

Feedforward Neural Networks in Depth, Part 2: Activation Functions

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This is the second post of a three-part series in which we derive the mathematics behind feedforward neural networks. We worked our way through forward and backward propagations in [the first post](#), but if you remember, we only mentioned activation functions in passing. In particular, we did not derive an analytic expression for $\partial a_{j,i}^{[l]} / \partial z_{j,i}^{[l]}$ or, by extension, $\partial J / \partial z_{j,i}^{[l]}$. So let us pick up the derivations where we left off.

ReLU

The rectified linear unit, or ReLU for short, is given by

$$\begin{aligned} a_{j,i}^{[l]} &= g_j^{[l]}(z_{1,i}^{[l]}, \dots, z_{j,i}^{[l]}, \dots, z_{n^{[l]},i}^{[l]}) \\ &= \max(0, z_{j,i}^{[l]}) \\ &= \begin{cases} z_{j,i}^{[l]} & \text{if } z_{j,i}^{[l]} > 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In other words,

$$\mathbf{A}^{[l]} = \max(0, \mathbf{Z}^{[l]}). \tag{1}$$

Next, we compute the partial derivatives of the activations in the current layer:

$$\begin{aligned} \frac{\partial a_{j,i}^{[l]}}{\partial z_{j,i}^{[l]}} &:= \begin{cases} 1 & \text{if } z_{j,i}^{[l]} > 0, \\ 0 & \text{otherwise,} \end{cases} \\ &= I(z_{j,i}^{[l]} > 0), \\ \frac{\partial a_{p,i}^{[l]}}{\partial z_{j,i}^{[l]}} &= 0, \quad \forall p \neq j. \end{aligned}$$

It follows that

$$\begin{aligned}
\frac{\partial J}{\partial z_{j,i}^{[l]}} &= \sum_p \frac{\partial J}{\partial a_{p,i}^{[l]}} \frac{\partial a_{p,i}^{[l]}}{\partial z_{j,i}^{[l]}} \\
&= \frac{\partial J}{\partial a_{j,i}^{[l]}} \frac{\partial a_{j,i}^{[l]}}{\partial z_{j,i}^{[l]}} + \sum_{p \neq j} \frac{\partial J}{\partial a_{p,i}^{[l]}} \frac{\partial a_{p,i}^{[l]}}{\partial z_{j,i}^{[l]}} \\
&= \frac{\partial J}{\partial a_{j,i}^{[l]}} I(z_{j,i}^{[l]} > 0),
\end{aligned}$$

which we can vectorize as

$$\frac{\partial J}{\partial \mathbf{Z}^{[l]}} = \frac{\partial J}{\partial \mathbf{A}^{[l]}} \odot I(\mathbf{Z}^{[l]} > 0), \quad (2)$$

where \odot denotes element-wise multiplication.

Sigmoid

The sigmoid activation function is given by

$$\begin{aligned}
a_{j,i}^{[l]} &= g_j(z_{1,i}^{[l]}, \dots, z_{j,i}^{[l]}, \dots, z_{n^{[l]},i}^{[l]}) \\
&= \sigma(z_{j,i}^{[l]}) \\
&= \frac{1}{1 + \exp(-z_{j,i}^{[l]})}.
\end{aligned}$$

Vectorization yields

$$\mathbf{A}^{[l]} = \frac{1}{1 + \exp(-\mathbf{Z}^{[l]})}. \quad (3)$$

To practice backward propagation, first, we construct a computation graph:

$$\begin{aligned}
u_0 &= z_{j,i}^{[l]}, \\
u_1 &= -u_0, \\
u_2 &= \exp(u_1), \\
u_3 &= 1 + u_2, \\
u_4 &= \frac{1}{u_3} = a_{j,i}^{[l]}.
\end{aligned}$$

