First we define known quantities

In[∘]:=

$$\Delta := r^2 - 2 M r + a^2$$

$$\Sigma := r^2 + a^2 \text{Cos}[\theta]^2$$

$$J := C\, \text{DiracDelta}[r - r0]\, \text{DiracDelta}\left[\theta - \frac{\pi}{2}\right]$$

$$Jm\dagger := \left(\sqrt{2}\left(r - i\, a \,\text{Cos}[\theta]\right)\right)^{-1} + i\left(r^2 + a^2\right) \text{Sin}[\theta]\, J$$

$$Jn := -\frac{a\,\Delta}{\Sigma} \text{Sin}[\theta]^2\, J$$

Now for J2, we can ignore $\partial_\varphi$ since Jm and Jn are independent of it

$$J_2 = \frac{-\Delta}{2\sqrt{2}\,\Sigma\,(r - ia\cos\theta)^2}\left[\sqrt{2}\left(\frac{\partial}{\partial r} - \frac{a}{\Delta}\frac{\partial}{\partial\varphi} + \frac{1}{r - ia\cos\theta}\right)(r - ia\cos\theta)^2 J_{\overline{m}}\right.$$
$$\left. + 2\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} + \frac{ia\sin\theta}{r - ia\cos\theta}\right)\frac{\Sigma\,(r - ia\cos\theta)}{\Delta} J_n\right] \qquad (2.12)$$

In[∘]:=

$$J2 := \frac{-\Delta}{2\sqrt{2}\,\Sigma\left(r - i\, a\, \text{Cos}[\theta]\right)^2}$$
$$\left(\sqrt{2}\left(\partial_r \# + \frac{1}{r - i\, a\, \text{Cos}[\theta]} \#\right) \&@\left(\left(r - i\, a\, \text{Cos}[\theta]\right)^2 Jm\dagger\right) + 2\left(\partial_\theta \# + \frac{i\, a\, \text{Sin}[\theta]}{r - i\, a\, \text{Cos}[\theta]} \#\right) \&@\right.$$
$$\left.\left(\frac{\Sigma\left(r - i\, a\, \text{Cos}[\theta]\right)}{\Delta} Jn\right)\right)$$

Finally for the integration we need to first look at the definition of the SpinWeighted spherical harmonics.

Since all SpinWeighted spherical harmonics, here denoted ${}_sY_{\text{lm}}(\theta,\phi)$= Y[s,l,$\theta$,$\phi$] behave in the $\phi$ argument as Exp[$i$ m $\phi$] and since all other variables are independent of $\phi$ in the equation above We have

$\int$Y[s,l,m,$\theta$,$\phi$]= 2 $\pi$ DiracDelta[m,0] Y[s,l,m,$\theta$,0]. Then we can integrate over $\theta$.

e the source term is given by

$${}^2J_{lm}(r) = \int_0^{2\pi}\int_0^{\pi} \frac{(r - ia\cos\theta)^2}{(r_+ - r_-)^2}\; \Sigma\, J_2\;\; {}_{-1}\overline{Y}_{lm} \sin\theta\, d\theta\, d\varphi$$

In[∘]:=

We can split the integrand into two parts depending if we have DiracDelta$\left[\theta - \frac{\pi}{2}\right]$ or the derivative DiracDelta'$\left[\theta - \frac{\pi}{2}\right]$.

and we will use the standard properties

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$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,d\,x = f(a)$$

$$\int f(x)\,\delta^{(n)}(x)\,d\,x \equiv -\int \frac{\partial f}{\partial x}\,\delta^{(n-1)}(x)\,d\,x.$$

First part

In[∘]:=
$$\frac{(r - i\, a\, \mathrm{Cos}[\theta])^2}{(rp - rm)^2}\, \Sigma\, J2\, \mathrm{Conjugate}[Y[-1, l, 0, \theta, 0]];$$

$$\%\ /.\ \mathrm{DiracDelta}[-\pi + 2\theta] \to 1\ /.\ \mathrm{DiracDelta}'[-\pi + 2\theta] \to 0;$$

$$\mathrm{FirstPart} = \%\ /.\ \theta \to \frac{\pi}{2}$$

Out[∘]=
$$-\frac{1}{2\sqrt{2}\,(-rm + rp)^2}\left(a^2 - 2\,M\,r + r^2\right)\mathrm{Conjugate}\left[Y\left[-1, l, 0, \frac{\pi}{2}, 0\right]\right]\left(-8\,i\,a^2\,C\,\mathrm{DiracDelta}[r - r0] + \sqrt{2}\left(3\,r\left(\frac{1}{\sqrt{2}\,r} + 2\,i\,C\left(a^2 + r^2\right)\mathrm{DiracDelta}[r - r0]\right) + r^2\left(-\frac{1}{\sqrt{2}\,r^2} + 4\,i\,C\,r\,\mathrm{DiracDelta}[r - r0] + 2\,i\,C\left(a^2 + r^2\right)\mathrm{DiracDelta}'[r - r0]\right)\right)\right)$$

