Stat 134 lec 19

warmup

Suppose customers are arriving at a ticket booth at rate of five per minute, according to a Poisson arrival process. Find the probability that:

At least one customer arrives within 40 seconds after the arrival of the 13th customer

$$P(\underbrace{T_{14}-T_{13}}_{W \sim Exp(\lambda=5)} < 2/3) = 1 - P(\underbrace{T_{14}-T_{13}}_{W} \geq 2/3) = \boxed{1 - e^{-5 \cdot 2/3}}$$

alternatively

$= 1 - P(\text{no arrivals in time } t)$

$= 1 - e^{-5 \cdot 2/3}$ by Poisson formula.

Announcement:

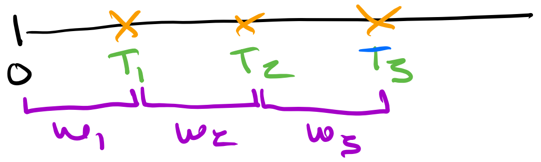
Wednesday (lec 21) is a special lecture on moment generating functions (not in textbook).

Last time

Sec 4.2 Exponential Distribution

$f(t) = \lambda e^{-\lambda t}$ where $\lambda = 1/E(T)$

Sec 4.2 Gamma Distribution



$T_1 \sim Exp(\lambda)$, $\lambda > 0$ $\qquad f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{else} \end{cases}$ ← Variable part

$T_r \sim Gamma(r, \lambda)$, $\lambda > 0$, $r > 0$ $\qquad f(t) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t} & t \geq 0 \\ 0 & \text{else} \end{cases}$ where $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} \, dt$

$r \in \{1, 2, 3, \ldots\}$ then $\Gamma(r) = (r-1)!$

$T_r = W_1 + W_2 + \cdots + W_r$, $W_i \overset{iid}{\sim} Exp(\lambda)$

$E(W_1) = \frac{1}{\lambda} \Rightarrow E(T_r) = \frac{r}{\lambda}$

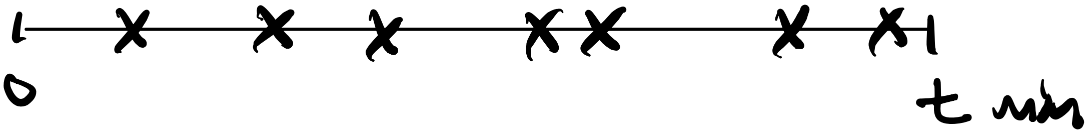
$Var(W_1) = \frac{1}{\lambda^2} \Rightarrow Var(T_r) = \frac{r}{\lambda^2}$

ex A random variable X has non negative values and density $cx^4 e^{-3x}$ for $0 \leq x < \infty$, and some constant c.
What distribution is X? $X \sim Gamma(5, 3)$ $\quad c = \frac{3^5}{4!}$
Find $Var(X) = \frac{5}{9}$

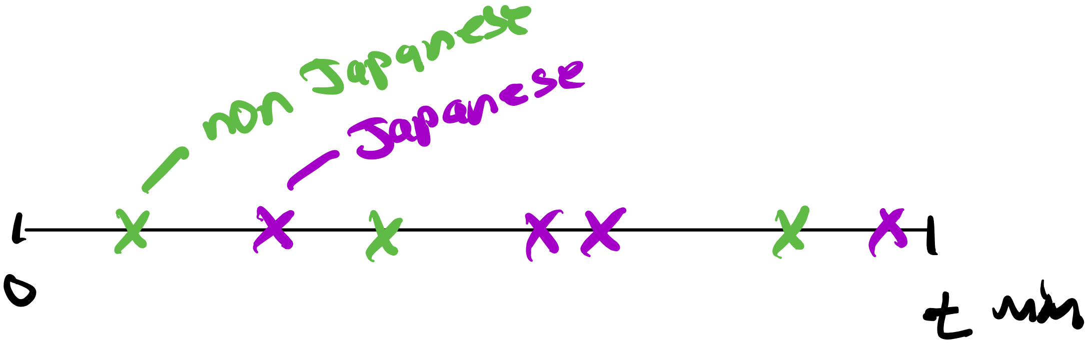
Sec 3.5 Poisson thinning

Cars arrive at a toll booth according to a Poisson process at a rate λ arrivals/min

