MATRICIAL GAUSSIAN QUADRATURE RULES: NONSINGULAR CASE

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ABSTRACT. Let L be a linear operator on univariate polynomials of bounded degree, mapping into real symmetric matrices, such that its moment matrix is positive definite. It is known that L admits a finitely atomic positive matrix-valued representing measure μ. Any μ with the smallest sum of the ranks of the matricial masses is called minimal. In this paper, we characterize the existence of a minimal representing measure containing a prescribed atom with prescribed rank of the corresponding mass, thus extending a recent result [BKRSV20] for the scalar-valued case. As a corollary, we obtain a constructive, linear algebraic proof of the strong truncated Hamburger matrix moment problem [Sim06] in the nonsingular case. The results will be important in the study of the truncated univariate rational matrix moment problem.

1. INTRODUCTION

In this paper we study matricial Gaussian quadrature rules for a linear operator L on univariate polynomials of bounded degree, mapping into real symmetric matrices, such that the corresponding moment matrix is positive definite. More precisely, we fix a real number t and a natural number $m \in \mathbb{N} \cup \{0\}$ and characterize, when there is a minimal representing measure for L containing t in the support with the rank of the corresponding mass equal to m. Apart from being interesting on its own extending a recent result [BKRSV20] from scalars to matrices, the results will be importantly used in the solution to the truncated univariate matrix rational moment problem, analogous to the scalar case [NZ25].

Let $k \in \mathbb{N} \cup \{0\}$ and $p \in \mathbb{N}$. We denote by $\mathbb{R}[x]_{\leq k}$ the vector space of univariate polynomials of degree at most k and by $\mathbb{S}_p(\mathbb{R})$ the set of real symmetric matrices of size $p \times p$. For a given linear operator

$$L : \mathbb{R}[x]_{\leq 2n} \to \mathbb{S}_p(\mathbb{R}), \tag{1.1}$$

denote by $S_i := L(x^i)$, $i = 0, 1, \ldots, 2n$, its **matricial moments** and by

$$M(n) := (S_{i+j-2})_{i,j=1}^{n+1} = \begin{array}{c|ccccc} & \mathit{1} & X & X^2 & \cdots & X^n \\ \hline \mathit{1} & S_0 & S_1 & S_2 & \cdots & S_n \\ X & S_1 & S_2 & \iddots & \iddots & S_{n+1} \\ X^2 & S_2 & \iddots & \iddots & \iddots & \vdots \\ \vdots & \vdots & \iddots & \iddots & \iddots & S_{2n-1} \\ X^n & S_n & S_{n+1} & \cdots & S_{2n-1} & S_{2n} \end{array} \tag{1.2}$$

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the corresponding n**–th truncated moment matrix.** Assume that $M(n)$ is positive definite. It is known (see Theorem 2.1 below), that L admits a positive $\mathbb{S}_p(\mathbb{R})$-valued measure μ (see (2.1)), such that

$$L(p) = \int_{\mathbb{R}} p \, d\mu \quad \text{for every } p \in \mathbb{R}[x]_{\leq 2n}. \tag{1.3}$$

Every measure μ satisfying (1.3) is a **representing measure for** L.

A representing measure $\mu = \sum_{j=1}^{\ell} A_j \delta_{x_j}$ for L, where each $0 \neq A_j \in \mathbb{S}_p(\mathbb{R})$ is positive semidefinite and δ_{x_j} stands for the Dirac measure supported in x_j, is **minimal**, if $\sum_{j=1}^{\ell} \operatorname{rank} A_j$ is minimal among all representing measures for L. In this case, (1.3) is equal to

$$L(p) = \sum_{j=1}^{\ell} A_j p(x_j), \tag{1.4}$$

and (1.4) is a **matricial Gaussian quadrature rule for** L. The points x_j are **atoms** of the measure μ. If $x_1, x_2, \ldots, x_\ell$ are pairwise distinct, then for each j, the matrix $A_j = \mu(\{x_j\})$ is the **mass** of μ at x_j and its rank is the **multiplicity** of x_j in μ, which we denote by $\operatorname{mult}_\mu x_j$. If x is not an atom of μ, then $\operatorname{mult}_\mu x := 0$.

The motivation of the paper is to settle the following problem.

Problem. Let L be as in (1.1) such that $M(n)$ (see (1.2)) is positive definite. Given $t \in \mathbb{R}$ and $m \in \mathbb{N} \cup \{0\}$, characterize when there exists a minimal representing measure μ for L such that $\operatorname{mult}_\mu t = m$.

In [BKRSV20, Theorem 1.4], the authors solved the scalar version (i.e., $p = 1$ in (1.1)) of the Problem in terms of symmetric determinantal representations involving moment matrices. They also showed how to determine other atoms of μ based on the determinant of some univariate matrix polynomial. Their proof uses convex analysis and algebraic geometry, while an alternative proof, using moment theory, and an extension to minimal measures with finitely many prescribed atoms, appears in [NZ+]. We mention that in [BKRSV20], a version of the Problem with an atom at ∞, called **evaluation at** ∞, is also studied. The corresponding quadrature rules are called *generalized Gaussian quadrature rules*. We also mention that in the scalar case, the restriction to the case where $M(n)$ is positive definite is natural. Namely, if $M(n)$ is positive semidefinite but not positive definite, then the minimal representing measure is uniquely determined [CF91, Theorems 3.9 and 3.10]. This fact does not generalize to the matrix case and a version of the Problem with positive semidefinite $M(n)$ is relevant. Moreover, it turns out that this version is technically more involved and will be treated in our forthcoming work [ZZ+].

The main result of the paper is the solution to the Problem above.

Theorem 1.1. *Let $n, p \in \mathbb{N}$ and $L : \mathbb{R}[x]_{\leq 2n} \to \mathbb{S}_p(\mathbb{R})$ be a linear operator such that $M(n)$ is positive definite. Fix $t \in \mathbb{R}$ and $m \in \mathbb{N} \cup \{0\}$. Let $\mathcal{H} := (S_{i+j-1} - tS_{i+j-2})_{i,j=1}^{n}$. Then the following statements are equivalent:*

(1) There exists a minimal representing measure μ for L such that $\operatorname{mult}_\mu t = m$.
(2) $m \leq \operatorname{rank} \mathcal{H} - (n-1)p$.

In the case $m = 0$, Theorem 1.1 simplifies to the following result.

Corollary 1.2. *Let $n, p \in \mathbb{N}$ and $L : \mathbb{R}[x]_{\leq 2n} \to \mathbb{S}_p(\mathbb{R})$ be a linear operator such that $M(n)$ is positive definite. Fix $t \in \mathbb{R}$. Then there exists a minimal representing measure μ for L such that $\operatorname{mult}_\mu t = 0$.*

A simple consequence of Corollary 1.2 is a solution to a strong truncated matrix Hamburger moment problem in the nonsingular case (i.e., the matrix in (1.5) below is invertible).

Corollary 1.3. *Let $n_1, n_2, p \in \mathbb{N}$ and $S_k \in \mathbb{S}_p(\mathbb{R})$ for $k = -2n_1, -2n_1+1, \ldots, 2n_2$. Assume that the matrix*

$$(S_{i+j-2-2n_1})_{i,j=1}^{n_1+n_2+1} \tag{1.5}$$

is positive definite. Then there exists a measure μ such that $S_k = \int_{\mathbb{R}} x^k d\mu$ for each k.

Corollary 1.3 is a special case of [Sim06, Theorem 3.3] under the assumption that the matrix in (1.5) is positive definite. The techniques in [Sim06] use involved operator theory, by studying self-adjoint extensions of certain, not necessarily everywhere defined, linear operator on the finite dimensional Hilbert space of vector-valued Laurent polynomials. Our contribution is a constructive, linear algebraic proof, in the sense that representing measures can be easily computed following the steps in the proof of Theorem 1.1 (see Examples 4.1 and 4.2). To extend Corollary 1.3 to the singular case (i.e., the matrix in (1.5) is only positive semidefinite and not necessarily definite), Theorem 1.1 needs to be extended to the case $M(n)$ is positive semidefinite [ZZ+]. In the scalar case an alternative proof of [Sim06, Theorem 3.3] is [Zal22, Theorem 3.1].

Matricial Gaussian quadrature rules have been studied by several authors (e.g., [DD02, DLR96, DS03]). These works address the question of computing atoms and masses of a representing measure, which is uniquely determined after the odd matricial moment S_{2n+1} is fixed. The formulas are in terms of the roots of the corresponding orthogonal matrix polynomial. A novelty of our results is that we do not specify S_{2n+1}, but characterize, when there is a suitable S_{2n+1}, that leads to a minimal measure containing a prescribed atom with prescribed multiplicity. In the proof, we essentially construct S_{2n+1} with the required properties such that the extended moment matrix $M(n+1)$, with $\operatorname{rank} M(n+1) = \operatorname{rank} M(n)$, has a suitable block column relation (see Section 2.5).

Recently, a question related to the Problem was studied in [FKM24]. Namely, the authors describe for $t \in \mathbb{R}$, the set of all possible masses at t over all representing measures for L. In particular, the maximal mass is determined. The focus of our work is on *minimal* representing measures with *fixed multiplicity* of the mass at t. In a multivariate setting, the set of possible atoms in a representing measure has been charactarized in [MS24a], while the question of possible masses in a given point was studied in [MS24b].

1.1. **Reader's guide.** In Section 2 we introduce the notation and some preliminary results. In Section 3 we present proofs of our main results, i.e., Theorem 1.1 and Corollaries 1.2, 1.3. In Section 4 we demonstrate the application of Theorem 1.1 on numerical examples (see Examples 4.1 and 4.2). In particular, we show that a minimal representing measure containing a prescribed atom with prescribed multiplicity is not unique and that a given atom can be a part of minimal representing measures with different multiplicities. Finally, in Section 5 we allow the evaluation at ∞ and prove a sufficient condition for the existence of a generalized matricial Gaussian quadrature rule containing $\operatorname{rank} M(n-1)$ real atoms, among which a prescribed atom has a prescribed multiplicity (see Theorem 5.1).

2. Preliminaries

Let $m, m_1, m_2 \in \mathbb{N}$. We write $M_{m_1,m_2}(\mathbb{R})$ for the set of $m_1 \times m_2$ real matrices and $M_m(\mathbb{R}) \equiv M_{m,m}(\mathbb{R})$ for short. For a matrix $A \in M_{m_1,m_2}(\mathbb{R})$ we call the linear span of its columns a **column space** and denote it by $\mathcal{C}(A)$. We denote by I_m the identity $m \times m$ matrix and by

$\mathbf{0}_{m_1,m_2}$ the zero $m_1 \times m_2$ matrix, while $\mathbf{0}_m \equiv \mathbf{0}_{m,m}$ for short. We use $M_m(\mathbb{R}[x])$ to denote $m \times m$ matrices over $\mathbb{R}[x]$. The elements of $M_m(\mathbb{R}[x])$ are called **matrix polynomials**.

Let $p \in \mathbb{N}$. For $A \in \mathbb{S}_p(\mathbb{R})$ the notation $A \succeq 0$ (resp. $A \succ 0$) means A is positive semidefinite (psd) (resp. positive definite (pd)). We use $\mathbb{S}_p^{\succeq 0}(\mathbb{R})$ for the subset of all psd matrices in $\mathbb{S}_p(\mathbb{R})$.

Given a polynomial $p(x) \in \mathbb{R}[x]$, we write $\mathcal{Z}(p(x)) := \{x \in \mathbb{R} \colon p(x) = 0\}$ for the set of its zeros.