Then, we perform an outside first traversal of the chain rule:

$$\begin{aligned}
\frac{\partial a_{j,i}^{[l]}}{\partial u_4} &= 1, \\
\frac{\partial a_{j,i}^{[l]}}{\partial u_3} &= \frac{\partial a_{j,i}^{[l]}}{\partial u_4} \frac{\partial u_4}{\partial u_3} = -\frac{1}{u_3^2} = -\frac{1}{(1 + \exp(-z_{j,i}^{[l]}))^2}, \\
\frac{\partial a_{j,i}^{[l]}}{\partial u_2} &= \frac{\partial a_{j,i}^{[l]}}{\partial u_3} \frac{\partial u_3}{\partial u_2} = -\frac{1}{u_3^2} = -\frac{1}{(1 + \exp(-z_{j,i}^{[l]}))^2}, \\
\frac{\partial a_{j,i}^{[l]}}{\partial u_1} &= \frac{\partial a_{j,i}^{[l]}}{\partial u_2} \frac{\partial u_2}{\partial u_1} = -\frac{1}{u_3^2} \exp(u_1) = -\frac{\exp(-z_{j,i}^{[l]})}{(1 + \exp(-z_{j,i}^{[l]}))^2}, \\
\frac{\partial a_{j,i}^{[l]}}{\partial u_0} &= \frac{\partial a_{j,i}^{[l]}}{\partial u_1} \frac{\partial u_1}{\partial u_0} = \frac{1}{u_3^2} \exp(u_1) = \frac{\exp(-z_{j,i}^{[l]})}{(1 + \exp(-z_{j,i}^{[l]}))^2}.
\end{aligned}$$

Let us simplify:

$$\begin{aligned}
\frac{\partial a_{j,i}^{[l]}}{\partial z_{j,i}^{[l]}} &= \frac{\exp(-z_{j,i}^{[l]})}{(1 + \exp(-z_{j,i}^{[l]}))^2} \\
&= \frac{1 + \exp(-z_{j,i}^{[l]}) - 1}{(1 + \exp(-z_{j,i}^{[l]}))^2} \\
&= \frac{1}{1 + \exp(-z_{j,i}^{[l]})} - \frac{1}{(1 + \exp(-z_{j,i}^{[l]}))^2} \\
&= a_{j,i}^{[l]}(1 - a_{j,i}^{[l]}).
\end{aligned}$$

We also note that

$$\frac{\partial a_{p,i}^{[l]}}{\partial z_{j,i}^{[l]}} = 0, \quad \forall p \neq j.$$

Consequently,

$$\begin{aligned}
\frac{\partial J}{\partial z_{j,i}^{[l]}} &= \sum_p \frac{\partial J}{\partial a_{p,i}^{[l]}} \frac{\partial a_{p,i}^{[l]}}{\partial z_{j,i}^{[l]}} \\
&= \frac{\partial J}{\partial a_{j,i}^{[l]}} \frac{\partial a_{j,i}^{[l]}}{\partial z_{j,i}^{[l]}} + \sum_{p \neq j} \frac{\partial J}{\partial a_{p,i}^{[l]}} \frac{\partial a_{p,i}^{[l]}}{\partial z_{j,i}^{[l]}} \\
&= \frac{\partial J}{\partial a_{j,i}^{[l]}} a_{j,i}^{[l]}(1 - a_{j,i}^{[l]}).
\end{aligned}$$

Lastly, no summations mean trivial vectorization:

$$\frac{\partial J}{\partial \mathbf{Z}^{[l]}} = \frac{\partial J}{\partial \mathbf{A}^{[l]}} \odot \mathbf{A}^{[l]} \odot (1 - \mathbf{A}^{[l]}). \quad (4)$$

Tanh

The hyperbolic tangent function, i.e., the tanh activation function, is given by

$$\begin{aligned} a_{j,i}^{[l]} &= g_j^{[l]}(z_{1,i}^{[l]}, \dots, z_{j,i}^{[l]}, \dots, z_{n^{[l]},i}^{[l]}) \\ &= \tanh(z_{j,i}^{[l]}) \\ &= \frac{\exp(z_{j,i}^{[l]}) - \exp(-z_{j,i}^{[l]})}{\exp(z_{j,i}^{[l]}) + \exp(-z_{j,i}^{[l]})}. \end{aligned}$$

By utilizing element-wise multiplication, we get

$$\mathbf{A}^{[l]} = \frac{1}{\exp(\mathbf{Z}^{[l]}) + \exp(-\mathbf{Z}^{[l]})} \odot (\exp(\mathbf{Z}^{[l]}) - \exp(-\mathbf{Z}^{[l]})). \quad (5)$$

Once again, let us introduce intermediate variables to practice backward propagation:

$$\begin{aligned} u_0 &= z_{j,i}^{[l]}, \\ u_1 &= -u_0, \\ u_2 &= \exp(u_0), \\ u_3 &= \exp(u_1), \\ u_4 &= u_2 - u_3, \\ u_5 &= u_2 + u_3, \\ u_6 &= \frac{1}{u_5}, \\ u_7 &= u_4 u_6 = a_{j,i}^{[l]}. \end{aligned}$$

