First we define known quantities

In[7]:=
$$\Delta := r^2 - 2 M r + a^2$$
$$\Sigma := r^2 + a^2 \mathrm{Cos}[\theta]^2$$
$$J := C\,\mathrm{DiracDelta}[r - r0]\,\mathrm{DiracDelta}\left[\theta - \frac{\pi}{2}\right]$$
$$Jm\dagger := \left(\sqrt{2}\,\left(r - i\,a\,\mathrm{Cos}[\theta]\right)\right)^{-1} + i\left(r^2 + a^2\right)\mathrm{Sin}[\theta]\,J$$
$$Jn := -\frac{a\,\Delta}{\Sigma}\,\mathrm{Sin}[\theta]^2\,J$$

Now for J2, we can ignore ∂_φ since Jm and Jn are independent of it

$$J_2 = \frac{-\Delta}{2\sqrt{2}\,\Sigma\,(r - ia\cos\theta)^2}\left[\sqrt{2}\left(\frac{\partial}{\partial r} - \frac{a}{\Delta}\frac{\partial}{\partial\varphi} + \frac{1}{r - ia\cos\theta}\right)(r - ia\cos\theta)^2 J_{\overline{m}} + 2\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} + \frac{ia\sin\theta}{r - ia\cos\theta}\right)\frac{\Sigma\,(r - ia\cos\theta)}{\Delta}J_n\right] \qquad (2.12)$$

In[12]:=
$$J2 := \frac{-\Delta}{2\,\sqrt{2}\,\Sigma\left(r - i\,a\,\mathrm{Cos}[\theta]\right)^2}\left(\sqrt{2}\left(\partial_r\# + \frac{1}{r - i\,a\,\mathrm{Cos}[\theta]}\#\right)\&@\left(\left(r - i\,a\,\mathrm{Cos}[\theta]\right)^2 Jm\dagger\right) + 2\left(\partial_\theta\# + \frac{i\,a\,\mathrm{Sin}[\theta]}{r - i\,a\,\mathrm{Cos}[\theta]}\#\right)\&@\left(\frac{\Sigma\left(r - i\,a\,\mathrm{Cos}[\theta]\right)}{\Delta}\,Jn\right)\right)$$

Finally for the integration we need to first look at the definition of the SpinWeighted spherical harmonics.

Since all SpinWeighted spherical harmonics, here denoted ${}_sY_{\mathrm{lm}}(\theta,\phi)$= Y[s,l,$\theta$,$\phi$] behave in the ϕ argument as Exp[i m ϕ] and since all other variables are independent of ϕ in the equation above We have

$\int$Y[s,l,m,θ,ϕ]= 2 π DiracDelta[m,0] Y[s,l,m,θ,0]. Then we can integrate over θ.

e the source term is given by

$${}^2J_{lm}(r) = \int_0^{2\pi}\int_0^{\pi}\frac{(r - ia\cos\theta)^2}{(r_+ - r_-)^2}\;\Sigma\,J_2\;{}_{-1}\overline{Y}_{lm}\,\sin\theta\,d\theta\,d\varphi$$

In[13]:=

We can split the integrand into two parts depending if we have DiracDelta$\left[\theta - \frac{\pi}{2}\right]$ or the derivative DiracDelta'$\left[\theta - \frac{\pi}{2}\right]$.

and we will use the standard properties

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$$

$$\int f(x)\,\delta^{(n)}(x)\,dx \equiv -\int \frac{\partial f}{\partial x}\,\delta^{(n-1)}(x)\,dx.$$

First part

In[14]:= $\frac{(r - i\, a\, \mathrm{Cos}[\theta])^2}{(rp - rm)^2}\, \Sigma\, J2\, \mathrm{Conjugate}[Y[-1, l, 0, \theta, 0]];$

% /. DiracDelta[$-\pi + 2\,\theta$] → 1 /. DiracDelta′[$-\pi + 2\,\theta$] → 0;

FirstPart = % /. $\theta \to \frac{\pi}{2}$

Out[16]= $-\frac{1}{2\sqrt{2}\,(-rm + rp)^2}\left(a^2 - 2\,M\,r + r^2\right)\mathrm{Conjugate}\left[Y\left[-1, l, 0, \frac{\pi}{2}, 0\right]\right]$
$\left(-8\, i\, a^2\, C\, \mathrm{DiracDelta}[r - r0] + \sqrt{2}\left(3\, r\left(\frac{1}{\sqrt{2}\, r} + 2\, i\, C\left(a^2 + r^2\right)\mathrm{DiracDelta}[r - r0]\right) +\right.\right.$
$\left.\left.r^2\left(-\frac{1}{\sqrt{2}\, r^2} + 4\, i\, C\, r\, \mathrm{DiracDelta}[r - r0] + 2\, i\, C\left(a^2 + r^2\right)\mathrm{DiracDelta}'[r - r0]\right)\right)\right)$