2.1. **Matrix measures.** Let $\mathrm{Bor}(\mathbb{R})$ be the Borel σ-algebra of $\mathbb{R}$. We call

$$\mu = (\mu_{ij})_{i,j=1}^p : \mathrm{Bor}(\mathbb{R}) \to \mathbb{S}_p(\mathbb{R})$$

a $p \times p$ **Borel matrix-valued measure** supported on $\mathbb{R}$ (or **positive** $\mathbb{S}_p(\mathbb{R})$**-valued measure**) if

(1) $\mu_{ij} : \mathrm{Bor}(\mathbb{R}) \to \mathbb{R}$ is a real measure for every $i, j = 1, 2, \ldots, p$ and

(2) $\mu(\Delta) \succeq 0$ for every $\Delta \in \mathrm{Bor}(\mathbb{R})$.

A positive $\mathbb{S}_p(\mathbb{R})$-valued measure μ is **finitely atomic**, if there exists a finite set $M \in \mathrm{Bor}(\mathbb{R})$ such that $\mu(\mathbb{R} \setminus M) = \mathbf{0}_p$ or equivalently, $\mu = \sum_{j=1}^{\ell} A_j \delta_{x_j}$ for some $\ell \in \mathbb{N}$, $x_j \in \mathbb{R}$, $A_j \in \mathbb{S}_p^{\succeq 0}(\mathbb{R})$. Let μ be a positive $\mathbb{S}_p(\mathbb{R})$-valued measure and $\tau := \mathrm{tr}(\mu) = \sum_{i=1}^p \mu_{ii}$ denote its trace measure. A polynomial $f \in \mathbb{R}[x]_{\leq k}$ is μ-integrable if $f \in L^1(\tau)$. We define its integral by

$$\int_{\mathbb{R}} f \, d\mu = \Big(\int_{\mathbb{R}} f \, d\mu_{ij} \Big)_{i,j=1}^p .$$

2.2. **Riesz mapping.** Equivalently, one can define L as in (1.1) by a sequence of its values on monomials x^i, $i = 0, 1, \ldots, 2n$. Throughout the paper we will denote these values by $S_i := L(x^i)$. If $\mathcal{S} := (S_0, S_1, \ldots, S_{2n}) \in (\mathbb{S}_p)^{2n+1}$ is given, then we denote the corresponding linear mapping on $\mathbb{R}[x]_{\leq 2n}$ by $L_{\mathcal{S}}$ and call it a **Riesz mapping of** $\mathcal{S}$.

2.3. **Moment matrix and localizing moment matrices.** For $n \in \mathbb{N}$ and

$$\mathcal{S} := (S_0, S_1, \ldots, S_{2n}) \in (\mathbb{S}_p)^{2n+1}, \tag{2.1}$$

we denote by $M(n) \equiv M_{\mathcal{S}}(n)$ as in (1.2) its n–th truncated moment matrix. For $i, j \in \mathbb{N} \cup \{0\}$, $i + j \leq 2n$, we also write

$$\mathbf{v}_i^{(j)} = \big(S_{i+r-1}\big)_{r=1}^{j+1} = \begin{pmatrix} S_i \\ S_{i+1} \\ \vdots \\ S_{i+j} \end{pmatrix} \tag{2.2}$$

Given $f \in \mathbb{R}[x]_{\leq 2n}$ and a linear operator $L : \mathbb{R}[x]_{\leq 2n} \to \mathbb{S}_p(\mathbb{R})$, we define an f**–localizing linear operator** by

$$L_f : \mathbb{R}[x]_{\leq 2n - \deg f} \to \mathbb{S}_p(\mathbb{R}), \quad L_f(g) := L(fg).$$

We call the ℓ-th truncated moment matrix of L_f the ℓ**–th truncated** f**–localizing moment matrix of** L and denote it by $\mathcal{H}_f(\ell)$. Defining

$$S_i^{(f)} := L_f(x^i) = L(f x^i),$$

we have

$$\mathcal{H}_f(\ell) := \left(S^{(f)}_{i+j-2}\right)_{i,j=1}^{\ell+1} = \begin{array}{c} \\ \mathit{1} \\ X \\ X^2 \\ \vdots \\ X^\ell \end{array} \begin{array}{c} \begin{array}{ccccc} \mathit{1} & X & X^2 & \cdots & X^\ell \end{array} \\ \begin{pmatrix} S^{(f)}_0 & S^{(f)}_1 & S^{(f)}_2 & \cdots & S^{(f)}_\ell \\ S^{(f)}_1 & S^{(f)}_2 & \iddots & \iddots & S^{(f)}_{\ell+1} \\ S^{(f)}_2 & \iddots & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & S^{(f)}_{2\ell-1} \\ S^{(f)}_\ell & S^{(f)}_{\ell+1} & \cdots & S^{(f)}_{2\ell-1} & S^{(f)}_{2\ell} \end{pmatrix} \end{array}$$

In particular, for $f(x) = x - t$ with $t \in \mathbb{R}$, we have

$$\mathcal{H}_{x-t}(\ell) = \begin{array}{c} \\ \mathit{1} \\ X \\ X^2 \\ \vdots \\ X^\ell \end{array} \begin{array}{c} \begin{array}{ccccc} \mathit{1} & X & X^2 & \cdots & X^\ell \end{array} \\ \begin{pmatrix} S_1 - tS_0 & S_2 - tS_1 & S_3 - tS_2 & \cdots & S_{\ell+1} - tS_\ell \\ S_2 - tS_1 & S_3 - tS_2 & \iddots & \iddots & S_{\ell+2} - tS_{\ell+1} \\ S_3 - tS_2 & \iddots & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & S_{2\ell} - tS_{2\ell-1} \\ S_{\ell+1} - tS_\ell & S_{\ell+2} - tS_{\ell+1} & \cdots & S_{2\ell} - tS_{2\ell-1} & S_{2\ell+1} - tS_{2\ell} \end{pmatrix} \end{array}.$$

For $i, j \in \mathbb{N} \cup \{0\}$, $i + j \leq 2n - \deg f$, we also write

$$(f \cdot \mathbf{v})^{(j)}_i := \left(S^{(f)}_{i+r-1}\right)_{r=1}^{j+1} = \begin{pmatrix} S^{(f)}_i \\ S^{(f)}_{i+1} \\ \vdots \\ S^{(f)}_{i+j} \end{pmatrix}. \tag{2.3}$$

2.4. **Solution to the truncated matrix Hamburger moment problem.**

Theorem 2.1 ([BW11, Theorem 2.7.6])**.** *Let $n, p \in \mathbb{N}$ and*

$$\mathcal{S} \equiv \mathcal{S}^{(2n)} = (S_0, S_1, \ldots, S_{2n}) \in (\mathbb{S}_p(\mathbb{R}))^{2n+1}$$

be a given sequence. Then the following statements are equivalent:

1. *There exists a representing measure for $\mathcal{S}$.*
2. *There exists a* $(\operatorname{rank} M(n))$*–atomic representing measure for $\mathcal{S}$.*
3. *$M(n)$ is positive semidefinite and $\mathcal{C}(\mathbf{v}^{(n-1)}_{n+1}) \subseteq \mathcal{C}(M(n))$, where $\mathbf{v}^{(n-1)}_{n+1}$ is as in* (2.2).

Remark 2.2. The truncated matrix Hamburger moment problem was also considered in [Bol96, Dym89, DFKM09].

2.5. **Support of the representing measure.** Given a matrix polynomial $P(x) = \sum_{i=0}^n x^i P_i \in M_p(\mathbb{R}[x])$, we define the **evaluation** $P(X)$ on the moment matrix $M(n)$ (see (1.2)) to be a matrix, obtained by replacing each monomial of P by the corresponding column of $M(n)$ and multiplying with the matrix coefficients P_i from the right, i.e.,

$$P(X) := \sum_{i=0}^n X^i P_i = \mathbf{v}^{(n)}_0 P_0 + \mathbf{v}^{(n)}_1 P_1 + \cdots + \mathbf{v}^{(n)}_n P_n \in M_{(n+1)p,p}(\mathbb{R}),$$

where $\mathbf{v}^{(j)}_i$ are as in (2.2) above. If $P(X) = \mathbf{0}_{(n+1)p,p}$, then P is a **block column relation** of $M(n)$.

The following lemma connects $\operatorname{supp}(\mu)$ of a representing measure μ for $\mathcal{S}$ as in (2.1) with a block column relation of $M(n)$.

Lemma 2.3 ([KT22, Lemma 5.53])**.** *Let $n, p \in \mathbb{N}$ and*

$$\mathcal{S} \equiv \mathcal{S}^{(2n)} = (S_0, S_1, \ldots, S_{2n}) \in (\mathbb{S}_p)^{2n+1}$$

be a given sequence with a representing measure μ. If $P(x) = \sum_{i=0}^n x^i P_i \in M_p(\mathbb{R}[x])$ is a block column relation of $M(n)$, then

$$\operatorname{supp}(\mu) \subseteq \mathcal{Z}(\det P(x)).$$

2.6. **Evaluation at ∞.** We recall the definition of the evaluation at ∞ from [BKRSV20, Definition 1.1]. The **evaluation at ∞** is the linear functional $\mathbf{ev}_\infty : \mathbb{R}[x]_{\leq 2n} \to \mathbb{R}$, defined by

$$\mathbf{ev}_\infty\left(\sum_{i=0}^{2n} a_i x^i\right) = a_{2n}. \tag{2.4}$$

Let L be as in (1.1). We say μ is a **finitely atomic** $(\mathbb{R} \cup \{\infty\})$**–representing measure for** L if it is of the form

$$\mu = \sum_{j=1}^{\ell} \delta_{x_j} A_j + \mathbf{ev}_\infty A,$$

where $\ell \in \mathbb{N}$, $x_j \in \mathbb{R}$, and $A_j \in \mathbb{S}_p^{\succeq 0}(\mathbb{R})$, $A \in \mathbb{S}_p^{\succeq 0}(\mathbb{R})$. If $\sum_{j=1}^{\ell} \operatorname{rank} A_j + \operatorname{rank} A = r$, we say that μ is an r**–atomic** $(\mathbb{R} \cup \infty)$**–representing measure** for L.

3. Proofs of Theorem 1.1 and Corollaries 1.2, 1.3

In the proof of Theorem 1.1 we will need the following lemma on the determinant of a matrix polynomial.

Lemma 3.1. *Let $p, n \in \mathbb{N}$, $t \in \mathbb{R}$, and let*

$$H(x) = (x - t) \sum_{i=0}^{n} x^i H_i + P_0$$

be a nonzero matrix polynomial, where $H_i, P_0 \in M_p(\mathbb{R})$. We define

$$m := \dim \operatorname{Ker} P_0 \qquad \text{and} \qquad s := \dim\Big(\operatorname{Ker} P_0 \bigcap \bigcap_{i=0}^{n} \operatorname{Ker} H_i\Big).$$

Then

$$\det H(x) = \begin{cases} (x-t)^m g(x), & \text{if } s = 0, \\ 0, & \text{if } s > 0, \end{cases} \tag{3.1}$$

where $0 \neq g(x) \in \mathbb{R}[x]$.

