FORMALITY CONJECTURE FOR K3 SURFACES

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Abstract. On a complex projective K3 surface, we apply the uniqueness of DG enhancement of the bounded derived category of coherent sheaves to give a proof of the formality conjecture of Kaledin and Lehn: the DG algebra $\mathrm{RHom}^\bullet(F,F)$ is formal for any sheaf F polystable with respect to an ample line bundle. We also extend the formality result to objects that are polystable with respect to a generic Bridgeland stability condition.

1. Introduction

1.1. **Background.** Given a complex K3 surface X with an ample line bundle H, the moduli space $\mathcal{M}_{X,H}(\mathbf{v})$ of H-semistable coherent sheaves of a fixed Mukai vector $\mathbf{v}$ has been extensively studied for a long time. It parametrizes S-equivalence classes of H-semistable sheaves E on X with Mukai vector $\mathbf{v}$. The points represented by strictly semistable sheaves form the singular locus of $\mathcal{M}_{X,H}(\mathbf{v})$. In each S-equivalence class, there is a unique H-polystable sheaf, hence there is a one-to-one correspondence between closed points of the $\mathcal{M}_{X,H}(\mathbf{v})$ and H-polystable sheaves of Mukai vector $\mathbf{v}$.

If we choose H and $\mathbf{v}$ so that every H-semistable sheaf of class $\mathbf{v}$ is H-stable, then the moduli space $\mathcal{M}_{X,H}(\mathbf{v})$ is a *holomorphic symplectic manifold* due to a celebrated theorem of Mukai [Muk84]. For the choice of H and $\mathbf{v}$ so that strictly semistable sheaves exists, one could ask whether $\mathcal{M}_{X,H}(\mathbf{v})$ admits a symplectic resolution. This question has already a complete answer when H is *generic* with respect to $\mathbf{v}$. On one hand, O'Grady constructed in [O'G99] a symplectic resolution of $\mathcal{M}_{X,H}(2,0,-2)$, which turned out to represent a new deformation type of holomorphic symplectic manifolds. This was later generalized to all Mukai vectors $\mathbf{v}=2\mathbf{v}_p$ with $\mathbf{v}_p^2=2$ in [LS06]. On the other hand, it was proven in [KLS06] that a symplectic resolution does not exist for any other singular $\mathcal{M}_{X,H}(\mathbf{v})$ with generic H. When H is not generic, a recent work [AS18] studied symplectic resolutions of $\mathcal{M}_{X,H}(\mathbf{v})$ when $\mathbf{v}$ is of rank 0 from the point of view of quiver varieties.

The search for symplectic resolutions of singular moduli spaces naturally leads to the study of the singularity types that appear in them. It was proven in [KLS06] that all singularities are *symplectic* in the sense of [Bea00] when H is generic. In all the above discussions, only the information of the normal cone at a singular point was essentially necessary. In a work of Kaledin and Lehn [KL07] that motivated the present paper, the authors started the investigation of the (analytic) local models of these singularities. We will follow their path and give an explicit description of such local models in the present paper.

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1.2. **First conjecture.** To motivate the ideas of Kaledin and Lehn, we follow [KL07, §3.1] and [KLS06, §2.6] to briefly recall some ingredients in the deformation theory of sheaves and the construction of moduli spaces; we refer the readers to [Man04, Man09] and [HL10] for more comprehensive discussions.

The moduli space $\mathcal{M}_{X,H}(\mathbf{v})$ is constructed as a GIT quotient of (a closed subscheme of) a Quot scheme by a group $\mathrm{PGL}(N)$ for some large N. Let F be an H-polystable sheaf of class $\mathbf{v}$, given in the form of

$$F = F_1^{\oplus n_1} \oplus \cdots \oplus F_k^{\oplus n_k}$$

where $F_1, \cdots, F_k$ are distinct H-stable summands. In the GIT-semistable locus of the Quot scheme, there is a closed $\mathrm{PGL}(N)$-orbit parametrizing different framings of F, in which we fix a point q. The stabilizer of the $\mathrm{PGL}(N)$ action at q is given by the group

$$G = \mathrm{Aut}(F)/\mathbb{C}^* = (\mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_k))/\mathbb{C}^*$$

where $\mathbb{C}^*$ acts as scalar multiplications. Luna's slice theorem implies the existence of a G-invariant locally closed subscheme Z containing q, with $T_q Z \cong \mathrm{Ext}^1(F,F)$, such that the canonical morphism

$$Z /\!\!/ G \longrightarrow \mathcal{M}_{X,H}(\mathbf{v})$$

is étale. More explicitly, an analytic (or formal) neighborhood of $q \in Z$ can be given as the zero fiber $\kappa^{-1}(0)$ of the *Kuranishi map*:

$$\kappa = \kappa_2 + \kappa_3 + \cdots : \mathrm{Ext}^1(F,F) \longrightarrow \mathrm{Ext}^2(F,F)_0$$

where $\mathrm{Ext}^2(F,F)_0 = \ker(\mathrm{Ext}^2(F,F) \xrightarrow{\mathrm{tr}} H^2(X, \mathcal{O}_X) \cong \mathbb{C})$, and κ_i is the sum of terms of degree i. The Kuranishi map κ is G-equivariant, hence an analytic (or formal) neighborhood of $[F] \in \mathcal{M}_{X,H}(\mathbf{v})$ can be given as $\kappa^{-1}(0)/G$. It is also known that κ_2 is given by the Yoneda pairing

$$\kappa_2 : \mathrm{Ext}^1(F,F) \longrightarrow \mathrm{Ext}^2(F,F)_0; \quad e \longmapsto e \cup e.$$

However, the higher order terms are usually hard to compute. In order to give an explicit description of a neighborhood of $q \in Z$, hence of $[F] \in \mathcal{M}_{X,H}(\mathbf{v})$, Kaledin and Lehn have essentially made the following conjecture in [KL07]:

Conjecture 1.1. *There is an isomorphism* $\kappa^{-1}(0) \cong \kappa_2^{-1}(0)$.

Indeed, this conjecture has been proven by Yoshioka in [Yos16, Corollary 0.6], along with a special case in [AS18, Theorem 3.9]. We will mention the idea of their proofs below. We point out that this conjecture implies immediately that an analytic neighborhood of $[F] \in \mathcal{M}_{X,H}(\mathbf{v})$ is given by $\kappa_2^{-1}(0)/G$ by [AS18, Proposition 4.4].

1.3. **Second conjecture.** Along with the above geometric description of the local model of the moduli space, there is a stronger conjecture from an algebraic point of view. The guiding principle of this algebraic formalism is the following: given a *differential graded* (DG for short) *(Lie) algebra*, say $A^\bullet$, there is a standard algebraic procedure to associate to it a deformation functor, which in turn determines a Kuranishi map $\kappa_{A^\bullet}$ via the so-called Maurer-Cartan equation. In our particular case, the deformations of F are governed by the DG algebra $\mathrm{RHom}^\bullet(F,F)$. It is also known that if two DG (Lie) algebras $A^\bullet$ and $B^\bullet$ are quasi-isomorphic, then we have an isomorphism of Kuranishi spaces $\kappa_{A^\bullet}^{-1}(0) \cong \kappa_{B^\bullet}^{-1}(0)$; see [GM90, Theorem

4.8]. (Indeed, we will see in Remark 2.25 that the DG algebra $\mathrm{RHom}^\bullet(F,F)$ is well-defined up to quasi-isomorphisms.)

For any DG (Lie) algebra $A^\bullet$, its homology algebra $H^\bullet(A^\bullet)$ is a DG (Lie) algebra with zero differentials. We say $A^\bullet$ is *formal* if it is quasi-isomorphic to $H^\bullet(A^\bullet)$. In such a situation, the Kuranishi space $\kappa_{A^\bullet}^{-1}(0)$ is particularly simple. Indeed, it is isomorphic to $\kappa_{H^\bullet(A^\bullet)}^{-1}(0)$, which is given by the zero locus of the cup product map

$$H^1(A^\bullet) \longrightarrow H^2(A^\bullet); \quad e \longmapsto e \cup e$$

due to the vanishing differential in $H^\bullet(A^\bullet)$; see [GM90, Theorem 5.3]. In our case, we have $H^i(\mathrm{RHom}^\bullet(F,F)) = \mathrm{Ext}^i(F,F)$, and the above cup product is precisely the Yoneda pairing κ_2. Using this language, Kaledin and Lehn have actually conjectured in [KL07] the following:

Conjecture 1.2. *The DG algebra* $\mathrm{RHom}^\bullet(F,F)$ *is formal if* F *is* H*-polystable.*

The above discussion shows that Conjecture 1.1 can be deduced from Conjecture 1.2. However, we point out that the latter is much stronger than the former. Indeed, the DG (Lie) algebra encodes very rich information of higher structures in the deformation problem that it governs, which is not captured by the (commutative and underived) moduli space. See e.g. [Tod17a] for its relation to non-commutative deformation theory and [Toe17] for its relation to derived deformation theory.

1.4. **Known results.** Various authors have studied Conjectures 1.1 and 1.2. In [KL07], Kaledin and Lehn gave an affirmative answer to both conjectures in a very first case, namely, $F = \mathcal{I}_Z^{\oplus n}$, where $\mathcal{I}_Z$ is the ideal sheaf of a 0-dimensional closed subscheme $Z \subset X$. Indeed, this is precisely the case that is needed for the discussion in the rest of [KL07]. However, their beautiful proof in this special case paved the way for the future development of theory.

We briefly explain how they proved the formality of $\mathrm{RHom}^\bullet(F,F)$ for $F = \mathcal{I}_Z^{\oplus n}$. The key ingredient in their proof is a formality criterion established by Kaledin in [Kal07] (see also [Lun10]). Given a DG algebra $A^\bullet$, Kaledin identified an obstruction class $K_{A^\bullet}$ in the obstruction space $HH^2(\widetilde{A}^\bullet)$, where $\widetilde{A}^\bullet$ is a DG algebra constructed from $A^\bullet$, such that the vanishing of $K_{A^\bullet}$ is a necessary and sufficient condition for the formality of $A^\bullet$.

