

L-EQUIVALENCE FOR DEGREE FIVE ELLIPTIC CURVES, ELLIPTIC FIBRATIONS AND K3 SURFACES

EVGENY SHINDER AND ZIYU ZHANG

ABSTRACT. We construct nontrivial L-equivalence between curves of genus one and degree five, and between elliptic surfaces of multisection index five. These results give the first examples of L-equivalence for curves (necessarily over non-algebraically closed fields) and provide a new bit of evidence for the conjectural relationship between L-equivalence and derived equivalence.

The proof of the L-equivalence for curves is based on Kuznetsov's Homological Projective Duality for $\mathrm{Gr}(2, 5)$, and L-equivalence is extended from genus one curves to elliptic surfaces using the Ogg–Shafarevich theory of twisting for elliptic surfaces.

Finally, we apply our results to K3 surfaces and investigate when the two elliptic L-equivalent K3 surfaces we construct are isomorphic, using Neron–Severi lattices, moduli spaces of sheaves and derived equivalence. The most interesting case is that of elliptic K3 surfaces of polarization degree ten and multisection index five, where the resulting L-equivalence is new.

1. INTRODUCTION

1.1. The Grothendieck ring of varieties and L-equivalence. Recall that the Grothendieck ring of varieties $K_0(\mathrm{Var}/k)$ is generated as an abelian group by isomorphism classes $[X]$ of schemes of finite type X/k modulo the scissor relations

$$[X] = [U] + [Z]$$

for every closed $Z \subset X$ with open complement $U = X \setminus Z$. The product structure on $K_0(\mathrm{Var}/k)$ is induced by product of schemes. We write $\mathbb{L} \in K_0(\mathrm{Var}/k)$ for the class of the affine line $[\mathbb{A}^1]$.

The concept of L-equivalence stems from the recently discovered fact that \mathbb{L} is a zero-divisor [Bor18]. Specifically, for Calabi-Yau threefolds X, Y in the so-called Pfaffian-Grassmannian correspondence, the classes satisfy $[X] \neq [Y]$ and

$$(1.1) \quad \mathbb{L}^n \cdot ([X] - [Y]) = 0,$$

where one can take any $n \geq 6$ [Bor18, Mar16]. Following [KS18], we say that smooth projective connected varieties X and Y are *L-equivalent* if the equation (1.1) holds for some $n \geq 1$, and we say that X and Y are *nontrivially L-equivalent* if in addition $[X] \neq [Y]$. If X and Y are not covered by rational

2010 *Mathematics Subject Classification.* Primary: 14F05; Secondary: 14H52, 14J28, 14D06.

Key words and phrases. genus one curve, elliptic K3 surface, L-equivalence.

curves and X and Y are not birational then an L-equivalence between them is automatically nontrivial (see e.g. [KS18, Proposition 2.2]).

There are at least two important reasons why one would want to study L-equivalence. Firstly, it seems to be closely related to derived equivalence [KS18, IMOU16, Kaw18]. As an evidence for this, the classes of derived categories of L-equivalent varieties in the Bondal-Larsen-Lunts ring of triangulated categories [BLL04] are equal, and since for Calabi-Yau varieties the derived categories are indecomposable, it is very likely that nontrivially L-equivalent Calabi-Yau varieties are actually derived equivalent (see [KS18, IMOU16] for an extended discussion of this relationship). In fact all currently known examples of pairs of nontrivially L-equivalent varieties are known to be derived equivalent. These examples include K3 surfaces [KS18, HL18, IMOU16, KKM17], Calabi-Yau threefolds [Bor18, IMOU19, BCP17], Calabi-Yau fivefolds [Man17] and Hilbert schemes of points on K3 surfaces [Ok18].

The second reason to study L-equivalence is the relation to rationality problems, specifically to that of cubic fourfolds. Namely, the approach of [GS14] can be used to show that very general cubic fourfolds are not rational as soon as one has sufficient control over the L-equivalence relation.

In this paper we study L-equivalence for genus one curves and elliptic surfaces, in particular for elliptic K3 surfaces.

1.2. Genus one curves. We work over a field of characteristic zero. Let X be a genus one curve with a line bundle of degree d . For every k coprime to d we can consider the Jacobian $Y = \text{Jac}^k(X)$ which is a fine moduli space parametrizing degree k line bundles on X . Of course, if X has a rational point, then all Jacobians $\text{Jac}^k(X)$ are isomorphic to X , however in general this is not the case, and X and Y are typically different torsors over the same elliptic curve $E = \text{Jac}^0(X)$.

Theorem 1.1. [AKW17] *If k and d are coprime, then genus one curves X and $\text{Jac}^k(X)$ are derived equivalent, and furthermore, every smooth projective variety Y derived equivalent to X will be of the form $Y = \text{Jac}^k(X)$ for some k coprime to d .*

In light of a conjectural relation between L-equivalence and derived equivalence we may ask the following:

Question 1.2. *When are genus one curves X and $Y = \text{Jac}^k(X)$ L-equivalent?*

Due to the periodicity relations $\text{Jac}^{k+d}(X) \simeq \text{Jac}^k(X)$, $\text{Jac}^{-k}(X) \simeq \text{Jac}^k(X)$ and the isomorphism $X \simeq \text{Jac}^1(X)$, the first nontrivial test case is $d = 5$. Furthermore, in the $d = 5$ case the only nontrivial coprime Jacobian is $Y = \text{Jac}^2(X) \simeq \text{Jac}^3(X)$. Our first main result is the following:

Theorem 1.3. *(see Theorem 2.9) If X is a genus one curve with a line bundle of degree 5 and $Y = \text{Jac}^2(X)$, then X and Y are L-equivalent, and in general this L-equivalence is nontrivial.*

More precisely, we show that (1.1) holds for X and Y when $n \geq 4$ (and does not hold for $n = 0$). This is the first existing construction of nontrivial L-equivalence for curves, as all the previous constructions were for K3 surfaces or Calabi-Yau varieties of higher dimension.

As it is often the case with proving L-equivalence we relate the geometry of X and Y to Homological Projective Duality of A. Kuznetsov [Kuz06]. Specifically, as one of the steps in the proof of the theorem above we prove the following:

Proposition 1.4. *(see Proposition 2.8 for the precise statement) If X is a genus one curve with a line bundle of degree 5 and $Y = \text{Jac}^2(X)$, then X and Y are homologically projectively dual codimension 5 linear sections of $\text{Gr}(2, 5)$.*

We note the interplay between the moduli space geometry and the Homological Projective Duality geometry, in particular either of the two approaches can be used to show derived equivalence of X and $Y = \text{Jac}^2(X)$. If one starts with the $Y = \text{Jac}^2(X)$ description, derived equivalence follows from Theorem 1.1 and if one starts with the Homological Projective Duality description of Proposition 1.4, derived equivalence follows from [Kuz06].

To generalize our work and to construct L-equivalence of genus one curves in degrees $d > 5$ it seems necessary to study explicit geometry of the moduli space of curves of genus one and degree d . To describe the geometry of such moduli spaces for small d we can use the classical projective models of genus one curves with a degree d divisor:

- $d = 2$: double covers of \mathbb{P}^1 branched in four points
- $d = 3$: cubic curves in \mathbb{P}^2
- $d = 4$: intersections of two quadrics in \mathbb{P}^3
- $d = 5$: one-dimensional linear sections of a Grassmannian $\text{Gr}(2, 5) \subset \mathbb{P}^9$

These explicit descriptions show in particular that for $d \leq 5$ the corresponding moduli spaces are rational. It is this description in the $d = 5$ case, together with the geometric characterization of the self-map Jac^2 on the moduli space of degree 5 genus one curves, given in Proposition 1.4 that allows us to prove L-equivalence.