Proof. Clearly, if $s > 0$, there exists a nonzero vector $v \in \mathbb{R}^p$ such that $H(x)v = \mathbf{0}_{p,1}$, which implies (3.1). From now on we assume that $s = 0$. Let $\mathcal{B} := \{b_1, b_2, \ldots, b_m, b_{m+1}, \ldots, b_p\}$ be a basis of $\mathbb{R}^p$ such that the set $\{b_1, b_2, \ldots, b_m\}$ is a basis of $\operatorname{Ker} P_0$. Let us define an invertible matrix

$$B := \begin{pmatrix} b_1 & b_2 & \cdots & b_p \end{pmatrix} \in M_p(\mathbb{R}).$$

For $i = 0, 1, \ldots, n$ define matrices

$$\widetilde{H}_i \equiv \begin{pmatrix} \widetilde{h}_i^{(1)} & \widetilde{h}_i^{(2)} & \cdots & \widetilde{h}_i^{(m)} & \cdots & \widetilde{h}_i^{(p)} \end{pmatrix} := H_i B \tag{3.2}$$

and let

$$\widetilde{P}_0 \equiv \begin{pmatrix} \underbrace{\mathbf{0}_{p,1} \quad \cdots \quad \mathbf{0}_{p,1}}_{m} & \widetilde{p}_0^{(m+1)} & \cdots & \widetilde{p}_0^{(p)} \end{pmatrix} := P_0 B, \tag{3.3}$$

where the last equality follows by $b_1, b_2, \ldots, b_m \in \operatorname{Ker} P_0$. Define a matrix polynomial

$$\widetilde{H}(x) := (x-t)\sum_{i=0}^{n} x^i \widetilde{H}_i + \widetilde{P}_0 = H(x)B \in M_p(\mathbb{R}[x]).$$

By (3.2) and (3.3), it follows that the first m columns of $\widetilde{H}(x)$ are of the form

$$(x-t)\sum_{i=0}^{n} x^i \begin{pmatrix} \widetilde{h}_i^{(1)} & \widetilde{h}_i^{(2)} & \cdots & \widetilde{h}_i^{(m)} \end{pmatrix},$$

while the last $p-m$ columns of $\widetilde{H}(x)$ are equal to

$$(x-t)\sum_{i=0}^{n} x^i \begin{pmatrix} \widetilde{h}_i^{(m+1)} & \widetilde{h}_i^{(m+2)} & \cdots & \widetilde{h}_i^{(p)} \end{pmatrix} + \begin{pmatrix} \widetilde{p}_0^{(m+1)} & \widetilde{p}_0^{(m+2)} & \cdots & \widetilde{p}_0^{(p)} \end{pmatrix}$$

Observe that the first m columns of $\widetilde{H}(x)$ have a common factor $(x-t)$. Using this observation and upon factoring the determinant of $\widetilde{H}(x)$ column-wise we obtain

$$\det H(x) = \frac{\det \widetilde{H}(x)}{\det B} = (x-t)^m g(x),$$

which proves (3.1). Since $s = 0$, $g(x) \neq 0$ also holds. $\square$

Proof of Theorem 1.1. Let $\mathbf{v}_i^{(j)}$ be as in (2.2) and $((x-t)\cdot \mathbf{v})_i^{(j)}$ as in (2.3) for $f(x) = x - t$. Note that $\mathcal{H}$ from the statement of the theorem is equal to $\mathcal{H}_{x-t}(n-1)$. First we establish the following claim.

Claim. $\operatorname{rank}\begin{pmatrix} \mathbf{v}_0^{(n)} & \begin{matrix} \mathcal{H}_{x-t}(n-1) \\ \left(((x-t)\cdot\mathbf{v})_n^{(n-1)}\right)^T \end{matrix} \end{pmatrix} = (n+1)p.$

Proof of Claim. We have

$$\begin{aligned}
&\operatorname{rank}\begin{pmatrix} \mathbf{v}_0^{(n)} & \begin{matrix} \mathcal{H}_{x-t}(n-1) \\ \left(((x-t)\cdot\mathbf{v})_n^{(n-1)}\right)^T \end{matrix} \end{pmatrix} \\
&= \operatorname{rank}\begin{pmatrix} \mathbf{v}_0^{(n)} & \mathbf{v}_1^{(n)} - t\mathbf{v}_0^{(n)} & \mathbf{v}_2^{(n)} - t\mathbf{v}_1^{(n)} & \cdots & \mathbf{v}_n^{(n)} - t\mathbf{v}_{n-1}^{(n)} \end{pmatrix} \\
&= \operatorname{rank}\begin{pmatrix} \mathbf{v}_0^{(n)} & \mathbf{v}_1^{(n)} & \mathbf{v}_2^{(n)} & \cdots & \mathbf{v}_n^{(n)} \end{pmatrix} \\
&= \operatorname{rank} M(n) = (n+1)p,
\end{aligned}$$

where we used that $M(n)$ is positive definite in the last equality. $\blacksquare$

Next we prove the implication (1) $\Rightarrow$ (2). Let $\mu = \sum_{j=1}^{\ell} \delta_{x_j} A_j$ be a representing measure for L such that the atoms $x_i \in \mathbb{R}$ are pairwise distinct, $A_i \in \mathbb{S}_p^{\succeq 0}(\mathbb{R})$, $x_1 = t$, $\operatorname{rank} A_1 = m$ and $\sum_{j=1}^{\ell} \operatorname{rank} A_j = (n+1)p$. We compute S_{2n+1} with respect to the measure μ, i.e., $S_{2n+1} := \int_{\mathbb{R}} x^{2n+1}\, d\mu = \sum_{j=1}^{\ell} x_j^{2n+1} A_j$. By the Claim,

$$np = \operatorname{rank}\begin{pmatrix} \mathcal{H}_{x-t}(n-1) \\ \left(((x-t)\cdot\mathbf{v})_n^{(n-1)}\right)^T \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \mathcal{H}_{x-t}(n-1) & ((x-t)\cdot\mathbf{v})_n^{(n-1)} \end{pmatrix}. \tag{3.4}$$

Define $\widetilde{\mu} := \sum_{j=2}^{\ell} \delta_{x_j} A_j$. Note that $\mu = \delta_t A_1 + \widetilde{\mu}$ and

$$\sum_{j=2}^{\ell} \operatorname{rank} A_j = \sum_{j=1}^{\ell} \operatorname{rank} A_j - \operatorname{rank} A_1 = (n+1)p - m.$$

Let $\widetilde{L} : \mathbb{R}[x] \to \mathbb{S}_p(\mathbb{R})$ be a linear operator, defined by

$$\widetilde{L}(x^i) \equiv \widetilde{S}_i := \int_{\mathbb{R}} x^i \, d\widetilde{\mu} \quad \text{for } i \in \mathbb{N} \cup \{0\}.$$

Let $\widetilde{\mathcal{S}} := (\widetilde{S}_0, \widetilde{S}_1, \ldots, \widetilde{S}_{2n+2})$. We have

$$\begin{aligned} (n+1)p - m = \operatorname{rank} M(n) - m \leq \operatorname{rank} M_{\widetilde{\mathcal{S}}}(n) \leq \operatorname{rank} M_{\widetilde{\mathcal{S}}}(n+1), \\ \operatorname{rank} M_{\widetilde{\mathcal{S}}}(n+1) \leq \sum_{j=2}^{\ell} \operatorname{rank} A_j = (n+1)p - m, \end{aligned} \tag{3.5}$$

where the first inequality follows from the fact that the difference $M(n) - M_{\widetilde{\mathcal{S}}}(n)$ is a sum of m matrices of rank 1. The inequalities (3.5) imply that $(n+1)p - m \leq M_{\widetilde{\mathcal{S}}}(n+1) \leq (n+1)p - m$, whence all inequalities in (3.5) must be equalities. In particular,

$$\operatorname{rank} M_{\widetilde{\mathcal{S}}}(n) = \operatorname{rank} M_{\widetilde{\mathcal{S}}}(n+1) = (n+1)p - m. \tag{3.6}$$

For every $i \in \mathbb{N} \cup \{0\}$ we have

$$\begin{aligned} \widetilde{S}_{i+1} - t\widetilde{S}_i &= \int_{\mathbb{R}} (x^{i+1} - tx^i) \, d\widetilde{\mu} \\ &= \int_{\mathbb{R}} (x^{i+1} - tx^i) \, d(\widetilde{\mu} + \delta_t A_1) \\ &= \int_{\mathbb{R}} (x^{i+1} - tx^i) \, d\mu \\ &= S_{i+1} - tS_i. \end{aligned} \tag{3.7}$$

Let $\widetilde{\mathcal{H}}_{x-t}(n-1)$ be the $(x-t)$–localizing moment matrix of $\widetilde{L}$,

$$\widetilde{\mathbf{v}}_i^{(n)} = \left(\widetilde{S}_{i+r-1}\right)_{r=1}^{n+1} = \begin{pmatrix} \widetilde{S}_i \\ \widetilde{S}_{i+1} \\ \vdots \\ \widetilde{S}_{i+n} \end{pmatrix} \quad \text{and} \quad ((x-t)\cdot\widetilde{\mathbf{v}})_n^{(n-1)} := \begin{pmatrix} \widetilde{S}_{n+1} - t\widetilde{S}_n \\ \widetilde{S}_{n+2} - t\widetilde{S}_{n+1} \\ \vdots \\ \widetilde{S}_{2n} - t\widetilde{S}_{2n-1} \end{pmatrix}.$$

By (3.7), it follows that

$$\widetilde{\mathcal{H}}_{x-t}(n-1) = \mathcal{H}_{x-t}(n-1) \quad \text{and} \quad ((x-t)\cdot\widetilde{\mathbf{v}})_n^{(n-1)} = ((x-t)\cdot\mathbf{v})_n^{(n-1)}. \tag{3.8}$$

Hence,

$$\begin{aligned} \operatorname{rank} \begin{pmatrix} \widetilde{\mathbf{v}}_0^{(n)} & \begin{matrix} \mathcal{H}_{x-t}(n-1) \\ \left(((x-t)\cdot\mathbf{v})_n^{(n-1)}\right)^T \end{matrix} \end{pmatrix} &= \operatorname{rank} \begin{pmatrix} \widetilde{\mathbf{v}}_0^{(n)} & \begin{matrix} \widetilde{\mathcal{H}}_{x-t}(n-1) \\ \left(((x-t)\cdot\widetilde{\mathbf{v}})_n^{(n-1)}\right)^T \end{matrix} \end{pmatrix} \\ &= \operatorname{rank} M_{\widetilde{\mathcal{S}}}(n) \underbrace{=}_{(3.6)} np + (p-m). \end{aligned} \tag{3.9}$$

