STABLE VECTOR BUNDLES ON GENERALIZED KUMMER VARIETIES

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Abstract. In this article we prove that smooth connected components of moduli spaces of stable sheaves on irreducible holomorphic symplectic manifolds of dimension $\geqslant 4$ need not be irreducible holomorphic symplectic. We do this by explicitly constructing two new families of stable vector bundles on the generalized Kummer variety $K_n(A)$ associated to an abelian surface A. The first is the family of tautological bundles associated to stable bundles on A, and the second is the family of the "wrong-way" fibers of a universal family of stable bundles on the dual abelian surface $\widehat{A}$ parametrized by $K_n(A)$. Each family exhibits a smooth connected component in the moduli space of stable bundles on $K_n(A)$, which is not irreducible holomorphic symplectic.

Introduction

Background. Irreducible holomorphic symplectic manifolds are a type of building blocks in the classification of compact Kähler manifolds with trivial first Chern class. In the very influential paper [3], Beauville constructed two classes of irreducible holomorphic symplectic manifolds, which are the Hilbert schemes $X^{[n]}$ of n-points on K3 surfaces X, and the generalized Kummer varieties $K_n(A)$ associated to abelian surfaces A, obtained as the zero fibers of the summation map $\Sigma : A^{[n+1]} \to A$. The second construction was later generalized by Yoshioka in [25], in which he proved that the fibers $K_H(v)$ of the Albanese morphism $\mathfrak{a}_v : M_H(v) \to A \times \widehat{A}$ for moduli spaces $M_H(v)$ of μ_H-stable sheaves on A with Mukai vector v are deformation equivalent to generalized Kummer varieties.

Main results. The present manuscript is a continuation of the authors' work [20, 21] on the construction of new stable sheaves on irreducible holomorphic symplectic manifolds. The same problem was also studied by various authors, such as in [22, 23, 24]. The main purpose of this manuscript is to construct new stable vector bundles on generalized Kummer varieties and study some of their properties. We achieved two different constructions.

A natural family of vector bundles on $K_n(A)$ for considering stability are the so-called tautological bundles. In [23] Wandel constructed some examples of tautological bundles on $K_n(A)$ for $n = 1, 2$. Following an idea of Stapleton [22], we generalize Wandel's results by proving that in fact all taulogical bundles on $K_n(A)$ for $n \geqslant 2$ are stable with respect to suitable amples classes. Moreover, under suitable numerical assumptions, we show that the tautological bundles form a connected component of the moduli space of stable bundles on $K_n(A)$. This in particular indicates that smooth components of moduli spaces of sheaves on generalized Kummer varieties need not be irreducible holomorphic symplectic manifolds, contrary to the result for K3 surfaces, see [14, Theorem 10.3.10].

For another family of vector bundles on $K_n(A)$, we use the standard Fourier-Mukai transform to construct a fine moduli space $M_{\widehat{H}}(w)$ of stable vector bundles

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of rank $r \geqslant n+2$ on the dual abelian variety $\widehat{A}$ for some suitable choice of the Mukai vector w, such that $M_{\widehat{H}}(w) \cong A^{[n+1]} \times \widehat{A}$. Then $K_n(A)$ is naturally isomorphic to the zero fiber of the Albanese morphism $\mathfrak{a}_w : M_{\widehat{H}}(w) \to A \times \widehat{A}$. Let $\mathcal{U}$ be the restriction of the universal family on $\widehat{A}\times M_{\widehat{H}}(w)$ to the closed subscheme $\widehat{A}\times K_n(A)$. For each closed point $\widehat{a} \in \widehat{A}$, the further restriction of $\mathcal{U}$ to the slice $\{\widehat{a}\} \times K_n(A)$ gives a vector bundle $\mathcal{U}_{\widehat{a}}$ on $K_n(A)$. Following our approach in [21], we show that each $\mathcal{U}_{\widehat{a}}$ is a stable bundle on $K_n(A)$, hence we obtain a family of stable bundles on $K_n(A)$ parametrized by $\widehat{A}$.

Our main results can be summarized as follows:

Theorem. *Let (A, H) be a polarized abelian surface, and $(\widehat{A}, \widehat{H})$ its dual.*

1. *(Theorem 1.7) Let E be a μ_H-stable vector bundle of class v on A with $E \neq \mathcal{O}_A$, then the tautological bundle $E^{(n)}$ is a μ_D-stable vector bundle on $K_n(A)$ with respect to some ample divisor D on $K_n(A)$. Moreover, under suitable numerical assumptions on v, the moduli space $M_H(v)$ of μ_H-stable vector bundles of class v on A can be embedded as a connected component of some moduli space of μ_D-stable vector bundles on $K_n(A)$.*
2. *(Theorem 2.12) Let $\mathcal{U}$ be the restriction of the universal vector bundle on $\widehat{A}\times M_{\widehat{H}}(w)$ to the closed subscheme $\widehat{A}\times K_n(A)$ as described above. Then for each closed point $\widehat{a} \in \widehat{A}$, the fiber $\mathcal{U}_{\widehat{a}}$ is a μ_D-stable bundle on $K_n(A)$ with respect to an ample divisor D. Moreover, $\widehat{A}$ can be embedded as a connected component of a moduli space of μ_D-stable vector bundles on $K_n(A)$.*

Sketch of proof. Let us give a quick overview on how we achieved the above results. Although the setup in both cases looks very different, we will follow a similar chain of ideas to prove the slope stability of the bundles $E^{(n)}$ (resp. $\mathcal{U}_{\widehat{a}}$) with respect to some ample divisor D on $K_n(A)$. The proof consists of the following three major steps.

STEP 1. To begin with, let $P_n(A)$ be the codimension 2 subvariety of A^{n+1} parametrizing $(n+1)$-tuples whose components add up to zero under the group law of A. Each bundle $E^{(n)}$ (resp. $\mathcal{U}_{\widehat{a}}$) defines uniquely a reflexive sheaf on $P_n(A)$. We adapt the technique developed by Stapleton in [22] to prove the slope stability of $E^{(n)}$ (resp. $\mathcal{U}_{\widehat{a}}$) with respect to a natural nef divisor H_K on $K_n(A)$ by showing that the corresponding reflexive sheaf cannot be destabilized by any $\mathfrak{S}_{n+1}$-invariant subsheaf on $P_n(A)$. (See Propositions 1.5 and 2.10.)

STEP 2. In order to show the slope stability of $E^{(n)}$ (resp. $\mathcal{U}_{\widehat{a}}$) with respect to an ample divisor on $K_n(A)$, we use the openness of stability to perturb H_K to a nearby ample divisor D. This perturbation argument was developed in [9, 22], and generalized in [21]. The main difficulty for our application is to show the existence of the ample divisor D independent of the choice of E in its own moduli $M_H(v)$ (resp. the choice of the fiber $\mathcal{U}_{\widehat{a}}$ in the family $\mathcal{U}$ parametrized by $\widehat{A}$). (See Propositions 1.6 and 2.11.)

STEP 3. Finally, in order to identify $M_H(v)$ (resp. $\widehat{A}$) as a smooth connected component of the moduli space of μ_D-stable sheaves on $K_n(A)$, we interpret $E^{(n)}$ (resp. $\mathcal{U}_{\widehat{a}}$) as the image of E (resp. a line bundle on A) under an integral functor Θ induced by the structure sheaf (resp. the ideal sheaf) of the universal subscheme for $K_n(A)$. By a result of Meachan [17], we can apply the technique of $\mathbb{P}$-functors invented in [1, 2] to compute the relevant cohomology groups, which lead to our conclusion. (See Theorems 1.7 and 2.12.)

The text is organized in two sections, which deal with the two cases mentioned above respectively. All objects are defined over the field of complex numbers $\mathbb{C}$.

1. Tautological Bundles

1.1. **Notations.** For any integer $n \geqslant 2$, let A be an abelian surface, $K_n(A)$ the generalized Kummer variety of dimension $2n$, and $\mathcal{Z} \subset A \times K_n(A)$ the corresponding universal family. The projections from $\mathcal{Z}$ to the two factors A and $K_n(A)$ are denoted by p and q respectively. The following diagram exhibits the relations of some relevant schemes:

$$\begin{tikzcd}
P_n(A)_\circ \arrow[r, "\sigma_\circ"] \arrow[d, "j_P"'] & S_n(A)_\circ \arrow[d, "j_S"'] & K_n(A)_\circ \arrow[l, "h_\circ"'] \arrow[d, "j_K"'] \\
P_n(A) \arrow[r, "\sigma"] \arrow[d, "\tau"'] & S_n(A) \arrow[d] & K_n(A) \arrow[l, "h"'] \arrow[d, "\iota"'] \\
A^{n+1} \arrow[r, "\overline{\sigma}"] & A^{(n+1)} & A^{[n+1]} \arrow[l, "\overline{h}"']
\end{tikzcd} \tag{1}$$

Each vertical arrow in the lower half of the diagram is the embedding of a zero fiber of the addition morphism to A; σ and $\overline{\sigma}$ are quotients by the symmetric group $\mathfrak{S}_{n+1}$; h and $\overline{h}$ are Hilbert-Chow morphisms. Moreover, we denote the projections from A^{n+1} to each individual factor by $q_0, q_1, \cdots, q_n$.

Each vertical arrow in the upper half of the diagram is the embedding of an open subscheme parametrizing $n+1$ distinct (ordered or unordered) points in A. It is clear that the complement of each of these embeddings is a closed subscheme of codimension 2. The morphisms $\sigma_\circ$ and $h_\circ$ are restrictions of the morphisms in the second row. Clearly $\sigma_\circ$ is a free $\mathfrak{S}_{n+1}$-quotient and $h_\circ$ is an isomorphism.