We note that the same explicit geometry of genus one and degree five curves has been used to study the average size of 5-Selmer groups and the average ranks of elliptic curves [BS13].

For $d > 5$ no such explicit description is known, and furthermore it not known whether the corresponding moduli spaces are rational or not for large d .

1.3. Elliptic surfaces. Let k be an algebraically closed field of characteristic zero. We work with elliptic surfaces without a section; by a *multisection index* of such a surface we mean the minimal fiber degree of a multisection. Our second main result is:

Theorem 1.5. *(see Theorem 3.2) If $X \rightarrow C$ is an elliptic surface of multisection index 5 and $Y = \text{Jac}^2(X/C)$, then X and Y are L -equivalent, and in general this L -equivalence is nontrivial.*

We note that the derived equivalence of X and Y had been proved by Bridgeland [Bri98].

We also investigate the case of elliptic K3 surfaces in detail, and answer the question when the L -equivalence constructed in the Theorem is in fact nontrivial. Here we take $k = \mathbb{C}$.

L -equivalence for K3 surfaces is one of the central open questions in the field. As a general structural result it is proved by Efimov [Ef18] that every L -equivalence class of K3 surfaces contains only finitely many isomorphism classes in it. Previously known cases when nontrivial L -equivalence of derived equivalent K3 surfaces has been constructed are K3 surfaces of degrees 8 and 2 and Picard rank two [KS18], K3 surfaces of degree 12 and Picard rank one [HL18, IMOU16], and K3 surfaces of degree 2 and Picard rank two [KKM17].

Let $X \rightarrow \mathbb{P}^1$ be an elliptic K3 surface of multisection index five, then $Y = \text{Jac}^2(X/\mathbb{P}^1)$ is also an elliptic K3 surface (see e.g. [Huy16, Proposition 11.4.5]), and by Theorem 1.5 these K3 surfaces are L -equivalent. The next Propositions explains when X and Y are not isomorphic.

Proposition 1.6. *(see Proposition 3.10) Let $X \rightarrow \mathbb{P}^1$ be an elliptic K3 surface of Picard rank two, multisection index 5 and polarization of degree $2d$ (d is well-defined modulo 5), and let $Y = \text{Jac}^2(X/\mathbb{P}^1)$.*

- (1) *If $d \equiv 2 \pmod{5}$ or $d \equiv 3 \pmod{5}$, then X and Y are isomorphic.*
- (2) *If $d \equiv 1 \pmod{5}$ or $d \equiv 4 \pmod{5}$, then X and Y are not isomorphic.*
- (3) *If $d \equiv 0 \pmod{5}$, and X is very general in moduli, then X and Y are not isomorphic.*

We note that for every d such K3 surfaces exist and form an 18-dimensional irreducible subvariety in the moduli space of degree $2d$ polarized K3 surfaces. Such elliptic K3 surfaces may have more than one elliptic fibrations (in fact a Picard rank two elliptic K3 has always one or two elliptic fibrations), and by an isomorphism of elliptic K3 surfaces we mean an isomorphism of K3 surfaces, regardless of the elliptic fibration structure. The above Proposition is proved by analyzing lattice theory of the corresponding K3 surfaces, along the lines of [St04, vG05].

Explicitly, the case (2) of the Proposition covers elliptic K3 surfaces of degrees 12 ($d \equiv 1 \pmod{5}$) and 8 ($d \equiv 4 \pmod{5}$), considered previously in [HL18, IMOU16] and [KS18] respectively.

The K3 surfaces in case (3) can be geometrically described as intersections of $\text{Gr}(2, 5)$, three hyperplanes and a quadric in \mathbb{P}^9 , and containing an elliptic quintic curve (see Example 3.7). This is a genuinely new instance of nontrivial L -equivalence between K3 surfaces.

Acknowledgements. We would like to thank Tom Bridgeland, Tom Fisher, Sergey Galkin, Daniel Huybrechts, Alexander Kuznetsov, Jayanta Manoharmayum, C.S. Rajan, Matthias Schütt, Constantin Shramov and Damiano Testa for helpful discussions and e-mail correspondences. In particular we thank Alexander Kuznetsov for explaining to us how to prove the duality in Proposition 2.8, and for his comments on a draft of the paper.

We thank the University of Sheffield, Leibniz University Hannover and the University of Bonn for opportunities for us to travel and collaborate, and the Max-Planck-Institut für Mathematik in Bonn for the excellent and inspiring working conditions, where much of this work has been written.

E.S. was partially supported by Laboratory of Mirror Symmetry NRU HSE, RF government grant, ag. N 14.641.31.0001.

2. DUAL ELLIPTIC QUINTICS

In this section we work over a field k of characteristic zero.

2.1. Hyperplane sections of the Grassmannian. We recall some standard facts about the Grassmannian $\mathrm{Gr}(2, 5)$ and its smooth and singular hyperplane sections.

Let V be a five-dimensional vector space; we consider the Plücker embedding $\mathrm{Gr}(2, V) \subset \mathbb{P}(\Lambda^2(V)) \simeq \mathbb{P}^9$ and the hyperplane sections $D_\theta := \mathrm{Gr}(2, V) \cap H_\theta$, parametrized by points of the dual projective space $[\theta] \in \mathbb{P}(\Lambda^2(V^\vee))$, where $\theta \in \Lambda^2(V^\vee)$ is a nonzero two-form.

By a kernel of a two-form $\theta \in \Lambda^2(V^\vee)$ we mean the subspace

$$\mathrm{Ker}(\theta) = \{v \in V : \theta(v \wedge u) = 0 \text{ for all } u \in V\}.$$

For a non-zero form there are two cases:

- (1) General case: $\mathrm{Ker}(\theta)$ is one-dimensional. Then θ can be written as $x_1 \wedge x_2 + x_3 \wedge x_4$ for some basis in V .
- (2) Special case: $\mathrm{Ker}(\theta)$ is three-dimensional. Then θ is decomposable and can be written as $x_1 \wedge x_2$ in some basis. In other words $[\theta] \in \mathrm{Gr}(2, V^\vee) \subset \mathbb{P}(\Lambda^2(V^\vee))$.

It is well-known that the two Grassmannians $\mathrm{Gr}(2, V)$ and $\mathrm{Gr}(2, V^\vee)$ are projectively dual in their Plücker embeddings. More precisely, we have the following well-known result:

Lemma 2.1. *Let $A \subset \Lambda^2(V^\vee)$ be a linear subspace, and consider its orthogonal subspace*

$$A^\perp = \{p \in \Lambda^2(V) : \theta(p) = 0 \text{ for all } \theta \in A\} \subset \Lambda^2(V).$$

Then $[U] \in \mathrm{Gr}(2, V)$ is a singular point of $X_A := \mathrm{Gr}(2, V) \cap \mathbb{P}(A^\perp)$ if and only if for every $\theta \in A$, $\theta(U) = 0$ and for some $\theta_0 \in A$, $U \subset \mathrm{Ker}(\theta_0)$.

In particular, the hyperplane section D_θ is singular if and only if $\theta \in \mathrm{Gr}(2, V^\vee)$, and in this case the singular locus of D_θ is isomorphic to \mathbb{P}^2 .

