# A SHORT INTRODUCTION TO THE RANDAL-WILLIAMS–WAHL MACHINE

ZOË POPE

These notes were prepared for a talk I gave at the Talbot 2025 workshop on homological stability, mentored by Alexander Kupers and Nathalie Wahl. The material for these notes is mainly based on the original article by Randal-Williams and Wahl [RWW17] and on Wahl's ICM survey [Wah23].

The Randal-Williams–Wahl machine is a tool for proving homological stability theorems for families of groups. Following classical results, the method axiomatizes conditions sufficient to apply Quillen's spectral sequence argument and obtain homological stability theorems.

## 1. The Setup

A sequence of groups with group homomorphisms

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \xrightarrow{f_3} \cdots$$

(called a *family of groups*) exhibits *homological stability* if the maps $(f_n)_* : H_i(G_n) \to H_i(G_{n+1})$ are isomorphisms for $n$ large relative to $i$. Motivated by classical families of groups—symmetric groups, general linear groups, etc-.–we assume that these families admit additional 'block sum' homomorphisms $\oplus_{n,m} : G_n \times G_m \to G_{n+m}$ that induce the maps $f_i$ :

$$f_i = - \oplus_{i,1} e : G_i \to G_{i+1}\,.$$

Viewing each $G_n$ as the category $BG_n$ with one object $n$ and automorphism group $G_n$, we assemble such a family into a *monoidal groupoid* $\mathcal{G} = \coprod_{n\geqslant 0} BG_n$ with unit object $BG_0 = *$ and monoidal product $\oplus : (n,m) \mapsto n+m$ induced by the block sum structure.

*Example.* The symmetric groups $\Sigma_n$ have block sums given by sticking two permutations side by side, with stabilization maps adjoining the trivial permutation of one letter to the right of any permutation. We then obtain $\Sigma = \coprod_{n\geqslant 0} B\Sigma_n$.

Though neither a necessary nor sufficient condition for a family of groups to exhibit homological stability, the Randal-Williams–Wahl machine employs a *braiding* on the monoidal groupoid to define the semisimplicial set needed for Quillen's spectral sequence argument.

**Definition.** *A monoidal groupoid* $(\mathcal{G}, \oplus, 0)$ *is* braided *if it has natural isomorphisms* $b_{X,Y} : X \oplus Y \to Y \oplus X$ *(called braiding isomorphisms) for all pairs of objects,*

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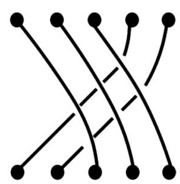
*satisfying the identity* $b_{X,Y}\, b_{X,Z}\, b_{Y,Z} = b_{Y,Z}\, b_{X,Z}\, b_{X,Y}$ *and such that the following square commutes for all* $f$ *and* $g$*.*

$$\begin{tikzcd} W \oplus X \arrow[r, "b_{W,X}"] \arrow[d, "f \oplus g"'] & X \oplus W \arrow[d, "g \oplus f"] \\ Y \oplus Z \arrow[r, "b_{Y,Z}"'] & Z \oplus Y \end{tikzcd} \tag{1}$$

*A braided monoidal category is* symmetric *if* $b_{Y,X} b_{X,Y} = 1_{X \oplus Y}$ *for* $X$ *and* $Y$*.*

*Example.* By considering the braid groups $B_n$ with block sum given by sticking two braids side by side, we obtain the *braided* monoidal groupoid $\mathcal{B} = \coprod_{n \geqslant 0} BB_n$. The braiding isomorphisms are conjugations $\mathrm{conj}_{b_{n,m}}$ by 'block braid' elements $b_{n,m} \in B_{n+m}$ such that the diagram in the following figure commutes.



$$\begin{tikzcd} B_n \times B_m \arrow[r, "\oplus_{n,m}"] \arrow[d, "\cong"'] & B_{n+m} \arrow[d, "conj_{b_{n,m}}"] \\ B_m \times B_n \arrow[r, "\oplus_{m,n}"] & B_{m+n} \end{tikzcd}$$

FIGURE 1. The block braid $b_{3,2}$ [Wah23, figure 1]; the required commuting square for $\mathrm{conj}\, b_{n+m}$ to be a braiding isomorphism.

More generally, braided monoidal groupoids of the form $\coprod_{n \geqslant 0} BG_n$ have braiding isomorphisms $\mathrm{conj}_{\phi_{n+m}(b_{n,m})}$ for $\phi_{n+m} : B_{n+m} \to G_{n+m}$ satisfying a similar square to the one in (1).