Let H be an ample divisor on A. For each $0 \leqslant i \leqslant n$, we define $h_i = \tau^* q_i^* H$. Then $H_P = \sum_{i=0}^n h_i$ on $P_n(A)$ is an $\mathfrak{S}_{n+1}$-invariant ample divisor on $P_n(A)$, which descends to an ample divisor H_S on $S_n(A)$, whose pullback $H_K = h^*(H_S)$ is a big and nef divisor on $K_n(A)$. For any $E \in \mathrm{Coh}(A)$, the corresponding *tautological sheaf* $E^{(n)}$ on $K_n(A)$ is defined by $E^{(n)} = q_* p^* E$. Moreover, we write $E_i = \tau^* q_i^* E$ for each $0 \leqslant i \leqslant n$. The goal of this section is to show that if E is a non-trivial μ_H-stable vector bundle on A, then $E^{(n)}$ is slope stable with respect to some ample divisor sufficiently close to H_K. Our approach will mainly follow the idea in [22].

1.2. **Pullback of stable bundles.** We aim to prove Proposition 1.4, which is an analogue of [22, Proposition 4.7] in the Kummer case. We first collect necessary notations and tools required in the course of the proof, following [22, §4].

For any normal projective variety X, let $\gamma \in N_1(X)_{\mathbb{R}}$ be a curve class and $\mathcal{E} \in \mathrm{Coh}(X)$ a torsion-free sheaf. The slope of $\mathcal{E}$ with respect to γ is defined as

$$\mu^\gamma(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cdot \gamma}{\operatorname{rk} \mathcal{E}}.$$

It is clear that the slope is linear with respect to γ, and the usual notion of slope $\mu_H(\mathcal{E}) = \mu^{H^{d-1}}(\mathcal{E})$ for any ample class $H \in N^1(X)_{\mathbb{R}}$, where $n = \dim X$. The new notion of slope defines a *slope stability* (resp. *semistability*) of $\mathcal{E}$ with respect to a curve class γ, by requiring any torsion-free quotient of $\mathcal{E}$ of a smaller rank to have a smaller (resp. smaller or equal) slope with respect to γ.

The main advantage of studying slope stability with respect to curve classes is the linearity of slopes with respect to the curve parameter. More precisely, we have

Lemma 1.1 ([22, Lemma 4.4])**.** *Let $\gamma, \delta \in N_1(X)_{\mathbb{R}}$ such that $\mathcal{E}$ is semistable with respect to γ and stable with respect to δ, then $\mathcal{E}$ is stable with respect to $a\gamma + b\delta$ for any $a, b > 0$.* □

Our main tool for determining the slope stability is the following observation

Lemma 1.2 ([22, Corollary 4.6])**.** *Let $\pi : C_T \to T$ be a family of smooth irreducible closed curves in X with class γ. Suppose that $\mathcal{E}$ is a vector bundle on X such that $\mathcal{E}|_{C_t}$ is stable for all $t \in T$, and that the curves in C_T are dense in X, then $\mathcal{E}$ is stable with respect to the curve class γ. Moreover, the statement also holds if stability is replaced by semistability.* □

The following lemma will be required in the proof of our main result

Lemma 1.3. *Let $S : A^3 \to A$ be the addition with respect to the group law on A, and $q_i : A^3 \to A$ the projection to the i-th factor for $1 \leqslant i \leqslant 3$. Assume H is a sufficiently ample divisor on A. Let $C_i \in |H|$ for $1 \leqslant i \leqslant 3$. Then*

(1) *For any fixed point $b \in A$, the scheme theoretic intersection of $S^{-1}(b)$, $q_1^{-1}(C_1)$, $q_2^{-1}(C_2)$, $q_3^{-1}(C_3)$ is a smooth curve C for a generic choice of C_i for $1 \leqslant i \leqslant 3$;*

(2) *Each projection $q_i : C \to C_i$ is a finite morphism for $1 \leqslant i \leqslant 3$.*

Proof. For the first statement we use Bertini theorem; see e.g. [7, Proposition 0.5]. We first observe that the addition morphism S is smooth, hence $Y_0 = S^{-1}(b)$ is smooth and irreducible. By assumption the complete linear system $|H|$ has no base point, so does $q_1^{-1}(H)|_{Y_0}$. Hence a generic choice of $C_1 \in |H|$ and $Y_1 = Y_0 \cap q_1^{-1}(C_1)$ are smooth and irreducible by Bertini theorem. For the same reason $q_2^{-1}(H)|_{Y_1}$ has no base point, hence a generic choice of $C_2 \in |H|$ and $Y_2 = Y_1 \cap q_2^{-1}(C_2)$ are smooth and irreducible. Similarly, a generic choice of $C_3 \in |H|$ and $C = Y_2 \cap q_3^{-1}(C_3)$ are smooth and irreducible. A dimension count shows that C is a curve.

For the second statement, since C is smooth irreducible, and the projection q_i is surjective, it follows by [11, Proposition II.6.8] that q_i is a finite morphism. □

It was proven in [22, Proposition 4.7] that the pullback of a slope stable bundle E from A to A^{n+1} via the projection to any factor is stable with respect to a natural $\mathfrak{S}_{n+1}$-invariant ample class. The following proposition shows that a further restriction to $P_n(A)$, the zero fiber of the addition morphism, remains stable.

Proposition 1.4. *Under the above notations, let E be a μ_H-stable bundle on A. Then E_i is a μ_{H_P}-stable bundle on $P_n(A)$ for each $0 \leqslant i \leqslant n$.*

Proof. Without loss of generality, we prove the result for $i = 0$. Moreover, by replacing H with a high tensor power of itself, we can assume that a generic element $C \in |H|$ in the linear system is a smooth curve such that $E|_C$ is slope stable.

Using the notation of slope stability with respect to a real curve class defined in the beginning of the subsection, we need to show that E_0 is slope stable with respect to the curve class H_P^{2n-1}. We expand the product to obtain

$$H_P^{2n-1} = \sum_{\substack{k_0+\cdots+k_n=2n-1 \\ 0\leqslant k_0,\cdots,k_n\leqslant 2}} c_{k_0\cdots k_n} h_0^{k_0} \cdots h_n^{k_n} \tag{2}$$

where each $c_{k_0\cdots k_n}$ is some positive integer. We will analyze the slope stability of E_0 with respect to each term on the right-hand side of (2). Without loss of generality, by permuting the indices $1 \leqslant i \leqslant n$, we need to consider the following 5 cases:

Case 1. $k_0 = 2$, $k_1 = 1$, $k_2 = 0$, and $k_i = 2$ for each $i \geqslant 3$. We consider the family of curves in $P_n(A)$ given by intersecting $\{a_0\} \times C_1 \times A \times \{a_3\} \times \cdots \times \{a_n\}$ with $P_n(A)$ in A^{n+1}, where $a_0, a_3, \cdots, a_n \in A$ and $C_1 \in |H|$. Each curve C in this family lies in the class $h_0^{k_0} \cdots h_n^{k_n}/(H^2)^{n-1}$, and is isomorphic to C_1 via the projection q_1. Therefore a generic choice of C is smooth, and it is clear that $E_0|_C$ is a trivial bundle, hence is slope semistable. We claim that all such curves C cover a dense subset of $P_n(A)$. Indeed, the projection $q' = (q_0, q_1, q_3, \cdots, q_n)$ identifies $P_n(A)$ with A^n, and C with $\{a_0\} \times C_1 \times \{a_3\} \times \cdots \times \{a_n\}$. Since we assume that H is very ample, there is a subset $F \subset |H|$ parametrizing smooth irreducible curves $C_1 \in |H|$, such that the union $U = \bigcup_{C_1 \in F} C_1$ is dense in A. By varying $a_0, a_3, \cdots, a_n$, we obtain a family of smooth and irreducible curves in A^n parametrized by $A \times F \times A^{n-2}$, whose union $A \times U \times A^{n-2}$ is dense in A^n. Then the preimages of these curves via q' are smooth irreducible curves in $P_n(A)$, whose union is dense in $P_n(A)$, as desired. Therefore E_0 is slope semistable with respect to the curve class $h_0^{k_0} \cdots h_n^{k_n}$ by Lemma 1.2.

Case 2. $k_0 = 2$, $k_1 = k_2 = k_3 = 1$, and $k_i = 2$ for each $i \geqslant 4$. We consider the family of curves in $P_n(A)$ given by intersecting $\{a_0\} \times C_1 \times C_2 \times C_3 \times \{a_4\} \times \cdots \{a_n\}$ with $P_n(A)$ in A^{n+1}, where $a_0, a_4, \cdots, a_n \in A$ and $C_1, C_2, C_3 \in |H|$. Each curve C in this family lies in the class $h_0^{k_0} \cdots h_n^{k_n}/(H^2)^{n-2}$. By Lemma 1.3, a generic choice of C is smooth, and it is clear that $E_0|_C$ is a trivial bundle, hence is slope semistable. All such curves C cover a dense subset of $P_n(A)$. Therefore E_0 is slope semistable with respect to the curve class $h_0^{k_0} \cdots h_n^{k_n}$ by Lemma 1.2.

Case 3. $k_0 = 1$, $k_1 = 0$, and $k_i = 2$ for each $i \geqslant 2$. We consider the family of curves in $P_n(A)$ given by intersecting $C_0 \times A \times \{a_2\} \times \cdots \{a_n\}$ with $P_n(A)$ in A^{n+1}, where $a_2, \cdots, a_n \in A$ and $C_0 \in |H|$. Each curve C in this family lies in the class $h_0^{k_0} \cdots h_n^{k_n}/(H^2)^{n-1}$, and is isomorphic to C_0 via the projection q_0. Therefore a generic choice of C is smooth, and $E_0|_C$ is isomorphic to $E|_{C_0}$, which by the assumption on H is slope stable for a generic choice of C_0. All such curves C cover a dense subset of $P_n(A)$. Therefore E_0 is slope stable with respect to the curve class $h_0^{k_0} \cdots h_n^{k_n}$ by Lemma 1.2.

