16811 A5

November 2022

1.

The surface area is the revolution is $A = \int_{x_0}^{x_1} 2\pi y\sqrt{1+(y')^2}dx$.
To minimize the surface area is equal to minimize $J = \int_{x_0}^{x_1} f(x,y,y')dx$, where $f(x,y,y') = y\sqrt{1+(y')^2}$, with $x_0, y_0 = y(x_0)$ and $x_1, y_1 = y(x_1)$.
From Euler-Lagrange differential equation, we know that $f_y - \frac{d}{dx}f_{y'} = 0$.
and since $f_x = 0$, we can simplify it to the Beltrami identity: $f - y'f_{y'} = C$, where C is a constant.
$\{$Proof of Beltrami identity:

$$f_y - \frac{d}{dx}f_{y_x} = 0 \tag{1}$$
$$\Rightarrow \frac{df}{dx} = f_y y_x + f_{y_x}y_{xx} + f_x \tag{2}$$
$$\Rightarrow f_y y_x = \frac{df}{dx} - f_{y_x}y_{xx} - f_x - - - (1) \tag{3}$$

$$f_y - \frac{d}{dx}f_{y_x} = 0 \tag{4}$$
$$\Rightarrow y_x f_y - y_x\frac{d}{dx}(f_{y_x}) = 0 - - - (2) \tag{5}$$
$$\tag{6}$$

From equation (1) and (2) we can get
$-f_x + \frac{d}{dx}(f - y_x f_{y_x}) = 0 \Rightarrow \frac{d}{dx}(f - y_x f_{y_x}) = 0 \Rightarrow f - y'f_{y'} = C.\}$

$$f - y'f_{y'} = C \tag{7}$$
$$y\sqrt{1+(y')^2} - \frac{y(y')^2}{\sqrt{1+(y')^2}} = C \tag{8}$$
$$y1 + (y')^2 - y(y')^2 = C\sqrt{1+(y')^2} \tag{9}$$
$$y = C\sqrt{1+(y')^2} \tag{10}$$
$$\frac{dy}{dx} = y'\frac{\sqrt{y^2 - C^2}}{C} \tag{11}$$
$$dx = \frac{C}{\sqrt{y^2 - C^2}}dy \tag{12}$$
$$x = \int \frac{C}{\sqrt{y^2 - C^2}}dy = C\cosh^{-1}(\frac{y}{C}) + C_1 \tag{13}$$
$$\tag{14}$$

Therefore, $y = C\cosh(\frac{x-C_1}{C})$, and $y_0 = C\cosh(\frac{x_0-C_1}{C})$, $y_1 = C\cosh(\frac{x_1-C_1}{C})$.
The optimal 'curve' forms the surface with breaks, like circular disks (rings) when the curve is not C^2
The endpoint conditions matter because when the endpoints are farther away, it is more likely that the curve is not C_2 and the surface has breaks, and in this case, we cannot use the Calculus of Variations to find the solutions.

2.

The 3D arclength is $dl = \sqrt{dx^2 + dy^2 + dz^2}$.
From the spherical (u, v) coordinates, we get:
$dx = Rcos(v)cos(u)dv - Rsin(v)sin(u)du$, $dy = Rcos(v)sin(u)dv + Rsin(v)cos(u)du$, $dz = -Rsin(v)dv$.
Substitute the arclength formula with the spherical (u, v) coordinates, we get:

$$
\begin{aligned}
dl &= \sqrt{dx^2 + dy^2 + dz^2} && (15)\\
&= \sqrt{(Rcos(v)cos(u)dv - Rsin(v)sin(u)du)^2 + (Rcos(v)sin(u)dv + Rsin(v)cos(u)du)^2 + (-Rsin(v)dv)^2} && (16)\\
&= R\sqrt{dv^2 + sin^2(v)du^2} && (17)\\
&= R\sqrt{dv^2 + sin^2(v)(u')^2dv^2}u' = \frac{du}{dv} && (18)\\
&= R\sqrt{1 + sin^2(v)(u')^2}dv && (19)
\end{aligned}
$$

Therefore, the curve length between endpoints (u_1, v_1) and (u_2, v_2) is

$$l = \int_v 1^(v_2)R\sqrt{1 + sin^2v(u')^2}dv$$

To find the shortest curve is equivalent to minimize $\int_v 1^(v_2)f(v, u, u')dv$, where $f(v, u, u') = \sqrt{1 + sin^2v(u')^2}$.
From Euler-Lagrange differential equation, we know that $f_u - \frac{d}{dv}f_{u'} = 0$.

