Exercise Nr. 1
(Deadline 25.04.2022)

1

a)

$$
\begin{aligned}
P(X = \text{Male}) &= 0.5 \\
P(X = \text{Female}) &= 0.5 \\
P(Y = \text{Small}) &= 0.4 \\
P(Y = \text{Medium}) &= 0.25 \\
P(Y = \text{Large}) &= 0.35
\end{aligned}
$$

X has no other categories than {Male, Female}. Therefore these two make up the entire population. Y does not play any role in this context.

b)

$$
\begin{aligned}
E(X) &= \sum_x x \cdot P(x) \\
&= \text{Male} \cdot 0.5 + \text{Female} \cdot 0.5 \\
\text{or if viewed as 0 and 1} \\
&= 0.5
\end{aligned}
$$

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$P\left(|\bar{X}_n - E(\bar{X}_n)| \geq \epsilon\right) \leq \frac{\text{Var}(\bar{X}_n)}{\epsilon^2}$$

$$\begin{aligned}
E(\bar{X}_n) &= E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \\
&= \frac{1}{n}\sum_{i=1}^{n} E\left(x_i\right) \\
&= \frac{1}{n}\sum_{i=1}^{n} E(X) \\
&= \frac{n}{n}E(X) = E(X)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(\bar{X}_n) &= \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}(x_i) \\
&= \frac{n}{n^2}\,\mathrm{Var}(X) = \frac{Var(X)}{n}
\end{aligned}$$

$$\begin{aligned}
P\left(|\bar{X}_n - E(\bar{X}_n)| \geq \epsilon\right) &\leq \frac{\mathrm{Var}(\bar{X}_n)}{\epsilon^2} \\
P\left(|\bar{X}_n - E(X)| \geq \epsilon\right) &\leq \frac{\mathrm{Var}(X)}{n\epsilon^2} \\
\lim_{n\to\infty} P\left(|\bar{X}_n - E(X)| \geq \epsilon\right) &\leq \lim_{n\to\infty} \frac{\mathrm{Var}(X)}{n\epsilon^2} \\
\text{as } \mathrm{Var}(X) \text{ is finite:}& \\
\lim_{n\to\infty} P\left(|\bar{X}_n - E(X)| \geq \epsilon\right) &\leq 0 \\
\text{as probabilities are positive:}& \\
\lim_{n\to\infty} P\left(|\bar{X}_n - E(X)| \geq \epsilon\right) &= 0
\end{aligned}$$

2

a)

$$P(\text{Medium} \mid \text{Female}) = \frac{0.1}{0.5} = 0.2$$

b)

$$\begin{aligned}
P(A \mid B) &= P(A) \\
P(A, B) &= P(A)P(B)
\end{aligned}$$

c)

$$\begin{aligned}P(\text{positive}) =&P(\text{positive} \mid \text{cancer})P(\text{cancer})\\&+P(\text{positive} \mid \text{no cancer})P(\text{no cancer})\\=&0.95 \cdot 0.01 + 0.05 \cdot 0.99 = 0.059\\P(\text{cancer} \mid \text{positive}) =&\frac{P(\text{positive} \mid \text{cancer})P(\text{cancer})}{P(\text{positive})}\\=&\frac{0.95 \cdot 0.01}{0.059} \approx 0.161\end{aligned}$$

The result is surprisingly low, as the test will produce many false positives. This is because of the sheer amount of people taking the test without having cancer (in comparison with the low amount that have cancer).

d)

$$\begin{aligned}O(\text{cancer}) \cdot (\text{Bayes Factor}) &= \frac{P(\text{cancer})}{P(\text{no cancer})}\frac{P(\text{positive} \mid \text{cancer})}{P(\text{positive} \mid \text{no cancer})}\\&= \frac{P(\text{positive, cancer})}{P(\text{positive, no cancer})}\\&= \frac{P(\text{cancer} \mid \text{positive})}{P(\text{positive})}\frac{P(\text{positive})}{P(\text{no cancer} \mid \text{positive})}\\&= O(\text{cancer} \mid \text{positive})\end{aligned}$$

This means the Bayes Factor tells you how much information you get from taking the test.

