STABILITY OF SOME VECTOR BUNDLES ON HILBERT SCHEMES OF POINTS ON K3 SURFACES

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Abstract. Let X be a projective K3 surfaces. In two examples where there exists a fine moduli space M of stable vector bundles on X, isomorphic to a Hilbert scheme of points, we prove that the universal family $\mathcal{E}$ on $X \times M$ can be understood as a complete flat family of stable vector bundles on M parametrized by X, which identifies X with a smooth connected component of some moduli space of stable sheaves on M.

Introduction

Let X be a projective K3 surface, and M a moduli space of semistable sheaves on X. By Mukai's seminal work [16], when M is smooth, it is an example of the so-called irreducible holomorphic symplectic manifolds, which are an important class of building blocks in the classification of compact Kähler manifolds with trivial first Chern class. It is then an interesting question to understand whether the moduli spaces $\mathcal{M}$ of semistable sheaves on M inherit any good properties from M. This paper grew out of an attempt to study this question. When $\dim M > 2$, we cannot expect $\mathcal{M}$ to carry a holomorphic symplectic structure in general, because the Serre duality does not induce a non-degenerate anti-symmetric pairing on the tangent space of $\mathcal{M}$ any more, as opposed to the case of K3 surfaces; however, some components of $\mathcal{M}$ may nevertheless be holomorphic symplectic.

In order to study this question, we need to classify all semistable sheaves on M with fixed Chern classes, which seems difficult in general when $\dim M > 2$; it is even a challenging question to construct any non-trivial examples of semistable sheaves on M, due to the fact that stability is difficult to check on higher dimensional varieties in general. When M is a Hilbert scheme of points on the K3 surface X, a natural family of vector bundles on M for considering stability are the so-called tautological bundles, which were proven to be stable with respect to a suitable choice of an ample line bundle on M by Schlickewei [20], Wandel [23] and Stapleton [21]. In fact, Wandel proved that, under some mild assumptions, the connected component of the moduli space containing the tautological bundles is isomorphic to some moduli space of vector bundles on the underlying K3 surface X.

There is another way to construct examples of stable sheaves on M. Assuming that M is a fine moduli space of stable sheaves on X with a universal family $\mathcal{E}$ on $X \times M$, and denoting the "wrong-way fiber" $\mathcal{E}|_{\{x\}\times M}$ by $\mathcal{E}_x$ for each closed point $x \in X$, we can ask the following questions:

- Is $\mathcal{E}$ also a flat family of coherent sheaves on M parametrized by X?
- If so, are the "wrong-way" fibers $\mathcal{E}_x$ stable sheaves on M with respect to some suitable choice of an ample line bundle for every closed point $x \in X$?
- If so, can we identify X with a connected component of the corresponding moduli space of stable sheaves on M?

This idea has also been explored in the literature. In [19], the authors studied some families of ideal sheaves and torsion sheaves of pure dimension 1, and obtained an affirmative answer to the above questions in these cases. A systematic study of the above questions in the case of locally free sheaves was carried out in the very interesting and inspiring thesis of Wray [24]. In order to get around the difficulty of proving stability directly, he invoked the very deep and powerful technique of Hitchin-Kobayashi correspondence to translate the stability problem to the existence of some Hermitian-Einstein metrics, which was then solved by analytic methods to give affirmative answers to the above questions.

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The present paper is devoted to study the above questions, in particular the stability of wrong-way fibers $\mathcal{E}_x$ with respect to a polarization near the boundary of the ample cone of M, in the very classical way by showing that every proper subsheaf of $\mathcal{E}_x$ of a smaller rank has a smaller slope. We will focus on two special cases, namely a projective K3 surface X along with a Mukai vector v such that either

- $\mathrm{NS}(X) = \mathbb{Z}h$ with $h^2 = 4k$ and $v = (k+1, -h, 1)$ for any $k \geqslant 1$; or
- $\mathrm{NS}(X) = \mathbb{Z}e \oplus \mathbb{Z}f$ with the intersection matrix given by $\begin{pmatrix} -2k & 2k+1 \\ 2k+1 & 0 \end{pmatrix}$ and $v = (2k-1, e + (2k-1)f, 2k)$ for any $k \geqslant 2$.

We summarize our main results in the following theorem:

Theorem. *For any projective K3 surface X satisfying either of the above conditions,*

(1) we can explicitly construct a fine moduli space M of stable vector bundles of Mukai vector v on X, isomorphic to the Hilbert scheme of k points on X, along with a universal family $\mathcal{E}$ (see Theorem 1.2 and Theorem 2.6);

(2) there exists an ample divisor H on M such that $\mathcal{E}$ can be regarded as a flat family of μ_H-stable vector bundles on M parametrized by X (see Proposition 1.8 and Proposition 2.15);

(3) the classifying morphism induced by the family $\mathcal{E}$ identifies X with a smooth connected component of a moduli space of μ_H-stable sheaves on M (see Theorem 1.10 and Theorem 2.21).

Let us briefly explain how we achieved the above results. Our choices of the K3 surfaces and the Mukai vectors, as well as the explicit constructions of the moduli space M and the universal family $\mathcal{E}$ in the above two cases, are motivated by [11, Example 5.3.7] and [17, Theorem 1.2] respectively. In fact, in both cases, the stable sheaves on X are given by the spherical twist (or its inverse) of the ideal sheaves of k points on X around $\mathcal{O}_X$, hence their corresponding moduli spaces M are isomorphic to the Hilbert scheme $X^{[k]}$ of k points on X. To show the slope stability of the wrong-way fibers $\mathcal{E}_x$ with respect to some ample divisor H on M, we apply the technique developed by Stapleton [21]; namely, we first prove the slope stability of $\mathcal{E}_x$ with respect to a natural nef divisor on M by passing to the k-fold product of X, then use the openness of stability to perturb the nef divisor to a nearby ample divisor. In fact, since the perturbation argument in [21] works only for individual sheaves, we need to generalize it so as to find an ample divisor H with respect to which all $\mathcal{E}_x$'s are simultaneously stable. Finally, to identify X as a smooth connected component of some moduli space of stable sheaves on M, we interpret $\mathcal{E}_x$'s as images of some sheaves or derived objects on X under the integral functor Φ induced by the universal ideal sheaf for $X^{[k]}$. By the fundamental result of Addington [1] that Φ is a $\mathbb{P}^{k-1}$-functor, we can obtain, by computing the relevant cohomology groups, that $\mathcal{E}_x$'s are distinct and the tangent space of deformations of each $\mathcal{E}_x$ is of dimension 2, which leads immediately to the conclusion.

The text is organized in two sections, which deal with the two cases mentioned above respectively. All objects in this text are defined over the field of complex numbers $\mathbb{C}$.

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1. K3 surfaces with Picard number one

Throughout this section we assume X is a K3 surface such that $\mathrm{NS}(X) = \mathbb{Z}h$, where h is an ample class with $h^2 = 4k$. We denote the line bundle associated to h by $\mathcal{O}_X(1)$ and the Hilbert scheme of length k subschemes of X by $X^{[k]}$.

1.1. Explicit construction of a universal family.

In this section we generalize [11, Example 5.3.7] to give an explicit construction of a universal family of stable vector bundles on X parametrized by the Hilbert scheme $X^{[k]}$ for $k \geqslant 1$. We consider the moduli space $M_h(v)$ of μ_h-stable sheaves on X with Mukai vector $v = (k+1, -h, 1)$. We note that μ_h-semistability implies μ_h-stability for any rank

$k+1$ sheaf E with $v(E)=v$ as $\gcd(k+1,1)=1$ by [11, Remark 4.6.8]. Thus $M_h(v)$ is a smooth projective variety. We compute:

$$\dim(M_h(v)) = v^2+2 = 4k-2(k+1)+2 = 2k.$$

Furthermore $v'=(k+1,-h,a)$ with $a \geqslant 2$ satisfies

$$v'^2+2 = 4k-2a(k+1)+2 \leqslant 4k-4(k+1)+2 = -2 < 0,$$

and thus the second Chern class is minimal (here $c_2(E)=3k$). This minimality implies that every point $[E]$ in $M_h(v)$ is locally free. The condition $\gcd(k+1,1)=1$ also implies that $M_h(v)$ is a fine moduli space.

The following lemma produces examples of elements in this moduli space:

Lemma 1.1. *For any $[Z] \in X^{[k]}$ the sheaf $I_Z(1)$ is globally generated, that is the evaluation morphism*

$$\mathrm{ev}: H^0(I_Z(1)) \otimes \mathcal{O}_X \to I_Z(1)$$

is surjective. Furthermore $E_Z := \ker(\mathrm{ev})$ is a μ_h-stable locally free sheaf with Mukai vector $v(E_Z)=(k+1,-h,1)$.

Proof. The standard exact sequence

$$0 \longrightarrow I_Z(1) \longrightarrow \mathcal{O}_X(1) \longrightarrow \mathcal{O}_Z(1) \longrightarrow 0 \tag{1}$$

shows

$$\chi(I_Z(1)) = \chi(\mathcal{O}_X(1)) - \chi(\mathcal{O}_Z(1)) = (2k+2)-k = k+2.$$

Since Z has codimension two in X, using Serre duality gives

$$H^2(I_Z(1)) \cong \mathrm{Hom}(I_Z(1),\mathcal{O}_X)^\vee \cong H^0(\mathcal{O}_X(-1))^\vee = 0.$$

By [6, Proposition 3.7], the line bundle $\mathcal{O}_X(1)$ is k-very ample which implies that the exact sequence of global sections attached to (1)

$$0 \longrightarrow H^0(I_Z(1)) \longrightarrow H^0(\mathcal{O}_X(1)) \longrightarrow H^0(\mathcal{O}_Z(1)) \longrightarrow 0$$

is still exact. This implies $H^1(I_Z(1)) \cong H^1(\mathcal{O}_X(1)) = 0$ and thus

$$\dim(H^0(I_Z(1))) = \chi(I_Z(1)) = k+2.$$

Now if the evaluation map is not surjective, let $Q := \mathrm{coker}(\mathrm{ev})$ and pick $x \in \mathrm{supp}(Q)$. Then we have an exact sequence

$$0 \longrightarrow I_{Z'}(1) \longrightarrow I_Z(1) \longrightarrow \mathcal{O}_x \longrightarrow 0$$

for a length $k+1$ subscheme Z' containing Z.

Since $I_Z(1)$ is not globally generated at x the last exact sequence gives isomorphisms

$$H^0(I_{Z'}(1)) \cong H^0(I_Z(1)) \text{ and } H^1(I_{Z'}(1)) \cong H^0(\mathcal{O}_x) \neq 0.$$

But $\mathcal{O}_X(1)$ is k-very ample so by definition

$$0 \longrightarrow H^0(I_{Z'}(1)) \longrightarrow H^0(\mathcal{O}_X(1)) \longrightarrow H^0(\mathcal{O}_{Z'}(1)) \longrightarrow 0$$

is still exact, which implies $H^1(I_{Z'}(1)) = 0$, a contradiction. So ev is indeed surjective and we have an exact sequence:

$$0 \longrightarrow E_Z \longrightarrow H^0(I_Z(1)) \otimes \mathcal{O}_X \longrightarrow I_Z(1) \longrightarrow 0.$$

Computing invariants shows $\mathrm{rk}(E_Z) = k+1$, $c_1(E_Z) = -h$ and $c_2(E_Z) = 3k$, hence $v(E_Z) = (k+1,-h,1)$. The sheaf E_Z is locally free as it is the kernel of a morphism between a locally free and a torsion free sheaf on a smooth surface. The stability of E_Z follows from [25, Lemma 2.1 (2-2)]. □

This shows that there is a map $X^{[k]} \to M_h(v)$. In fact we have:

Theorem 1.2. *There is an isomorphism $X^{[k]} \cong M_h(v)$.*

Proof. For any $[E] \in M_h(v)$ we have $h^0(E) = 0$ by stability. Hirzebruch-Riemann-Roch gives $\chi(E) = k + 2$ and hence

$$\hom(E, \mathcal{O}_X) = h^2(E) \geqslant k + 2.$$

Let $\varphi_i : E \to \mathcal{O}_X$ for $1 \leqslant i \leqslant k+2$ be linearly independent homomorphisms and define

$$\varphi := \bigoplus_{i=1}^{k+2} \varphi_i : E \to \mathcal{O}_X^{\oplus k+2}.$$

We claim φ fits into a short exact sequence of the form

$$0 \longrightarrow E \xrightarrow{\varphi} \mathcal{O}_X^{\oplus k+2} \longrightarrow I_Z(1) \longrightarrow 0$$

where I_Z is the ideal sheaf of some $[Z] \in X^{[k]}$. If φ were not injective, then we would have an exact sequence:

$$0 \longrightarrow \ker(\varphi) \longrightarrow E \longrightarrow \operatorname{im}(\varphi) \longrightarrow 0$$

with $\operatorname{im}(\varphi) \hookrightarrow \mathcal{O}_X^{\oplus k+2}$ of rank $1 \leqslant r \leqslant k$.

