Stat 134 Lec 31 (MT2 review)

Warmup

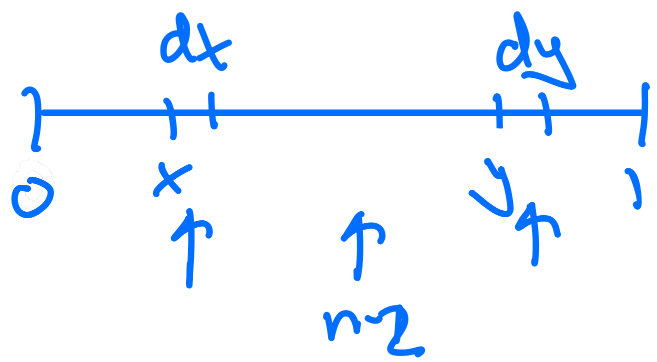
Let $U_1, \dots, U_n \overset{iid}{\sim} U(0,1)$

Let $X = U_{(1)}$ and $Y = U_{(n)}$

a) Find the joint density $f_{X,Y}(x,y)$

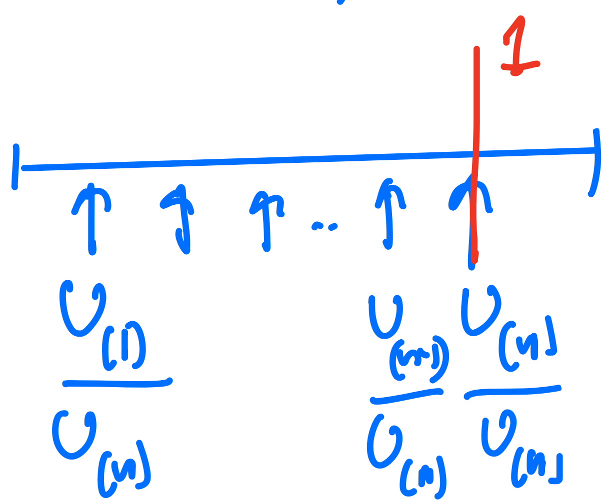
b) Identify the distribution $\frac{X}{Y}$

a)



$$f_{X,Y}(x,y) = \binom{n}{1,\, n-2,\, 1} (y-x)^{n-2}$$

b)



$$\frac{X}{Y} \sim \text{Beta}(1, n-1)$$

Announcement Please post your questions for Monday's review on bCourses/discussion board by Sunday night.

Let U_1 and U_2 be independent Uniform(0,1) distributions.

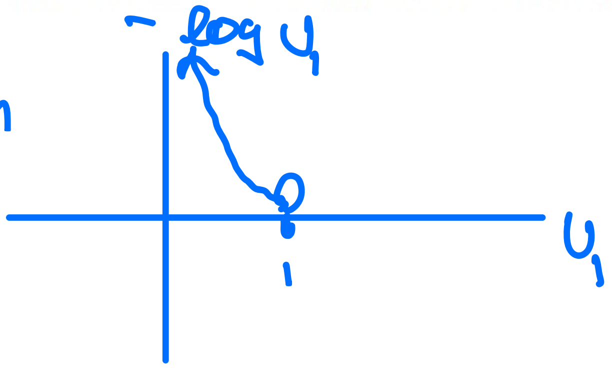
(a) Find the distribution of $Z = -log(U_1)$ **(2pts)**

(b) Find the distribution of $X = -log(U_1U_2)$ (Hint: log(ab) = log(a) + log(b)) **(3pts)**

(c) Let the joint distribution for variables X (from part B) and Y be $f(x,y) = \frac{1}{2}e^{-x}$ for $0 < |y| < x < \infty$. Find the marginal density of Y. **(3pts)**

(d) Are X and Y independent? Justify your answer. **(1pts)**

a) $z = -\log u_1$



so $0 < z < \infty$

$u = e^{-z}$

$\frac{du}{dz} = -e^{-z}$

$\left|\frac{du}{dz}\right| = e^{-z}$

$\Rightarrow f_Z(z) = \left|\frac{du}{dz}\right| f_U(u)$

$= e^{-z}$ for $0 < z < 1$

$\Rightarrow \boxed{Z \sim Exp(1)}$

b) $X = -\log u_1 \cdot u_2 = (-\log u_1) + (-\log u_2)$

$\Rightarrow$ X is sum of two inden Exp(1)

