16811 A1

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1.

(a) The implementation is in the q1.py. Run *python q1.py* to get the interpolation result.

(b) By running *python q1.py*, I got the interpolate result for $sin(2\pi x)$ at $x = \frac{1}{10}$ as 0.5877882222892551.

(c) By running *python q1.py*, I got the estimate results:
at n = 2, 1.0882;
at n = 4, 1.048946673207547;
at n = 40, 0.7157134629956571.
The actual value of $f(0.09)$ is 0.7158708568974157.

(d) the interpolation error for n = 2 is 1.9589807426941546
the interpolation error for n = 4 is 5.418917597832672
the interpolation error for n = 6 is 15.731031225174867
the interpolation error for n = 8 is 45.68235961893594
the interpolation error for n = 10 is 131.19950253756573
the interpolation error for n = 12 is 372.06087588229434
the interpolation error for n = 14 is 1043.1298787688495
the interpolation error for n = 16 is 2896.750208459296
the interpolation error for n = 18 is 7981.511500692966
the interpolation error for n = 20 is 21851.558191480788
the interpolation error for n = 40 is 427327258.30265874
The error estimate results show that the error increases as n increases. This demonstrates that larger n does not necessary leads to a better result. This make sense, because the accuracy of polynomial interpolation depends on how close the interpolated point is to the middle of the x values of the set of points used. In the case for example we are interpolating $f(x)$ where $x \in [-1, 1]$, as n increases, the middle becomes farther from the interpolated points, and thus the error gets larger.

2.

$$f^{'}(x) = 2\pi cos(2\pi x); f^{''}(x) = -(2\pi)^2 sin(2\pi x) f^{'''}(x) = -(2\pi)^3 cos(2\pi x)$$

1. linear interpolation

$$\max_{0\le x\le 1}|f^{''}(x)| \le (2\pi)^2$$
$$\max_{0\le x\le 1}(x-x_i)(x-x_{i+1}) = \max_{0\le y\le h}|y(y-h)| = \frac{h^2}{4}$$
$$\max_{0\le x\le 1}|e_1(x)| \le \frac{1}{4}\cdot(2\pi)^2\cdot\frac{h^2}{4} = \frac{(\pi)^2h^2}{4}$$
$$\text{want } \frac{(\pi)^2h^2}{4} \le 5\cdot 10^{-7} \Rightarrow h \le 0.000112539$$
$$\Rightarrow N = \frac{1}{h} = 8885.8$$

Therefore, need 8886 data points (or 8885 intervals).

2. quadratic interpolation

$$\max_{0\le x\le 1}|f^{'''}(x)| \le (2\pi)^3$$
$$\max_{0\le x\le 1}(x-x_{i-1})((x-x_i)(x-x_{i+1}) = \max_{-h\le y\le h}|(y+h)y(y-h)| = \frac{2h^3}{3\sqrt{3}}$$
$$\max_{0\le x\le 1}|e_1(x)| \le \frac{1}{6}\cdot(2\pi)^3\cdot\frac{2h^3}{3\sqrt{3}} = \frac{8\pi^3h^3}{9\sqrt{3}}$$
$$\text{want } \frac{8\pi^3h^3}{9\sqrt{3}} \le 5\cdot 10^{-7} \Rightarrow h \le 0.0031555702$$
$$\Rightarrow N = \frac{1}{h} = 316.9$$

Therefore, need 317 data points (or 316 intervals).

3.

The implementation is in the q3.py. Run *python q3.py* to get the result.
The solutions are: $x_{low} = 14.0661939, x_{high} = 17.2208$.

4.