Although it is generally hard to compute $K_{A^\bullet}$ explicitly, it is sometimes very helpful to consider how this class varies in families. Namely, if we fit $A^\bullet$ into a family of DG algebras $\mathcal{A}^\bullet \to S$ over an irreducible base S with a suitable flatness assumption, then the obstruction space $HH^2(\widetilde{\mathcal{A}}_s^\bullet)$ for each fiber glues into an obstruction bundle Ob_S over S, and the obstruction class $K_{\mathcal{A}_s^\bullet}$ glues into a global section of the obstruction bundle $\sigma_{\mathcal{A}^\bullet} \in \Gamma(S, \mathrm{Ob}_S)$. Kaledin made two nice observations [Kal07, Theorem 4.3] to make use of this point of view:

- if $K_{\mathcal{A}_s^\bullet} = 0$ for a generic choice of $s \in S$, then it holds for every $s \in S$; in other words, the formality of a generic fiber $\mathcal{A}_s^\bullet$ implies the formality of every fiber in the family $\mathcal{A}^\bullet$ over S;
- if Ob_S has no non-trivial global sections, then $K_{\mathcal{A}_s^\bullet} = 0$ for every $s \in S$; in such a case, $\mathcal{A}_s^\bullet$ is formal for every fiber in the family $\mathcal{A}^\bullet$ over S.

By virtue of [Kal07, Theorem 4.3], Kaledin and Lehn considered in [KL07, §6] the twistor space $\mathcal{X} \to \mathbb{P}^1$ associated to the polarized K3 surface (X, H). When Z comprises n distinct closed points of X, the sheaf $F = \mathcal{I}_Z^{\oplus n}$ and the DG algebra $\mathrm{RHom}^\bullet(F, F)$ can be both lifted to a flat family over the base $\mathbb{P}^1$ of the twistor space. A simple calculation shows that the obstruction bundle has no non-trivial global sections, which implies the formality of $\mathrm{RHom}^\bullet(F, F)$.

In [Zha12], the second named author combined the approach in [KL07] and a theorem of hyperholomorphic bundles in [Ver08], and proved the formality of $\mathrm{RHom}^\bullet(F, F)$ in a few more cases; see Proposition 3.1 for a precise statement. It was also pointed out in [Zha12] that the technique in [KL07] is no longer enough for tackling the remaining cases, because the obstruction bundle does acquire non-trivial global sections. Therefore new ideas are required.

Despite the silence on the formality Conjecture 1.2, several authors studied Conjecture 1.1 and gave positive answers. In [AS18, Theorem 3.7], Arbarello and Saccà proved the quadraticity Conjecture 1.1 for polystable sheaves of pure dimension 1. In [Yos16, Corollary 0.6], Yoshioka proved Conjecture 1.1 in full generality. Their approaches are very similar. Indeed, they used some integral functors to relate the unknown cases to the cases already treated in [Zha12]. By proving that these integral functors preserve stability, they obtain an isomorphism of the Kuranishi spaces, hence the validity of Conjecture 1.1 on the known side implies its validity on the other side.

1.5. **Our approach.** The goal of the present paper is to show how the techniques in DG category theory can be used to study formality questions. Our main observation is the following:

Proposition 1.3 (Proposition 2.26)**.** *For smooth projective varieties X and Y, assume that*

$$\Phi : D^b(\mathrm{Coh}(X)) \longrightarrow D^b(\mathrm{Coh}(Y)) \tag{1}$$

is a derived equivalence. Then for any object $E \in D^b(\mathrm{Coh}(X))$, the DG algebra $\mathrm{RHom}^\bullet(E, E)$ is formal if and only if $\mathrm{RHom}^\bullet(\Phi(E), \Phi(E))$ is formal.

Based on this observation, we can solve Conjecture 1.2 in general. Indeed, given any polystable sheaf F for which the formality of $\mathrm{RHom}^\bullet(F, F)$ is unknown previously, there exists some Fourier-Mukai transform Φ by [Yos09, Theorem 1.7], such that $\Phi(F)$ is a polystable sheaf that satisfies the assumption required in [Zha12] (more precisely, condition (ii) in Proposition 3.1). Proposition 1.3 then immediately implies our main result:

Theorem 1.4 (Theorem 3.7)**.** *Let (X, H) be a complex projective K3 surface, and F a H-polystable sheaf on X. Then the DG algebra $\mathrm{RHom}^\bullet(F, F)$ is formal. In other words, Conjecture 1.2 holds in general.*

As explained in [Tod17b], we can combine [Tod17b, Theorem 1.3] and Theorem 1.4 to recover the main theorem (Theorem 1.1) of [AS18] on symplectic resolutions via variations of GIT quotients of quiver varieties. Combining [BS16, Proposition 1.2] and Theorem 1.4, we also obtain that $\mathcal{M}_{X,H}(\mathbf{v})$ has symplectic singularities, generalizing slightly [KLS06, Theorem 6.2]; see Remark 3.9.

Moreover, given a Mukai vector $\mathbf{v}$ and a generic Bridgeland stability condition σ (in the distinguished component $\mathrm{Stab}^\dagger(X)$ of the stability manifold) on X, by applying the results from [Bri08, Yos09, BM14a, BM14b] in various cases, we can find an autoequivalence Θ of $D^b(\mathrm{Coh}(X))$, which induces a bijection between σ-polystable objects of Mukai vector $\mathbf{v}$ and L-polystable sheaves of some Mukai vector $\mathbf{u}$ for some ample line bundle L. As another application of Proposition 1.3, we obtained:

Corollary 1.5 (Corollary 3.12)**.** *Let X be a projective K3 surface, $\mathbf{v}$ a Mukai vector, and $\sigma \in \mathrm{Stab}^\dagger(X)$ a Bridgeland stability condition that is generic with respect to $\mathbf{v}$. For any σ-polystable object E of Mukai vector $\mathbf{v}$, the DG algebra $\mathrm{RHom}^\bullet(E,E)$ is formal.*

We briefly explain the main idea in the proof of Proposition 1.3, which requires some language in DG category theory. Given a triangulated category $\mathscr{T}$, a *DG enhancement* of $\mathscr{T}$ is a pre-triangulated DG category $\mathscr{A}$, whose homotopy category is equivalent to $\mathscr{T}$. As an example, we consider $\mathscr{T} = D^b(\mathrm{Coh}(X))$ for a smooth projective variety X. A familiar DG enhancement of $D^b(\mathrm{Coh}(X))$ is given by the category $\mathscr{A} = \mathbf{Inj}(X)$, consisting of all injective resolutions of objects in $D^b(\mathrm{Coh}(X))$. For any injective resolution $\mathcal{I}^\bullet$ of an object $E \in D^b(\mathrm{Coh}(X))$, the DG algebra $\mathrm{RHom}^\bullet(E,E)$ is precisely given by the space of self-homomorphisms $\mathrm{Hom}_{\mathbf{Inj}(X)}(\mathcal{I}^\bullet, \mathcal{I}^\bullet)$ in the DG category $\mathbf{Inj}(X)$.

The main ingredient in the proof of Proposition 1.3 is a deep theorem of Lunts and Orlov [LO10, Theorem 2.14], which states that the bounded derived category $D^b(\mathrm{Coh}(X))$ of a smooth projective variety X admits a strongly unique DG enhancement. Roughly speaking, given an equivalence Φ as in (1), [LO10, Theorem 2.14] implies that Φ can be lifted to DG level, which gives a *quasi-equivalence* between the DG categories $\mathbf{Inj}(X)$ and $\mathbf{Inj}(Y)$ (see Definition 2.13 for a precise statement). This in particular implies that the spaces of self-homomorphisms $\mathrm{Hom}_{\mathbf{Inj}(X)}(\mathcal{I}^\bullet, \mathcal{I}^\bullet)$ and $\mathrm{Hom}_{\mathbf{Inj}(Y)}(\mathcal{J}^\bullet, \mathcal{J}^\bullet)$ are quasi-isomorphic DG algebras, for injective resolutions $\mathcal{I}^\bullet$ and $\mathcal{J}^\bullet$ of $E \in D^b(\mathrm{Coh}(X))$ and $\Phi(E) \in D^b(\mathrm{Coh}(Y))$ respectively, which verifies Proposition 1.3.

Finally, we point out that our results should have a symplectic counterpart via Homological Mirror Symmetry. Kontsevich's HMS conjecture says that $D^b(\mathrm{Coh}(X))$ is equivalent to another triangulated category constructed from the symplectic geometry of the mirror of a Calabi-Yau manifold X. Corollary 2.20 below guarantees that formality would be mirrored. For K3 surfaces, HMS is known for (the mirror of) the smooth quartic in $\mathbb{P}^3$ by [Sei15]. However, before any mirror conclusions of our formality results for K3 surfaces are drawn, a technical hurdle must be passed: we use $\mathbb{C}$ coefficients, whereas HMS uses Novikov rings as coefficients.

1.6. **Structure of the Paper.** Our presentation will be divided into two main parts with very different flavors.

§2 is the categorical part of the paper, in which we will establish Proposition 1.3. After reviewing some necessary terminologies of DG categories in §2.1, we will show that the preservation of formality holds for equivalences of triangulated categories with strongly unique DG enhancement in §2.2. We will then apply this result to bounded derived categories of smooth projective varieties to prove Proposition 1.3.

§3 is the geometric part of the paper, in which we will prove Conjecture 1.2 in its full generality. We will first recall some necessary terminologies in the theory of moduli spaces of sheaves on K3 surfaces in §3.1. The proof of Theorem 1.4 on the formality for polystable sheaves will be given in §3.2. Then we extend the result to derived objects and prove Corollary 1.5 in §3.3. Two technical lemmas required for the proof of Corollary 1.5 will be proven in §3.4.

We remark that we will allow arbitrary ground field $\Bbbk$ in §2. However we assume $\Bbbk = \mathbb{C}$ in the entire §3.

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2. Formality via Uniqueness of DG Enhancement

2.1. **Generalities on DG categories.** We collect some classical concepts which will be used in the discussion. We follow mainly [LO10]. Other well-written references include [Dri04, Kel06, Toe11, CS16]. We always work over a fixed field $\Bbbk$. All categories are assumed to be small and $\Bbbk$-linear.

Definition 2.1. A *DG category* is a $\Bbbk$-linear category $\mathscr{A}$ whose morphism spaces $\mathrm{Hom}(A_1, A_2)$ are DG $\Bbbk$-modules (aka complexes of $\Bbbk$-vector spaces), such that

$$\mathrm{Hom}_{\mathscr{A}}(A_1, A_2) \otimes \mathrm{Hom}_{\mathscr{A}}(A_2, A_3) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A_1, A_3) \tag{2}$$

are morphisms of DG $\Bbbk$-modules for any objects $A_1, A_2, A_3 \in \mathrm{Ob}(\mathscr{A})$. Moreover, for any $A \in \mathrm{Ob}(\mathscr{A})$, there is an identity morphism $1_A \in \mathrm{Hom}_{\mathscr{A}}(A, A)$ which is closed of degree 0 and compatible with the composition.

