RELATIVE VGIT AND AN APPLICATION TO DEGENERATIONS OF HILBERT SCHEMES

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Abstract. We generalize the classical semi-continuity theorem for GIT (semi)stable loci under variations of linearizations to a relative situation of an equivariant projective morphism $X \to S$ over an affine base S. As an application to moduli problems, we consider degenerations of Hilbert schemes, and give a conceptual interpretation of the (semi)stable loci of the degeneration families constructed in [GHH19].

1. Introduction

Geometric invariant theory (GIT) is a very versatile method for constructing moduli spaces. The classical GIT concerns the action of a reductive group G on a projective variety X along with an ample line bundle L on X. The action of G on L determines a semistable locus X^{ss} on X, with a quotient X^{ss}/G defined as the GIT quotient $X//G$ with respect to L. The main tool for determining the (semi)stability of a point in X is the Hilbert-Mumford criterion, which reduces the problem to a purely numerical computation of the weights of the actions of some 1-parameter subgroups of G on the fibres of L at fixed points. For a much more detailed account of the classical theory, we recommend [MFK94].

One important aspect of the theory is the semi-continuity property of the GIT (semi)stable loci under a variation of the G-linearized ample line bundle L. Roughly, it says the following: let L_t be a family of G-linearized ample line bundles on X, then for any t sufficiently close to 0, we have $X^s(L_0) \subseteq X^s(L_t) \subseteq X^{ss}(L_t) \subseteq X^{ss}(L_0)$; see e.g. [Laz13, Lemma 3.10]. This property has been used to study birational transformations of certain moduli spaces constructed as GIT quotients; see e.g. [Tha96, DH98].

In [GHH15], M. G. Gulbrandsen and the first two named authors generalized the GIT technique to the relative situation. Namely, instead of a single projective variety X, they considered a projective morphism $X \to S$ over an affine base S. The classical GIT theory can be therefore thought of as the special case when the base scheme S is a point. For a reductive group G which acts equivariantly on the family $X \to S$ along with a relative ample line bundle L on X, they developed a version of the Hilbert-Mumford criterion which can be used to determine the (semi)stable locus in this setting. In [GHH19], they also applied this criterion to construct a degeneration family of Hilbert schemes of points on surfaces.

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In the present paper, we will generalize the semi-continuity property of the (semi)stable loci under a variation of the G-linearized line bundle L. Our generalization is made in two aspects: first of all, instead of the absolute case of a single projective variety, we consider the relative case of a projective morphism $X \to S$ over an affine base S; secondly, instead of requiring all G-linearized line bundles L_t to be ample, we require a weaker assumption on the limit line bundle L_0. More precisely, our main result is:

Theorem 1.1 (Theorem 3.23). *Let $f\colon X \to S$ be a projective morphism over an affine variety S, and G a reductive group acting equivariantly on $f\colon X \to S$. Let L_0 and L_1 be G-linearized line bundles on X and $L_t = L_0^{1-t} \otimes L_1^t$ for any $0 \leqslant t \leqslant 1$. Assume that L_t is ample for $0 < t \leqslant 1$, and that L_0 satisfies the Hilbert-Mumford criterion. Then we have*

$$X^s(L_0) \subseteq X^s(L_t) \subseteq X^{ss}(L_t) \subseteq X^{ss}(L_0)$$

for $0 < t \ll 1$.

We note that this generalizes the known results even in the absolute case, as in the existing literature X is assumed to be irreducible and normal, whereas we only require X to be a projective scheme.

The idea of the proof follows from [DH98, Res00]. It requires two main ingredients, which were both established only for a projective variety X in [DH98], hence require new proofs in our setting. The first ingredient is the continuity of the M-function, as defined in (2), with respect to the choice of the line bundle; see Proposition 3.7. The second ingredient states that for a fixed G-action on X with respect to different ample G-linearized line bundles, there are only finitely many possible semistable loci; see Proposition 3.8. We combine them with the relative Hilbert-Mumford criterion proved in [GHH15] to conclude Theorem 1.1.

We would next like to mention an application of these results, which also served as the main motivation for developing VGIT in a relative setting. To do this, we first need to briefly recall the degeneration family of Hilbert schemes of points constructed in [GHH19]. Let $f\colon X \to C$ be a projective strict simple degeneration over an affine curve C. Using Jun Li's technique of expanded degenerations (see e.g. [Li13]), one obtains an expanded family $f[n]\colon X[n] \to C[n]$ with an action of $G[n] = (\mathbb{G}_m)^n$. By functoriality, the relative Hilbert scheme of points $\mathrm{Hilb}^n(X[n]/C[n]) \to C[n]$ is also equipped with a $G[n]$-action. By carefully choosing a suitable $G[n]$-linearized ample line bundle, one obtains a GIT quotient $I^n_{X/C} = \mathrm{Hilb}^n(X[n]/C[n]) /\!/ G[n]$ such that the (proper) Hilbert scheme degeneration $I^n_{X/C} \to C$ has good geometric properties. In fact, this family is isomorphic to $\mathrm{Hilb}^n(X/C) \to C$ over the open subset of C where f is smooth. Moreover, when the relative dimension of f is at most two, it was proved in [GHHZ18, Theorem 5.9, Corollary 5.16] that $I^n_{X/C} \to C$ is a dlt model with reduced special fibre, which is even a minimal model if $X \to C$ is minimal. The detailed analysis of the (semi)stable locus $\mathrm{Hilb}^n(X[n]/C[n])^{ss}$ was carried out in [GHH19, §2]. Intriguingly, one finds that the property whether a point $[Z] \in \mathrm{Hilb}^n(X[n]/C[n])$ is (semi)stable or not only depends on its underlying 0-cycle; see [GHH19, Theorem 2.9]. In the present paper, we will use Theorem 1.1 and Lemma 2.4 to give a conceptual proof of this fact.

The paper is organized as follows. In §2 we give our definition of the (semi)stability of a point on a variety X under the action of a reductive group G via a linearization L, and prove functoriality of the formation of the (semi)stable locus along certain types of projective morphisms. In §3, we first establish the two main ingredients required in the proof of Theorem 1.1. Their proofs in the case of torus actions are given in §3.2 and §3.3, respectively; and in the case of general reductive group actions in §3.4. We prove Theorem 1.1 and two other variants of it in §3.5. Finally, §4 is devoted to the main application of Theorem 1.1. After recalling the necessary notations required in the setting of expanded degenerations in §4.1, we explain in §4.2 and §4.3 in two steps why the (semi)stability for a point in $\mathrm{Hilb}^n(X[n]/C[n])$ only depends on the underlying cycle.

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2. Linearizations on line bundles

GIT works very well for projective quotients with respect to ample linearizations. However, for our purposes, we need to generalize some standard results in GIT to include also *semi-ample* linearizations. We point out that, although the classical result [MFK94, Theorem 1.10] holds without any ampleness assumption, it is not exactly what we need. The following example illustrates a potential issue for the purpose of our application:

Example 2.1. Let $\mathbb{G}_m$ act on $(\mathbb{P}^1, \mathscr{O}(1))$ in the standard way; namely, $\lambda \cdot [x_0 : x_1] = [\lambda^{-1}x_0 : \lambda x_1]$. Let $\mathbb{G}_m$ act on $(\mathbb{P}^2, \mathscr{O}(1))$ trivially. Then we get an action of $\mathbb{G}_m$ on $(\mathbb{P}^1 \times \mathbb{P}^2, \mathscr{O}(1,1))$. The stable locus is $(\mathbb{P}^1 \backslash \{0, \infty\}) \times \mathbb{P}^2$ with quotient $\mathbb{P}^2$.

Let $\pi : W \to \mathbb{P}^1 \times \mathbb{P}^2$ be the blowup along $\mathbb{P}^1 \times \{p\}$, where $p \in \mathbb{P}^2$ is any closed point. Then W still carries a $\mathbb{G}_m$-action on the first factor and $L = \pi^*\mathscr{O}(1,1)$ carries a $\mathbb{G}_m$-linearization. Let E be the exceptional divisor of π. We claim that no point in E is semistable in the usual sense. Indeed, E is a trivial $\mathbb{P}^1$ bundle along the center of the blowup. Let F be any such $\mathbb{P}^1$-fibre. Then for any $\mathbb{G}_m$-invariant section s of a given tensor power of L, s is constant along F. If s does not vanish along F, then the entire $F \subseteq W_s$, where $W_s = W \backslash s^{-1}(0)$. This implies that W_s cannot be affine, hence all points in F are unstable.

This means that the entire exceptional divisor E is excluded from the semistable locus. Indeed, the semistable locus is isomorphic to $(\mathbb{P}^1 \backslash \{0, \infty\}) \times (\mathbb{P}^2 \backslash \{p\})$ via π, and the quotient is $\mathbb{P}^2 \backslash \{p\}$.

We will see that this is not ideal for our application. Indeed, we will show that, for the family of expanded degenerations considered in §4, the (semi)stable locus of the relative Hilbert scheme is precisely the pullback of the (semi)stable locus of the relative symmetric product under the Hilbert-Chow morphism; see Proposition 4.1. However, Example 2.1 exhibits a potential issue, namely the preimage of some semistable points could become unstable. In the situation of our interest, this means that the entire exceptional locus of the Hilbert-Chow morphism will be excluded from the semistable locus. Therefore we need to modify the definition of semistability for our purpose. The requirement we need to drop is the affineness of the complement of an invariant section.

Definition 2.2. Assume a reductive group G acts on a quasi-projective scheme X over an algebraically closed field $\Bbbk$. Let L be a G-linearized line bundle on X. Then a point $x \in X$ is said to be

- *semistable* if $x \in X_s$ for some G-invariant section $s \in \Gamma(X, L^{\otimes n})^G$, where n is a non-negative integer and $X_s = X \backslash s^{-1}(0)$;
- *stable* if x is semistable, G_x is finite, and $G \cdot x$ is closed in the semistable locus of X, where G_x is the stabilizer of x, and $G \cdot x$ is the G-orbit containing x.

We denote the locus of semistable and stable points by $X^{ss}(L)$ and $X^s(L)$ respectively.

When L is an ample line bundle on X, the above definition is equivalent to the classical one [MFK94, Definition 1.7]. But in general, this definition is weaker, therefore allows more (semi)stable points. As a result, unexpected phenomena can happen. For example, the closure of a non-closed orbit could contain more than one closed orbit. For this reason we will not use our definition to construct a quotient of X by G, unless L is an ample linearization.

However, the notion of the (semi)stable locus in Definition 2.2 is very convenient for two types of results we need: functoriality of (semi)stable loci along some projective morphisms and semi-continuity of (semi)stable loci in families. We start with the following lemma concerning the functoriality under finite group quotients.

