First we define known quantities

In[49]:=

$$\Delta := r^2 - 2\,M\,r + a^2$$

$$\Sigma := r^2 + a^2\,\text{Cos}[\theta]^2$$

$$\text{J} := \text{C}\;\text{DiracDelta}[r - r0]\;\text{DiracDelta}\left[\theta - \frac{\pi}{2}\right]$$

$$\text{Jm}\dagger := \left(\sqrt{2}\,\left(r - i\,a\,\text{Cos}[\theta]\right)\right)^{-1}\; i\,\left(r^2 + a^2\right)\text{Sin}[\theta]\;\text{J}$$

$$\text{Jn} := -\frac{a\,\Delta}{\Sigma}\,\text{Sin}[\theta]^2\;\text{J}$$

Now for J2, we can ignore ∂_φ since Jm and Jn are independent of it

$$J_2 = \frac{-\Delta}{2\sqrt{2}\,\Sigma\,(r - ia\cos\theta)^2}\left[\sqrt{2}\left(\frac{\partial}{\partial r} - \frac{a}{\Delta}\frac{\partial}{\partial\varphi} + \frac{1}{r - ia\cos\theta}\right)(r - ia\cos\theta)^2 J_{\overline{m}}\right.$$
$$\left. + 2\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} + \frac{ia\sin\theta}{r - ia\cos\theta}\right)\frac{\Sigma\,(r - ia\cos\theta)}{\Delta}J_n\right] \qquad (2.12)$$

In[54]:=

$$\text{J2} := \frac{-\Delta}{2\,\sqrt{2}\,\Sigma\left(r - i\,a\,\text{Cos}[\theta]\right)^2}\left(\sqrt{2}\left(\partial_r\#+\frac{1}{r - i\,a\,\text{Cos}[\theta]}\,\#\right)\&@\left(\left(r - i\,a\,\text{Cos}[\theta]\right)^2\,\text{Jm}\dagger\right) + 2\left(\partial_\theta\#+\frac{i\,a\,\text{Sin}[\theta]}{r - i\,a\,\text{Cos}[\theta]}\,\#\right)\&@\left(\frac{\Sigma\left(r - i\,a\,\text{Cos}[\theta]\right)}{\Delta}\,\text{Jn}\right)\right)$$

Finally for the integration we need to first look at the definition of the SpinWeighted spherical harmonics.

Since all SpinWeighted spherical harmonics, here denoted ${}_sY_{\text{lm}}(\theta,\phi)$= Y[s,l,$\theta$,$\phi$] behave in the ϕ argument as Exp[i m ϕ] and since all other variables are independent of ϕ in the equation above We have

$\int$Y[s,l,m,θ,ϕ]= 2 π DiracDelta[m,0] Y[s,l,m,θ,0]. Then we can integrate over θ.

e the source term is given by

$$^2J_{lm}(r) = \int_0^{2\pi}\int_0^{\pi}\frac{(r - ia\cos\theta)^2}{(r_+ - r_-)^2}\;\Sigma\,J_2\;\;{}_{-1}\overline{Y}_{lm}\sin\theta\,d\theta\,d\varphi$$

In[55]:=

We can split the integrand into two parts depending if we have DiracDelta$\left[\theta - \frac{\pi}{2}\right]$ or the derivative DiracDelta'$\left[\theta - \frac{\pi}{2}\right]$.

and we will use the standard properties

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$$

$$\int f(x)\,\delta^{(n)}(x)\,dx \equiv -\int \frac{\partial f}{\partial x}\,\delta^{(n-1)}(x)\,dx.$$

First part

In[56]:= $\frac{(r - i\, a\, \text{Cos}[\theta])^2}{(rp - rm)^2}\, \Sigma\, J2\, \text{Conjugate}[Y[-1, l, 0, \theta, 0]];$

```
% /. DiracDelta[-π + 2 θ] → 1 /. DiracDelta'[-π + 2 θ] → 0;
FirstPart = % /. θ → π/2
```

Out[58]= $-\frac{1}{2\sqrt{2}\,(-rm + rp)^2}\left(a^2 - 2\,M\,r + r^2\right)\text{Conjugate}\left[Y\left[-1, l, 0, \frac{\pi}{2}, 0\right]\right]$
$\left(-8\,i\,a^2\,C\,\text{DiracDelta}[r - r0] + \sqrt{2}\left(2\,i\,\sqrt{2}\,C\,r^2\,\text{DiracDelta}[r - r0] + 2\,i\,\sqrt{2}\,C\left(a^2 + r^2\right)\text{DiracDelta}[r - r0] + i\,\sqrt{2}\,C\,r\left(a^2 + r^2\right)\text{DiracDelta}'[r - r0]\right)\right)$

Second part

In[59]:= $\frac{(r - i\, a\, \text{Cos}[\theta])^2}{(rp - rm)^2}\, \Sigma\, J2\, \text{Conjugate}[Y[-1, l, 0, \theta, 0]];$

```
% /. DiracDelta[-π + 2 θ] → 0 /. DiracDelta'[-π + 2 θ] → 1;
SecondPart = -D[%, θ] /. θ → π/2 /. Conjugate'[Y[-1, l, 0, π/2, 0]] → 1
```