Remark 2.2. In particular, the definition implies that the graded Leibniz rule holds and $\mathrm{Hom}_{\mathscr{A}}(A, A)$ is a DG algebra for any $A \in \mathrm{Ob}(\mathscr{A})$.

Definition 2.3. The *homotopy category* $H^0(\mathscr{A})$ of a DG category $\mathscr{A}$ is a $\Bbbk$-linear category with the same objects as in $\mathscr{A}$ and morphism spaces

$$\mathrm{Hom}_{H^0(\mathscr{A})}(A_1, A_2) = H^0(\mathrm{Hom}_{\mathscr{A}}(A_1, A_2))$$

for any $A_1, A_2 \in \mathrm{Ob}(\mathscr{A})$.

Definition 2.4. A *DG functor* $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ between two DG categories is given by a map of sets

$$\mathcal{F} : \mathrm{Ob}(\mathscr{A}) \to \mathrm{Ob}(\mathscr{B})$$

and morphisms of DG $\Bbbk$-modules

$$\mathcal{F}(A_1, A_2) : \mathrm{Hom}_{\mathscr{A}}(A_1, A_2) \longrightarrow \mathrm{Hom}_{\mathscr{B}}(\mathcal{F}(A_1), \mathcal{F}(A_2))$$

for any $A_1, A_2 \in \mathrm{Ob}(\mathscr{A})$, compatible with compositions (2) and units.

Definition 2.5. A DG functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ is called a *quasi-equivalence* if $\mathcal{F}(X, Y)$ is a quasi-isomorphism for any $X, Y \in \mathrm{Ob}(\mathscr{A})$ and the induced functor on the homotopy categories

$$H^0(\mathcal{F}) : H^0(\mathscr{A}) \longrightarrow H^0(\mathscr{B})$$

is an equivalence of categories.

Remark 2.6. In fact, instead of requiring $H^0(\mathcal{F})$ to be an equivalence, it is sufficient to require it to be essentially surjective, as the full faithfulness is already encoded in the quasi-isomorphisms of morphism spaces. See e.g. [Toe11, Definition 2, §2.3].

We denote the category of small DG categories with DG functors as morphisms by **dgCat**, and its localization with respect to quasi-equivalences by **Hqe**. It was proven in [Tab05] that **dgCat** has the structure of a model category, with quasi-equivalences being the weak equivalences in the model structure. Then **Hqe** is the homotopy category of this model category. One special property of this model structure on **dgCat** is that every small DG category is a fibrant object.

Definition 2.7. A morphism between two DG categories in **Hqe** is called a *quasi-functor*. We say two DG categories are *quasi-equivalent* if they are isomorphic in **Hqe**.

By the above definition, two quasi-equivalent DG categories can be connected by a zig-zag chain of DG functors with alternative arrow directions. In fact, we have the following simpler presentation for a quasi-functor.

Lemma 2.8. *Let $\mathscr{A}$ and $\mathscr{B}$ be DG categories. Any quasi-functor from $\mathscr{A}$ to $\mathscr{B}$ can be represented by the diagram*

$$\mathscr{A} \xleftarrow{f} \mathscr{C} \xrightarrow{g} \mathscr{B} \tag{3}$$

where $\mathscr{C}$ is a DG category, f and g are DG functors, with f being a quasi-equivalence. Moreover, $\mathscr{A}$ and $\mathscr{B}$ are quasi-equivalent if and only if g is also a quasi-equivalence.

Proof. By the fundamental theorem of model categories [Hov99, Theorem 1.2.10 (ii)], we can represent a quasi-functor from $\mathscr{A}$ to $\mathscr{B}$ by a DG functor $\mathscr{C} \xrightarrow{g} \mathscr{D}$, where $\mathscr{C} \xrightarrow{f} \mathscr{A}$ is a cofibrant replacement of $\mathscr{A}$, and $\mathscr{B} \xrightarrow{h} \mathscr{D}$ is a fibrant replacement of $\mathscr{B}$. Since $\mathscr{B}$ itself is a fibrant object, we can choose $\mathscr{D} = \mathscr{B}$ and h the identity functor. Hence we get (3). The other statement follows from [Hov99, Theorem 1.2.10 (iv)]. $\square$

Remark 2.9. The above result shows that a quasi-functor between DG categories can be represented by a single roof of DG functors. This result must be well-known to experts; e.g. it is mentioned in [LO10, p.858]. We supply a proof for the sake of completeness.

For any DG category $\mathscr{A}$, it was constructed in [BK91] the *pre-triangulated hull* $\mathscr{A}^{\text{pre-tr}}$ of $\mathscr{A}$ by formally adding to $\mathscr{A}$ all shifts, all cones of morphisms, and cones of morphisms between cones, etc. There is a canonical embedding of DG categories $\mathscr{A} \hookrightarrow \mathscr{A}^{\text{pre-tr}}$.

Definition 2.10. A DG category $\mathscr{A}$ is said to be *pre-triangulated* if for every object $A \in \mathscr{A}$ and $n \in \mathbb{Z}$, the object $X[n] \in \mathscr{A}^{\text{pre-tr}}$ is homotopy equivalent to an object in $\mathscr{A}$, and for every closed morphism f in $\mathscr{A}$ of degree 0, the object $\mathrm{Cone}(f) \in \mathscr{A}^{\text{pre-tr}}$ is homotopy equivalent to an object in $\mathscr{A}$.

Remark 2.11. In other words, a DG category $\mathscr{A}$ is pre-triangulated if and only if the DG functor $\mathscr{A} \hookrightarrow \mathscr{A}^{\text{pre-tr}}$ is a quasi-equivalence; equivalently, the embedding of the homotopy categories $H^0(\mathscr{A}) \hookrightarrow H^0(\mathscr{A}^{\text{pre-tr}})$ is an equivalence. In such a case, $H^0(\mathscr{A})$ is naturally a triangulated category.

Definition 2.12. A *DG enhancement* of a triangulated category $\mathscr{T}$ is a pair $(\mathscr{B}, e)$, where $\mathscr{B}$ is a pre-triangulated DG category and $e : H^0(\mathscr{B}) \to \mathscr{T}$ is an equivalence of triangulated categories.

Definition 2.13. We say a triangulated category $\mathscr{T}$ has a *unique DG enhancement* if, given two DG enhancements $(\mathscr{B}, e)$ and $(\mathscr{B}', e')$ of $\mathscr{T}$, there exists a quasi-functor $\mathcal{F} : \mathscr{B} \to \mathscr{B}'$ such that $H^0(\mathcal{F}) : H^0(\mathscr{B}) \to H^0(\mathscr{B}')$ is an equivalence of triangulated categories. We say $\mathscr{T}$ has a *strongly unique DG enhancement* if moreover $\mathcal{F}$ can be chosen so that the functors e and $e' \circ H^0(\mathcal{F})$ are isomorphic.

Remark 2.14. Under the assumption of Definition 2.13, $\mathscr{B}$ and $\mathscr{B}'$ are quasi-equivalent, due to the pretriangulation. See e.g. [CS16, Definition 1.15].

2.2. **Preservation of formality.** In this part we explain why the uniqueness of DG enhancement of a triangulated category can help with formality problems. The following observation will be the key to our discussions. It might also be well-known to experts; however, we are not aware of a detailed proof in the literature. Since it plays a fundamental role in our discussion, we supply a proof with details.

Proposition 2.15. *Let $\mathscr{T}$ be a triangulated category, and $(\mathscr{B}_1, e_1)$ and $(\mathscr{B}_2, e_2)$ DG enhancements of $\mathscr{T}$. Let $T \in \mathrm{Ob}(\mathscr{T})$, $B_1 \in \mathrm{Ob}(\mathscr{B}_1)$, $B_2 \in \mathrm{Ob}(\mathscr{B}_2)$, such that T, $e_1(B_1)$ and $e_2(B_2)$ are isomorphic in $\mathscr{T}$. Assume that $\mathscr{T}$ has a strongly unique DG enhancement, then $\mathrm{Hom}_{\mathscr{B}_1}(B_1, B_1)$ and $\mathrm{Hom}_{\mathscr{B}_2}(B_2, B_2)$ are quasi-isomorphic DG algebras.*

Proof. For $i = 1$ and 2, we construct a full subcategory $\mathscr{C}_i$ of $\mathscr{B}_i$, whose objects are given by

$$\mathrm{Ob}(\mathscr{C}_i) = \mathrm{Ob}(\mathscr{B}_i) \backslash \{B \in \mathrm{Ob}(\mathscr{B}_i) \mid e_i(B) \cong T \text{ in } \mathscr{T}, B \neq B_i\}.$$

Clearly $\mathscr{C}_i$ is also a DG category. We claim it is pre-triangulated. Indeed, let $\mathscr{B}_i^{\text{pre-tr}}$ and $\mathscr{C}_i^{\text{pre-tr}}$ be the pre-triangulated hulls of $\mathscr{B}_i$ and $\mathscr{C}_i$ respectively. Then all functors in the commutative diagram

$$\begin{tikzcd} H^0(\mathscr{B}_i) \arrow[r, hook] & H^0(\mathscr{B}_i^{\text{pre-tr}}) \\ H^0(\mathscr{C}_i) \arrow[r, hook] \arrow[u, hook] & H^0(\mathscr{C}_i^{\text{pre-tr}}) \arrow[u, hook] \end{tikzcd}$$

are fully faithful. By the assumption that $\mathscr{B}_i$ is pre-triangulated, the upper horizontal arrow is an equivalence. By the construction of $\mathscr{C}_i$, the left vertical arrow is also an equivalence. In particular, they are essentially surjective. Hence the bottom horizontal arrow must be essentially surjective, hence an equivalence, which proves that $\mathscr{C}_i$ is a pre-triangulated DG category. Moreover, since the composition

$$H^0(\mathscr{C}_i) \hookrightarrow H^0(\mathscr{B}_i) \xrightarrow{e_i} \mathscr{T}$$

is an equivalence of categories, we conclude that $\mathscr{C}_i$ is a DG enhancement of $\mathscr{T}$.