It follows from (3.4) and (3.9) that

$$\begin{aligned}\operatorname{rank}&\begin{pmatrix}\widetilde{\mathbf{v}}_0^{(n)} & \begin{matrix}\mathcal{H}_{x-t}(n-1)\\ \left(((x-t)\cdot\mathbf{v})_n^{(n-1)}\right)^T\end{matrix}\end{pmatrix}\\ &=\operatorname{rank}\begin{pmatrix}\widetilde{a}_1 & \widetilde{a}_2 & \cdots & \widetilde{a}_{p-m} & \begin{matrix}\mathcal{H}_{x-t}(n-1)\\ \left(((x-t)\cdot\mathbf{v})_n^{(n-1)}\right)^T\end{matrix}\end{pmatrix},\end{aligned}\tag{3.10}$$

where $\widetilde{a}_1,\widetilde{a}_2,\ldots,\widetilde{a}_{p-m}$ are $p-m$ columns of the block $\widetilde{\mathbf{v}}_0^{(n)}$. We have

$$\begin{aligned}\operatorname{rank}&\begin{pmatrix}\widetilde{\mathbf{v}}_0^{(n)} & \begin{matrix}\mathcal{H}_{x-t}(n-1)\\ \left(((x-t)\cdot\mathbf{v})_n^{(n-1)}\right)^T\end{matrix}\end{pmatrix}\\ \underbrace{=}_{(3.9)}\operatorname{rank}&\begin{pmatrix}\widetilde{\mathbf{v}}_0^{(n)} & \begin{matrix}\widetilde{\mathcal{H}}_{x-t}(n-1)\\ \left(((x-t)\cdot\widetilde{\mathbf{v}})_n^{(n-1)}\right)^T\end{matrix}\end{pmatrix}\\ =\operatorname{rank}&\begin{pmatrix}\widetilde{\mathbf{v}}_0^{(n)} & \widetilde{\mathbf{v}}_1^{(n)}-t\widetilde{\mathbf{v}}_0^{(n)} & \widetilde{\mathbf{v}}_2^{(n)}-t\widetilde{\mathbf{v}}_1^{(n)} & \cdots & \widetilde{\mathbf{v}}_n^{(n)}-t\widetilde{\mathbf{v}}_{n-1}^{(n)}\end{pmatrix}\\ =\operatorname{rank}&\begin{pmatrix}\widetilde{\mathbf{v}}_0^{(n)} & \widetilde{\mathbf{v}}_1^{(n)} & \widetilde{\mathbf{v}}_2^{(n)} & \cdots & \widetilde{\mathbf{v}}_n^{(n)}\end{pmatrix}\\ \underbrace{=}_{(3.6)}\operatorname{rank}&\begin{pmatrix}\widetilde{\mathbf{v}}_0^{(n)} & \widetilde{\mathbf{v}}_1^{(n)} & \widetilde{\mathbf{v}}_2^{(n)} & \cdots & \widetilde{\mathbf{v}}_n^{(n)} & \widetilde{\mathbf{v}}_{n+1}^{(n)}\end{pmatrix}\\ =\operatorname{rank}&\begin{pmatrix}\widetilde{\mathbf{v}}_0^{(n)} & \widetilde{\mathbf{v}}_1^{(n)}-t\widetilde{\mathbf{v}}_0^{(n)} & \widetilde{\mathbf{v}}_2^{(n)}-t\widetilde{\mathbf{v}}_1^{(n)} & \cdots & \widetilde{\mathbf{v}}_{n+1}^{(n)}-t\widetilde{\mathbf{v}}_n^{(n)}\end{pmatrix}\\ =\operatorname{rank}&\begin{pmatrix}\widetilde{\mathbf{v}}_0^{(n)} & \widetilde{\mathcal{H}}_{x-t}(n)\end{pmatrix}\\ \underbrace{=}_{(3.8)}\operatorname{rank}&\begin{pmatrix}\widetilde{\mathbf{v}}_0^{(n)} & \mathcal{H}_{x-t}(n)\end{pmatrix}.\end{aligned}\tag{3.11}$$

Write

$$((x-t)\cdot\mathbf{v})_n^{(n)} =: \begin{pmatrix}k_1 & k_2 & \cdots & k_p\end{pmatrix}.$$

By (3.10) and (3.11), for every $j=1,2,\ldots,p$, there exist $\alpha_1^{(j)},\alpha_2^{(j)},\ldots,\alpha_{p-m}^{(j)}\in\mathbb{R}$ and $v_j\in M_{np,1}(\mathbb{R})$, such that

$$k_j=\alpha_1^{(j)}\widetilde{a}_1+\alpha_2^{(j)}\widetilde{a}_2+\ldots+\alpha_{p-m}^{(j)}\widetilde{a}_{p-m}+\begin{pmatrix}\mathcal{H}_{x-t}(n-1)\\ \left(((x-t)\cdot\mathbf{v})_n^{(n-1)}\right)^T\end{pmatrix}v_j.$$

Hence, we have

$$\mathcal{H}_{x-t}(n)=\begin{pmatrix}\widetilde{a}_1 & \widetilde{a}_2 & \cdots & \widetilde{a}_{p-m} & \begin{matrix}\mathcal{H}_{x-t}(n-1)\\ \left(((x-t)\cdot\mathbf{v})_n^{(n-1)}\right)^T\end{matrix}\end{pmatrix}\begin{pmatrix}\mathbf{0}_{p-m,np} & W\\ I_{np} & V\end{pmatrix},$$

where

$$W:=\begin{pmatrix}\alpha_1^{(1)} & \alpha_1^{(2)} & \cdots & \alpha_1^{(p)}\\ \alpha_2^{(1)} & \alpha_2^{(2)} & \cdots & \alpha_2^{(p)}\\ \vdots & \vdots & \ddots & \vdots\\ \alpha_{p-m}^{(1)} & \alpha_{p-m}^{(2)} & \cdots & \alpha_{p-m}^{(p)}\end{pmatrix}\quad\text{and}\quad V:=\begin{pmatrix}v_1 & v_2 & \cdots & v_p\end{pmatrix}.$$

Writing $\widetilde{a}_j =: \begin{pmatrix} b_j \\ c_j \end{pmatrix} \in \begin{pmatrix} M_{np,1}(\mathbb{R}) \\ M_{p,1}(\mathbb{R}) \end{pmatrix}$ for each $j = 1, 2, \dots, p-m$, we have

$$\begin{pmatrix} \mathcal{H}_{x-t}(n-1) & ((x-t)\cdot \mathbf{v})_n^{(n-1)} \end{pmatrix} = \begin{pmatrix} b_1 & b_2 & \cdots & b_{p-m} & \mathcal{H}_{x-t}(n-1) \end{pmatrix} \begin{pmatrix} \mathbf{0}_{p-m,np} & W \\ I_{np} & V \end{pmatrix}$$

and hence,

$$\begin{aligned} &\operatorname{rank} \begin{pmatrix} \mathcal{H}_{x-t}(n-1) & ((x-t)\cdot \mathbf{v})_n^{(n-1)} \end{pmatrix} \\ &= \operatorname{rank} \begin{pmatrix} \mathcal{H}_{x-t}(n-1) & \begin{pmatrix} b_1 & b_2 & \cdots & b_{p-m} \end{pmatrix} W + \mathcal{H}_{x-t}(n-1) V \end{pmatrix} \\ &= \operatorname{rank} \begin{pmatrix} \mathcal{H}_{x-t}(n-1) & \begin{pmatrix} b_1 & b_2 & \cdots & b_{p-m} \end{pmatrix} W \end{pmatrix} \\ &\leq \operatorname{rank} \mathcal{H}_{x-t}(n-1) + \operatorname{rank} \begin{pmatrix} b_1 & b_2 & \cdots & b_{p-m} \end{pmatrix} W \\ &\leq \operatorname{rank} \mathcal{H}_{x-t}(n-1) + p - m. \end{aligned} \tag{3.12}$$

Using (3.4) in (3.12) concludes the proof of the implication (1) $\Rightarrow$ (2).

It remains to prove the implication (2) $\Rightarrow$ (1). Let us first describe the main idea of the proof. The aim is to construct a matrix $S_{2n+1} \in \mathbb{S}_p(\mathbb{R})$ such that

$$\dim \operatorname{Ker} \mathcal{H}_{x-t}(n) = m. \tag{3.13}$$

It will then follow from

$$\operatorname{Ker} \begin{pmatrix} \mathcal{H}_{x-t}(n-1) \\ \left(((x-t)\cdot \mathbf{v})_n^{(n-1)}\right)^T \end{pmatrix} = \{0\},$$

that there are m columns in $((x-t)\cdot \mathbf{v})_n^{(n)}$, which are in the span of the other columns of $\mathcal{H}_{x-t}(n)$. By the Claim, the matrix

$$\mathcal{M} := \begin{pmatrix} \mathbf{v}_0^{(n)} & \begin{matrix} \mathcal{H}_{x-t}(n-1) \\ \left(((x-t)\cdot \mathbf{v})_n^{(n-1)}\right)^T \end{matrix} \end{pmatrix} \tag{3.14}$$

is invertible. This fact and (3.13) will imply that there exists a polynomial

$$P_L(x) = (x-t)\Big(x^n I_p - \sum_{i=0}^{n-1} x^i P_{i,L}\Big) - P_0,$$

where $P_{i,L}, P_0 \in M_p(\mathbb{R})$ and $\operatorname{rank} P_0 = p - m$, which is a block column relation of the matrix $M(n+1) = \begin{pmatrix} M(n) & \mathbf{v}_{n+1}^{(n)} \\ (\mathbf{v}_{n+1}^{(n)})^T & S_{2n+2} \end{pmatrix}$, where S_{2n+2} is uniquely determined by $\operatorname{rank} M(n) = \operatorname{rank} M(n+1)$. Hence,

$$\det P_L(x) = (x-t)^m g(x), \tag{3.15}$$

for some polynomial $g(x) \in \mathbb{R}[x]$ of degree $(n+1)p - m$ with $g(t) \neq 0$. Thus, Theorem 1.1.(1) will follow from (3.15), Theorem 2.1 and Lemma 2.3. Moreover, the constructed measure μ will satisfy $\operatorname{mult}_\mu t = m$.

Let

$$\begin{aligned} k &:= \operatorname{rank} \begin{pmatrix} \mathcal{H}_{x-t}(n-1) & ((x-t)\cdot \mathbf{v})_n^{(n-1)} \end{pmatrix} - \operatorname{rank} \mathcal{H}_{x-t}(n-1) \\ &= np - \operatorname{rank} \mathcal{H}_{x-t}(n-1), \end{aligned} \tag{3.16}$$

where we used that $\mathcal{M}$ from (3.14) is invertible in the last equality. By assumption (2), we have

$$np - \operatorname{rank} \mathcal{H}_{x-t}(n-1) + m = k + m \leq p,$$

or equivalently

$$k \le p - m.$$

In order to simplify the technical structure of the proof, we make the following modification. We permute the columns of $((x-t)\cdot \mathbf{v})_n^{(n-1)}$ using a permutation matrix $P \in M_p(\mathbb{R})$ to obtain a matrix

$$\begin{aligned}\widehat{((x-t)\cdot \mathbf{v})}_n^{(n-1)} &\equiv \Big(\underbrace{\widehat{((x-t)\cdot \mathbf{v})}_{n;1}^{(n-1)}}_{\substack{p-m\\ \text{columns}}}\ \underbrace{\widehat{((x-t)\cdot \mathbf{v})}_{n;2}^{(n-1)}}_{\substack{m\\ \text{columns}}}\Big)\\ &:= ((x-t)\cdot \mathbf{v})_n^{(n-1)}P,\end{aligned} \tag{3.17}$$

such that

$$\begin{aligned}\operatorname{rank}\Big(\mathcal{H}_{x-t}(n-1)\quad \widehat{((x-t)\cdot \mathbf{v})}_{n;1}^{(n-1)}\Big) &= \operatorname{rank}\Big(\mathcal{H}_{x-t}(n-1)\quad ((x-t)\cdot \mathbf{v})_n^{(n-1)}\Big)\\ &= \operatorname{rank}\mathcal{H}_{x-t}(n-1)+k = np.\end{aligned} \tag{3.18}$$

By (3.18), it follows that

$$\widehat{((x-t)\cdot \mathbf{v})}_{n;2}^{(n-1)} = \Big(\mathcal{H}_{x-t}(n-1)\quad \widehat{((x-t)\cdot \mathbf{v})}_{n;1}^{(n-1)}\Big)J \tag{3.19}$$

for some $J \in M_{(n+1)p-m,m}(\mathbb{R})$ or equivalently

$$\Big(\mathcal{H}_{x-t}(n-1)\quad \widehat{((x-t)\cdot \mathbf{v})}_{n;1}^{(n-1)}\quad \widehat{((x-t)\cdot \mathbf{v})}_{n;2}^{(n-1)}\Big)\begin{pmatrix}-J\\ I_m\end{pmatrix} = \mathbf{0}_{np,m}. \tag{3.20}$$