*Non-Example.* Usually, families of groups with non-braided monoidal groupoid structures fail to exhibit homological stability. For example, the groups $\mathbb{Z}^n$ assemble into a monoidal groupoid. However, $\mathrm{conj}_{\phi_{n+m}(b_{n,m})}$ fails to make the square in (1) commute for any choice of homomorphism $\phi_{n+m} : B_{n+m} \to \mathbb{Z}^{n+m}$ since $\mathbb{Z}^{n+m}$ is abelian. Indeed, the maps $H_1(\mathbb{Z}^n) \to H_1(\mathbb{Z}^{n+1})$ are never isomorphisms as they correspond to the inclusions $\mathbb{Z}^n \hookrightarrow \mathbb{Z}^{n+1}$ .

Viceversa, given a braided monoidal groupoid $(\mathcal{G}, \oplus, 0)$ we obtain a family of groups as follows. Fixing a 'starting point' $A \in \mathcal{G}$ and a 'stabilization direction' $X \in \mathcal{G}$, the monoidal product and stabilization direction give an endofunctor $- \oplus X : \mathcal{G} \to \mathcal{G}$. Applying this functor to the starting point repeatedly results in a sequence of objects of $\mathcal{G}$

$$A \oplus X \xrightarrow{-\oplus X} A \oplus X^2 \to A \oplus X^3 \to \cdots$$

which induces the sequence of groups

$$\mathrm{Aut}_{\mathcal{G}}(A \oplus X) \xrightarrow{-\oplus 1_X} \mathrm{Aut}_{\mathcal{G}}(A \oplus X^2) \to \mathrm{Aut}_{\mathcal{G}}(A \oplus X^3) \to \cdots .$$

For conciseness, we denote $- \oplus 1_X$ as $- \oplus X$.

*Example.* Consider the symmetric monoidal groupoid $(\mathrm{FinSet}^{\simeq}, \sqcup, \varnothing)$ of finite sets with respect to disjoint union. Letting $A = \varnothing$ and $X = pt$, we obtain up to isomorphism the sequence of symmetric groups $\Sigma_1 \to \Sigma_2 \to \cdots$.

*Example.* Consider the symmetric monoidal groupoid $(R\text{-mod}^{\simeq}, \oplus, 0)$ of $R$-modules. Letting $A = 0$ and $X = R$, up to isomorphism we recover the sequence of general linear groups $GL_1(R) \to GL_2(R) \to \cdots$.

*Example.* Consider the symmetric monoidal groupoid $(\mathrm{Group}^{\simeq}, *, \{e\})$ of groups under the free product. Letting $A = \{e\}$ and $X = \mathbb{Z}$, up to isomorphism we get the sequence of automorphism groups of free groups $\mathrm{Aut}(F_1) \to \mathrm{Aut}(F_2) \to \cdots$ .

Work by Manuel Krannich further generalizes the Randal-Williams–Wahl machine to the setting of modules over braided monoidal groupoids [Kra19, Section 7].

**Definition.** *A (right) module $\mathcal{M}$ over a braided monoidal groupoid $\mathcal{G}$ is a groupoid $\mathcal{M}$ with a strictly associative and unital (right) action*

$$\oplus : \mathcal{M} \times \mathcal{G} \to \mathcal{M}.$$

Just as before, picking a starting point $A \in \mathcal{M}$ and stabilization direction $X \in \mathcal{G}$ produces an endofunctor $- \oplus X : \mathcal{M} \to \mathcal{M}$ that induces a sequence of groups

$$\mathrm{Aut}_{\mathcal{M}}(A \oplus X) \xrightarrow{-\oplus X} \mathrm{Aut}_{\mathcal{M}}(A \oplus X^2) \to \mathrm{Aut}_{\mathcal{M}}(A \oplus X^3) \to \cdots .$$

*Remark.* The Randal-Williams–Wahl machinery requires braided monoidal groupoids (or more generally modules over braided monoidal groupoids) to be *strictly* associative and unital. We can always overcome this issue by instead considering the strict braided monoidal groupoid $\coprod_{n\geqslant 0} B\mathrm{Aut}(A \oplus X^{n+1})$ (viewed as a module over itself).

*Example.* All braided monoidal groupoids $\mathcal{G}$ are modules over themselves.

*Example.* Mapping class groups of surfaces $S_{g,r}$ with genus $g$ and $r$ boundary components form a family of groups with respect to the genus, by considering maps $S_{g,r} \to S_{g,r+1} \to S_{g+1,r}$ via first gluing a pair of pants to a boundary component and then gluing a pair of pants to two boundary components. We prove homological stability theorems by considering the groupoid of 'decorated surfaces' (surfaces with boundary where two boundaries are marked). Though not braided monoidal, this groupoid is a module over the braided monoidal groupoid $\mathcal{B} = \coprod_{n\geqslant 0} BB_n$. This is studied in [HVW24].