By assumption, $\mathscr{T}$ has a strongly unique DG enhancement. Therefore by Lemma 2.8, there exists some DG category $\mathscr{C}_0$, such that both functors f_1 and f_2 in the roof

$$\mathscr{C}_1 \xleftarrow{f_1} \mathscr{C}_0 \xrightarrow{f_2} \mathscr{C}_2$$

are quasi-equivalences. In particular, all functors in the diagram

$$\begin{tikzcd} H^0(\mathscr{C}_0) \arrow[r, "H^0(f_2)"] \arrow[d, "H^0(f_1)"'] & H^0(\mathscr{C}_2) \arrow[d] \\ H^0(\mathscr{C}_1) \arrow[r] & \mathscr{T} \end{tikzcd}$$

are equivalences of categories, and the diagram is 2-commutative (the two compositions from $H^0(\mathscr{C}_0)$ to $\mathscr{T}$ are isomorphic functors).

By the essential surjectivity of f_1, there exists some $B_0 \in \mathrm{Ob}(\mathscr{C}_0)$, such that $f_1(B_0) \cong B_1$ in $H^0(\mathscr{C}_1)$. By the construction of $\mathscr{C}_1$, we know that B_1 is the only object in its isomorphism class of objects in $H^0(\mathscr{C}_1)$, hence $f_1(B_0) = B_1$. By the 2-commutativity of the diagram, the images of $f_2(B_0)$ and B_2 are both isomorphic to T in $\mathscr{T}$, hence $f_2(B_0)$ and B_2 themselves are in the same isomorphism class of objects in $H^0(\mathscr{C}_2)$, which implies $f_2(B_0) = B_2$ by the construction of the category $\mathscr{C}_2$.

Since f_1 and f_2 are quasi-equivalences, the morphism

$$f_i(B_0, B_0) : \mathrm{Hom}_{\mathscr{C}_0}(B_0, B_0) \longrightarrow \mathrm{Hom}_{\mathscr{C}_i}(B_i, B_i)$$

is a quasi-isomorphism of DG algebras for $i = 1$ and 2. Since $\mathscr{C}_i$ is a full subcategory of $\mathscr{B}_i$ for $i = 1$ and 2, we conclude that $\mathrm{Hom}_{\mathscr{B}_1}(B_1, B_1)$ and $\mathrm{Hom}_{\mathscr{B}_2}(B_2, B_2)$ are quasi-isomorphic DG algebras. □

Remark 2.16. Under the assumption of the above proposition, we can associate canonically to any $T \in \mathrm{Ob}(\mathscr{T})$ a DG algebra $\mathrm{Hom}_{\mathscr{B}_1}(B_1, B_1)$ (for any lift B_1 of T in any DG enhancement $\mathscr{B}_1$ of $\mathscr{T}$), which is well-defined up to quasi-isomorphisms. For convenience, we will denote this (quasi-isomorphism class of) DG algebra by $\mathrm{RHom}^\bullet(T, T)$.

The following alternative formulation of the proposition will be useful for us.

Corollary 2.17. *Let $\Phi : \mathscr{T}_1 \to \mathscr{T}_2$ be an equivalence of triangulated categories. Assume that $\mathscr{T}_2$ (hence $\mathscr{T}_1$) has a strongly unique DG enhancement. Then for any object $T \in \mathrm{Ob}(\mathscr{T}_1)$, the (quasi-isomorphism classes of) DG algebras $\mathrm{RHom}^\bullet(T, T)$ and $\mathrm{RHom}^\bullet(\Phi(T), \Phi(T))$ are quasi-isomorphic.*

Proof. Let $(\mathscr{B}_i, e_i)$ be a DG enhancement of $\mathscr{T}_i$ for $i = 1$ and 2. Let $B_i \in \mathrm{Ob}(\mathscr{B}_i)$ such that $e_1(B_1) \cong T$ in $\mathscr{T}_1$ and $e_2(B_2) \cong \Phi(T)$ in $\mathscr{T}_2$. Then it is clear that

$(\mathscr{B}_1, \varphi \circ e_1)$ is also a DG enhancement of $\mathscr{T}_2$ and $(\Phi \circ e_1)(B_1) \cong \varphi(T)$. We apply Proposition 2.15 on the enhancements $(\mathscr{B}_1, \Phi \circ e_1)$ and $(\mathscr{B}_2, e_2)$ and conclude that $\mathrm{Hom}_{\mathscr{B}_1}(B_1, B_1)$ and $\mathrm{Hom}_{\mathscr{B}_2}(B_2, B_2)$ are quasi-isomorphic DG algebras. By Remark 2.16, this precisely means that $\mathrm{RHom}^\bullet(T, T)$ and $\mathrm{RHom}^\bullet(\Phi(T), \Phi(T))$ are quasi-isomorphic. ☐

Definition 2.18. A DG algebra $A^\bullet$ is *formal* if it is quasi-isomorphic to its homology algebra $H^\bullet(A^\bullet)$.

The following result is elementary.

Lemma 2.19. *Assume that DG algebras $A^\bullet$ and $B^\bullet$ are quasi-isomorphic. Then their homology algebras $H^\bullet(A^\bullet)$ and $H^\bullet(B^\bullet)$ are isomorphic.*

Proof. Since quasi-isomorphic DG algebras can be connected by a chain of quasi-isomorphic morphisms with alternating arrow directions, without loss of generality, it suffices to show the result with an extra assumption that there is a morphism $\varphi : A^\bullet \to B^\bullet$ of DG algebras that induces the given quasi-isomorphism. It is easy to see that φ induces a morphism of algebras $H^\bullet(\varphi) : H^\bullet(A^\bullet) \to H^\bullet(B^\bullet)$. The condition that φ is a quasi-isomorphism implies that $H^\bullet$ is an isomorphism of algebras. The general case follows by an induction argument. ☐

Corollary 2.20. *In Corollary 2.17, the DG algebra* $\mathrm{RHom}(T, T)$ *is formal if and only if* $\mathrm{RHom}(\Phi(T), \Phi(T))$ *is formal.*

Proof. By assumption, $\mathrm{RHom}^\bullet(T, T)$ and $\mathrm{RHom}^\bullet(\Phi(T), \Phi(T))$ are quasi-isomorphic. By Lemma 2.19, $H^\bullet(\mathrm{RHom}^\bullet(T, T))$ and $H^\bullet(\mathrm{RHom}^\bullet(\Phi(T), \Phi(T)))$ are isomorphic. Since quasi-isomorphism is an equivalence relation, we conclude that $\mathrm{RHom}^\bullet(T, T)$ and $H^\bullet(\mathrm{RHom}^\bullet(T, T))$ are quasi-isomorphic if and only if $\mathrm{RHom}^\bullet(\Phi(T), \Phi(T))$ and $H^\bullet(\mathrm{RHom}^\bullet(\Phi(T), \Phi(T)))$ are quasi-isomorphic, as required. ☐

2.3. **Geometric situation.** From now on we focus on the geometric situation; namely, the DG enhancements of derived categories of coherent sheaves on smooth projective varieties. The following result establishes the applicability of the above theory.

Theorem 2.21. [LO10, Theorem 2.14] *Let X be a smooth projective variety. Then the triangulated category $D^b(\mathrm{Coh}(X))$ has a strongly unique DG enhancement.* ☐

Remark 2.22. For any object $E \in D^b(\mathrm{Coh}(X))$, the above theorem implies in particular that the DG algebra $\mathrm{RHom}^\bullet(E, E)$ is well-defined up to quasi-isomorphisms by Remark 2.16.

Indeed, there are several explicit DG enhancements for $D^b(\mathrm{Coh}(X))$ in the literature, which should be equivalent in the sense of Definition 2.13 by Theorem 2.21. We give one such example; see more details in [CS16, §1.2] and [BLL04, §6.1].

Consider the category $\mathbf{Inj}(X)$, whose objects are bounded below complexes of injective quasi-coherent sheaves on X with bounded coherent cohomologies. Given two such complexes $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$, the space of morphisms between them is given by a complex $\mathrm{Hom}_{\mathbf{Inj}(X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$, whose term in degree n is given by

$$\mathrm{Hom}^n_{\mathbf{Inj}(X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Qcoh}(X)}(\mathcal{E}^i, \mathcal{F}^{n+i}).$$

The differential defined on any homogeneous element $f \in \mathrm{Hom}^n_{\mathbf{Inj}(X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ of degree n is given by

$$d(f) = d_{\mathcal{F}^\bullet} \circ f - (-1)^n f \circ d_{\mathcal{E}^\bullet}.$$

It is easy to check that $d \circ d = 0$. The following result is mentioned in [BLL04, §6.1] and [CS16, §1.2].

Lemma 2.23. *The category* $\mathbf{Inj}(X)$ *is a pre-triangulated DG category.* $\square$

We turn to its homotopy category $H^0(\mathbf{Inj}(X))$, whose objects are the same as the objects in $\mathbf{Inj}(X)$. The space of morphisms between two objects $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ in the homotopy category is given by

$$\mathrm{Hom}_{H^0(\mathbf{Inj}(X))}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = H^0(\mathrm{Hom}_{\mathbf{Inj}(X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)).$$

Such a category $H^0(\mathbf{Inj}(X))$ is a triangulated category. Moreover we have

Lemma 2.24. *There exists an exact equivalence* $e : H^0(\mathbf{Inj}(X)) \to D^b(\mathrm{Coh}(X))$*. In other words,* $\mathbf{Inj}(X)$ *is a DG enhancement of* $D^b(\mathrm{Coh}(X))$*.*

Proof. This result is also in [BLL04, §6.1] and [CS16, §1.2]. More explicitly, let $D^b_{\mathrm{Coh}}(\mathrm{Qcoh}(X))$ be the derived category of bounded below complexes of quasi-coherent sheaves on X with bounded coherent cohomologies, then the natural functor

$$\varphi : H^0(\mathbf{Inj}(X)) \longrightarrow D^b_{\mathrm{Coh}}(\mathrm{Qcoh}(X))$$

is an equivalence by [Huy06, Proposition 2.40]. Moreover the natural functor

$$\psi : D^b(\mathrm{Coh}(X)) \longrightarrow D^b_{\mathrm{Coh}}(\mathrm{Qcoh}(X))$$

is also an equivalence by [Huy06, Proposition 3.5]. By abuse of notation we write ψ^{-1} for any quasi-inverse of ψ. Then

$$e = \psi^{-1} \circ \varphi : H^0(\mathbf{Inj}(X)) \longrightarrow D^b(\mathrm{Coh}(X))$$

is an equivalence as required. $\square$

An explicit quasi-inverse of φ, say φ^{-1}, is given by choosing an injective resolution for each object in $D^b_{\mathrm{Coh}}(\mathrm{Qcoh}(X))$, as explained in [Huy06, Proposition 2.35]. In particular, this holds for every object in $D^b(\mathrm{Coh}(X))$.