Proof. The projective tangent space to $\mathrm{Gr}(2, V)$ at a point $[U]$ is $\mathbb{P}(U \wedge V) \subset \mathbb{P}(\Lambda^2(V))$, and it follows that the hyperplane H_θ is tangent to $\mathrm{Gr}(2, V)$ if and only if $\theta|_{U \wedge V} = 0$, that is $U \subset \mathrm{Ker}(\theta)$. Thus if $\mathrm{Ker}(\theta)$ is one-dimensional, D_θ is smooth, and if $\theta \in \mathrm{Gr}(2, V^\vee)$ so that the $\mathrm{Ker}(\theta)$ is three-dimensional, D_θ is singular along $\mathrm{Gr}(2, \mathrm{Ker}(\theta)) \simeq \mathbb{P}^2$.

More generally, if $\theta_1, \dots, \theta_k$ form a basis of A , and $[U] \in X_A = \mathrm{Gr}(2, V) \cap H_{\theta_1} \cap \dots \cap H_{\theta_k}$ so that all θ_i vanish on U , then the projective tangent space to $[U]$ at X_A is

$$\mathbb{P}(U \wedge V) \cap H_{\theta_1} \cap \dots \cap H_{\theta_k} \subset \mathbb{P}(\Lambda^2(V)),$$

and this intersection is not transverse if and only if $\theta_1, \dots, \theta_k$ are linearly dependent when restricted to $\mathbb{P}(U \wedge V)$, which is equivalent to existence of a nonzero form $\theta \in A$ vanishing on $U \wedge V$, or equivalently $U \subset \mathrm{Ker}(\theta)$. \square

Lemma 2.2. *The class in the Grothendieck ring of the Grassmannian is*

$$[\mathrm{Gr}(2, V)] = 1 + \mathbb{L} + 2\mathbb{L}^2 + 2\mathbb{L}^3 + 2\mathbb{L}^4 + \mathbb{L}^5 + \mathbb{L}^6$$

and the classes of its smooth and singular hyperplane sections $D_\theta = \mathrm{Gr}(2, V) \cap H_\theta$ are given by:

$$D_\theta = \begin{cases} 1 + \mathbb{L} + 2\mathbb{L}^2 + 2\mathbb{L}^3 + \mathbb{L}^4 + \mathbb{L}^5, & \theta \notin \mathrm{Gr}(2, V^\vee) \\ 1 + \mathbb{L} + 2\mathbb{L}^2 + 2\mathbb{L}^3 + 2\mathbb{L}^4 + \mathbb{L}^5, & \theta \in \mathrm{Gr}(2, V^\vee) \end{cases}$$

Proof. The computation for $\mathrm{Gr}(2, 5)$ is standard: it is a variety with an affine cell decomposition whose cells are parametrized by Young diagrams fitting into a 3×2 rectangle, the codimension of a cell given by the number of blocks in the diagram [GH78, Chapter 1.5].

We know that a nonzero 2-form θ on a five-dimensional space has kernel of dimension 1 (general case) or 3 (special case) and this distinguishes smooth hyperplane sections from singular ones. Let K be the kernel of θ .

In the smooth case, when $\dim(K) = 1$, the two-dimensional subspace $U \subset V$ can either contain K or intersect it trivially; thus the subspace $(U + K)/K$ of the four-dimensional space V/K can have dimension 2 or 1. The space V/K is endowed with a symplectic form $\bar{\theta}$, and the subspace $(U + K)/K$ is isotropic by construction. Using the relations in the Grothendieck ring we compute

$$\begin{aligned} [\mathrm{Gr}(2, 5) \cap H_\theta] &= [\mathbb{P}^3] + ([\mathbb{P}^2] - [\mathbb{P}^1])[\mathrm{LG}(2, 4)] = \\ &= 1 + \mathbb{L} + 2\mathbb{L}^2 + 2\mathbb{L}^3 + \mathbb{L}^4 + \mathbb{L}^5, \end{aligned}$$

where we used that the Lagrangian Grassmannian $\mathrm{LG}(2, 4)$ is isomorphic to a three-dimensional split quadric so that $[\mathrm{LG}(2, 4)] = 1 + \mathbb{L} + \mathbb{L}^2 + \mathbb{L}^3$ (see e.g. [KS18, Example 2.8]).

Similarly in the singular case, when $\dim(K) = 3$, the two-dimensional subspace $U \subset V$ can either be contained in K or intersect it along a line and considering $(U + K)/K \subset V/K$ yields

$$\begin{aligned} [\mathrm{Gr}(2, 5) \cap H] &= [\mathbb{P}^2] + ([\mathbb{P}^3] - [\mathbb{P}^1])[\mathbb{P}^2] = \\ &= 1 + \mathbb{L} + 2\mathbb{L}^2 + 2\mathbb{L}^3 + 2\mathbb{L}^4 + \mathbb{L}^5, \end{aligned}$$

which finishes the proof. \square

Proposition 2.3. *For any locally closed subset $S \subset \mathbb{P}(\Lambda^2(V^\vee))$, consider the universal hyperplane section of $\mathrm{Gr}(2, V)$:*

$$\mathcal{H}_S := \{([U] \in \mathrm{Gr}(2, V), [\theta] \in S) : [U] \in H_\theta\} \subset \mathrm{Gr}(2, V) \times S.$$

Then we have

$$[\mathcal{H}_S] = [S](1 + \mathbb{L} + 2\mathbb{L}^2 + 2\mathbb{L}^3 + \mathbb{L}^4 + \mathbb{L}^5) + \mathbb{L}^4 \cdot [S \cap \mathrm{Gr}(2, V^\vee)].$$

Proof. Presenting S as $(S \setminus \mathrm{Gr}(2, V^\vee)) \cup (S \cap \mathrm{Gr}(2, V^\vee))$, we see that it suffices to show the statement when either $S \subset \mathbb{P}(\Lambda^2(V^\vee)) \setminus \mathrm{Gr}(2, V^\vee)$ or $S \subset \mathrm{Gr}(2, V^\vee)$.

Let $S \subset \mathrm{Gr}(2, V^\vee)$. The family of kernels $\mathrm{Ker}(\theta)$, $\theta \in S$ forms a locally-free sheaf of rank three over \mathcal{H}_S , and considering the relative position of the fibers of this sheaf with respect to the fibers of the tautological bundle coming from $\mathrm{Gr}(2, V)$ allows to repeat the proof of Lemma 2.2 and to deduce that

$$[\mathcal{H}_S] = [S](1 + \mathbb{L} + 2\mathbb{L}^2 + 2\mathbb{L}^3 + 2\mathbb{L}^4 + \mathbb{L}^5),$$

which is what we had to prove in this case.

The other case is proved analogously. \square

We need one more result regarding incidence rank one sheaves on hyperplane sections of Grassmannians. Let V be an n -dimensional space, and let $D \subset \mathrm{Gr}(k, n)$ be the Schubert divisor $\sigma_{1,0,\dots,0}$ corresponding to a fixed $(n - k)$ -dimensional linear subspace $W \subset V$, that is

$$D := \{[U] \in \mathrm{Gr}(k, n) : \dim(U \cap W) \geq 1\} \subset \mathrm{Gr}(k, n).$$

See [GH78, Chapter 1.5] for the basic properties of the Schubert cycles σ_{a_1,\dots,a_k} .

Consider the resolution $\tilde{D} \rightarrow D$ defined as

$$(2.1) \quad \tilde{D} := \{([U], [l]) \in \mathrm{Gr}(k, n) \times \mathbb{P}(W) : l \subset U \cap W\}.$$

Then \tilde{D} is a Grassmannian bundle over $\mathbb{P}(W)$. We write h for the hyperplane section on $\mathbb{P}(W)$, as well as for its class on \tilde{D} , and we write H for the hyperplane section on $\mathrm{Gr}(k, V) \subset \mathbb{P}(\Lambda^k(V))$ and its class on \tilde{D} .