3

a)

$$Ax = \begin{pmatrix}1 & 1 & 2\\1 & 2 & 1\\5 & 7 & 8\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}1\\1\\5\end{pmatrix}x_1 + \begin{pmatrix}1\\2\\7\end{pmatrix}x_2 + \begin{pmatrix}2\\1\\8\end{pmatrix}x_3 = \begin{pmatrix}1x_1 + 1x_2 + 2x_3\\1x_1 + 2x_2 + 1x_3\\5x_1 + 7x_2 + 8x_3\end{pmatrix}$$

b)

$$\det(A) \neq 0 \iff \text{columns are lin. indep.} \iff \text{rows are lin. indep.}$$

Here: $\det(A) = 16 + 5 + 14 - 20 - 7 - 8 = 0$ therefore neither rows nor columns are linearly independent and thus can not be a basis of $\mathbb{R}^3$

c)

Consider $\tilde{b} = \begin{pmatrix}-3\\1\\2\end{pmatrix}$

Clearly, there is no real valued solution for x with

$$\begin{pmatrix}1 & 1 & 2\\1 & 2 & 1\\5 & 7 & 8\end{pmatrix} x = \begin{pmatrix}-3\\1\\2\end{pmatrix}$$

as $ranks(A) = 2$ and therfore $Span(A)$ is 2-Dimensional. Furthermore $\begin{pmatrix}-3\\1\\1\end{pmatrix}$ is an Eigenvector (and therefore a on the span).

$$\begin{pmatrix}1 & 1 & 2\\1 & 2 & 1\\5 & 7 & 8\end{pmatrix} x = \begin{pmatrix}2\\3\\12\end{pmatrix}$$
$$\begin{pmatrix}1 & 1 & 2\\1 & 2 & 1\\5 & 7 & 8\end{pmatrix} \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\3\\12\end{pmatrix}$$

d)

Row rank, column rank, and rank of the matrix are the same.
In this case it is 2, as they are not linearly independent but span a plane.

4

a)

Matrix C has the Eigenvalue 1
and Eigenspace $\text{Eig}(C, 1) = \{\begin{pmatrix}x\\0\end{pmatrix} \mid x \in \mathbb{R}\}$

b)

(1)

$$A + \alpha I = (V(\Lambda + \alpha I)V^T)$$
$$Au_k = \lambda u_k \implies (A + \alpha I)u_k = (\lambda + \alpha)u_k$$

In other words: the eigenvectors don't change, while the eigenvalues increase by α.

(2)

$$A^T = A \implies A^T A = AA^T = AA$$
$$AA = V\Lambda V^T V\Lambda V^T = V\Lambda^2 V^T$$
$$Au_k = \lambda u_k \implies AAu_k = \lambda^2 u_k$$

(3)

same as (2)

(4)

$$
\begin{aligned}
Au_k &= \lambda u_k \\
A^{-1}Au_k &= A^{-1}\lambda u_k \\
u_k &= \lambda A^{-1} u_k \\
\frac{1}{\lambda}u_k &= A^{-1}u_k
\end{aligned}
$$

$$Au_k = \lambda u_k \implies A^{-1}u_k = \frac{1}{\lambda}u_k$$

c)

$$
\begin{aligned}
S &= U\Sigma V^T \\
SS^T &= U\Sigma V^T(U\Sigma V^T)^T \\
&= U\Sigma V^T V\Sigma^T U^T \\
&= U\Sigma\Sigma^T U^T \\
S^TS &= (U\Sigma V^T)^T U\Sigma V^T \\
&= V\Sigma^T U^T U\Sigma V^T \\
&= V\Sigma^T\Sigma V^T
\end{aligned}
$$

$\implies U$ are eigenvectors of SS^T and V are eigenvectors of S^TS
since $\Sigma\Sigma^T = \Sigma^T\Sigma$ are the eigenvalue matrices of SS^T and S^TS. Therefore the the singular values of S are the positives square roots of these eigenvalues.