Assignment 6

Statistical Machine Learning

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Summer term 2022 — due to **May 30th at 12:00**

Exercise 1 (Building new kernels, 1+1+1+1 points)
Assume that $k_1, k_2 : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ are kernel functions with $k_1(x,x) > 0$ for all $x \in \mathcal{X}$. Are the following functions kernel functions, too? Prove your claim.

(a) $k = \alpha_1 k_1 + k_2$ for $\alpha_1 \geq 0$

(b) $k = k_1 - k_2$

(c) $k(x,y) = f(x)k_1(x,y)f(y)$ for any function $f : \mathcal{X} \to \mathbb{R}$

(d) $k(x,y) = \frac{k_1(x,y)}{\sqrt{k_1(x,x)k_1(y,y)}}$

Exercise 2 (Kernelization, moved to assignment 7)

Exercise 3 (Kernels on Graphs, 1+2+2+3 points)
Let (V,E) be a connected undirected graph with n nodes $V = \{1, \ldots, n\}$. We write $i \sim j$, when there is an edge between i and j.
In this exercise, we show that the set of zero-sum node weightings $\mathcal{H} = \{f \in \mathbb{R}^n \mid \sum_{i=1}^n f_i = 0\}$ endowed with the norm $\Omega(f) = \sqrt{\frac{1}{2}\sum_{i \sim j}(f_i - f_j)^2}$ is an RKHS and find its reproducing kernel $k : V \times V \to \mathbb{R}$ without knowing the feature map $\Phi : V \to \mathcal{H}$!
Note that every graph induces a different norm and scalar product on $\mathcal{H}$ and a different kernel over V, resulting in different measures of similarity between the nodes.

(a) To which problems could this setting be applied? State two examples.

(b) Now we introduce the graph Laplacian. The *adjacency matrix* is defined as

$$A_{ij} = \begin{cases} 1, & \text{when } i \sim j, \\ 0, & \text{else.} \end{cases}$$

The *degree matrix* D is a diagonal matrix with $D_{ii} = \sum_{j=1}^n A_{ij}$. $L = D - A$ is called the *graph Laplacian matrix*.
Show that $\Omega(f)^2 = f^T L f$.

(c) Show that $< f,\ g >_{\mathcal{H}} := f^T L g$ defines a scalar product on $\mathcal{H}$. By (b) it induces the norm Ω and makes $(\mathcal{H}, \Omega)$ a Hilbert space.

(d) Given the scalar product $< \cdot, \cdot >_{\mathcal{H}}$, find the reproducing kernel k that yields $< f,\ k(i,\cdot) >_{\mathcal{H}} = f_i$ for all $i \in V$ and show that $k(i,\cdot) \in \mathcal{H}$ for all $i \in V$.

What is the corresponding feature map $\Phi : V \to \mathcal{H}$?

Hint: Here a vector $v \in \mathbb{R}^n$ and its correponding element in the dual of $\mathbb{R}^n$ $< v, \cdot >$ are treated 'as the same thing'. You can also think of column and row vectors. Formally, $v \mapsto < v, \cdot >$ is an isomorphism from $\mathbb{R}^n$ to its dual space.

Exercise 4 (Kernel SVM, moved to assignment 7)

Exercise 5 (Feedback Poll: Week 6, 0 bonus points but 1 gratitude)
Every week there will be a feedback survey on Ilias about the lecture, tutorials and exercise sheets. If you participate, it helps us in understanding what we can improve. Future polls will be anonymous again to ensure open feedback. The poll asks about the lectures that cover the topics treated in the respective sheet.