Lemma 2.3. *Let W be a quasi-projective scheme over an algebraically closed field $\Bbbk$ of characteristic zero. Let H be a finite group acting on W, and $f : W \to V$ the quotient morphism. Assume G acts on both W and V, such that the actions of G and H on W commute. Assume further that f is G-equivariant. Let L be a G-linearized line bundle on V. Then we have*

$$W^{ss}(f^*L) = f^{-1}(V^{ss}(L));$$
$$W^{s}(f^*L) = f^{-1}(V^{s}(L)).$$

Proof. The proof is literally the same as that of [GHHZ18, Lemma 3.5]. □

The second lemma below concerns the functoriality of the (semi)stable locus under certain projective morphisms. This will be important for our application to degenerations of Hilbert schemes.

Lemma 2.4. *Assume W and V are schemes of finite type over a field $\Bbbk$ of characteristic 0. Let $f : W \to V$ be a G-equivariant projective morphism and L a G-linearized line bundle on V. We furthermore assume that*

- *V is irreducible and normal;*
- *the fibres of f are connected;*
- *there exist G-equivariant open subsets $U_W \subseteq W$ and $U_V \subseteq V$, such that the complement of U_V in V is of codimension at least 2, and the restriction $f|_{U_W} : U_W \to U_V$ is an isomorphism.*

Then we have

$$W^{ss}(f^*L) = f^{-1}(V^{ss}(L));$$
$$W^{s}(f^*L) = f^{-1}(V^{s}(L)).$$

Proof. We first prove the equality for semistable loci.

The inclusion $W^{ss}(f^*L) \supseteq f^{-1}(V^{ss}(L))$ is obvious. Let $v \in V^{ss}(L)$ and $w \in f^{-1}(v)$. Then there exists a section $t \in \Gamma(V, L^{\otimes n})^G$ with $t(v) \neq 0$. Hence we have $f^*t \in \Gamma(W, f^*L^{\otimes n})^G$ with $(f^*t)(w) \neq 0$.

For the inclusion $W^{ss}(f^*L) \subseteq f^{-1}(V^{ss}(L))$, let $w \in W^{ss}(f^*L)$ and $v = f(w)$. There exists a section $s \in \Gamma(W, f^*L^{\otimes n})^G$ with $s(w) \neq 0$. By identifying U_W and U_V, the restriction $s|_{U_W}$ defines a section $s' \in \Gamma(U_V, L^{\otimes n})^G$. By normality we can extend s' to a section $t \in \Gamma(V, L^{\otimes n})^G$, which is also G-invariant by the uniqueness of the extension. We claim that $t(v) \neq 0$. Assume on the contrary that $t(v) = 0$. It is easy to see that $f|_{\overline{U}_W} : \overline{U}_W \to V$ is surjective. Let $w' \in \overline{U}_W$ such that $f(w') = v$. By construction $(f^*t)|_{U_W} = s|_{U_W}$, hence $(f^*t)|_{\overline{U}_W} = s|_{\overline{U}_W}$. It follows that $s(w') = (f^*t)(w') = t(v) = 0$. On the other hand, since the fibres of f are connected, and the restriction of f^*L on each fibre is trivial, we have $s(w) = s(w') = 0$, a contradiction.

Now we prove the equality for the stable loci. For any $w \in W^{ss}(f^*L)$, we write $v = f(w)$, then by the first statement $v \in V^{ss}(L)$. We will show that w is stable in W if and only if v is stable in V.

Let $w \in W^s(f^*L)$. We need to show that $v \in V^s(L)$. For simplicity we write $f^{ss} : W^{ss}(f^*L) \to V^{ss}(L)$ for the restriction of f on the semistable locus. Then the projectivity of f implies that f^{ss} is closed. Since the orbit $G \cdot w$ is closed in $W^{ss}(f^*L)$, it follows that $G \cdot v = f^{ss}(G \cdot w)$ is closed in $V^{ss}(L)$. Now we consider the commutative diagram

(1)
$$\begin{tikzcd} G \times \{w\} \arrow[r] \arrow[d, "\cong"'] & G \cdot w \arrow[d, "h := f|_{G\cdot w}"] \\ G \times \{v\} \arrow[r] & G \cdot v. \end{tikzcd}$$

By [Hum75, Proposition 8.3], the orbits $G \cdot w$ and $G \cdot v$ are smooth, hence the morphism $h := f|_{G \cdot w}$ is generically smooth. Furthermore G acts transitively on $G \cdot w$, hence h is smooth. In particular, $h^{-1}(v)$ is smooth. Moreover it is also projective since f is a projective morphism.

We pull back the diagram (1) along the inclusion $\{v\} \hookrightarrow G \cdot v$ to get

$$\begin{tikzcd} G_v \times \{w\} \arrow[r, "\varphi"] \arrow[d, "\cong"'] & h^{-1}(v) \arrow[d, "h|_{h^{-1}(v)} = f|_{h^{-1}(v)}"] \\ G_v \times \{v\} \arrow[r] & \{v\} \end{tikzcd}$$

where G_v denotes the stabilizer of v in G. Notice that the action map φ factors through the quotient group

$$\begin{tikzcd} G_v \times \{w\} \arrow[rr, "\varphi"] \arrow[dr, "\alpha"'] & & f^{-1}(v) \\ & G_v/G_w \times \{w\}, \arrow[ur, "\beta"'] & \end{tikzcd}$$

where $G_v/G_w = \{gG_w \mid g \in G_v\}$ is a smooth quasi-projective variety; see [BB02, Theorem 4.4.1]. It is clear that β is a bijection of points. Since $f^{-1}(v)$ is smooth, we conclude by Zariski's main theorem that β is an isomorphism, hence G_v/G_w is a projective variety. By assumption w is a stable point, hence α is a finite group quotient which is a finite morphism. Thus G_v is proper. However, as a closed subvariety of an affine variety G, G_v is affine itself. It follows that G_v is finite.

For the other inclusion, assume $w \in W^{ss}(f^*L)$ and $v = f(w) \in V^s(L)$. We claim that $G \cdot w$ must be closed in $W^{ss}(f^*L)$. Otherwise, there is another orbit $G \cdot w'$ in the closure of of $G \cdot w$, which also maps to $G \cdot v$ under f. Moreover, $\dim(G \cdot w') < \dim(G \cdot w) \leqslant \dim G = \dim(G \cdot v)$, which is a contradiction. To show G_w is finite, it suffices to realize that G_w is a subgroup of G_v which is finite itself. We conclude that w is stable. This finishes the proof. $\square$

3. Relative VGIT

The main goal for this section is to prove the semi-continuity result Theorem 3.23. After introducing the general assumptions and notations in §3.1, we will establish two key ingredients, which will be proven for tori in §3.2 and §3.3, and generalized to arbitrary reductive groups in §3.4. The proof of the main result will be given in §3.5.

3.1. **General assumptions.** We state some assumptions and notations which are valid throughout this section.

Let $\Bbbk$ be an algebraically closed field. Let S be an affine scheme of finite type over $\Bbbk$, and X a projective S-scheme with structure morphism

$$f : X \longrightarrow S.$$

Let G be a linearly reductive group acting equivariantly on X and S. The G-fixed loci are denoted by X^G and S^G respectively. In §3.2 and §3.3, we will assume that $G = (\mathbb{G}_m)^n$ is a torus. In such a case, we write $\mathcal{X}_*(G)$ for the lattice of 1-PS's of G, and $\mathcal{X}^*(G)$ for the lattice of characters of G, which is canonically the dual of $\mathcal{X}_*(G)$. We also write $\mathcal{X}_*(G)_{\mathbb{R}} = \mathcal{X}_*(G) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathcal{X}^*(G)_{\mathbb{R}} = \mathcal{X}^*(G) \otimes_{\mathbb{Z}} \mathbb{R}$. In §3.4 and §3.5, G will be an arbitrary linearly reductive group. In such a case, we will still use $\mathcal{X}_*(G)$ for the set of 1-PS's of G.

Next we define some numerical functions.

For an arbitrary reductive group G, let L be a G-linearized line bundle on X. For every $x \in X$ and any 1-PS $\lambda : \mathbb{G}_m \to G$, we define $\mu^L(x, \lambda)$ as follows:

- if $x_0 = \lim_{t\to 0} \lambda(t) \cdot x$ exists, then $\lambda(\mathbb{G}_m)$ acts on the fibre $L|_{x_0}$ via a character $t \to t^{-\mu^L(x,\lambda)}$;

- if $\lim_{t\to 0}\lambda(t)\cdot x$ does not exist, then we formally define $\mu^L(x,\lambda) = +\infty$.

We fix a norm $\|\cdot\|$ on $\mathcal{X}_*(G)$ which is invariant under the action of the Weyl group; see [DH98, §1.1.3]. Then we define

$$M^L(x) = \inf_{\lambda\in\mathcal{X}_*(G)\setminus\{0\}} \frac{\mu^L(x,\lambda)}{\|\lambda\|}. \tag{2}$$

The next two subsections will be devoted to the two key ingredients required for the proof of the main result, under the extra assumption that $G = (\mathbb{G}_m)^n$ is a torus. We will remove this assumption in §3.4.

3.2. **Continuity of the function $M^\bullet(x)$.** In this subsection, we assume

$$G = (\mathbb{G}_m)^n$$

is a torus. We will establish the first key result Proposition 3.7, which will be required for the main result in this section.

We recall some notations from [GHH15]. We write $S = \operatorname{Spec} A$ for some finitely generated $\Bbbk$-algebra A. Assume L is an f-ample line bundle on X. By replacing L with a tensor power, which does not affect stability, we can assume L is very ample. Let $V = H^0(L)$; this is a finitely generated A-module. Then we obtain a closed embedding

$$X \subseteq \mathbb{P}(V) = \operatorname{Proj}\operatorname{Sym} V.$$

The relative projective space $\mathbb{P}(V)$ can be realized as the projectivization of the relative affine space $\mathbb{V}(V) = \operatorname{Spec}\operatorname{Sym} V$.

We first extend some results in [GHH15, §5] to higher dimensional tori.

Lemma 3.1. *Let $G = (\mathbb{G}_m)^n$. Let A be a ring and V an A-module. Then a G-action on $S = \operatorname{Spec} A$ is given by a weight decomposition*

$$A = \bigoplus_{\chi\in\mathcal{X}^*(G)} A_\chi. \tag{3}$$

A compatible G-action on V is also given by a weight decomposition

$$V = \bigoplus_{\chi\in\mathcal{X}^*(G)} V_\chi, \tag{4}$$

such that the action of A on V respects the multi-grading.