Next, we compute the partial derivatives:

$$\begin{aligned}
\frac{\partial a_{j,i}^{[l]}}{\partial u_7} &= 1, \\
\frac{\partial a_{j,i}^{[l]}}{\partial u_6} &= \frac{\partial a_{j,i}^{[l]}}{\partial u_7} \frac{\partial u_7}{\partial u_6} = u_4 = \exp(z_{j,i}^{[l]}) - \exp(-z_{j,i}^{[l]}), \\
\frac{\partial a_{j,i}^{[l]}}{\partial u_5} &= \frac{\partial a_{j,i}^{[l]}}{\partial u_6} \frac{\partial u_6}{\partial u_5} = -u_4 \frac{1}{u_5^2} = -\frac{\exp(z_{j,i}^{[l]}) - \exp(-z_{j,i}^{[l]})}{(\exp(z_{j,i}^{[l]}) + \exp(-z_{j,i}^{[l]}))^2}, \\
\frac{\partial a_{j,i}^{[l]}}{\partial u_4} &= \frac{\partial a_{j,i}^{[l]}}{\partial u_7} \frac{\partial u_7}{\partial u_4} = u_6 = \frac{1}{\exp(z_{j,i}^{[l]}) + \exp(-z_{j,i}^{[l]})}, \\
\frac{\partial a_{j,i}^{[l]}}{\partial u_3} &= \frac{\partial a_{j,i}^{[l]}}{\partial u_4} \frac{\partial u_4}{\partial u_3} + \frac{\partial a_{j,i}^{[l]}}{\partial u_5} \frac{\partial u_5}{\partial u_3} \\
&= -u_6 - u_4 \frac{1}{u_5^2} \\
&= -\frac{1}{\exp(z_{j,i}^{[l]}) + \exp(-z_{j,i}^{[l]})} - \frac{\exp(z_{j,i}^{[l]}) - \exp(-z_{j,i}^{[l]})}{(\exp(z_{j,i}^{[l]}) + \exp(-z_{j,i}^{[l]}))^2} \\
&= -\frac{2 \exp(z_{j,i}^{[l]})}{(\exp(z_{j,i}^{[l]}) + \exp(-z_{j,i}^{[l]}))^2}, \\
\frac{\partial a_{j,i}^{[l]}}{\partial u_2} &= \frac{\partial a_{j,i}^{[l]}}{\partial u_4} \frac{\partial u_4}{\partial u_2} + \frac{\partial a_{j,i}^{[l]}}{\partial u_5} \frac{\partial u_5}{\partial u_2} \\
&= u_6 - u_4 \frac{1}{u_5^2} \\
&= \frac{1}{\exp(z_{j,i}^{[l]}) + \exp(-z_{j,i}^{[l]})} - \frac{\exp(z_{j,i}^{[l]}) - \exp(-z_{j,i}^{[l]})}{(\exp(z_{j,i}^{[l]}) + \exp(-z_{j,i}^{[l]}))^2} \\
&= \frac{2 \exp(-z_{j,i}^{[l]})}{(\exp(z_{j,i}^{[l]}) + \exp(-z_{j,i}^{[l]}))^2}, \\
\frac{\partial a_{j,i}^{[l]}}{\partial u_1} &= \frac{\partial a_{j,i}^{[l]}}{\partial u_3} \frac{\partial u_3}{\partial u_1} \\
&= \left(-u_6 - u_4 \frac{1}{u_5^2}\right) \exp(u_1) \\
&= -\frac{2 \exp(z_{j,i}^{[l]}) \exp(-z_{j,i}^{[l]})}{(\exp(z_{j,i}^{[l]}) + \exp(-z_{j,i}^{[l]}))^2}, \\
\frac{\partial a_{j,i}^{[l]}}{\partial u_0} &= \frac{\partial a_{j,i}^{[l]}}{\partial u_1} \frac{\partial u_1}{\partial u_0} + \frac{\partial a_{j,i}^{[l]}}{\partial u_2} \frac{\partial u_2}{\partial u_0} \\
&= \left(-\frac{2 \exp(z_{j,i}^{[l]}) \exp(-z_{j,i}^{[l]})}{(\exp(z_{j,i}^{[l]}) + \exp(-z_{j,i}^{[l]}))^2}\right) \exp(u_1) + \frac{2 \exp(-z_{j,i}^{[l]})}{(\exp(z_{j,i}^{[l]}) + \exp(-z_{j,i}^{[l]}))^2} \exp(u_2)
\end{aligned}$$

$$= \frac{4 \exp(z_{j,i}^{[l]}) \exp(-z_{j,i}^{[l]})}{(\exp(z_{j,i}^{[l]}) + \exp(-z_{j,i}^{[l]}))^2}.$$