We will now define $\widehat{Z} \in \mathbb{S}_p(\mathbb{R})$ such that

$$\begin{pmatrix}\mathcal{H}_{x-t}(n-1) & \widehat{((x-t)\cdot \mathbf{v})}_n^{(n-1)}\\ \Big(\widehat{((x-t)\cdot \mathbf{v})}_n^{(n-1)}\Big)^T & \widehat{Z}\end{pmatrix}\begin{pmatrix}-J\\ I_m\end{pmatrix} = \mathbf{0}_{(n+1)p,m}.$$

By (3.20), it suffices to establish the equality

$$\Big(\Big(\widehat{((x-t)\cdot \mathbf{v})}_n^{(n-1)}\Big)^T\quad \widehat{Z}\Big)\begin{pmatrix}-J\\ I_m\end{pmatrix} = \mathbf{0}_{p,m}. \tag{3.21}$$

Let us decompose $\widehat{Z}$ as

$$\widehat{Z} := \begin{pmatrix}\widehat{Z}_1 & \widehat{Z}_2\\ \widehat{Z}_2^T & \widehat{Z}_3\end{pmatrix}, \tag{3.22}$$

where $\widehat{Z}_1$, $\widehat{Z}_2$ and $\widehat{Z}_3$ are of sizes $(p-m)\times(p-m)$, $(p-m)\times m$ and $m\times m$, respectively. In this notation, (3.21) becomes

$$\begin{pmatrix}\Big(\widehat{((x-t)\cdot \mathbf{v})}_{n;1}^{(n-1)}\Big)^T & \widehat{Z}_1 & \widehat{Z}_2\\ \Big(\widehat{((x-t)\cdot \mathbf{v})}_{n;2}^{(n-1)}\Big)^T & \widehat{Z}_2^T & \widehat{Z}_3\end{pmatrix}\begin{pmatrix}-J\\ I_m\end{pmatrix} = \mathbf{0}_{p,m}. \tag{3.23}$$

We choose $\widehat{Z}_1 \in \mathbb{S}_{p-m}(\mathbb{R})$ so that

$$\operatorname{rank}\begin{pmatrix}\mathcal{H}_{x-t}(n-1) & \widehat{((x-t)\cdot \mathbf{v})}_{n;1}^{(n-1)}\\ \Big(\widehat{((x-t)\cdot \mathbf{v})}_{n;1}^{(n-1)}\Big)^T & \widehat{Z}_1\end{pmatrix} = np+(p-m) \tag{3.24}$$

and define

$$\widehat{Z}_2 := \Big(\Big(\widehat{((x-t)\cdot \mathbf{v})}_{n;1}^{(n-1)}\Big)^T\quad \widehat{Z}_1\Big)J \quad\text{and}\quad \widehat{Z}_3 := \Big(\Big(\widehat{((x-t)\cdot \mathbf{v})}_{n;2}^{(n-1)}\Big)^T\quad \widehat{Z}_2^T\Big)J. \tag{3.25}$$

By (3.25), it is clear that $\widehat{Z}$ satisfies (3.21). It remains to show that $\widehat{Z}$ is symmetric. Since $\widehat{Z}_1 \in \mathbb{S}_{p-m}(\mathbb{R})$, we only need to show that $\widehat{Z}_3 \in \mathbb{S}_m(\mathbb{R})$. But this follows by the following computation:

$$\begin{aligned}
\widehat{Z}_3 &= \begin{pmatrix} \big((\widehat{(x-t)\cdot \mathbf{v})}^{(n-1)}_{n;2}\big)^T & \widehat{Z}_2^T \end{pmatrix} J = \begin{pmatrix} (\widehat{(x-t)\cdot \mathbf{v})}^{(n-1)}_{n;2} \\ \widehat{Z}_2 \end{pmatrix}^T J \\
&\underbrace{=}_{(3.19),\,(3.25)} \left(\begin{pmatrix} \mathcal{H}_{x-t}(n-1) & (\widehat{(x-t)\cdot \mathbf{v})}^{(n-1)}_{n;1} \\ \big((\widehat{(x-t)\cdot \mathbf{v})}^{(n-1)}_{n;1}\big)^T & \widehat{Z}_1 \end{pmatrix} J \right)^T J \\
&= J^T \begin{pmatrix} \mathcal{H}_{x-t}(n-1) & (\widehat{(x-t)\cdot \mathbf{v})}^{(n-1)}_{n;1} \\ \big((\widehat{(x-t)\cdot \mathbf{v})}^{(n-1)}_{n;1}\big)^T & \widehat{Z}_1 \end{pmatrix}^T J.
\end{aligned}$$

Defining the vectors $c_1, c_2, \ldots, c_m$ by

$$C \equiv \begin{pmatrix} c_1 & c_2 & \cdots & c_m \end{pmatrix} := (I_{np} \oplus P) \begin{pmatrix} -J \\ I_m \end{pmatrix}$$

and the matrix Z by

$$Z := P\widehat{Z}P^T \in \mathbb{S}_p(\mathbb{R}), \tag{3.26}$$

we have

$$c_1, c_2, \ldots, c_m \in \operatorname{Ker} \begin{pmatrix} \mathcal{H}_{x-t}(n-1) & ((x-t)\cdot \mathbf{v})^{(n-1)}_{n} \\ \big(((x-t)\cdot \mathbf{v})^{(n-1)}_{n}\big)^T & Z \end{pmatrix}$$

and $c_1, c_2, \ldots, c_m$ are linearly independent. Indeed,

$$\begin{aligned}
&\begin{pmatrix} \mathcal{H}_{x-t}(n-1) & ((x-t)\cdot \mathbf{v})^{(n-1)}_{n} \\ \big(((x-t)\cdot \mathbf{v})^{(n-1)}_{n}\big)^T & Z \end{pmatrix} C \\
&= \begin{pmatrix} \mathcal{H}_{x-t}(n-1) & (\widehat{(x-t)\cdot \mathbf{v})}^{(n-1)}_{n} P^T \\ P\big((\widehat{(x-t)\cdot \mathbf{v})}^{(n-1)}_{n}\big)^T & P\widehat{Z}P^T \end{pmatrix} (I_{np} \oplus P) \begin{pmatrix} -J \\ I_m \end{pmatrix} \\
&= (I_{np} \oplus P) \begin{pmatrix} \mathcal{H}_{x-t}(n-1) & (\widehat{(x-t)\cdot \mathbf{v})}^{(n-1)}_{n} \\ \big((\widehat{(x-t)\cdot \mathbf{v})}^{(n-1)}_{n}\big)^T & \widehat{Z} \end{pmatrix} (I_{np} \oplus P^T)(I_{np} \oplus P) \begin{pmatrix} -J \\ I_m \end{pmatrix} \\
&= (I_{np} \oplus P) \begin{pmatrix} \mathcal{H}_{x-t}(n-1) & (\widehat{(x-t)\cdot \mathbf{v})}^{(n-1)}_{n} \\ \big((\widehat{(x-t)\cdot \mathbf{v})}^{(n-1)}_{n}\big)^T & \widehat{Z} \end{pmatrix} \begin{pmatrix} -J \\ I_m \end{pmatrix} \\
&\underbrace{=}_{(3.20),\,(3.21)} (I_{np} \oplus P)\mathbf{0}_{(n+1)p,m} = \mathbf{0}_{(n+1)p,m}.
\end{aligned}$$

Defining

$$S_{2n+1} := Z + tS_{2n} \in \mathbb{S}_p(\mathbb{R}),$$

the equality (3.13) holds. Since $\mathcal{M}$ from (3.14) is invertible, it follows that

$$\widehat{\mathcal{M}} := (I_{np} \oplus P^T)\mathcal{M} = \begin{pmatrix} \widehat{\mathbf{v}}^{(n)}_0 & \begin{matrix} \mathcal{H}_{x-t}(n-1) \\ \big((\widehat{(x-t)\cdot \mathbf{v})}^{(n-1)}_{n}\big)^T \end{matrix} \end{pmatrix}$$

is also invertible, where

$$\widehat{\mathbf{v}}_0^{(n)} := (I_{np} \oplus P^T)\mathbf{v}_0^{(n)}. \tag{3.27}$$

Therefore

$$\begin{pmatrix} \widehat{((x-t)\cdot\mathbf{v})}_{n;1}^{(n-1)} \\ \widehat{Z}_1 \\ \widehat{Z}_2^T \end{pmatrix} = \widehat{\mathcal{M}} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

for some real matrices $U_1 \in M_{p,p-m}(\mathbb{R})$ and $U_2 \in M_{np,p-m}(\mathbb{R})$ with

$$\operatorname{rank} U_1 = p-m \quad (\text{see } (3.24)), \tag{3.28}$$

or equivalently

$$\begin{pmatrix} \widehat{\mathcal{M}} & \begin{matrix} \widehat{((x-t)\cdot\mathbf{v})}_{n;1}^{(n-1)} \\ \widehat{Z}_1 \\ \widehat{Z}_2^T \end{matrix} \end{pmatrix} \begin{pmatrix} -U_1 \\ -U_2 \\ I_{p-m} \end{pmatrix} = \mathbf{0}_{(n+1)p,(p-m)}. \tag{3.29}$$

By (3.20), (3.25) and (3.29), we have that

$$\begin{pmatrix} \widehat{\mathbf{v}}_0^{(n)} & \begin{matrix} \mathcal{H}_{x-t}(n-1) \\ \left(\widehat{((x-t)\cdot\mathbf{v})}_n^{(n-1)}\right)^T \end{matrix} & \begin{matrix} \widehat{((x-t)\cdot\mathbf{v})}_n^{(n-1)} \\ \widehat{Z} \end{matrix} \end{pmatrix} \begin{pmatrix} -U_1 & \mathbf{0}_{p,m} \\ -U_2 & -J_1 \\ I_{p-m} & -J_2 \\ \mathbf{0}_{m,p-m} & I_m \end{pmatrix} = \mathbf{0}_{(n+1)p,p},$$

where $J =: \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} \in \begin{pmatrix} M_{np,m}(\mathbb{R}) \\ M_{p-m,m}(\mathbb{R}) \end{pmatrix}$. Therefore

$$\begin{aligned} &\begin{pmatrix} \widehat{((x-t)\cdot\mathbf{v})}_n^{(n-1)} \\ \widehat{Z} \end{pmatrix} \begin{pmatrix} I_{p-m} & -J_2 \\ \mathbf{0}_{m,p-m} & I_m \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{H}_{x-t}(n-1) \\ \left(\widehat{((x-t)\cdot\mathbf{v})}_n^{(n-1)}\right)^T \end{pmatrix} \begin{pmatrix} U_2 & J_1 \end{pmatrix} + \widehat{\mathbf{v}}_0^{(n)} \begin{pmatrix} U_1 & \mathbf{0}_{p,m} \end{pmatrix} \end{aligned} \tag{3.30}$$