Remark 2.25. By Remark 2.16 and Theorem 2.21, we immediately know that for any object object $F \in D^b(\mathrm{Coh}(X))$, the DG algebra $\mathrm{RHom}^\bullet(F, F)$ is well-defined up to quasi-isomorphisms. In other words, if we choose two different injective resolutions $\mathcal{I}^\bullet$ and $\mathcal{J}^\bullet$ of F, the DG algebras $\mathrm{Hom}_{\mathrm{Inj}(X)}(\mathcal{I}^\bullet, \mathcal{I}^\bullet)$ and $\mathrm{Hom}_{\mathrm{Inj}(X)}(\mathcal{J}^\bullet, \mathcal{J}^\bullet)$ are quasi-isomorphic. Notice that it is relatively easy to show that they are quasi-isomorphic as complexes (see e.g. [Huy06, Corollary 3.14]); but it may not be easy to see that they are quasi-isomorphic DG algebras without invoking Theorem 2.21.

The following consequence is our main observation:

Proposition 2.26. *Let X and Y be smooth projective varieties. Assume*

$$\Phi : D^b(\mathrm{Coh}(X)) \longrightarrow D^b(\mathrm{Coh}(Y))$$

is an equivalence of triangulated categories. Then for any object $E \in D^b(\mathrm{Coh}(X))$, the DG algebra $\mathrm{RHom}^\bullet(E, E)$ is formal if and only if $\mathrm{RHom}^\bullet(\Phi(E), \Phi(E))$ is formal.

Proof. This follows immediately from Corollary 2.20 and Theorem 2.21. □

3. Proof of Formality Conjecture on K3 Surfaces

3.1. **Generalities on moduli of sheaves.** From this point on, we will always assume that the ground field $\Bbbk = \mathbb{C}$. We first recall some terminologies in the theory of moduli spaces of semistable sheaves; see [HL10] for details.

Throughout this section, we assume that (X, H) is a complex projective K3 surface. Let F be a coherent sheaf on X. We say $\mathbf{v}$ is the *Mukai vector* of F if

$$\mathbf{v} = \mathrm{ch}(F) \cdot \sqrt{\mathrm{td}(X)}.$$

If we write the components of $\mathbf{v}$ as

$$\mathbf{v} = (\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2) \in H^0(X, \mathbb{Z}) \oplus \mathrm{NS}(X) \oplus H^4(X, \mathbb{Z}) = H^*_{alg}(X, \mathbb{Z}),$$

then the *dual* of $\mathbf{v}$ is defined by

$$\mathbf{v}^\vee = (\mathbf{v}_0, -\mathbf{v}_1, \mathbf{v}_2).$$

Given two classes $\mathbf{v}, \mathbf{w} \in H^*_{alg}(X, \mathbb{Z})$, their *Mukai pairing* is defined by

$$\mathbf{v} \cdot \mathbf{w} = -\mathbf{v}_0\mathbf{w}_2 + \mathbf{v}_1\mathbf{w}_1 - \mathbf{v}_2\mathbf{w}_0,$$

where the products on the right hand side are Poincaré pairings. For any non-zero class $\mathbf{v} \in H^*_{alg}(X, \mathbb{Z})$, we can always decompose it canonically as

$$\mathbf{v} = m\mathbf{v}_p,$$

where $\mathbf{v}_p$ is a primitive class and m is a positive integer.

The (semi)stability of a coherent sheaf F with respect to the ample class H is in the Gieseker sense, defined by comparing the reduced Hilbert polynomial of F and those of its proper subsheaves. Occasionally in our discussion, we also need to consider the μ_H-(semi)stability of a torsion free sheaf F, which is defined by comparing the H-slope of F and those of its proper subsheaves.

We say an H-semistable sheaf F is *H-polystable* if it can be written in the form of

$$F = F_1^{\oplus n_1} \oplus \cdots \oplus F_k^{\oplus n_k} \tag{4}$$

where $F_1, \cdots, F_k$ are pairwise non-isomorphic H-stable summands, and $n_1, \cdots, n_k$ are positive integers.

The moduli space of H-semistable coherent sheaves on X of Mukai vector $\mathbf{v}$ is written as $\mathcal{M}_{X,H}(\mathbf{v})$. Its closed points are one-to-one correspondent to S-equivalence classes of semistable sheaves. Since there is a unique H-polystable sheaf in each S-equivalence class, the closed points of $\mathcal{M}_{X,H}(\mathbf{v})$ are also one-to-one correspondent to polystable sheaves it parametrizes.

3.2. **Formality property for sheaves.** In this part, we will apply Proposition 2.26 to prove Conjecture 1.2. We first recall the cases when the formality conjecture is already known to be true. The following result was proven in [KL07, Zha12]:

Proposition 3.1. *Let (X, H) be a complex projective K3 surface, and $\mathbf{v}$ a Mukai vector of positive rank. Assume H is generic with respect to $\mathbf{v}$, and there is at least one μ_H-stable sheaf of Mukai vector $\mathbf{v}$. Let F be a H-polystable sheaf with a decomposition given by (4). Assume either*

(i) $\operatorname{rk} F_i = 1$ *for all* $i = 1, \cdots, k$*; or*
(ii) $\operatorname{rk} F_i \geqslant 2$ *for all* $i = 1, \cdots, k$*,*

then the DG algebra $\operatorname{RHom}^\bullet(F, F)$ *is formal.*

Proof. See [KL07, Proposition 3.1] and [Zha12, Proposition 1.3]. □

Remark 3.2. It was observed in [AS18, Remark 3.4 (2)] that the assumption of H being generic with respect to $\mathbf{v}$ is not necessary for the proof of case (ii) in Proposition 3.1. However, we also point out that the proof of case (i) does require that H is generic.

Remark 3.3. As explained in [Zha12, §2], instead of requiring the existence of a μ_H-stable sheaf of Mukai vector $\mathbf{v}$, it suffices to require that each stable summand F_i has a μ_H-stable deformation in its own moduli.

We will prove Conjecture 1.2 in full generality, even without the assumption that the polarization H is generic. The main idea is to apply Proposition 2.26 to reduce all the remaining cases, namely, either when the ranks of stable summands of E are arbitrary combination of positive integers, or when E is torsion, or when there is no μ_H-stable sheaf of class $\mathbf{v}$, to case (ii) in Proposition 3.1.

To apply Proposition 2.26, we need to find suitable derived equivalences. In fact, we only need the following two types of (auto)equivalences:

- Tensoring with a line bundle $-\otimes H : D^b(\operatorname{Coh}(X)) \to D^b(\operatorname{Coh}(X))$ is a derived equivalence;
- Let $\mathcal{I}_\Delta$ be the ideal sheaf of the diagonal embedding $X \hookrightarrow X \times X$, then the integral functor $\Phi : D^b(\operatorname{Coh}(X)) \to D^b(\operatorname{Coh}(X))$ with kernel $\mathcal{I}_\Delta$ is a derived equivalence.

They are both classical examples of (auto)equivalences; see e.g. [Huy06, Examples 10.9]. We elaborate the latter in more details.

We write the projections from $X \times X$ to both factors by p and q respectively. Let $\mathcal{I}_\Delta$ be the ideal sheaf of the diagonal embedding $X \hookrightarrow X \times X$, then the integral functor Φ is defined by

$$\begin{aligned} \Phi : D^b(\operatorname{Coh}(X)) &\longrightarrow D^b(\operatorname{Coh}(X)) \\ F &\longmapsto \mathbf{R}q_*(p^*F \overset{\mathbf{L}}{\otimes} \mathcal{I}_\Delta). \end{aligned} \tag{5}$$

The integral functor Φ is a Fourier-Mukai transform; indeed, it is the spherical twist induced by the spherical object $\mathcal{O}_X$. First we describe how it transforms Mukai vectors. This is given by the following result of Yoshioka:

Lemma 3.4. *For any $F \in D^b(\mathrm{Coh}(X))$, the Mukai vector of F can be uniquely written as*

$$\mathbf{v} = (r, dH + D, a)$$

for some $r, a \in \mathbb{Z}$, $d \in \mathbb{Q}$ and $D \in \mathrm{NS}(X)_{\mathbb{Q}} \cap H^{\perp}$. And the Mukai vector of $\Phi(F)$ is given by

$$\widehat{\mathbf{v}} = (a, -(dH + \widehat{D}), r)$$

for some $\widehat{D} \in \mathrm{NS}(X)_{\mathbb{Q}} \cap H^{\perp}$.

Proof. This is a special case of [Yos09, Proposition 1.5] for the functor (5). Notice that in this case we have $\widehat{H} = H$ due to [Yos09, Remark 1.3]. □

Remark 3.5. An explicit formula for $\widehat{D}$ is given in [Yos09, (1.4)]. However it will be irrelevant to our discussion.

Next we discuss the preservation of stability under the Fourier-Mukai transform Φ as defined in (5). The following result of Yoshioka [Yos09, Theorem 1.7] is crucial:

Theorem 3.6. *Let (X, H) be a projective K3 surface. Let $\mathbf{v} = (r, dH + D, a)$ be a Mukai vector with $r \geqslant 0$ and $D \in \mathrm{NS}(X)_{\mathbb{Q}} \cap H^{\perp}$, and $\widehat{\mathbf{v}} = (a, -(dH + \widehat{D}), r)$. Then in either of the two following cases:*

(1) *$r > 0$, $a > 0$ and*

$$d > \max\left\{4r^2 + \frac{1}{H^2},\ 2r(\mathbf{v}^2 - D^2)\right\},$$

(2) *$r = 0$ and*

$$a > \max\left\{3,\ \frac{1}{2}(\mathbf{v}^2 - D^2) + 1\right\},$$

the Fourier-Mukai transform Φ in (5) induces an isomorphism of moduli spaces of H-semistable sheaves $\mathcal{M}_{X,H}(\mathbf{v}) \cong \mathcal{M}_{X,H}(\widehat{\mathbf{v}})$, which preserves H-polystability and S-equivalence classes. Moreover, for every H-polystable sheaf F given by (4), the image $\Phi(F_i)$ of each H-stable summand F_i is μ_H-stable.