It follows that

$$\begin{aligned} \frac{\partial a_{j,i}^{[l]}}{\partial z_{j,i}^{[l]}} &= \frac{4 \exp(z_{j,i}^{[l]}) \exp(-z_{j,i}^{[l]})}{(\exp(z_{j,i}^{[l]}) + \exp(-z_{j,i}^{[l]}))^2} \\ &= \frac{\exp(z_{j,i}^{[l]})^2 + 2 \exp(z_{j,i}^{[l]}) \exp(-z_{j,i}^{[l]}) + \exp(-z_{j,i}^{[l]})^2}{(\exp(z_{j,i}^{[l]}) + \exp(-z_{j,i}^{[l]}))^2} \\ &\quad - \frac{\exp(z_{j,i}^{[l]})^2 - 2 \exp(z_{j,i}^{[l]}) \exp(-z_{j,i}^{[l]}) + \exp(-z_{j,i}^{[l]})^2}{(\exp(z_{j,i}^{[l]}) + \exp(-z_{j,i}^{[l]}))^2} \\ &= 1 - \frac{(\exp(z_{j,i}^{[l]}) - \exp(-z_{j,i}^{[l]}))^2}{(\exp(z_{j,i}^{[l]}) + \exp(-z_{j,i}^{[l]}))^2} \\ &= 1 - a_{j,i}^{[l]} a_{j,i}^{[l]}. \end{aligned}$$

Similar to the sigmoid activation function, we also have

$$\frac{\partial a_{p,i}^{[l]}}{\partial z_{j,i}^{[l]}} = 0, \quad \forall p \neq j.$$

Thus,

$$\begin{aligned} \frac{\partial J}{\partial z_{j,i}^{[l]}} &= \sum_p \frac{\partial J}{\partial a_{p,i}^{[l]}} \frac{\partial a_{p,i}^{[l]}}{\partial z_{j,i}^{[l]}} \\ &= \frac{\partial J}{\partial a_{j,i}^{[l]}} \frac{\partial a_{j,i}^{[l]}}{\partial z_{j,i}^{[l]}} + \sum_{p \neq j} \frac{\partial J}{\partial a_{p,i}^{[l]}} \frac{\partial a_{p,i}^{[l]}}{\partial z_{j,i}^{[l]}} \\ &= \frac{\partial J}{\partial a_{j,i}^{[l]}} (1 - a_{j,i}^{[l]} a_{j,i}^{[l]}), \end{aligned}$$

which implies that

$$\frac{\partial J}{\partial \mathbf{Z}^{[l]}} = \frac{\partial J}{\partial \mathbf{A}^{[l]}} \odot (1 - \mathbf{A}^{[l]} \odot \mathbf{A}^{[l]}). \quad (6)$$

Softmax

The softmax activation function is given by

$$\begin{aligned} a_{j,i}^{[l]} &= g_j^{[l]}(z_{1,i}^{[l]}, \dots, z_{j,i}^{[l]}, \dots, z_{n^{[l]},i}^{[l]}) \\ &= \frac{\exp(z_{j,i}^{[l]})}{\sum_p \exp(z_{p,i}^{[l]})}. \end{aligned}$$

Vectorization results in

$$\mathbf{A}^{[l]} = \frac{1}{\underbrace{\text{broadcast}(\sum_{\text{axis}=0} \exp(\mathbf{Z}^{[l]}))}_{\text{row vector}}} \odot \exp(\mathbf{Z}^{[l]}). \quad (7)$$