Since E is μ_h-stable and $\mathcal{O}_X^{\oplus k+2}$ is μ_h-semistable we have the following inequalities:

$$-\frac{4k}{k+1} = \mu(E) < \mu(\operatorname{im}(\varphi)) \leqslant \mu(\mathcal{O}_X^{\oplus k+2}) = 0.$$

If $c_1(\operatorname{im}(\varphi)) = \mathcal{O}_X(a)$ these inequalities read:

$$-\frac{4k}{k+1} < \frac{4ak}{r} \leqslant 0 \Leftrightarrow -\frac{r}{k+1} < a \leqslant 0$$

which implies $a = 0$. The same arguments as in [9, Theorem 5.4 (ii), Step 3.] show that the torsion free sheaf $\operatorname{im}(\varphi)$ embeds into $\mathcal{O}_X^{\oplus r}$ with $1 \leqslant r \leqslant k$. But then

$$\varphi : E \to \operatorname{im}(\varphi) \subset \mathcal{O}_X^{\oplus r} \subset \mathcal{O}_X^{\oplus k+2}$$

and hence the φ_i would only span an $r \leqslant k$-dimensional subspace of $\operatorname{Hom}(E, \mathcal{O}_X)$, which contradicts our choice. Therefore φ is injective.

A Chern class computation shows $c_1(\operatorname{coker}(\varphi)) = h$ and $c_2(\operatorname{coker}(\varphi)) = k$. Since $\operatorname{rk}(\operatorname{coker}(\varphi)) = 1$ it is enough to show that $\operatorname{coker}(\varphi)$ is torsion free. If not, let E' be the saturation of E in $\mathcal{O}_X^{\oplus k+2}$. Then E' is a rank $k+1$ vector bundle as well and

$$\det(E) \subset \det(E') \cong \mathcal{O}_X(b) \subset \Lambda^{n+1}(\mathcal{O}_X^{\oplus k+2})$$

for some $-1 \leqslant b \leqslant 0$. Since both E and E' are locally free, $\det(E) \ncong \det(E')$; hence $b = 0$. The quotient $\mathcal{O}_X^{\oplus k+2}/E'$ then is necessarily of the form I_W for a codimension two subscheme W. But $\operatorname{Hom}(\mathcal{O}_X, I_W) = 0$ unless $W = \emptyset$ which then implies $E' \cong \mathcal{O}_X^{\oplus k+1}$ contradicting again the linear independence of the φ_i. Eventually, we see that indeed any $[E] \in M_h(v)$ is part of a short exact of the form:

$$0 \longrightarrow E \longrightarrow \mathcal{O}_X^{\oplus k+2} \longrightarrow I_Z(1) \longrightarrow 0.$$

Since by stability $H^0(E) = 0$ the map $H^0(\mathcal{O}_X^{\oplus k+2}) \to H^0(I_Z(1)) \cong \mathbb{C}^{\oplus k+2}$ is bijective, thus $\operatorname{Ext}^1(E, \mathcal{O}_X) = H^1(E) = 0$. So by Hirzebruch-Riemann-Roch we have $\hom(E, \mathcal{O}_X) = k+2$ and thus φ (and the exact sequence) is uniquely determined by $[E]$ (up to the action of $\mathrm{GL}(k+2)$).

On the other hand, if we start with $[Z] \in X^{[k]}$ and denote the kernel of the (surjective) evaluation morphism

$$H^0(I_Z(1)) \otimes \mathcal{O}_X \to I_Z(1)$$

by E_Z, then $[E_Z] \in M_h(v)$ by Lemma 1.1.

In order to globalize this construction let $\mathcal{Z} \subset X \times X^{[k]}$ denote the universal length k subscheme, $\mathcal{I}_{\mathcal{Z}}$ its ideal sheaf. We have projections $p : X \times X^{[k]} \to X^{[k]}$ as well as $q : X \times X^{[k]} \to X$. Define a sheaf $\mathcal{E}$ on $X \times X^{[k]}$ by the exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow p^*(p_*(\mathcal{I}_{\mathcal{Z}} \otimes q^*\mathcal{O}_X(1))) \longrightarrow \mathcal{I}_{\mathcal{Z}} \otimes q^*\mathcal{O}_X(1) \longrightarrow 0. \quad (2)$$

Then $\mathcal{E}$ is p-flat and $\mathcal{E}_{|p^{-1}(Z)} \cong E_Z$ (which implies $\mathcal{E}$ is locally free on $X \times X^{[k]}$). Thus $\mathcal{E}$ defines a classifying morphism

$$X^{[k]} \to M_h(v),\ [Z] \mapsto [E_Z]$$

The considerations above show that this map is surjective, because any E is part of an exact sequence of this form, and injective, because φ is uniquely determined by E. Since both spaces are smooth and irreducible of dimension $2k$ the morphism $X^{[k]} \to M_h(v)$ is an isomorphism. □

Remark 1.3. The locally free sheaf E_Z is nothing but $T_{\mathcal{O}_X}(I_Z(1))[1]$ the shift of the spherical twist of $I_Z(1)$ around $\mathcal{O}_X$, as noted in [10, Example 10.3.6] for the case $k = 1$. See also [24, Section 4.5] for a proof of this theorem using the spherical twist.

1.2. **Stability of wrong-way fibers.**

In the above section, we explicitly constructed a universal family $\mathcal{E}$, which is a locally free sheaf on $X \times X^{[k]}$. In this section we take the alternative point of view and consider $\mathcal{E}$ as a family of vector bundles on $X^{[k]}$ parametrized by X. A "wrong-way fiber" of $\mathcal{E}$ is just the restriction of $\mathcal{E}$ over a point $x \in X$ which gives a locally free sheaf on $X^{[k]}$.

More precisely, we first note that by standard cohomology and base change arguments

$$p_*(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1)) \otimes \mathcal{O}_{[Z]} \to H^0(I_Z(1))$$

is an isomorphism. Hence

$$K := p_*(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1)) \tag{3}$$

is a locally free sheaf of rank $k+2$ on $X^{[k]}$. This implies that $\mathcal{E}$ is not only p-flat, but also q-flat since $\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1)$ is both p- and q-flat by [14, Theorem 2.1.]. Thus we can restrict the exact sequence (2) to the fiber over a point $x \in X$ and get the following description of the fiber $E_x := \mathcal{E}_{|q^{-1}(x)}$:

$$0 \longrightarrow E_x \longrightarrow K \longrightarrow I_{S_x} \longrightarrow 0, \tag{4}$$

where $S_x := \{[Z] \in X^{[k]} \mid x \in \operatorname{supp}(Z)\}$ is a codimension 2 subscheme of $X^{[k]}$. Hence E_x is a locally free sheaf of rank $k+1$ on $X^{[k]}$.

Before proving the stability of E_x with respect to some ample class $H \in \mathrm{NS}(X^{[k]})$, we recall that for any coherent sheaf F on X there is the associated coherent *tautological sheaf* $F^{[k]}$ on $X^{[k]}$ defined by

$$F^{[k]} := p_*\left(q^* F \otimes \mathcal{O}_{\mathcal{Z}}\right). \tag{5}$$

If F is locally free of rank r then $F^{[k]}$ is locally free of rank kr.

Also recall the well-known fact that $\mathrm{NS}(X^{[k]}) = \mathrm{NS}(X)_k \oplus \mathbb{Z}\delta$. Here d_k is the divisor class on $X^{[k]}$ induced by the divisor class d on X and δ is a divisor class on $X^{[k]}$ such that $2\delta = [E]$ where E is the exceptional divisor of the Hilbert-Chow morphism $X^{[k]} \to X^{(k)}$. In our case this reads

$$\mathrm{NS}(X^{[k]}) = \mathbb{Z}h_k \oplus \mathbb{Z}\delta.$$

Lemma 1.4. *We have $c_1(E_x) = -h_k + \delta$, $H^0(E_x) = 0$ and E_x is a subsheaf of a free sheaf.*

Proof. We have the exact sequence:

$$0 \longrightarrow p_*(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1)) \longrightarrow p_* q^* \mathcal{O}_X(1) \longrightarrow p_*(\mathcal{O}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1)) \longrightarrow 0$$

as $R^1 p_*(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1)) = 0$ since $H^1(I_Z(1)) = 0$ for all $[Z] \in X^{[k]}$.

We also have

$$p_* q^* \mathcal{O}_X(1) \cong H^0(\mathcal{O}_X(1)) \otimes \mathcal{O}_{X^{[k]}}$$

and the sheaf $p_*(\mathcal{O}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1))$ is nothing but the tautological sheaf $\mathcal{O}_X(1)^{[k]}$ associated to $\mathcal{O}_X(1)$ on $X^{[k]}$. By [13, Remark 3.20.] we also have $H^0(\mathcal{O}_X(1)^{[k]}) = H^0(\mathcal{O}_X(1))$. The last exact sequence thus gets

$$0 \longrightarrow K \longrightarrow H^0(\mathcal{O}_X(1)^{[k]}) \otimes \mathcal{O}_{X^{[k]}} \longrightarrow \mathcal{O}_X(1)^{[k]} \longrightarrow 0. \tag{6}$$

Using [23, Lemma 1.5.] we get

$$c_1(K) = -c_1(\mathcal{O}_X(1)^{[k]}) = -h_k + \delta$$

and since K is the kernel of the evaluation morphism of $\mathcal{O}_X(1)^{[k]}$ we have

$$H^0(K) = 0.$$

Using (4) we see

$$c_1(E_x) = c_1(K) = -h_k + \delta$$

as well as $H^0(E_x) = 0$ and realize E_x is a subsheaf of the free sheaf $H^0(\mathcal{O}_X(1)) \otimes \mathcal{O}_{X^{[k]}}$. □

To compute slopes on $X^{[k]}$ we need the following intersection numbers, which can, for example, be found in [23, Lemma 1.10]:

Lemma 1.5. *For the classes h_k and δ from $\mathrm{NS}(X^{[k]})$ we have:*

- $h_k^{2k} = \frac{(2k-1)!}{(k-1)!2^{k-1}}(h^2)^k = \frac{(2k-1)!2^{k+1}}{(k-1)!}k^k > 0$
- $h_k^{2k-1}\delta = 0.$

We also recall the notations introduced in [21, §1]. The ample divisor h on X naturally induces an ample divisor

$$h_{X^k} = \bigoplus_{i=1}^{k} q_i^* h$$

on X^k, where q_i are projections from X^k to the factors, as well as a semi-ample divisor h_k on $X^{[k]}$.

Moreover, we write $X^k_\circ$, $\mathrm{Sym}^k(X)_\circ$ and $X^{[k]}_\circ$ for the loci of the relevant spaces parametrizing distinct points. Then the natural map

$$\overline{\sigma}_\circ : X^k_\circ \to X^{[k]}_\circ$$

is an étale cover and $j : X^k_\circ \to X^k$ is an open embedding. For any coherent sheaf F on $X^{[k]}$, we denote by $F_\circ$ the restriction of F on $X^{[k]}_\circ$, and define

$$(F)_{X^k} = j_*(\overline{\sigma}^*_\circ(F_\circ))$$

which is a torsion free coherent sheaf if F is.

Proposition 1.6. *The vector bundle K defined in* (3) *is slope stable with respect to h_k.*

Proof. We follow the idea in the proof of [21, Theorem 1.4].