Proof. One can either use a proof similar to that of [GHH15, Lemma 5.2], or use the statement of [GHH15, Lemma 5.2] repeatedly n times, each time for a $\mathbb{G}_m$ factor of G. $\square$

Let $p \in \mathbb{P}(V)$ be a closed point and $p^* \in \mathbb{V}(V)$ be any of its lifts. As mentioned in [GHH15, §5], the image $f(p) \in S$ is given by a $\Bbbk$-algebra homomorphism

$$A \to \Bbbk,$$

which is the evaluation map of functions on S at the point $f(p)$ (where we have extended f in the obvious way to all of $\mathbb{P}(V)$). Moreover, $p^* \in \mathbb{V}(V)$ itself is given by an A-module homomorphism

$$V \to \Bbbk,$$

which is the evaluation map of sections of L at the point p^*. For each character $\chi \in \mathcal{X}^*(G)$, we write the restrictions of the above homomorphisms to the corresponding component as given in Lemma 3.1 by

$$[f(p)]_\chi : A_\chi \to \Bbbk$$

and

$$[p^*]_\chi : V_\chi \to \Bbbk.$$

In fact, it is worth pointing out that, whether a given section vanishes at p can be checked on any of its lifts p^*. Therefore, without ambiguity, in the following discussions, we can simply write the condition $[p^*]_\chi = 0$ (resp. $[p^*]_\chi \neq 0$) by $[p]_\chi = 0$ (resp. $[p]_\chi \neq 0$).

For any 1-PS $\lambda \in \mathcal{X}_*(G)$, we give a more detailed analysis on when $\mu^L(x, \lambda)$ achieves a finite value, and give another interpretation of this value. The following two results are essentially [GHH15, Lemma 5.3].

Lemma 3.2. *Assume L is very ample. Let x be a closed point in X. Under the above notations, the following statements are equivalent:*

(i) $\mu^L(x, \lambda)$ *is finite;*
(ii) $\lim_{t\to 0} \lambda(t) \cdot x$ *exists in X;*
(iii) $\lim_{t\to 0} \lambda(t) \cdot f(x)$ *exists in S;*
(iv) $[f(x)]_\chi = 0$ *for each $\chi \in \mathcal{X}^*(G)$ satisfying $\langle \lambda, \chi \rangle < 0$;*
(v) *There exists some $d_0 \in \mathbb{Z}$, such that $[x]_\chi = 0$ for each $\chi \in \mathcal{X}^*(G)$ satisfying $\langle \lambda, \chi \rangle < d_0$.*

Proof. (i)⇔(ii) is by definition.

(ii)⇔(iii) is due to the assumption that f is projective.

(iii)⇔(iv) and (iv)⇔(v) are both in [GHH15, Lemma 5.3], except that every component V_d in the single grading used in [GHH15, Lemma 5.3] should be understood as the direct sum of V_χ for all χ satisfying $\langle \lambda, \chi \rangle = d$. □

We will now rephrase this in a manner which will be useful later. By [Kem78, Lemma 1.1(a)], we can find a G-equivariant closed embedding

$$\iota : S \longrightarrow W$$

where W is an affine space on which the G-action is linear. As a vector space, W can be decomposed into weight spaces associated to distinct characters of G; namely

$$W = \bigoplus_{\chi \in \Gamma} W_\chi,$$

where Γ is a non-empty finite subset of the character lattice $\mathcal{X}^*(G)$. Then each point $w \in W$ can be written as

$$w = \sum_{\chi \in \Gamma} w_\chi$$

where $w_\chi \in W_\chi$ can be thought as the coordinate of w in the subspace W_χ.

For every subset $I \subseteq \Gamma$, we write

$$\begin{aligned} W_I &= \{w \in W \mid w_\chi \neq 0 \Leftrightarrow \chi \in I\}, \\ S_I &= \iota^{-1}(W_I), \\ X_I &= f^{-1}(S_I). \end{aligned}$$

Then

$$S = \bigcup_{I \subseteq \Gamma} S_I \tag{5}$$

is a finite stratification indexed by the power set of Γ.

Without loss of generality, we can assume that the coordinate w_χ does not vanish identically on S; otherwise we can remove the summand W_χ from W and embed S equivariantly in a smaller affine space. Under this assumption, the set of characters χ with non-trivial A_χ in the decomposition (3) is precisely the submonoid of $\mathcal{X}^*(G)$ generated by Γ.

Recall that in our case $V = H^0(L)$ is a finitely generated A-module, and let $v_1, v_2, \cdots, v_n$ be a set of generators. Then we have

$$V = Av_1 + Av_2 + \cdots + Av_n. \tag{6}$$

Without loss of generality, we can assume that each generator v_i is of pure G-weight; otherwise we can replace v_i by its components of pure weights, which still gives a finite set of generators for V. To be more precise, we write $v_i \in V_{\chi_i}$ for each $1 \leqslant i \leqslant n$.

Now we take an arbitrary point $x \in X$. We write

$$\Gamma_{f(x)} = \{\chi \in \mathcal{X}^*(G) \mid [f(x)]_\chi \neq 0\},$$
$$\Gamma_x = \{\chi \in \mathcal{X}^*(G) \mid [x]_\chi \neq 0\}.$$

We assume $x \in X_I$ for some $I \subseteq \Gamma$, or equivalently, $f(x) \in S_I$. Then we see that $\Gamma_{f(x)}$ is the submonoid of $\mathcal{X}^*(G)$ generated by I. The following lemma gives a description of the structure of Γ_x:

Lemma 3.3. *Without loss of generality, we assume that*

$$[x]_{\chi_i}(v_i) \begin{cases} \neq 0 & \text{if } 1 \leqslant i \leqslant k; \\ = 0 & \text{if } k+1 \leqslant i \leqslant n \end{cases}$$

for some integer k with $1 \leqslant k \leqslant n$. Then we have

$$\Gamma_x = \bigcup_{1 \leqslant i \leqslant k} \left(\chi_i + \Gamma_{f(x)}\right).$$

Proof. We first show "$\supseteq$".

For each $1 \leqslant i \leqslant k$, the condition $[x]_{\chi_i}(v_i) \neq 0$ implies that $v_i(x) \neq 0$. For every $\chi' \in \Gamma_{f(x)}$, there exists some $a' \in A_{\chi'}$, such that $a'(f(x)) \neq 0$. Then the section $a' \cdot v_i \in V$ does not vanish at x, which is of pure weight $\chi_i + \chi'$. It follows immediately that $\chi_i + \chi' \in \Gamma_x$.

We then show "$\subseteq$".

For any $\chi' \in \Gamma_x$, there exists some $v' \in V_{\chi'}$ such that $v'(x) \neq 0$. By (6) we can write

$$v' = a_1 v_1 + \cdots + a_n v_n \tag{7}$$

for some $a_1, \cdots, a_n \in A$. Without loss of generality, we can assume that $a_i v_i \in V_{\chi'}$ for every $1 \leqslant i \leqslant n$; otherwise, we can remove all components in the right-hand side of (7) that are not of weight χ' and the equality still holds. Then the condition $v'(x) \neq 0$ implies that $a_{i_0} v_{i_0}$ does not vanish at x for some $1 \leqslant i_0 \leqslant n$, which in particular implies that $v_{i_0}(x) \neq 0$. By assumption, $v_i(x) = 0$ for each $k+1 \leqslant i \leqslant n$. Therefore we have $1 \leqslant i_0 \leqslant k$.

Assume $a_{i_0} \in A$ is of weight χ_0, then we have $\chi' = \chi_{i_0} + \chi_0 \in \chi_{i_0} + \Gamma_{f(x)}$, as desired. $\square$

For $x \in X$ we define $\mathcal{C}_x$ to be the convex cone in $\mathcal{X}_*(G)_{\mathbb{R}}$ generated by the set

$$\{\lambda \in \mathcal{X}_*(G) \mid \langle \lambda, \chi \rangle \geqslant 0 \text{ for all } \chi \in \Gamma_{f(x)}\}. \tag{8}$$

Since $\Gamma_{f(x)}$ is a finitely generated monoid, $\mathcal{C}_x$ is a closed rational polyhedral cone. Moreover, by the equivalence of (i) and (iv) in Lemma 3.2, $\mu^L(x, \lambda)$ is finite for any λ in the set (8). Indeed, we have the following alternative description of $\mu^L(x, \lambda)$:

Lemma 3.4. *Under the assumption of Lemma 3.2, for any* 1*-PS* $\lambda \in \mathcal{X}_*(G)$*, if* $\mu^L(x, \lambda)$ *is finite, then it is given by*

$$\mu^L(x, \lambda) = -\min\{\langle \lambda, \chi \rangle \mid \chi \in \Gamma_x\}. \tag{9}$$

Proof. We can understand V_d in [GHH15, Lemma 5.3] as the direct sum of V_χ for all χ satisfying $\langle \lambda, \chi \rangle = d$, then the claim follows from [GHH15, Lemma 5.3] immediately. $\square$

One of the main benefits of Lemma 3.4 is that we can use (9) to extend the definition of $\mu^L(x, \lambda)$ to all $\lambda \in \mathcal{C}_x$ that are not necessarily integral. Moreover, it also gives us a geometric interpretation of the value of $\mu^L(x, \lambda)$.

Indeed, let $\mathrm{Conv}(\Gamma_x)$ be the convex hull of Γ_x in $\mathcal{X}^*(G)_{\mathbb{R}}$. Then the quotient $\frac{\mu^L(x,\lambda)}{\|\lambda\|}$ is equal to the signed distance from the origin to the boundary of the projection of $\mathrm{Conv}(\Gamma_x)$ to the positive ray spanned by the vector $-\lambda$. In particular, the quotient $\frac{\mu^L(x,\lambda)}{\|\lambda\|}$ is a continuous function defined on the closed subset of the unit sphere

$$\mathcal{C}_x \cap \{\lambda \in \mathcal{X}_*(G)_{\mathbb{R}} \mid \|\lambda\| = 1\}$$

which is compact, hence achieves a finite maximum and a minimum if $\mathcal{C}_x \supsetneq \{0\}$. This observation allows us to obtain the following result; see [DH98, Proposition 1.1.6].

Proposition 3.5. *For any G-linearized line bundle L on X, we have*

- *if* $\lim_{t\to 0} \lambda(t) \cdot x$ *does not exist for any non-trivial* 1*-PS* $\lambda \in \mathcal{X}_*(G)$*, then* $M^L(x) = +\infty$*;*
- *otherwise,* $M^L(x)$ *has a finite value.*

Proof. In the first case we have $\mu^L(x, \lambda) = +\infty$ for all $\lambda \in \mathcal{X}_*(G)$, hence $M^L(x) = +\infty$.