## 2. The Machinery

Homological stability theorems are usually proved using Quillen's spectral sequence argument. Summarily, for a family of groups $G_{n+1} = \mathrm{Aut}_{\mathcal{M}}(A \oplus X^{n+1})$ with maps $- \oplus X$, we want to construct a semisimplicial set $W_{n+1}$ with an action of $G_{n+1}$ satisfying the following three properties:

(1) the action is transitive levelwise;
(2) $- \oplus X^{q+1} : G_{n-q} \to G_{n+1}$ is an isomorphism onto $\mathrm{Stab}(\sigma_q)$ for some $q$-simplex $\sigma_q$;
(3) $W_{n+1}$ is homologically highly connected.

In particular, (1) and (2) imply that $(W_{n+1})_q \cong G_{n+1}/G_{n-q}$. These requirements suggest how to define $W_{n+1}$. Informally, $q$-simplices are equivalence classes of isomorphisms $B \oplus X^{q+1} \xrightarrow{\cong} A \oplus X^{n+1}$ made by identifying isomorphic 'complements' $B$. Motivated by this, we introduce the following definition:

**Definition.** [Wah23, Definition 2.4] *Let $\mathcal{M}$ be a module over a braided monoidal groupoid $\mathcal{G}$. Fix $A \in \mathcal{M}$ and $X \in \mathcal{G}$. The* space of destabilizations $W_{n+1} = W_{n+1}(A, X)$ *is the semisimplicial set with q-simplices*

$$(W_{n+1})_q := {\{(B,f)\,|\,B \in \mathcal{M},\ f : B \oplus X^{q+1} \xrightarrow{\cong} A \oplus X^{n+1}\}}\big/{\sim}$$

*where $(B, f) \sim (B', f')$ if there exists an isomorphism $g : B \xrightarrow{\cong} B'$ in $\mathcal{M}$ such that the following triangle commutes.*

$$\begin{tikzcd}
B \oplus X^{q+1} \arrow[d, "g \oplus X^{q+1}"'] \arrow[dr, "f"] & \\
& A \oplus X^{n+1} \\
B' \oplus X^{q+1} \arrow[ur, "f'"'] &
\end{tikzcd}$$

*$W_{n+1}(A, X)$ has face maps $d_i[B, f] = [B \oplus X, d_i f]$ where*

$$d_i f : (B \oplus X) \oplus X^q \xrightarrow{B \oplus b^{-1}_{X^i,X} \oplus X^{q-i}} B \oplus X^{q+1} \xrightarrow{f} A \oplus X^{n+1}$$

*and where $b^{-1}_{X^i,X}$ is the braiding isomorphism in $\mathcal{G}$.*

*Remark.* $G_{n+1}$ acts on this semisimplicial set via postcomposition. Also notice that the braiding isomorphisms in the face maps are needed for the semisimplicial relations to hold for $W_{n+1}$.

*Example.* Consider $\mathrm{FinSet}^{\simeq}$ as a module over itself. The $q$-simplices of $W_{n+1}(\varnothing, pt)$ are represented by isomorphisms $B \sqcup pt^{q+1} \xrightarrow{\cong} pt^{n+1}$. Since all possible complements $B$ are isomorphic, equivalence classes are in bijection with injections $pt^{q+1} \hookrightarrow pt^{n+1}$. The face maps $d_i : pt^q \hookrightarrow pt^{q+1}$ are then given by 'forgetting' the $i^{\text{th}}$ point. This is exactly the *complex of injective words* used to study homological stability of the symmetric groups [RWW17, Section 5.1].

*Example.* Consider $\text{R-mod}^{\simeq}$ as a module over itself. The $q$-simplices of $W_{n+1}(0, R)$ are represented by split injective homomorphisms $R^{q+1} \hookrightarrow R^{n+1}$ with *choice of complement* $B$ such that $B \oplus R^{q+1} \xrightarrow{\cong} R^{n+1}$.

*Example.* Consider $\mathrm{Group}^{\simeq}$ as a module over itself. The $q$-simplices of $W_{n+1}(e, \mathbb{Z})$ are represented by pairs $(f, H)$ for $f : A \to B$ injective and $H \leqslant B$ such that $B \cong H * f(A)$.