Lemma 2.4. *The H -degree of the $c_1(\mathcal{O}(h)) \in \mathrm{Pic}(\tilde{D})$ is equal to the degree of the Schubert cycle $\sigma_{2,0,\dots,0}$ on $\mathrm{Gr}(k, n)$, that is*

$$c_1(\mathcal{O}(h)) \cdot H^{n(n-k)-2} = \sigma_{2,0,\dots,0} \cdot H^{n(n-k)-2}.$$

Proof. A codimension one linear subspace $W' \subset W$ gives rise to an irreducible divisor representing $c_1(\mathcal{O}(h))$:

$$Z = \{([U], [l]) \in \tilde{D} : l \subset U \cap W'\} \subset \tilde{D},$$

and this divisor maps birationally onto its image

$$\{[U] \in \mathrm{Gr}(k, n) : \dim(U \cap W') \geq 1\} \subset D \subset \mathrm{Gr}(k, n).$$

This subvariety represents the class $\sigma_{2,0,\dots,0}$ in the Chow groups of the Grassmannian, and it follows that H -degree of $c_1(\mathcal{O}(h))$ is equal to the H -degree of $\sigma_{2,0,\dots,0}$. \square

2.2. Elliptic quintics, Jacobians and duality.

Definition 2.5. *An elliptic quintic is a smooth projective genus one curve which admits a line bundle of degree five.*

By Riemann-Roch theorem a degree five line bundle \mathcal{L} on an elliptic quintic X is very ample and defines an embedding $X \subset \mathbb{P}H^0(X, \mathcal{L})^\vee = \mathbb{P}^4$.

Lemma 2.6. *Let V be a 5-dimensional k -vector space, and $A \subset \Lambda^2(V^\vee)$ be a 5-dimensional subspace. If $X = \text{Gr}(2, V) \cap \mathbb{P}(A^\perp)$ is a transverse intersection, then X is an elliptic quintic and every elliptic quintic is obtained in this way.*

Proof. The first claim follows from the adjunction formula, while the second one is a classical fact known as existence of a Pfaffian representation for an elliptic quintic, see [F13] for a modern exposition. \square

For any smooth projective curve X and an integer $k \in \mathbb{Z}$ we consider the degree k Jacobian $\text{Jac}^k(X)$, defined as the moduli space of degree k line bundles on X . If X is an elliptic quintic, then by tensoring with the degree 5 line bundle and by dualizing we obtain the isomorphisms

$$\text{Jac}^{k+5}(X) \simeq \text{Jac}^k(X), \quad \text{Jac}^{-k}(X) \simeq \text{Jac}^k(X).$$

Thus in this case all Jacobians are isomorphic to one of the

$$E := \text{Jac}^0(X), \quad X = \text{Jac}^1(X) \simeq \text{Jac}^4(X), \quad Y = \text{Jac}^2(X) \simeq \text{Jac}^3(X).$$

Here E is an elliptic curve, that is a genus one curve with a rational point and X and Y are E -torsors. E -torsors are parametrized by the Weil-Chatelet group $H^1(k, E)$ [Sil86, X.3]. If $[X] \in H^1(k, E)$ is the class of the torsor X , it is well-known that for any $k \in \mathbb{Z}$, $d \cdot [X] = [\text{Jac}^k(X)]$ (see e.g. [Huy16, Remark 11.5.2]).

In particular, we see that since X has degree five, then the order of $[X]$ equals five unless X has a rational point in which case $[X] = 0$. Let $Y = \text{Jac}^2(X)$, then $X \simeq \text{Jac}^2(Y) \simeq \text{Jac}^3(Y)$. We call X and Y the *dual elliptic quintics*. It is clear that if X has a rational point, which is always the case when the base field k is algebraically closed, then X and Y are isomorphic.

We have the following almost converse result.

Lemma 2.7. *If X has no rational points and the j -invariant satisfies $j(E) \neq 1728$ then X and Y are not isomorphic.*

Proof. The dual elliptic quintics X and Y give rise to elements $[X], [Y] \in H^1(k, E)$ of order five, and $[Y] = 2[X]$.

The classes $[X], [Y]$ correspond to isomorphic genus one curves if and only if $[Y]$ lies in the $\text{Aut}(E)$ -orbit of $[X]$ in $H^1(k, E)$ [Sil86, Exercise 10.4].

If we assume that for an automorphism $\sigma \in \text{Aut}(E)$ we have $\sigma([X]) = [Y] = 2[X]$, the action of σ on $H^1(k, E)$ preserves the subgroup $\mathbb{Z}/5$ generated by $[X]$ and we get a surjective group homomorphism $\langle \sigma \rangle \rightarrow (\mathbb{Z}/5)^\times \simeq \mathbb{Z}/4$. In particular the order of σ should be a multiple of 4. On the other hand since

$j(E) \neq 1728$ and $\text{char}(k) = 0$, we have $\text{Aut}(E) = \mathbb{Z}/2$ or $\text{Aut}(E) = \mathbb{Z}/6$, and no such σ exists.

Thus X and Y are not isomorphic. \square

We now explain duality between elliptic quintics in terms of projective duality.

Proposition 2.8. *Let V be a 5-dimensional k -vector space and let $A \subset \Lambda^2(V^\vee)$ be a 5-dimensional subspace. We consider the Grassmannian $\text{Gr}(2, V) \subset \mathbb{P}(\Lambda^2(V))$ and the dual Grassmannian $\text{Gr}(2, V^\vee) \subset \mathbb{P}(\Lambda^2(V^\vee))$. For a five-dimensional linear subspace $A \subset \Lambda^2(V^\vee)$ let*

$$X := \text{Gr}(2, V) \cap \mathbb{P}(A^\perp)$$

$$Y := \text{Gr}(2, V^\vee) \cap \mathbb{P}(A).$$

Assume that X is a smooth transverse intersection, so that X is a genus one curve. Then Y is also a smooth transverse intersection and X and Y are dual elliptic quintics, that is we have

$$Y \simeq \text{Jac}^3(X), \quad X \simeq \text{Jac}^2(Y).$$

Proof. By [DK18, Proposition 2.24] if X is a smooth transverse intersection, then the same is true for Y .

We construct a line bundle \mathcal{M} on $X \times Y$. At each point $([U], [\theta]) \in X \times Y$ we consider the vector space $\mathcal{M}_{[U], \theta} := U \cap \text{Ker}(\theta)$. Let us show that this space is one-dimensional. On the one hand we have $\theta(U) = 0$ so that U can not have trivial intersection with $\text{Ker}(\theta)$, otherwise dimension of $\text{Ker}(\theta)$ would be greater than 3. On the other hand U can not be contained in $\text{Ker}(\theta)$, otherwise $[U]$ would be a singular point of X by Lemma 2.1.

Thus \mathcal{M} , considered as a sheaf given by the kernel of

$$p_1^*(\mathcal{U}|_X) \oplus p_2^*(\mathcal{K}|_Y) \rightarrow V \otimes \mathcal{O}_{X \times Y}$$

on $X \times Y$, where p_1, p_2 are the projections from $X \times Y$ on the two factors, $\mathcal{U} \subset V \otimes \mathcal{O}_{\text{Gr}(2, V)}$ is the tautological rank two subbundle on $\text{Gr}(2, V)$ and $\mathcal{K} \subset V \otimes \mathcal{O}_{\text{Gr}(2, V^\vee)}$ is the rank three subbundle of kernels of 2-forms, is a locally free sheaf of rank one.

We now compute the bidegree of \mathcal{M} . For any $\theta \in Y$, since X does not intersect the singular locus of D_θ (otherwise X would have been singular), X can be considered as a curve on the resolution \widetilde{D}_θ defined by (2.1).