(a) Define $h(x) = \frac{f(x)}{f'(x)}$.
By Taylor's series, $h(\xi+\varepsilon) = h(\xi) + h^{'}(\xi)\varepsilon + \frac{1}{2}h^{''}(\xi)\varepsilon^2....$

$$
\begin{aligned}
h(\xi) &= \frac{f(\xi)}{f'(\xi)} = 0 \\
h'(\xi) &= 1 - f(\xi)\frac{f''(\xi)}{(f'(\xi))^2} = 1 - \frac{f'(\xi)f''(\xi) + f(\xi)f'''(\xi)}{2f'(\xi)f''(\xi)} \\
&= 1 - \frac{f''(\xi)f''(\xi) + f'(\xi)f'''(\xi) + f(\xi)f''''(\xi) + f'(\xi)f'''(\xi)}{2f''(\xi)f''(\xi) + 2f'(\xi)f'''(\xi)} = \frac{f^{''}(\xi)f^{''}(\xi)}{2f''(\xi)f''(\xi)} = \frac{1}{2}
\end{aligned}
$$

Rewrite $x_{n+1} = x_n - h(x_n)$ as $\xi + \varepsilon_{n+1} = \xi + \varepsilon_n - h(x_n)$.
$\Rightarrow \varepsilon_{n+1} = \varepsilon_n - h(x_n)$.
Then when $n \to \infty, x \to \xi$, we can get

$$
\begin{aligned}
\varepsilon_{n+1} &= \varepsilon_n - h(\xi) \\
&= \varepsilon_n - (h(\xi) + h'(\xi)\varepsilon_n \\
&= \varepsilon_n - (0 + \frac{1}{2}\varepsilon_n) \\
&= \varepsilon_n - (0 + \frac{1}{2}\varepsilon_n) \\
&= \frac{1}{2}\varepsilon_n
\end{aligned}
$$

Then we can get $\lim_{n\to\infty} \frac{\varepsilon_{n+1}}{\varepsilon_n} = \frac{1}{2}$
Thus, in this case ξ is a root of order 2 of $f(x)$, the Newton's method converges linearly, and no longer converges quadratically.

(b) From (a), we get $h(\xi) = 0, h'(\xi) = \frac{1}{2}$.

$$
\begin{aligned}
h''(\xi) &= -\frac{(f'(\xi))^2f''(\xi) + f(\xi)f'(\xi)f'''(\xi) - 2f(\xi)(f''(\xi))^2}{(f'(\xi))^3} \\
&= -\frac{(2f'(\xi))^2f'''(\xi) + f(\xi)f'(\xi)f'''(\xi) - 3f(\xi)f''(\xi)f'''(\xi)}{3(f'(\xi))^2f''(\xi)} \\
&= -\frac{(3f'(\xi))^2f''''(\xi) + f'(\xi)f''(\xi)f'''(\xi) - 3f(\xi)(f'''(\xi))^2 - 2f'(\xi)f''(\xi)f''''(\xi) + f(\xi)f'(\xi)f'''''(\xi)}{3(f'(x))^2f'''(\xi) + f'(\xi)(f''(\xi))^2} \\
&= -\frac{1}{6}\frac{f'''(\xi)}{f''(\xi)}
\end{aligned}
$$

Similarly, if modify Netwon as $x_{n+1} = x_n - 2h(x_n)$, then rewrite as $\varepsilon_{n+1} = \varepsilon_n - 2h(x_n)$.

Then when $n \to \infty, x \to \xi, \varepsilon_n \to 0$, we can get

$$\begin{aligned}
\varepsilon_{n+1} &= \varepsilon_n - 2h(\xi) \\
&= \varepsilon_n - 2(h(\xi) + h'(\xi)\varepsilon_n + \frac{1}{2}h''(\xi)(\varepsilon_n)^2) \\
&= \varepsilon_n - (0 + \frac{1}{2}\varepsilon_n + \frac{1}{2}(-\frac{1}{6}\frac{f'''(\xi)}{f''(\xi)})(\varepsilon_n)^2) \\
&= -\frac{1}{6}\frac{f'''(\xi)}{f''(\xi)}(\varepsilon_n)^2
\end{aligned}$$

Then we can get $\lim_{n\to\infty} \frac{\varepsilon_{n+1}}{(\varepsilon_n)^2} = -\frac{1}{6}\frac{f'''(\xi)}{f''(\xi)}$
Since $f''(\xi)) \neq 0$, in this case ξ is a root of order 2 of $f(x)$, the Newton's method does converge quadratically.

