# 16811 A3

October 2022

# 1.

(a) Taylor expansion:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n \\ &= f^0(0) + \frac{f^1(0)}{1} x^1 + \frac{f^2(0)}{2!} x^2 + \frac{f^3(0)}{3!} x^3 + ... \end{aligned}$$

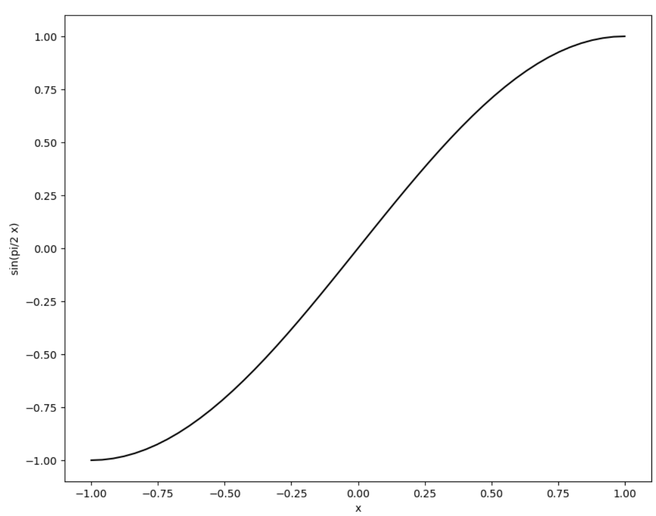
Since

$$\begin{aligned} f(x) &= sin(\frac{\pi}{2} x) \\ f^0(0) &= 0 \\ f^1(0) &= \frac{\pi}{2} cos(0) = \frac{\pi}{2} \\ f^2(0) &= -(\frac{\pi}{2})^2 sin(0) = 0 \\ f^3(0) &= -(\frac{\pi}{2})^3 cos(0) = -(\frac{\pi}{2})^3 \\ &... \end{aligned}$$

we can get that

$$\begin{aligned} f(x) &= f^0(0) + \frac{f^1(0)}{1} x^1 + \frac{f^2(0)}{2!} x^2 + \frac{f^3(0)}{3!} x^3 + ... \\ &= 0 + \frac{\pi}{2} x^1 + 0 - \frac{1}{3!} (\frac{\pi}{2})^3 x^3 + 0 + \frac{1}{5!} (\frac{\pi}{2})^5 x^5 + 0... \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n-1)!} (\frac{\pi}{2})^{2n-1} x^{2n-1} \end{aligned}$$

(b) Run *python q1.py* to get the plot.



(c)

$$n = 2, p(x) = a + bx + cx^2$$
$$e(x) = sin(\frac{\pi}{2}x) - a - bx - cx^2$$

From (a) we know that the coefficient of $x^2$ is $0,$ therefore $x^2 \in$ null space of $f(x)$ and $c = 0$

$$\Rightarrow p(x) = a + bx, e(x) = sin(\frac{\pi}{2}x) - a - bx$$

$n + 2 = 3$, let $-1 \le x_0 < x_1 < x_2 \le x_3 \le 1$, and since $f^{n+1}$ does not change the sign on $[-1, 1]$, then $x_0 = -1, x_3 = 1$.
We need to find $a, b, x_1, x_2$.

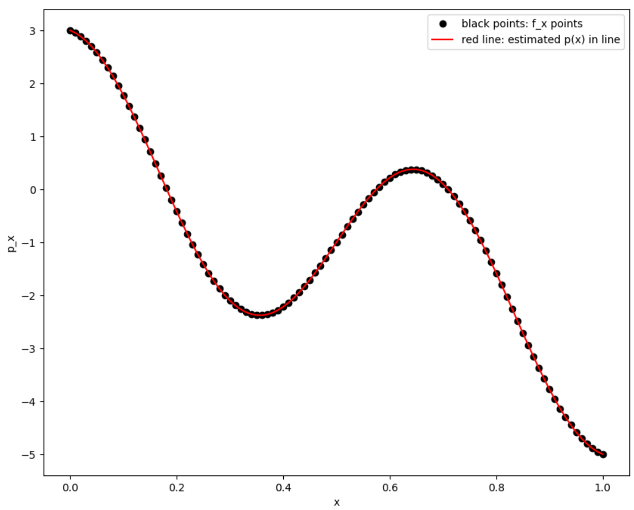
$$e(-1) = -e(x_1) = e(x_2) = -e(1)$$
$$e(-1) = -e(1) \Rightarrow sin(-\frac{\pi}{2}) - a + b = -sin(\frac{\pi}{2}) + a + b \Rightarrow a = 0$$
$$e'(x_1) = 0 \Rightarrow \frac{\pi}{2}cos(\frac{\pi}{2}x_1) - b = 0 \ (1)$$
$$e'(x_2) = 0 \Rightarrow \frac{\pi}{2}cos(\frac{\pi}{2}x_2) - b = 0 \ (2)$$
$$e(-1) = -e(x_1) \Rightarrow -1 + b = -sin(\frac{\pi}{2}x_1) + bx_1 \ (3)$$
$$e(-1) = e(x_2) \Rightarrow -1 + b = sin(\frac{\pi}{2}x_2) - bx_2 \ (4)$$