It follows from definitions that the restriction $\mathcal{M}|_{X \times \theta}$ is isomorphic to the restriction of the line bundle $\mathcal{O}(-h)$ from \widetilde{D}_θ to X , and thus by Lemma 2.4 the degree of $\mathcal{M}|_{X \times \theta}$ is equal up to sign to the degree of $\sigma_{2,0}$ in $\text{Gr}(2, 5)$. The latter degree is equal to three, as can be computed using the Pieri formula [GH78, Chapter 1.5].

The Fourier-Mukai transform defined by \mathcal{M} is a derived equivalence between X and Y by [Kuz06, Section 4.1, Section 6.1], which by a standard argument implies that X and Y are moduli spaces of line bundles on each other with \mathcal{M} playing the role of the universal bundle.

Thus we see that

$$Y \simeq \text{Jac}^{-3}(X),$$

and we have $\text{Jac}^{-3}(X) \simeq \text{Jac}^3(X)$ by taking dual bundles.

Finally, $X \simeq \text{Jac}^2(Y)$ follows by symmetry by repeating the last part of the above argument with the roles of X and Y switched, as the degree of the Schubert cycle $\sigma_{2,0,0}$ on $\text{Gr}(3, 5)$ is equal to two. \square

We now deduce L-equivalence of the dual elliptic quintics from their projective duality construction.

Theorem 2.9. *Let X and Y be smooth projective dual elliptic quintics. Then X and Y are L-equivalent, more precisely we have*

$$\mathbb{L}^4([X] - [Y]) = 0,$$

and in general $[X] \neq [Y]$.

Proof. By Lemma 2.6 and Proposition 2.8 there exists a five-dimensional subspace $A \subset \Lambda^2(V^\vee)$ such that

$$X \simeq \text{Gr}(2, V) \cap \mathbb{P}(A^\perp)$$

$$Y \simeq \text{Gr}(2, V^\vee) \cap \mathbb{P}(A).$$

We consider the universal hyperplane section $\mathcal{H} \subset \text{Gr}(2, V) \times \mathbb{P}(A)$:

$$\mathcal{H} := \{U \in \text{Gr}(2, V), \theta \in \mathbb{P}(A) : \theta(U) = 0\}$$

and compute its class in the Grothendieck ring of varieties in two ways.

We apply Proposition 2.3 to $S := \mathbb{P}(A) \subset \mathbb{P}(\Lambda^2(V^\vee))$ to obtain

$$(2.2) \quad [\mathcal{H}] = [\mathbb{P}^4](1 + \mathbb{L} + 2\mathbb{L}^2 + 2\mathbb{L}^3 + \mathbb{L}^4 + \mathbb{L}^5) + \mathbb{L}^4 \cdot [Y].$$

On the other hand, the morphism $\mathcal{H} \rightarrow \text{Gr}(2, V)$ is Zariski locally-trivial over locally-closed subset $\text{Gr}(2, V) \setminus X$ and X with fibers \mathbb{P}^3 and \mathbb{P}^4 respectively so that we have

$$(2.3) \quad [\mathcal{H}] = [\text{Gr}(2, 5)][\mathbb{P}^3] + \mathbb{L}^4 \cdot [X].$$

We compare (2.2) and (2.3). An easy computation shows that both $[\mathbb{P}^4](1 + \mathbb{L} + 2\mathbb{L}^2 + 2\mathbb{L}^3 + \mathbb{L}^4 + \mathbb{L}^5)$ and $[\text{Gr}(2, 5)][\mathbb{P}^3]$ are equal to

$$\mathbb{L}^9 + 2\mathbb{L}^8 + 4\mathbb{L}^7 + 6\mathbb{L}^6 + 7\mathbb{L}^5 + 7\mathbb{L}^4 + 6\mathbb{L}^3 + 4\mathbb{L}^2 + 2\mathbb{L} + 1$$

(for $[\text{Gr}(2, 5)]$ see Lemma 2.2). Thus (2.2) and (2.3) together give

$$\mathbb{L}^4 \cdot ([X] - [Y]) = 0.$$

Finally X and Y are in general not isomorphic by Lemma 2.7, and since X and Y are not uniruled, the standard argument shows that $[X] \neq [Y]$ [KS18, Proposition 2.2]. \square

3. ELLIPTIC SURFACES OF INDEX FIVE

In this section k is an algebraically closed field of characteristic zero, and we assume $k = \mathbb{C}$ when discussing Hodge lattices of K3 surfaces.

3.1. L-equivalence of elliptic surfaces. We refer to [Dol10, Chapter 2] for general discussion of elliptic surfaces and their Jacobians. We recall the basic concepts. By an *elliptic surface* we mean a smooth projective surface X with a morphism $\pi : X \rightarrow C$ to a smooth projective curve C such that the general fiber of π is a genus one curve. We always assume that X is relatively minimal, that is the fibers of π do not contain (-1) -curves.

We do not assume that π admits a section. By the *index* of an elliptic surface we mean the minimal positive degree of a multisection of π .

For every $k \in \mathbb{Z}$ one can consider the relative Jacobian $Y = \text{Jac}^k(X/C)$; Y is another elliptic surface over the same base curve C defined as the unique minimal regular model with the generic fiber $\text{Jac}^k(X_{k(C)})$. As in the genus one curve case, if X admits a section, then all Jacobians $\text{Jac}^k(X/C)$ are isomorphic to X over C .

Lemma 3.1. *If $X \rightarrow C$ is an elliptic surface and $Y = \text{Jac}^k(X/C)$, then for every point $c \in C$, the reduced fibers $(X_c)_{\text{red}}$ and $(Y_c)_{\text{red}}$ are isomorphic.*

Proof. This follows from [Dol10, Chapter 2, Proposition 1 and 2]. \square

We now consider the case when the multisection index of an elliptic surface $X \rightarrow C$ is equal to five, and analogously to the genus one curve case we call X and $Y = \text{Jac}^2(X/C) \simeq \text{Jac}^3(X/C)$ the dual elliptic fibrations.

Theorem 3.2. *Let $X \rightarrow C$ be an elliptic fibration of index five over an algebraically closed field of characteristic zero, and let $Y = \text{Jac}^2(X/C)$. Then X and Y are L-equivalent, more precisely we have*

$$\mathbb{L}^4([X] - [Y]) = 0.$$

Proof. Let $X_{k(C)}$, $Y_{k(C)}$ be the generic fibers of X and Y . By Theorem 2.9, we have $\mathbb{L}^4([X_{k(C)}] - [Y_{k(C)}]) = 0$ in $K_0(\text{Var}/k(C))$. Therefore by [NS11, Proposition 3.4] there exists a non-empty open set $U \subset C$ such that

$$\mathbb{L}^4([X_U] - [Y_U]) = 0,$$

in $K_0(\text{Var}/k)$, where X_U , Y_U are preimages of U in X and Y respectively. Let $C \setminus U = \{c_1, \dots, c_n\}$, then X_{c_i} and Y_{c_i} are isomorphic for each i by Lemma 3.1. In particular $\mathbb{L}^4([X_{c_i}] - [Y_{c_i}]) = 0$; summing everything together we obtain the desired L-equivalence statement. \square

In the next section we show that elliptic K3 surfaces of index five and Picard rank two provide examples when X and Y are not isomorphic, see Proposition 3.10, so that $[X] \neq [Y]$ (see e.g. [KS18, Proposition 2.8]).