5.

The implementation is in the q5.py. Run *python q5.py* to get the approximation result.
The approximation result for $x^5+x^4+x^3+x^2+x+1$ is -1.0, $-0.5+0.866i$, $-0.5-0.866i$, $0.5+0.866i$, $0.5-0.866i$.

6.

(a) Define Q = $\begin{pmatrix} 1 & -3 & 1 & -3 & 0 \\ 0 & 1 & -3 & 1 & -3 \\ 1 & -1 & -6 & 0 & 0 \\ 0 & 1 & -1 & -6 & 0 \\ 0 & 0 & 1 & -1 & -6 \end{pmatrix}$

x = $\begin{pmatrix} x^4 \\ x^3 \\ x^2 \\ x \\ 1 \end{pmatrix}$

Using the method of resultants, and $Q\mathbf{x} = 0$ iff det(Q) $= 0$. Since $det(Q) = 0$, then $p(x)$ and $q(x)$ share common root.

(b) Using the ratio method, let Q be the an $(n-1)*n$ matrix of rank $n-1$, then a solution to Qx $= 0$ satisfies:

$$x = \frac{x^4}{x^3} = (-1)^{1+2}\frac{det(Q_i)}{det(Q_j)} = (-1)\frac{405}{-135} = 3$$

where Q_i is the (n-1)*(n-1) matrix obtained by deleting the i_th column.

7.

Treat x as constant, and rewrite:

$$p(y) = 1 \cdot y^2 + 0 \cdot y + (x^2 - 1)$$
$$q(y) = 0 \cdot y^2 + 1 \cdot y + x^2$$

Then using the method of resultants, define Q =
$\begin{pmatrix} 1 & 0 & x^2-1 & 0 \\ 0 & 1 & 0 & x^2-1 \\ 1 & x^2 & 0 & 0 \\ 0 & 1 & x^2 & 0 \end{pmatrix}$
let $det(Q) = (x^4 + x^2 - 1)(x^2 - 1) = 0$, we find the projection points $x = -1, 1, \pm\sqrt{\frac{-1+\sqrt{5}}{2}}, \pm\sqrt{\frac{-1-\sqrt{5}}{2}}$.
By plugging into $p(y) = q(y)$, we see roots $y = \frac{1}{2}(1+\sqrt{5}), \frac{1}{2}(1-\sqrt{5})$.

8.

