Stat 134 lec 35

Stat 134

Friday October 21 2022

1. Let N have a $Poisson(\mu)$ distribution. Suppose that given $N = n$, random variable X follows a $Binomial(n, p)$ distribution. $E(X)$ is:

   a np

   b μ

   c $n\mu$

   d none of the above

$N \sim Pois(\mu)$

$X|N \sim Bin(N, p)$

$E(X|N) = Np$

$$E(X) = E(E(X|N)) = E(Np) = pE(N) = p\mu$$

$$= \boxed{p\mu}$$

By warmup last time,

$X \sim Pois(p\mu)$

$E(X) = p\mu$

Last time

Sec 6.1 Conditional probability (discrete case)

Rule of average conditional prob

$$P(Y=y) = \sum_{x \in X} P(Y=y \mid X=x) P(X=x)$$

ex

if Y = 2nd card of deck

X = 1st card of deck

we have

$$P(Y=3\heartsuit) = \sum_{x \in X} P(Y=3\heartsuit \mid X=x) P(X=x)$$

$$= \sum_{\substack{x \in X \\ x \neq 3\heartsuit}} P(Y=3\heartsuit \mid X=x) P(X=x)$$

$$= 51 \cdot \left(\frac{1}{51} \cdot \frac{1}{52}\right) = \boxed{\frac{1}{52}}$$

Sec 6.2 Conditional expectation (discrete case)

let T, S be discrete RVs.

$$E(T|S=s) = \sum_{t \in T} t \cdot P(T=t|S=s)$$

Two main points:

1. $E(T|S)$ is a function of S.
2. $E(T|S)$ is a RV so it has an expectation.

Properties

1. $E(Y) = E(E(Y|X))$ iterated expectation.
2. $E(aY+b|X) = aE(Y|X)+b$
3. $E(Y+Z|X) = E(Y|X)+E(Z|X)$
4. $E(g(X)|X) = g(X)$
5. $E(g(X)Y|X) = g(X)E(Y|X)$
6. $Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$ total variance decomposition (sec 6.2.18)

For example lets prove Property 4

$E(g(X)|X) = g(X)$.

For clarity suppose we are given $X=5$.

$$E(g(X)|X=5) = \sum_{i \in X} g(i)P(X=i|X=5)$$

Note that $P(X=i \mid X=5) = \begin{cases} 1 & \text{if } i=5 \\ 0 & \text{else} \end{cases}$

so

$$E(g(X) \mid X=5) = g(5)\underbrace{P(X=5 \mid X=5)}_{=1} = g(5)$$

More generally

$$E(g(X) \mid X=x) = g(x) \quad \text{for all } x \in \mathcal{X}$$

which we write as $E(g(X) \mid X) = g(X)$.