Out[61]= $-\frac{2\,i\,\sqrt{2}\,a^2\,C\left(a^2 - 2\,M\,r + r^2\right)\text{Conjugate}\left[Y\left[-1, l, 0, \frac{\pi}{2}, 0\right]\right]\text{DiracDelta}[r - r0]}{(-rm + rp)^2} - \frac{2\,\sqrt{2}\,a\,C\,r\left(a^2 - 2\,M\,r + r^2\right)\text{DiracDelta}[r - r0]\,Y^{(0,0,0,1,0)}\left[-1, l, 0, \frac{\pi}{2}, 0\right]}{(-rm + rp)^2}$

We get the solution

In[62]:=
```
solution = FirstPart + SecondPart // Simplify
```

Out[62]=

$$-\frac{2 i \sqrt{2}\, a^2\, C\, (a^2 - 2 M r + r^2)\, \mathrm{Conjugate}[Y[-1, l, 0, \frac{\pi}{2}, 0]]\, \mathrm{DiracDelta}[r - r0]}{(-rm + rp)^2} - \frac{1}{2\sqrt{2}\,(-rm+rp)^2} (a^2 - 2 M r + r^2)\, \mathrm{Conjugate}[Y[-1, l, 0, \frac{\pi}{2}, 0]] \left(-8 i a^2 C\, \mathrm{DiracDelta}[r - r0] + \sqrt{2}\left(2 i \sqrt{2}\, C r^2\, \mathrm{DiracDelta}[r - r0] + 2 i \sqrt{2}\, C (a^2 + r^2)\, \mathrm{DiracDelta}[r - r0] + i \sqrt{2}\, C r (a^2 + r^2)\, \mathrm{DiracDelta}'[r - r0]\right)\right) - \frac{2\sqrt{2}\, a C r (a^2 - 2 M r + r^2)\, \mathrm{DiracDelta}[r - r0]\, Y^{(0,0,0,1,0)}[-1, l, 0, \frac{\pi}{2}, 0]}{(-rm + rp)^2}$$

Finally, we can simplify the results by using identities for SpinWeightedSpherical Harmoncis

In[63]:=
```
HarmonicRules =
  {Y^(0,0,0,1,0)[s_, l_, m_, θ_, ϕ_] → s Cot[θ] Y[s, l, m, θ, ϕ] - Sqrt[l + l^2 - s - s^2] Y[1 + s, l, m, θ, ϕ] -
     I Csc[θ] Y^(0,0,0,0,1)[s, l, m, θ, ϕ], Y^(0,0,0,0,1)[s_, l_, m_, θ_, ϕ_] → I m Y[s, l, m, θ, ϕ],
   Conjugate[Y[s_, l_, m_, θ_, ϕ_]] → (-1)^(-m+s) Y[-s, l, -m, θ, ϕ]};
```

Applying these rules we get the final solution

In[64]:=
```
solution /. HarmonicRules /. Y^(0,0,0,0,1)[-1, l, 0, π/2, 0] → 0;
% /. {Y[1, l, 0, π/2, 0] → -Conjugate[Y[-1, l, 0, π/2, 0]]} // Simplify;
Print["Solution == ", %]
```

Solution ==

$$-\frac{2 i \sqrt{2}\, a^2\, C\, (a^2 - 2 M r + r^2)\, \mathrm{Conjugate}[Y[-1, l, 0, \frac{\pi}{2}, 0]]\, \mathrm{DiracDelta}[r - r0]}{(-rm + rp)^2} + \frac{2\sqrt{2}\, a C \sqrt{l + l^2}\, r (a^2 - 2 M r + r^2)\, \mathrm{DiracDelta}[r - r0]\, Y[0, l, 0, \frac{\pi}{2}, 0]}{(-rm + rp)^2} - \frac{1}{2\sqrt{2}\,(-rm+rp)^2} (a^2 - 2 M r + r^2)\, \mathrm{Conjugate}[Y[-1, l, 0, \frac{\pi}{2}, 0]] \left(-8 i a^2 C\, \mathrm{DiracDelta}[r - r0] + \sqrt{2}\left(2 i \sqrt{2}\, C r^2\, \mathrm{DiracDelta}[r - r0] + 2 i \sqrt{2}\, C (a^2 + r^2)\, \mathrm{DiracDelta}[r - r0] + i \sqrt{2}\, C r (a^2 + r^2)\, \mathrm{DiracDelta}'[r - r0]\right)\right)$$

We should further utilize identities for the Y function but we can already see that we must have made a mistake

Since the solution given in the article ($C = I\left(\frac{\Delta 0}{P0}\right)$) and ($e = 0$)

$$
\begin{aligned}
{}^2J_{lm} = {}& -\frac{\Delta\delta_{m0}}{\sqrt{2}\,(r_+ - r_-)^2}\left[(Mae/\mathcal{A}_0) + \pi\,\mathfrak{J}(\Delta_0/\mathcal{A}_0)^{1/2}\right] \\
& \cdot\left[i(r_0^2 + a^2)\,{}_{-1}\overline{Y}_{l0}\left(\frac{\pi}{2}, 0\right)\delta'(r - r_0)\right. \\
& \left. + \left\{ir_0\,{}_{-1}\overline{Y}_{l0}\left(\frac{\pi}{2}, 0\right) - a\,[l(l+1)]^{1/2}\,{}_{0}\overline{Y}_{l0}\left(\frac{\pi}{2}, 0\right)\right\}\delta(r - r_0)\right]
\end{aligned}
$$

Is only quadratic in r, but our solution is at least quartic in r.