Case 4. $k_0 = k_1 = k_2 = 1$, and $k_i = 2$ for each $i \geqslant 3$. We consider the family of curves in $P_n(A)$ given by intersecting $C_0 \times C_1 \times C_2 \times \{a_3\} \times \cdots \times \{a_n\}$ with $P_n(A)$ in A^{n+1}, where $a_3, \cdots, a_n \in A$ and $C_0, C_1, C_2 \in |H|$. Each curve C in this family lies in the class $h_0^{k_0} \cdots h_n^{k_n}/(H^2)^{n-2}$. By Lemma 1.3, a generic choice of C is smooth, and the projection gives a finite morphism $\varphi : C \to C_0$ such that $E_0|_C = \varphi^*(E|_{C_0})$. We know $E|_{C_0}$ is slope stable by the assumption on H for a generic choice of C_0. It follows by [15, Lemma 3.2.3] that $E_0|_C$ is slope semistable. All such curves C cover a dense subset of $P_n(A)$. Therefore E_0 is slope semistable with respect to the curve class $h_0^{k_0} \cdots h_n^{k_n}$ by Lemma 1.2.

Case 5. $k_0 = 0$, $k_1 = 1$, and $k_i = 2$ for each $i \geqslant 2$. We consider the family of curves in $P_n(A)$ given by intersecting $A \times C_1 \times \{a_2\} \times \cdots \{a_n\}$ with $P_n(A)$ in A^{n+1}, where $a_2, \cdots, a_n \in A$ and $C_1 \in |H|$. Each curve C in this family lies in the class $h_0^{k_0} \cdots h_n^{k_n}/(H^2)^{n-1}$, and is isomorphic to C_1 via the projection q_1. Therefore a generic choice of C is smooth. For any fixed choice of $a_2, \cdots, a_n \in A$, let ι be the inverse morphism on A, $b = -(a_2 + \cdots + a_n)$ (with respect to the group law on A), and $t_b : A \to A$ is the corresponding translation. Then $E_0|_C$ is isomorphic to $(t_b^* \iota^* E)|_{C_1}$. Since E is μ_H-stable, $t_b^* \iota^* E$ is also μ_H-stable. Hence $(t_b^* \iota^* E)|_{C_1}$ is slope stable for a generic choice of C_1, and the corresponding curves C cover a dense subset of the intersection of $A \times A \times \{a_2\} \times \cdots \times \{a_n\}$ with $P_n(A)$. When we allow $a_2, \cdots, a_n$ to move in A, it follows that $E_0|_C$ is slope stable for a generic choice of the curve C as described above, and such curves cover a dense subset of $P_n(A)$. Therefore E_0 is slope stable with respect to the curve class $h_0^{k_0} \cdots h_n^{k_n}$ by Lemma 1.2.

Applying Lemma 1.1, the above cases together implies that E_0 is slope stable with respect to the curve class H_P^{2n-1}; in other words, E_0 is μ_{H_P}-stable. □

1.3. **Tautological bundles.** For any torsion free coherent sheaf F on $K_n(A)$, we follow [22, §1] to define an $\mathfrak{S}_{n+1}$-invariant coherent sheaf on $P_n(A)$ by

$$(F)_P = (j_P)_* \sigma_\circ^* (h_\circ^{-1})^* j_K^* F,$$

which is reflexive if F itself is reflexive. Moreover, we observe that an analogue of [22, Lemma 1.2], namely

$$(n+1)!\int_{K_n(A)} c_1(F)\cdot(H_K)^{2n-1} = \int_{P_n(A)} c_1((F)_P)\cdot(H_P)^{2n-1} \tag{3}$$

holds due to the relevant diagonals having codimension 2. The following result is an analogue of [22, Theorem 1.4] in the Kummer case.

Proposition 1.5. *Let E be a μ_H-stable bundle on A not isomorphic to $\mathcal{O}_A$, then $E^{(n)}$ is a μ_{H_K}-stable bundle on $K_n(A)$.*

Proof. It suffices to show that every reflexive subsheaf of $E^{(n)}$ of smaller rank has a smaller slope. Let F be such a subsheaf of $E^{(n)}$, then $(F)_P$ is an $\mathfrak{S}_{n+1}$-invariant reflexive subsheaf of $(E^{(n)})_P$. Using equation (3), it is enough to prove $\mu_{H_P}((F)_P) < \mu_{H_P}((E^{(n)})_P)$. Let G be a non-zero (not necessarily $\mathfrak{S}_{n+1}$-invariant) μ_{H_P}-stable subsheaf of $(F)_P$ of maximal slope; e.g., we can take G to be the first factor in a Jordan-Hölder filtration of $(F)_P$. A similar argument as in [22, Lemma 1.1] shows that $(E^{(n)})_P = E_0 \oplus \cdots \oplus E_n$. (Both sides are reflexive sheaves and isomorphic on $P_n(A)_\circ$ whose complement is of codimension 2.) Therefore, there exists some i such that the composition of the embedding and projection

$$G \hookrightarrow (F)_P \hookrightarrow (E^{(n)})_P \twoheadrightarrow E_i \tag{4}$$

is non-zero. Since E_i is also μ_{H_P}-stable for each $0 \leqslant i \leqslant n$ by Proposition 1.4, we must have $\mu_{H_P}(G) \leqslant \mu_{H_P}(E_i)$.

Case 1. If $\mu_{H_P}(G) < \mu_{H_P}(E_i)$, then

$$\mu_{H_P}((F)_P) \leqslant \mu_{H_P}(G) < \mu_{H_P}(E_i) = \mu_{H_P}((E^{(n)})_P),$$

hence $(F)_P$ does not destabilize $(E^{(n)})_P$.

Case 2. If $\mu_{H_P}(G) = \mu_{H_P}(E_i)$, then the composition map (4) must be an isomorphism. Since $E \not\cong \mathcal{O}_A$, we have $E_i \not\cong E_j$ for $i \neq j$. (Choose any k different from i and j, then the projection $q'' = (q_0, \cdots, q_{k-1}, q_{k+1}, \cdots, q_n)$ identifies $P_n(A)$ with A^n. The pullback of a non-trivial sheaf via projections to two distinct factors are not isomorphic.) It follows that the composition

$$G \hookrightarrow (F)_P \hookrightarrow (E^{(n)})_P \twoheadrightarrow E_j$$

must be zero for any $j \neq i$. It follows that G is the direct summand E_i of $(E^{(n)})_P$. Since $(F)_P$ is an $\mathfrak{S}_{n+1}$-invariant subsheaf of $(E^{(n)})_P$ containing the direct summand E_i, we obtain $(F)_P = (E^{(n)})_P$, which cannot happen since the left-hand side has a smaller rank than the right-hand side. This concludes the proof. □

In the following we will apply the perturbation argument in [22, §4] to show the slope stability of $E^{(n)}$ with respect to an ample divisor on $K_n(A)$, which stays constant when we deform E in its moduli.

1.4. **The family of stable bundles.** In this subsection we study families of stable tautological bundles. We assume that

$$v = (v_0, v_1, v_2) \in H^0(A,\mathbb{Z}) \oplus \mathrm{NS}(A) \oplus H^4(A,\mathbb{Z})$$

is a Mukai vector satisfying the condition

(†) the projective moduli space $M_H(v)$ of H-semistable sheaves of class v is non-empty and contains only μ_H-stable locally free sheaves.

This condition is easy to achieve: first of all we require $v_0 > 0$; in order for every μ_H-semistable sheaf of class v to be stable, it suffices to require that v_0 and $H\cdot v_1$ are coprime; the nonemptiness can be achieved by requiring

$$\langle v^2\rangle = v_1^2 - 2v_0v_2 \geqslant 0; \tag{5}$$

finally the local freeness of all μ_H-stable sheaves holds when v_2 takes the largest possible value satisfying (5) for any fixed v_0 and v_1. For instance, if A is an abelian surface with a primitive ample divisor H such that $H^2 = 16$, then the Mukai vector $v = (5, H, 1)$ satisfies the condition (†).

Under the condition (†), we have seen by Proposition 1.5 that the tautological bundle $E^{(n)}$ is μ_{H_K}-stable for each $E \in M_H(v)$. However, H_K lies in the boundary of the ample cone of $K_n(A)$. In order to establish the stability of the tautological bundle with respect to some ample class, we need the following result

Proposition 1.6. *Under condition* (†)*, there exists an ample class $H' \in \mathrm{NS}(K_n(A))$ near H_K, such that $E^{(n)}$ is $\mu_{H'}$-stable for all $[E] \in M_H(v)$.*

Proof. By replacing H with a high tensor power of itself if necessary, we assume the complete linear system $|H_K|$ defines (the restriction of) the Hilbert-Chow morphism $h : K_n(A) \to S_n(A)$ as shown in (1). We claim that h is semismall. Indeed, for any partition ξ given by $n+1 = 1 \cdot n_1 + 2 \cdot n_2 + \cdots + r \cdot n_r$, we consider the locally closed subscheme $Y_\xi \subset S_n(A)$ parametrizing $n_1+n_2+\cdots+n_r$ distinct points, among which are n_i points of multiplicity i for $1 \leqslant i \leqslant r$. We have $\dim Y_\xi = 2(n_1+n_2+\cdots+n_r)-2$ and $\dim h^{-1}(y) = 0 \cdot n_1 + 1 \cdot n_2 + \cdots + (p-1) \cdot n_p$ for any closed point $y \in Y$. It follows that $\dim Y_\xi + 2 \dim h^{-1}(y) = 2n = \dim K_n(A)$ which implies that h is semismall. Therefore H_K is lef by [6, Definition 2.1.3] and satisfies the hard Lefschetz property by [6, Theorem 2.3.1]. It then follows from [22, Proposition 4.8] that each $E^{(n)}$ is $\mu_{H'}$-stable with respect to some ample class H' near H_K. However, in order to find a single H' that works simultaneously for all $E^{(n)}$, we can apply the entire proof of [22, Proposition 4.8] except one step; namely, we need to find a convex open set U such that $\alpha \coloneqq H_K^{2n-1}$ is in the closure of U, and for each $\gamma \in U$, the tautological bundle $E^{(n)}$ is stable with respect to γ for all $[E] \in M_H(v)$.

We follow the notations in [9, Definition 3.1]. For each $[E] \in M_H(v)$, $\mathrm{SStab}(E^{(n)})$ is a convex closed set containing α. Hence the intersection

$$\overline{U} \coloneqq \bigcap_{[E] \in M_H(v)} \mathrm{SStab}(E^{(n)})$$

is also a convex closed set containing α.