<u>Today</u> sec 6.3 conditional density

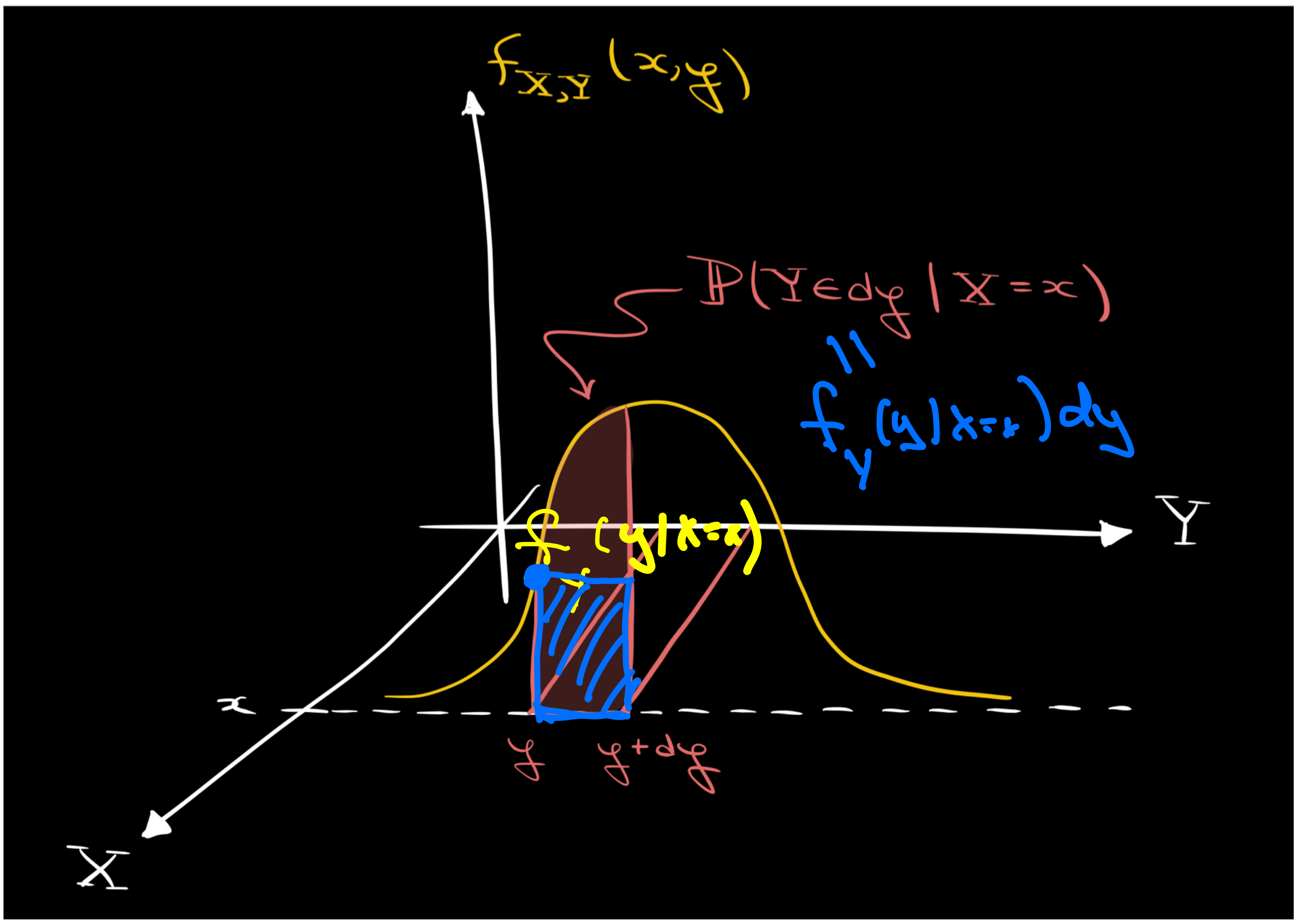
1

sec 6.3 Conditional Density:

Let X, Y both be continuous RVs.

Let $f_Y(y|X=x)$ be a slice of $\dfrac{f_{X,Y}(x,y)}{f_X(x)}$ through $X=x$. ← constant

Define $P(Y \in dy \mid X=x)$ as the area under $f_Y(y|X=x)$ for $Y \in dy$



By Baye's rule,

$$P(y \in dY \mid X = x) = \frac{P(y \in dY, X \in dX)}{P(x \in dX)}, \text{ for small } dX$$

$$\underset{=\, f_Y(y \mid X=x)\,dy}{P(y \in dY \mid X = x)} \qquad P(y \in dY, X \in dX) = f_{X,Y}(x,y)\,dx\,dy \qquad P(x \in dX) \approx f_X(x)\,dx$$

$$\boxed{f_Y(y \mid X=x) = \frac{f_{X,Y}(x,y)}{f_X(x)}}$$

Conditional density of Y given X=x

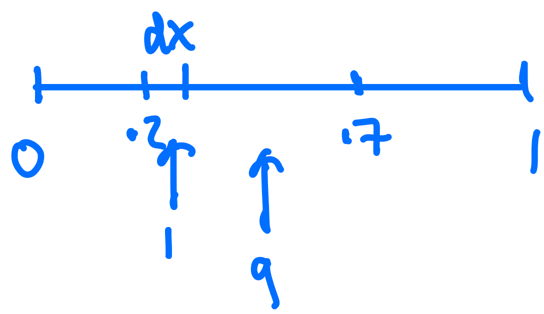
The multiplication rule is

$$\boxed{f_{X,Y}(x,y) = f_Y(y \mid X=x)\, f_X(x)}$$

$\underline{\underline{ex}}$ Let $U_1, \dots, U_{10} \overset{iid}{\sim} U(0,1)$

$X = U_{(1)}$, $Y = U_{(10)}$

Find $P(Y > .7 \mid X = .2) = 1 - P(Y \le .7 \mid X = .2)$



$$= 1 - \frac{P(Y \le .7,\ X \in dx)}{P(X \in dx)}$$

$$= 1 - \frac{\cancel{\binom{10}{1,9}}\, 1\, \cancel{dx}\, (.5)^9}{\cancel{\binom{10}{1,9}}\, 1\, \cancel{dx}\, (.8)^9}$$

$$= \boxed{1 - \left(\frac{.5}{.8}\right)^9}$$

Alternatively,

$f(x,y) = 90\,(y-x)^8$ and $f_X(x) = \binom{10}{1,9}(1-x)^9$

$= 10\,(1-x)^9$

$$\boxed{f_Y(y \mid X = x) = \frac{f_{X,Y}(x,y)}{f_X(x)}}$$

a) Find $f_Y(y \mid X = .2) = \dfrac{f_{X,Y}(.2, y)}{f_X(.2)}$

$$= \frac{90\,(y - .2)^8}{10\,(.8)^9}$$

b) Find $\int_{.7}^{1} f_Y(y|X=.2)\,dy = \frac{9}{(.8)^9}\int_{y=.7}^{y=1}(y-.2)^8\,dy$

let $u = y-.2$

$du = dy$

$= \frac{9}{(.8)^9}\int_{u=.5}^{u=.8} u^8\,du = \frac{9}{(.8)^9}\frac{u^9}{9}\Big|_{.5}^{.8}$