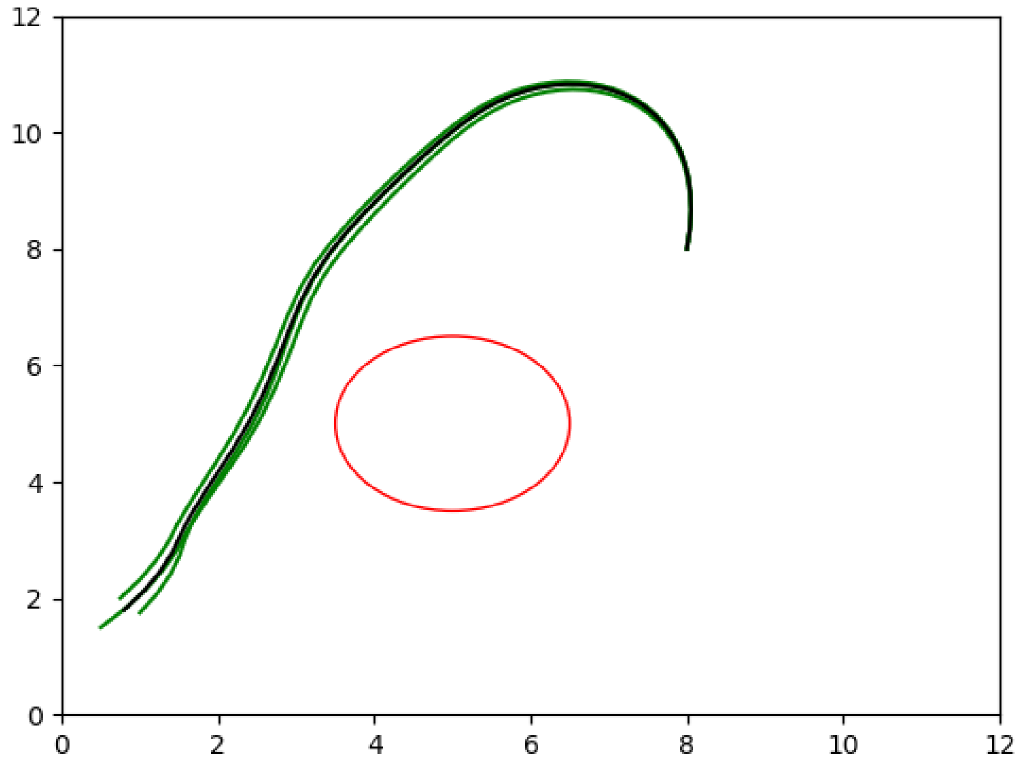
(a) The implementation is in the *get_w* function at q8.py.
Explanation:
To determine whether a 2D point (x, y) falls within the triangle formed by three 2D points:
Define A =
$$\begin{pmatrix} x^{(i)} & x^{(j)} & x^{(k)} \\ y^{(i)} & y^{(j)} & y^{(k)} \\ 1 & 1 & 1 \end{pmatrix}$$
b =
$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$
Then calculate v such that $Av = b$. If the components of vector v are all positive, then point (x, y) falls within the triangle formed by $(x^{(i)}, y^{(i)}), (x^{(j)}, y^{(j)}), (x^{(k)}, y^{(k)})$.

(b) The implementation is in q8.py. Run *python q8.py* to get the result and plots.

(c) I first load the data from paths.txt. Then picked the 3 optimal paths: 1. pick 3 paths from left or right side, and the picking criteria is that the starting points of the 3 paths should be closet to the starting point; the starting point falls within the triangle formed by the three starting points of the path selected.
2. Decide left or right 3 paths, which are the better choice. Calculate the weights using the method in (a): Av = b, and v contains the three weights we want to solve.
3. The time scale for t is [0, 1, .. 49], the same as the interpolated paths.
4. I used linear interpolation and found it already gave a good result, therefore, there's no need to use a more complex method for simplicity

(d) the green lines are the selected 3 paths, and back line the interpolated result. Figures are in the last pages of this pdf.