We claim that [9, Theorem 3.4] holds for all $E^{(n)}$ simultaneously; namely, we will show that for any $\beta \in \mathrm{Mov}(K_n(A))^\circ$ (see [9, Definition 2.1] for the notation), there exists some $e \in \mathbb{Q}^+$, such that $(\alpha + \varepsilon\beta) \in \bigcap_{[E] \in M_H(v)} \mathrm{Stab}(E^{(n)})$ for any real $\varepsilon \in [0, e]$.

To prove the claim, we first note by [8, p.87, Lemma 5(v)] that E is μ_H-stable of class $v = (v_0, v_1, v_2)$ if and only if $E^\vee$ is μ_H-stable of class $v^\vee = (v_0, -v_1, v_2)$. Since μ_H-stable sheaves of class $v^\vee$ are bounded, there exists some positive integer m such that $E^\vee(mH)$ is globally generated and $H^i(A, E^\vee(mH)) = 0$ for all $i > 0$, hence there exists a surjective map $\mathcal{O}_A(-mH)^{\oplus N} \twoheadrightarrow E^\vee$, where m and N are independent of E. Since E is locally free, we can take the dual of the above surjective map to obtain an injective map $E \hookrightarrow \mathcal{O}_A(mH)^{\oplus N}$, and complete it to an exact sequence

$$0 \longrightarrow E \longrightarrow \mathcal{O}_A(mH)^{\oplus N} \longrightarrow Q_E \longrightarrow 0.$$

We apply the functor $q_* \circ p^*$ on the above sequence. It was proven in [20, Lemma 3.1] that the morphism p is flat for $n \geqslant 2$, hence the functor p^* is exact. The morphism q is finite thus q_* is also exact. Therefore we obtain an exact sequence

$$0 \longrightarrow E^{(n)} \longrightarrow q_*p^*\mathcal{O}_A(mH)^{\oplus N} \longrightarrow q_*p^*Q_E \longrightarrow 0. \tag{6}$$

We note that the slope $c \coloneqq \mu_\beta(E^{(n)})$ is independent of the choice of $[E] \in M_H(v)$, and redefine the set S in the proof of [9, Theorem 3.4] to be

$$S \coloneqq \{c_1(F) \mid F \subseteq E^{(n)} \text{ for some } [E] \in M_H(v) \text{ such that } \mu_\beta(F) \geqslant c\}.$$

By (6) we see that S is a subset of

$$T := \{c_1(F) \mid F \subseteq q_* p^* \mathcal{O}_A(mH)^{\oplus N} \text{ such that } \mu_\beta(F) \geqslant c\},$$

which is finite by [9, Theorem 2.29]. Thus S is a also a finite set. We can then apply the rest of the proof of [9, Theorem 3.4] literally to conclude the claim.

Now we can show that $\overline{U}$ is of full dimension $r := \operatorname{rk} N_1(K_n(A))$. If not, then we have $\alpha \in \overline{U} \subseteq L$ for some hyperplane $L \subset N_1(K_n(A))_\mathbb{R}$. Since $\operatorname{Mov}(K_n(A))$ is of full dimension, we can choose some $\beta \in \operatorname{Mov}(K_n(A))^\circ \setminus L$. It follows that $(\alpha + \varepsilon\beta) \in \overline{U} \setminus L$ for some small $\varepsilon > 0$ by the above claim and the choice of β. Contradiction.

We define U to be the interior of $\overline{U}$ and claim that U is non-empty. Indeed, since $\overline{U}$ is of full dimension r, we can choose $r+1$ points of $\overline{U}$ in general positions, which form an r-simplex. By the convexity of $\overline{U}$, the entire simplex is in $\overline{U}$ hence any interior point of the simplex is also an interior point of $\overline{U}$. The convexity of U follows from the convexity of $\overline{U}$. And it is clear from the construction that $\alpha = H_K^{2n-1}$ is in the closure of U.

We finally prove that $U \subseteq \bigcap_{[E]\in M_H(v)} \operatorname{Stab}(E^{(n)})$. If not, suppose that there is some $\gamma_0 \in U$ and some $[E] \in M_H(v)$, such that $\gamma_0 \in \operatorname{SStab}(E^{(n)}) \setminus \operatorname{Stab}(E^{(n)})$; namely, $\mu_{\gamma_0}(F) = \mu_{\gamma_0}(E^{(n)})$ for some proper subsheaf F of $E^{(n)}$. Since the slope function is linear with respect to the curve class, and $\mu_\alpha(F) < \mu_\alpha(E^{(n)})$ by Proposition 1.4, one can find a hyperplane in $N^1(K_n(A))_\mathbb{R}$ through γ_0, such that $\mu_\gamma(E^{(n)}) - \mu_\gamma(F)$ takes opposite signs for γ in the two open half-spaces separated by the hyperplane. In particular, F destabilizes $\mathcal{U}_{\widehat{a}_0}$ in one of the half-spaces. Since U has non-empty intersection with both half-spaces, this contradicts the condition $U \subseteq \operatorname{SStab}(E^{(n)})$. Therefore we have $U \subseteq \bigcap_{[E]\in M_H(v)} \operatorname{Stab}(E^{(n)})$, as desired. $\square$

1.5. **A component of the moduli space.** In this subsection, we show that under some favorable numerical conditions, $M_H(v)$ is isomorphic to a connected component of a moduli space of stable sheaves on $K_n(A)$.

Indeed, we still assume that v satisfies condition (†); or more precisely, the numerical conditions in the paragraph below (†) that ensure its validity. We further assume that

(‡) for every $[E] \in M_H(v)$, we have $H^i(A, E) = 0$ for $i > 0$.

This condition is also easy to achieve. Since all stable sheaves are bounded, there exists some positive integer m independent of the choice of E, such that $H^i(A, E(mH)) = 0$ for all $i > 0$. By replacing v with $v \cdot \operatorname{ch}(mH)$, we obtain a Mukai vector v satisfying both (†) and (‡).

Under the above assumptions, let H' be the ample line bundle constructed in Proposition 1.6, and $\mathcal{M}$ the moduli space of $\mu_{H'}$-stable sheaves on $K_n(A)$ with the same numerical invariants as $E^{(n)}$. By applying Proposition 1.6, the integral functor $q_* \circ p^*$ induces a morphism

$$f : M_H(v) \longrightarrow \mathcal{M}, \quad [E] \longmapsto [E^{(n)}]. \tag{7}$$

In fact the morphism f can be described as follows:

Theorem 1.7. *Under the assumptions* (†) *and* (‡)*, the classifying morphism* (7) *identifies $M_H(v)$ with a smooth connected component of $\mathcal{M}$.*

Proof. By [20, Lemma 1.6.] we have to prove that f is injective on closed points and that $\dim(T_{[E^{(n)}]}\mathcal{M}) = \dim(T_{[E]}M_H(v))$ for all $[E] \in M_H(v)$.

The main tool for achieving this is [17, Theorem 6.9], which is a formula for computing various extension groups between tautological sheaves. More exactly

[17, Theorem 6.9] implies for any $[E_1], [E_2] \in M_H(v)$ that

$$\mathrm{Hom}_{K_n(A)}(E_1^{(n)}, E_2^{(n)}) \cong \mathrm{Hom}_A(E_1, E_2) = \begin{cases} \mathbb{C}, & \text{when } E_1 \cong E_2; \\ 0, & \text{when } E_1 \ncong E_2. \end{cases}$$

In particular, the case of $E_1 \ncong E_2$ implies that f is injective on closed points. Moreover, [17, Theorem 6.9] also implies

$$\begin{aligned} &\mathrm{Ext}^1_{K_n(A)}(E_1^{(n)}, E_2^{(n)}) \\ \cong\ &\mathrm{Ext}^1_A(E_1, E_2) \oplus H^1(A, E_1^\vee) \otimes H^0(A, E_2) \oplus H^0(A, E_1^\vee) \otimes H^1(A, E_2) \\ =\ &\mathrm{Ext}^1_A(E_1, E_2), \end{aligned}$$

where the last equality follows from (‡) and the Serre duality on A. In particular, when $[E_1]$ and $[E_2]$ represent the same closed point $[E] \in M_H(v)$, we obtain that $\dim T_{E^{(n)}}(\mathcal{M}) = \dim T_E(M_H(v))$ as desired. □

2. Universal Bundles

In this section we want to construct a second type of stable bundles on $K_n(A)$. The basic idea is to use the Fourier-Mukai transform to find a fine moduli space of stable sheaves $M_{\widehat{H}}(w)$ on $\widehat{A}$ such that the generalized Kummer $K_{\widehat{H}}(w)$ in $M_{\widehat{H}}(w)$ is isomorphic to $K_n(A)$. We restrict the universal family of $M_{\widehat{H}}(w)$ to $K_{\widehat{H}}(w)$ and study its fibers over a point $\widehat{a} \in \widehat{A}$, which is a sheaf on $K_{\widehat{H}}(w) \cong K_n(A)$.

2.1. **Stable sheaves on abelian surfaces.** Pick $n, r \in \mathbb{N}$ with $n \geqslant 2$ as well as $r \geqslant n+2$ and let A be an abelian surface satisfying

$$\mathrm{NS}(A) = \mathbb{Z}H \text{ such that } H^2 = 2(n+r+1).$$

We denote the dual abelian surface by $\widehat{A}$. We have the Poincare line bundle $\mathcal{P}$ on $A \times \widehat{A}$ which defines the classical Fourier-Mukai transform

$$\Phi : \mathrm{D^b}(A) \to \mathrm{D^b}(\widehat{A}), \ \ E \mapsto Rp_*(\mathcal{P} \otimes q^*(E))$$

where $p : A \times \widehat{A} \to \widehat{A}$ and $q : A \times \widehat{A} \to A$ are the projections.