In the second case we have $\mathcal{C}_x \supsetneq \{0\}$. If L is ample, the above discussion shows that $\frac{\mu^L(x,\lambda)}{\|\lambda\|}$ is a continuous function, which, using the fact that the rational points lie dense, in particular leads to

$$\begin{aligned} M^L(x) &= \inf_{\lambda \in \mathcal{C}_x \setminus \{0\}} \frac{\mu^L(x, \lambda)}{\|\lambda\|} \\ &= \inf_{\substack{\lambda \in \mathcal{C}_x \\ \|\lambda\| = 1}} \frac{\mu^L(x, \lambda)}{\|\lambda\|} \end{aligned} \tag{10}$$

which achieves a finite minimum on the compact set $\mathcal{C}_x \cap \{\lambda \in \mathcal{X}_*(G)_{\mathbb{R}} \mid \|\lambda\| = 1\}$. If L is not necessarily ample, we can always find G-linearized ample line bundles L_1 and L_2, such that $L = L_1 \otimes L_2^{-1}$. Notice that $\mu^L(x, \lambda)$ is linear with respect to L, hence we have

$$\begin{aligned} M^L(x) &= \inf_{\lambda \in \mathcal{C}_x \backslash \{0\}} \frac{\mu^L(x, \lambda)}{\|\lambda\|} \\ &= \inf_{\lambda \in \mathcal{C}_x \backslash \{0\}} \left(\frac{\mu^{L_1}(x, \lambda)}{\|\lambda\|} - \frac{\mu^{L_2}(x, \lambda)}{\|\lambda\|} \right) \\ &= \inf_{\substack{\lambda \in \mathcal{C}_x \\ \|\lambda\|=1}} \left(\frac{\mu^{L_1}(x, \lambda)}{\|\lambda\|} - \frac{\mu^{L_2}(x, \lambda)}{\|\lambda\|} \right). \end{aligned}$$

Similar to the above discussion, since a continuous function defined on a compact set $\mathcal{C}_x \cap \{\lambda \in \mathcal{X}_*(G)_{\mathbb{R}} \mid \|\lambda\| = 1\}$ achieves a finite minimum, we conclude that $M^L(x)$ has a finite value for an arbitrary L. $\square$

Remark 3.6. We emphasize that the condition for $M^L(x)$ to be finite (resp. infinite) is independent of the choice of L. Indeed, it depends only on the G-action on X; more precisely, it depends on whether any 1-PS of G gives a limit point when it acts on the point $x \in X$.

Let $\mathrm{Pic}^G(X)$ be the group of G-linearized line bundles on X and let $\mathrm{Pic}^G(X)_{\mathbb{R}}$ be the vector space obtained by tensoring with $\mathbb{R}$. In analogy to [Res00, §2.1] we also denote by $\mathrm{NS}^G(X)_{\mathbb{R}}$ the group of G-linearized line bundles on X modulo algebraic equivalence, tensored with $\mathbb{R}$. Here we call two G-linearized line bundles algebraically equivalent if they can be deformed into each other (as G-linearized line bundles). We do not claim that in our situation $\mathrm{NS}^G(X)_{\mathbb{R}}$ is a finite-dimensional vector space, nor do we need this in the subsequent arguments. The argument of [Res00, Proposition 2] goes through unchanged and shows that the function $M^\bullet(x) : \mathrm{Pic}^G(X) \to \mathbb{R} \cup \{+\infty\}$ factors through $M^\bullet(x) : \mathrm{NS}^G(X)_{\mathbb{R}} \to \mathbb{R} \cup \{+\infty\}$. The above remark then says that for any point $x \in X$, either $M^L(x) = +\infty$ for all L or $M^\bullet(x) : \mathrm{NS}^G(X)_{\mathbb{R}} \to \mathbb{R}$ is a well defined function.

We are ready to state the following key result; see [Res00, Lemma 2].

Proposition 3.7. *For any fixed point $x \in X$, we have*

- *either $M^L(x) = +\infty$ for every G-linearized line bundle L; or*
- *the function*

$$M^\bullet(x) : \mathrm{NS}^G(X)_{\mathbb{R}} \longrightarrow \mathbb{R}; \quad L \longmapsto M^L(x)$$

  is well-defined, and continuous with respect to L over any finite dimensional subspace of $\mathrm{NS}^G(X)_{\mathbb{R}}$.

Proof. Under the assumption we have $\mathcal{C}_x \supsetneq \{0\}$. Notice that for every fixed 1-PS $\lambda \in \mathcal{C}_x \backslash \{0\}$, the function $\frac{\mu^L(x,\lambda)}{\|\lambda\|}$ is a linear function in L. As the infimum of a family of linear functions, $M^L(x)$ is continuous with respect to L as well. $\square$

3.3. **Finiteness of possible semistable loci.** In this subsection, we will still work under the general assumptions given in Subsection 3.1. In particular, we still assume $G = (\mathbb{G}_m)^n$ to be a torus. The goal is to establish our second key result, namely

Proposition 3.8. *For any given G-actions on X and S, there are only finitely many subsets of X which can be realized as $X^{ss}(L)$ for some G-linearized ample line bundle L.*

The strategy for proving Proposition 3.8 is inspired by that of [DH98, Theorem 1.3.9 (ii)], but the situation turns out to be substantially more involved due to the action of the group G on both X and the base S. More precisely, we will construct a finite stratification of X, and show that for any G-linearized ample line bundle L, the semistable locus $X^{ss}(L)$ is always the union of a subset of the strata.

Recall that we have a stratification (5) of S indexed by the power set of Γ. For each subset $I \subseteq \Gamma$, we write L_I for the sublattice of $\mathcal{X}^*(G)$ spanned by I, $\operatorname{Conv}(I)$ for the convex hull of I in $\mathcal{X}^*(G)_{\mathbb{R}}$ and $V_I = L_I \otimes_{\mathbb{Z}} \mathbb{R}$ the linear subspace of $\mathcal{X}^*(G)_{\mathbb{R}}$ spanned by I. Then we have

$$\operatorname{Conv}(I) \subseteq V_I \subseteq \mathcal{X}^*(G)_{\mathbb{R}}.$$

Using the duality between $\mathcal{X}^*(G)$ and $\mathcal{X}_*(G)$, we can realize $L_I^{\perp}$ as a sublattice of $\mathcal{X}_*(G)$. Then there exists a unique subtorus G_I of G, such that $\mathcal{X}_*(G_I) = L_I^{\perp}$. We start with the following observation:

Lemma 3.9. *For any point $s \in S_I$, the torus G_I is the identity component G_s° of the stabilizer group $G_s \subseteq G$.*

Proof. By [Hum75, Theorem in §16.2], the identity component G_s° is a subtorus of G. We have $\mathcal{X}_*(G_s^\circ) \subseteq L_I^{\perp}$ by the construction of $L_I^{\perp}$ and $\mathcal{X}_*(G_s^\circ) \supseteq L_I^{\perp}$ by the maximality of G_s°. Hence $\mathcal{X}_*(G_s^\circ) = L_I^{\perp}$, which implies $G_s^\circ = G_I$. □

Lemma 3.10. *Let $I \subseteq \Gamma$ and $s \in S_I$ be a closed point, then the following conditions are equivalent:*

(i) *$G \cdot s$ is a closed G-orbit in S;*
(ii) *for every $\lambda \in \mathcal{X}_*(G)$, either λ fixes s or $\lim_{t\to 0} \lambda(t) \cdot s$ does not exist;*
(iii) *$\operatorname{Conv}(I)$ as a subset of V_I contains 0 as an interior point.*

Proof. First we show (i) $\Rightarrow$ (ii). If there exists some $\lambda \in \mathcal{X}_*(G)$ such that $\lim_{t\to 0} \lambda(t) \cdot s = s_0 \neq s$, then $s_0 \notin G \cdot s$ since s_0 has a larger stabilizer than s. Therefore $G \cdot s_0 \subseteq \overline{G \cdot s}$, which contradicts (i).

Next we show (ii) $\Rightarrow$ (i). Assume on the contrary that there is a G-orbit $G \cdot y \subseteq (\overline{G \cdot s}) \backslash (G \cdot s)$, then by [Bir71, Theorem 3.6], there exists some 1-PS $\lambda \in \mathcal{X}_*(G)$, such that $\lim_{t\to 0} \lambda(t) \cdot s \in \overline{G \cdot y}$, which contradicts (ii).

Now we show (ii) $\Rightarrow$ (iii). If (iii) does not hold, then there exists a (rational) hyperplane in V_I, such that $\operatorname{Conv}(I)$ is contained in the closed half space on one side of this hyperplane in V_I, with some elements of I not on the hyperplane itself. Since V_I is a (rational) linear subspace of $\mathcal{X}^*(G)_{\mathbb{R}}$, there exists a (rational) hyperplane in $\mathcal{X}^*(G)_{\mathbb{R}}$, such that $\operatorname{Conv}(I)$ is contained in the closed half space on one side of this hyperplane in $\mathcal{X}^*(G)_{\mathbb{R}}$, with some elements of I not on the hyperplane itself. In other words, there exists some

$\lambda \in \mathcal{X}_*(G)$, such that $\langle \lambda, \chi \rangle \geqslant 0$ for all $\chi \in I$, with strict inequalities for some $\chi \in I$. This implies $\lim_{t\to 0} \lambda(t) \cdot s$ exists and is not equal to s itself, which contradicts (ii).

Finally we show (iii) $\Rightarrow$ (ii). Consider an arbitrary $\lambda \in \mathcal{X}_*(G)$. If $\lambda \in L_I^\perp$, then $\langle \lambda, \chi \rangle = 0$ for every $\chi \in I$, hence λ fixes s. Otherwise, the value of $\langle \lambda, \chi \rangle$ is positive for some $\chi \in I$ and negative for some other $\chi \in I$, therefore $\lim_{t\to 0} \lambda(t) \cdot s$ is divergent. $\square$

Remark 3.11. As explained in the proof above, we can make condition (ii) in Lemma 3.10 more precise. In fact, when this condition holds, for any $\lambda \in \mathcal{X}_*(G)$, we have

$$\begin{aligned} \lambda \in L_I^\perp &\iff \lambda \text{ fixes } s; \\ \lambda \notin L_I^\perp &\iff \lim_{t\to 0} \lambda(t) \cdot s \text{ does not exist.} \end{aligned}$$

For every $I \subseteq \Gamma$ satisfying condition (iii) in Lemma 3.10, since G_I fixes any closed point $s \in S_I$, every fibre $f^{-1}(s)$ is G_I-invariant. Since moreover the morphism $f : X \to S$ is projective, X_I contains a non-empty G_I-fixed locus $(X_I)^{G_I}$. We have the following observation:

Lemma 3.12. *The union of $(X_I)^{G_I}$ for all $I \subseteq \Gamma$ satisfying condition (iii) in Lemma 3.10 is precisely the union of closed G-orbits in X.*

Proof. First of all we observe that, if $G \cdot x$ is a closed orbit for some $x \in X$, then $G \cdot f(x)$ is also a closed orbit in S, since the morphism $f : X \to S$ is projective. By Lemma 3.10, $f(x) \in S_I$ for some $I \subseteq \Gamma$ satisfying condition (iii), therefore $x \in X_I$ for the same I.

It remains to show that, for any $I \subseteq \Gamma$ satisfying condition (iii) and any point $x \in X_I$, $G \cdot x$ is a closed G-orbit if and only if $x \in (X_I)^{G_I}$.

For one direction, we assume that $G \cdot x$ is a closed G-orbit. Then we see $(G \cdot x) \cap f^{-1}(f(x)) = G_{f(x)} \cdot x$ is also closed in the fibre $f^{-1}(f(x))$. By Lemma 3.9, $G_I \cdot x$ is a connected component of $G_{f(x)} \cdot x$, it follows that $G_I \cdot x$ is also closed in the fibre $f^{-1}(f(x))$. Since the fibre is projective, every 1-PS of G_I must fix x, hence $x \in (X_I)^{G_I}$.