3.2. Elliptic K3 surfaces of Picard rank two. We consider elliptic K3 surfaces over $k = \mathbb{C}$. Recall that for a K3 surface $\text{NS}(X) \simeq \text{Pic}(X)$ is a free finitely generated abelian group whose rank is called the Picard rank of X . Intersection pairing gives $\text{NS}(X)$ a structure of a lattice. See [Huy16, Chapter 14] for an introduction to lattices. We write U for the hyperbolic plane, and $N(X)$ for the extended Neron–Severi lattice $N(X) = U \oplus \text{NS}(X)$ under the

Mukai pairing. We say that two indefinite lattices have the same genus if they have the same rank, signature and discriminant groups.

We only consider projective K3 surfaces, that is the ones admitting a polarization. We think of polarization as a class of an ample divisor in $\text{NS}(X)$. Since by degree reasons the class of a polarization is linearly independent to the class of the fiber of an elliptic fibration, the minimal Picard rank of an elliptic K3 surface is equal to two. Good references about such K3 surfaces are papers of Stellari [St04] and van Geemen [vG05], and [Huy16, Chapter 11].

Lemma 3.3. [vG05, Remark 4.2] *Let X be an elliptic K3 surface of index $t > 0$ and of Picard rank two. Let $F \in \text{NS}(X)$ be the class of the fiber. Then there exists a polarization H such that $H \cdot F = t$, and H, F form a basis of $\text{NS}(X)$.*

Proof. Let us show that $F \in \text{NS}(X)$ is a primitive class. Indeed, if $F = mC$, for $m \geq 1$, then C will be an effective divisor contained in a fiber. Since we assume that Picard rank of X is two, all fibers are irreducible, and $m = 1$.

Since F is a primitive class, there exists $D \in \text{NS}(X)$ such that D, F form a basis of $\text{NS}(X)$. Up to replacing D by $-D$ we may assume that $D \cdot F = t$. A simple computation shows that the only possible (-2) -classes in $\text{NS}(X)$ are given by $\pm(D + \frac{2-D^2}{2t}F)$, hence there is at most one (-2) -curve in X .

We consider $H = D + nF$. It is clear that

$$H^2 = D^2 + 2nt > 0$$

for $n \gg 0$. If C is a (-2) -curve, then

$$H \cdot C = D \cdot C + nF \cdot C > 0$$

for $n \gg 0$ since C is not in any fiber (otherwise the Picard rank of X would be at least three). Hence H is ample for $n \gg 0$ by [Huy16, Proposition 2.1.4]. \square

For a pair of integers $t > 0$ and $d \in \mathbb{Z}$ we consider a rank two lattice $\Lambda_{t,d}$ with basis H, F and pairing defined by

$$(3.1) \quad \begin{pmatrix} 2d & t \\ t & 0 \end{pmatrix}$$

There always exist projective K3 surfaces with $\text{NS}(X) \simeq \Lambda_{t,d}$ [Huy16, Corollary 14.3.1]. Any such K3 surface is elliptic because $\text{NS}(X)$ contains a square-zero class [Huy16, Proposition 11.1.3]. Furthermore since the embedding of $\Lambda_{t,d}$ into a K3 lattice is unique up to isomorphism by [Huy16, Corollary 14.3.1] the locus of these K3 surfaces is an irreducible locally closed subset of dimension 18 in the moduli space of all degree $2d$ polarized K3 surfaces.

Note that t is a well-defined invariant of $\Lambda_{d,t}$, as the discriminant of (3.1) is $-t^2$. The following result describes the complete set of invariants of $\Lambda_{d,t}$ in the case when t is an odd prime.

Proposition 3.4 (van Geemen, Stellari). *Let $t > 0$ be an odd prime, and let $d, d' \in \mathbb{Z}$.*

- (1) $\Lambda_{d,t}$ is isomorphic to $\Lambda_{d',t}$ if and only if $d \equiv d' \pmod{t}$ or $dd' \equiv 1 \pmod{t}$.
- (2) $O(\Lambda_{d,t}) = \{\pm 1\}$ if $d \not\equiv \pm 1 \pmod{t}$ and $O(\Lambda_{d,t}) = \mathbb{Z}/2 \times \mathbb{Z}/2 = \{\pm 1, \pm J\}$ where J is the isometry swapping the two isotropic classes if $d \equiv \pm 1 \pmod{t}$.
- (3) The discriminant group $A_{d,t} = \Lambda_{d,t}^*/\Lambda_{d,t}$ is $(\mathbb{Z}/t)^2$ if t divides d , and for $\gcd(d,t) = 1$, it is $A_{d,t} = \mathbb{Z}/t^2$ with the square of the generator given by $\frac{-2d}{t^2}$.
- (4) $\Lambda_{d,t}, \Lambda_{d',t}$ are in the same genus if and only if $d' \equiv k^2 d \pmod{t}$ for some integer k coprime to t .

Proof. (1) is [vG05, Proposition 3.7]. and (2) is [vG05, Lemma 4.6]. The result in (3) is easy for $t|d$ as we can assume $d = 0$. For $\gcd(d,t) = 1$, (3) is the computation in the proof of [St04, Lemma 3.2 (ii)]. (4) is [St04, Lemma 3.2 (ii)]. \square

Example 3.5. If $t = 5$, then there are four isomorphism classes of lattices $\Lambda_{5,d}$:

$$\Lambda_{5,0}, \Lambda_{5,1}, \Lambda_{5,2} \simeq \Lambda_{5,3}, \Lambda_{5,4}.$$

The discriminant group $A_{d,t} = \Lambda_{d,t}^*/\Lambda_{d,t}$ for $\Lambda_{5,0}$ is $\mathbb{Z}/5 \oplus \mathbb{Z}/5$, and it is $\mathbb{Z}/25$ in the other cases.

The lattices $\Lambda_{5,1}$ and $\Lambda_{5,4}$ are in the same genus, whereas the other lattices have only one isomorphism class in each genus.

Finally, the lattices $\Lambda_{1,5}, \Lambda_{4,5}, \Lambda_{5,5}$ admit an isometry J permuting the two isotropic classes, and the isometry group is $\mathbb{Z}/2 \times \mathbb{Z}/2 = \{\pm 1\} \times \{\pm J\}$, whereas the lattice $\Lambda_{2,5} \simeq \Lambda_{3,5}$ has the isometry group $\mathbb{Z}/2 = \{\pm 1\}$.

Explicitly one can get a K3 surface with $\text{NS}(X) = \Lambda_{t,d}$ by taking a general K3 surface containing a degree t elliptic curve.

Example 3.6. A very general degree 8 K3 surface $X \subset \mathbb{P}^5$ which contains a normal rational curve C of degree three, has $H^2 = 8$, $C \cdot H = 3$, $C^2 = -2$, so that

$$\text{NS}(X) \simeq \begin{pmatrix} 8 & 3 \\ 3 & -2 \end{pmatrix}.$$

Such a K3 surface admits an elliptic fibration provided by the pencil $F = H - C$, which consists of the residual elliptic quintics in the hyperplane sections of X through C and it is easy to compute that we have

$$\text{NS}(X) \simeq \Lambda_{5,4}.$$

We note that X admits a unique elliptic fibration [vG05, 4.7].

Example 3.7. A general degree 10 K3 surface X is a complete intersection of a Grassmannian $\text{Gr}(2, 5) \subset \mathbb{P}^9$ with three hyperplanes and a quadric [Muk88, Corollary 0.3].

As soon as X contains a normal elliptic quintic curve $F \subset \mathbb{P}^4$, it will admit an elliptic fibration of index five, and generically we have

$$\text{NS}(X) \simeq \begin{pmatrix} 10 & 5 \\ 5 & 0 \end{pmatrix}.$$

in the basis H, F . In fact if we write $F' = H - F$, we see that $\mathrm{NS}(X)$ is isomorphic to $\Lambda_{5,0}$. We note that F' gives rise to a second elliptic fibration structure on X , cf. [vG05, 4.7].