From (1) and (2), we can get $cos(\frac{\pi}{2}x_1) = cos(\frac{\pi}{2}x_2) \Rightarrow x_1 = -x_2$ (5)
From (1) and (3), solve two unkowns using 2 equations, we can get $b = 1.138, x_1 = \frac{2}{\pi}arccos(\frac{2.276}{\pi})$.
From (5), we get $x_2 = -\frac{2}{\pi}arccos(\frac{2.276}{\pi})$.

$L_\infty = e(-1) = -1 + 1.138 = 0.138$
$L_2 = \sqrt{\int_{-1}^{1} |e(x)|^2 dx} = \sqrt{\int_{-1}^{1} |e(x)|^2 dx} = \sqrt{\int_{-1}^{1} |sin(\frac{\pi}{2}x) - 1.138x|^2 dx} = 0.0185$

# 2.

Run *python q2.py* to get the estimated p(x), and the graph below: the black points represents the original f_x points, and the red line represents the estimated p(x) in line.
Explanation:
I first plotted the graph of f_x.
From the observation of start and points $f\_(x_0) = 3, f\_(x_{100}) = -5$, and the magnitude of frequency, I guess that '$cos(2\pi x)$ and $x$' might be in f_x.
Therefore I chose the basis as $1, cos(2\pi x), x$.
In the code, in order to find the estimated coefficients c, I used the SVD to solve the equation $\mathbf{Ac} = \mathbf{f\_x}$, where $\mathbf{A} = [cos(3\pi \mathbf{x}), \mathbf{x}, \mathbf{1}].T$



## 3.

(a) $T_0(x) = cos(0) = 1$
$T_1(x) = cos(\theta) = x$
$\Rightarrow T_2(x) = 2xT_1(x) - T_0(x) = 2x^2 - 1$
$\Rightarrow T_3(x) = 2xT_2(x) - T_1(x) = 2x(2x^2 - 1) - x = 4x^3 - 3x$

(b) $T_2(x)T_3(x) = (2x^2 - 1) * (4x^3 - 3x)$
The multiplication result is an odd function, and $(1 - x^2)^{-1/2}$ is an even function.
Therefore, $(1 - x^2)^{-1/2} g(x)h(x)$ is an odd function, and the integral of an odd function over $[-1, 1]$ is always zero.
Thus, $T_2(x)$ and $T_3(x)$ are orthogonal polynomials relative to the inner product.

(c)

$$\begin{aligned}
< T_n, T_n > &= \int_{-1}^{1} (1 - x^2)^{-1/2} (T_n(x))^2 dx \\
&= \int_{-1}^{1} (1 - cos(\theta))^2)^{-1/2} (cos(n\theta))^2 d(cos(\theta)) \\
&= \int_{\pi}^{2\pi} (1 - (cos(\theta))^2)^{-1/2} (cos(n\theta))^2 (-sin(\theta)) d\theta \\
&= \int_{\pi}^{2\pi} (-sin(\theta))^{-1} (cos(n\theta))^2 (-sin(\theta)) d\theta \\
&= \int_{\pi}^{2\pi} \frac{1}{2} \cdot 2(cos(n\theta))^2 d\theta \\
&= \int_{\pi}^{2\pi} \frac{1}{2} (cos(2n\theta) + 1) d\theta \\
&= \frac{\pi}{2}
\end{aligned}$$

Therefore, the length of $T_n(x) = \sqrt{< T_n, T_n >} = \sqrt{\frac{\pi}{2}}$ is a constant. We proved that all $T_n(x), n > 0$, have the same length.

(d)

$$\begin{aligned}
< T_i, T_j > &= \int_{-1}^{1} (1 - x^2)^{-1/2} T_i(x) T_j(x) dx \\
&= \int_{-1}^{1} (1 - cos(\theta))^2)^{-1/2} cos(i\theta) cos(j\theta) d(cos(\theta)) \\
&= \int_{\pi}^{2\pi} (1 - (cos(\theta))^2)^{-1/2} cos(i\theta) cos(j\theta) (-sin(\theta)) d\theta \\
&= \int_{\pi}^{2\pi} (-sin(\theta))^{-1} cos(i\theta) cos(j\theta) (-sin(\theta)) d\theta \\
&= \int_{\pi}^{2\pi} cos(i\theta) cos(j\theta) d\theta \\
&= \int_{\pi}^{2\pi} cos((i + j)\theta) cos((i - j)\theta) d\theta \\
&= 0
\end{aligned}$$

Therefore, we proved that $< T_i, T_j >= 0$ for all $i, j$ such that $i \geq 0, j \geq 0, i \neq j$.