Since $(-)_\circ$ and $\overline{\sigma}^*_\circ(-)$ are exact, and $j_*(-)$ is left exact, by applying these functors to (6) we obtain an exact sequence of $\mathfrak{S}_n$-invariant reflexive sheaves on X^k as follows

$$0 \longrightarrow (K)_{X^k} \longrightarrow (H^0(\mathcal{O}_X(1)) \otimes \mathcal{O}_{X^{[k]}})_{X^k} \xrightarrow{\varphi} (\mathcal{O}_X(1)^{[k]})_{X^k}$$

where φ is not necessarily surjective. It is clear that

$$(H^0(\mathcal{O}_X(1)) \otimes \mathcal{O}_{X^{[k]}})_{X^k} = H^0(\mathcal{O}_X(1)) \otimes \mathcal{O}_{X^k},$$

and we also have

$$(\mathcal{O}_X(1)^{[k]})_{X^k} = \bigoplus_{i=1}^{k} q_i^*\,\mathcal{O}_X(1)$$

by [21, Lemma 1.1]. Hence the above sequence becomes

$$0 \longrightarrow (K)_{X^k} \longrightarrow H^0(\mathcal{O}_X(1)) \otimes \mathcal{O}_{X^k} \xrightarrow{\varphi} \bigoplus_{i=1}^{k} q_i^*\,\mathcal{O}_X(1) \tag{7}$$

where φ is the evaluation map on $X^k_\circ$.

More precisely, for any set of closed points $(x_1,\dots,x_n)\in X^k$ with $x_i\neq x_j$, the morphism of fibers can be identified as

$$\begin{aligned}\varphi_{(x_1,\dots,x_k)}: H^0(\mathcal{O}_X(1)) &\longrightarrow \bigoplus_{i=1}^k \mathcal{O}_X(1)_{x_i}\\ s &\longmapsto (s(x_1),\dots,s(x_k))\end{aligned}$$

Since for any non-trivial $s\in H^0(\mathcal{O}_X(1))$, there are always (many choices of) distinct points $(x_1,\dots x_k)\in X^k$ such that $(s(x_1),\dots,s(x_k))\neq(0,\dots,0)$, we conclude that the map of global sections

$$H^0(\varphi): H^0(\mathcal{O}_X(1))\longrightarrow H^0(\bigoplus_{i=1}^k q_i^*\,\mathcal{O}_X(1))$$

is injective. It follows by (7) that $H^0((K)_{X^k})=0$.

Note that φ is surjective on $X^k_\circ$, hence $\operatorname{coker}(\varphi)$ is supported on the big diagonal of X^k which is of codimension 2. It follows that

$$c_1((K)_{X^k})=-\sum_{i=1}^k q_i^*h.$$

We claim that $(K)_{X^k}$ has no $\mathfrak{S}_k$-invariant subsheaf which is destabilizing with respect to h_{X^k}. Indeed, assume F is an $\mathfrak{S}_k$-invariant subsheaf of $(K)_{X^k}$, then for some $a\in\mathbb{Z}$:

$$c_1(F)=a(\sum_{i=1}^k q_i^*h).$$

If $a\leqslant -1$, then

$$c_1(F)\cdot h_{X^k}^{2k-1}\leqslant c_1((K)_{X^k})\cdot h_{X^k}^{2k-1}<0$$

Since $1\leqslant \operatorname{rk}(F)<\operatorname{rk}((K)_{X^k})$, it follows that $\mu_{h_{X^k}}(F)<\mu_{h_{X^k}}((K)_{X^k})$, hence F is not destabilizing.

If $a=0$, we choose a (not necessarily $\mathfrak{S}_k$-invariant) non-zero stable subsheaf $F'\subseteq F$ which has maximal slope with respect to h_{X^k} (e.g. one can take a stable factor in the first Harder-Narasimhan factor of F). Without loss of generality, we can assume F and F' are both reflexive. Since F' is also a subsheaf of $H^0(\mathcal{O}_X(1))\otimes\mathcal{O}_{X^k}$, there must be a projection from $H^0(\mathcal{O}_X(1))\otimes\mathcal{O}_{X^k}$ to a certain direct summand of it, such that the composition of the embedding and projection $F'\to H^0(\mathcal{O}_X(1))\otimes\mathcal{O}_{X^k}\to\mathcal{O}_{X^k}$ is non-zero. Since $\mu_{X^k}(F')\geqslant\mu_{X^k}(F)=0=\mu_{X^k}(\mathcal{O}_{X^k})$, and $\mathcal{O}_{X^k}$ is also stable with respect to h_{X^k}, the map $F'\to\mathcal{O}_{X^k}$ must be injective, and its cokernel is supported on a locus of codimension at least 2. Since both are reflexive, we must have $F'=\mathcal{O}_{X^k}$. Therefore F, and consequently $(K)_{X^k}$, have non-trivial global sections. Contradiction.

If $a\geqslant 1$, F would be a subsheaf of the trivial bundle $H^0(\mathcal{O}_X(1))\otimes\mathcal{O}_{X^k}$ of positive slope. Contradiction.

Finally, assume G is a reflexive subsheaf of K. Then $(G)_{X^k}$ is an $\mathfrak{S}_k$-invariant reflexive subsheaf of $(K)_{X^k}$. By the above claim we have $\mu_{h_{X^k}}((G)_{X^k})<\mu_{h_{X^k}}((K)_{X^k})$. It follows by [21, Lemma 1.2] that $\mu_{h_k}(G)<\mu_{h_k}(K)$. Therefore K is slope stable with respect to h_n, as desired. □

Proposition 1.7. *For any closed point $x\in X$, the vector bundle E_x is slope stable with respect to h_k.*

Proof. By Lemma 1.4, we have $c_1(E_x)=c_1(K)=-h_k+\delta$. Therefore by Lemma 1.5

$$c_1(E_x)\cdot h_k^{2k-1}=c_1(K)\cdot h_k^{2k-1}=(-h_k+\delta)\cdot h_k^{2k-1}=-h_k^{2k}<0.$$

Assume F is a destabilizing subsheaf of E_x with $1\leqslant\operatorname{rk}(F)\leqslant k$ and $c_1(F)=ah_k+b\delta$ for some $a,b\in\mathbb{Z}$. Then

$$c_1(F)\cdot h_k^{2k-1}=ah_k^{2k}.$$

By the assumption and Proposition 1.6, we have the inequality

$$\mu_{h_k}(E_x)\leqslant\mu_{h_k}(F)<\mu_{h_k}(K),$$

which can be written as

$$\frac{-h_k^{2k}}{k+1} \leqslant \frac{ah_k^{2k}}{\mathrm{rk}(F)} < \frac{-h_k^{2k}}{k+2} \Leftrightarrow -\frac{\mathrm{rk}(F)}{k+1} \leqslant a < -\frac{\mathrm{rk}(F)}{k+2} \text{ as } h_k^{2k} > 0.$$

Such an integer a cannot exist. Contradiction. Hence E_x is stable with respect to h_k. □

1.3. **A smooth connected component.**

In this section, we will interpret the universal sheaf $\mathcal{E}$ defined in (2) as a family of stable sheaves on $X^{[k]}$ whose base is a smooth connected component of the corresponding moduli space. We have shown above that each wrong-way fiber E_x of the family $\mathcal{E}$ is μ_{h_k}-stable; however, it would be more preferable to establish the stability with respect to some ample class on $X^{[k]}$. Although the perturbation technique in [21, Proposition 4.8] can be used to achieve this for every single E_x, for our purpose we will have to extend this technique to prove that all sheaves E_x are slope stable with respect to the same ample class near h_k.

Proposition 1.8. *There exists some ample class $H \in \mathrm{NS}(X^{[k]})$ near h_k, such that E_x is μ_H-stable for all $x \in X$ simultaneously.*

Proof. Proposition 1.7 and [5, Theorem 2.3.1] guarantees that the assumptions in [21, Proposition 4.8] are satisfied for each E_x, hence every E_x is slope stable with respect to some ample class near h_k by [21, Proposition 4.8]. In order to find a single ample class H that is independent of the choice of E_x, we can literally use the entire proof of [21, Proposition 4.8] except that we need to reconstruct the non-empty convex open set U so that $\alpha := h_k^{2k-1}$ is in the closure of U, and for every $\gamma \in U$, E_x is stable with respect to γ for all $x \in X$.

We follow the notations in [8, Definition 3.1]. For each $x \in X$, $\mathrm{SStab}(E_x)$ is a convex closed set containing α. Hence the intersection

$$\overline{U} := \cap_{x\in X} \mathrm{SStab}(E_x)$$

is also a convex closed set containing α. We first claim that [8, Theorem 3.4] holds for all E_x simultaneously; namely, we will show that for any $\beta \in \mathrm{Mov}(X)^\circ$ (see [8, Definition 2.1] for the notation), there exists a number $e \in \mathbb{Q}^+$, such that $(\alpha + \varepsilon\beta) \in \cap_{x\in X} \mathrm{Stab}(E_x)$ for any real $\varepsilon \in [0, e]$.

To prove the claim, we first note that the slope $c := \mu_\beta(E_x)$ is independent of the choice of $x \in X$. We redefine the set S in the proof of [8, Theorem 3.4] to be

$$S := \{c_1(F) \mid F \subseteq E_x \text{ for some } x \in X \text{ such that } \mu_\beta(F) \geqslant c\}.$$

Since $E_x \subseteq K$ for all $x \in X$ by (4), we obtain that S is a subset of

$$T := \{c_1(F) \mid F \subseteq K \text{ such that } \mu_\beta(F) \geqslant c\},$$

which is finite by [8, Theorem 2.29], hence S is also finite. We can then use the rest of the proof of [8, Theorem 3.4] literally to conclude the claim.

We then claim that $\overline{U}$ is of full dimension $r := \mathrm{rk}\, N_1(X)$. If not, then we have $\alpha \in \overline{U} \subseteq L$ for some hyperplane $L \subset N_1(X)_\mathbb{R}$. Since $\mathrm{Mov}(X)$ is of full dimension, we can choose some $\beta \in \mathrm{Mov}(X)^\circ \setminus L$. It follows that $(\alpha + \varepsilon\beta) \in \overline{U} \setminus L$ for some small $\varepsilon > 0$ by the previous claim and the choice of β. Contradiction.

We define U to be the interior of $\overline{U}$ and claim that U is non-empty. Indeed, since $\overline{U}$ is of full dimension r, we can choose $r+1$ points of $\overline{U}$ in general positions, which form an r-simplex. By the convexity of $\overline{U}$, the entire simplex is in $\overline{U}$ hence any interior point of the simplex is also an interior point of $\overline{U}$. The convexity of U follows from the convexity of $\overline{U}$. And it is clear from the construction that $\alpha = h_k^{2k-1}$ is in the closure of U. We finally claim that every $\gamma \in U$ is in $\cap_{x\in X} \mathrm{Stab}(E_x)$. If not, suppose that there exists some $\gamma_0 \in U$ and some $x_0 \in X$, such that $\gamma_0 \in \mathrm{SStab}(E_{x_0}) \setminus \mathrm{Stab}(E_{x_0})$; namely, $\mu_{\gamma_0}(F) = \mu_{\gamma_0}(E_{x_0})$ for some proper subsheaf F of E_{x_0}. Since the slope function is linear with respect to the curve class, and $\mu_\alpha(F) < \mu_\alpha(E_{x_0})$ by Proposition 1.7, one can find a hyperplane in $N^1(X)_\mathbb{R}$ through γ_0, such that $\mu_\gamma(E_{x_0}) - \mu_\gamma(F)$ takes opposite signs for γ in the two open half-spaces separated by the hyperplane. In particular, F destabilizes E_{x_0} in one of the half-spaces. Since U has non-empty intersection with both half-spaces, this contradicts the condition $U \subseteq \mathrm{SStab}(E_x)$. Therefore we have $U \subseteq \cap_{x\in X} \mathrm{Stab}(E_x)$, as desired. □

Before proceeding to the main result of the section, we make a brief digression to consider the integral functor

$$\Phi\colon \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(X^{[k]}) \tag{8}$$

whose kernel is the universal ideal sheaf $\mathcal{I}_{\mathcal{Z}}$ on $X \times X^{[k]}$. Recall that Φ is a $\mathbb{P}^{k-1}$-functor by [1, Theorem 3.1, Example 4.2(2)], which further implies by [2, §2.1] that for any $E, F \in \mathrm{D^b}(X)$ we have an isomorphism of graded vector spaces

$$\mathrm{Ext}^*_{X^{[k]}}(\Phi(E), \Phi(F)) \cong \mathrm{Ext}^*_X(E, F) \otimes H^*(\mathbb{P}^{k-1}, \mathbb{C}). \tag{9}$$