To begin with, we construct a computation graph for the j th activation of the current layer:

$$\begin{aligned} u_{-1} &= z_{j,i}^{[l]}, \\ u_{0,p} &= z_{p,i}^{[l]}, & \forall p \neq j, \\ u_1 &= \exp(u_{-1}), \\ u_{2,p} &= \exp(u_{0,p}), & \forall p \neq j, \\ u_3 &= u_1 + \sum_{p \neq j} u_{2,p}, \\ u_4 &= \frac{1}{u_3}, \\ u_5 &= u_1 u_4 = a_{j,i}^{[l]}. \end{aligned}$$

By applying the chain rule, we get

$$\begin{aligned}
\frac{\partial a_{j,i}^{[l]}}{\partial u_5} &= 1, \\
\frac{\partial a_{j,i}^{[l]}}{\partial u_4} &= \frac{\partial a_{j,i}^{[l]}}{\partial u_5} \frac{\partial u_5}{\partial u_4} = u_1 = \exp(z_{j,i}^{[l]}), \\
\frac{\partial a_{j,i}^{[l]}}{\partial u_3} &= \frac{\partial a_{j,i}^{[l]}}{\partial u_4} \frac{\partial u_4}{\partial u_3} = -u_1 \frac{1}{u_3^2} = -\frac{\exp(z_{j,i}^{[l]})}{(\sum_p \exp(z_{p,i}^{[l]}))^2}, \\
\frac{\partial a_{j,i}^{[l]}}{\partial u_1} &= \frac{\partial a_{j,i}^{[l]}}{\partial u_3} \frac{\partial u_3}{\partial u_1} + \frac{\partial a_{j,i}^{[l]}}{\partial u_5} \frac{\partial u_5}{\partial u_1} \\
&= -u_1 \frac{1}{u_3^2} + u_4 \\
&= -\frac{\exp(z_{j,i}^{[l]})}{(\sum_p \exp(z_{p,i}^{[l]}))^2} + \frac{1}{\sum_p \exp(z_{p,i}^{[l]})}, \\
\frac{\partial a_{j,i}^{[l]}}{\partial u_{-1}} &= \frac{\partial a_{j,i}^{[l]}}{\partial u_1} \frac{\partial u_1}{\partial u_{-1}} \\
&= \left(-u_1 \frac{1}{u_3^2} + u_4 \right) \exp(u_{-1}) \\
&= -\frac{\exp(z_{j,i}^{[l]})^2}{(\sum_p \exp(z_{p,i}^{[l]}))^2} + \frac{\exp(z_{j,i}^{[l]})}{\sum_p \exp(z_{p,i}^{[l]})}.
\end{aligned}$$

Next, we need to take into account that $z_{j,i}^{[l]}$ also affects other activations in the same layer:

$$\begin{aligned}
u_{-1} &= z_{j,i}^{[l]}, \\
u_{0,p} &= z_{p,i}^{[l]}, & \forall p \neq j, \\
u_1 &= \exp(u_{-1}), \\
u_{2,p} &= \exp(u_{0,p}), & \forall p \neq j, \\
u_3 &= u_1 + \sum_{p \neq j} u_{2,p}, \\
u_4 &= \frac{1}{u_3}, \\
u_5 &= u_{2,p} u_4 = a_{p,i}^{[l]}, & \forall p \neq j.
\end{aligned}$$

Backward propagation gives us the remaining partial derivatives:

$$\begin{aligned}
\frac{\partial a_{p,i}^{[l]}}{\partial u_5} &= 1, \\
\frac{\partial a_{p,i}^{[l]}}{\partial u_4} &= \frac{\partial a_{p,i}^{[l]}}{\partial u_5} \frac{\partial u_5}{\partial u_4} = u_{2,p} = \exp(z_{p,i}^{[l]}), \\
\frac{\partial a_{p,i}^{[l]}}{\partial u_3} &= \frac{\partial a_{p,i}^{[l]}}{\partial u_4} \frac{\partial u_4}{\partial u_3} = -u_{2,p} \frac{1}{u_3^2} = -\frac{\exp(z_{p,i}^{[l]})}{(\sum_p \exp(z_{p,i}^{[l]}))^2}, \\
\frac{\partial a_{p,i}^{[l]}}{\partial u_1} &= \frac{\partial a_{p,i}^{[l]}}{\partial u_3} \frac{\partial u_3}{\partial u_1} = -u_{2,p} \frac{1}{u_3^2} = -\frac{\exp(z_{p,i}^{[l]})}{(\sum_p \exp(z_{p,i}^{[l]}))^2}, \\
\frac{\partial a_{p,i}^{[l]}}{\partial u_{-1}} &= \frac{\partial a_{p,i}^{[l]}}{\partial u_1} \frac{\partial u_1}{\partial u_{-1}} = -u_{2,p} \frac{1}{u_3^2} \exp(u_{-1}) = -\frac{\exp(z_{p,i}^{[l]}) \exp(z_{j,i}^{[l]})}{(\sum_p \exp(z_{p,i}^{[l]}))^2}.
\end{aligned}$$