Using (3.27) and

$$\begin{pmatrix} \widehat{((x-t)\cdot\mathbf{v})}_n^{(n-1)} \\ \widehat{Z} \end{pmatrix} = \begin{pmatrix} ((x-t)\cdot\mathbf{v})_n^{(n-1)} P \\ P^T Z P \end{pmatrix} = (I_{np} \oplus P^T)((x-t)\cdot\mathbf{v})_n^{(n)} P$$

in (3.30), we get

$$((x-t)\cdot\mathbf{v})_n^{(n)} G_n = \sum_{i=0}^{n-1} ((x-t)\cdot\mathbf{v})_i^{(n)} G_i + \mathbf{v}_0^{(n)} \begin{pmatrix} U_1 & \mathbf{0}_{p,m} \end{pmatrix}, \tag{3.31}$$

where $G_n := P \begin{pmatrix} I_{p-m} & -J_2 \\ \mathbf{0}_{m,p-m} & I_m \end{pmatrix}$ and $G_0, G_1, \ldots, G_{n-1} \in M_p(\mathbb{R})$. Since G_n is invertible, (3.31) is equivalent to

$$((x-t)\cdot\mathbf{v})_n^{(n)} = \sum_{i=0}^{n-1} ((x-t)\cdot\mathbf{v})_i^{(n)} G_i G_n^{-1} + \mathbf{v}_0^{(n)} \begin{pmatrix} U_1 & \mathbf{0}_{p,m} \end{pmatrix} G_n^{-1}.$$

We now define the matrix polynomial

$$\begin{aligned} H(x) &:= (x^{n+1}-tx^n)I_p - \sum_{i=0}^{n-1}(x^{i+1}-tx^i)G_iG_n^{-1} - \begin{pmatrix} U_1 & \mathbf{0}_{p,m} \end{pmatrix} G_n^{-1} \\ &= (x-t)\left(x^n I_p - \sum_{i=0}^{n-1} x^i G_i G_n^{-1}\right) - \begin{pmatrix} U_1 & \mathbf{0}_{p,m} \end{pmatrix} G_n^{-1}. \end{aligned} \tag{3.32}$$

Observe that $H(x)$ is monic of degree $n+1$ and represents the block column relation $H(X) = \mathbf{0}_{(n+1)p,p}$ in the matrix

$$M(n+1) = \begin{pmatrix} M(n) & \mathbf{v}_{n+1}^{(n)} \\ \left(\mathbf{v}_{n+1}^{(n)}\right)^T & S_{2n+2} \end{pmatrix}, \tag{3.33}$$

where S_{2n+2} is uniquely determined by $\operatorname{rank} M(n) = \operatorname{rank} M(n+1)$. Note that

$$\operatorname{rank} \begin{pmatrix} U_1 & \mathbf{0}_{p,m} \end{pmatrix} G_n^{-1} = \operatorname{rank} \begin{pmatrix} U_1 & \mathbf{0}_{p,m} \end{pmatrix} \underbrace{=}_{(3.28)} p - m,$$

whence

$$\dim \left(\operatorname{Ker} \begin{pmatrix} U_1 & \mathbf{0}_{p,m} \end{pmatrix} G_n^{-1} \right) = m.$$

By Lemma 3.1 used for $H(x)$ from (3.32), we get that

$$\det H(x) = (x-t)^m g(x),$$

for some polynomial $g(x) \in \mathbb{R}[x]$ of degree $(n+1)p - m$. By Theorem 2.1, there exists a representing measure for L of the form $\mu = \sum_{j=1}^{\ell} \delta_{x_j} A_j$, where $\ell \in \mathbb{N}$, $x_j \in \mathbb{R}$ are pairwise distinct, $A_j \in \mathbb{S}_p^{\succeq 0}(\mathbb{R})$ and $(n+1)p = \operatorname{rank} M(n) = \sum_{j=1}^{\ell} \operatorname{rank} A_j$. By Lemma 2.3, the atoms $x_1, x_2, \ldots, x_\ell$ are exactly pairwise distinct zeros of $\det H(x)$. Hence, $t = x_{j'}$ for some $j' \in \{1, 2, \ldots, \ell\}$, and $\operatorname{rank} A_{j'} \geq m$. We now need to show that $\operatorname{rank} A_{j'} = m$. Suppose on the contrary that $\operatorname{rank} A_{j'} = m'$ for some

$$m' > m. \tag{3.34}$$

Let us define the measure $\widetilde{\mu} := \mu - \delta_t A_{j'}$. By analogous reasoning as for μ in the proof of implication (1) $\Rightarrow$ (2), we obtain an equality of type (3.11), where m is replaced by m', and $\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_{p-m'}$ are $p - m'$ columns of the block $\widetilde{\mathbf{v}}_0^{(n)} := \begin{pmatrix} \widetilde{S}_0 & \widetilde{S}_1 & \cdots & \widetilde{S}_n \end{pmatrix}$, where $\widetilde{S}_i := \int_{\mathbb{R}} x^i \, d\widetilde{\mu}$. The equality is equivalent to

$$\mathcal{H}_{x-t}(n) = \begin{pmatrix} \widetilde{a}_1 & \widetilde{a}_2 & \cdots & \widetilde{a}_{p-m'} & \begin{matrix} \mathcal{H}_{x-t}(n-1) \\ \left(((x-t)\cdot \mathbf{v})_n^{(n-1)}\right)^T \end{matrix} \end{pmatrix} \begin{pmatrix} \mathbf{0}_{p-m',np} & \widetilde{W} \\ I_{np} & \widetilde{V} \end{pmatrix},$$

for some matrices $\widetilde{W} \in M_{p-m',p}(\mathbb{R})$ and $\widetilde{V} \in M_{np,p}(\mathbb{R})$. Hence, we have

(3.35)
$$\begin{aligned}
&\operatorname{rank} \mathcal{H}_{x-t}(n) \\
=&\operatorname{rank} \left(\begin{pmatrix} \mathcal{H}_{x-t}(n-1) \\ \left(((x-t)\cdot \mathbf{v})_n^{(n-1)}\right)^T \end{pmatrix} \begin{pmatrix} \widetilde{a}_1 & \widetilde{a}_2 & \cdots & \widetilde{a}_{p-m'} \end{pmatrix} \widetilde{W} + \begin{pmatrix} \mathcal{H}_{x-t}(n-1) \\ \left(((x-t)\cdot \mathbf{v})_n^{(n-1)}\right)^T \end{pmatrix} \widetilde{V} \right) \\
=&\operatorname{rank} \left(\begin{pmatrix} \mathcal{H}_{x-t}(n-1) \\ \left(((x-t)\cdot \mathbf{v})_n^{(n-1)}\right)^T \end{pmatrix} \begin{pmatrix} \widetilde{a}_1 & \widetilde{a}_2 & \cdots & \widetilde{a}_{p-m'} \end{pmatrix} \widetilde{W} \right) \\
\leq&\operatorname{rank} \begin{pmatrix} \mathcal{H}_{x-t}(n-1) \\ \left(((x-t)\cdot \mathbf{v})_n^{(n-1)}\right)^T \end{pmatrix} + \operatorname{rank} \begin{pmatrix} \widetilde{a}_1 & \widetilde{a}_2 & \cdots & \widetilde{a}_{p-m'} \end{pmatrix} \widetilde{W} \\
\leq&\operatorname{rank} \begin{pmatrix} \mathcal{H}_{x-t}(n-1) \\ \left(((x-t)\cdot \mathbf{v})_n^{(n-1)}\right)^T \end{pmatrix} + p - m' \\
=&np + p - m'.
\end{aligned}$$

On the other hand, we have

(3.36)
$$\begin{aligned}
\operatorname{rank} \mathcal{H}_{x-t}(n) &= \operatorname{rank} \begin{pmatrix} \mathcal{H}_{x-t}(n-1) & ((x-t)\cdot \mathbf{v})_{n;1}^{(n-1)} & ((x-t)\cdot \mathbf{v})_{n;2}^{(n-1)} \\ \left(((x-t)\cdot \mathbf{v})_{n;1}^{(n-1)}\right)^T & \widehat{Z}_1 & \widehat{Z}_2 \\ \left(((x-t)\cdot \mathbf{v})_{n;2}^{(n-1)}\right)^T & \widehat{Z}_2^T & \widehat{Z}_3 \end{pmatrix} \\
&\underbrace{=}_{(3.19),\,(3.25)} \operatorname{rank} \begin{pmatrix} \mathcal{H}_{x-t}(n-1) & ((x-t)\cdot \mathbf{v})_{n;1}^{(n-1)} \\ \left(((x-t)\cdot \mathbf{v})_{n;1}^{(n-1)}\right)^T & \widehat{Z}_1 \\ \left(((x-t)\cdot \mathbf{v})_{n;2}^{(n-1)}\right)^T & \widehat{Z}_2^T \end{pmatrix} \\
&\underbrace{=}_{(3.19),\,(3.25)} \operatorname{rank} \begin{pmatrix} \mathcal{H}_{x-t}(n-1) & ((x-t)\cdot \mathbf{v})_{n;1}^{(n-1)} \\ \left(((x-t)\cdot \mathbf{v})_{n;1}^{(n-1)}\right)^T & \widehat{Z}_1 \end{pmatrix} \\
&\underbrace{=}_{(3.24)} np + p - m
\end{aligned}$$

Combining (3.35) and (3.36), we get $m \geq m'$, which is a contradiction with (3.34). Therefore $\operatorname{mult}_\mu t = m$ and $g(t) \neq 0$. This completes the proof. □

Proof of Corollary 1.2. Let $\mathbf{v}_i^{(j)}$ be as in (2.2) and $((x-t)\cdot \mathbf{v})_i^{(j)}$ as in (2.3). Since $M(n)$ is invertible, it follows that the matrix

$$\mathcal{M} := \begin{pmatrix} \mathbf{v}_0^{(n)} & \begin{matrix} \mathcal{H}_{x-t}(n-1) \\ \left(((x-t)\cdot \mathbf{v})_n^{(n-1)}\right)^T \end{matrix} \end{pmatrix}$$

is also invertible. Therefore

$$\operatorname{rank} \begin{pmatrix} \mathcal{H}_{x-t}(n-1) \\ \left(((x-t)\cdot \mathbf{v})_n^{(n-1)}\right)^T \end{pmatrix} = np,$$

whence $\operatorname{rank} \mathcal{H}_{x-t}(n-1) \geq (n-1)p$. By Theorem 1.1, the corollary follows. □

Proof of Corollary 1.3. Define a sequence $\widetilde{\mathcal{S}} = (\widetilde{S}_0, \widetilde{S}_1, \ldots, \widetilde{S}_{2n_1+2n_2})$, where $\widetilde{S}_i := S_{i-2n_1}$. By assumption (1.5), $M_{\widetilde{\mathcal{S}}}(n_1+n_2) := (\widetilde{S}_{i+j-2})_{i,j=1}^{n_1+n_2+1}$ is positive definite. By Corollary 1.2, $\widetilde{\mathcal{S}}$ has a minimal representing measure $\widetilde{\mu} = \sum_{j=1}^{\ell} A_j\delta_{x_j}$ for some $x_j \in \mathbb{R}\setminus\{0\}$ and $A_j \in S_p^{\succeq 0}(\mathbb{R})$. Namely, $\widetilde{S}_i = \sum_{j=1}^{\ell} A_j x_j^i$ for each $i = 0, 1, \ldots, 2n_1 + 2n_2$. But then

$$S_i = \widetilde{S}_{i+2n_1} = \sum_{j=1}^{\ell} A_j x_j^{i+2n_1} = \sum_{j=1}^{\ell} (A_j x_j^{2n_1}) x_j^i,$$

whence $\mu := \sum_{j=1}^{\ell} (A_j x_j^{2n_1})\delta_{x_j}$ is a representing measure in Corollary 1.3. □

Remark 3.2. (1) The polynomial $H(x)$ (see (3.32)), which is a block column relation of the matrix in (3.33), can also be obtained by computing

$$\begin{pmatrix} H_0^T & H_1^T & \cdots & H_n^T \end{pmatrix}^T := M(n)^{-1}\mathbf{v}_{n+1}^{(n)},$$

to obtain $H(x) = x^{n+1}I_p - \sum_{i=0}^{n} x^i H_i$.