Proof. This is a special case of [Yos09, Theorem 1.7] for the functor Φ as defined in (5). Notice that the G_1-twisted and G_2-twisted (semi)stabilities in the statement of [Yos09, Theorem 1.7] are simply the classical Gieseker (semi)stability in our situation, since we have $\mathbf{v}(G_1) = \mathbf{v}(G_2) = (1, 0, 0)$.

Although the preservation of H-polystability was not explicitly stated in [Yos09, Theorem 1.7], it was also proven along with the other statements. Indeed, for an H-polystable sheaf F with decomposition (4), the equation

$$\Phi(F) = \Phi(F_1)^{\oplus n_1} \oplus \cdots \oplus \Phi(F_k)^{\oplus n_k}. \tag{6}$$

gives the decomposition of $\Phi(F)$ into a direct sum of stable summands; see [Yos09, p.132, p.134]. It follows from (the proof of) [Yos09, Corollary 2.14] that each summand $\Phi(F_i)$ is μ_H-stable. □

We are now ready to prove Conjecture 1.2, even without the assumption that the polarization H is generic.

Theorem 3.7. *Let (X, H) be a complex projective K3 surface, and F an H-polystable sheaf of Mukai vector $\mathbf{v}$. Then the DG algebra $\mathrm{RHom}^\bullet(F, F)$ is formal. In other words, Conjecture 1.2 holds in general.*

Proof. We write $\mathbf{v} = (r, dH + D, a)$ as in Lemma 3.4. For the moment we assume $(r, dH + D) \neq (0, 0)$. We observe that $r \geqslant 0$, and that $r = 0$ would imply $d > 0$ since otherwise $dH + D$ is not effective. Then for any positive integer m we have

$$\mathbf{v} \cdot e^{mH} = \left(r, dH + D + rmH, a + dmH^2 + \frac{1}{2}rm^2H^2\right). \tag{7}$$

For simplicity, we denote the right hand side of (7) by $\mathbf{v}' = (\mathbf{v}'_0, \mathbf{v}'_1, \mathbf{v}'_2)$. We observe that for $m \gg 0$, the following two conditions hold:

(†) The vector $\mathbf{v}'$ satisfies (either of) the conditions in Theorem 3.6 (depending on whether the rank of $\mathbf{v}$ is positive or zero); c.f. [Yos09, Remark 1.4].

(‡) Either $0 < \mathbf{v}'_0 < \mathbf{v}'_2$ or $0 < H \cdot \mathbf{v}'_1 < \mathbf{v}'_2$ (depending on whether the rank of $\mathbf{v}$ is positive or zero).

Now we consider the composition of autoequivalences

$$D^b(\mathrm{Coh}(X)) \xrightarrow{-\otimes H^m} D^b(\mathrm{Coh}(X)) \xrightarrow{\Phi} D^b(\mathrm{Coh}(X)),$$

where Φ is defined by (5). For any given H-polystable sheaf F with a decomposition (4), the Mukai vector of $F \otimes H^m$ is $\mathbf{v}' = \mathbf{v} \cdot e^{mH}$. The condition (†) guarantees that $\Phi(F \otimes H^m)$ is an H-polystable sheaf by Theorem 3.6, which can be decomposed into stable summands in the form of

$$\Phi(F \otimes H^m) = \Phi(F_1 \otimes H^m)^{\oplus n_1} \oplus \cdots \oplus \Phi(F_k \otimes H^m)^{\oplus n_k}.$$

For each i, the condition (‡) guarantees that the last component of the Mukai vector of $F_i \otimes H^m$ is at least 2, which implies by Lemma 3.4 that the rank of $\Phi(F_i \otimes H^m)$ is at least 2, hence $\Phi(F \otimes H^m)$ satisfies the condition (ii) in Proposition 3.1. Moreover each $\Phi(F_i)$ is μ_H-stable by Theorem 3.6. By Proposition 3.1 and Remarks 3.2, 3.3, the DG algebra $\mathrm{RHom}^\bullet(\Phi(F \otimes H^m), \Phi(F \otimes H^m))$ is formal. It follows that $\mathrm{RHom}^\bullet(F, F)$ is also formal by Proposition 2.26.

The case of $(r, dH + D) = (0, 0)$ can be reduced to the above general situation by first applying (5) once. □

The following result, which is proven by Yoshioka in [Yos16, Corollary 0.6] using a different approach, is an immediate consequence of Theorem 3.7.

Corollary 3.8. *We follow the assumptions in Theorem 3.7. Let $G = \mathrm{Aut}(F)/\mathbb{C}^*$, $V = \mathrm{Ext}^1(F, F)$ and $W = \ker(\mathrm{Ext}^2(F, F) \xrightarrow{\mathrm{tr}} \mathbb{C})$. Let $\kappa_2 : V \to W$ be the Yoneda pairing. Then there exists an analytic open neighborhood of $[F]$ in the moduli space $\mathcal{M}_{X,H}(\mathbf{v})$ of H-semistable sheaves on X, which is isomorphic to an analytic open neighborhood of 0 in the quotient $\kappa_2^{-1}(0)/G$.*

Proof. It follows from Theorem 3.7, [GM90, Theorem 5.3] and [AS18, Proposition 4.4]; see also the discussion in [KL07, §3.1]. □

Remark 3.9. More precisely, for a polystable sheaf F given in the form of (4), we have $G = (\mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_k))/\mathbb{C}^*$. As described in [KLS06, §2.7], there is a natural G-action on V, preserving the symplectic form on V, such that κ_2 is precisely the symplectic moment map for this G-action. Based on this description, the quotient $\kappa_2^{-1}(0)/G$ was interpreted as a Nakajima quiver variety in [AS18, Proposition 6.1, Theorem 6.5]. Combining with the above Corollary 3.8, we can conclude that, analytically locally near the point $[F]$, the moduli space $\mathcal{M}_{X,H}(\mathbf{v})$ is isomorphic to the normal cone at $[F]$, which is a Nakajima quiver variety. By [BS16, Proposition 1.2], we immediately obtain that $\mathcal{M}_{X,H}(\mathbf{v})$ has symplectic singularities in the sense of [Bea00]. Thus we recover a result in [KLS06, Theorem 6.2] previously stated only for generic polarizations H.

3.3. **Formality property for derived objects.** As another application of Proposition 2.26, we will show that Conjecture 1.2 is valid not only for moduli spaces of Gieseker semistable sheaves, but also for moduli spaces of semistable objects in the derived category of X under a generic Bridgeland stability condition σ. To state our results rigorously, we write $\mathrm{Stab}^\dagger(X)$ for the connected component of the space of stability conditions on X which contains the geometric ones; see [Bri08, Definition 11.4].

In order to apply Proposition 2.26, we will first find explicit autoequivalences of $D^b(\mathrm{Coh}(X))$ to identify Bridgeland semistable objects with Gieseker semistable objects. Due to the different flavors in the proof, we split this step into two results: the first one handles the case $\mathbf{v}^2 > 0$ while the second handles the case $\mathbf{v}^2 \leqslant 0$.

In the case of $\mathbf{v}^2 > 0$, we have the following Proposition 3.10 due to Kōta Yoshioka, which has appeared in [MZ16, Remark 3.15]:

Proposition 3.10. *Let X be a complex projective K3 surface, $\mathbf{v}$ a Mukai vector with $\mathbf{v}^2 > 0$, and $\sigma \in \mathrm{Stab}^\dagger(X)$ a Bridgeland stability condition which is generic with respect to $\mathbf{v}$. Then there exists an autoequivalence*

$$\Theta : D^b(\mathrm{Coh}(X)) \longrightarrow D^b(\mathrm{Coh}(X))$$

which induces an isomorphism $\mathcal{M}_{X,\sigma}(\mathbf{v}) \cong \mathcal{M}_{X,L}(\mathbf{u})$ between the moduli space $\mathcal{M}_{X,\sigma}(\mathbf{v})$ of σ-semistable objects of class $\mathbf{v}$, and the moduli space $\mathcal{M}_{X,L}(\mathbf{u})$ of L-semistable coherent sheaves of class $\mathbf{u}$ for some generic ample class L on X. Moreover, this isomorphism preserves S-equivalence classes.

This observation due to Yoshioka is essentially a combination of [BM14a, Lemma 7.3] and [Yos09, Theorem 1.7]. In fact, a sketch of the proof is already provided in [MZ16, Remark 3.15]. We supply a full proof here for the sake of completeness.

Proof. We follow the sketch in [MZ16, Remark 3.15].

Step 1. By [BM14a, Lemma 7.3] and the discussion thereafter, we know that for some K3 surface Y, an ample class H on Y and a Brauer class $\alpha \in \mathrm{Br}(Y)$, there exists a Fourier-Mukai transform

$$\Phi : D^b(\mathrm{Coh}(X)) \longrightarrow D^b(\mathrm{Coh}(Y, \alpha))$$

which induces an isomorphism of moduli spaces

$$\Phi \circ [-1] : \mathcal{M}_{X,\sigma}(\mathbf{v}) \xrightarrow{\sim} \mathcal{M}_{Y,\alpha,H}(-\Phi(\mathbf{v})) \tag{8}$$

where $\mathcal{M}_{Y,\alpha,H}(-\Phi(\mathbf{v}))$ parametrizes α-twisted H-semistable bundles. Moreover, this isomorphism preserves S-equivalence classes.

