discrete points instead of the algebraic function,the numerical integration is needed,while the limited data will come out spurious result.