For the other direction, we assume that x is fixed by G_I. We claim that the restriction of f to the orbit $G \cdot x$ is an étale map onto its image $G \cdot f(x)$. Indeed, this map can be understood as the natural map from $(G/G_I) \cdot x$ to $(G/G_I) \cdot f(x)$, and the stabilizer of $f(x)$ in G/G_I is finite by Lemma 3.9, hence the claim follows. If the orbit $G \cdot x$ were not closed, i.e. there were another orbit $G \cdot y \subseteq (\overline{G \cdot x}) \backslash (G \cdot x)$, then $G \cdot f(y) \subseteq G \cdot f(x)$ since $G \cdot f(x)$ is closed by Lemma 3.10. However we have $\dim(G \cdot f(y)) \leqslant \dim(G \cdot y) < \dim(G \cdot x) = \dim(G \cdot f(x))$. This is a contradiction because $G \cdot f(x)$ does not contain any strictly smaller G-orbit. $\square$

Regarding closed G-orbits, the following lemma will be helpful later:

Lemma 3.13. *For any closed point $x \in X$, the closure $\overline{G \cdot x}$ of the G-orbit of x contains some closed G-orbit.*

Proof. We proceed by induction on the dimension of the orbit. A 0-dimensional G-orbit is a point, hence always closed. Let $x \in X$ be an arbitrary closed point. If the orbit $G \cdot x$ itself is closed then the result holds. Otherwise the boundary of the orbit $(\overline{G\cdot x})\backslash(G\cdot x)$ contains an orbit $G\cdot y$ of lower dimension. By the induction hypothesis, $\overline{G\cdot y}$ contains a closed G-orbit, which is also in $\overline{G\cdot x}$, as desired. □

Now we are ready to construct the required stratification. Since X_I is quasi-projective, the G_I-fixed locus $(X_I)^{G_I}$ has finitely many connected components. We write Λ for the set of all connected components of $(X_I)^{G_I}$ for all $I\subseteq\Gamma$ satisfying condition (iii) in Lemma 3.10. Then Λ is a finite set of quasi-projective subschemes of X.

For any pair of subsets $I\subseteq\Gamma$ and $J\subseteq\Lambda$, we define

$$X_I^J=\{x\in X_I \mid \overline{G\cdot x}\cap Y\neq\varnothing \Leftrightarrow Y\in J,\ \text{for every } Y\in\Lambda\}.$$

In other words, X_I^J contains points in X_I with the closure of the corresponding G-orbits meeting only the connected components indexed by J. Since both Γ and Λ are finite sets, we obtain a finite stratification

$$X=\bigcup_{\substack{I\subseteq\Gamma\\ J\subseteq\Lambda}} X_I^J \tag{11}$$

which will be used in the proof of Proposition 3.8.

Remark 3.14. In (11) we consider each stratum X_I^J simply as a subset of closed points of X. It is not clear to us whether it is locally closed, but this is irrelevant to the subsequent discussion.

From now on the line bundle will come into play. Assume L is a G-linearized line bundle on X. Let $\lambda\in\mathcal{X}_*(G)$ be a 1-PS of G and $Y\in\Lambda$ a connected component of $(X_I)^{G_I}$ for some $I\subseteq\Gamma$ satisfying condition (iii) of Lemma 3.10. By Remark 3.11, there are two possibilities:

- if $\lambda\notin L_I^{\perp}$, then $\lim_{t\to0}\lambda(t)\cdot x$ does not exist for any $x\in X_I$;
- if $\lambda\in L_I^{\perp}$, then λ fixes all points in $(X_I)^{G_I}$; since Y is a connected component, the weight of the λ-action on the fibre L_y has to be constant for all points $y\in Y$, which we denoted by $w^L(Y,\lambda)$.

The following observation is essential.

Lemma 3.15. *For any fixed stratum X_I^J in the stratification* (11)*, let $x\in X_I^J$ be a closed point. Then for any G-linearized line bundle L and any* 1*-PS $\lambda\in\mathcal{X}_*(G)$, we have*

$$\mu^L(x,\lambda)=\begin{cases}\infty & \text{if } \langle\lambda,\chi\rangle<0 \text{ for some } \chi\in I;\\ -\min\{w^L(Y,\lambda)\mid Y\in J\} & \text{if } \langle\lambda,\chi\rangle\geqslant0 \text{ for all } \chi\in I.\end{cases}$$

Proof. When $\langle\lambda,\chi\rangle<0$ for some $\chi\in I$, the condition $x\in X_I$ implies that $\lim_{t\to0}\lambda(t)\cdot x$ does not exist, hence $\mu^L(x,\lambda)=\infty$ by definition.

When $\langle\lambda,\chi\rangle\geqslant0$ for all $\chi\in I$, let $x_0=\lim_{t\to0}\lambda(t)\cdot x$. We compute the value of $\mu^L(x,\lambda)$ in the following two steps.

Step 1. We show that $-\mu^L(x,\lambda)=w^L(Y,\lambda)$ for some $Y\in J$.

By Lemma 3.13, $\overline{G\cdot x_0}$ contains a closed G-orbit, say, $G\cdot y$. Since $G\cdot y\subseteq\overline{G\cdot x}$, there exists some $Y\in J$, such that $G\cdot y\subseteq Y$ by Lemma 3.12. Notice that x_0 is a λ-fixed point, hence every point in $\overline{G\cdot x_0}$ is a λ-fixed point. Therefore λ acts on each fibre of $L|_{\overline{G\cdot x_0}}$ via an integral weight, which has to be constant over the entire orbit closure $\overline{G \cdot x_0}$. This in particular implies that

$$-\mu^L(x, \lambda) = \mathrm{wt}_\lambda(L_{x_0}) = \mathrm{wt}_\lambda(L_y) = w^L(Y, \lambda),$$

where $\mathrm{wt}_\lambda(-)$ is the weight of the λ-action on the corresponding line. This finishes Step 1.

Step 2. We show that $-\mu^L(x, \lambda) \leqslant w^L(Y, \lambda)$ for every $Y \in J$.

For this purpose, it suffices to show that $-\mu^L(x, \lambda) \leqslant -\mu^L(z, \lambda)$ for every $z \in \overline{G \cdot x}$ such that $G \cdot z$ is a closed G-orbit.

In order to apply Lemma 3.4, we write

$$\Gamma_x = \{\chi \in \mathcal{X}^*(G) \mid [x]_\chi \neq 0\},$$
$$\Gamma_z = \{\chi \in \mathcal{X}^*(G) \mid [z]_\chi \neq 0\}.$$

Then we get by Lemma 3.4 that

$$-\mu^L(x, \lambda) = \min\{\langle \lambda, \chi \rangle \mid \chi \in \Gamma_x\};$$
$$-\mu^L(z, \lambda) = \min\{\langle \lambda, \chi \rangle \mid \chi \in \Gamma_z\}.$$

Therefore, in order to show $-\mu^L(x, \lambda) \leqslant -\mu^L(z, \lambda)$, it suffices to show that $\Gamma_z \subseteq \Gamma_x$.

For any $\chi \in \Gamma_z$, since $[z]_\chi : V_\chi \to \Bbbk$ is non-zero, there exists some section $\sigma \in V_\chi$, such that $\sigma(z) \neq 0$. We claim that $\sigma(x) \neq 0$. Otherwise, we have $\sigma(x) = 0$, and for any $g \in G$, $\sigma(g \cdot x) = (g^*\sigma)(x) = \chi(g) \cdot \sigma(x) = 0$. This means that $\sigma = 0$ on the entire orbit $G \cdot x$, hence also on its closure $\overline{G \cdot x}$. This is a contradiction since $\sigma(z) \neq 0$. This verifies that $\sigma(x) \neq 0$, hence $[x]_\chi : V_\chi \to \Bbbk$ is non-zero, so $\chi \in \Gamma_x$. This finishes Step 2.

The above two steps conclude the second case in the lemma. □

We are now ready to prove the main result of this subsection.

Proof of Proposition 3.8. By Lemma 3.15, we see that for any G-linearized line bundle L on X, the function on the lattice of 1-PS's

$$\mu^L(x, -) : \mathcal{X}_*(G) \longrightarrow \mathbb{Z} \cup \{\infty\}$$

is the same function for all points $x \in X_I^J$.

Assume L is an ample line bundle. By the relative Hilbert-Mumford criterion [GHH15, Corollary 1.1], the stability of points in X_I^J are all the same, hence $X^{ss}(L)$ must be a union of a subset of strata in (11), as desired. □

Corollary 3.16. *The finiteness statement in Proposition 3.8 also holds for stable loci and unstable loci with respect to G-linearized ample line bundles.*

Proof. For any fixed semistable locus, the corresponding stable locus consists of semistable points with closed orbits and finite stabilizers by Definition 2.2, and the corresponding unstable locus is simply the complement of the semistable locus, which are both independent of the choice of the G-linearized line bundle. Therefore the finiteness of possible semistable loci implies the finiteness of possible stable loci and unstable loci. □

3.4. **Generalization to arbitrary reductive groups.** The main goal in this subsection is to generalize the results in §3.2 and §3.3; more precisely, we will show that Proposition 3.5, Proposition 3.7, Proposition 3.8 and Corollary 3.16 are all valid for arbitrary reductive groups. We follow mostly the arguments in [DH98, §1.1] and [Res00, §1.2].

Let G be an arbitrary reductive group, and T a fixed maximal torus of G. Then every maximal torus of G can be given by $g^{-1}Tg$ for some $g \in G$; see e.g. [Mil17, Theorem 17.87]. We also assume that $\|\cdot\|$ is a norm on $\mathcal{X}_*(G)$ that is invariant under conjugation; see §3.1.

Proposition 3.17. *Proposition 3.5 and Proposition 3.7 hold for an arbitrary reductive group G.*

Proof. In this proof we need to emphasize the dependence of the function $M^L(x)$ on the group G. So the function defined in (2) will be denoted by $M^L_G(x)$. On the other hand, if we only consider the nontrivial 1-PS's of the maximal torus T, then the corresponding function will be denoted by $M^L_T(x)$.