(2) The zeroes of the polynomial $g(x)$ from (3.15) correspond to the other atoms in the representing measure, while the multiplicity of the atom as the zero of $g(x)$ coincides with the multiplicity of the atom.

(3) Assume a linear operator $L : \mathbb{R}[x]_{\leq 2n} \to \mathbb{S}_p(\mathbb{R})$ has a representing measure $\mu = \sum_{j=1}^{\ell} \delta_{x_j} A_j$, where the atoms $x_j \in \mathbb{R}$ are pairwise distinct, $A_j \in \mathbb{S}_p^{\succeq 0}(\mathbb{R})$ and $\sum_{j=1}^{\ell} \operatorname{rank} A_j = (n+1)p$. Assume that we know the atoms $x_1, x_2, \ldots, x_\ell$. It remains to compute the masses A_j. We denote by $V \equiv V_{(x_1,x_2,\ldots,x_\ell)} := \left(x_j^{i-1}\right)_{i,j=1}^{\ell}$ the Vandermonde matrix. Since $x_1, x_2, \ldots, x_\ell$ are pairwise distinct, it follows that V is invertible. The masses A_j are obtained via

$$\begin{pmatrix} A_1 & A_2 & \cdots & A_\ell \end{pmatrix}^T = \left(V^{-1} \otimes I_p\right) \mathbf{v}_0^{(\ell-1)},$$

where $\otimes$ denotes the Kronecker product of two matrices, i.e., $V^{-1} \otimes I_p = (V \otimes I_p)^{-1} = \left((x_j^{i-1} I_p)_{i,j=1}^{\ell}\right)^{-1}$. Note that if $\ell > 2n+2$, then not all S_j are given. In particular, $S_{2n+2}, S_{2n+3}, \ldots, S_{\ell-1}$ need to be computed recursively by

$$S_j = \begin{pmatrix} S_{j-n-1} & S_{j-n} & \cdots & S_{j-1} \end{pmatrix} \begin{pmatrix} H_0^T & H_1^T & \cdots & H_n^T \end{pmatrix}^T,$$

for $j = 2n+2, 2n+3, \ldots, \ell-1$, where H_i are as in (1) above.

(4) If $m = p$ in Theorem 1.1, then k must be 0 in (3.16) and there are no blocks $(\widehat{(x-t)\cdot \mathbf{v})}_{n;1}^{(n-1)}$ (see (3.17)) and $\widehat{Z}_1$, $\widehat{Z}_2$ (see (3.22)). Moreover, $k = 0$ implies that

$$\operatorname{rank}\begin{pmatrix} \mathcal{H}_{x-t}(n-1) & ((x-t)\cdot\mathbf{v})_n^{(n-1)} \end{pmatrix} = \operatorname{rank}\mathcal{H}_{x-t}(n-1) = np,$$

whence $\mathcal{H}_{x-t}(n-1)$ is invertible. Further, J in (3.19) is equal to

$$J = \mathcal{H}_{x-t}(n-1)^{-1}((x-t)\cdot\mathbf{v})_n^{(n-1)},$$

while Z in (3.19) is equal to

$$Z = \left(((x-t)\cdot\mathbf{v})_n^{(n-1)}\right)^T (\mathcal{H}_{x-t}(n-1))^{-1}((x-t)\cdot\mathbf{v})_n^{(n-1)}.$$

Therefore, the measure μ for L, with $\operatorname{mult}_\mu t = m$, is unique.

(5) If $m < p$ in Theorem 1.1, then $k > 0$ in (3.16) and we have a free choice of selecting $\widehat{Z}_1 \in \mathbb{S}_{p-m}(\mathbb{R})$ and different possibilities for J in (3.19). To be precise, J can be chosen arbitrarily from the set

$$\Big\{ \begin{pmatrix} \mathcal{H}_{x-t}(n-1) & (\widehat{(x-t)\cdot\mathbf{v})}_{n;1}^{(n-1)} \end{pmatrix}^{\dagger} (\widehat{(x-t)\cdot\mathbf{v})}_{n;2}^{(n-1)} + U \colon U \in M_{(n+1)p-m,m}(\mathbb{R})$$
$$\text{such that } \begin{pmatrix} \mathcal{H}_{x-t}(n-1) & (\widehat{(x-t)\cdot\mathbf{v})}_{n;1}^{(n-1)} \end{pmatrix} U = \mathbf{0}_{np,m} \Big\},$$

where $(*)^\dagger$ denotes the Moore-Penrose pseudoinverse of the matrix $(*)$. Therefore, in this case, a measure μ for L such that $\operatorname{mult}_\mu t = m$ is not unique, as can be seen in Example 4.1 below.

4. Examples

In this section we demonstrate the application of Theorem 1.1 on numerical examples.

The following example considers a moment sequence $\mathcal{S}$ with $k > 0$ as defined in (3.16). We construct two distinct $(n+1)p$–atomic representing measures for $\mathcal{S}$. In both cases, the measures include 0 in the support with largest multiplicity allowed by Theorem 1.1, namely $m = p - k$, demonstrating that a representing measure for $\mathcal{S}$ containing an atom t with $\operatorname{mult}_\mu t = m$ is not unique whenever $m < p$.

Example 4.1. [1] Let $p = 2$, $n = 1$ and

$$S_0 = \begin{pmatrix} 18 & 10 \\ 10 & 7 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 50 & 26 \\ 26 & 14 \end{pmatrix}.$$

We can easily check that $M(1) \succ 0$. Let $t = 0$. We have that

$$\mathcal{H}_x(0) = \begin{pmatrix} S_1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \quad \text{and} \quad (x \cdot \mathbf{v})_1^{(0)} = \begin{pmatrix} S_2 \end{pmatrix} = \begin{pmatrix} 50 & 26 \\ 26 & 14 \end{pmatrix}.$$

We observe that $\operatorname{rank} \mathcal{H}_x(0) = 1$ and $\operatorname{rank} \begin{pmatrix} \mathcal{H}_x(0) & (x \cdot \mathbf{v})_1^{(0)} \end{pmatrix} = 2$, therefore $k = 1$ in (3.16). In this case, we can take a trivial permutation $P = I_2$ in (3.17) since $(x \cdot \mathbf{v})_1^{(0)} = \begin{pmatrix} (x \cdot \mathbf{v})_{1;1}^{(0)} & (x \cdot \mathbf{v})_{1;2}^{(0)} \end{pmatrix}$, where $(x \cdot \mathbf{v})_{1;1}^{(0)} = \begin{pmatrix} 50 \\ 26 \end{pmatrix}$, satisfies

$$\operatorname{rank} \begin{pmatrix} \mathcal{H}_x(0) & (x \cdot \mathbf{v})_1^{(0)} \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \mathcal{H}_x(0) & (x \cdot \mathbf{v})_{1;1}^{(0)} \end{pmatrix}.$$

Let $J := \begin{pmatrix} -\frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}^T$. We check that

$$(x \cdot \mathbf{v})_{1;2}^{(0)} = \begin{pmatrix} 26 \\ 14 \end{pmatrix} = \begin{pmatrix} \mathcal{H}_x(0) & (x \cdot \mathbf{v})_{1;1}^{(0)} \end{pmatrix} J.$$

We will now construct the matrix $Z = \widehat{Z}$ (see (3.22)), which is used in the proof of Theorem 1.1 to obtain a polynomial $H(x)$ (see (3.32)), being a block column relation of $\begin{pmatrix} M(1) & \mathbf{v}_2^{(1)} \end{pmatrix}$ and such that $\mathcal{Z}(\det H(x))$ is precisely the set of atoms in some minimal representing measure for $\mathcal{S} := (S_0, S_1, S_2)$. Note that since $t = 0$, we have $S_3 = Z$. For every $Z_1 \in \mathbb{R}$, the matrix

$$S_3 = Z = \begin{pmatrix} Z_1 & Z_2 \\ Z_2^T & Z_3 \end{pmatrix},$$

where $Z_2 = \begin{pmatrix} ((x \cdot \mathbf{v})_{1;1}^{(0)})^T & Z_1 \end{pmatrix} J$ and $Z_3 = \begin{pmatrix} ((x \cdot \mathbf{v})_{1;2}^{(0)})^T & Z_2^T \end{pmatrix} J$, is symmetric and satisfies (3.21).

Let $Z_1^{(1)} := 2$ and $Z_1^{(2)} := 98$. Computing

$$Z_2^{(i)} = \begin{pmatrix} ((x \cdot \mathbf{v})_{1;1}^{(0)})^T & Z_1^{(i)} \end{pmatrix} J \quad \text{and} \quad Z_3^{(i)} = \begin{pmatrix} ((x \cdot \mathbf{v})_{1;2}^{(0)})^T & (Z_2^{(i)})^T \end{pmatrix} J$$

[1] The *Mathematica* file with numerical computations can be found on the link https://github.com/ZobovicIgor/Matricial-Gaussian-Quadrature-Rules/tree/main.

for $i = 1, 2$, we get $S_3^{(1)} = Z^{(1)} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ and $S_3^{(2)} = Z^{(2)} = \begin{pmatrix} 98 & 50 \\ 50 & 26 \end{pmatrix}$. We can obtain the coefficients $H_0^{(i)}, H_1^{(i)}$ of the corresponding matrix polynomials

$$H^{(i)}(x) = x^2 I_2 - xH_1^{(i)} - H_0^{(i)}$$

by computing (see Remark 3.2.(1))

$$\begin{pmatrix} H_0^{(i)} \\ H_1^{(i)} \end{pmatrix} = M(1)^{-1} \cdot \begin{pmatrix} S_2 \\ S_3^{(i)} \end{pmatrix}$$

for $i = 1, 2$. The polynomials are the following:

$$H^{(1)}(x) = x^2 I_2 - x \begin{pmatrix} 2 & \frac{1}{2} \\ -4 & -1 \end{pmatrix} - \begin{pmatrix} 3 & \frac{3}{2} \\ 0 & 0 \end{pmatrix},$$
$$H^{(2)}(x) = x^2 I_2 - x \begin{pmatrix} 6 & \frac{5}{2} \\ -8 & -3 \end{pmatrix} - \begin{pmatrix} 3 & \frac{3}{2} \\ 0 & 0 \end{pmatrix},$$

with the determinants

$$\det H^{(1)}(x) = x(x-1)(x-\sqrt{3})(x+\sqrt{3}),$$
$$\det H^{(2)}(x) = x(x-3)(x-1)(x+1).$$

Therefore the sets $\{0, 1, \sqrt{3}, -\sqrt{3}\}$ and $\{0, 3, 1, -1\}$ represent the atoms of two distinct matrix measures for (S_0, S_1, S_2). Note that both determinants only have zeroes of multiplicity 1, therefore the multiplicities of all the atoms from both sets are 1. We confirm this by computing the corresponding masses for both sets of atoms. It turns out (using Remark 3.2.(3)) that the masses for the atoms $0, 1, \sqrt{3}, -\sqrt{3}$ in the first measure are $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$, $\begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}$, $\begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}$, respectively, and the masses for the atoms $0, 3, 1, -1$ in the second measure are $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$, $\begin{pmatrix} 12 & 6 \\ 6 & 3 \end{pmatrix}$, respectively.