Second part

In[17]:= $\frac{(r - i\, a\, \mathrm{Cos}[\theta])^2}{(rp - rm)^2}\, \Sigma\, J2\, \mathrm{Conjugate}[Y[-1, l, 0, \theta, 0]];$

% /. DiracDelta[$-\pi + 2\,\theta$] → 0 /. DiracDelta′[$-\pi + 2\,\theta$] → 1;

SecondPart = -D[%, θ] /. $\theta \to \frac{\pi}{2}$ /. Conjugate′[Y[-1, l, 0, $\frac{\pi}{2}$, 0]] → 1

Out[19]= $-\frac{2\, i\, \sqrt{2}\, a^2\, C\left(a^2 - 2\,M\,r + r^2\right)\mathrm{Conjugate}\left[Y\left[-1, l, 0, \frac{\pi}{2}, 0\right]\right]\mathrm{DiracDelta}[r - r0]}{(-rm + rp)^2} +$
$\frac{\left(a^2 - 2\,M\,r + r^2\right)\left(2 - 8\, a\, C\, r\, \mathrm{DiracDelta}[r - r0]\right) Y^{(0,0,0,1,0)}\left[-1, l, 0, \frac{\pi}{2}, 0\right]}{2\sqrt{2}\,(-rm + rp)^2}$

We get the solution

In[20]:= **solution = FirstPart + SecondPart // Simplify**

Out[20]=
$$-\frac{2\,i\,\sqrt{2}\,a^2\,C\left(a^2-2\,M\,r+r^2\right)\text{Conjugate}\left[Y\left[-1,l,0,\frac{\pi}{2},0\right]\right]\text{DiracDelta}[r-r0]}{(-rm+rp)^2}-\frac{1}{2\sqrt{2}\,(-rm+rp)^2}\left(a^2-2\,M\,r+r^2\right)\text{Conjugate}\left[Y\left[-1,l,0,\frac{\pi}{2},0\right]\right]\left(-8\,i\,a^2\,C\,\text{DiracDelta}[r-r0]+\sqrt{2}\left(3\,r\left(\frac{1}{\sqrt{2}\,r}+2\,i\,C\left(a^2+r^2\right)\text{DiracDelta}[r-r0]\right)+r^2\left(-\frac{1}{\sqrt{2}\,r^2}+4\,i\,C\,r\,\text{DiracDelta}[r-r0]+2\,i\,C\left(a^2+r^2\right)\text{DiracDelta}'[r-r0]\right)\right)\right)+\frac{\left(a^2-2\,M\,r+r^2\right)\left(2-8\,a\,C\,r\,\text{DiracDelta}[r-r0]\right)Y^{(0,0,0,1,0)}\left[-1,l,0,\frac{\pi}{2},0\right]}{2\sqrt{2}\,(-rm+rp)^2}$$

Finally, we can simplify the results by using identities for SpinWeightedSpherical Harmoncis

HarmonicRules =

$$\left\{Y^{(0,0,0,1,0)}[s\_,l\_,m\_,\theta\_,\phi\_]\to s\,\text{Cot}[\theta]\,Y[s,l,m,\theta,\phi]-\sqrt{l+l^2-s-s^2}\,Y[1+s,l,m,\theta,\phi]-i\,\text{Csc}[\theta]\,Y^{(0,0,0,0,1)}[s,l,m,\theta,\phi],\ Y^{(0,0,0,0,1)}[s\_,l\_,m\_,\theta\_,\phi\_]\to i\,m\,Y[s,l,m,\theta,\phi],\ \text{Conjugate}[Y[s\_,l\_,m\_,\theta\_,\phi\_]]\to(-1)^{-m+s}\,Y[-s,l,-m,\theta,\phi]\right\};$$