We only need to show that $M^L_G(x) > -\infty$. Indeed, we have

$$\begin{aligned} M^L_G(x) &= \inf_{\lambda \in \mathcal{X}_*(G)\setminus\{0\}} \frac{\mu^L(x,\lambda)}{\|\lambda\|} \\ &= \inf_{g\in G} \left(\inf_{\lambda \in \mathcal{X}_*(T)\setminus\{0\}} \frac{\mu^L(x, g^{-1}\lambda g)}{\|g^{-1}\lambda g\|} \right) \\ &= \inf_{g\in G} \left(\inf_{\lambda \in \mathcal{X}_*(T)\setminus\{0\}} \frac{\mu^L(gx, \lambda)}{\|\lambda\|} \right) \\ &= \inf_{g\in G} M^L_T(gx). \end{aligned}$$

By Proposition 3.5, $M^L_T(y) > -\infty$ for every point $y \in X$. Moreover, by Proposition 3.8, for any fixed G-linearized line bundle L (which in particular is T-linearized), the function $\mu^L(y, -) : \mathcal{X}_*(T) \to \mathbb{Z} \cup \{\infty\}$ stays the same when y runs over all closed points of any fixed stratum X^J_I. It follows that $M^L_T(y)$ is a constant function on each stratum X^J_I, hence takes only finitely many different values in $\mathbb{R} \cup \{\infty\}$ when y runs over all closed points of X, which implies that $M^L_G(x) = \inf_{g\in G} M^L_T(gx) > -\infty$. □

Proposition 3.18. *Proposition 3.8 and Corollary 3.16 hold for an arbitrary reductive group G.*

Proof. As before we have to emphasize the dependence of the semistable locus on the chosen group and therefore the G-semistable and T-semistable loci of X with respect to a fixed ample line bundle L will be denoted by $X^{ss}_G(L)$ and $X^{ss}_T(L)$ respectively.

This proof follows [DH98, Remark 1.3.10]. By the relative Hilbert-Mumford criterion [GHH15, Corollary 1.1], we have

$$X^{ss}_G(L) = \bigcap_{g\in G} X^{ss}_{g^{-1}Tg}(L).$$

Moreover, since $\mu^L(x, g^{-1}\lambda g) = \mu^L(gx, \lambda)$ for every $\lambda \in \mathcal{X}_*(T)$, it follows that the $g^{-1}Tg$-(semi)stability of the point x is equivalent to T-(semi)stability of the point gx, again by the relative Hilbert-Mumford criterion [GHH15, Corollary 1.1]. In other words, we have

$$X^{ss}_{g^{-1}Tg}(L) = g^{-1} \cdot X^{ss}_T(L).$$

It follows that

$$X^{ss}_G(L) = \bigcap_{g \in G} g^{-1} \cdot X^{ss}_T(L).$$

Since there are only finitely many possibilities for $X^{ss}_T(L)$ by Proposition 3.8, it follows that there are only finitely possibilities for $X^{ss}_G(L)$, as desired. □

3.5. **Proof of the main semi-continuity result.** In this subsection we will prove our main result Theorem 3.23. Before getting there, we use the notations in Subsection 3.1 to rewrite the Hilbert-Mumford criterion in a slightly different language.

Lemma 3.19. *Assume L is a G-linearized ample line bundle. Then we have*

$$X^{ss}(L) = \{x \in X \mid M^L(x) \geqslant 0\};$$
$$X^{s}(L) = \{x \in X \mid M^L(x) > 0\}.$$

Proof. For the first identity, by [GHH15, Corollary 1.1], a point $x \in X$ is semistable with respect to a G-linearized ample line bundle L if and only if $\mu^L(x, \lambda) \geqslant 0$ for every non-trivial 1-PS λ of G. We claim that it is equivalent to $M^L(x) \geqslant 0$. This follows from the definition (2) of $M^L(x)$ being an infimum.

For the second identity, by [GHH15, Corollary 1.1], a point $x \in X$ is stable with respect to a G-linearized ample line bundle L if and only if $\mu^L(x, \lambda) > 0$ for every non-trivial 1-PS λ of G. It follows immediately that $M^L(x) > 0$ implies $x \in X^s(L)$.

Now we assume that $x \in X^s(L)$, then it is clear that $M^L(x) \geqslant 0$. We claim that the inequality has to be strict. Assume otherwise that $M^L(x) = 0$. Then there exists some $\lambda_0 \in \mathcal{X}_*(G)_{\mathbb{R}}$ such that $\mu^L(x, \lambda_0) = 0$ by Proposition 3.5 and the discussion above it. We claim that we can replace λ_0 by a rational class λ_0' which still satisfies the condition $\mu^L(x, \lambda_0') = 0$.

Without loss of generality, we can assume G is a torus; otherwise we can replace G by a maximal torus which contains λ_0. By Lemma 3.4, we have $\langle \lambda_0, \chi \rangle \geqslant 0$ for every $\chi \in \Gamma_x$, with equality being achieved by characters in a non-empty subset of Γ_x. We notice that $\langle \lambda_0, \chi' \rangle \geqslant 0$ for every $\chi' \in \Gamma_{f(x)}$. We conclude from Lemma 3.3 that $\langle \lambda_0, \chi_i \rangle \geqslant 0$ for every $1 \leqslant i \leqslant k$ with equality being achieved by some i.

Every such character χ_i satisfying $\langle \lambda_0, \chi_i \rangle = 0$ defines a rational hyperplane in $\mathcal{X}_*(G)_{\mathbb{R}}$. The intersection of all these hyperplanes is a linear subspace of $\mathcal{X}_*(G)_{\mathbb{R}}$ containing λ_0, in which rational points are dense. Let λ_0' be a rational point in this subspace that is sufficiently close to λ_0, then we claim that we have $\mu^L(x, \lambda_0') = 0$.

Indeed, the choice of λ_0' guarantees that $\langle \lambda_0', \chi_i \rangle = 0$ for every χ_i satisfying $\langle \lambda_0, \chi_i \rangle = 0$. For every χ_i satisfying $\langle \lambda_0, \chi_i \rangle > 0$, we still obtain $\langle \lambda_0', \chi_i \rangle > 0$ since λ_0' is sufficiently close to λ_0. It follows again by Lemma 3.3 that $\langle \lambda_0', \chi \rangle \geqslant 0$ for every $\chi \in \Gamma_x$ with equality being achieved by characters in a non-empty subset of Γ_x. Hence the claim holds.

Therefore we have found a rational class λ_0' such that $\mu^L(x, \lambda_0') = 0$, which leads to a contradiction to the assumption that $x \in X^s(L)$. It follows that $M^L(x) > 0$, which finishes the proof of the second identity. $\square$

The following lemma is not strictly required in the proof of Theorem 3.23. However, this lemma will be helpful for applying the proposition in various situations; in particular, our application in Section 4.

Lemma 3.20. *Let X and Y be projective S-schemes. We assume $\pi : X \to Y$ is a G-equivariant projective morphism of S-schemes satisfying the assumptions in Lemma 2.4. Let $\widetilde{L}$ be a G-linearized ample line bundle on Y and $L = \pi^*\widetilde{L}$. Then we have*

$$X^{ss}(L) = \{x \in X \mid M^L(x) \geqslant 0\};$$
$$X^s(L) = \{x \in X \mid M^L(x) > 0\}.$$

Proof. Let $x \in X$ and $y = f(x) \in Y$ be closed points. On the one hand, by Lemma 2.4 we know that $x \in X^{ss}(L)$ if and only if $y \in Y^{ss}(\widetilde{L})$, which is further equivalent to $M^{\widetilde{L}}(y) \geqslant 0$ by Lemma 3.19. On the other hand, we claim $\mu^L(x, \lambda) = \mu^{\widetilde{L}}(y, \lambda)$ for every non-trivial 1-PS λ. Indeed, since X and Y are both projective over S, it is clear that $\lim_{t\to 0} \lambda(t) \cdot x$ exists if and only if $\lim_{t\to 0} \lambda(t) \cdot y$ exists, hence $\mu^L(x, \lambda) = \infty$ if and only if $\mu^{\widetilde{L}}(y, \lambda) = \infty$. In the case when they are both finite, the equation follows from [MFK94, p.49 (iii)]. This claim implies that $M^L(x) = M^{\widetilde{L}}(y)$ by (2). In other words, $M^{\widetilde{L}}(y) \geqslant 0$ is equivalent to $M^L(x) \geqslant 0$. The above equivalences together conclude the first assertion. The proof for the second assertion is the same. $\square$

The semi-continuity of the (semi)stable loci is a very useful result in VGIT; see e.g. [Laz13, Lemma 3.10] and the references therein. In the following result, we will generalize this theorem by relaxing the assumption on the ampleness of the line bundles. Subsequently we will give two further formulations of this result, as these can be useful for applications.

Theorem 3.21. *Let L_t be a G-linearized line bundle on X for every t satisfying $0 \leqslant t \leqslant 1$. We assume that for every point $x \in X$ the function $M^{L_t}(x)$ is continuous in t. Assume further that L_t is ample for $0 < t \leqslant 1$, and that*

$$X^{ss}(L_0) = \{x \in X \mid M^{L_0}(x) \geqslant 0\};$$
$$X^s(L_0) = \{x \in X \mid M^{L_0}(x) > 0\}.$$

Then we have

$$X^s(L_0) \subseteq X^s(L_t) \subseteq X^{ss}(L_t) \subseteq X^{ss}(L_0)$$

for $0 < t \ll 1$.

We point out that the condition on L_0 is fulfilled, for instance, when L_0 is a G-linearized ample line bundle (by Lemma 3.19), or the pullback of a G-linearized ample line bundle along a projective morphism satisfying the three assumptions in Lemma 2.4 (by Lemma 3.20).

Proof. We follow the proof of [Res00, Proposition 4]. The middle inclusion is obvious. We prove the other two.

Step 1. We first prove $X^s(L_0) \subseteq X^s(L_t)$.

By Proposition 3.18, there are finitely many possible subsets of X, say, $X_1^s, X_2^s, \cdots, X_n^s$, which could be realized as the stable loci for some G-linearized ample line bundle on X. Namely, for each $0 < t \leqslant 1$, $X^s(L_t)$ must be one of them.

We assume that $X^s(L_0) \nsubseteq X_i^s$ for each $1 \leqslant i \leqslant p$ and $X^s(L_0) \subseteq X_j^s$ for each $p+1 \leqslant j \leqslant n$. Then for each $1 \leqslant i \leqslant p$, we can find some point $x_i \in X^s(L_0) \backslash X_i^s$.

By assumption, $x_i \in X^s(L_0)$ implies $M^{L_0}(x_i) > 0$. Since $M^{L_t}(x_i)$ is continuous in t, there exists some $\varepsilon_i > 0$, such that $M^{L_t}(x_i) > 0$ for every $0 < t < \varepsilon_i$. By Lemma 3.19, we have $x_i \in X^s(L_t)$ for every $0 < t < \varepsilon_i$. Since $x_i \notin X_i^s$, we conclude that $X^s(L_t) \neq X_i^s$ for every $0 < t < \varepsilon_i$.

Let $\varepsilon = \min\{\varepsilon_1, \cdots, \varepsilon_p\}$. Then for every $0 < t < \varepsilon$, $X^s(L_t)$ must be one of $X_{p+1}^s, \cdots X_n^s$. In particular, we have $X^s(L_0) \subseteq X^s(L_t)$ for every $0 < t < \varepsilon$.

