16811 A1

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1.

Instruction: Run *python q1.py*
Examples verified in code and their results:

Example 1: $\begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix}$

PA = LDU

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 2: $\begin{bmatrix} 1 & -2 & 3 \\ 3 & 5 & 2 \\ -1 & 3 & -4 \end{bmatrix}$

PA = LDU

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 3 & 5 & 2 \\ -1 & 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 0.091 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & -0.364 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -0.636 \\ -0 & -0 & 1 \end{bmatrix}$$

Example 3: $\begin{bmatrix} 4 & -20 & -12 \\ -8 & 45 & 44 \\ 20 & -105 & 79 \end{bmatrix}$

PA = LDU

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & -20 & -12 \\ -8 & 45 & 44 \\ 20 & -105 & 79 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 159 \end{bmatrix} \begin{bmatrix} 1 & -5 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

2.

Instruction: Run *python q2.py* to do SVD and test.
SVD results:

A1. U, S, V =

$$\begin{bmatrix} -0.24990784 & 0.19618951 & -0.9481855 \\ -0.82664436 & -0.55313889 & 0.1034237 \\ -0.50418764 & 0.8096586 & 0.3004127 \end{bmatrix}$$
$$\begin{bmatrix} 6.44727392 & 0 & 0 \\ 0 & 2.28171784 & 0 \\ 0 & 0 & 0.47583894 \end{bmatrix}$$
$$\begin{bmatrix} -0.83624567 & -0.37339567 & -0.40158292 \\ -0.41643526 & -0.04401494 & 0.90809931 \\ 0.356756 & -0.92662741 & 0.11868782 \end{bmatrix}$$

A2. U, S, V =

$$\begin{bmatrix} 0.27871723 & -0.05034398 & 0.22252444 & -0.67991678 \\ 0.56705659 & 0.3743265 & 0.6658971 & 0.22265812 \\ 0.28635903 & 0.1666676 & -0.33914231 & -0.60599065 \\ 0.37218833 & -0.89935675 & 0.10299055 & 0.07285736 \\ 0.61663432 & 0.1439601 & -0.61760778 & 0.3400054 \end{bmatrix}$$
$$\begin{bmatrix} 6.08508058 & 0 & 0 & 0 \\ 0 & 4.94768011 & 0 & 0 \\ 0 & 0 & 2.41235125 & 0 \\ 0 & 0 & 0 & 1.2933744 \end{bmatrix}$$
$$\begin{bmatrix} 0.09286258 & 0.32536742 & 0.82482759 & 0.45295919 \\ 0.02351074 & 0.21679565 & 0.40055361 & -0.8899459 \\ -0.04834199 & 0.92035343 & -0.38489419 & 0.04968988 \\ -0.99422675 & -0.00923354 & 0.10522698 & 0.01884637 \end{bmatrix}$$

A3. U, S, V =

$$\begin{bmatrix} -0.31557817 & 0.89972292 & -0.30151134 \\ -0.20373088 & -0.37458231 & -0.90453403 \\ -0.92677082 & -0.22402403 & 0.30151134 \end{bmatrix}$$
$$\begin{bmatrix} 1.68621374e+01 & 0 & 0 & \\ 0 & 0 & 3.95832300e+00 & 0 \\ 0 & 0 & 0 & 6.02104524e-16 \end{bmatrix}$$
$$\begin{bmatrix} -0.61902812 & -0.50128785 & -0.6045781 \\ 0.7598013 & -0.18743064 & -0.6225526 \\ 0.1987616 & -0.84473679 & 0.49690399 \end{bmatrix}$$

For LDU:

A1.

$$A|L$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} | \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -4 \\ 0 & -1 & 1 \end{bmatrix} | \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -4 \\ 0 & 0 & \frac{7}{3} \end{bmatrix} | \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & \frac{1}{3} & 0 \end{bmatrix}$$

Then we can D from A's diagonal: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & \frac{7}{3} \end{bmatrix}$

and U by normalizing A: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{4}{3} \\ 0 & 0 & 1 \end{bmatrix}$

PA = LDU

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & \frac{7}{3} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{4}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

Verification: PA = $\begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ which is equal to LDU = $\begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

A2. Similarly we can get the LDU decompose of $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 3 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 4 & 1 \end{bmatrix}$

PA = LDU

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 3 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{1}{3} & \frac{1}{2} & -\frac{3}{10} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A3. Similarly we can get the LDU decompose of $\begin{bmatrix} 6 & 2 & 1 \\ 1 & 2 & 3 \\ 9 & 8 & 10 \end{bmatrix}$

PA = LDU

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 2 & 1 \\ 1 & 2 & 3 \\ 9 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.167 & 1 & 0 \\ 1.5 & 3 & 1 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 1.67 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0.333 & 0.167 \\ 0 & 1 & 1.7 \\ 0 & 0 & 0 \end{bmatrix}$$

3.

(a) There is exact on solution:

$$x = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

since A is square and invertible matrix. The solution can be verified easily by

$$\begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$$

(b) There has more than one exact solutions.