Step 2. Let $(0,0,1)_X$ be the Mukai vector of a skyscraper sheaf on X, and $\mathbf{w} = \Phi((0,0,1)_X)^\vee$. Then $\mathbf{w}$ is a $(-\alpha)$-twisted Mukai vector on Y. For the moment we assume $\operatorname{rk}\mathbf{w} > 0$, then we can perturb H to a sufficient close H', so that H' is generic with respect to $\mathbf{w}$. Since $\langle \Phi((1,0,1)_X)^\vee, \mathbf{w}\rangle = -1$, the moduli space $Z := \mathcal{M}_{Y,-\alpha,H'}(\mathbf{w})$ is an untwisted K3 surface with a universal family $\mathcal{F}$, which gives a Fourier-Mukai transform

$$\Psi : D^b(\mathrm{Coh}(Y,\alpha)) \longrightarrow D^b(\mathrm{Coh}(Z)).$$

Since H and H' are sufficiently close, we have $\mathcal{M}_{Y,\alpha,H}(-\Phi(\mathbf{v})) = \mathcal{M}_{Y,\alpha,H'}(-\Phi(\mathbf{v}))$, and H' is still generic with respect to $-\Phi(\mathbf{v})$. By [Yos09, Theorem 1.7, Remark 1.6], we can choose $n \gg 0$, such that the composition

$$D^b(\mathrm{Coh}(Y,\alpha)) \xrightarrow{-\otimes H^n} D^b(\mathrm{Coh}(Y,\alpha)) \xrightarrow{\Psi} D^b(\mathrm{Coh}(Z)).$$

induces an isomorphism of moduli spaces

$$\Psi \circ (-\otimes H^n) : \mathcal{M}^{G_1}_{Y,\alpha,H'}(-\Phi(\mathbf{v})) \xrightarrow{\sim} \mathcal{M}^{G_2}_{Z,L}(\mathbf{u}) \tag{9}$$

where $G_1 = \mathcal{F}^\vee|_{Y\times\{z\}}$, $G_2 = \mathcal{F}|_{\{y\}\times Z}$, $L = \widehat{H'}$ is an ample class on Z, and $\mathbf{u} = \Psi(-\Phi(\mathbf{v})\cdot e^{nH'})$ is a Mukai vector. Since both H' and L are generic, and since both $\mathcal{M}_{Y,\alpha,H'}(-\Phi(\mathbf{v}))$ and $\mathcal{M}_{Z,L}(\mathbf{u})$ parametrize torsion free sheaves, the G_1- and G_2-twists are irrelevant by an argument that is similar to [Yos01, Lemma 1.1]. Hence (9) reduces to

$$\Psi \circ (-\otimes H^n) : \mathcal{M}_{Y,\alpha,H'}(-\Phi(\mathbf{v})) \xrightarrow{\sim} \mathcal{M}_{Z,L}(\mathbf{u}). \tag{10}$$

Then the composition of (8) and (10) identifies the Bridgeland moduli space $\mathcal{M}_{X,\sigma}(\mathbf{v})$ with an untwisted Gieseker moduli space $\mathcal{M}_{Z,L}(\mathbf{u})$ via a derived equivalence.

Step 3. We claim that we in fact have $X \cong Z$. We apply [HS05, Proposition 4.3] on the Fourier-Mukai transforms Φ and Ψ to get Hodge isometries

$$\widetilde{H}(X,\mathbb{Z}) \overset{\Phi}{\cong} \widetilde{H}(Y,B,\mathbb{Z}) \overset{\Psi}{\cong} \widetilde{H}(Z,\mathbb{Z})$$

where B is a B-field lift of the Brauer class α. Under these Hodge isometries we get

$$(\Psi\circ\Phi)((0,0,1)_X) = \Psi(\mathbf{w}^\vee) = (0,0,1)_Z,$$

which implies that

$$\begin{aligned} H^2(X,\mathbb{Z}) &\cong (0,0,1)_X^\vee/\mathbb{Z}(0,0,1)_X \\ &\cong (0,0,1)_Z^\vee/\mathbb{Z}(0,0,1)_Z \cong H^2(Z,\mathbb{Z}) \end{aligned}$$

is a Hodge isometry. It follows by the global Torelli theorem that $X \cong Z$.

Step 4. We have finished the proof under the additional assumption $\operatorname{rk}\mathbf{w} > 0$. It remains to reduce other cases to this case. Indeed, if $\operatorname{rk}\mathbf{w} < 0$, we can replace $\mathbf{w}$ by $-\mathbf{w}$, and the same argument applies. Now we assume $\operatorname{rk}\mathbf{w} = 0$. After a small perturbation of H if necessary, we can assume that either $H\cdot c_1(\mathbf{w}) \neq 0$ or $c_1(\mathbf{w}) = 0$. Let

$$\Xi : D^b(\mathrm{Coh}(Y,\alpha)) \longrightarrow D^b(\mathrm{Coh}(Y_0,\alpha_0))$$

be any Fourier-Mukai transform induced by the universal family of a 2-dimensional moduli space Y_0 of $(-\alpha)$-twisted stable torsion free sheaves on Y, where Y_0 itself is a K3 surface. We consider the composition

$$D^b(\mathrm{Coh}(Y,\alpha)) \xrightarrow{-\otimes H^m} D^b(\mathrm{Coh}(Y,\alpha)) \xrightarrow{\Xi} D^b(\mathrm{Coh}(Y_0,\alpha_0)).$$

Let $\mathbf{w}_0 = \Xi(\mathbf{w}\cdot e^{mH})$, then a straightforward calculation shows that $\operatorname{rk}\mathbf{w}_0 \neq 0$ for $m \gg 0$. Moreover, by [Yos09, Theorem 1.7, Remark 1.6] and an argument similar to that for (10), we get an isomorphism

$$\Xi\circ(-\otimes H^m) : \mathcal{M}_{Y,\alpha,H}(-\Phi(\mathbf{v})) \xrightarrow{\sim} \mathcal{M}_{Y_0,\alpha_0,H_0}(\mathbf{u}_0)$$

for $m \gg 0$, where $H_0 = \widehat{H}$, $\mathbf{u}_0 = \Xi(-\Phi(\mathbf{v})\cdot e^{mH})$ and $\mathcal{M}_{Y_0,\alpha_0,H_0}(\mathbf{u}_0)$ is moduli space of α_0-twisted H_0-semistable torsion free sheaves of class $\mathbf{u}_0$ on Y_0. Now we can run Steps 2 and 3 for the twisted K3 surface (Y_0,α_0) instead of (Y,α) to finish the proof. □

In the case of $\mathbf{v}^2 \leqslant 0$, we need the following Proposition 3.11. The idea of its proof, along with that of Lemmas 3.13 and 3.14 it relies on, was communicated to us by Arend Bayer:

Proposition 3.11. *Let X be a complex projective K3 surface, $\mathbf{v}$ a Mukai vector with $\mathbf{v}^2 \leqslant 0$, and $\sigma \in \mathrm{Stab}^\dagger(X)$ a Bridgeland stability condition which is generic with respect to $\mathbf{v}$. Then there exists an autoequivalence*

$$\Theta : D^b(\mathrm{Coh}(X)) \longrightarrow D^b(\mathrm{Coh}(X))$$

which induces an isomorphism $\mathcal{M}_{X,\sigma}(\mathbf{v}) \cong \mathcal{M}_{X,L}(\mathbf{u})$ between the moduli space $\mathcal{M}_{X,\sigma}(\mathbf{v})$ of σ-semistable objects of class $\mathbf{v}$, and the moduli space $\mathcal{M}_{X,L}(\mathbf{u})$ of L-semistable coherent sheaves of class $\mathbf{u}$ for some generic ample class L on X. Moreover, this isomorphism preserves S-equivalence classes.

Proof. We write as before $\mathbf{v} = m\mathbf{v}_p$ for some positive integer m and primitive Mukai vector $\mathbf{v}_p$. Then we have $\mathbf{v}_p^2 = -2$ or 0 by a theorem of Yoshioka and Toda; see e.g. [BM14b, Theorem 2.15 (a)]. As a first reduction, we claim that it suffices to assume $m = 1$. Indeed, [BM14a, Lemma 7.1, Lemma 7.2 (b)] imply that every σ-semistable object of class $\mathbf{v}$ is given by extensions of σ-stable objects of class $\mathbf{v}_p$. Hence it suffices to find an autoequivalence of $D^b(\mathrm{Coh}(X))$, that sends all σ-stable objects of class $\mathbf{v}_p$ to stable coherent sheaves.

We also write $\mathbf{v} = (\mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_2)$ as before. As a second reduction, we claim that we can assume $\mathbf{v}_0 > 0$ without loss of generality. Indeed, if $\mathbf{v}_0 < 0$, we can apply the shift functor [1], so that $\mathbf{v}$ gets replaced by $-\mathbf{v}$. If $\mathbf{v}_0 = 0$, after tensoring with a line bundle if necessary, we can assume $\mathbf{v}_2 \neq 0$. Then we can apply the Fourier-Mukai transform (5) to obtain $\widehat{\mathbf{v}}$ whose leading component is non-zero by Lemma 3.4.

From now on we assume $\mathbf{v}$ is primitive and $\mathbf{v}_0 > 0$. The rest of the proof makes use of the wall-crossing technique in [BM14b]. We know by [Bri08, §9] that the stability manifold $\mathrm{Stab}^\dagger(X)$ admits a wall and chamber structure. There is one chamber which contains σ as an interior point, and another "Gieseker chamber" in which we can pick a stability condition τ, such that the τ-stability for class $\mathbf{v}$ is the same as the Gieseker β-twisted ω-stability for some generic $\beta \in \mathrm{NS}(X)_{\mathbb{Q}}$ and $\omega \in \mathrm{Amp}(X)_{\mathbb{Q}}$; see [Bri08, §14]. The assumptions $\mathbf{v}_0 > 0$ and ω being generic imply further that the β-twisted ω-stability for class $\mathbf{v}$ is the same as the untwisted ω-stability by an argument similar to [Yos01, Lemma 1.1]. We can move σ to τ in $\mathrm{Stab}^\dagger(X)$ along a path that never meets two walls simultaneously. Similar to [BM14b, Theorem 1.1], we will show that, each time we cross a wall, we can find an autoequivalence of $D^b(\mathrm{Coh}(X))$, which induces an isomorphism of the moduli spaces of stable objects with respect to generic stability conditions in the neighboring chambers separated by the wall. The explicit construction of such autoequivalences in the case of $\mathbf{v}^2 = -2$ will be given in Lemma 3.13, and in the case of $\mathbf{v}^2 = 0$ given in Lemma 3.14, which finishes the proof. □

The proofs of the technical Lemmas 3.13 and 3.14 will be postponed to §3.4. The above results give us immediately the following formality statement:

Corollary 3.12. *Let X be a complex projective K3 surface, $\mathbf{v}$ a Mukai vector, and $\sigma \in \mathrm{Stab}^\dagger(X)$ a Bridgeland stability condition which is generic with respect to $\mathbf{v}$. Assume that E is a σ-polystable object of Mukai vector $\mathbf{v}$. Then the DG algebra $\mathrm{RHom}^\bullet(E, E)$ is formal.*

Proof. By Propositions 3.10 and 3.11, $\Theta(E)$ is an L-polystable coherent sheaf on X. By Theorem 3.7, the DG algebra $\mathrm{RHom}^\bullet(\Theta(E), \Theta(E))$ is formal, which implies that $\mathrm{RHom}^\bullet(E, E)$ is formal by Proposition 2.26. □

3.4. **Constructions of derived equivalences.** In this part we prove two technical results which are required in the proof of Proposition 3.11.