The next example illustrates that the inequality in Theorem 1.1.(2) can be strict. Namely, starting from a measure whose atom 0 has multiplicity strictly smaller than $\operatorname{rank} \mathcal{H}_x(n-1) - (n-1)p$, we build a new representing measure in which the multiplicity of the atom 0 is the highest possible, i.e., equal to $\operatorname{rank} \mathcal{H}_x(n-1) - (n-1)p$.

Example 4.2. [2] Let $\mu = \sum_{j=1}^4 \delta_{x_j} A_j$ be a finitely atomic matrix measure with $(x_1, x_2, x_3, x_4) = (0, 1, -1, -2)$ and $(A_1, A_2, A_3, A_4) = \left(\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right)$, and let L be a linear operator, defined by $L(p) = \int_{\mathbb{R}} p \, d\mu$ for every $p \in \mathbb{R}[x]_{\leq 2}$. We define

$$S_0 := L(1) = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}, \quad S_1 := L(x) = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad S_2 := L(x^2) = \begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix}.$$

The measure μ contains the atom 0 with $\operatorname{mult}_\mu 0 = \operatorname{rank} A_1 = 1$. However, the localizing matrix $\mathcal{H}_x(0) = \begin{pmatrix} S_1 \end{pmatrix}$ is invertible, therefore

$$\operatorname{rank} \begin{pmatrix} \mathcal{H}_x(0) & S_2 \end{pmatrix} = \operatorname{rank} \mathcal{H}_x(0) = 2 < \operatorname{rank} \mathcal{H}_x(0) + 2 - \operatorname{mult}_\mu 0 = 3.$$

[2]The *Mathematica* file with numerical computations can be found on the link `https://github.com/ZobovicIgor/Matricial-Gaussian-Quadrature-Rules/tree/main`.

Since $\operatorname{rank}\begin{pmatrix}\mathcal{H}_x(0) & S_2\end{pmatrix} = \operatorname{rank}\mathcal{H}_x(0)$, it follows from Theorem 1.1 that there exists a 4–atomic representing matrix measure $\widetilde{\mu}$ for L which contains the atom 0 with $\operatorname{mult}_{\widetilde{\mu}} 0 = 2$. Such measure $\widetilde{\mu}$ is unique (see Remark 3.2.(4)) and we will now find its atoms. We first compute

$$S_3^{(\widetilde{\mu})} := S_2^T S_1^{-1} S_2 = \begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix}^T \cdot \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 11 & 13 \\ 13 & 8 \end{pmatrix}.$$

Then we obtain the polynomial $H(x)$, which is a block column relation of $\begin{pmatrix} M(1) & \mathbf{v}_2^{(1)} \end{pmatrix}$ and such that $\mathcal{Z}(\det H(x))$ is precisely the set of atoms in some minimal represenitng measure for L. Namely, $H(x) = x^2 I_2 - xH_1 - H_0$ where

$$\begin{pmatrix} H_0 \\ H_1 \end{pmatrix} = M(1)^{-1} \begin{pmatrix} S_2 \\ S_3^{(\widetilde{\mu})} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 \\ 6 & 3 \end{pmatrix} \end{pmatrix}.$$

Thus, it follows that

$$H(x) = x^2 I_2 - x \begin{pmatrix} 1 & 2 \\ 6 & 3 \end{pmatrix}.$$

The atoms of the measure $\widetilde{\mu}$ are the zeroes of

$$\det H(x) = x^2(x^2 - 4x - 9) = x^2(x - 2 + \sqrt{13})(x - 2 - \sqrt{13}),$$

therefore

$$\widetilde{\mu} = \delta_0 B_1 + \delta_{2-\sqrt{13}} B_2 + \delta_{2+\sqrt{13}} B_3,$$

where $\operatorname{rank} B_1 = 2$ and $\operatorname{rank} B_2 = \operatorname{rank} B_3 = 1$. By Remark 3.2.(3), the masses of the atoms are

$$B_1 = \begin{pmatrix} 3 & \frac{10}{3} \\ \frac{10}{3} & \frac{34}{9} \end{pmatrix}, \quad B_2 = \begin{pmatrix} \frac{13+3\sqrt{13}}{26} & \frac{-13-5\sqrt{13}}{78} \\ \frac{-13-5\sqrt{13}}{78} & \frac{13+2\sqrt{13}}{117} \end{pmatrix}, \quad B_3 = \begin{pmatrix} \frac{13-3\sqrt{13}}{26} & \frac{5\sqrt{13}-13}{78} \\ \frac{5\sqrt{13}-13}{78} & \frac{13-2\sqrt{13}}{117} \end{pmatrix}.$$

5. Generalized matricial Gaussian quadrature rules with prescribed atom

In this section we allow the evaluation at ∞ (see (2.6)) as a measure and prove a sufficient condition for the existence of a generalized matricial Gaussian quadrature rule for a linear operator $L : \mathbb{R}[x]_{\leq 2n} \to \mathbb{S}_p(\mathbb{R})$, containing $\operatorname{rank} M(n-1)$ real atoms, among which a prescribed atom has a prescribed multiplicity (see Theorem 5.1).

Let $m, n \in \mathbb{N}$ and

$$\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{n+m}(\mathbb{R}),$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$. The **Schur complement** [Zha05] of D in $\mathcal{M}$ is defined by $\mathcal{M}/D = A - BD^{-1}C$.

Theorem 5.1. *Let $n, p \in \mathbb{N}$ and $L : \mathbb{R}[x]_{\leq 2n} \to \mathbb{S}_p(\mathbb{R})$ be a linear operator such that $M(n-1)$ is positive definite. Fix $t \in \mathbb{R}$ and $m \in \mathbb{N} \cup \{0\}$. Assume the notation from §2. If*

$$m = \operatorname{rank}\begin{pmatrix} \mathcal{H}_{x-t}(n-2) & ((\mathbf{x}-\mathbf{t})\cdot\mathbf{v})_{n-1}^{(n-2)} \end{pmatrix} - \operatorname{rank}\mathcal{H}_{x-t}(n-2) \tag{5.1}$$

and

$$M(n)/M(n-1) \succeq 0, \tag{5.2}$$

then there exists a $(\operatorname{rank} M(n))$*–atomic* $(\mathbb{R}\cup\{\infty\})$*–representing measure μ for L such that* $\operatorname{mult}_\mu t = m$ *and* $\operatorname{mult}_\mu \infty = \operatorname{rank} M(n) - np$.

Proof. By the same proof as for the implication (2) $\Rightarrow$ (1) of Theorem 1.1, (5.1) implies that the sequence $\mathcal{S}^{(2n-1)} := (S_0, S_1, \ldots, S_{2n-1})$ has a $(\operatorname{rank} M(n-1))$–atomic $\mathbb{R}$–representing measure $\widetilde{\mu}$ such that $\operatorname{mult}_{\widetilde{\mu}} t = m$. Let $\widetilde{S}_{2n} = \int_{\mathbb{R}} x^{2n} d\widetilde{\mu}$ and $\widetilde{\mathcal{S}} = (S_0, S_1, \ldots, S_{2n-1}, \widetilde{S}_{2n})$. Since $\operatorname{rank} M_{\widetilde{\mathcal{S}}}(n) = \operatorname{rank} M_{\widetilde{\mathcal{S}}}(n-1)$, it follows that $M_{\widetilde{\mathcal{S}}}(n)/M_{\widetilde{\mathcal{S}}}(n-1) = \mathbf{0}_p$. Moreover,

$$M(n)/M(n-1) = S_{2n} - \widetilde{S}_{2n} + M_{\widetilde{\mathcal{S}}}(n)/M_{\widetilde{\mathcal{S}}}(n-1).$$

By (5.2), $S_{2n} - \widetilde{S}_{2n} \succeq 0$ and thus $\mathcal{S} = (S_0, S_1, \ldots, S_{2n})$ has a $(\mathbb{R} \cup \{\infty\})$–representing measure

$$\mu := \widetilde{\mu} + \delta_\infty \big(M(n)/M(n-1)\big).$$

This concludes the proof of Theorem 5.1. $\square$

References

[BKRSV20] G. Blekherman, M. Kummer, C. Riener, M. Schweighofer, C. Vinzant, *Generalized eigenvalue methods for Guassian quadrature rules*. Ann. H. Lebesgue **3** (2020), 1327–1341.

[BW11] M. Bakonyi, H.J. Woerdeman: *Matrix Completions, Moments, and Sums of Hermitian Squares*, Princeton University Press, Princeton, 2011

[Bol96] V. Bolotnikov, *On degenerate Hamburger moment problem and extensions of nonnegative Hankel block matrices*, Integral Equ. Oper. Theory **25** (1996), 253–276.

[CF91] R. Curto, L. Fialkow, *Recursiveness, positivity, and truncated moment problems.* Houston J. Math. **17** (1991), 603–635.

[DS03] H. Dette, W.J. Studden, *Quadrature formulas for matrix measures—a geometric approach*, Linear Algebra Appl. **364** (2003), 33–64.

[DD02] A.J. Durán, E. Defez, *Orthogonal matrix polynomials and quadrature formulas*, Linear Algebra Appl. **345** (2002), 71–84.

[DLR96] A.J. Duran, P. Lopez-Rodriguez, *Orthogonal Matrix Polynomials: Zeros and Blumenthal's Theorem*, J. Approx. Theory **84** (1996), 96–118.

[Dym89] H. Dym, *On Hermitian block Hankel matrices, matrix polynomials, the Hamburger moment problem, interpolation and maximum entropy*, Integral Equ. Oper. Theory **12** (1989) 757–812.

[DFKM09] Y.M. Dyukarev, B. Fritzsche, B. Kirstein, B., C. Mädler, H.C. Thiele, *On distinguished solutions of truncated matricial Hamburger moment problems*, Complex Anal. Oper. Theory **3** (2009) 759–834.

[FKM24] B. Fritzsche, B. Kirstein, C. Mädler, *Description of the set of all possible masses at a fixed point of solutions of a truncated matricial Hamburger moment problem*, Linear Algebra Appl. **697** (2024), 639–687.

[KT22] D.P. Kimsey, M. Trachana, *On a solution of the multidimensional truncated matrix-valued moment problem*, Milan J. Math. **90** (2022), 17–101.

[MS24a] C. Mädler, K. Schmüdgen, *On the Truncated Matricial Moment Problem. I*, J. Math. Anal. Appl. **540** (2024), 36 pp.

[MS24b] C. Mädler, K. Schmüdgen: *On the Truncated Matricial Moment Problem. II*, Linear Algebra Appl. **649** (2024), 63–97.

[NZ25] R. Nailwal, A. Zalar, *The truncated univariate rational moment problem*, Linear Algebra Appl. **708** (2025), 280–301.

[NZ+] R. Nailwal, A. Zalar, *Moment theory approach to Gaussian quadratures with prescribed nodes*, arXiv preprint https://arxiv.org/abs/2412.20849.

[Sim06] K.K. Simonov, *Strong truncated matrix moment problem of Hamburger*, Sarajevo J. Math. **2** (2006), 181–204.

[Wol] Wolfram Research, Inc., Mathematica, Version 14.0, Wolfram Research, Inc., Champaign, IL, 2025.

[Zal22] A. Zalar, *The strong truncated Hamburger moment problem with and without gaps*. J. Math. Anal. Appl. **516** (2022), 21pp.

[ZZ+] A. Zalar, I. Zobovič, *Matricial Gaussian quadrature rules: singular case*, in preparation.

[Zha05] F. Zhang, *The Schur Complement and Its Applications*. Springer-Verlag, New York, 2005.

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