We prepare to address the question when $\mathrm{Jac}^k(X/\mathbb{P}^1)$ and X are isomorphic.

Lemma 3.8. *If X is a K3 surface with $\mathrm{NS}(X) = \Lambda_{t,d}$, and $\gcd(t, k) = 1$, then $\mathrm{NS}(\mathrm{Jac}^k(X/\mathbb{P}^1)) = \Lambda_{t,d \cdot k^2}$ for any elliptic fibration on X .*

Proof. Let $N(X) = U \oplus \mathrm{NS}(X)$ be the extended Neron-Severi lattice and let e_1, e_2 be a basis of U consisting of two isotropic vectors with $e_1 \cdot e_2 = -1$. Then $v = F + ke_2 \in N(X)$ is the Mukai vector giving rise to the moduli space $Y = \mathrm{Jac}^k(X/\mathbb{P}^1)$ [Huy16, Example 16.2.4]. Using [Muk87, Theorem 1.4] we have

$$\mathrm{NS}(Y) = v^\perp / v.$$

Explicitly we have

$$v^\perp = \langle F, e_2, kH + te_1 \rangle = \langle v, e_2, kH + te_1 \rangle,$$

so that

$$v^\perp / v = \langle e_2, kH + te_1 \rangle,$$

and the intersection form on this lattice is isomorphic to $\Lambda_{t,d \cdot k^2}$. \square

We need the following result, which describes the group of Hodge isometries of the transcendental lattice for a sufficiently general K3 surface. This group is important for studying derived equivalence between K3 surfaces. In particular it appears in the counting formula for the number of Fourier-Mukai partners [HLOY04, Theorem 2.3]. In the proof we follow the strategy of [Og02, Proposition B.1].

Lemma 3.9. *If X has Picard rank $\rho < 20$ and X is very general in the moduli space of K3 surfaces polarized by a fixed sublattice $\mathrm{NS}(X)$ of the K3 lattice, then the group of Hodge isometries of the transcendental lattice T_X is $\{\pm 1\}$.*

Proof. By the Torelli theorem for K3 surfaces, a (marked) K3 surface polarized by $\mathrm{NS}(X)$ is determined by a holomorphic 2-form $\sigma_X \in T_X \otimes \mathbb{C}$, considered up to scalar. Since the choice of the form σ is given by the condition

$$(3.2) \quad \sigma_X^2 = 0 \quad \text{and} \quad \sigma_X \bar{\sigma}_X > 0,$$

and for a very general choice of σ_X satisfying (3.2), σ_X^\perp in $T_X \otimes \mathbb{C}$ contains no non-trivial integral class, we conclude that the moduli of (marked) K3 surface polarized by $\mathrm{NS}(X)$ has dimension $\mathrm{rk}(T_X) - 2$.

Let us fix an isometry g of T_X , and assume that g induces a Hodge isometry of T_X for the K3 surface X corresponding to σ_X . We use [HLOY04, Proposition B.1]. For any choice of σ_X , the group of Hodge isometries of T_X is a finite cyclic group of even order $2m$, and without loss of generality we may assume that g is a generator of this group. Furthermore in this case g acts on σ_X via

multiplication by a primitive $2m$ -th root of unity. Finally, $T_X \otimes \mathbb{C}$ decomposes into a direct sum of eigenspaces of g as

$$(3.3) \quad T_X \otimes \mathbb{C} = \bigoplus_{\xi} V_{\xi},$$

where ξ runs over all primitive $2m$ -th roots of unity, and the dimension of each eigenspace V_{ξ} is $\text{rk } T_X / \varphi(2m)$ with $\varphi(-)$ being the Euler function (see [HLOY04, Steps 4, 5 in the proof of Proposition B.1]). Since σ_X is an eigenvector for g , we have $\sigma_X \in V_{\xi}$ for some ξ . It follows that the moduli of such K3 surfaces has dimension at most $\text{rk}(T_X) / \varphi(2m) - 1$.

By assumption $\rho < 20$, so that we have $\text{rk}(T_X) > 2$. If $m > 1$, then $\varphi(2m) \geq 2$ and

$$\text{rk}(T_X) / \varphi(2m) - 1 \leq \text{rk}(T_X) / 2 - 1 < \text{rk}(T_X) - 2,$$

where the right-hand-side is the dimension of the moduli space of K3 surfaces polarized by $\text{NS}(X)$ and the left-hand-side is the dimension of the closed subvariety in the moduli where g becomes the generator for the group of Hodge isometries. This means that unless $g = \pm 1$, g is not a Hodge isometry of T_X of a general K3 surface in the moduli.

Since the group of isometries of T_X is countable, very general choices of σ_X would give K3 surfaces with the group of Hodge isometries of T_X equal to $\{\pm 1\}$. \square

We now consider the multisection index 5 case. According to Lemma 3.3, an elliptic K3 surface with Picard rank two will have Neron–Severi lattice isomorphic to one of the $\Lambda_{5,d}$, where d is considered modulo 5. See Example 3.5 for more details about these lattices.

Proposition 3.10. *Let X be an elliptic K3 surface with $\text{NS}(X) = \Lambda_{5,d}$, and let $Y = \text{Jac}^2(X/\mathbb{P}^1)$.*

- (1) *If $d = 2$ or $d = 3$, then X and Y are isomorphic.*
- (2) *If $d = 1$ or $d = 4$, then X and Y are not isomorphic.*
- (3) *If $d = 0$, and X is very general in moduli, then X and Y are not isomorphic.*

Proof. (1) It suffices to show that X does not have nontrivial Fourier–Mukai partners. We note that by Proposition 3.4 (1) and (4), $\Lambda_{5,2} \simeq \Lambda_{5,3}$ is the only isometry class of a lattice in its genus. Hence the counting formula for Fourier–Mukai partners [HLOY04, Theorem 2.3] has only one term and since by Proposition 3.4 (2) the orthogonal group $O(\Lambda_{5,2})$ consists of ± 1 , this term is equal to one.

(2) By Lemma 3.8, taking Jac^2 interchanges the Neron–Severi lattices $\Lambda_{5,1}$ and $\Lambda_{5,4}$, and since these lattices are not isomorphic, X and Y are not isomorphic.

(3) If X and Y are isomorphic, then the Fourier–Mukai transform $\Phi : \mathcal{D}^b(X) \simeq \mathcal{D}^b(\text{Jac}^2(X/\mathbb{P}^1))$ corresponding to the moduli space $\text{Jac}^2(X/C)$ on X induces a Hodge isometry of $H^*(X, \mathbb{Z})$ taking one Mukai vector to the other [Huy16, Section 16.3].

Consider the extended Neron–Severi lattice $N(X) = U \oplus NS(X)$, where we choose a basis e_1, e_2 for U consisting of two isotropic vectors satisfying $e_1 \cdot e_2 = -1$. The action of Φ takes e_1 (Mukai vector for moduli space X on X) to $F + 2e_2$ (Mukai vector for moduli space Y on X).

We note that one such isometry $g_0 \in O(N(X))$ is

$$(3.4) \quad \begin{aligned} e_1 &\mapsto 2e_2 + F \\ e_2 &\mapsto -2e_1 - H \\ H &\mapsto 2H + 5e_1 \\ F &\mapsto -2F - 5e_2 \end{aligned}$$

and any other isometry g mapping e_1 to $F + 2e_2$ will have the form

$$g = g_0 \cdot h,$$

where $h \in O(N(X), e_1)$ is an isometry of $N(X)$ fixing e_1 .