We first introduce some notations following [BM14b]. For a Mukai vector $\mathbf{v}$ with $\mathbf{v}^2 = -2$, let $\mathcal{W}$ be a wall of the chamber decomposition of $\mathrm{Stab}^\dagger(X)$ with respect to $\mathbf{v}$. Let $\sigma_0 = (Z_0, \mathcal{P}_0) \in \mathcal{W}$ be a generic stability condition on the wall. Let σ_+ and σ_- be generic stability conditions in the two chambers separated by $\mathcal{W}$ respectively. By [BM14a, Lemma 7.1], the moduli spaces $\mathcal{M}_{X,\sigma_+}(\mathbf{v})$ and $\mathcal{M}_{X,\sigma_-}(\mathbf{v})$ are both a single point represented by a spherical object, say E^+ and E^- respectively. We have the following result:

Lemma 3.13. *There exists an autoequivalence Φ of $D^b(\mathrm{Coh}(X))$ with $\Phi(E^-) = E^+$.*

Indeed, the proof of Lemma 3.13 is essentially contained in [BM14b, Proposition 6.8]. However, due to different contexts, the proof there has to be adjusted to accommodate our situation. We will point out the differences carefully, and be very brief with the arguments in [BM14b] that are still valid in our setting.

Proof. The proof will be given in a few steps.

Step 1. We determine the distribution of effective spherical classes.

Following [BM14b, §5], we can associate to $\mathcal{W}$ a primitive rank two lattice $\mathcal{H} \subset H^*_{alg}(X, \mathbb{Z})$, which contains $\mathbf{v}$ and the Mukai vectors of all Jordan-Hölder factors of $E_\pm$ under the stability condition σ_0. Without loss of generality, we assume $Z_0(\mathcal{H}) \subset \mathbb{R}$. Notice that $\mathcal{H}$ does not have to be hyperbolic in our setting, due to the lack of a square positive class.

Since $E_\pm$ are strictly semistable under the stability condition σ_0, the lattice $\mathcal{H}$ has at least two linearly independent effective spherical classes, which are represented by σ_0-stable spherical objects by [BM14b, Lemma 6.2]. Hence [BM14b, Proposition 6.3 (a)(b)] cannot happen in our setting. A similar calculation as in [BM14b, Proposition 6.3 (c)] shows that there exist exactly two σ_0-stable spherical objects $S, T \in \mathcal{P}_0(1)$ with Mukai vectors $\mathbf{s}$ and $\mathbf{t}$ in $\mathcal{H}$ respectively, where $(\mathbf{s}, \mathbf{t}) \geqslant 1$. Without loss of generality, we assume that $\phi^+(S) < \phi^+(T)$, where ϕ^+ is the phase with respect to σ_+. Depending on the signature of $\mathcal{H}$, all effective spherical classes in $\mathcal{H}$ are given in two ways:

(i) When $(\mathbf{s}, \mathbf{t}) \geqslant 2$, there are infinitely many spherical classes, given by $\mathbf{t}_i$ for $i \geqslant 1$ and $\mathbf{s}_i$ for $i \leqslant 0$ as in [BM14b, p.539]. They lie on two branches of a hyperbola $\mathbf{v}^2 = -2$; see [BM14b, Fig.2, p.539]. Notice that the hyperbola degenerates to two parallel lines when $(\mathbf{s}, \mathbf{t}) = 2$.

(ii) When $(\mathbf{s}, \mathbf{t}) = 1$, the only effective spherical classes in $\mathcal{H}$ are $\mathbf{s}$, $\mathbf{t}$ and $\mathbf{s} + \mathbf{t}$. Notice that $\mathbf{s} + \mathbf{t}$ can be understood as $\mathbf{t}_2$ or $\mathbf{s}_{-1}$.

Step 2. We determine the unique stable object under the stability condition σ_+ and σ_- for each effective spherical class.

We follow the notations in [BM14b, Proposition 6.8]. For each class $\mathbf{t}_i$ with $i \geqslant 1$, we write T_i^+ and T_i^- for the unique stable object under σ_+ and σ_-. Since T is σ_0-stable, it is clear that $T_1^+ = T_1^- = T$. We claim that

$$T_2^+ = \mathrm{ST}_T(S), \tag{11}$$

$$T_{i+1}^+ = \mathrm{ST}_{T_i^+} T_{i-1}^+[-1] \qquad \text{for any } i \geqslant 3, \tag{12}$$

where ST_A is the spherical twist associated to a spherical object A. Indeed, (11) follows immediately from [BM14b, Lemma 6.9]. For (12), it was proven in the proof of [BM14b, Proposition 6.8] that T_i^+ is a simple object of $\mathcal{A}_{i-1}$. Using a "dual" version of [BM14b, Lemma 6.10], we can similarly prove that $T_{i-1}[-1]$ is also a simple object of $\mathcal{A}_{i-1}$. Then (12) follows again from [BM14b, Lemma 6.9]. Analogously, we can show that

$$T_2^- = \mathrm{ST}_T^{-1}(S)$$

$$T_{i+1}^- = \mathrm{ST}_{T_i^-}^{-1} T_{i-1}^-[1] \qquad \text{for any } i \geqslant 3.$$

The stable objects for the classes $\mathbf{s}_i$ with $i \leqslant 0$ can be analogously described.

Step 3. We write down the autoequivalence Φ explicitly.

Assume that $\mathbf{v} = \mathbf{t}_i$ for some $i \geqslant 2$. For any positive integer k, we define

$$\Phi = \begin{cases} \mathrm{ST}_{T_{2k}^+} \circ \cdots \circ \mathrm{ST}_{T_2^+} \circ \mathrm{ST}_{T_2^-} \circ \cdots \circ \mathrm{ST}_{T_{2k}^-}[-2k] & \text{if } i = 2k+1; \\ \mathrm{ST}_{T_{2k-1}^+} \circ \cdots \circ \mathrm{ST}_{T_1^+} \circ \mathrm{ST}_{T_1^-} \circ \cdots \circ \mathrm{ST}_{T_{2k-1}^-}[2-2k] & \text{if } i = 2k. \end{cases}$$

It is clear that Φ is an autoequivalence of $D^b(\mathrm{Coh}(X))$. Moreover, it is easy to check by the formula in the previous step that

$$E^+ = T_i^+ = \Phi(T_i^-) = \Phi(E^-).$$

The analogous statement holds in the case of $\mathbf{v} = \mathbf{s}_i$ for some $i \leqslant -1$. □

Now we assume that $\mathbf{v}$ is a primitive Mukai vector with $\mathbf{v}^2 = 0$. Let $\mathcal{W}$ be a wall of the chamber decomposition of $\mathrm{Stab}^\dagger(X)$ with respect to $\mathbf{v}$. Let $\sigma_0 = (Z_0, \mathcal{P}_0) \in \mathcal{W}$ be a generic stability condition on the wall. Let σ_+ and σ_- be generic stability conditions in the two chambers separated by $\mathcal{W}$ respectively. By [BM14a, Lemma 7.2 (a)], the moduli spaces $\mathcal{M}_{X,\sigma_+}(\mathbf{v})$ and $\mathcal{M}_{X,\sigma_-}(\mathbf{v})$ are both smooth projective K3 surfaces.

Lemma 3.14. *There exists an autoequivalence Φ of $D^b(\mathrm{Coh}(X))$ which induces an isomorphism $\Phi : \mathcal{M}_{X,\sigma_-}(\mathbf{v}) \cong \mathcal{M}_{X,\sigma_+}(\mathbf{v})$.*

Proof. Let $Y = \mathcal{M}_{X,\sigma_-}(\mathbf{v})$. By [BM14a, Lemma 7.2 (a)], for some Brauer class $\alpha \in \mathrm{Br}(Y)$, there exists a derived equivalence

$$\Phi_0 : D^b(\mathrm{Coh}(X)) \xrightarrow{\sim} D^b(\mathrm{Coh}(Y, \alpha)),$$

which induces isomorphisms of moduli spaces

$$\Phi_0 : \mathcal{M}_{X,\sigma_\pm}(\mathbf{v}) \xrightarrow{\sim} \mathcal{M}_{Y,\alpha,\tau_\pm}(\mathbf{w})$$

where $\mathbf{w}$ is the Mukai vector of the skyscraper sheaf $\mathcal{O}_y$ for any closed point $y \in Y$. Moreover, $\tau_0 = \Phi_0(\sigma_0)$ is a generic stability condition on the wall $\Phi_0(\mathcal{W})$ that separates the chambers containing $\tau_\pm = \Phi_0(\sigma_\pm)$ respectively. Due to the construction of Φ_0, the τ_--stable objects of class $\mathbf{w}$ are precisely the skyscraper sheaves, which implies that the chamber containing τ_- is the geometric chamber of Y.

An explicit description of the wall $\Phi_0(\mathcal{W})$ is given by [Bri08, Theorem 12.1]; see also [BM14a, Remark 6.4]. As a consequence, there exists an autoequivalence Φ_1 of $D^b(\mathrm{Coh}(Y, \alpha))$, which induces an isomorphism of moduli spaces

$$\Phi_1 : \mathcal{M}_{Y,\alpha,\tau_-}(\mathbf{w}) \xrightarrow{\sim} \mathcal{M}_{Y,\alpha,\tau_+}(\mathbf{w}).$$

In the language of [Bri08, Theorem 12.1], we have more precisely that $\Phi_1 = \mathrm{ST}_A^2$ when $\Phi_0(\mathcal{W})$ is a wall of type (A^+); $\Phi_1 = \mathrm{ST}_A^{-2}$ for type (A^-); and $\Phi_1 = \mathrm{ST}_{\mathcal{O}_C(k)}^{-1}$ for type (C_k). It follows that the composition

$$\Phi = \Phi_0^{-1} \circ \Phi_1 \circ \Phi_0$$

is an autoequivalence of $D^b(\mathrm{Coh}(X))$ that induces the desired isomorphism of moduli spaces $\mathcal{M}_{X,\sigma_-}(\mathbf{v}) \cong \mathcal{M}_{X,\sigma_+}(\mathbf{v})$. ☐

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