$$
\begin{aligned}
& f_u - \frac{d}{dv}f_{u'} = 0\\
& \Rightarrow \frac{d}{dv}\frac{sin^2vu'}{\sqrt{1 + sin^2v(u')^2}} = 0\\
& \Rightarrow \frac{sin^2vu'}{\sqrt{1 + sin^2v(u')^2}} = C\\
& \Rightarrow \frac{du}{dv} = u' = \frac{C}{\sqrt{sin^4v - C^2sin^2v}}\\
& \Rightarrow u = \int \frac{C}{\sqrt{sin^4v - C^2sin^2v}}dv\\
& \Rightarrow u = -sin^{-1}(\frac{cotv}{\sqrt{\frac{1}{C^2} - 1}}) + k
\end{aligned}
$$

, where C and k are appropriate constants

$$
\begin{aligned}
& \Rightarrow sin(k - u) = \frac{cotv}{\sqrt{\frac{1}{C^2} - 1}}\\
& \Rightarrow sin(k)cos(u) - cos(k)sin(u) = \frac{cotv}{\sqrt{\frac{1}{C^2} - 1}}\\
& \Rightarrow Rsin(v)cos(u)sin(k) - Rsin(v)sin(u)cos(k) = R\frac{cosv}{\sqrt{\frac{1}{C^2} - 1}}\\
& \Rightarrow Rsin(v)cos(u)sin(k) - Rsin(v)sin(u)cos(k) = R\frac{cosv}{\sqrt{\frac{1}{C^2} - 1}}\\
& \Rightarrow xsin(k) - ycos(k) = \frac{z}{\sqrt{\frac{1}{C^2} - 1}}
\end{aligned}
$$

Since $xsin(k) - ycos(k) = \frac{z}{\sqrt{\frac{1}{C^2} - 1}}$ is a plane passing through the origin, and its intersection with a sphere is an arc of a great circle. Therefore we proved that the shortest curve between two points on a sphere is an arc of a great circle.

3.

In the brachistochrone problem, we want to minimize $t = \int f(x, y, y')dx$, where $f(x, y, y') = \sqrt{\frac{1+(y')^2}{y_0 - y}}$. Since the right endpoint (x_1, y_1) merely constrained to satisfy an equation of the form $g(x, y) = 0$, we can get the new Euler-Lagrange differential equation as: $f_{y'} - \frac{g_y f}{g_x + g_y y'} = 0$.

$$
\begin{aligned}
& f_{y'} - \frac{g_y f}{g_x + g_y y'} = 0 \\
& \frac{y'}{\sqrt{(y_0 - y)(1 + (y')^2)}} - \frac{g_y}{g_x + g_y y'} \sqrt{\frac{1 + (y')^2}{y_0 - y}} = 0 \\
& y' - \frac{g_y}{g_x + g_y y'}(1 + (y')^2) = 0 \\
& (g_x + g_y y')y' - g_y(1 + (y')^2) = 0 \\
& y' = \frac{g_y}{g_x}
\end{aligned}
$$

The gradient of $y(x)$ is the same as $\frac{g_y}{g_x}$. Therefore we have proved that the optimizing curve $y(x)$ for this brachistochrone problem is perpendicular to the iso-contour $g(x, y) = 0$.

4.

(a) Since there is no gravity, the potential energy is 0.
Therefore, the Lagrangian is:

$$\begin{aligned}L &= T - U \\ &= T \\ &= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_{21}^2 + \frac{1}{2}m_2v_{22}^2\end{aligned}$$

We want to rewrite L in (τ, θ):

$$\begin{aligned}&v_1 = l_1\dot{\theta}_1 \\ &x_{21} = l_1cos(\theta_1) + l_2cos(\theta_1+\theta_2) \\ &y_{21} = l_1sin(\theta_1) + l_2sin(\theta_1+\theta_2) \\ &x_{22} = l_1cos(\theta_1) - l_2cos(\theta_1+\theta_2) \\ &y_{22} = l_1sin(\theta_1) - l_2sin(\theta_1+\theta_2) \\ &\Rightarrow \dot{x_{21}} = -l_1sin(\theta_1)\dot{\theta}_1 - l_2sin(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) \\ &\Rightarrow \dot{y_{21}} = l_1cos(\theta_1)\dot{\theta}_1 + l_2cos(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) \\ &\Rightarrow \dot{x_{22}} = -l_1sin(\theta_1)\dot{\theta}_1 + l_2sin(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) \\ &\Rightarrow \dot{y_{22}} = l_1cos(\theta_1)\dot{\theta}_1 - l_2cos(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2)\end{aligned}$$