$\Rightarrow \boxed{X \sim Gamma(r=2, \lambda=1)}$

c) $f_{X,Y}(x,y) = \frac{1}{2}e^{-x}, \quad 0<|y|<x<1$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_{x=|y|}^{x=\infty} \frac{1}{2}e^{-x}\,dx = -\frac{1}{2}e^{-x}\Big|_{x=|y|}^{x=\infty}$$

$$= \boxed{\frac{1}{2}e^{-|y|}}$$

d) $f_{X,Y}(x,y) \overset{?}{=} f_X(x)\,f_Y(y)$

$$\frac{1}{2}e^{-x} \neq \left(x e^{-x}\right)\left(\frac{1}{2}e^{-|y|}\right)$$

$\boxed{\text{not indep}}$

Review MGF $M_X(t) = E(e^{tX})$

Main properties

① $M_X(0) = 1$

② $M_{aX}(t) = M_X(at)$

③ $M_X'(0) = E(X)$

$M_X''(0) = E(X^2)$

$\vdots$

$M_X^{(k)}(0) = E(X^k)$

④ If $X_1, \dots, X_n$ are independent then

$M_{X_1 + \dots + X_n}(t) = M_{X_1}(t) \cdots M_{X_n}(t)$

⑤ $M_X(t)$ is unique for t in a neighborhood of 0. So if $M_X(t) = e^{t^2/2}$, for t around 0, then $X \sim N(0,1)$.

Taylor series around 0:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

ex

CLT

Let $X_1, \dots, X_n \overset{iid}{\sim} F$, mean μ, SD σ ← any distribution

$$S_n = \sum_{i=1}^{n} X_i$$

$$S_n \longrightarrow N(n\mu, n\sigma^2) \quad \text{as } n \to \infty$$

Pf/ We show that

$$\frac{S_n - n\mu}{\sqrt{n}\,\sigma} \longrightarrow Z \sim N(0,1) \quad \text{as } n \to \infty$$

Let $Y_i = \dfrac{X_i - \mu}{\sigma}$

$$\sum_{i=1}^{n} Y_i = \frac{\overbrace{\sum_{i=1}^{n} X_i}^{S_n} - n\mu}{\sigma}$$

So $$\sum_{i=1}^{n} \frac{Y_i}{\sqrt{n}} = \frac{S_n - n\mu}{\sqrt{n}\,\sigma}$$

We will show that for n large,

$\sum_{i=1}^{n} \frac{Y_i}{\sqrt{n}}$ and Z have the same MGF.

Note that

$$E(Y_i) = E\left(\frac{X_i - \mu}{\sigma}\right) = \frac{1}{\sigma} E(X_i - \mu) = 0$$

$$Var(Y_i) = \frac{1}{\sigma^2} Var(X_i - \mu) = \frac{1}{\sigma^2} \cdot \sigma^2 = 1$$

So $E(Y_i^2) = Var(Y_i) + E(Y_i)^2 = 1$

Make a Taylor series of $M_{\frac{Y_i}{\sqrt{n}}}(t)$ around 0:

$$M_{\frac{Y_i}{\sqrt{n}}}(t) = M_{Y_i}\left(\frac{t}{\sqrt{n}}\right)$$

Property ②.

$$\frac{d}{dt} M_{\frac{Y_i}{\sqrt{n}}}(t) = \frac{d}{dt} M_{Y_i}\left(\frac{t}{\sqrt{n}}\right) = M'_{Y_i}\left(\frac{t}{\sqrt{n}}\right) \cdot \frac{1}{\sqrt{n}}$$

$$\frac{d^2}{dt^2} M_{\frac{Y_i}{\sqrt{n}}}(t) = M''_{Y_i}\left(\frac{t}{\sqrt{n}}\right) \cdot \frac{1}{n}$$

$$\frac{d^3}{dt^3} M_{\frac{Y_i}{\sqrt{n}}}(t) = M'''_{Y_i}\left(\frac{t}{\sqrt{n}}\right) \cdot \frac{1}{n^{3/2}}$$

$\vdots$

$$M_{\frac{Y_i}{\sqrt{n}}}(t) = M_{Y_i}(0) + M'_{Y_i}(0)\frac{t}{\sqrt{n}} + \frac{M''_{Y_i}(0)\frac{t^2}{n}}{2!} + \frac{M'''(0)\frac{t^3}{n^{3/2}}}{3!} + \cdots$$

$$= \underline{1} + \underbrace{E(Y_i)}_{=0}\frac{t}{\sqrt{n}} + \frac{\overbrace{E(Y_i^2)}^{=1}}{2!}\frac{t^2}{n} + \frac{E(Y_i^3)}{3!\, n^{3/2}} + \cdots$$

$$= 1 + 0 + \frac{1}{n}\left[\frac{t^2}{2} + \frac{t^3 E(Y_i^3)}{3!\, n^{1/2}} + \cdots\right]$$

