Computations supporting Example ??? in the paper ??? by Igor Zobovič and Aljaž Zalar.

This example shows studies a linear operator L which is represented by a measure μ which contains the atom t = 0 with multiplicity 1. However, since rank(HH K) = rank(HH), it follows that L is also represented by a measure $\tilde{\mu}$ which contains the atom t = 0 atom with multiplicity p = 2.
Let $\mu = \sum_{j=1}^{4} x_j A_j$ be a finitely atomic matrix measure with (x_1, x_2, x_3, x_4) = (0, 1, -1, 2) and $(A_1, A_2, A_3, A_4) = (\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix})$, and let L be a linear operator, defined by $L(f) = \int f \, d\mu$.

In[1]:=
```
x1 = 0
x2 = 1
x3 = -1
x4 = -2
atoms = {x1, x2, x3, x4};
A1 = {{2, 2}, {2, 2}};
MatrixForm[A1]
A2 = {{1, 1}, {1, 1}};
MatrixForm[A2]
A3 = {{0, 0}, {0, 1}};
MatrixForm[A3]
A4 = {{1, 0}, {0, 0}};
MatrixForm[A4]
masses = {A1, A2, A3, A4};
```

Out[1]= 0

Out[2]= 1

Out[3]= −1

Out[4]= −2

Out[7]//MatrixForm=
$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

Out[9]//MatrixForm=
$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Out[11]//MatrixForm=
$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Out[13]//MatrixForm=
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

For every i ∈ {0, 1, 2} we define Si := L(x^i).

In[15]:=
```
f0[x_] := 1;
f1[x_] := x;
f2[x_] := x^2;
S0 = Total[masses * (f0 /@ atoms)];
MatrixForm[S0]
S1 = Total[masses * (f1 /@ atoms)];
MatrixForm[S1]
S2 = Total[masses * (f2 /@ atoms)];
MatrixForm[S2]
```

Out[19]//MatrixForm=

$$\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$$

Out[21]//MatrixForm=

$$\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

Out[23]//MatrixForm=

$$\begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix}$$

We define the moment matrix M := M(1) , the (x - t)-localizing matrix HH, and the matrix K.

In[24]:=
```
M = ArrayFlatten[{{S0, S1}, {S1, S2}}];
MatrixForm[M]
HH = S1;
MatrixForm[HH]
K = S2;
MatrixForm[K]
```

Out[25]//MatrixForm=

$$\begin{pmatrix} 4 & 3 & -1 & 1 \\ 3 & 4 & 1 & 0 \\ -1 & 1 & 5 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}$$

Out[27]//MatrixForm=

$$\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

Out[29]//MatrixForm=

$$\begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix}$$

We check that M is positive definite by computing its eigenvalues.

In[30]:=
```
N[Eigenvalues[M]]
```

Out[30]=

```
{7.10792, 5.59462, 2.27536, 0.0221038}
```

Although the measure contains the atom t = 0, its multiplicity is rank(A1) = 1. However, the localizing matrix HH is invertible, therefore rank(HH K) =

rank(HH), and thus there exists a 4-atomic $\mathbb{R}$-representing measure $\tilde{\mu}$ for L which contains the atom t = 0 with multiplicity 2. Such measure $\tilde{\mu}$ is unique and we will now find its atoms. We first compute $Z := S_3^{(\tilde{\mu})} = K^T HH^{-1} K$.

In[31]:=
```
Z = Transpose[K].Inverse[HH].K;
MatrixForm[Z]
```

Out[32]//MatrixForm=

$$\begin{pmatrix} 11 & 13 \\ 13 & 8 \end{pmatrix}$$

We now obtain the generating polynomial polH(x).

In[33]:=
```
H = Inverse[M].ArrayFlatten[{{S2}, {Z}}];
MatrixForm[H]
H0 = H[[1 ;; 2]];
MatrixForm[H0]
H1 = H[[3 ;; 4]];
MatrixForm[H1]
polH[x_] := x^2 IdentityMatrix[2] - (H0 + x H1)
polH[x_] // Simplify // MatrixForm
```

Out[34]//MatrixForm=

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \\ 6 & 3 \end{pmatrix}$$

Out[36]//MatrixForm=

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Out[38]//MatrixForm=

$$\begin{pmatrix} 1 & 2 \\ 6 & 3 \end{pmatrix}$$

Out[40]//MatrixForm=

$$\begin{pmatrix} (-1 + x\_)\, x\_ & -2\, x\_ \\ -6\, x\_ & (-3 + x\_)\, x\_ \end{pmatrix}$$

We compute the determinant from which we get the atoms.

In[41]:=
```
Factor[Det[polH[x]]]
Solve[Det[polH[x]] == 0, x]
```

Out[41]=

$$x^2 \left(-9 - 4x + x^2\right)$$

Out[42]=

$$\left\{\{x \to 0\}, \{x \to 0\}, \left\{x \to 2 - \sqrt{13}\right\}, \left\{x \to 2 + \sqrt{13}\right\}\right\}$$

We will now find the corresponding masses. First we compute the Vandermonde matrix and its inverse.

In[43]:=
```
newAtoms = DeleteDuplicates[x /. Solve[Det[polH[x]] == 0, x]]
V = Table[root^i, {i, 0, 2}, {root, newAtoms}] /. Indeterminate → 1;
MatrixForm[V]
InvV = Inverse[V];
MatrixForm[InvV]
```

Out[43]=

$$\{0, 2-\sqrt{13}, 2+\sqrt{13}\}$$

Power: Indeterminate expression 0^0 encountered.

Out[45]//MatrixForm=

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2-\sqrt{13} & 2+\sqrt{13} \\ 0 & \left(2-\sqrt{13}\right)^2 & \left(2+\sqrt{13}\right)^2 \end{pmatrix}$$

Out[47]//MatrixForm=

$$\begin{pmatrix} 1 & \frac{4}{9} & -\frac{1}{9} \\ 0 & -\frac{17+4\sqrt{13}}{18\sqrt{13}} & -\frac{-2-\sqrt{13}}{18\sqrt{13}} \\ 0 & -\frac{-17+4\sqrt{13}}{18\sqrt{13}} & -\frac{2-\sqrt{13}}{18\sqrt{13}} \end{pmatrix}$$

We compute the corresponding masses by InvV {S0, S1, S2}.

In[48]:=
```
newMasses = Table[InvV〚r〛.{S0, S1, S2}, {r, 1, Length[InvV]}];
newMassesSimplified = FullSimplify[ArrayFlatten[Transpose[{newMasses}]]];
rationalizeExpr[expr_] := PowerExpand[Together[Rationalize[expr, 0]]];
newMassesRationalized = Map[rationalizeExpr, newMassesSimplified, {2}];
MatrixForm[newMassesRationalized]
```

Out[52]//MatrixForm=

$$\begin{pmatrix} 3 & \frac{10}{3} \\ \frac{10}{3} & \frac{34}{9} \\ \frac{1}{26}\left(13+3\sqrt{13}\right) & \frac{1}{78}\left(-13-5\sqrt{13}\right) \\ \frac{1}{78}\left(-13-5\sqrt{13}\right) & \frac{1}{117}\left(13+2\sqrt{13}\right) \\ \frac{1}{26}\left(13-3\sqrt{13}\right) & \frac{1}{78}\left(-13+5\sqrt{13}\right) \\ \frac{1}{78}\left(-13+5\sqrt{13}\right) & \frac{1}{117}\left(13-2\sqrt{13}\right) \end{pmatrix}$$