Using the canonical isomorphism $A \cong \widehat{\widehat{A}}$ (given by the Poincare bundle), we can also understand $\mathcal{P}$ as the Poincare bundle on $\widehat{A} \times A$ up to switching the factors, see [13, p.198, Remark 9.12]. This gives rise to the Fourier-Mukai transform

$$\widehat{\Phi} : \mathrm{D^b}(\widehat{A}) \to \mathrm{D^b}(A), \ \ F \mapsto Rq_*(\mathcal{P} \otimes p^*(F))$$

It is well known that $\det(\Phi(\mathcal{O}_A(H)))^{-1}$ defines the canonical polarization $\widehat{H}$ on $\widehat{A}$ and $\mathrm{NS}(\widehat{A}) = \mathbb{Z}\widehat{H}$, see for example [4].

Now we look at the Mukai vector

$$v = (1, H, r)$$

and denote the moduli space of μ_H-semistable sheaves on A with Mukai vector v by $M_H(v)$. Then there is an isomorphism

$$\epsilon : A^{[n+1]} \times \widehat{A} \xrightarrow{\cong} M_H(v), \ \ (Z, \widehat{a}) \longmapsto I_Z(H) \otimes \mathcal{P}_{\widehat{a}}\,.$$

We compute $\langle v^2 \rangle = H^2 - 2r = 2(n+r+1) - 2r = 2n+2$ and thus

$$\dim(M_H(v)) = 2n + 4. \tag{8}$$

Furthermore by the choice of r we have $r > \frac{\langle v^2 \rangle}{2}$, which by [25, Corollary 3.3] implies that every $E \in M_H(v)$ satisfies IT_0 with respect to Φ and that $\Phi(E)$ is a $\mu_{\widehat{H}}$-stable locally free sheaf on $\widehat{A}$ with Mukai vector

$$w = (r, -\widehat{H}, 1).$$

By [25, Prop. 3.2, Cor. 3.3] we get that the Fourier-Mukai transform induces an isomorphism

$$\Phi : M_H(v) \xrightarrow{\cong} M_{\widehat{H}}(w).$$

Remark 2.1. The moduli space $M_{\widehat{H}}(w)$ is fine as $\gcd(r, \widehat{H}^2, 1) = 1$. Furthermore [25, Corollary 3.3] also shows that all sheaves classified by $M_{\widehat{H}}(w)$ are $\mu_{\widehat{H}}$-stable locally free sheaves.

2.2. **Generalized Kummer varieties.** We recall the original construction of the generalized Kummer due to Beauville, see [3, Sect. 7]: the group law on A defines, via the symmetric power and the Hilbert-Chow morphism, a summation morphism:

$$\Sigma : A^{[n+1]} \to A^{(n+1)} \to A.$$

The generalized Kummer variety is then defined by $K_n(A) := \Sigma^{-1}(0_A)$.

This construction was generalized by Yoshioka to moduli spaces of stable sheaves $M_H(v)$ on A, see [25, Theorem 4.1., Definition 4.1.]. We quickly summarize his main results: let v be a primitive Mukai vector with $\langle v^2 \rangle + 2 \geqslant 6$ and H be a generic polarization, i.e. $\overline{M_H(v)} = M_H(v)$. One finds that the Albanese morphism of $M_H(v)$ is given by

$$\mathfrak{a}_v : M_H(v) \to A \times \widehat{A}$$

with

$$\mathfrak{a}_v(E) = \big(\det(\Phi(E)) \otimes \det(\Phi(E_0))^{-1}, \det(E) \otimes \det(E_0)^{-1}\big)$$

for some fixed $E_0 \in M_H(v)$. Then one can give the following:

Definition 2.2. The generalized Kummer variety $K_H(v)$ in $M_H(v)$ is defined to be the fiber of $\mathfrak{a}_v$ over the point $(0_A, 0_{\widehat{A}})$, i.e. $K_H(v) = \mathfrak{a}_v^{-1}((0_A, 0_{\widehat{A}}))$.

Note that we have $\dim(K_H(v)) = 2n$ by (8). Now assume that v also satisfies all conditions from [25, Corollary 3.3], that is the Fourier-Mukai transform induces an isomorphism $\Phi : M_H(v) \xrightarrow{\cong} M_{\widehat{H}}(w)$. Under these circumstances not only are the moduli spaces are isomorphic, but also the induced generalized Kummer varieties:

Lemma 2.3. *The isomorphism $\Phi : M_H(v) \xrightarrow{\cong} M_{\widehat{H}}(w)$ restricts to an isomorphism between generalized Kummer varieties $K_H(v) \xrightarrow{\cong} K_{\widehat{H}}(w)$.*

Proof. We first note that the Albanese morphism $\mathfrak{a}_w : M_{\widehat{H}}(w) \to \widehat{A} \times \widehat{\widehat{A}}$ can be understood as a morphism $\mathfrak{a}_w : M_{\widehat{H}}(w) \to A \times \widehat{A}$ after identifying $A \cong \widehat{\widehat{A}}$ and switching the factors. It is then given by

$$\mathfrak{a}_w(F) = (\det(F) \otimes \det(F_0)^{-1}, \det(\widehat{\Phi}(F)) \otimes \det(\widehat{\Phi}(F_0))^{-1})$$

with $F_0 = \Phi(E_0) \in M_{\widehat{H}}(w)$.

Using the isomorphism $\varphi : A \times \widehat{A} \to A \times \widehat{A}$ given by $\varphi := 1_A \times (-1_A)^*$ we claim that the following diagram commutes:

$$\begin{array}{ccc} M_H(v) & \xrightarrow{\Phi} & M_{\widehat{H}}(w) \\ \Big\downarrow{\scriptstyle \mathfrak{a}_v} & & \Big\downarrow{\scriptstyle \mathfrak{a}_w} \\ A \times \widehat{A} & \xrightarrow{\varphi} & A \times \widehat{A}. \end{array}$$

To see this we simply note that since every $E \in M_H(v)$ is IT_0 with respect to Φ, we have the following isomorphism by [18, Corollary 2.4.]:

$$\widehat{\Phi}(\Phi(E)) \cong (-1_A)^* E.$$

Using $\varphi((0_A, 0_{\widehat{A}})) = (0_A, 0_{\widehat{A}})$ and the commutativity, we see that Φ restricts to an isomorphism $K_H(v) \cong K_{\widehat{H}}(w)$. $\square$

2.3. **Construction of a universal family.** In this section we want to construct a universal family for the generalized Kummer variety $K_{\widehat{H}}(w)$. For this we first note that $M_H(v)$ is a fine moduli space, that is there is a universal family on the product $A \times M_H(v)$. Denote the restriction of the universal family along the closed immersion $A \times K_H(v) \hookrightarrow A \times M_H(v)$ by $\mathcal{E}$.

Remark 2.4. For the Mukai vector $v = (1, H, r)$ we have the following isomorphism:

$$K_n(A) \xrightarrow{\cong} K_H(v), \quad [Z] \longmapsto I_Z(H).$$

By making a careful choice of the line bundles on A and $\widehat{A}$ representing $\det(E_0)$ and $\det(\Phi(E_0))$ as in [10, §3.1], an explicit computation similar to [10, Lemma 3.2] shows that there is a commutative diagram

$$\begin{tikzcd} A^{[n+1]} \times \widehat{A} \arrow[r, "\epsilon"] \arrow[d, "\Sigma\times 1_{\widehat{A}}"'] & M_H(v) \arrow[d, "\mathfrak{a}_v"] \\ A \times \widehat{A} \arrow[r, "\rho"] & A \times \widehat{A} \end{tikzcd}$$

with the isomorphism

$$\rho : A \times \widehat{A} \xrightarrow{\cong} A \times \widehat{A}, \quad (a, \widehat{a}) \longmapsto (-a + \phi_{\widehat{H}^{-1}}(\widehat{a}), \widehat{a}).$$

Again, as $\rho(0_A, 0_{\widehat{A}}) = (0_A, 0_{\widehat{A}})$, we find that the isomorphism ϵ restricts to an isomorphism between the fibers of $\Sigma \times 1_{\widehat{A}}$ and $\mathfrak{a}_v$ over $(0_A, 0_{\widehat{A}})$. It remains to note that these fibers are $K_n(A)$ and $K_H(v)$ by definition.

Using the last remark we will, from now on, understand the universal family $\mathcal{E}$ on $A \times K_H(v)$ as a family on $A \times K_n(A)$, which is easily seen to be given by

$$\mathcal{E} = \mathcal{I}_{\mathcal{Z}} \otimes \pi_1^* \mathcal{O}_A(H)$$

where $\pi_1 : A \times K_n(A) \to A$ is the projection and $\mathcal{I}_{\mathcal{Z}}$ is the universal ideal sheaf on $A \times K_n(A)$.

We now define a family $\mathcal{U}$ on $\widehat{A} \times K_n(A)$ using the Fourier-Mukai transform relative to $K_n(A)$ following [19, Sect. 1]. For this we introduce some notation:

$$\begin{tikzcd} & & A \times \widehat{A} \times K_n(A) \arrow[dl, "p_A"'] \arrow[d, "q"] \arrow[dr, "p_{\widehat{A}}"] & \\ A & A \times K_n(A) \arrow[l, "\pi_1"'] & A \times \widehat{A} & \widehat{A} \times K_n(A) \end{tikzcd}$$

Then the relative Fourier-Mukai transform is defined by:

$$\Psi : \mathrm{D^b}(A \times K_n(A)) \to \mathrm{D^b}(\widehat{A} \times K_n(A)), \quad \mathcal{F} \mapsto Rp_{\widehat{A}*}(q^* \mathcal{P} \otimes p_A^*(\mathcal{F}))$$

Using this we define the following family on $\widehat{A} \times K_n(A)$:

$$\mathcal{U} := \Psi(\mathcal{E}).$$

The restriction of $\mathcal{E}$ to the fiber over $[Z] \in K_n(A)$ is just $I_Z(H)$ which is IT_0 with respect to Φ, implying that $\mathcal{U}$ is WIT_0 and that $\Psi(\mathcal{E})$ commutes with arbitrary base change $T \to K_n(A)$ by [18, Theorem 1.6.]. By choosing $T = \{[Z]\}$ for some $[Z] \in K_n(A)$ we see that there is an isomorphism

$$\mathcal{U} \otimes \mathcal{O}_{[Z]} = \Psi(\mathcal{E}) \otimes \mathcal{O}_{[Z]} \cong \Phi(\mathcal{E} \otimes \mathcal{O}_{[Z]}) \cong \Phi(I_Z(H)).$$

As $\Phi(I_Z(H))$ is locally free the last equation shows that $\mathcal{U}$ is locally free by [15, Lemma 2.1.7].