$= \frac{\cancel{9}}{\cancel{(.8)^9}}\frac{\cancel{(.8)^9}}{\cancel{9}} - \frac{\cancel{9}}{(.8)^9}\frac{(.5)^9}{\cancel{9}}$

$= \boxed{1-\left(\frac{.5}{.8}\right)^9}$

ex

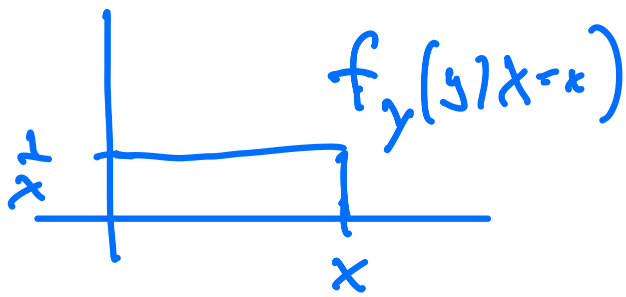
let $X \sim \text{Gamma}(2,\lambda)$

$Y|X=x \sim \text{Unif}(0,x)$

$X \sim \text{Gamma}(r,\lambda)$

$f_X(x) = \frac{\lambda^r}{\Gamma(r)}x^{r-1}e^{-\lambda x}, \quad x>0$

a) Find $f_Y(y|X=x) = \frac{1}{x}$



b) Find $f_{X,Y}(x,y)$

$= f_Y(y|X=x)\,f_X(x)$

$= \frac{1}{x}\cdot\frac{\lambda^2}{\Gamma(2)}x^1 e^{-\lambda x} = \boxed{\lambda^2 e^{-\lambda x} \text{ for } x>0}$

P425 Pitman

Conditioning Formulae: Density Case

Multiplication rule: The joint density is the product of the marginal and the conditional

$$f(x, y) = f_X(x) f_Y(y \,|\, X = x)$$

Division rule: The conditional density of Y at y given $X = x$ is

$$f_Y(y \,|\, X = x) = \frac{f(x, y)}{f_X(x)}$$

Bayes' rule:

$$f_X(x|Y = y) = \frac{f_Y(y \,|\, X = x) f_X(x)}{f_Y(y)}$$

Conditional distribution of Y given $X = x$: Integrate the conditional density

$$P(Y \in B \,|\, X = x) = \int_B f_Y(y \,|\, X = x) dy$$

Conditional expectation of $g(Y)$ given $X = x$: Integrate g against the conditional density:

$$E\left(g(Y) \,|\, X = x\right) = \int g(y) f_Y(y \,|\, X = x) dy$$

Average conditional probability:

$$P(B) = \int P(B \,|\, X = x) f_X(x)\, dx$$

$$f_Y(y) = \int f_Y(y \,|\, X = x) f_X(x)\, dx$$

Average conditional expectation:

'rule of iterated expectation $E(Y) = E(E(Y|X))$

$$E(Y) = \int E(Y \,|\, X = x) f_X(x)\, dx$$

ex

Let I be discrete and X continuous
the rule of average conditional probabilities says

$$P(\underbrace{I=i}_{\text{event } B}) = \int_{x \in X} P(\underbrace{I=i}_{B} \mid X=x) \cdot f_X(x)\,dx$$

For example,

let $X \sim \text{Unif}(0,1)$ and $I_1 \mid X=x,\ I_2 \mid X=x \overset{iid}{\sim} \text{Ber}(x)$

a) Find $P(I_1=1)$ $= \int_{x=0}^{x=1} \underbrace{P(I_1=1 \mid X=x)}_{=x} \underbrace{f_X(x)}_{=1}dx = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \boxed{1/2}$

b) Find $P(I_2=1 \mid I_1=1)$ $= \dfrac{P(I_2=1, I_1=1)}{P(I_1=1)}$

$$P(I_2=1, I_1=1) = \int_{x=0}^{x=1} \underbrace{P(I_2=1, I_1=1 \mid X=x)}_{= P(I_2=1 \mid X=x)\,P(I_1=1 \mid X=x)} f_X(x)\,dx$$

$$P(I_2=1 \mid X=x) = x, \quad P(I_1=1 \mid X=x) = x$$

$$= \int_0^1 x^2\,dx = \boxed{1/3}$$

$$P(I_2=1 \mid I_1=1) = \frac{1/3}{1/2} = \boxed{2/3}$$

notice

$P(I_2=1 \mid I_1=1) \neq P(I_2=1)$

$\Rightarrow I_1$ and I_2 are <u>dependent</u>!