X = # cars arriving at a toll booth in t min. $X \sim Pois(\lambda t)$



The chance of a car arriving being a Japanese import is p.



cars $\sim Pois(\lambda t)$

Japanese imports $\sim Pois(p \lambda t)$

non Japanese $\sim Pois(q \lambda t)$

indep

Japanese cars is a thinned Poisson process $Pois(p \lambda t)$

Today

1. more practice
2. Sec 4.2 Competing Exponentials
3. Sec 4.2 Memoryless property

① More practice

A family is getting ready for their trip to Yosemite. Each person is in their room, packing their bags. For each person, the time it takes them to pack their bag is exponentially distributed and independent of the time it takes any other person. On average, it takes each parent 1 hour and each child 2 hours to get ready. In a family with 2 parents and 4 children, what is the probability that it takes the family more than 2 hours to get ready?

let $P_1, P_2, C_3, C_4, C_5, C_6$ be the packing time of the parents and children.

$P_1, P_2 \overset{iid}{\sim} \text{Exp}(1)$

$C_1, C_2, C_3, C_4 \overset{iid}{\sim} \text{Exp}(1/2)$ since $\lambda = \frac{1}{E(C_1)}$

$M = \max(P_1, P_2, C_3, C_4, C_5, C_6)$

$P(M>2)$ is the probability at least one person isn't packed in 2 hrs.

$$P(M>2) = 1 - P(M \le 2)$$

probability everyone is packed in 2 hrs.

$$= 1 - P(P_1 \le 2)P(P_2 \le 2)P(C_3 \le 2) \cdots P(C_6 \le 2)$$

$$\underset{1-e^{-1\cdot 2}}{\parallel} \quad \underset{1-e^{-1\cdot 2}}{\parallel} \quad \underset{1-e^{-\frac{1}{2}\cdot 2}}{\parallel} \quad \underset{1-e^{-\frac{1}{2}\cdot 2}}{\parallel}$$

$$= \boxed{1 - (1-e^{-2})^2(1-e^{-1})^4}$$

② Competing Exponentials

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Stat 134

1. GSI Brian and Yiming are each helping a student. Brian and Yiming see students at a rate of λ_B and λ_Y students per hour respectively.

   Let

   $$\left.\begin{aligned} B &= \text{wait time for Brian } \sim Exp(\lambda_B) \\ Y &= \text{wait time for Yiming } \sim Exp(\lambda_Y) \end{aligned}\right\} \text{ indep}$$

   What distribution is $T = \min(B, Y)$?
   Hint: compute $P(T > t)$

   a $Exp(\max(\lambda_B, \lambda_Y))$

   b $Exp(\lambda_B - \lambda_Y)$

   c $Exp(\lambda_B + \lambda_Y)$

   d none of the above

$$P(T>t) = \underbrace{P(B>t)}_{= e^{-\lambda_B t}}\,\underbrace{P(Y>t)}_{= e^{-\lambda_Y t}} = e^{-(\lambda_B+\lambda_Y)t}$$

$$\Rightarrow \boxed{T \sim Exp(\lambda_B+\lambda_Y)}$$

<u>so</u>

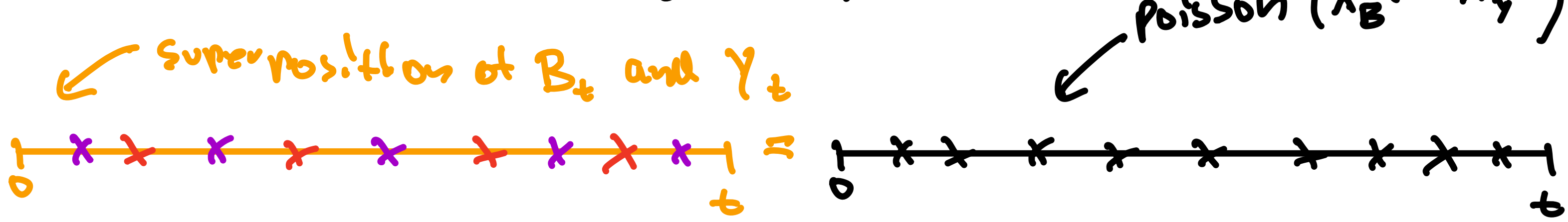
If $X_1, \dots, X_n$ are indep exponentials with rates $\lambda_1, \dots, \lambda_n$ then $\min(X_1, \dots, X_n) \sim \text{Exp}(\lambda_1 + \dots + \lambda_n)$