We give the following alternative description of E_x using the above integral functor:

Lemma 1.9. *For each $x \in X$, let I_x be the ideal sheaf of $x \in X$, then $E_x = \Phi(I_x(1))$.*

Proof. We start with the exact sequence

$$0 \longrightarrow E_x \longrightarrow K \longrightarrow I_{S_x} \longrightarrow 0. \tag{10}$$

We note that $I_{S_x} = \Phi(\mathcal{O}_x)$ as $\mathcal{I}_{\mathcal{Z}}$ is flat over X. Furthermore we have $K = \Phi(\mathcal{O}_X(1))$ since $R^i p_*(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1)) = 0$ for $i = 1, 2$ as this is true for $H^i(I_Z(1))$ for any $[Z] \in X^{[k]}$. These two facts imply that

$$\mathrm{Hom}_{X^{[k]}}(K, I_{S_x}) = \mathrm{Hom}_{X^{[k]}}(\Phi(\mathcal{O}_X(1)), \Phi(\mathcal{O}_x)) \cong \mathrm{Hom}_X(\mathcal{O}_X(1), \mathcal{O}_x)$$

by (9). Thus the exact sequence (10) is induced by the exact sequence

$$0 \longrightarrow I_x(1) \longrightarrow \mathcal{O}_X(1) \longrightarrow \mathcal{O}_x \longrightarrow 0.$$

As $K \to I_{S_x}$ is surjective, applying Φ to the last exact sequence shows $E_x = \Phi(I_x(1))$. □

We return to the main result of the section. Let H be an ample class that satisfies Proposition 1.8, and $\mathcal{M}$ the moduli space of μ_H-stable sheaves on $X^{[k]}$ with the same numerical invariants as E_x. Then the universal family $\mathcal{E}$ defines a classifying morphism

$$f\colon X \longrightarrow \mathcal{M}, \quad x \longmapsto [E_x] \tag{11}$$

In fact the morphism f can be described as follows:

Theorem 1.10. *The classifying morphism* (11) *defined by the family $\mathcal{E}$ identifies X with a smooth connected component of $\mathcal{M}$.*

Proof. By [19, Lemma 1.6.] we have to prove that f is injective on closed points and that $\dim(T_{[E_x]}\mathcal{M}) = 2$ for all $x \in X$.

Now by Lemma 1.9 we know $E_x = \Phi(I_x(1))$, so for $x \neq y$ we find

$$\begin{aligned}\mathrm{Hom}_{X^{[k]}}(E_x, E_y) &= \mathrm{Hom}_{X^{[k]}}(\Phi(I_x(1)), \Phi(I_y(1)))\\ &\cong \mathrm{Hom}_X(I_x(1), I_y(1))\\ &\cong \mathrm{Hom}_X(\mathcal{O}_x, \mathcal{O}_y) = 0\end{aligned}$$

by (9) again. This implies f is injective on closed points.

A similar computation shows

$$\begin{aligned}\mathrm{Ext}^1_{X^{[k]}}(E_x, E_x) &= \mathrm{Ext}^1_{X^{[k]}}(\Phi(I_x(1)), \Phi(I_x(1)))\\ &\cong \mathrm{Ext}^1_X(I_x(1), I_x(1))\\ &\cong \mathrm{Ext}^1_X(\mathcal{O}_x, \mathcal{O}_x) \cong T_x X\end{aligned}$$

Using $T_{[E_x]}\mathcal{M} \cong \mathrm{Ext}^1_{X^{[k]}}(E_x, E_x)$ we thus find $\dim(T_{[E_x]}\mathcal{M}) = 2$ as desired. □

2. K3 surfaces with Picard number two

In this section we assume X is a K3 surface with

$$\mathrm{NS}(X) = \mathbb{Z}e \oplus \mathbb{Z}f$$

such that $e^2 = -2k$, $f^2 = 0$ and $ef = 2k + 1$ for some integer $k \geqslant 2$. The existence of such K3 surfaces is guaranteed by [10, Corollary 14.3.1]. Since $f^2 = 0$, either f or $-f$ is effective. Without loss of generality, we will assume that the divisor class f is effective, after possibly replacing the pair (e, f) by $(-e, -f)$.

2.1. **Properties of the K3 surface.**

In this section, we collect some helpful properties of the K3 surface X which will be used in the construction of some moduli spaces of stable sheaves in the next section.

Lemma 2.1. *We have $D^2 \geqslant 0$ for all effective divisors on X. Especially there are no smooth curves C on X with $C \cong \mathbb{P}^1$.*

Proof. Any irreducible curve C on S satisfies

$$C^2 = C(C + K_X) = 2p_a(C) - 2 \geqslant -2.$$

So assume $C^2 = -2$ and write $C = me + nf$. Then we have

$$\begin{aligned} C^2 &= (me + nf)^2 = m^2e^2 + 2mnef \\ &= -2km^2 + 2(2k+1)mn \\ &= -2m(km - (2k+1)n). \end{aligned}$$

The equation $C^2 = -2$ translates into $m(km - (2k+1)n) = 1$. This implies $m = \pm 1$ but then one can see that there is no $n \in \mathbb{Z}$ satisfying this equation. $\square$

Lemma 2.2. *The divisor classes $h = e + (2k-1)f$ and $\widehat{h} = (2k)e + (2k-1)f$ are ample.*

Proof. We have

$$\begin{aligned} h^2 &= (e + (2k-1)f)^2 = e^2 + 2(2k-1)ef \\ &= -2k + 2(2k-1)(2k+1) = 8k^2 - 2k - 2. \end{aligned}$$

So $h^2 > 0$ as $k \geqslant 2$. Since also $hf = ef = 2k+1 > 0$ we see that h is ample by the remark after [10, Corollary 8.1.7].

A similar computation shows $\widehat{h}^2 > 0$ and $\widehat{h}f > 0$. $\square$

Lemma 2.3. *Let m and n be integers. If the class $me + nf$ is effective, then $n \geqslant 0$ and $0 \leqslant m \leqslant \frac{2k+1}{k}n$. Furthermore $h(me + nf) \geqslant ((2k-1)(2k+1) - k)m$.*

Proof. Let D be an effective divisor with class $me + nf$. Since the claim is additive in m and n, we may assume w.l.o.g. that D is an irreducible curve C.

By the previous results we may assume $C^2 \geqslant 0$. We have:

$$\begin{aligned} C^2 &= 2m\{-km + (2k+1)n\} \geqslant 0 \\ hC &= (4k^2 - k - 1)m + \{-km + (2k+1)n\} > 0 \end{aligned}$$

which implies $m \geqslant 0$ and $-km + (2k+1)n \geqslant 0$. The last inequality can also be read as

$$(2k+1)n \geqslant km \Leftrightarrow m \leqslant \frac{2k+1}{k}n.$$

Putting everything together shows

$$0 \leqslant m \leqslant \frac{2k+1}{k}n$$

as well as $hC \geqslant ((2k-1)(2k+1) - k)m$. $\square$

Corollary 2.4. *There is a surjective morphism $\pi : X \to \mathbb{P}^1$ such that all fibers are integral curves of arithmetic genus $p_a(C) = 1$, that is X is elliptically fibered.*

Proof. Since $f^2 = 0$ it is known that the linear system $|f|$ induces a surjective map $\pi : X \to \mathbb{P}^1$ with $\pi^* \mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_X(f)$. By the previous lemma the class f cannot be the sum of two effective divisors, hence all fibers C of π are integral and have $p_a(C) = 1$. $\square$

Lemma 2.5. *Let $[Z] \in X^{[k]}$. Assume R is a torsion quotient of $I_Z(e)$ with $c_1(R) = nf$ for some $n \geqslant 0$, then $H^1(R) = 0$.*

Proof. The quotient defines the following exact sequence:

$$0 \longrightarrow K \longrightarrow I_Z(e) \longrightarrow R \longrightarrow 0.$$

Now K is torsion free of rank one, so its double dual K^{**} is locally free of rank one and the natural map $K \to K^{**}$ is injective and the cokernel T has finite support. Especially $c_1(T) = 0$ so

$$c_1(K^{**}) = c_1(K) = c_1(I_Z(e)) - c_1(R) = e - nf$$

and thus $K^{**} \cong \mathcal{O}_X(e - nf)$. The embedding $K \hookrightarrow I_Z(e)$ induces an embedding

$$K^{**} \cong \mathcal{O}_X(e - nf) \hookrightarrow \mathcal{O}_X(e).$$

This embedding is given by a global section of $\mathcal{O}_X(nf)$, that is by an effective divisor $D = \sum_i a_i C_i$ with class nf.

This global section is the pullback along the elliptic fibration π of a global section of $\mathcal{O}_{\mathbb{P}^1}(n)$, with corresponding effective divisor $\sum_i a_i z_i$ on $\mathbb{P}^1$, here $C_i = \pi^{-1}(z_i)$.

Denote by $D \subset X$ also the corresponding closed subscheme (which maybe non-reduced, if $a_i \geqslant 2$ for some i). We get the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & \mathcal{O}_X(e-nf) \arrow[r] \arrow[d] & T \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & I_Z(e) \arrow[r] \arrow[d] & \mathcal{O}_X(e) \arrow[r] \arrow[d] & \mathcal{O}_Z \arrow[r] & 0 \\
 & R \arrow[r, "\beta"] \arrow[d] & \mathcal{O}_D(e) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

The snake lemma gives an exact sequence

$$0 \longrightarrow \ker(\alpha) \longrightarrow R \xrightarrow{\beta} \mathcal{O}_D(e) \longrightarrow \operatorname{coker}(\alpha) \longrightarrow 0.$$

Let $R' \subset \mathcal{O}_D(e)$ be the image of β. Since the torsion sheaf $\mathcal{O}_{\sum_i a_i z_i}$ on $\mathbb{P}^1$ has a composition series by skyscraper sheaves $\mathcal{O}_{z_i}$ as composition factors, $\mathcal{O}_D$ has a composition series with composition factors $\mathcal{O}_{C_i}$, thus $\mathcal{O}_D(e)$ has a composition series with composition factors $\mathcal{O}_{C_i}(e)$. The latter is a line bundle of degree

$$e \cdot C_i = e \cdot f = 2k + 1$$

on C_i. The quotient $\mathcal{O}_D(e)/R'$ is isomorphic to $\operatorname{coker}(\alpha)$, that is to a quotient Q of $\mathcal{O}_Z$. By intersecting with R' we get a composition series for R' with composition factors which are kernels of a surjection $\mathcal{O}_{C_i}(e) \twoheadrightarrow Q'$ with Q' of length $\leqslant k$. Thus we have exact sequences:

$$0 \longrightarrow L \longrightarrow \mathcal{O}_{C_i}(e) \longrightarrow Q' \longrightarrow 0,$$

with a torsion free sheaf L of rank one on the integral projective curve C_i of arithmetic genus one. Using $\chi(\mathcal{O}_{C_i}) = 0$ and

$$\chi(L) = \chi(\mathcal{O}_{C_i}(e)) - \chi(Q') \geqslant k + 1,$$

gives

$$\deg(\mathcal{O}_{C_i}(e)) \geqslant \deg(L) \geqslant k + 1.$$

By [7, Proposition 4.6.] all of these composition factors have trivial H^1. By constructing short exact sequences out of the composition series and use the induced exact sequences for H^1, it follows

$$H^1(R') = 0.$$

As $\ker(\beta) = \ker(\alpha) \subseteq T$ has finite support, we also have $H^1(\ker(\beta)) = 0$. Hence

$$H^1(R) = 0.$$

□

2.2. **Explicit construction of a universal family.**

In this section we want to generalize [17, Theorem 1.2]. Let h be the ample line bundle defined in Lemma 2.2. We consider the moduli space $M_h(v)$ of μ_h-stable sheaves on X with Mukai vector $v = (2k-1, h, 2k)$. We note that μ_h-semistability implies μ_h-stability for any rank $2k-1$ sheaf E with $v(E) = v$ as $\gcd(2k-1, 2k) = 1$. Thus $M_h(v)$ is a smooth projective variety. We compute:

$$\dim(M_h(v)) = v^2 + 2 = (8k^2 - 2k - 2) - 2(2k-1)(2k) + 2 = 2k.$$

Furthermore $v' = (2k-1, h, a)$ with $a \geqslant 2k+1$ satisfies

$$v'^2 + 2 = h^2 - 2a(2k-1) + 2 \leqslant (8k^2 - 2k - 2) - 2(2k-1)(2k+1) + 2 = 2 - 2k < 0,$$

so again every point $[E]$ in $M_h(v)$ is locally free. The condition $\gcd(2k-1, 2k) = 1$ also implies that $M_h(v)$ is a fine moduli space.