We now consider the action of g on the discriminant group $N(X)^*/N(X) \simeq NS(X)^*/NS(X) \simeq A_{5,0} = \mathbb{Z}/5 \oplus \mathbb{Z}/5$ generated by $\frac{1}{5}H, \frac{1}{5}F$ (cf. Proposition 3.4 (3)). Since we assume that the action of g is induced by a Hodge isometry of $H^*(X, \mathbb{Z})$, the action of g on the discriminant group is the same as the action induced by a Hodge isometry of T_X . By Lemma 3.9 for general X this action on the discriminant group is ± 1 .

We note that the action of $O(N(X), e_1)$ on the discriminant group factors through $O(e_1^\perp/e_1) = O(NS(X))$, so by [vG05, Lemma 4.6] its action is given by one of the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

On the other hand we see from (3.4) that the action of $\pm g_0$ on $A_{5,0}$ does not belong to the subgroup above. Therefore there is no element $g \in O(N(X))$ which maps e_1 to $F + 2e_2$ and is induced by a Hodge isometry of $H^*(X, \mathbb{Z})$. \square

REFERENCES

- [AKW17] B. Antieau, D. Krashen, M. Ward: *Derived categories of torsors for abelian schemes*. Adv. Math. 306 (2017), 1–23.
- [BS13] M. Bhargava, A. Shankar: *The average size of the 5-Selmer group of elliptic curves is 6, and the average rank is less than 1*, [arXiv:1312.7859](#).
- [Bor18] L.A. Borisov: *The class of the affine line is a zero divisor in the Grothendieck ring*, J. Algebraic Geom. 27 (2018), no. 2, 203–209.
- [BCP17] L.A. Borisov, A. Caldararu, A. Perry: *Intersections of two Grassmannians in \mathbb{P}^9* , [arXiv:1707.00534](#).
- [BL04] A.I. Bondal, M. Larsen, V.A. Lunts: *Grothendieck ring of pretriangulated categories*, Int. Math. Res. Not. 2004, no. 29, 1461–1495.
- [Bri98] T. Bridgeland: *Fourier-Mukai transforms for elliptic surfaces*, J. Reine Angew. Math. 498 (1998), 115–133.
- [Dol10] I. Dolgachev: *Algebraic surfaces with $q = p_g = 0$* . Algebraic surfaces, 97–215, C.I.M.E. Summer Sch., 76, Springer, Heidelberg, 2010.
- [DK18] O. Debarre, A. Kuznetsov: *Gushel–Mukai varieties: Classification and birationalities*, Algebr. Geom. 5 (2018), no. 1, 15–76.

- [Ef18] A.I. Efimov: *Some remarks on L -equivalence of algebraic varieties*, Selecta Math. (N.S.) 24 (2018), no. 4, 3753–3762.
- [F13] T. Fisher: *Invariant theory for the elliptic normal quintic, I. Twists of $X(5)$* . Math. Ann. 356 (2013), no. 2, 589–616.
- [GS14] S. Galkin, E. Shinder: *The Fano variety of lines and rationality problem for a cubic hypersurface*, [arXiv:1405.5154](#).
- [GH78] P. Griffiths, J. Harris: *Principles of algebraic geometry*, Pure and Applied Mathematics. Wiley–Interscience [John Wiley & Sons], New York, 1978.
- [HL18] B. Hassett, K.-W. Lai: *Cremona transformations and derived equivalences of $K3$ surfaces* Compos. Math. 154 (2018), no. 7, 1508–1533.
- [HLOY04] S. Hosono, B.H. Lian, K. Oguiso, S.-T. Yau: *Fourier-Mukai number of a $K3$ surface*, Algebraic structures and moduli spaces, 177–192, CRM Proc. Lecture Notes, 38, Amer. Math. Soc., Providence, RI, 2004.
- [Huy16] D. Huybrechts: *Lectures on $K3$ surfaces*. Cambridge Studies in Advanced Mathematics, 158. Cambridge University Press, Cambridge, 2016.
- [IMOU16] A. Ito, M. Miura, S. Okawa, K. Ueda: *Derived equivalence and Grothendieck ring of varieties: the case of $K3$ surfaces of degree 12 and abelian varieties*, [arXiv:1612.08497](#).
- [IMOU19] A. Ito, M. Miura, S. Okawa, K. Ueda: *The class of the affine line is a zero divisor in the Grothendieck ring: via $G2$ -Grassmannians*, J. Algebraic Geom. 28 (2019), no. 2, 245–250.
- [KKM17] G. Kapustka, M. Kapustka, R. Moschetti: *Equivalence of $K3$ surfaces from Verra threefolds*, [arXiv:1712.06958](#).
- [Kaw18] Y. Kawamata: *Birational geometry and derived categories*, Surveys in differential geometry 2017. Celebrating the 50th anniversary of the Journal of Differential Geometry, 291–317, Surv. Differ. Geom., 22, Int. Press, Somerville, MA, 2018.
- [Kuz06] A.G. Kuznetsov: *Hyperplane sections and derived categories*, Izv. Ross. Akad. Nauk Ser. Mat. 70 (2006), no. 3, 23–128; translation in Izv. Math. 70 (2006), no. 3, 447–547.
- [KS18] A. Kuznetsov, E. Shinder: *Grothendieck ring of varieties, D - and L -equivalence, and families of quadrics*, Selecta Math. (N.S.) 24 (2018), no. 4, 3475–3500.
- [Man17] L. Manivel: *Double spinor Calabi-Yau varieties*, [arXiv:1709.07736](#).
- [Mar16] N. Martin: *The class of the affine line is a zero divisor in the Grothendieck ring: an improvement*, C. R. Math. Acad. Sci. Paris 354 (2016), no. 9, 936–939.
- [Muk87] S. Mukai: *On the moduli space of bundles on $K3$ surfaces, I*, Vector bundles on algebraic varieties (Bombay, 1984), 341–413, Tata Inst. Fund. Res. Stud. Math., 11, Tata Inst. Fund. Res., Bombay, 1987.
- [Muk88] S. Mukai: *Curves, $K3$ surfaces and Fano 3-folds of genus ≤ 10* , Algebraic geometry and commutative algebra, Vol. I, 357–377, Kinokuniya, Tokyo, 1988.
- [NS11] J. Nicaise, J. Sebag: *The Grothendieck ring of varieties*, Motivic integration and its interactions with model theory and non-Archimedean geometry. Volume I, 145–188, London Math. Soc. Lecture Note Ser., 383, Cambridge Univ. Press, Cambridge, 2011.
- [Og02] K. Oguiso: *$K3$ surfaces via almost-primes*, Math. Res. Lett. 9 (2002), no. 1, 47–63.
- [Ok18] S. Okawa: *An example of birationally inequivalent projective symplectic varieties which are D -equivalent and L -equivalent* [arXiv:1801.09385](#).
- [Sil86] J.H. Silverman: *The arithmetic of elliptic curves*, Graduate Texts in Mathematics, 106. Springer–Verlag, New York, 1986.
- [St04] P. Stellari: *Some remarks about the FM-partners of $K3$ surfaces with Picard numbers 1 and 2*, Geom. Dedicata 108 (2004), 1–13.
- [vG05] B. van Geemen: *Some remarks on Brauer groups of $K3$ surfaces*, Adv. Math. 197 (2005), no. 1, 222–247.

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SHEFFIELD, THE HICKS BUILDING, HOUNSFIELD ROAD, SHEFFIELD S3 7RH, UK

Email address: `e.shinder@sheffield.ac.uk`

INSTITUTE FOR ALGEBRAIC GEOMETRY, LEIBNIZ UNIVERSITY HANNOVER, WELFENGARTEN 1, 30167 HANNOVER, GERMANY

Email address: `zhangzy@math.uni-hannover.de`