Step 2. We now prove $X^{ss}(L_t) \subseteq X^{ss}(L_0)$, which can be reformulated in terms of unstable loci as $X^{us}(L_0) \subseteq X^{us}(L_t)$. Indeed, the proof in this step is exactly the same as in Step 1, if we replace every occurrence of X^s in the proof by X^{us}, and replace every occurrence of $M^L(x_i) > 0$ in the proof by $M^L(x_i) < 0$. This finishes the proof of the statement. □

The same proof can be used for the following discrete version of the above result:

Theorem 3.22. *For every positive integer m, let L_m be a G-linearized ample line bundle on X and let L_∞ be a G-linearized line bundle X satisfying the condition*

$$X^{ss}(L_\infty) = \{x \in X \mid M^{L_\infty}(x) \geqslant 0\};$$
$$X^s(L_\infty) = \{x \in X \mid M^{L_\infty}(x) > 0\}.$$

We assume that

$$\lim_{m\to\infty} M^{L_m}(x) = M^{L_\infty}(x)$$

for every point $x \in X$. Then we have

$$X^s(L_\infty) \subseteq X^s(L_m) \subseteq X^{ss}(L_m) \subseteq X^{ss}(L_\infty)$$

for $m \gg 0$. □

Finally, the following result is a more familiar reformulation of Theorem 3.21:

Theorem 3.23. *Let L_0 and L_1 be G-linearized line bundles on X, and*

$$L_t = L_0^{1-t} \otimes L_1^t$$

for any $0 \leqslant t \leqslant 1$. Assume further that L_t is ample for $0 < t \leqslant 1$, and that

$$X^{ss}(L_0) = \{x \in X \mid M^{L_0}(x) \geqslant 0\};$$
$$X^s(L_0) = \{x \in X \mid M^{L_0}(x) > 0\}.$$

Then we have

$$X^s(L_0) \subseteq X^s(L_t) \subseteq X^{ss}(L_t) \subseteq X^{ss}(L_0)$$

for $0 < t \ll 1$.

Proof. We note that the line bundles L_t are all contained in a 2-dimensional subspace of $\mathrm{NS}^G(X)_{\mathbb{R}}$. By Proposition 3.7 the function $M^{L_t}(x)$ is continuous in t for every point $x \in X$. The claim now follows immediately from Theorem 3.21. □

4. Application to degeneration of Hilbert schemes

In this section, we will apply the general theory of the previous sections to study degenerations of Hilbert schemes. In particular, we will consider relative Hilbert schemes of strict simple degenerations in the framework developed in [GHH19, GHHZ18].

4.1. **Setup.** We first fix some notation which will be used throughout this section. Let $\Bbbk$ be an algebraically closed field of characteristic 0. Let C be a smooth affine curve over $\Bbbk$ and $f\colon X \to C$ a projective strict simple degeneration. Roughly speaking, this means that f has a unique (mildly) singular fiber, which moreover forms a strict normal crossing divisor on the non-singular variety X; we refer to [GHH19] for a precise definition. We emphasize however, that we do not make any assumption on the relative dimension of f.

By applying Jun Li's technique of expanded degenerations, one obtains, as explained in [GHH19], a new family

$$f[n] : X[n] \longrightarrow C[n]. \tag{12}$$

The fibers of $f[n]$ are certain degenerations (called "expansions") of the fibers of f. Therefore, the relative dimension is preserved when passing from f to $f[n]$. On the other hand, we have increased the dimension of the base, in fact, $C[n]$ can be seen to be étale over $\mathbb{A}^{n+1}$. The total space $X[n]$ is again a non-singular variety. Lastly, we mention that an important feature of this construction is that the group $G = (\mathbb{G}_m)^n$ acts equivariantly on $X[n]$ and $C[n]$. Next, let

$$g\colon \operatorname{Hilb}^n(X[n]/C[n]) \longrightarrow C[n]$$

be the relative Hilbert scheme and

$$\mathcal{Z} \subseteq X[n] \times_{C[n]} \operatorname{Hilb}^n(X[n]/C[n])$$

the universal closed subscheme with natural projections to the two factors denoted by q and p respectively. Then we obtain the commutative diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "q"] \arrow[d, "p"'] & X[n] \arrow[d, "{f[n]}"] \\ \operatorname{Hilb}^n(X[n]/C[n]) \arrow[r, "g"'] & C[n]. \end{tikzcd}$$

Let $\mathcal{L}$ be the G-linearized ample line bundle on $X[n]$ constructed in [GHH19], then for every positive integer ℓ, we can define

$$\mathcal{L}_m = \det p_* q^* \mathcal{L}^m,$$

which is an ample line bundle on $\operatorname{Hilb}^n(X[n]/C[n])$ for $\ell \gg 0$ by [GHH19, §2.2.1]. On the other hand, the line bundle $\mathcal{L}^{\boxtimes n}$ on the n-fold product $X[n] \times_{C[n]} \cdots \times_{C[n]} X[n]$ descends to an ample line bundle $\widetilde{\mathcal{L}}$ on the relative symmetric product $\mathrm{Sym}^n(X[n]/C[n])$ by [GHHZ18, Lemma 3.1]. Via the relative Hilbert-Chow morphism (see e.g. [Ryd08, Paper III, §4.3])

$$\pi\colon \mathrm{Hilb}^n(X[n]/C[n]) \longrightarrow \mathrm{Sym}^n(X[n]/C[n]),$$

we obtain a semi-ample line bundle

$$\mathcal{L}_\infty = \pi^* \widetilde{\mathcal{L}}$$

on $\mathrm{Hilb}^n(X[n]/C[n])$. Notice that $\mathcal{L}_\infty$ is not ample, because it is not positive on any curve that is contracted by π. The relative Hilbert scheme $\mathrm{Hilb}^n(X[n]/C[n])$, along with the line bundles $\mathcal{L}_m$ and $\mathcal{L}_\infty$, carries a naturally induced G-action.

Recall that in [GHH19, Theorem 2.9], the (semi)stable locus of the relative Hilbert scheme $\mathrm{Hilb}^n(X[n]/C[n])$ with respect to the G-linearized line bundle $\mathcal{L}_m$ for sufficiently large ℓ was computed. One striking phenomenon is that, whether a point $[Z] \in \mathrm{Hilb}^n(X[n]/C[n])$ is (semi)stable only depends on its underlying cycle. In fact, this is not a coincidence. The goal of this section is to determine the (semi)stable locus of $\mathrm{Hilb}^n(X[n]/C[n])$ from an alternative perspective, which gives a conceptual interpretation for the irrelevance of the scheme structure of $[Z]$ to (semi)stability.

More precisely, our starting point is the relation

$$\mathrm{Sym}^n(X[n]/C[n])^{ss}(\widetilde{\mathcal{L}}) = \mathrm{Sym}^n(X[n]/C[n])^{s}(\widetilde{\mathcal{L}}) \tag{13}$$

which was proved in [GHHZ18, Proposition 3.3]. We will relate this to the (semi)stable locus of $\mathrm{Hilb}^n(X[n]/C[n])$ with respect to $\mathcal{L}_m$ for $m \gg 0$ in two steps using the general theory developed in previous sections.

4.2. **From symmetric product to Hilbert scheme.** The goal of this subsection is to show the following result:

Proposition 4.1. *The following equalities hold:*

$$\begin{aligned}\mathrm{Hilb}^n(X[n]/C[n])^{ss}(\mathcal{L}_\infty) &= \mathrm{Hilb}^n(X[n]/C[n])^{s}(\mathcal{L}_\infty)\\ = \pi^{-1}(\mathrm{Sym}^n(X[n]/C[n])^{ss}(\widetilde{\mathcal{L}})) &= \pi^{-1}(\mathrm{Sym}^n(X[n]/C[n])^{s}(\widetilde{\mathcal{L}})).\end{aligned}$$

We will apply Lemma 2.4 to deduce this result. In order to verify that this lemma is applicable, we need the following observation:

Lemma 4.2. *The scheme* $\mathrm{Sym}^n(X[n]/C[n])$ *is normal.*

Proof. Without loss of generality, we can assume that $C[n] = \mathbb{A}^{n+1}$. We prove the statement in four steps:

STEP 1. We claim that the singular locus of the n-fold product $X[n] \times_{C[n]} \cdots \times_{C[n]} X[n]$ has codimension 2.

Consider the natural morphism

$$\varphi : X[n] \times_{C[n]} \cdots \times_{C[n]} X[n] \longrightarrow C[n]. \tag{14}$$

Note that any point in $X[n] \times_{C[n]} \cdots \times_{C[n]} X[n]$ representing a tuple of n points in the smooth locus of the morphism $f[n] : X[n] \to C[n]$ is a smooth point in $X[n] \times_{C[n]} \cdots \times_{C[n]} X[n]$.

For any closed point $q \in C[n] = \mathbb{A}^{n+1}$, the fibre $(f[n])^{-1}(q)$ is smooth if no coordinate of q vanishes and singular in codimension 1 if q has at least one vanishing coordinate. Hence the intersection of the singular locus of

$X[n] \times_{C[n]} \cdots \times_{C[n]} X[n]$ with any closed fibre $\varphi^{-1}(q)$ is empty if q has no vanishing coordinate; and of codimension 1 otherwise. Therefore the claim follows.

STEP 2. We claim that $X[n] \times_{C[n]} \cdots \times_{C[n]} X[n]$ is a local complete intersection in a smooth variety.

Consider the closed embedding

$$\iota : X[n] \times_{C[n]} \cdots \times_{C[n]} X[n] \longrightarrow X[n] \times \cdots \times X[n].$$

Since $X[n]$ is smooth, the target variety of ι is a smooth variety. It remains to show that the number of equations required for the closed embedding ι agrees with the codimension of the closed embedding ι.

Assume the relative dimension of $f[n] : X[n] \to C[n]$ is k, then we have $\dim X[n] = n + k$, hence we obtain

$$\dim X[n] \times_{C[n]} \cdots \times_{C[n]} X[n] = n + kn,$$
$$\dim X[n] \times \cdots \times X[n] = n(n + k).$$

It follows that the codimension of the closed embedding ι is $n^2 - n$.

Now we take any closed point $(p_1, \cdots, p_n) \in X[n] \times_{C[n]} \cdots \times_{C[n]} X[n]$. Since $X[n]$ is smooth of dimension $n + k$, in a neighbourhood of each point $p_i \in X[n]$, we pick a local chart of coordinates $\{x_{i,1}, \cdots, x_{i,n+k}\}$ on $X[n]$. The morphism $f[n] : X[n] \to C[n]$ is then given by the equations

$$t_j = f_{ij}(x_{i,1}, \cdots, x_{i,n+k})$$

for $j = 1, 2, \cdots, n$, where $t_1, \cdots, t_n$ are the coordinates of $C[n] \cong \mathbb{A}^{n+1}$. It follows that locally near the point $(p_1, \cdots, p_n)$, the closed embedding ι is defined by the equations

$$f_{1j}(x_{1,1}, \cdots, x_{1,n+k}) = \cdots = f_{nj}(x_{n,1}, \cdots, x_{n,n+k})$$

for $j = 1, 2, \cdots, n$, which is a total of $n(n-1)$ equations.

We see that the codimension agrees with the number of equations, hence the closed embedding ι is a local complete intersection.

STEP 3. We show that $X[n] \times_{C[n]} \cdots \times_{C[n]} X[n]$ is normal. Indeed, this follows from the above two steps and Serre's $R_1 + S_2$ criterion, see [Har77, Proposition II.8.23].