We want to construct μ_h-stable locally free sheaves E with $v(E) = v$. For this we observe that by Lemma 2.3 we have

$$h^0(\mathcal{O}_X(e)) = 0 \text{ and } h^0(\mathcal{O}_X(-e)) = 0. \tag{12}$$

Using Riemann-Roch and Serre duality we see:

$$h^1(\mathcal{O}_X(\pm e)) = -\chi(\mathcal{O}_X(\pm e)) = k - 2. \tag{13}$$

For any $[Z] \in X^{[k]}$ with ideal sheaf I_Z, we have by (12) and (13):

$$\dim(\mathrm{Ext}^1(I_Z(e+f), \mathcal{O}_X(f))) = h^1(I_Z(e)) = 2k - 2. \tag{14}$$

We want to study extension classes in $\mathrm{Ext}^1(I_Z(e+f), \mathcal{O}_X(f) \otimes_{\mathbb{C}} G^*)$ for the vector space $G := \mathrm{Ext}^1(I_Z(e+f), \mathcal{O}_X(f))$. There are the following isomorphisms:

$$\begin{aligned} \mathrm{Ext}^1(I_Z(e+f), \mathcal{O}_X(f) \otimes_{\mathbb{C}} G^*) &\cong \mathrm{Ext}^1(I_Z(e+f), \mathcal{O}_X(f)) \otimes_{\mathbb{C}} G^* \\ &= G \otimes G^* \\ &\cong \mathrm{Hom}(G, G). \end{aligned} \tag{15}$$

Let $\mathfrak{e} \in \mathrm{Ext}^1(I_Z(e+f), \mathcal{O}_X(f) \otimes_{\mathbb{C}} G^*)$ be the class corresponding to $\mathrm{id}_G \in \mathrm{Hom}(G, G)$ under the isomorphism (15). This extension class $\mathfrak{e}$ gives the exact sequence:

$$0 \longrightarrow \mathcal{O}_X(f) \otimes_{\mathbb{C}} G^* \longrightarrow E_Z \longrightarrow I_Z(e+f) \longrightarrow 0. \tag{16}$$

The sheaf E_Z is called the *universal extension* of $I_Z(e+f)$ by $\mathcal{O}_X(f)$ and satisfies $\mathrm{Hom}(E_Z, \mathcal{O}_X(f)) = 0$. The goal of this section is to prove the following theorem:

Theorem 2.6. *For any $[Z] \in X^{[k]}$, the sheaf E_Z defined in* (16) *is a μ_h-stable locally free sheaf with Mukai vector $v(E_Z) = (2k-1, h, 2k)$.*

Proof. The subscheme $[Z] \in X^{[k]}$ has the property of being e-stable for $e \in \mathrm{NS}(X)$ since

$$h^1(I_{Z'}(e)) < h^1(I_Z(e))$$

for every subscheme $Z' \subset Z$ of length $< k$; see [22, Definition 1.2.] for the definition of this stability condition. It then follows from [22, Lemma 2.2.] that E_Z is locally free. The exact sequence (16) shows that $\mathrm{rk}(E_Z) = 2k-1$ as well as $c_1(E_Z) = e + (2k-1)f$. Furthermore we have

$$\begin{aligned} \chi(E_Z) &= (2k-2)\chi(\mathcal{O}_X(f)) + \chi(I_Z(e+f)) \\ &= (2k-2)\chi(\mathcal{O}_X(f)) + \chi(\mathcal{O}_X(e+f)) - k \\ &= 4k - 1 = (2k-1) + 2k. \end{aligned}$$

Thus the locally free sheaf E_Z has Mukai vector $v(E_Z) = (2k-1, h, 2k)$.

It then remains to prove that $E_Z(-f)$ is μ_h-stable, which is defined by:

$$0 \longrightarrow \mathcal{O}_X^{\oplus 2k-2} \longrightarrow E_Z(-f) \longrightarrow I_Z(e) \longrightarrow 0. \tag{17}$$

By the construction of E_Z as a universal extension (16), we have $\mathrm{Hom}(E_Z(-f), \mathcal{O}_X) = \mathrm{Hom}(E_Z, \mathcal{O}_X(f)) = 0$. Furthermore

$$\mu_h(E_Z(-f)) = \frac{eh}{2k-1} = \frac{-2k + (2k-1)(2k+1)}{2k-1} = 2k + 1 - \frac{2k}{2k-1} > 0.$$

Pick a torsion free quotient F of $E_Z(-f)$ with $1 \leqslant \mathrm{rk}(F) \leqslant 2k-2$. We have

$$E_Z(-f) \longrightarrow F \longrightarrow 0$$

with $\mathrm{Hom}(F, \mathcal{O}_X) \hookrightarrow \mathrm{Hom}(E_Z(-f), \mathcal{O}_X) = 0$.

We want to show that we always have $\mu_h(F) > \mu_h(E_Z(-f))$. For this, define the torsion free sheaf F_0 as the image of the composition

$$\mathcal{O}_X^{\oplus 2k-2} \hookrightarrow E_Z(-f) \longrightarrow F.$$

We get a surjection

$$\mathcal{O}_X^{\oplus 2k-2} \longrightarrow F_0 \longrightarrow 0.$$

This implies $c_1(F_0)$ is effective and we have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_0 \arrow[r] \arrow[d] & K_1 \arrow[r] \arrow[d] & K_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X^{\oplus 2k-2} \arrow[r] \arrow[d] & E_Z(-f) \arrow[r] \arrow[d] & I_Z(e) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_0 \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{18}$$

Due to the diagram $\mathrm{rk}(F_1) \in \{0, 1\}$.

Case 1: $\mathrm{rk}(F_1) = 1$. Then $\mathrm{rk}(F_0) = \mathrm{rk}(F) - 1$ and $F_1 \cong I_Z(e)$. We conclude

$$c_1(F) = c_1(F_0) + c_1(I_Z(e)) \Rightarrow c_1(F) = c_1(F_0) + e.$$

Using this we find:

$$\mu_h(F) = \frac{c_1(F)h}{\mathrm{rk}(F)} = \underbrace{\frac{c_1(F_0)h}{\mathrm{rk}(F)}}_{\geqslant 0} + \frac{eh}{\mathrm{rk}(F)} > \frac{eh}{2k-1} = \mu_h(E_Z(-f)).$$

So we indeed have $\mu_h(F) > \mu_h(E_x(-f))$.

Case 2: $\mathrm{rk}(F_1) = 0$. Now $\mathrm{rk}(F_0) = \mathrm{rk}(F)$. Write $c_1(F) = me + nf$. Since $c_1(F_0)$ and $c_1(F_1)$ are effective, so is their sum $c_1(F)$, which by Lemma 2.3 implies, that $m \geqslant 0$ as well as

$$\mu_h(F) = \frac{(me+nf)h}{\mathrm{rk}(F)} \geqslant \frac{m((2k-1)(2k+1) - k)}{\mathrm{rk}(F)} \geqslant m(2k + 1 - \frac{k}{2k-1}).$$

For $m \geqslant 1$ we have

$$\begin{aligned}
\mu_h(F) &\geqslant m(2k + 1 - \frac{k}{2k-1}) \\
&\geqslant 2k + 1 - \frac{k}{2k-1} \\
&> 2k + 1 - \frac{2k}{2k-1} = \mu_h(E_Z(-f))
\end{aligned}$$

So only the case $m = 0$ remains, i.e. $c_1(F) = nf$. We have

$$\mu_h(F) = \frac{n(2k+1)}{\mathrm{rk}(F)}.$$

If we can prove $n \geqslant \mathrm{rk}(F)$ we are done since

$$\mu_h(F) \geqslant 2k+1 > 2k+1 - \frac{2k}{2k-1} = \mu_h(E_Z(-f)).$$

As $c_1(F) = nf$ is the sum of the two effective divisors $c_1(F_0)$ and $c_1(F_1)$, it follows from Lemma 2.3 that $c_1(F_0) = n_0 f$ and $c_1(F_1) = n_1 f$ with $n_0, n_1 \geqslant 0$ and $n_0 + n_1 = n$.

By Lemma 2.5 we have $H^1(F_1) = 0$ which implies $\mathrm{Ext}^1(F_1, \mathcal{O}_X) = 0$ using Serre duality. So the restriction map

$$\mathrm{Hom}(F, \mathcal{O}_X) \to \mathrm{Hom}(F_0, \mathcal{O}_X)$$

surjective. But we know $\mathrm{Hom}(F, \mathcal{O}_X) = 0$. So

$$\mathrm{Hom}(F_0, \mathcal{O}_X) = 0. \tag{19}$$

Using the elliptic fibration $\pi : X \to \mathbb{P}^1$ we have:

$$h^0(\det(F_0)) = h^0(\mathcal{O}_X(n_0 f)) = n_0 + 1. \tag{20}$$

Now there is a trivial sub-bundle in $\mathcal{O}_X^{\oplus 2k-2}$ of rank $\mathrm{rk}(F)+1$ such that

$$\mathcal{O}_X^{\oplus(\mathrm{rk}(F)+1)} \xrightarrow{\varphi} F_0$$

is surjective outside a finite subset of X by [3, Lemma 4.60.].

Define $R := \mathrm{coker}(\varphi)$. Then there is the exact sequence:

$$0 \longrightarrow F_0' \longrightarrow F_0 \longrightarrow R \longrightarrow 0.$$

As R has finite support, we get:

$$\det(F_0) = \det(F_0') \text{ as well as } H^2(F_0') \cong H^2(F_0).$$

We also have the exact sequence

$$0 \longrightarrow \det(F_0)^{-1} \longrightarrow \mathcal{O}_X^{\oplus(\mathrm{rk}(F)+1)} \longrightarrow F_0' \longrightarrow 0.$$

The end of the induced long cohomology sequence gives:

$$H^1(F_0') \longrightarrow H^2(\det(F_0)^{-1}) \longrightarrow H^2(\mathcal{O}_X^{\oplus(\mathrm{rk}(F)+1)}) \longrightarrow H^2(F_0') \longrightarrow 0. \tag{21}$$

It follows from (19) by Serre duality that

$$H^2(F_0') \cong H^2(F_0) \cong \mathrm{Hom}(F_0, \mathcal{O}_X)^\vee = 0.$$

Since $H^2(F_0') = 0$, we apply Serre duality again and obtain from (21) that

$$0 \longrightarrow H^0(\mathcal{O}_X^{\oplus(\mathrm{rk}(F)+1)}) \longrightarrow H^0(\det(F_0)).$$

We conclude

$$h^0(\det(F_0)) \geqslant \mathrm{rk}(F) + 1.$$

Using this inequality together with (20) we get:

$$n_0 + 1 = h^0(\det(F_0)) \geqslant \mathrm{rk}(F) + 1 \Rightarrow n_0 \geqslant \mathrm{rk}(F) \Rightarrow n \geqslant \mathrm{rk}(F).$$

We thus find the desired inequality between n and $\mathrm{rk}(F)$, so $E_Z(-f)$ is stable. □

Remark 2.7. Similarly to Remark 1.3 one can see that in this example E_Z can be described as the inverse spherical twist $T_{\mathcal{O}_X}^{-1}(I_Z(e)) \otimes \mathcal{O}_X(f)$ of $I_Z(e)$ tensored by $\mathcal{O}_X(f)$.

We want to globalize the previous construction. For this we denote the two projections from $X \times X^{[k]}$ by $p : X \times X^{[k]} \to X^{[k]}$ and $q : X \times X^{[k]} \to X$ respectively, and the universal closed subscheme of length n by $\mathcal{Z} \subset X \times X^{[k]}$. Also recall the integral functor $\Phi : \mathrm{D^b}(X) \to \mathrm{D^b}(X^{[k]})$ from (8).