Second part

In[∘]:=
$$\frac{(r - i\, a\, \mathrm{Cos}[\theta])^2}{(rp - rm)^2}\, \Sigma\, J2\, \mathrm{Conjugate}[Y[-1, l, 0, \theta, 0]];$$

$$\%\ /.\ \mathrm{DiracDelta}[-\pi + 2\theta] \to 0\ /.\ \mathrm{DiracDelta}'[-\pi + 2\theta] \to 1;$$

$$\mathrm{SecondPart} = -D[\%, \theta]\ /.\ \theta \to \frac{\pi}{2}\ /.\ \mathrm{Conjugate}'\left[Y\left[-1, l, 0, \frac{\pi}{2}, 0\right]\right] \to 1$$

Out[∘]=
$$-\frac{2\,i\,\sqrt{2}\,a^2\,C\left(a^2 - 2\,M\,r + r^2\right)\mathrm{Conjugate}\left[Y\left[-1, l, 0, \frac{\pi}{2}, 0\right]\right]\mathrm{DiracDelta}[r - r0]}{(-rm + rp)^2} + \frac{\left(a^2 - 2\,M\,r + r^2\right)\left(2 - 8\,a\,C\,r\,\mathrm{DiracDelta}[r - r0]\right)Y^{(0,0,0,1,0)}\left[-1, l, 0, \frac{\pi}{2}, 0\right]}{2\sqrt{2}\,(-rm + rp)^2}$$

We get the solution

In[∘]:= `solution = FirstPart + SecondPart // Simplify`

Out[∘]=

$$-\frac{2\,i\,\sqrt{2}\;a^2\,C\left(a^2-2\,M\,r+r^2\right)\text{Conjugate}\left[Y\left[-1,\,l,\,0,\,\frac{\pi}{2},\,0\right]\right]\text{DiracDelta}[r-r0]}{(-rm+rp)^2}-$$

$$\frac{1}{2\,\sqrt{2}\;(-rm+rp)^2}\left(a^2-2\,M\,r+r^2\right)\text{Conjugate}\left[Y\left[-1,\,l,\,0,\,\frac{\pi}{2},\,0\right]\right]$$

$$\left(-8\,i\,a^2\,C\,\text{DiracDelta}[r-r0]+\sqrt{2}\left(3\,r\left(\frac{1}{\sqrt{2}\;r}+2\,i\,C\left(a^2+r^2\right)\text{DiracDelta}[r-r0]\right)+\right.\right.$$

$$\left.\left.r^2\left(-\frac{1}{\sqrt{2}\;r^2}+4\,i\,C\,r\,\text{DiracDelta}[r-r0]+2\,i\,C\left(a^2+r^2\right)\text{DiracDelta}'[r-r0]\right)\right)\right)+$$

$$\frac{\left(a^2-2\,M\,r+r^2\right)\left(2-8\,a\,C\,r\,\text{DiracDelta}[r-r0]\right)Y^{(0,0,0,1,0)}\left[-1,\,l,\,0,\,\frac{\pi}{2},\,0\right]}{2\,\sqrt{2}\;(-rm+rp)^2}$$

Finally, we can simplify the results by using identities for SpinWeightedSpherical Harmoncis

**HarmonicRules =**

$$\boldsymbol{\left\{Y^{(0,0,0,1,0)}[s\_,\,l\_,\,m\_,\,\theta\_,\,\phi\_]\to s\,\text{Cot}[\theta]\,Y[s,\,l,\,m,\,\theta,\,\phi]-\sqrt{l+l^2-s-s^2}\;Y[1+s,\,l,\,m,\,\theta,\,\phi]-\right.}$$

$$\boldsymbol{i\,\text{Csc}[\theta]\,Y^{(0,0,0,0,1)}[s,\,l,\,m,\,\theta,\,\phi],\;Y^{(0,0,0,0,1)}[s\_,\,l\_,\,m\_,\,\theta\_,\,\phi\_]\to i\,m\,Y[s,\,l,\,m,\,\theta,\,\phi],}$$

$$\boldsymbol{\left.\text{Conjugate}[Y[s\_,\,l\_,\,m\_,\,\theta\_,\,\phi\_]]\to(-1)^{-m+s}\,Y[-s,\,l,\,-m,\,\theta,\,\phi]\right\};}$$