Furthermore, since $H^1(A, I_Z(H)) = 0$ for all $[Z] \in K_n(A)$ we see, using standard results from the theory of cohomology and base change, that for every morphism $\alpha : S \to \widehat{A} \times K_n(A)$ we get a diagram

$$\begin{tikzcd} A & A \times S \arrow[l, "t_1"'] \arrow[r, "\beta"] \arrow[d, "t_2"'] & A \times \widehat{A} \times K_n(A) \arrow[r, "p_A"] \arrow[d, "p_{\widehat{A}}"] & A \times K_n(A) \\ & S \arrow[r, "\alpha"] & \widehat{A} \times K_n(A) & \end{tikzcd} \tag{9}$$

together with an isomorphism:

$$\begin{aligned} \alpha^*\mathcal{U} &= \alpha^*(Rp_{\widehat{A}*}(q^*\mathcal{P} \otimes p_A^*\mathcal{E})) \\ &\cong Rt_{2*}\beta^*(q^*\mathcal{P} \otimes p_A^*\mathcal{E}). \end{aligned}$$

We sum up the results from this subsection in the following:

Lemma 2.5. *The family $\mathcal{U} = \Psi(\mathcal{E})$ on $\widehat{A} \times K_n(A)$ is a locally free universal family for $K_{\widehat{H}}(w)$, namely, its classifying morphism $K_n(A) \to M_{\widehat{H}}(w)$ induces the isomorphism*

$$K_n(A) \xrightarrow{\cong} K_{\widehat{H}}(w), \ \ [Z] \longmapsto \Phi(I_Z(H)).$$

2.4. **Stability of the wrong-way fibers.** In this section we want to study the stability of the wrong-way fibers of $\mathcal{U}$, that is the fibers over points $\widehat{a} \in \widehat{A}$. For this we choose in the diagram (9) the base change along the inclusion $j_{\widehat{a}}$ of the fiber over $\widehat{a}$ of the projection $\widehat{A} \times K_n(A) \to \widehat{A}$, that is

$$\begin{tikzcd} A & A \times K_n(A) \arrow[l, "t_1"'] \arrow[r, "i_{\widehat{a}}"] \arrow[d, "t_2"'] & A \times \widehat{A} \times K_n(A) \arrow[r, "p_A"] \arrow[d, "p_{\widehat{A}}"] & A \times K_n(A) \\ & K_n(A) \arrow[r, "j_{\widehat{a}}"] & \widehat{A} \times K_n(A) & \end{tikzcd} \tag{10}$$

where the morphisms $j_{\widehat{a}}$ and $i_{\widehat{a}}$ are given on closed points by

$$\begin{aligned} j_{\widehat{a}} &: K_n(A) \hookrightarrow \widehat{A} \times K_n(A), \ \ [Z] \mapsto (\widehat{a}, [Z]) \\ i_{\widehat{a}} &: A \times K_n(A) \hookrightarrow A \times \widehat{A} \times K_n(A), \ \ (a, [Z]) \mapsto (a, \widehat{a}, [Z]). \end{aligned}$$

Going through the base change we see that we can describe the wrong-way fibers in the following way:

$$\begin{aligned} \mathcal{U}_{\widehat{a}} = j_{\widehat{a}}^*\mathcal{U} &= j_{\widehat{a}}^*(Rp_{\widehat{A}*}(q^*\mathcal{P} \otimes p_A^*\mathcal{E}))) \\ &\cong Rt_{2*}i_{\widehat{a}}^*(q^*\mathcal{P} \otimes p_A^*(\mathcal{I}_{\mathcal{Z}} \otimes \pi_1^*\mathcal{O}_A(H))) \\ &\cong Rt_{2*}(\mathcal{I}_{\mathcal{Z}} \otimes t_1^*(\mathcal{P}_{\widehat{a}}(H))). \end{aligned}$$

We recall the integral functor

$$\Theta : \mathrm{D^b}(A) \to \mathrm{D^b}(K_n(A)), \ \ E \mapsto Rt_{2*}(\mathcal{I}_{\mathcal{Z}} \otimes t_1^*E),$$

which is a $\mathbb{P}^{n-1}$-functor by [17, Theorem 4.1].

We see that the wrong-way fiber is given by

$$\mathcal{U}_{\widehat{a}} = \Theta(\mathcal{P}_{\widehat{a}}(H)) \tag{11}$$

and sits in the exact sequence:

$$0 \longrightarrow \mathcal{U}_{\widehat{a}} \longrightarrow Rt_{2*}(t_1^*(\mathcal{P}_{\widehat{a}}(H))) \longrightarrow Rt_{2*}(\mathcal{O}_{\mathcal{Z}} \otimes t_1^*(\mathcal{P}_{\widehat{a}}(H))) \longrightarrow 0 \tag{12}$$

We also have:

$$Rt_{2*}(t_1^*(\mathcal{P}_{\widehat{a}}(H))) \cong H^0(\mathcal{P}_{\widehat{a}}(H)) \otimes \mathcal{O}_{K_n(A)}$$

by cohomology and base change. Furthermore

$$Rt_{2*}(\mathcal{O}_{\mathcal{Z}} \otimes t_1^*(\mathcal{P}_{\widehat{a}}(H))) = (\mathcal{P}_{\widehat{a}}(H))^{(n)}$$

is the tautological bundle of rank $n+1$ on $K_n(A)$ induced by $\mathcal{P}_{\widehat{a}}(H)$.

A quick diagram chase shows that we have

$$(\mathcal{P}_{\widehat{a}}(H))^{(n)} \cong \iota^*((\mathcal{P}_{\widehat{a}}(H))^{[n+1]})$$

where $\iota : K_n(A) \hookrightarrow A^{[n+1]}$ is the inclusion and $(\mathcal{P}_{\widehat{a}}(H))^{[n+1]}$ is the tautological bundle induced by $\mathcal{P}_{\widehat{a}}(H)$ on $A^{[n+1]}$.

For the next results we recall that we have $\mathrm{NS}(K_n(A)) = \mathrm{NS}(A)_K \oplus \mathbb{Z}\delta$. Here D_K is the divisor class on $K_n(A)$ induced by the divisor class D on A and δ is a divisor class on $K_n(A)$ such that $2\delta = [E]$ where E is the exceptional divisor of the Hilbert-Chow morphism $K_n(A) \to S_n(A)$. In our case this reads

$$\mathrm{NS}(K_n(A)) = \mathbb{Z}H_K \oplus \mathbb{Z}\delta.$$

Remark 2.6. Note that we can also write $\mathrm{NS}(K_n(A)) = \iota^* \mathrm{NS}(A^{[n+1]})$, with

$$\mathrm{NS}(A^{[n+1]}) = \mathrm{NS}(A)_{n+1} \oplus \mathbb{Z}\Delta \oplus \Sigma^* \mathrm{NS}(A).$$

where $\mathrm{NS}(A)_{n+1}$ are the divisor classes on $A^{[n+1]}$ induced from A and Δ is the class such that 2Δ is the class of the exceptional divisor of $A^{[n+1]} \to A^{(n+1)}$. We have $\iota^* H_{n+1} = H_K$ and $\iota^*\Delta = \delta$.

Lemma 2.7. *We have* $c_1(\mathcal{U}_{\widehat{a}}) = -H_K + \delta$.

Proof. By the exact sequence (12) we get:

$$\begin{aligned} c_1(\mathcal{U}_{\widehat{a}}) &= -c_1((\mathcal{P}_{\widehat{a}})^{(n)}) \\ &= -c_1(\iota^*((\mathcal{P}_{\widehat{a}}(H))^{[n+1]})) \\ &= -\iota^* c_1((\mathcal{P}_{\widehat{a}}(H))^{[n+1]}) \\ &= -\iota^*(H_{n+1} - \Delta) = -H_K + \delta \end{aligned}$$

where we use [23, Lemma 1.5] in the second to last step. □

To compute slopes on $K_n(A)$ we need the following intersection numbers, which can, for example, be found in [5, 1.2., 1.4.]:

Lemma 2.8. *For the classes H_K and δ from $\mathrm{NS}(K_n(A))$ we have:*

- $H_K^{2n} = \frac{(n+1)(2n)!}{(n)!2^n}(H^2)^n > 0$
- $H_K^{2n-1}\delta = 0$.

Lemma 2.9. *There is an isomorphism*

$$\mathrm{NS}(A) \xrightarrow{\cong} \mathrm{NS}(P_n(A))^{\mathfrak{S}_{n+1}}, \quad H \longmapsto \sum_{i=0}^{n} \tau^* q_i^* H.$$

Proof. We note that $P_n(A)$ is itself an abelian variety (isomorphic to A^n via projection) hence its integral cohomology is torsion free. This implies especially that its Neron-Severi group $\mathrm{NS}(P_n(A))$ is torsion free and hence so is $\mathrm{NS}(P_n(A))^{\mathfrak{S}_{n+1}}$.

Furthermore by [12, Lemma 3] we have an isomorphism

$$\mathrm{NS}(P_n(A))^{\mathfrak{S}_{n+1}} \otimes \mathbb{Q} \cong (\mathrm{NS}(P_n(A)) \otimes \mathbb{Q})^{\mathfrak{S}_{n+1}}.$$

and so it is enough to prove the lemma over the field of rational numbers $\mathbb{Q}$.

We start with the morphisms

$$P_n(A) \xhookrightarrow{\tau} A^{n+1} \xrightarrow{S} A$$

where $S = \sum_{i=0}^{n} q_i$ is the summation morphism using the group law on A.