We now know that

$$\begin{aligned}
\frac{\partial a_{j,i}^{[l]}}{\partial z_{j,i}^{[l]}} &= -\frac{\exp(z_{j,i}^{[l]})^2}{(\sum_p \exp(z_{p,i}^{[l]}))^2} + \frac{\exp(z_{j,i}^{[l]})}{\sum_p \exp(z_{p,i}^{[l]})} \\
&= a_{j,i}^{[l]}(1 - a_{j,i}^{[l]}), \\
\frac{\partial a_{p,i}^{[l]}}{\partial z_{j,i}^{[l]}} &= -\frac{\exp(z_{p,i}^{[l]}) \exp(z_{j,i}^{[l]})}{(\sum_p \exp(z_{p,i}^{[l]}))^2} \\
&= -a_{p,i}^{[l]} a_{j,i}^{[l]}, \quad \forall p \neq j.
\end{aligned}$$

I, Deep Learning

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$$\begin{aligned}
\frac{\partial J}{\partial z_{j,i}^{[l]}} &= \sum_p \frac{\partial J}{\partial a_{p,i}^{[l]}} \frac{\partial a_{p,i}^{[l]}}{\partial z_{j,i}^{[l]}} \\
&= \frac{\partial J}{\partial a_{j,i}^{[l]}} \frac{\partial a_{j,i}^{[l]}}{\partial z_{j,i}^{[l]}} + \sum_{p \neq j} \frac{\partial J}{\partial a_{p,i}^{[l]}} \frac{\partial a_{p,i}^{[l]}}{\partial z_{j,i}^{[l]}} \\
&= \frac{\partial J}{\partial a_{j,i}^{[l]}} a_{j,i}^{[l]}(1 - a_{j,i}^{[l]}) - \sum_{p \neq j} \frac{\partial J}{\partial a_{p,i}^{[l]}} a_{p,i}^{[l]} a_{j,i}^{[l]} \\
&= a_{j,i}^{[l]} \left(\frac{\partial J}{\partial a_{j,i}^{[l]}} (1 - a_{j,i}^{[l]}) - \sum_{p \neq j} \frac{\partial J}{\partial a_{p,i}^{[l]}} a_{p,i}^{[l]} \right) \\
&= a_{j,i}^{[l]} \left(\frac{\partial J}{\partial a_{j,i}^{[l]}} (1 - a_{j,i}^{[l]}) - \sum_p \frac{\partial J}{\partial a_{p,i}^{[l]}} a_{p,i}^{[l]} + \frac{\partial J}{\partial a_{j,i}^{[l]}} a_{j,i}^{[l]} \right) \\
&= a_{j,i}^{[l]} \left(\frac{\partial J}{\partial a_{j,i}^{[l]}} - \sum_p \frac{\partial J}{\partial a_{p,i}^{[l]}} a_{p,i}^{[l]} \right),
\end{aligned}$$

which we can vectorize as

$$\frac{\partial J}{\partial \mathbf{z}_{:,i}^{[l]}} = \mathbf{a}_{:,i}^{[l]} \odot \left(\frac{\partial J}{\partial \mathbf{a}_{:,i}^{[l]}} - \underbrace{\mathbf{a}_{:,i}^{[l]\top} \frac{\partial J}{\partial \mathbf{a}_{:,i}^{[l]}}}_{\text{scalar}} \right).$$

Let us not stop with the vectorization just yet:

$$\frac{\partial J}{\partial \mathbf{Z}^{[l]}} = \mathbf{A}^{[l]} \odot \left(\frac{\partial J}{\partial \mathbf{A}^{[l]}} - \underbrace{\text{broadcast} \left(\sum_{\text{axis}=0} \frac{\partial J}{\partial \mathbf{A}^{[l]}} \odot \mathbf{A}^{[l]} \right)}_{\text{row vector}} \right). \quad (8)$$