Lemma 2.8. *The sheaf of relative extensions* $\mathcal{G} := \mathcal{E}xt^1_p(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e), \mathcal{O}_{X\times X^{[k]}})$ *is a locally free sheaf of rank* $2k-2$ *on* $X^{[k]}$*, whose fiber at any point* $[Z] \in X^{[k]}$ *is given by* $\operatorname{Ext}^1(I_Z(e), \mathcal{O}_X)$*. Moreover, its dual* $\mathcal{G}^\vee$ *fits in an exact sequence*

$$0 \longrightarrow \mathcal{O}_X(e)^{[k]} \longrightarrow \mathcal{G}^\vee \longrightarrow H^1(\mathcal{O}_X(e)) \otimes \mathcal{O}_{X^{[k]}} \longrightarrow 0. \tag{22}$$

Furthermore we have $\mathcal{G}^\vee = \Phi(\mathcal{O}_X(e))[1]$ *as well as* $\operatorname{Hom}_{X^{[k]}}(\mathcal{O}_{X^{[k]}}, \mathcal{G}^\vee) = 0$*.*

Proof. The morphism p is proper and flat, and the dimension function

$$X^{[k]} \longrightarrow \mathbb{N}, \quad [Z] \longmapsto \operatorname{ext}^1(I_Z(e), \mathcal{O}_X)$$

is constant by (14). So by [4, Satz 3.] the first relative Ext-sheaf is locally free of rank $2k-2$ on $X^{[k]}$ and commutes with base change, that is for every $[Z] \in X^{[k]}$ we have

$$\mathcal{E}xt^1_p(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e), \mathcal{O}_{X\times X^{[k]}}) \otimes \mathcal{O}_{[Z]} \cong \operatorname{Ext}^1(I_Z(e), \mathcal{O}_X).$$

The relative Serre duality [12, Corollary (24)] gives an isomorphism

$$\mathcal{G} = \mathcal{E}xt^1_p(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e), \mathcal{O}_{X\times X^{[k]}}) \cong (R^1p_*(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e)))^\vee. \tag{23}$$

By (12), (13) and standard cohomology and base change results $R^1p_*(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e))$ is locally free and $R^ip_*(q^* \mathcal{O}_X(e)) = R^ip_*(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e)) = 0$ for $i = 0$ and $i = 2$, see for example [18, II.5.]. This shows that there is an exact sequence

$$0 \longrightarrow p_*(\mathcal{O}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e)) \longrightarrow R^1p_*(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e)) \longrightarrow R^1p_*(q^* \mathcal{O}_X(e)) \longrightarrow 0.$$

Using the definition of tautological sheaves from (5), dualizing (23) and applying cohomology and base change again, the last exact sequence turns into (22).

As $R^ip_*(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e)) = 0$ for $i = 0, 2$ we have by definition

$$\mathcal{G}^\vee = \Phi(\mathcal{O}_X(e))[1].$$

Recall the codimension two subscheme $S_x \subset X^{[k]}$ from (4) with its exact sequence:

$$0 \longrightarrow I_{S_x} \longrightarrow \mathcal{O}_{X^{[k]}} \longrightarrow \mathcal{O}_{S_x} \longrightarrow 0.$$

Applying $\operatorname{Hom}(-, \mathcal{G}^\vee)$ and using that S_x has codimension 2, we can obtain by using (9):

$$\begin{aligned}\operatorname{Hom}(\mathcal{O}_{X^{[k]}}, \mathcal{G}^\vee) &\cong \operatorname{Hom}_{X^{[k]}}(I_{S_x}, \mathcal{G}^\vee)\\ &= \operatorname{Hom}_{X^{[k]}}(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_X(e))[1])\\ &= \operatorname{Ext}^1_{X^{[k]}}(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_X(e)))\\ &\cong \operatorname{Ext}^1_X(\mathcal{O}_x, \mathcal{O}_X(e)) \otimes H^0(\mathbb{P}^{k-1}, \mathbb{C}) = 0.\end{aligned}$$

□

Theorem 2.9. *There exists a flat family of locally free sheaves on* X *parametrized by* $X^{[k]}$*, whose universal object* $\mathcal{E}$ *fits in an exact sequence*

$$0 \longrightarrow p^*\mathcal{G}^\vee \otimes q^* \mathcal{O}_X(f) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e+f) \longrightarrow 0, \tag{24}$$

such that the fiber of $\mathcal{E}$ *over each point* $[Z] \in X^{[k]}$ *is the* μ_h*-stable vector bundle* E_Z *defined in* (16)*.*

Proof. For every $[Z] \in X^{[k]}$ we have $\operatorname{Hom}(I_Z(e), \mathcal{O}_X) = 0$, so

$$\mathcal{E}xt^0_p(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e), \mathcal{O}_{X\times X^{[k]}}) = p_*\mathcal{H}om(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e), \mathcal{O}_{X\times X^{[k]}}) = 0.$$

Using this fact and the projection formula for relative Ext-sheaves [15, Lemma 4.1.], the five term exact sequence of the spectral sequence

$$H^i(X^{[k]}, \mathcal{E}xt^j_p(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e), p^*\mathcal{G}^\vee)) \Rightarrow \operatorname{Ext}^{i+j}(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e), p^*\mathcal{G}^\vee)$$

reduces to an isomorphism

$$\begin{aligned}\operatorname{Ext}^1(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e), p^*\mathcal{G}^\vee) &\cong H^0(X^{[k]}, \mathcal{E}xt^1_p(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e), p^*\mathcal{G}^\vee))\\ &\cong H^0(X^{[k]}, \mathcal{E}xt^1_p(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e), \mathcal{O}_{X\times X^{[k]}}) \otimes \mathcal{G}^\vee)\\ &\cong \operatorname{Hom}(\mathcal{G}, \mathcal{G}).\end{aligned}$$

The identity $\mathrm{id}_{\mathcal{G}}$ gives rise to an extension on $X \times X^{[k]}$:

$$0 \longrightarrow p^*\mathcal{G}^\vee \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e) \longrightarrow 0$$

with $\mathcal{F}$ flat over $X^{[k]}$, since both other terms are. Restricting to the fiber over a point $[Z] \in X^{[k]}$ defines by flatness of $\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e)$ a map

$$\mathrm{Ext}^1(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e), p^*\mathcal{G}^\vee) \to \mathrm{Ext}^1(I_Z(e), G^\vee \otimes \mathcal{O}_X).$$

By [15, Lemma 2.1.] the extension defined by $\mathrm{id}_{\mathcal{G}}$ restricts to the extension given by id_G on the fiber over $[Z] \in X^{[k]}$, which is just the exact sequence (17).

Defining $\mathcal{E} := \mathcal{F} \otimes q^* \mathcal{O}_X(f)$ gives the desired exact sequence. The pullback of (24) to the fiber over every $[Z] \in X^{[k]}$ is exactly the exact sequence (16), which defines a μ_h-stable locally free sheaf of class $v = (2k-1, h, 2k)$. □

Lemma 2.10. *If for $[Z], [Z'] \in X^{[k]}$ there is an isomorphism $E_Z \cong E_{Z'}$, then $[Z] = [Z']$.*

Proof. We look at the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X^{\oplus 2k-2} \arrow[r, "\iota"] & E_Z(-f) \arrow[r] \arrow[d, "\cong"', "\alpha"] & I_Z(e) \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_X^{\oplus 2k-2} \arrow[r] & E_{Z'}(-f) \arrow[r, "q"] & I_{Z'}(e) \arrow[r] & 0.
\end{tikzcd}$$

Since $h^0(I_Z(e)) = 0$ the composition $q \circ \alpha \circ \iota$ is zero. Consequently the free submodule of $E_Z(-f)$ maps injectively to the free submodule of $E_{Z'}(-f)$, which must be an isomorphism then, so we get in fact the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X^{\oplus 2k-2} \arrow[r] \arrow[d, "\cong"'] & E_Z(-f) \arrow[r] \arrow[d, "\cong"', "\alpha"] & I_Z(e) \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & \mathcal{O}_X^{\oplus 2k-2} \arrow[r] & E_{Z'}(-f) \arrow[r] & I_{Z'}(e) \arrow[r] & 0.
\end{tikzcd}$$

Therefore there is an induced isomorphism $I_Z(e) \cong I_{Z'}(e)$ and so $[Z] = [Z']$. □

Corollary 2.11. *There is an isomorphism $X^{[k]} \cong M_h(v)$.*

Proof. The family $\mathcal{E}$ from Theorem 2.9 gives a classifying morphism

$$\varphi : X^{[k]} \longrightarrow M_h(v), \quad [Z] \longmapsto [E_Z].$$

Since $X^{[k]}$ and $M_h(v)$ are both of dimension $2k$, it is an open embedding by Lemma 2.10. But $X^{[k]}$ is projective, so φ is also closed. Since $X^{[k]}$ and $M_h(v)$ are both irreducible, the classifying morphism φ must be an isomorphism. □

2.3. **Stability of wrong fibers.**

Recall that the $X^{[k]}$-family $\mathcal{E}$ is defined by (24) where $\mathcal{G}^\vee$ is defined by (22). We note that $\mathcal{E}$ is not only q-flat, but also p-flat since $\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(e)$ is both p- and q-flat. Thus we can restrict the exact sequence (24) to the fiber over a point $x \in X$ and get a description of the "wrong" fiber $E_x := \mathcal{E}_{|q^{-1}(x)}$ which is thus a locally free sheaf of rank $2k-1$ on $X^{[k]}$.

The pullback of (24) to the fiber over $x \in X$ gives the exact sequence:

$$0 \longrightarrow \mathcal{G}^\vee \longrightarrow E_x \longrightarrow I_{S_x} \longrightarrow 0, \tag{25}$$

where again $S_x := \left\{[Z] \in X^{[k]} \,|\, x \in \mathrm{supp}(Z)\right\}$ is a codimension 2 subscheme of $X^{[k]}$.

We prove the stability of E_x with respect to an ample class $H \in \mathrm{NS}(X^{[k]})$ in the rest of this section. Similar to Lemma 1.5 we have

$$\mathrm{NS}(X^{[k]}) = \mathbb{Z}e_k \oplus \mathbb{Z}f_k \oplus \mathbb{Z}\delta$$

and the following intersection numbers on $X^{[k]}$, again by [23, Lemma 1.10]:

Lemma 2.12. *For an ample class h and an arbitrary class l from $\mathrm{NS}(X)$ we have:*

- $l_k h_k^{2k-1} = \frac{(2k-1)!}{(k-1)!2^{k-1}}(lh)(h^2)^{k-1}$,

- $\delta h_k^{2k-1} = 0$.

Lemma 2.13. *We have* $c_1(E_x) = e_k - \delta$.

Proof. The exact sequences (25) and (22) show $c_1(E_x) = c_1(\mathcal{G}^\vee) = c_1(\mathcal{O}_X(e)^{[k]})$.
Using [23, Lemma 1.5.] we get

$$c_1(E_x) = c_1(\mathcal{O}_X(e)^{[k]}) = e_k - \delta. \qquad \square$$

Again let h be any ample class on X. We can compute

$$\begin{aligned} c_1(E_x)h_k^{2k-1} &= (e_k - \delta)h_k^{2k-1} \\ &= e_k h_k^{2k-1} - \delta h_k^{2k-1} \\ &= \frac{(2k-1)!}{(k-1)!2^{k-1}}(eh)(h^2)^{k-1}. \end{aligned}$$

Thus we have

$$\mu_{h_k}(E_x) = \frac{c_1(E_x)h_k^{2k-1}}{2k-1} = \frac{(2k-2)!}{(k-1)!2^{k-1}}(eh)(h^2)^{k-1}.$$

But $\mathcal{O}_X(e)^{[k]}$ is a subsheaf of E_x with $c_1(\mathcal{O}_X(e)^{[k]}) = c_1(E_x)$. So for this subsheaf to not be destabilizing we need $eh < 0$ since $h^2 > 0$. For the ample class $\widehat{h}$ from Lemma 2.2 we compute

$$\begin{aligned} e\widehat{h} &= (2k)e^2 + (2k-1)ef \\ &= -(4k^2) + (4k^2 - 1) = -1. \end{aligned}$$

Thus to prove the stability of the locally free sheaf E_x defined in (25) with respect to some ample class on $X^{[k]}$, we first prove that it is $\mu_{\widehat{h}_k}$-stable. We use the same notation as in Section 1.2 and also need the following formula: Assume F is a coherent sheaf on X^k with $\mathfrak{S}_k$-invariant Chern class

$$c_1(F) = \sum_{i=1}^{k} q_i^* c$$

where $c \in \mathrm{NS}(X)$, then the intersection number

$$c_1(F) \cdot \widehat{h}_{X^k}^{2k-1} = \frac{k(2k-1)!}{2^{k-1}}(c \cdot \widehat{h})(\widehat{h} \cdot \widehat{h})^{k-1}$$

where the left-hand side is an intersection on X^k, both factors on the right-hand side are intersections on X, see [23, Lemma 1.10]. In the following we will abbreviate the factor $\frac{k(2k-1)!}{2^{k-1}}$ by a_k to make formulas more readable.