Out[21]=
$$\left\{Y^{(0,0,0,1,0)}[s\_,l\_,m\_,\theta\_,\phi\_]\to s\,\text{Cot}[\theta]\,Y[s,l,m,\theta,\phi]-\sqrt{l+l^2-s-s^2}\,Y[1+s,l,m,\theta,\phi]-i\,\text{Csc}[\theta]\,Y^{(0,0,0,0,1)}[s,l,m,\theta,\phi],\ Y^{(0,0,0,0,1)}[s\_,l\_,m\_,\theta\_,\phi\_]\to i\,m\,Y[s,l,m,\theta,\phi],\ \text{Conjugate}[Y[s\_,l\_,m\_,\theta\_,\phi\_]]\to(-1)^{-m+s}\,Y[-s,l,-m,\theta,\phi]\right\}$$

Applying these rules we get the final solution

In[40]:=
```
solution /. HarmonicRules /. Y^(0,0,0,0,1)[-1, l, 0, π/2, 0] → 0;
Collect[%, {DiracDelta[r - r0], DiracDelta'[r - r0]}] /.
  {Y[1, l, 0, π/2, 0] → -Conjugate[Y[-1, l, 0, π/2, 0]]};
Print["Solution == ", %]
```

$$\text{Solution} == -\frac{\left(a^2-2\,M\,r+r^2\right)\text{Conjugate}\left[Y\left[-1,l,0,\frac{\pi}{2},0\right]\right]}{\sqrt{2}\,(-rm+rp)^2}-\frac{\sqrt{l+l^2}\,\left(a^2-2\,M\,r+r^2\right)Y\left[0,l,0,\frac{\pi}{2},0\right]}{\sqrt{2}\,(-rm+rp)^2}+$$
$$\text{DiracDelta}[r-r0]\left(-\frac{2\,i\,C\,r^3\left(a^2-2\,M\,r+r^2\right)\text{Conjugate}\left[Y\left[-1,l,0,\frac{\pi}{2},0\right]\right]}{(-rm+rp)^2}-\right.$$
$$\frac{3\,i\,C\,r\left(a^2+r^2\right)\left(a^2-2\,M\,r+r^2\right)\text{Conjugate}\left[Y\left[-1,l,0,\frac{\pi}{2},0\right]\right]}{(-rm+rp)^2}+$$
$$\left.\frac{2\,\sqrt{2}\,a\,C\,\sqrt{l+l^2}\,r\left(a^2-2\,M\,r+r^2\right)Y\left[0,l,0,\frac{\pi}{2},0\right]}{(-rm+rp)^2}\right)-$$
$$\frac{i\,C\,r^2\left(a^2+r^2\right)\left(a^2-2\,M\,r+r^2\right)\text{Conjugate}\left[Y\left[-1,l,0,\frac{\pi}{2},0\right]\right]\text{DiracDelta}'[r-r0]}{(-rm+rp)^2}$$

We should further utilize identities for the Y function but we can already see that we must have made a mistake
Since the solution given in the article (C = I $\left(\frac{\Delta 0}{P0}\right)$) and (e = 0)

$$
\begin{aligned}
{}^2J_{lm} = -\frac{\Delta\delta_{m0}}{\sqrt{2}\,(r_+-r_-)^2}\ &\left[(Mae/\mathcal{P}_0)+\pi\,\mathcal{I}(\Delta_0/\mathcal{P}_0)^{1/2}\right] \\
&\cdot\left[i(r_0^2+a^2)\ {}_{-1}\overline{Y}_{l0}\left(\frac{\pi}{2},0\right)\ \delta'(r-r_0)\right. \\
&\left.+\left\{ir_0\ {}_{-1}\overline{Y}_{l0}\left(\frac{\pi}{2},0\right)-a\,[l(l+1)]^{1/2}\ {}_0\overline{Y}_{l0}\left(\frac{\pi}{2},0\right)\right\}\delta(r-r_0)\right]
\end{aligned}
$$

Is only quadratic in r, but our solution is at least quartic in r.

```
In[30]:= Conjugate[Y[s, l, m, θ, ϕ]] → (-1)^(-m+s) Y[-s, l, -m, θ, ϕ] /. s → -1 /. m → 0 /. ϕ → 0
         % /. Rule → Equal
         Solve[%, Y[1, l, 0, θ, 0]]

Out[30]= Conjugate[Y[-1, l, 0, θ, 0]] → -Y[1, l, 0, θ, 0]

Out[31]= Conjugate[Y[-1, l, 0, θ, 0]] == -Y[1, l, 0, θ, 0]

Out[32]= {{Y[1, l, 0, θ, 0] → -Conjugate[Y[-1, l, 0, θ, 0]]}}
```