The natural inclusion τ has the following retract:

$$A^{n+1} \to P_n(A), \ \ (a_0, \ldots, a_n) \mapsto (a_0, \ldots, a_{n-1}, -\sum_{i=0}^{n-1} a_i),$$

which shows that we have a surjection

$$H^2(A^{n+1}, \mathbb{Q}) \xrightarrow{\tau^*} H^2(P_n(A), \mathbb{Q}) \longrightarrow 0$$

As we work over $\mathbb{Q}$ and $\mathfrak{S}_{n+1}$ is finite we get an induced surjection:

$$H^2(A^{n+1}, \mathbb{Q})^{\mathfrak{S}_{n+1}} \xrightarrow{\tau^*} H^2(P_n(A), \mathbb{Q})^{\mathfrak{S}_{n+1}} \longrightarrow 0.$$

It is well known, see for example [16, Theorem 2.15], that:

$$H^2(A^{n+1}, \mathbb{Q})^{\mathfrak{S}_{n+1}} \cong H^2(A, \mathbb{Q}) \oplus \Lambda^2 \left(H^1(A, \mathbb{Q})\right)$$

where the maps are given by:

$$H^2(A, \mathbb{Q}) \hookrightarrow H^2(A^{n+1}, \mathbb{Q})^{\mathfrak{S}_{n+1}}, \ \ c \mapsto \sum_{i=0}^{n} q_i^* c$$

as well as (using $\Lambda^2 \left(H^1(A, \mathbb{Q})\right) \cong H^2(A, \mathbb{Q})$):

$$\Lambda^2 \left(H^1(A, \mathbb{Q})\right) \hookrightarrow H^2(A^{n+1}, \mathbb{Q})^{\mathfrak{S}_{n+1}}, \ \ c \wedge d \mapsto \sum_{i,j} \left(q_i^* c \wedge q_j^* d\right)$$

Now since $S = \sum_{i=0}^{n} q_i$ we get similar to Beauville in [3, Proposition 8.]:

$$\sum_{i,j} \left(q_i^* c \wedge q_j^* d\right) = \left(\sum_{i=0}^{n} q_i^* c\right) \wedge \left(\sum_{j=0}^{n} q_j^* d\right) = S^*(c \wedge d).$$

This implies $\Lambda^2 \left(H^1(A, \mathbb{Q})\right) \cong \mathrm{Im}(S^*)$. But then

$$\tau^*(S^*(c \wedge d)) = (S \circ \tau)^*(c \wedge d) = 0$$

which shows that we have

$$H^2(P_n(A), \mathbb{Q})^{\mathfrak{S}_{n+1}} \cong \tau^* H^2(A, \mathbb{Q}). \tag{13}$$

Using the Lefschetz $(1,1)$-theorem gives

$$(\mathrm{NS}(P_n(A)) \otimes \mathbb{Q})^{\mathfrak{S}_{n+1}} \cong \tau^* \left(\mathrm{NS}(A) \otimes \mathbb{Q}\right),$$

which is what we wanted to prove. □

Proposition 2.10. *The vector bundle* $\mathcal{U}_{\widehat{a}}$ *defined in* (11) *is slope stable with respect to* H_K*.*

Proof. We follow the idea in the proof of [22, Theorem 1.4].

Since $j_K^*(-)$, $(h_\circ^{-1})^*(-)$ and $\sigma_\circ^*(-)$ are exact, and $(j_P)_*$ is left exact, by applying these functors to (12) we obtain an exact sequence of $\mathfrak{S}_{n+1}$-invariant reflexive sheaves on $P_n(A)$ as follows:

$$0 \longrightarrow (\mathcal{U}_{\widehat{a}})_P \longrightarrow (H^0(\mathcal{P}_{\widehat{a}}(H)) \otimes \mathcal{O}_{K_n(A)})_P \xrightarrow{\varphi} (\mathcal{P}_{\widehat{a}}(H))^{(n)})_P$$

where φ is not necessarily surjective. It is clear that

$$(H^0(\mathcal{P}_{\widehat{a}}(H)) \otimes \mathcal{O}_{K_n(A)})_P = H^0(\mathcal{P}_{\widehat{a}}(H)) \otimes \mathcal{O}_{P_n(A)},$$

and we also have

$$((\mathcal{P}_{\widehat{a}}(H))^{(n)})_P = \bigoplus_{i=0}^{n} \tau^* q_i^* \left(\mathcal{P}_{\widehat{a}}(H)\right)$$

by a similar argument as in [22, Lemma 1.1] (see also Proposition 1.5). Hence the above sequence becomes

$$0 \longrightarrow (\mathcal{U}_{\widehat{a}})_P \longrightarrow H^0(\mathcal{P}_{\widehat{a}}(H)) \otimes \mathcal{O}_{P_n(A)} \xrightarrow{\varphi} \bigoplus_{i=0}^{n} \tau^* q_i^* \left(\mathcal{P}_{\widehat{a}}(H)\right) \tag{14}$$

where φ is the evaluation map on $P_n(A)_\circ$.

More precisely, for any set of closed points $(a_0, \dots, a_n) \in P_n(A)$ with $a_i \neq a_j$, the morphism of fibers can be identified as

$$\begin{aligned} \varphi_{(a_0,\dots,a_n)} : H^0(\mathcal{P}_{\widehat{a}}(H)) &\longrightarrow \bigoplus_{i=0}^{n} (\mathcal{P}_{\widehat{a}}(H))_{x_i} \\ s &\longmapsto (s(a_0), \dots, s(a_n)) \end{aligned}$$

Since for any non-trivial $s \in H^0(\mathcal{P}_{\widehat{a}}(H))$, there are always (many choices of) distinct points $(a_0, \dots a_n) \in P_n(A)$ such that $(s(a_0), \dots, s(a_n)) \neq (0, \dots, 0)$, we conclude that the map of global sections

$$H^0(\varphi) : H^0(\mathcal{P}_{\widehat{a}}(H)) \longrightarrow H^0(\bigoplus_{i=0}^{n} \tau^* q_i^* \, \mathcal{P}_{\widehat{a}}(H))$$

is injective. It follows by (14) that $H^0((\mathcal{U}_{\widehat{a}})_P) = 0$.

Note that φ is surjective on $P_n(A)_\circ$, hence $\operatorname{coker}(\varphi)$ is supported on the big diagonal of $P_n(A)$ which is of codimension 2. It follows that

$$c_1((\mathcal{U}_{\widehat{a}})_P) = -\sum_{i=0}^{n} \tau^* q_i^* H.$$

We claim that $(\mathcal{U}_{\widehat{a}})_P$ has no $\mathfrak{S}_{n+1}$-invariant subsheaf which is destabilizing with respect to H_P. Indeed, assume F is an $\mathfrak{S}_{n+1}$-invariant subsheaf of $(\mathcal{U}_{\widehat{a}})_P$, then $c_1(F) \in \mathrm{NS}(P_n(A))^{\mathfrak{S}_{n+1}}$ and thus by Lemma 2.9 we have:

$$c_1(F) = a(\sum_{i=0}^{n} \tau^* q_i^* H) \text{ for some } a \in \mathbb{Z}.$$

If $a \leqslant -1$, then

$$c_1(F) \cdot H_P^{2n-1} \leqslant c_1((\mathcal{U}_{\widehat{a}})_P) \cdot H_P^{2n-1} < 0$$

Since $1 \leqslant \operatorname{rk}(F) < \operatorname{rk}((\mathcal{U}_{\widehat{a}})_P)$, it follows that $\mu_{H_P}(F) < \mu_{H_P}((\mathcal{U}_{\widehat{a}})_P)$, hence F is not destabilizing.

If $a = 0$, we choose a (not necessarily $\mathfrak{S}_{n+1}$-invariant) non-zero stable subsheaf $F' \subseteq F$ which has maximal slope with respect to H_P (e.g. one can take a stable factor in the first Harder-Narasimhan factor of F). Without loss of generality, we can assume F and F' are both reflexive. Since F' is also a subsheaf of the trivial bundle $H^0(\mathcal{P}_{\widehat{a}}(H)) \otimes \mathcal{O}_{P_n(A)}$, there must be a projection from $H^0(\mathcal{P}_{\widehat{a}}(H)) \otimes \mathcal{O}_{P_n(A)}$ to a certain direct summand of it, such that the composition of the embedding and projection $F' \to H^0(\mathcal{P}_{\widehat{a}}(H)) \otimes \mathcal{O}_{P_n(A)} \to \mathcal{O}_{P_n(A)}$ is non-zero. Since $\mu_{P_n(A)}(F') \geqslant \mu_{P_n(A)}(F) = 0 = \mu_{P_n(A)}(\mathcal{O}_{P_n(A)})$, and $\mathcal{O}_{P_n(A)}$ is also stable with respect to H_P, the map $F' \to \mathcal{O}_{P_n(A)}$ must be injective, and its cokernel is supported on a locus of codimension at least 2. Since both are reflexive, we must have $F' = \mathcal{O}_{P_n(A)}$. Therefore F, and consequently $(\mathcal{U}_{\widehat{a}})_P$, have non-trivial global sections. Contradiction.

If $a \geqslant 1$, F would be a subsheaf of the trivial bundle $H^0(\mathcal{P}_{\widehat{a}}(H)) \otimes \mathcal{O}_{P_n(A)}$ of positive slope. Contradiction.

Finally, assume G is a reflexive subsheaf of $\mathcal{U}_{\widehat{a}}$. Then $(G)_P$ is an $\mathfrak{S}_{n+1}$-invariant reflexive subsheaf of $(\mathcal{U}_{\widehat{a}})_P$. By the above claim we have $\mu_{H_P}((G)_P) < \mu_{H_P}((\mathcal{U}_{\widehat{a}})_P)$. It follows from equation (3) that $\mu_{H_K}(G) < \mu_{H_K}(\mathcal{U}_{\widehat{a}})$. Therefore $\mathcal{U}_{\widehat{a}}$ is slope stable with respect to H_K, as desired. $\square$

Proposition 2.11. *There exists some ample class $H' \in \mathrm{NS}(K_n(A))$ near H_K, such that $\mathcal{U}_{\widehat{a}}$ is $\mu_{H'}$-stable for all $\widehat{a} \in \widehat{A}$ simultaneously.*

Proof. By Proposition 1.6 the divisor H_K is lef so that Proposition 2.10 and [6, Theorem 2.3.1] guarantee that the assumptions in [22, Proposition 4.8] are satisfied for each $\mathcal{U}_{\widehat{a}}$. Hence every $\mathcal{U}_{\widehat{a}}$ is slope stable with respect to some ample class near H_K. In order to find a single ample class H' that is independent of the choice of $\mathcal{U}_{\widehat{a}}$, we can use the entire proof of [22, Proposition 4.8] except that we need to reconstruct the non-empty convex open set U so that $\alpha := H_K^{2n-1}$ is in the closure of U, and for every $\gamma \in U$, $\mathcal{U}_{\widehat{a}}$ is stable with respect to γ for all $\widehat{a} \in \widehat{A}$.