Proposition 2.14. *E_x is μ-stable with respect to $\widehat{h}_k$.*

Proof. Assume that F is a reflexive subsheaf of E_x of rank $1 \leqslant r \leqslant 2k-2$. We need to show that $\mu_{\widehat{h}_k}(F) < \mu_{\widehat{h}_k}(E_x)$. By [21, Lemma 1.2], it suffices to check that

$$\mu_{\widehat{h}_{X^k}}((F)_{X^k}) < \mu_{\widehat{h}_{X^k}}((E_x)_{X^k}),$$

where $(F)_{X^k}$ is an $\mathfrak{S}_k$-invariant subsheaf of $(E_x)_{X^k}$.

We apply the functor $j_*(\overline{\sigma}_{k,\circ}^*((-)_\circ))$ to (25) and (22). Since the functor is left exact, we obtain with the help of [21, Lemma 1.1] that

$$0 \longrightarrow (\mathcal{G}^\vee)_{X^k} \longrightarrow (E_x)_{X^k} \longrightarrow (I_{S_x})_{X^k} \longrightarrow Q \longrightarrow 0, \tag{26}$$

as well as

$$0 \longrightarrow \bigoplus_{i=1}^{k} q_i^* \mathcal{O}_X(e) \longrightarrow (\mathcal{G}^\vee)_{X^k} \longrightarrow H^1(\mathcal{O}_X(e)) \otimes \mathcal{O}_{X^k} \longrightarrow T \longrightarrow 0, \tag{27}$$

where $\mathrm{supp}(Q), \mathrm{supp}(T) \subseteq D \subset X^k$, here D is the big diagonal. It is also clear that

$$\overline{\sigma}^*_{k,\circ}((I_{S_x})_\circ) = (\bigotimes_{i=1}^{k} q_i^* I_x)|_{X^k \setminus D}.$$

Since D is of codimension 2 in X^k, we have that $c_1((I_{S_x})_{X^k}) = 0$. It follows that

$$c_1((E_x)_{X^k}) = c_1((\mathcal{G}^\vee)_{X^k}).$$

A similar argument shows

$$c_1((\mathcal{G}^\vee)_{X^k}) = \sum_{i=1}^{k} q_i^* e.$$

Therefore

$$\begin{aligned} c_1((E_x)_{X^k}) \cdot \widehat{h}^{2k-1}_{X^k} &= c_1((\mathcal{G}^\vee)_{X^k}) \cdot \widehat{h}^{2k-1}_{X^k} \\ &= a_k(e \cdot \widehat{h})(\widehat{h} \cdot \widehat{h})^{k-1} \\ &= a_k(-1)(\widehat{h} \cdot \widehat{h})^{k-1}. \end{aligned}$$

Since $(F)_{X^k}$ is $\mathfrak{S}_k$-invariant, we have $c_1((F)_{X^k}) = \sum_{i=1}^{k} q_i^* c$ for some $c \in \mathrm{NS}(X)$, and

$$c_1((F)_{X^k}) \cdot \widehat{h}^{2k-1}_{X^k} = a_k(c \cdot \widehat{h})(\widehat{h} \cdot \widehat{h})^{k-1}.$$

We have the following two cases:

If $c \cdot \widehat{h} \leqslant -1$, then we have

$$c_1((F)_{X^k}) \cdot \widehat{h}^{2k-1}_{X^k} \leqslant c_1((E_x)_{X^k}) \cdot \widehat{h}^{2k-1}_{X^k} < 0.$$

Since $\mathrm{rk}((F)_{X^k}) < \mathrm{rk}((E_x)_{X^k})$, it follows that

$$\mu_{\widehat{h}_{X^k}}((F)_{X^k}) < \mu_{\widehat{h}_{X^k}}((E_x)_{X^k}).$$

If $c \cdot h \geqslant 0$, then $c_1((F)_{X^k}) \cdot \widehat{h}^{2k-1}_{X^k} \geqslant 0$.

We choose a (not necessarily $\mathfrak{S}_k$-invariant) non-zero $\mu_{\widehat{h}_{X^k}}$-stable reflexive subsheaf of maximal slope $F' \subseteq (F)_{X^k}$, then $\mu_{\widehat{h}_{X^k}}(F') \geqslant 0$. However $q_i^* \mathcal{O}_X(e)$ is $\mu_{\widehat{h}_{X^k}}$-stable for $i = 1, \ldots, k$, and

$$c_1(q_i^* \mathcal{O}_X(e)) \cdot \widehat{h}^{2k-1}_{X^k} = a_k(e \cdot \widehat{h})(\widehat{h} \cdot \widehat{h})^{k-1} = a_k(-1)(\widehat{h} \cdot \widehat{h})^{k-1} < 0.$$

Hence the only map from F' to $q_i^* \mathcal{O}_X(e)$ is zero.

By (26) we obtain a morphism $F' \xrightarrow{\alpha} (I_{S_x})_{X^k}$. It is clear that $(I_{S_x})_{X^k}$ is torsion free, so it is a subsheaf of its double dual $(I_{S_x})^{\vee\vee}_{X^k}$. We also note that the restriction of $(I_{S_x})_{X^k}$ on $X^k \setminus (D \cup (X \times \ldots \times \{x\}) \cup \cdots \cup (\{x\} \times X \times \ldots \times X))$ is the trivial line bundle, hence

$$(I_{S_x})^{\vee\vee}_{X^k} = \mathcal{O}_{X^k}.$$

Therefore we have

$$F' \xrightarrow{\alpha} (I_{S_x})_{X^k} \hookrightarrow \mathcal{O}_{X^k}.$$

If $\alpha \neq 0$, then the composition of both maps is non-zero, hence the stability forces

$$\mu_{\widehat{h}_{X^k}}(F') = 0 = \mu_{\widehat{h}_{X^k}}(\mathcal{O}_{X^k}).$$

Since F' is reflexive, the composition must be the identity map. It is a contradiction since $(I_{S_x})_{X^k} \neq \mathcal{O}_{X^k}$. It follows that $\alpha = 0$, which implies by (26) that F' is a subsheaf of $(\mathcal{G}^\vee)_{X^k}$. By (27) and the above discussion, we can furthermore conclude that F' is isomorphic to a subsheaf of the trivial bundle $H^1(\mathcal{O}_X(e)) \otimes \mathcal{O}_{X^k}$. The stability forces again that

$$\mu_{\widehat{h}_{X^k}}(F') = 0 = \mu_{\widehat{h}_{X^k}}(\mathcal{O}_{X^k})$$

and $F' \cong \mathcal{O}_{X^k}$. Moreover, we see from the construction of (27) that all global sections of the trivial bundle $H^1(\mathcal{O}_X(e)) \otimes \mathcal{O}_{X^k}$ are invariant under the permutation of $\mathfrak{S}_k$, hence F' itself is also $\mathfrak{S}_k$-invariant.

Now we perform an operation on the morphism

$$F'(\cong \mathcal{O}_{X^k}) \hookrightarrow (\mathcal{G}^\vee)_{X^k}$$

which is reverse to the one above. First we restrict the morphism to $X^k_\circ$. As the morphism is $\mathfrak{S}_k$-invariant, it descends to a morphism on $X^{[k]}_\circ$ (which is the free $\mathfrak{S}_k$-quotient of $X^k_\circ$. Since the diagonal D is of codimension 2 in $X^{[k]}$, the morphism extends to the entire $X^{[k]}$ to give a morphism $\mathcal{O}_{X^{[k]}} \to \mathcal{G}^\vee$. This morphism is injective on $X^{[k]}_\circ$, hence is an injective morphism itself. This contradicts Lemma 2.8. Hence the case of $c \cdot \widehat{h} \geqslant 0$ cannot happen.

This concludes that $(E_x)_{X^k}$ cannot be destabilized by any $\mathfrak{S}_k$-invariant subsheaf. Therefore E_x is $\mu_{\widehat{h}_k}$-stable. □

2.4. **A smooth connected component.**

In this section, we will interpret the universal sheaf $\mathcal{E}$ defined in (24) as a family of stable sheaves on $X^{[n]}$ whose base is a smooth connected component of the corresponding moduli space. We have shown above that each wrong-way fiber E_x of the family $\mathcal{E}$ is $\mu_{\widehat{h}_k}$-stable. We follow the idea in Proposition 1.8 to show the stability of all E_x with respect to a certain ample class near $\widehat{h}_k$.

Proposition 2.15. *There exists some ample class $H \in \mathrm{NS}(X^{[k]})$ near $\widehat{h}_k$, such that E_x is μ_H-stable for all $x \in X$ simultaneously.*

Proof. The same as in Proposition 1.8, the value of $c = \mu_\beta(E_x)$ is independent of the choice of $x \in X$. We still define

$$S := \{c_1(F) \mid F \subseteq E_x \text{ for some } x \in X \text{ such that } \mu_\beta(F) \geqslant c\}.$$

The proof of the present result is literally the same as the proof of Proposition 1.8, except that the step which shows that S is a finite set has to be modified.

For this purpose we make a few auxiliary definitions. Let $E'_x = \mathcal{G}^\vee \oplus I_{S_x}$ for each $x \in X$. We also define the set

$$S' := \{c_1(F') \mid F' \subseteq E'_x \text{ for some } x \in X \text{ such that } \mu_\beta(F') \geqslant c\}.$$

We claim that $S \subseteq S'$.

Indeed, by (25), every subsheaf $F \subseteq E_x$ is an extension of some subsheaf $F_2 \subseteq I_{S_x}$ by another subsheaf $F_1 \subseteq \mathcal{G}^\vee$. It is then clear that $F' = F_1 \oplus F_2$ is a subsheaf of E'_x, and that $c_1(F) = c_1(F')$. If F destabilizes E_x, then F' also destabilizes E'_x, which means that every element of S is also in S', as desired.

It remains to show that S' is finite. In fact, since $E'_x \subseteq (\mathcal{G}^\vee \oplus \mathcal{O}_{X^{[k]}})$ for all $x \in X$, we obtain that S' is a subset of

$$T' := \{c_1(F') \mid F' \subseteq (\mathcal{G}^\vee \oplus \mathcal{O}_{X^{[k]}}) \text{ such that } \mu_\beta(F') \geqslant c\},$$

which is finite by [8, Theorem 2.29], hence S' is also finite, which further implies the finiteness of S. This concludes the proof. □

Before proceeding to the main result of the section, we need to give the following alternative description of E_x using the integral functor Φ defined in (8):

Lemma 2.16. *E_x is the only non-trivial extension of I_{S_x} by $\mathcal{G}^\vee$. Moreover, there exists some $F_x \in \mathrm{D^b}(X)$ which fits in an exact triangle*

$$\mathcal{O}_x[-1] \xrightarrow{\alpha} \mathcal{O}_X(e)[1] \longrightarrow F_x \longrightarrow \mathcal{O}_x \tag{28}$$

such that $E_x = \Phi(F_x)$.

Proof. By Lemma 2.8 we have that $\mathcal{G}^\vee = \Phi(\mathcal{O}_X(e))[1]$. Also note that $I_{S_x} = \Phi(\mathcal{O}_x)$. As Φ is a $\mathbb{P}^{k-1}$-functor, using (9) we get

$$\mathrm{Ext}^*_{X^{[k]}}(I_{S_x}, \mathcal{G}^\vee) \cong \mathrm{Ext}^*_X(\mathcal{O}_x, \mathcal{O}_X(e)[1]) \otimes H^*(\mathbb{P}^{k-1}, \mathbb{C}).$$

It is easy to find that

$$\mathrm{Ext}^i_X(\mathcal{O}_x, \mathcal{O}_X(e)[1]) = \mathrm{Ext}^{i+1}_X(\mathcal{O}_x, \mathcal{O}_X(e)) = \begin{cases} \mathbb{C}, & i = 1; \\ 0, & i \neq 1. \end{cases}$$

Therefore $\mathrm{Ext}^1_{X^{[k]}}(I_{S_x}, \mathcal{G}^\vee) = \mathbb{C}$. Since E_x is locally free, the exact sequence (25) does not split. It follows that E_x is the only non-trivial extension of I_{S_x} by $\mathcal{G}^\vee$ up to isomorphism.