What is $P(B < Y)$?

superposition of Poisson random scatters:

let $B_t \sim \text{Pois}(\lambda_B t)$ and $Y_t \sim \text{Pois}(\lambda_Y t)$ be independent PRS corresponding to the number of arrivals of Brian and Yiming in time t.

Then $B_t + Y_t \sim \text{Pois}(\lambda_B t + \lambda_Y t)$



competing exponentials:

Let X = time until the first Brian arrival
Y = time until the first Yiming arrival

What is the chance, p, the first arrival is Brian?

$p = \text{Prob}(B < Y)$

$= \text{Prob}(\text{an arrival is Brian})$

By defn, $B_t \sim \text{Pois}(\lambda_B t)$

By Poisson thinning

$B_t \sim \text{Pois}(p(\lambda_B t + \lambda_Y t))$

Prob an arrival is Brian

$$\Rightarrow \lambda_B t = p(\lambda_B t + \lambda_Y t)$$

$$\Rightarrow p = \frac{\lambda_B t}{\lambda_B t + \lambda_Y t} = \boxed{\frac{\lambda_B}{\lambda_B + \lambda_Y}}$$

$P(B < Y)$

③ The memoryless property

Recall $P(T>k) = e^{-\lambda k}$

Let $T \sim Exp(\lambda)$

a) Find $P(T>5) = e^{-5\lambda}$

b) Find $P(T>13 \mid T>8) = \frac{P(T>13, T>8)}{P(T>8)} = \frac{P(T>13)}{P(T>8)}$

$$= \frac{e^{-13\lambda}}{e^{-8\lambda}} = e^{-5\lambda}$$

The Exponential distribution has the memoryless property

Version 1

$$P(T>j+k \mid T>j) = P(T>k)$$

Version 2

small interval after time t.

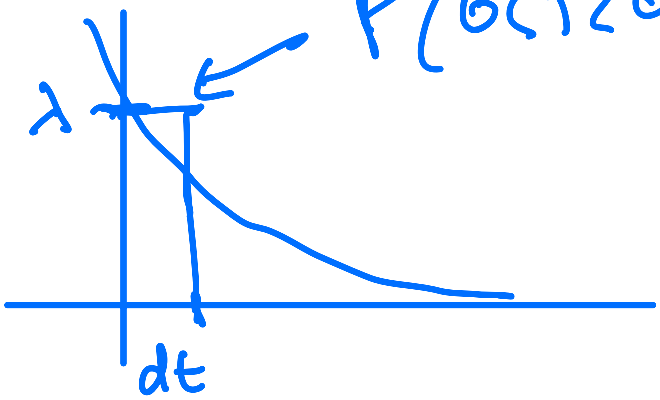
$$P(T \in dt \mid T>t) = P(0 < T < dt)$$

small interval after time 0

Proof that $T \sim Exp(\lambda)$ satisfies version 2:

RHS

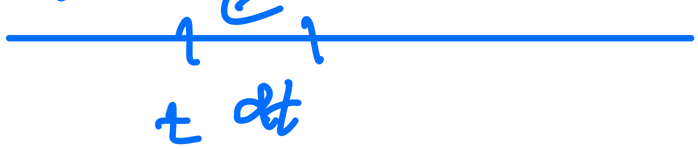
$P(0<T<dt) \approx \lambda dt$



LHS

no arrivals

1st arrival



$P(T \in dt \mid T>t)$

$= P(1 \text{ arrival in } dt \mid \text{no arrival in } t)$

$= P(1 \text{ arrival in } dt)$ since arrivals are indep.

$= e^{-\lambda dt} \lambda dt \approx \lambda dt$ for small λ.

Fact ← see appendix of lecture 12 for a proof.

Only two kinds of distributions are memoryless:

geometric distribution of nonnegative integers and the exponential distribution of nonnegative real numbers.