(e) If there are more obstacles or different shapes of obstacles, then I need to modify my code that I can no longer just devide to left or right cases. There would be more complicated cases to consider, e.g. the paths to be selected will have more complex shapes.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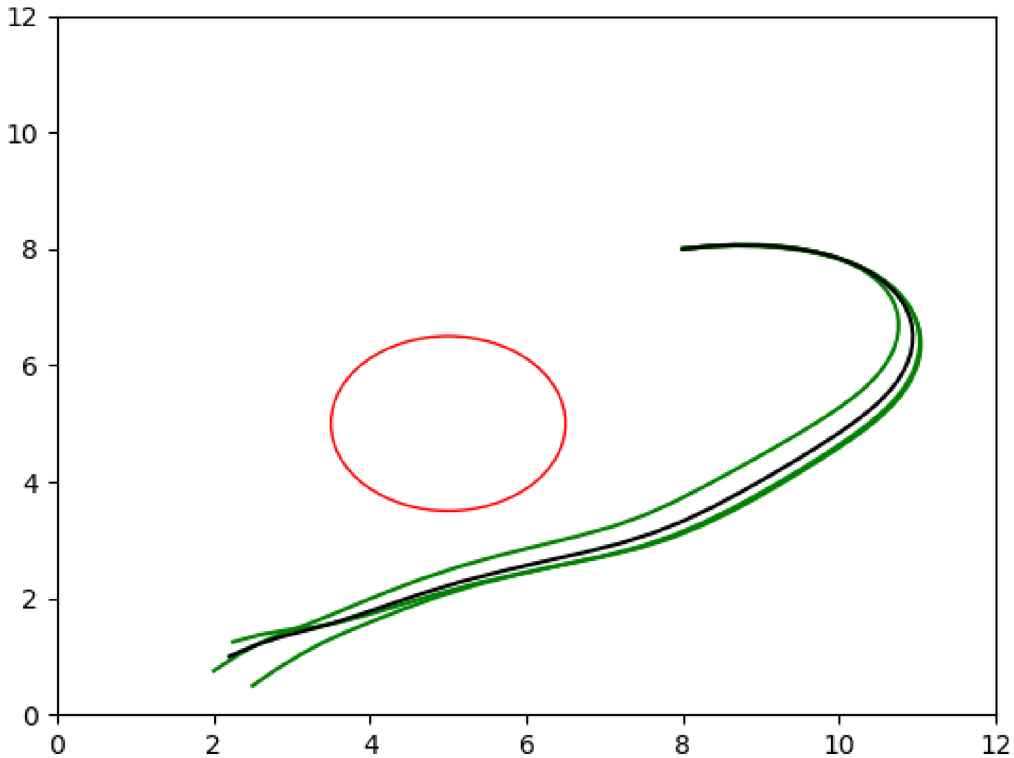
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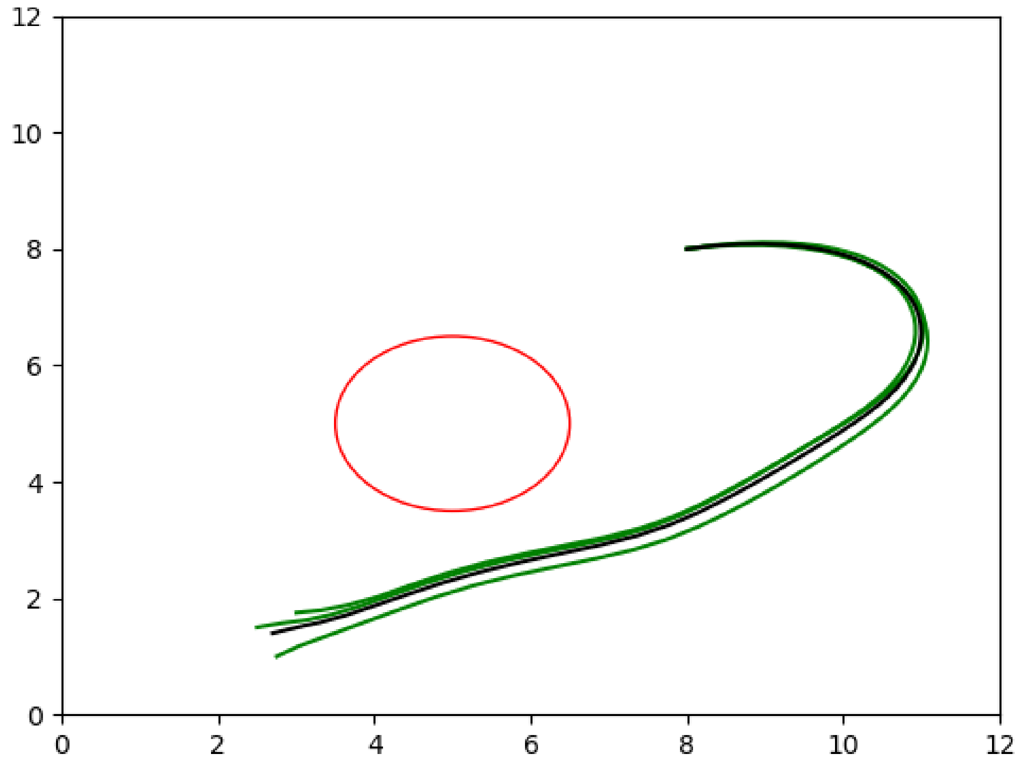
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