Out[∘]=

$$\left\{Y^{(0,0,0,1,0)}[s\_,\,l\_,\,m\_,\,\theta\_,\,\phi\_]\to\right.$$

$$s\,\text{Cot}[\theta]\,Y[s,\,l,\,m,\,\theta,\,\phi]-\sqrt{l+l^2-s-s^2}\;Y[1+s,\,l,\,m,\,\theta,\,\phi]-i\,\text{Csc}[\theta]\,Y^{(0,0,0,0,1)}[s,\,l,\,m,\,\theta,\,\phi],$$

$$Y^{(0,0,0,0,1)}[s\_,\,l\_,\,m\_,\,\theta\_,\,\phi\_]\to i\,m\,Y[s,\,l,\,m,\,\theta,\,\phi],$$

$$\left.\text{Conjugate}[Y[s\_,\,l\_,\,m\_,\,\theta\_,\,\phi\_]]\to(-1)^{-m+s}\,Y[-s,\,l,\,-m,\,\theta,\,\phi]\right\}$$

Applying these rules we get the final solution

```
In[•]:= solution /. HarmonicRules /. Y^(0,0,0,0,1)[-1, l, 0, π/2, 0] → 0;
        Collect[%, {DiracDelta[r - r0], DiracDelta'[r - r0]}] /.
          {Y[1, l, 0, π/2, 0] → -Conjugate[Y[-1, l, 0, π/2, 0]]};
        Print["Solution == ", %]
```

$$\text{Solution == } -\frac{\left(a^2-2\,M\,r+r^2\right)\text{Conjugate}\left[Y\left[-1,l,0,\frac{\pi}{2},0\right]\right]}{\sqrt{2}\,(-rm+rp)^2}-\frac{\sqrt{l+l^2}\,\left(a^2-2\,M\,r+r^2\right)Y\left[0,l,0,\frac{\pi}{2},0\right]}{\sqrt{2}\,(-rm+rp)^2}+$$

$$\text{DiracDelta}[r-r0]\left(-\frac{2\,i\,C\,r^3\left(a^2-2\,M\,r+r^2\right)\text{Conjugate}\left[Y\left[-1,l,0,\frac{\pi}{2},0\right]\right]}{(-rm+rp)^2}-\right.$$

$$\frac{3\,i\,C\,r\left(a^2+r^2\right)\left(a^2-2\,M\,r+r^2\right)\text{Conjugate}\left[Y\left[-1,l,0,\frac{\pi}{2},0\right]\right]}{(-rm+rp)^2}+$$

$$\left.\frac{2\sqrt{2}\,a\,C\,\sqrt{l+l^2}\,r\left(a^2-2\,M\,r+r^2\right)Y\left[0,l,0,\frac{\pi}{2},0\right]}{(-rm+rp)^2}\right)-$$

$$\frac{i\,C\,r^2\left(a^2+r^2\right)\left(a^2-2\,M\,r+r^2\right)\text{Conjugate}\left[Y\left[-1,l,0,\frac{\pi}{2},0\right]\right]\text{DiracDelta}'[r-r0]}{(-rm+rp)^2}$$

We should further utilize identities for the Y function but we can already see that we must have made a mistake
Since the solution given in the article ($C = I\left(\frac{\Delta 0}{P0}\right)$) and ($e = 0$)

$$ {}^2J_{lm} = -\frac{\Delta\delta_{m0}}{\sqrt{2}\,(r_+-r_-)^2}\left[(Mae/\mathcal{P}_0)+\pi\,\mathcal{I}(\Delta_0/\mathcal{P}_0)^{1/2}\right] \cdot \left[i(r_0^2+a^2)\;{}_{-1}\overline{Y}_{l0}\left(\frac{\pi}{2},0\right)\delta'(r-r_0) + \left\{ir_0\;{}_{-1}\overline{Y}_{l0}\left(\frac{\pi}{2},0\right) - a\,[l(l+1)]^{1/2}\;{}_0\overline{Y}_{l0}\left(\frac{\pi}{2},0\right)\right\}\delta(r-r_0)\right]$$

Is only quadratic in r, but our solution is at least quartic in r.