To be completed individually or in groups of two people. **Please be sure matriculation numbers are clearly included at the top of your submission.** Submissions can be handwritten or in LaTeX formatting, but hard-to-read handwritten submissions will not be graded.

Please submit via Ilias. Submissions should be a single PDF document (note that Jupyter notebooks can and should also be downloaded as PDFs, and not submitted as .ipynb files).

Each question will be graded "pass" (full points) or "fail" (no points). We award 0.5 bonus points for the exam for each theory and practical question solved. You must complete 50% of all exercises to enter the final exam.

1. **EXAMple Question** *(Monte-Carlo sampling)*

   a) Provide the formula for Monte-Carlo estimation of integrals of the form $\int p(x)f(x)dx$.

   b) The entropy of a random variable is defined as $H[X] = \int -p(x)\log p(x)dx$. Provide the formula for a Monte-Carlo estimate of the entropy.

   c) At what rate does the average error of Monte-Carlo samples decay as more samples are used?

   d) Why would one use Monte-Carlo estimation to estimate integrals instead of computing the integrals numerically (e.g., with the trapezoidal rule for integration)

2. **Theory Question, Part 1** *(Rejection sampling an unfair coin toss)*

   Suppose we want to simulate an unfair coin toss. The result of this random variable X should be 0 with probability $p < 0.5$ and 1 with probability $1 - p$. A friend provides us with a fair coin, i.e. a coin that comes out heads (0) with probability 50% and tails (1) with probability 50%.

   Our goal is to simulate the unfair coin toss with rejection sampling.

   a) Provide the probability mass function of the fair coin $q(x)$.

   b) Provide the probability mass function of the unfair coin $p(x)$.

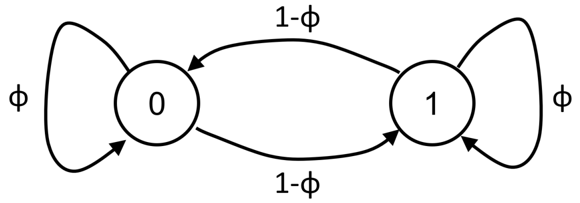
   c) Compute the smallest value c for which $c \cdot q(x) > p(x)$ for all x.

   d) Compute the rejection rate of the rejection sampler for the optimal choice of c. Evaluate this term for $p = 0.5$ and $p = 1.0$.

3. **Theory Question, Part 2** *(MCMC sampling an unfair coin toss)*

   We are interested in the same setup as in the previous question. We now aim to simulate the unfair coin toss with MCMC.

   To do so, we are setting up an MCMC sampler with two states: Heads (0) and tails (1). We are using a transition kernel which remains in its current state with probability ϕ and proposes to change the state with probability $1 - \phi$.



   a) Assume that the chain is currently in state 0. What is the probability that the chain will be in state 1 after the next step?

   b) Assume that the chain is currently in state 1. What is the probability that the chain will be in state 0 after the next step?

c) Using your results from a) and b), provide the transition matrix of a single step.

d) For $\phi = 0.9$ and $p = 0.4$, the transition matrix is $[[0.1, 0.6], [0.9, 0.4]]$. Assume that, initially, the states follow the distribution $x^* = [1.0, 0.0]$. Compute the distribution of the states after 2, 5, 10, and 20 steps. Interpret the result with respect to the unfair coin toss.

4. **Practical Question** See `Exercise_07.ipynb`.