We follow the notations in [9, Definition 3.1]. For each $\widehat{a} \in \widehat{A}$, $\mathrm{SStab}(\mathcal{U}_{\widehat{a}})$ is a convex closed set containing α. Hence the intersection

$$\overline{U} := \bigcap_{\widehat{a} \in \widehat{A}} \mathrm{SStab}(\mathcal{U}_{\widehat{a}})$$

is also a convex closed set containing α. We first claim that [9, Theorem 3.4] holds for all $\mathcal{U}_{\widehat{a}}$ simultaneously; namely, we will show that for any $\beta \in \mathrm{Mov}(K_n(A))^{\circ}$ (see [9, Definition 2.1] for the notation), there exists a number $e \in \mathbb{Q}^+$, such that $(\alpha + \varepsilon\beta) \in \cap_{x \in X} \mathrm{Stab}(\mathcal{U}_{\widehat{a}})$ for any real $\varepsilon \in [0, e]$.

To prove the claim, we first note that the slope $c := \mu_\beta(\mathcal{U}_{\widehat{a}})$ is independent of the choice of $\widehat{a} \in \widehat{A}$. We redefine the set S in the proof of [9, Theorem 3.4] to be

$$S := \{c_1(F) \mid F \subseteq \mathcal{U}_{\widehat{a}} \text{ for some } \widehat{a} \in \widehat{A} \text{ such that } \mu_\beta(F) \geqslant c\}.$$

Since $\mathcal{U}_{\widehat{a}} \subseteq H^0(\mathcal{P}_{\widehat{a}}(H)) \otimes \mathcal{O}_{K_n(A)}$ for all $\widehat{a} \in \widehat{A}$, we obtain that S is a subset of

$$T := \{c_1(F) \mid F \subseteq H^0(\mathcal{P}_{\widehat{a}}(H)) \otimes \mathcal{O}_{K_n(A)} \text{ such that } \mu_\beta(F) \geqslant c\},$$

which is finite by [9, Theorem 2.29], hence S is also finite. We can then use the rest of the proof of [9, Theorem 3.4] literally to conclude the claim.

We then claim that $\overline{U}$ is of full dimension $r := \mathrm{rk}\, N_1(K_n(A))$. If not, then we have $\alpha \in \overline{U} \subseteq L$ for some hyperplane $L \subset N_1(K_n(A))_{\mathbb{R}}$. Since $\mathrm{Mov}(K_n(A))$ is of full dimension, we can choose some $\beta \in \mathrm{Mov}(K_n(A))^{\circ} \setminus L$. It follows that $(\alpha + \varepsilon\beta) \in \overline{U} \setminus L$ for some small $\varepsilon > 0$ by the previous claim and the choice of β. Contradiction.

We define U to be the interior of $\overline{U}$ and claim that U is non-empty. Indeed, since $\overline{U}$ is of full dimension r, we can choose $r+1$ points of $\overline{U}$ in general positions, which form an r-simplex. By the convexity of $\overline{U}$, the entire simplex is in $\overline{U}$ hence any interior point of the simplex is also an interior point of $\overline{U}$. The convexity of U follows from the convexity of $\overline{U}$. And it is clear from the construction that $\alpha = H_K^{2n-1}$ is in the closure of U. We finally claim that every $\gamma \in U$ is in $\bigcap_{\widehat{a} \in \widehat{A}} \mathrm{Stab}(\mathcal{U}_{\widehat{a}})$. If not, suppose that there exists some class $\gamma_0 \in U$ and some closed point $\widehat{a}_0 \in \widehat{A}$, such that $\gamma_0 \in \mathrm{SStab}(\mathcal{U}_{\widehat{a}_0}) \setminus \mathrm{Stab}(\mathcal{U}_{\widehat{a}_0})$; namely, $\mu_{\gamma_0}(F) = \mu_{\gamma_0}(\mathcal{U}_{\widehat{a}_0})$ for some proper subsheaf F of $\mathcal{U}_{\widehat{a}_0}$. Since the slope function is linear with respect to the curve class, and $\mu_\alpha(F) < \mu_\alpha(\mathcal{U}_{\widehat{a}_0})$ by Proposition 2.10, one can find a hyperplane in $N^1(K_n(A))_{\mathbb{R}}$ through γ_0, such that $\mu_\gamma(\mathcal{U}_{\widehat{a}_0}) - \mu_\gamma(F)$ takes opposite signs for γ in the two open half-spaces separated by the hyperplane. In particular, F destabilizes $\mathcal{U}_{\widehat{a}_0}$ in one of the half-spaces. Since U has non-empty intersection with both half-spaces, this contradicts the condition $U \subseteq \mathrm{SStab}(\mathcal{U}_{\widehat{a}})$. Therefore we have $U \subseteq \bigcap_{\widehat{a}\in\widehat{A}} \mathrm{Stab}(\mathcal{U}_{\widehat{a}})$, as desired. □

2.5. **A component of the moduli space.** We start this subsection by making a brief digression to consider again the integral functor

$$\Theta\colon \mathrm{D^b}(A) \longrightarrow \mathrm{D^b}(K_n(A))$$

whose kernel is the universal ideal sheaf $\mathcal{I}_{\mathcal{Z}}$ on $A \times K_n(A)$. Recall that Θ is a $\mathbb{P}^{n-1}$-functor, which implies by [2, §2.1] that for any $E, F \in \mathrm{D^b}(A)$ we have an isomorphism of graded vector spaces

$$\mathrm{Ext}^*_{K_n(A)}(\Theta(E), \Theta(F)) \cong \mathrm{Ext}^*_A(E,F) \otimes H^*(\mathbb{P}^{n-1}, \mathbb{C}). \tag{15}$$

We now turn to the main result of the section. Let H' be an ample class that satisfies Proposition 2.11, and $\mathcal{M}$ the moduli space of $\mu_{H'}$-stable sheaves on $K_n(A)$ with the same numerical invariants as $\mathcal{U}_{\widehat{a}}$. Then the universal family $\mathcal{U}$ defines a classifying morphism

$$f\colon \widehat{A} \longrightarrow \mathcal{M}, \quad \widehat{a} \longmapsto [\mathcal{U}_{\widehat{a}}] \tag{16}$$

In fact the morphism f can be described as follows:

Theorem 2.12. *The classifying morphism* (16) *defined by the family* $\mathcal{U}$ *identifies* $\widehat{A}$ *with a smooth connected component of* $\mathcal{M}$.

Proof. By [20, Lemma 1.6.] we have to prove that f is injective on closed points and that $\dim(T_{[\mathcal{U}_{\widehat{a}}]}\mathcal{M}) = 2$ for all $\widehat{a} \in \widehat{A}$.

Now we know $\mathcal{U}_{\widehat{a}} = \Theta(\mathcal{P}_{\widehat{a}}(H))$, so for $\widehat{a}_1 \neq \widehat{a}_2$ we find by (15) that

$$\begin{aligned}\mathrm{Hom}_{K_n(A)}(\mathcal{U}_{\widehat{a}_1}, \mathcal{U}_{\widehat{a}_2}) &= \mathrm{Hom}_{K_n(A)}(\Theta(\mathcal{P}_{\widehat{a}_1}(H)), \Theta(\mathcal{P}_{\widehat{a}_2}(H))) \\ &\cong \mathrm{Hom}_A(\mathcal{P}_{\widehat{a}_1}(H), \mathcal{P}_{\widehat{a}_2}(H)) \\ &\cong H^0(A, \mathcal{P}^\vee_{\widehat{a}_1} \otimes \mathcal{P}_{\widehat{a}_2}) = 0,\end{aligned}$$

where the last step follows from [13, Lemma 9.9]. This implies f is injective on closed points.

A similar computation shows

$$\begin{aligned}\mathrm{Ext}^1_{K_n(A)}(\mathcal{U}_{\widehat{a}}, \mathcal{U}_{\widehat{a}}) &= \mathrm{Ext}^1_{K_n(A)}(\Theta(\mathcal{P}_{\widehat{a}}(H)), \Theta(\mathcal{P}_{\widehat{a}}(H))) \\ &\cong \mathrm{Ext}^1_A(\mathcal{P}_{\widehat{a}}(H), \mathcal{P}_{\widehat{a}}(H)) \\ &\cong \mathrm{Ext}^1_A(\mathcal{P}_{\widehat{a}}, \mathcal{P}_{\widehat{a}}) \\ &\cong \mathrm{Ext}^1_{\widehat{A}}(\mathcal{O}_{\widehat{a}}, \mathcal{O}_{\widehat{a}}) \cong T_{\widehat{a}}\widehat{A}\end{aligned}$$

where the second to last isomorphism uses $\mathcal{P}_{\widehat{a}} \cong \widehat{\Phi}(\mathcal{O}_{\widehat{a}})$ and the fact that $\widehat{\Phi}$ is an equivalence from $\mathrm{D^b}(\widehat{A})$ to $\mathrm{D^b}(A)$.

Using $T_{[\mathcal{U}_{\widehat{a}}]}\mathcal{M} \cong \mathrm{Ext}^1_{K_n(A)}(\mathcal{U}_{\widehat{a}}, \mathcal{U}_{\widehat{a}})$ we find $\dim(T_{[\mathcal{U}_{\widehat{a}}]}\mathcal{M}) = 2$ as desired. □

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