STEP 4. We show that $\mathrm{Sym}^n(X[n]/C[n])$ is normal. This follows immediately from the fact that it is a finite group quotient of the normal variety $X[n] \times_{C[n]} \cdots \times_{C[n]} X[n]$. □

We are now ready to relate the (semi)stable loci on the Hilbert scheme and the symmetric product.

Proof of Proposition 4.1. The claim follows from (13) and Lemma 2.4. It remains to check that all three assumptions in Lemma 2.4 are satisfied.

First of all, since the morphism $f[n]$ in (12) is flat, the structure morphism ϕ from the n-fold product (14) is also flat. Moreover, since the generic fibre of ϕ is irreducible, it follows that the n-fold product $X[n] \times_{C[n]} \cdots \times_{C[n]} X[n]$ is irreducible by [Liu02, Proposition 4.3.8]. Therefore as a finite group quotient, $\mathrm{Sym}^n(X[n]/C[n])$ is also irreducible. The normality of $\mathrm{Sym}^n(X[n]/C[n])$ has been proved in Lemma 4.2.

The connectivity of fibres of π follows from [Fog68, Proposition 2.3].

Finally, let U_H and U_S be the open subset in $\mathrm{Hilb}^n(X[n]/C[n])$ and $\mathrm{Sym}^n(X[n]/C[n])$ respectively, parametrising n-tuples of pairwise distinct points in the smooth locus of the morphism $f[n]$ in (12). Since the smooth locus of the morphism $f[n]$ is $G[n]$-invariant, both U_H and U_S are also $G[n]$-invariant. It is easy to see that the complement of U_S in $\mathrm{Sym}^n(X[n]/C[n])$ is of codimension 2, and the restriction $\pi|_{U_H} : U_H \to U_S$ is an isomorphism. Hence the desired statement follows. $\square$

4.3. **From semi-ample to ample line bundles.** The goal of this subsection is to relate the (semi)stable loci on $\mathrm{Hilb}^n(X[n]/C[n])$ with respect to $\mathcal{L}_\infty$ and $\mathcal{L}_m$ for $m \gg 0$. Namely

Proposition 4.3. *For $m \gg 0$ the following equalities hold:*

$$\begin{aligned}\mathrm{Hilb}^n(X[n]/C[n])^{ss}(\mathcal{L}_\infty) &= \mathrm{Hilb}^n(X[n]/C[n])^{s}(\mathcal{L}_\infty)\\ = \mathrm{Hilb}^n(X[n]/C[n])^{ss}(\mathcal{L}_m) &= \mathrm{Hilb}^n(X[n]/C[n])^{s}(\mathcal{L}_m).\end{aligned}$$

Before we go into the details of the proof of Proposition 4.3, we would like to point out to the reader that Proposition 4.3 would follow immediately from Theorem 3.23, if we could prove

$$\lim_{m\to\infty} \frac{\mathcal{L}_m}{m} = \mathcal{L}_\infty \tag{15}$$

in $\mathrm{NS}^G(\mathrm{Hilb}^n(X[n]/C[n]))_{\mathbb{Q}}$. Although we strongly believe that (15) should hold, a rigorous proof is not known to us.

As an alternative approach, we will instead use Theorem 3.22 to avoid proving (15). We will establish the following two continuity lemmas. The first one concerns the continuity of the weights:

Lemma 4.4. *For any point $z \in \mathrm{Hilb}^n(X[n]/C[n])$, and for any 1-PS $\lambda \in \mathcal{X}_*(G)$, we have*

$$\lim_{m\to\infty} \mu^{\frac{\mathcal{L}_m}{m}}(z,\lambda) = \mu^{\mathcal{L}_\infty}(z,\lambda).$$

Proof. There are two possible cases. If $\lim_{t\to 0} \lambda(t)\cdot z$ does not exist, then we have

$$\mu^{\frac{\mathcal{L}_m}{m}}(z,\lambda) = \infty = \mu^{\mathcal{L}_\infty}(z,\lambda)$$

for every positive integer m. The statement follows automatically.

Otherwise, assume that

$$\lim_{t\to 0} \lambda(t)\cdot z = z_0 \in \mathrm{Hilb}^n(X[n]/C[n]) \tag{16}$$

and assume further that z_0 is represented by a closed subscheme $Z_0 \subseteq X[n]$ of length n. We can decompose the corresponding cycle as a sum of positive multiples of distinct points

$$[Z_0] = \sum_p n_p[p]. \tag{17}$$

By [GHH19, §2.2.2], the fibre of $\mathcal{L}_m$ at the point z_0 is given by

$$\begin{aligned}\mathcal{L}_m(z_0) &= \wedge^n H^0(\mathscr{O}_{Z_0} \otimes \mathcal{L}^{\otimes m})\\ &= \wedge^n H^0(\mathscr{O}_{Z_0}) \otimes \left(\otimes_p \mathcal{L}(p)^{\otimes n_p}\right)^{\otimes m}.\end{aligned}$$

By definition we have

$$\begin{aligned}\mu^{\mathcal{L}_m}(z,\lambda) &= -\mathrm{wt}_\lambda(\mathcal{L}_m(z_0))\\ &= -\mathrm{wt}_\lambda(\wedge^n H^0(\mathscr{O}_{Z_0})) - m\cdot\sum_p n_p\cdot \mathrm{wt}_\lambda(\mathcal{L}(p)),\end{aligned}$$

where $\mathrm{wt}_\lambda(-)$ represents the weight of the λ-action on the corresponding line. This immediately gives

$$\mu^{\frac{\mathcal{L}_m}{m}}(z,\lambda) = -\frac{1}{m}\cdot \mathrm{wt}_\lambda(\wedge^n H^0(\mathscr{O}_{Z_0})) - \sum_p n_p\cdot \mathrm{wt}_\lambda(\mathcal{L}(p)). \tag{18}$$

On the other hand, from the construction of the line bundle $\mathcal{L}_\infty$, c.f. [GHHZ18, Remark 3.2], it is immediately clear that the fibre of $\mathcal{L}_\infty$ at the point z_0 is given by

$$\mathcal{L}_\infty(z_0) = \otimes_p \mathcal{L}(p)^{\otimes n_p}.$$

Therefore we have

$$\mu^{\mathcal{L}_\infty}(z,\lambda) = -\sum_p n_p\cdot \mathrm{wt}_\lambda(\mathcal{L}(p)). \tag{19}$$

Comparing (18) and (19), we have

$$\lim_{m\to\infty}\mu^{\frac{\mathcal{L}_m}{m}}(z,\lambda) = \mu^{\mathcal{L}_\infty}(z,\lambda),$$

as desired. □

The following second continuity lemma is a modified version of Proposition 3.7 with a similar proof.

Lemma 4.5. *For any point $z\in \mathrm{Hilb}^n(X[n]/C[n])$, we have*

$$\lim_{m\to\infty} M^{\frac{\mathcal{L}_m}{m}}(z) = M^{\mathcal{L}_\infty}(z).$$

Proof. Without loss of generality, we assume there exists some $\lambda\in\mathcal{X}_*(G)\backslash\{0\}$, such that $\lim_{t\to 0}\lambda(t)\cdot z$ exists; otherwise both sides are $+\infty$. For any such λ, we use the notations in (16) and (17) to define

$$f_\lambda(t) = \frac{1}{\|\lambda\|}\left(-t\cdot \mathrm{wt}_\lambda(\wedge^n H^0(\mathscr{O}_{Z_0})) - \sum_p n_p\cdot \mathrm{wt}_\lambda(\mathcal{L}(p))\right).$$

We further define

$$f(t) = \inf_{\lambda\in\mathcal{X}_*(G)\backslash\{0\}} f_\lambda(t).$$

By Proposition 3.5, the function $f(t)$ achieves finite values. Moreover, since $f_\lambda(t)$ is a linear function in t for each λ, as the infimum of a collection of linear functions, $f(t)$ is also a continuous function in t.

Moreover, by (18) and (19) we have

$$M^{\frac{\mathcal{L}_m}{m}}(z) = \inf_{\lambda\in\mathcal{X}_*(G)\backslash\{0\}}\frac{\mu^{\frac{\mathcal{L}_m}{m}}(z,\lambda)}{\|\lambda\|} = \inf_{\lambda\in\mathcal{X}_*(G)\backslash\{0\}} f_\lambda\left(\frac{1}{m}\right) = f\left(\frac{1}{m}\right)$$

and

$$M^{\mathcal{L}_\infty}(z) = \inf_{\lambda\in\mathcal{X}_*(G)\backslash\{0\}}\frac{\mu^{\mathcal{L}_\infty}(z,\lambda)}{\|\lambda\|} = \inf_{\lambda\in\mathcal{X}_*(G)\backslash\{0\}} f_\lambda(0) = f(0).$$

The continuity of $f(t)$ immediately implies that

$$\lim_{m\to\infty} M^{\frac{\mathcal{L}_m}{m}}(z) = M^{\mathcal{L}_\infty}(z)$$

as desired. $\square$

We are now ready to prove Proposition 4.3.

Proof of Proposition 4.3. We notice that the conclusion in Lemma 3.19 holds for $\mathcal{L}_m$ for $m \gg 0$ since $\mathcal{L}_m$ is ample by [GHH19, §2.2.1]. Moreover, we have seen in the proof of Proposition 4.1 that all three assumptions in Lemma 2.4 are satisfied by $\mathcal{L}_\infty$, hence the conclusion in Lemma 3.20 holds for $\mathcal{L}_\infty$. Together with Lemma 4.5, all conditions required for Theorem 3.22 hold. Hence

$$\begin{aligned} &\mathrm{Hilb}^n(X[n]/C[n])^s(\mathcal{L}_\infty) \subseteq \mathrm{Hilb}^n(X[n]/C[n])^s(\mathcal{L}_m) \\ &\subseteq \mathrm{Hilb}^n(X[n]/C[n])^{ss}(\mathcal{L}_m) \subseteq \mathrm{Hilb}^n(X[n]/C[n])^{ss}(\mathcal{L}_\infty). \end{aligned}$$

Since the first and the last set are equal by Proposition 4.1, it follows that all of them have to be equal. $\square$

To summarize, we obtain an alternative and more conceptual proof of [GHH19, Theorem 2.9], namely

Corollary 4.6. *For sufficiently large m the equalities*

$$\begin{aligned} &\mathrm{Hilb}^n(X[n]/C[n])^{ss}(\mathcal{L}_m) = \pi^{-1}(\mathrm{Sym}^n(X[n]/C[n])^{ss}(\widetilde{\mathcal{L}})) \\ &= \mathrm{Hilb}^n(X[n]/C[n])^{s}(\mathcal{L}_m) = \pi^{-1}(\mathrm{Sym}^n(X[n]/C[n])^{s}(\widetilde{\mathcal{L}})) \end{aligned}$$

hold.

Proof. This is a combination of Propositions 4.1 and 4.3. $\square$

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