Moreover, the above argument also shows that there is a unique object $F_x \in \mathrm{D^b}(X)$, up to isomorphism, which is the unique "non-trivial extension" of $\mathcal{O}_x$ by $\mathcal{O}_X(e)[1]$. More precisely, let the morphism

$$\alpha : \mathcal{O}_x[-1] \longrightarrow \mathcal{O}_X(e)[1] \tag{29}$$

represent a non-trivial element $\alpha \in \mathrm{Hom}_X(\mathcal{O}_x[-1], \mathcal{O}_X(e)[1]) = \mathrm{Ext}^1_X(\mathcal{O}_x, \mathcal{O}_X(e)[1])$, unique up to a scalar, then $F_x = \mathrm{Cone}(\alpha)$. Since Φ is an exact functor, $\Phi(F_x)$ is an extension of $\Phi(\mathcal{O}_x)$ by $\Phi(\mathcal{O}_X(e))[1]$. This extension must be non-trivial, since F_x can be recovered from $\Phi(F_x)$ by first applying the right adjoint functor of Φ, and then truncating to degree -1 and 0; see [1, Theorem 3.1(a)]. It follows that $E_x = \Phi(F_x)$. $\square$

Equipped with the above lemma, in order to compute $\mathrm{Hom}_{X^{[k]}}(E_x, E_y)$ for $x \neq y$ and $\mathrm{Ext}^1_{X^{[k]}}(E_x, E_x)$, it suffices by (9) to compute $\mathrm{Ext}^*_X(F_x, F_y)$ for $x \neq y$ and $\mathrm{Ext}^*_X(F_x, F_x)$. This computation will be accomplished by the following series of lemmas.

Lemma 2.17. *For any $x \in X$, we have*

$$\mathrm{Ext}^i_X(F_x, \mathcal{O}_X(e)[1]) = \begin{cases} \mathbb{C}, & i = 2; \\ 0, & i \neq 2. \end{cases}$$

Proof. We apply the functor $\mathrm{Hom}_X(-, \mathcal{O}_X(e)[1])$ on (28) and write down the associated long exact sequence of extension groups

$$\begin{aligned}
\cdots \longrightarrow \mathrm{Hom}_X(\mathcal{O}_x, \mathcal{O}_X(e)[1]) &\longrightarrow \mathrm{Hom}_X(F_x, \mathcal{O}_X(e)[1]) \longrightarrow \mathrm{Hom}_X(\mathcal{O}_X(e)[1], \mathcal{O}_X(e)[1]) \\
\xrightarrow{\beta} \mathrm{Ext}^1_X(\mathcal{O}_x, \mathcal{O}_X(e)[1]) &\longrightarrow \mathrm{Ext}^1_X(F_x, \mathcal{O}_X(e)[1]) \longrightarrow \mathrm{Ext}^1_X(\mathcal{O}_X(e)[1], \mathcal{O}_X(e)[1]) \\
\longrightarrow \mathrm{Ext}^2_X(\mathcal{O}_x, \mathcal{O}_X(e)[1]) &\longrightarrow \mathrm{Ext}^2_X(F_x, \mathcal{O}_X(e)[1]) \longrightarrow \mathrm{Ext}^2_X(\mathcal{O}_X(e)[1], \mathcal{O}_X(e)[1]) \\
\longrightarrow \mathrm{Ext}^3_X(\mathcal{O}_x, \mathcal{O}_X(e)[1]) &\longrightarrow \cdots
\end{aligned}$$

By computing the left column and the right column, we obtain

$$\begin{aligned}
\cdots \longrightarrow 0 &\longrightarrow \mathrm{Hom}_X(F_x, \mathcal{O}_X(e)[1]) \longrightarrow \mathbb{C} \\
\xrightarrow{\beta} \mathbb{C} &\longrightarrow \mathrm{Ext}^1_X(F_x, \mathcal{O}_X(e)[1]) \longrightarrow 0 \\
\longrightarrow 0 &\longrightarrow \mathrm{Ext}^2_X(F_x, \mathcal{O}_X(e)[1]) \longrightarrow \mathbb{C} \\
\longrightarrow 0 &\longrightarrow \cdots
\end{aligned}$$

We note that the map $\beta(-) = \alpha \cup (-)$ for the class $\alpha \in \mathrm{Ext}^1_X(\mathcal{O}_x, \mathcal{O}_X(e)[1])$ defined in (29), which sends the element 1 to α, hence β is a non-zero map, and the conclusion follows. $\square$

Lemma 2.18. *For any $x \in X$, we have*

$$\mathrm{Ext}^i_X(F_x, \mathcal{O}_x) = \begin{cases} \mathbb{C}, & i = 0; \\ \mathbb{C}^2, & i = 1; \\ 0, & \textit{otherwise}, \end{cases}$$

and

$$\mathrm{Ext}^i_X(F_x, F_x) = \begin{cases} \mathbb{C}, & i = 0 \textit{ or } 2; \\ \mathbb{C}^2, & i = 1; \\ 0, & \textit{otherwise}, \end{cases}$$

Proof. For the first claim, we apply the functor $\mathrm{Hom}_X(-, \mathcal{O}_x)$ on (28) and write down the associated long exact sequence of extension groups

$$\begin{aligned}
\cdots \longrightarrow \mathrm{Hom}_X(\mathcal{O}_x, \mathcal{O}_x) &\longrightarrow \mathrm{Hom}_X(F_x, \mathcal{O}_x) \longrightarrow \mathrm{Hom}_X(\mathcal{O}_X(e)[1], \mathcal{O}_x) \\
\longrightarrow \mathrm{Ext}^1_X(\mathcal{O}_x, \mathcal{O}_x) &\longrightarrow \mathrm{Ext}^1_X(F_x, \mathcal{O}_x) \longrightarrow \mathrm{Ext}^1_X(\mathcal{O}_X(e)[1], \mathcal{O}_x) \\
\xrightarrow{\gamma} \mathrm{Ext}^2_X(\mathcal{O}_x, \mathcal{O}_x) &\longrightarrow \mathrm{Ext}^2_X(F_x, \mathcal{O}_x) \longrightarrow \mathrm{Ext}^2_X(\mathcal{O}_X(e)[1], \mathcal{O}_x) \\
\longrightarrow \cdots &
\end{aligned}$$

By computing the left column and the right column, we obtain

$$\begin{aligned}\cdots \longrightarrow \mathbb{C} \longrightarrow \operatorname{Hom}_X(F_x, \mathcal{O}_x) \longrightarrow 0\\ \longrightarrow \mathbb{C}^2 \longrightarrow \operatorname{Ext}^1_X(F_x, \mathcal{O}_x) \longrightarrow \mathbb{C}\\ \xrightarrow{\gamma} \mathbb{C} \longrightarrow \operatorname{Ext}^2_X(F_x, \mathcal{O}_x) \longrightarrow 0\\ \longrightarrow \cdots\end{aligned}$$

We note that the map $\gamma(-) = \alpha \cup (-)$ for the class $\alpha \in \operatorname{Ext}^1_X(\mathcal{O}_x, \mathcal{O}_X(e)[1])$ defined in (29), where the cup product

$$\cup : \operatorname{Ext}^1_X(\mathcal{O}_x, \mathcal{O}_X(e)[1]) \times \operatorname{Ext}^1_X(\mathcal{O}_X(e)[1], \mathcal{O}_x) \longrightarrow \operatorname{Ext}^2_X(\mathcal{O}_x, \mathcal{O}_x)$$

is in fact the standard pairing between two vector spaces which are dual to each other (via Serre duality). It follows that γ is a non-zero map, and the first claim follows.

For the second claim, we apply the functor $\operatorname{Hom}_X(F_x, -)$ on (28) and write down the associated long exact sequence of extension groups

$$\begin{aligned}\cdots \longrightarrow \operatorname{Hom}_X(F_x, \mathcal{O}_X(e)[1]) \longrightarrow \operatorname{Hom}_X(F_x, F_x) \longrightarrow \operatorname{Hom}_X(F_x, \mathcal{O}_x)\\ \longrightarrow \operatorname{Ext}^1_X(F_x, \mathcal{O}_X(e)[1]) \longrightarrow \operatorname{Ext}^1_X(F_x, F_x) \longrightarrow \operatorname{Ext}^1_X(F_x, \mathcal{O}_x)\\ \longrightarrow \operatorname{Ext}^2_X(F_x, \mathcal{O}_X(e)[1]) \longrightarrow \operatorname{Ext}^2_X(F_x, F_x) \longrightarrow \operatorname{Ext}^2_X(F_x, \mathcal{O}_x)\\ \longrightarrow \cdots\end{aligned}$$

By applying Lemma 2.17 and the above result we obtain

$$\begin{aligned}\cdots \longrightarrow 0 \longrightarrow \operatorname{Hom}_X(F_x, F_x) \longrightarrow \mathbb{C}\\ \longrightarrow 0 \longrightarrow \operatorname{Ext}^1_X(F_x, F_x) \longrightarrow \mathbb{C}^2\\ \longrightarrow \mathbb{C} \longrightarrow \operatorname{Ext}^2_X(F_x, F_x) \longrightarrow 0\\ \longrightarrow \cdots\end{aligned}$$

It is clear that $\operatorname{Hom}_X(F_x, F_x) = \mathbb{C}$. By Serre duality we obtain $\operatorname{Ext}^2_X(F_x, F_x) = \mathbb{C}$, which forces $\operatorname{Ext}^1(F_x, F_x) = \mathbb{C}^2$. This proves the second claim. □

Lemma 2.19. *For any $x, y \in X$ with $x \neq y$, we have*

$$\operatorname{Ext}^i_X(F_x, \mathcal{O}_y) = \begin{cases} \mathbb{C}, & i = 1;\\ 0, & i \neq 1,\end{cases}$$

and

$$\operatorname{Ext}^i_X(F_x, F_y) = \begin{cases} \mathbb{C}, & i = 1;\\ 0, & i \neq 1.\end{cases}$$

Proof. We omit this proof, since it is completely parallel to that of Lemma 2.18. □

Corollary 2.20. *For any $x \in X$, we have $\operatorname{Ext}^1_{X^{[k]}}(E_x, E_x) = \mathbb{C}^2$; for any $x, y \in X$ with $x \neq y$, we have $\operatorname{Hom}_{X^{[k]}}(E_x, E_y) = 0$.*

Proof. By (9) we have

$$\begin{aligned}\operatorname{Ext}^*_{X^{[k]}}(E_x, E_x) = \operatorname{Ext}^*_X(F_x, F_x) \otimes H^*(\mathbb{P}^{k-1}, \mathbb{C});\\ \operatorname{Ext}^*_{X^{[k]}}(E_x, E_y) = \operatorname{Ext}^*_X(F_x, F_y) \otimes H^*(\mathbb{P}^{k-1}, \mathbb{C}).\end{aligned}$$

Then the claim follows from Lemma 2.18 and Lemma 2.19. □

We are ready to prove the main result of the section. Let H be an ample class that satisfies Proposition 2.15, and $\mathcal{M}$ the moduli space of μ_H-stable sheaves on $X^{[k]}$ with the same numerical invariants as E_x. Then the universal family $\mathcal{E}$ defines a classifying morphism

$$f \colon X \longrightarrow \mathcal{M}, \quad x \longmapsto [E_x]. \tag{30}$$

Similar as Theorem 1.10, we obtain

Theorem 2.21. *The classifying morphism* (30) *defined by the family $\mathcal{E}$ identifies X with a smooth connected component of $\mathcal{M}$.*

Proof. Similar as in the proof of Theorem 1.10, we need to prove that f is injective on closed points and that $\dim(T_{[E_x]}\mathcal{M}) = 2$ for all $x \in X$, both of which follow immediately from Corollary 2.20. □

Remark 2.22. The stable vector bundles constructed in Proposition 1.8 as well as Proposition 2.15 are not tautological bundles as the rank of a tautological bundle is always divisible by k, but in our cases the ranks are $k+1$ and $2k-1$.

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