Note $\left[\frac{t^2}{2} + \frac{t^3 E(Y_i^3)}{3!\, n^{1/2}} + \cdots\right] \approx \frac{t^2}{2}$ for large n

so

$$M_{\frac{Y_1}{\sqrt{n}}}(t) \approx 1 + \frac{1}{n}\frac{t^2}{2} \quad \text{for large } n$$

$\frac{Y_i}{\sqrt{n}}$ are independent,

$$M_{\frac{S_n - n\mu}{\sqrt{n}\sigma}}(t) = M_{\frac{Y_1}{\sqrt{n}}}(t) \cdots M_{\frac{Y_n}{\sqrt{n}}}(t)$$

$\left(1 + \frac{x}{n}\right)^n \approx e^x$ for large n

$= \sum_{i=1}^{n} \frac{Y_i}{\sqrt{n}}$

$$\to \left[1 + \frac{1}{n}\frac{t^2}{2}\right]^n \approx e^{\frac{t^2}{2}}$$

which is MGF of $N(0,1)$

Hence $\frac{S_n - n\mu}{\sqrt{n}\sigma} \to N(0,1)$

$\square$

(10 pts) [Convolution, Continuous distributions] Let X, Y be iid random variables each with density $f_X(x) = \frac{1}{x^2}$ for $x > 1$ and zero otherwise.

(a) (5 pts) Find the density of $Z = Y/X$.

(b) (5 pts) Find $\mathbb{E}[\sqrt{Z}]$.

(a) Using our convolution formula,

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} |x| f_X(x) f_Y(zx)\, dx \\ &= \int_{-\infty}^{\infty} x \cdot \frac{1}{x^2} 1_{\{x>1\}} \cdot \frac{1}{(zx)^2} 1_{\{zx>1\}}\, dx \end{aligned}$$

Case 1: When $z \geq 1$, the limits are $[1, \infty)$.

$$\begin{aligned} &= \frac{1}{z^2} \int_1^{\infty} \frac{1}{x^3}\, dx \\ &= \frac{1}{z^2} \left[-\frac{1}{2x^2} \right]_1^{\infty} \\ &= \frac{1}{2z^2}. \end{aligned}$$

Case 2: When $0 < z < 1$, the limits are $[1/z, \infty)$.

$$\begin{aligned} &= \frac{1}{z^2} \int_{1/z}^{\infty} \frac{1}{x^3}\, dx \\ &= \frac{1}{z^2} \left[-\frac{1}{2x^2} \right]_{1/z}^{\infty} \\ &= \frac{1}{2}. \end{aligned}$$

In total, we get

$$f_Z(z) = \begin{cases} \frac{1}{2} & 0 < z < 1 \\ \frac{1}{2z^2} & z \geq 1. \end{cases}$$

(b) The expectation of $\sqrt{Z}$ is

$$
\begin{aligned}
\mathbb{E}[\sqrt{Z}] &= \int_{-\infty}^{\infty} \sqrt{z} \cdot f_Z(z)\, dz \\
&= \int_0^1 \sqrt{z} \cdot \frac{1}{2}\, dz + \int_1^{\infty} \sqrt{z} \cdot \frac{1}{2z^2}\, dz \\
&= \left[\frac{z^{3/2}}{3}\right]_0^1 + \left[-\frac{1}{\sqrt{z}}\right]_1^{\infty} \\
&= \frac{1}{3} + 1 \\
&= \frac{4}{3}.
\end{aligned}
$$

extra

1. Consider independent $U, V \sim \text{Exp}(3)$. Let $X = \min\{U, V\}, Y = \max\{U, V\}$.

   (a) Name the distribution of X and give its parameter; (4 points)

   (b) Are X and Y independent? Explain briefly; (3 points)

   (c) Find the distribution of $Y - X$. (3 points)

(a) Name the distribution of X and give its parameter; (4 points)

Answer. By competing Exponentials, $X \sim \text{Exp}(6)$.

(b) Are X and Y independent? Explain briefly; (3 points)

Answer. No. Because (X, Y) has range $0 < x < y$.

(c) Find the distribution of $Y - X$. (3 points)

Answer. We can find $f(X, Y) = 18e^{-3(X+Y)}$ by our usual method of finding the joint of ordered statistics of two exponentials. Using the general convolution formula given in lecture 34 we see $f_Z(z) = \int_0^\infty 18e^{-3(2X+Z)}dx = 3e^{-3z}$, indicating $Z = Y - X$ is Exp(3).