Additional assumptions on $W_{n+1}(A, X)$ are needed to apply Quillen's spectral sequence argument:

(1) *Cancellation*: $B \oplus X^{q+1} \cong A \oplus X^{n+1}$ implies $B \cong A \oplus X^{n-q}$;
(2) *Injectivity*: $- \oplus X^{q+1} : G_{n-q} \to G_{n+1}$ is injective;
(3) *Connectivity With Slope $k$*: for all $n$, $W_{n+1}(A, X)$ is homologically $(\frac{n-1}{k})$-connected.

Note for assumption (2) that the map $- \oplus X^{q+1}$ already has image the stabilizer of $\sigma_q := [A \oplus X^{n-q}, 1_{A \oplus X^{n+1}}]$, as both are automorphisms that fix the last $X^{q+1}$.

*Remark.* In applications, showing connectivity is the crux and usually requires case-by-case approaches. Cancellation and injectivity are easier to establish. Further, injectivity can be disregarded if one works within an even more general setting estabilished in [Kra19].

## 3. The Theorem

Quillen's spectral sequence argument produces a homological stability theorem for $(f_n)_*: H_i(G_n) \to H_i(G_{n+1})$ by constructing a spectral sequence whose convergence implies that the maps $(f_n)_*$ are isomorphisms. We first make a double complex whose vertical spectral sequence converges to 0 using the space of destabilizations and its assumed connectivity property. We then use the cancellation and injectivity properties to identify the maps $(f_n)_*$ with differentials in the horizontal spectral sequence. The argument concludes by showing that the only way the horizontal spectral sequence collapses is that the maps $(f_n)_*$ are isomorphisms.

**Theorem.** [RWW17, Theorem 3.1; Kra19, Theorem A] *Let $\mathcal{M}$ be a right module over a braided monoidal groupoid $\mathcal{G}$. Pick $A \in \mathcal{M}$, $X \in \mathcal{G}$, and fix $k \geqslant 2$. Assume that cancellation, injectivity, and connectivity with slope $k$ are satisfied. Then the map*

$$H_p(G_n;\mathbb{Z}) \xrightarrow{(-\oplus X)_*} H_p(G_{n+1};\mathbb{Z})$$

*is an epimorphism for $p \leqslant \frac{n}{k}$ and an isomorphism for $p \leqslant \frac{n-1}{k}$.*

*Proof.* Consider the double complex

$$E_\bullet G_{n+1} \bigotimes_{G_{n+1}} \widetilde{C}_*(W_{n+1})$$

where $E_\bullet G_{n+1}$ is a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}G_{n+1}$-module and $\widetilde{C}_*(W_{n+1})$ is the augmented cellular chain complex of $W_{n+1}$. Fixing $p$, for all $q \leqslant \frac{n-1}{k}$ we have that

$${}^vE^1_{pq} = H_q(E_pG_{n+1} \otimes_{G_{n+1}} \widetilde{C}_*(W_{n+1})) \cong 0$$

as $E_pG_{n+1}$ is free and $H_q(\widetilde{C}_*(W_{n+1})) \cong 0$ for $q \leqslant \frac{n-1}{k}$ by the connectivity assumption. Thus ${}^hE_{pq}$ converges to 0 in the same range.

Fixing $q$, we have

$$\begin{aligned}
{}^hE^1_{pq} &\cong H_p(E_\bullet G_{n+1} \otimes_{G_{n+1}} \mathbb{Z}[G_{n+1}/\operatorname{Stab}(\sigma_q)]) && \text{cancellation, orbit-stabilizer thm.}\\
&= H_p(G_{n+1};\mathbb{Z}[G_{n+1}/\operatorname{Stab}(\sigma_q)]) && \text{def. of homology with coefficients}\\
&\cong H_p(\operatorname{Stab}(\sigma_q)) && \text{Shapiro's Lemma}\\
&\cong H_p(G_{n-q}) && \text{injectivity}
\end{aligned}$$

for $\sigma_q = [A \oplus X^{n-q}, 1_{A\oplus X^{n+1}}]$. To show that the differential $d^1 : {}^hE^1_{pq} \to {}^hE^1_{p,q-1}$ corresponds to the stabilization map $(-\oplus X)_*$, it suffices to show that the following square commutes:

$$\begin{tikzcd}
{}^hE^1_{p,q} \arrow[r, "d_i"] \arrow[d, "\cong"'] & {}^hE^1_{p,q-1} \arrow[d, "\cong"] \\
H_p(G_{n-q}) \arrow[r, "(-\oplus X)_*"] & H_p(G_{n-q+1})
\end{tikzcd} \tag{2}$$

where the vertical arrows are the sequence of isomorphisms above. To show this we introduce the intermediary map

