SEMINAR ON CUBIC HYPERSURFACES
TENTATIVE PLAN (WS 2016/17)

The goal for this reading seminar is to learn something about cubic threefolds and fourfolds, with an emphasis on the rationality questions. By popular request, we will focus more on Hodge theory rather than derived categories. The following plan is tentative, and open for discussion. Participants of the seminar should feel free to suggest other topics related to the main theme to be covered in the seminar. We will loosely follow the survey papers [Bea15, Has16]. However at various stages we might also want to check other references where the theorems originally appeared. All meetings take place at room f128 in the main building on Thursday from 10:15 to 11:45 unless otherwise announced.

1. Introduction

We need to define various notions of rationality, in particular rational, stably rational and unirational varieties. There are many examples of unirational varieties, including all smooth cubic hypersurfaces; see [Has16, Proposition 10]. However it is in general difficult to find unirational varieties that are not rational (the so-called Lüroth problem). We will study the classical example due to Artin and Mumford; see [AM72], also [Bea15, Section 6].

- Talk 1: Definitions; unirationality of smooth cubic hypersurfaces; and the example of Artin–Mumford. (27th October, Speaker: Ziyu Zhang)

2. Cubic Threefolds

Clemens and Griffiths proved in [CG72] that a smooth cubic threefold is not rational. They established a necessary condition for a smooth projective threefold to be rational, namely its intermediate Jacobian has to be the Jacobian of a curve, or a product of them. For a smooth cubic threefold X, it remains to show that its intermediate Jacobian $J(X)$ is not of this form.

Using the intermediately Jacobian $J(X)$, Clemens and Griffiths also proved a global Torelli theorem for smooth cubic threefolds. Namely, a smooth cubic threefold X is completely determined by $J(X)$. Moreover, we will see that $J(X)$ is in fact the Albanese of the Fano variety of lines $F(X)$, which relates the Hodge theory of X and $F(X)$.

The main references for this topic are [Bea81] and [Bea15, Section 3]; see also [CG72].

- Talk 2: Introduce the intermediate Jacobian; prove the Clemens–Griffiths criterion; following [Bea15, Section 3.1] (maybe also [Bea15, Section 3.3]). (3rd November, Speaker: Tobias Heckel)
- Talk 3: Prove that the Θ-divisor in $J(X)$ has a unique singular point, which fails the Clemens–Griffiths criterion; see [Bea15, Section 3.2] and [Bea81, Sections 2–3]. (10th November, Speaker: Víctor González-Alonso)
- Talk 4: Use the tangent cone of the Θ-divisor to prove the global Torelli theorem for cubic threefolds; see [Bea15, Section 3.2] and [Bea81, Sections 4–5]. (17th November, Speaker: Roberto Laface)

3. Cubic Fourfolds

The (ir)rationality of smooth cubic fourfolds is an open question and much more difficult. There are two main conjectures in this direction: Hassett's conjecture in [Has00] using Hodge theory and Kuznetsov's conjecture in [Kuz10] using derived categories. It was proved in a recent paper [AT14] that the two conjectures are almost equivalent. We will mostly focus on Hassett's approach.

Roughly speaking, Hassett characterized some special cubic fourfolds by the Hodge structure on their middle cohomology. These special cubic fourfolds form countably many irreducible divisors $\mathcal{C}_d$ in the

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moduli space of cubic fourfolds, labeled by a single integer parameter d, satisfying the condition

$$d > 6 \quad \text{and} \quad d \equiv 0 \text{ or } 2 \pmod 6. \tag{1}$$

Indeed, some of these divisors are more special than others. Namely, when d further satisfies the condition

$$d \text{ is not divisible by } 4, 9, \text{ or any prime } p \equiv 2 \pmod 3, \tag{2}$$

then every cubic fourfold in $\mathcal{C}_d$ admits an associated K3 surface; and vice verse. Hassett's conjecture states that a smooth cubic fourfold X is rational if and only if it admits an associated K3 surface; or equivalently, when $X \in \mathcal{C}_d$ for some d satisfying (1) and (2). This conjecture is still widely open: many examples of rational cubic fourfolds have been found, but no cubic fourfold has yet been shown to be irrational.

On the other hand, Kuznetsov's conjecture is parallel. Given a smooth cubic fourfold X, he showed that a certain component $\mathcal{A}(X)$ of its derived category $D^b(X)$ is a K3-type category, but not necessarily the derived category of a genuine K3 surface. He conjectured that this holds if and only if X is rational.

We will spend most of talks on this fascinating topic of cubic fourfolds. We will mainly follow the presentation in [Has16] but include more details, which can be found in Hassett's original papers [Has99, Has00].

- Talk 5: Some examples of rational cubic fourfolds; follow [Has16, Section 1]. (24th November, Speaker: Piotr Pokora)
- Talk 6: Hodge diamond of cubic fourfolds and special cubic fourfolds; follow [Has16, Section 2.1–2.2]. (1st December, Speaker: TBA)
- Talk 7: Moduli space of cubic fourfolds and loci of special cubic fourfolds (establish condition (1) as above); follow [Has16, Section 2.3–2.4]. (8th December, Speaker: TBA)
- Talk 8: Define the K3 surface associated to a cubic fourfold (using Hodge structure). Which cubic fourfolds admit associated K3 surfaces (establish condition (2) as above); follow [Has16, Section 3.1–3.2] or more. (15th December, Speaker: Carsten Liese)
- Talk 9: In some examples, the associated K3 surfaces can be explicitly constructed in a rather geometric way without using Hodge structure, instead, using Fano variety of lines; follow [Has16, Section 3.5], or even some content in [Has16, Section 3.6–3.7] if the speaker is enthusiastic enough. (Speaker: TBA)
- Talk 10: A brief introduction to Kuznetsov's conjecture, and the relation between Hassett and Kuznetsov's conjectures. (Speaker: TBA)
- Talk 11: A very recent development is the decomposition of the diagonal due to Voisin in a series of papers. It would be nice if someone could give an introduction to this; see [Has16, Section 5] and the references therein. (Speaker: TBA)

References

[AM72] Artin, M., Mumford, D., *Some elementary examples of unirational varieties which are not rational.* Proc. London Math. Soc. (3) 25 (1972), 75–95. http://plms.oxfordjournals.org/content/s3-25/1/75.full.pdf

[AT14] Nicolas Addington, Richard Thomas, *Hodge theory and derived categories of cubic fourfolds.* Duke Math. J. 163 (2014), no. 10, 1885–1927. http://projecteuclid.org/euclid.dmj/1404824304

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