The SVD solution is $\begin{bmatrix} 1.32592593 \\ -0.38518519 \\ -1.18518519 \end{bmatrix}$, and is in the row space of A, and the solution closest to the origin

The all solutions can be represented by

$$\begin{bmatrix} 1.32592593 \\ -0.38518519 \\ -1.18518519 \end{bmatrix} + a \begin{bmatrix} \frac{2}{5} \\ -\frac{17}{10} \\ 1 \end{bmatrix}$$

, where $\begin{bmatrix} \frac{2}{5} \\ -\frac{17}{10} \\ 1 \end{bmatrix} \in$ nullspace of A, and $a \in \mathbf{R}$.

This can be verified by let $a = 1$, and

$$A(\begin{bmatrix} 1.32592593 \\ -0.38518519 \\ -1.18518519 \end{bmatrix} + \begin{bmatrix} \frac{2}{5} \\ -\frac{17}{10} \\ 1 \end{bmatrix}) = b$$

(c) There has zero exact solutions.

The SVD solution is $\begin{bmatrix} 1.32592593 \\ -0.38518519 \\ -1.18518519 \end{bmatrix}$

which was calculated by running *python q3.py*,

and be verified by $Ax \neq b$.

Here b is not in the column space of A, and the SVD solution x represents a "least squares solution", i.e. it minimizes $||Ax - b||$.

4.

(a) Let x be a vector, and the action of matrix A on vector x will be:

$$
\begin{aligned}
Ax &= (I - uu^T)x \\
Ax &= Ix - uu^Tx \\
Ax &= x - uu^Tx
\end{aligned}
$$

If x is in the same direction of u, then $Ax = x - uu^Tx = 0$, which means x will be collapsed to 0;
If x is perpendicular to u, then $Ax = x - uu^Tx = x$, which means x will not be affected.

In sum, the effect will be that x will disappear on the direction of u, while not changed on the other directions.

(b) From (a), we know that the direction of u will disappear, while other directions unchanged, therefore, there are 1 zero and $(n-1)$ numbers of ones in the eigenvalues of A.

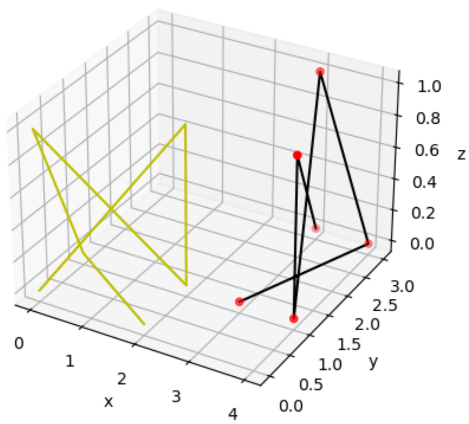
(c) The null space of A has less or equal to 1 dimension, since there are 1 zero and $(n-1)$ numbers of ones in the eigenvalues of A, and thus rank(A) $= n - 1$.
In addition, since $Au = 0$, $u \in$ nullspace(A).
Therefore, the nullspace of A is a 1 dimensional space spanned by vector u.

(d)

$$
\begin{aligned}
A^2 &= (I - uu^T)(I - uu^T) \\
&= I - 2uu^T + uu^Tuu^T \\
&= I - 2uu^T + uu^T \\
&= I - uu^T \\
&= A
\end{aligned}
$$

5.

Instruction: run *Python q5.py*
I have tested with examples from $n = 3$ to $n = 6$. Here is a test result for $n = 6$:
(The yellow lines connect $p1, p2, .., pn$, and black lines connect $q1, q2, .., qn$. The red dots represent the inferred $q'1, q'2, .., q'n$.



Prove "$trans = CentQ - rotat@CentP$" and "$rotat = (VH)U^T$" in the code (where CentQ and CentP are the centroids of Q and P. U, VH comes from the SVD decompose of PQ).
Let T be the translation, and R be the rotation. To find the best transformation:
$\Rightarrow$ minimize $func = (RP + T - Q)^2$
$\Rightarrow$ find the solution to $\frac{\partial func}{\partial T} = 0$ and $\frac{\partial func}{\partial R} = 0$.
1. We can easily find that $T = CentQ - RCentP$ from $\frac{\partial func}{\partial T} = 0$.
2. Substitute optimal T to $\frac{\partial func}{\partial R} = 0$:
we can replace minimize $func$ as maximize $(Q_1)^T RP_1$, where $Q_1 = Q - CentQ, P_1 = P - CentP$.
From the fact that $x^T R^T y = Tr(Rxy^T)$, we can further derive $func = (Q_1)^T RP_1 = Tr(RP_1(Q_1)^T)$.
Substitute $P_1(Q_1)^T$ with SVD decompose (U, S, and (VH)), we can get the function we need to maximize as $Tr(SVH^T RU)$.
Since $(VH)^T, R, U$ are orthogonal, therefore we can get maximum when $I = (VH)^T RU$, which gives us the optimal solution R as $(VH)U^T$.