$$\partial_i : \mathrm{Stab}(\sigma_q) \hookrightarrow \mathrm{Stab}(d_i\sigma_q) \xrightarrow{\mathrm{conj}\,h_i} \mathrm{Stab}(\sigma_{q-1})$$

where $h_i := A \oplus X^{n-q} \oplus b_{X^i,X} \oplus X^{q-i}$. By construction of $h_i$, the map $\partial_i$ is an inclusion of subgroups of $G_{n+1}$. This implies that the two following diagrams commute

$$\begin{tikzcd}
(W_{n+1})_q \arrow[r, "d_i"] \arrow[d, "\cong"'] & (W_{n+1})_{q-1} \arrow[d, "\cong"] \\
G_{n+1}/\mathrm{Stab}(\sigma_q) \arrow[r, "\partial_i"] & G_{n+1}/\mathrm{Stab}(\sigma_{q-1})
\end{tikzcd}
\qquad
\begin{tikzcd}
G_{n-q} \arrow[r, "-\oplus X"] \arrow[d, "-\oplus X^{q+1}"'] & G_{n-q+1} \arrow[d, "-\oplus X^{q}"] \\
\mathrm{Stab}(\sigma_q) \arrow[r, "\partial_i"'] & \mathrm{Stab}(\sigma_{q-1})
\end{tikzcd}$$

and therefore (2) commutes. Taking alternating sums, we see that $d^1$ corresponds to

$$\sum_{i=0}^{q}(-\oplus X)_* = \begin{cases}(-\oplus X)_* & q \text{ even}\\ 0 & q \text{ odd}.\end{cases}$$

Assuming $q = 0$, we recover $H_p(G_n) \xrightarrow{(-\oplus X)_*} H_p(G_{n+1})$ within the ${}^hE^1$-page as the differential $d^1$. The proof concludes as explained in [RWW17, Theorem 3.1] by showing that, for the horizontal spectral sequence to converge to 0, the $d^1$ differentials must be epimorphisms or isomorphisms in the wanted ranges. □

The theorem allows us to recover classical homological stability results. Note that, while the induction argument prevents $k < 2$ without further assumptions, we can improve the isomorphism range to $\frac{n-a}{k}$ for any $a$ by proving a different homological connectivity result for $W_{n+1}$.

*Example.* To show homological stability for the symmetric groups $\Sigma_n$, we consider $(\mathrm{FinSet}^{\simeq}, \sqcup, \varnothing)$ as a module over itself. It is straightforward to show that $W_{n+1}(\varnothing, pt)$ satisfies cancellation and injectivity. As explained in [RWW17, Section 5.1], the complex of injective words $W_{n+1}$ is $(n-1)$-connected. Letting $k = 2$, the theorem implies that

$$H_p(\Sigma_n) \xrightarrow{(-\sqcup pt)_*} H_p(\Sigma_{n+1})$$

is an isomorphism for $p \leqslant \frac{n-1}{2}$. This was first shown by [Nak60].

*Example.* To show homological stability for the general linear groups $GL_n(R)$, we consider $(\text{R-mod}^{\simeq}, \oplus, 0)$ as a module over itself. Injectivity is satisfied, but cancellation fails for general rings $R$. However, rings with finite stable rank $s$ have the invariant basis number property ($R^n \cong R^m \implies n = m$ for $n, m \geqslant s$). Considering only such rings, $W_{n+1}(R^s, R)$ satisfies injectivity, cancellation, and $(\frac{n-1}{2})$-connectivity [RWW17, Section 5.3]. The theorem then implies that

$$H_p(GL_{s+n}(R)) \xrightarrow{(-\oplus R)_*} H_p(GL_{s+n+1}(R))$$

is an isomorphism for $p \leqslant \frac{n-1}{2}$. This was first shown by [vdK80].

*Example.* To show homological stability for the automorphism groups of free groups $\mathrm{Aut}(F_n)$, we consider $(\mathrm{Group}^{\simeq}, *, e)$ as a module over itself. Injectivity holds for $W_{n+1}(e, \mathbb{Z})$, and we can obtain cancellation via Grushko's Decomposition Theorem.

It can also be shown that $W_{n+1}(e, \mathbb{Z})$ is $(\frac{n-3}{2})$-connected [RWW17, Section 5.2]. The theorem then implies that

$$H_p(\mathrm{Aut}(F_n)) \xrightarrow{(-*\mathbb{Z})_*} H_p(\mathrm{Aut}(F_{n+1}))$$

is an isomorphism for $p \leqslant \frac{n-3}{2}$. This was first shown by [Gal11].

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University at Albany, State University of New York, USA
*Email address*: zoe@worknotes.org
*URL*: https://zoe-pope.github.io