Therefore,

$$\begin{aligned}v_{21}^2 = \dot{x_{21}}^2 + \dot{y_{21}}^2 = l_1^2\dot{\theta_1}^2 + l_2^2(\dot{\theta}_1+\dot{\theta}_2)^2 + 2l_1l_2cos(\theta_2)\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2) \\ v_{22}^2 = \dot{x_{22}}^2 + \dot{y_{22}}^2 = l_1^2\dot{\theta_1}^2 + l_2^2(\dot{\theta}_1+\dot{\theta}_2)^2 - 2l_1l_2cos(\theta_2)\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2)\end{aligned}$$

Then we rewrite Lagrangian $L = \frac{1}{2}m_1(l_1\dot{\theta}_1)^2 + m_2(l_1^2\dot{\theta_1}^2 + l_2^2(\dot{\theta}_1+\dot{\theta}_2)^2)$
From the Euler-Lagrange we know that $\tau_i = \frac{\partial}{\partial t}(\frac{\partial L}{\partial \dot{\theta}_i}) - \frac{\partial L}{\partial \theta_i}$. We want to solve $\frac{\partial}{\partial t}(\frac{\partial L}{\partial \dot{\theta}_i})$ and $\frac{\partial L}{\partial \theta_i}$ for each $i = 1, i = 2$.
When $i = 1$,

$$\begin{aligned}&L = \frac{1}{2}m_1(l_1\dot{\theta}_1)^2 + m_2(l_1^2\dot{\theta_1}^2 + l_2^2(\dot{\theta}_1+\dot{\theta}_2)^2) \\ &\Rightarrow \frac{\partial L}{\partial \theta_1} = 0 \\ &\Rightarrow \frac{\partial L}{\partial \dot{\theta}_1} = m_1l_1^2\dot{\theta}_1 + 2m_2(l_1^2+l_2^2)\dot{\theta}_1 + 2m_2l_2^2\dot{\theta}_2 \\ &\Rightarrow \frac{\partial}{\partial t}(\frac{\partial L}{\partial \dot{\theta}_1}) = m_1l_1^2\ddot{\theta}_1 + 2m_2(l_1^2+l_2^2)\ddot{\theta}_1 + 2m_2l_2^2\ddot{\theta}_2\end{aligned}$$

Therefore, we get $\tau_1 = \frac{\partial}{\partial t}(\frac{\partial L}{\partial \dot{\theta}_1}) - \frac{\partial L}{\partial \theta_1} = m_1l_1^2\ddot{\theta}_1 + 2m_2(l_1^2+l_2^2)\ddot{\theta}_1 + 2m_2l_2^2\ddot{\theta}_2$.

When $i = 2$,

$$
\begin{aligned}
L &= \frac{1}{2}m_1(l_1\dot{\theta}_1)^2 + m_2(l_1^2\dot{\theta_1}^2 + l_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2) \\
&\Rightarrow \frac{\partial L}{\partial \theta_2} = 0 \\
&\Rightarrow \frac{\partial L}{\partial \dot{\theta}_2} = 2m_2 l_2^2\dot{\theta}_2 + 2m_2 l_2^2\dot{\theta}_1 \\
&\Rightarrow \frac{\partial}{\partial t}(\frac{\partial L}{\partial \dot{\theta}_2}) = 2m_2 l_2^2\ddot{\theta}_2 + 2m_2 l_2^2\ddot{\theta}_1
\end{aligned}
$$

Therefore, we get $\tau_2 = \frac{\partial}{\partial t}(\frac{\partial L}{\partial \dot{\theta}_2}) - \frac{\partial L}{\partial \theta_2} = 2m_2 l_2^2\ddot{\theta}_2 + 2m_2 l_2^2\ddot{\theta}_1$
Thus, the relationship between joint torques and the angular state is:
$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} m_1 l_1^2 + 2m_2(l_1^2 + l_2^2) & 2m_2 l_2^2 \\ 2m_2 l_2^2 & 2m_2 l_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$

(b) When $\ddot{\theta}_2 = 0$, $\tau_1 = \frac{\partial}{\partial t}(\frac{\partial L}{\partial \dot{\theta}_1}) - \frac{\partial L}{\partial \theta_1} = m_1 l_1^2\ddot{\theta}_1 + 2m_2(l_1^2 + l_2^2)\ddot{\theta}_1$.
Since $\ddot{\theta}_2 = 0$, then on the joint of link 1 and link 2, there is no angular acceleration by $\ddot{\theta}_2$.
The τ_1 should be equal to the moment of inertia times the angular acceleration, and the moment of inertia here is $m_1 l_1^2 + 2m_2(l_1^2 + l_2^2)$ by definition, and angular acceleration is $\ddot{\theta}_1$.