Warmup:

Let $X \sim Ber(p)$, $X = \begin{cases} 1 & \text{with prob } p \\ 0 & \text{with prob } 1-p \end{cases}$

a) Find $E(X) = 1 \cdot p + 0 \cdot (1-p) = p$

$E(X^2) = 1^2 p + 0^2 (1-p) = p$

$\vdots$

$E(X^k) = 1^k p + 0^k (1-p) = p$

think of e^{tX} as a function $g(X)$ of $X \sim Ber(p)$

b) Find $E(e^{tX})$, $t \in \mathbb{R}$

$E(g(X)) = g(1)p + g(0)(1-p)$

$= e^{t \cdot 1} \cdot p + \underbrace{e^{t \cdot 0}}_{=1} \cdot (1-p) = pe^t + 1 - p = \boxed{1 + p(e^t - 1)}$ for all t

$\frac{d}{dt} E(e^{tX}) \Big|_{t=0} = \frac{d}{dt}\left[1 + p(e^t - 1)\right] = pe^t \Big|_{t=0} = \boxed{p}$

$\frac{d^2}{dt^2} E(e^{tX}) \Big|_{t=0} = \frac{d}{dt} pe^t \Big|_{t=0} = \boxed{p}$

$\vdots$

$\frac{d^k}{dt^k} E(e^{tX}) \Big|_{t=0} = \boxed{p}$

To find the moments of X we take the derivatives of $E(e^{tX})$ and evaluate at $t=0$.

For X a RV, $M_X(t) = E(e^{tX})$ is called the moment generating function (MGF) of X.

Special lecture

Moment Generating Function of X

Not in book. See bCourses/pages/Daily Reading for reference

Moment Generating Function (MGF) of X

The k^{th} moment of a RV X is the number

$E(X^k)$ defined for $k = 0, 1, 2, 3, \dots$

$E(X^0) = E(1) = 1$

$E(X)$

$E(X^2)$

moments describe your distribution

ex 1st moment is mean

2nd moment relates to variance

$E(X^2) = Var(X) + [E(X)]^2$

3rd moment relates to how skewed the distribution is.

Recall $E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\,dx$

We define the MGF of X to be

$$M_X(t) = E(e^{tX}) = \begin{cases} \int_{-\infty}^{\infty} e^{tx} f(x)\,dx & \text{if } X \text{ continuous} \\ \sum_{x=-\infty}^{\infty} e^{tx} P(x) & \text{if } X \text{ discrete} \end{cases}$$

$M_X(t)$ is sometimes written $\psi_X(t)$ in HW 8

In the warmup we saw for $X \sim Ber(p)$

k^{th} derivative evaluated at $t=0$

$$m_X^{(k)}(t)\Big|_{t=0} = E(X^k)$$

Lets show that is true for most RV X.

Recall the Taylor series for e^y:

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$$

let $t \in \mathbb{R}$, X RV.

You can do this ~~for~~ the RV tX

$$e^{tX} = 1 + tX + \frac{(tX)^2}{2!} + \cdots$$

$$\underbrace{E(e^{tX})}_{M_X(t)} = E(1) + E(tX) + E\left(\frac{(tX)^2}{2!}\right) + \cdots$$

$$= E(1) + tE(X) + \frac{t^2}{2!}E(X^2) + \cdots$$

may only exist for a range of t values.

$$\frac{d}{dt} M_X(t)\Big|_{t=0} = E(X)$$

$$\frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = E(X^2)$$

The 1st property of MGF

1. If a MGF exists in an open interval around zero, $M^{(k)}(t)\Big|_{t=0} = E(X^k)$

Note: An MGF doesn't always exist in an open interval around zero. (see appendix for an example)

ex: let $X \sim \text{Gamma}(r, \lambda)$

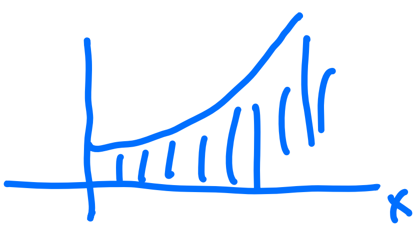
$$f(x) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x}, & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Recall $\Gamma(r) = \int_{u=0}^{u=\infty} u^{r-1} e^{-u}\, du$

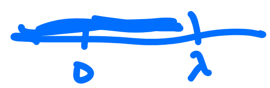
Find $M_X(t)$.

Step 1 Write $M_X(t)$ as an integral or a sum

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} \left(\frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} \right) dx$$



$$= \boxed{\frac{\lambda^r}{\Gamma(r)} \int_0^\infty x^{r-1} e^{-(\lambda - t)x}\, dx \quad \text{for } t < \lambda}$$



Step 2 Solve the integral

Hint: make a u substitution

let

$$u = \underbrace{(\lambda - t)}_{>0} x$$

so $x = \frac{u}{\lambda - t}$, $dx = \frac{1}{\lambda - t} du$

$$\frac{\lambda^r}{\Gamma(r)} \int_0^\infty x^{r-1} e^{-(\lambda-t)x}\, dx =$$

$$\frac{\lambda^r}{\Gamma(r)} \int_{u=0}^{u=\infty} \frac{u^{r-1}}{(\lambda-t)^{r-1}}\, e^{-u}\, \frac{1}{\lambda-t}\, du$$

$$\frac{\lambda^r}{\cancel{\Gamma(r)}} \frac{1}{(\lambda-t)^r} \underbrace{\int_{u=0}^{u=\infty} u^{r-1} e^{-u}\, du}_{= \cancel{\Gamma(r)}} = \boxed{\frac{\lambda^r}{(\lambda-t)^r} \text{ for } t<\lambda}$$

Recall If a MGF exists in an interval around zero, $M^{(r)}(t)\Big|_{t=0} = E(X^r)$

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Stat 134

Monday October 10 2022

1. Let $X \sim \text{Gamma}(r, \lambda)$. Using the MGF $M_X(t) = (\frac{\lambda}{\lambda - t})^r$ for $t < \lambda$ we calculate the second moment of X is:

   a $E(X^2) = \frac{r(r+1)}{\lambda}$

   b $E(X^2) = \frac{r(r-1)}{\lambda^2}$

   c $E(X^2) = \frac{r(r+1)}{\lambda^2}$

   d none of the above

$$M_X(t) = \lambda^r (\lambda - t)^{-r}$$

$$M_X'(t) = \lambda^r(-r)(\lambda - t)^{-r-1}(-1) = \lambda^r r(\lambda - t)^{-r-1}$$

$$M_X''(t) = r\lambda^r(-r-1)(\lambda - t)^{-r-2}(-1)$$

$$= r(r+1)\lambda^r \frac{1}{(\lambda - t)^{r+2}}$$

$$M_X''(0) = r(r+1)\lambda^r \frac{1}{\lambda^{r+2}} = \boxed{\frac{r(r+1)}{\lambda^2}}$$

Two more important properties of MGF:

② If X and Y are independent RVs,

$$M_{X+Y}(t) = M_X(t) M_Y(t).$$

Proved in MGF HW.8

③ If $M_X(t) = M_Y(t)$ for all t in an interval around 0 then $F_X(z) = F_Y(z)$ (i.e. X and Y have the same distribution).

ex

$$\left.\begin{array}{l} X_1 \sim \text{Pois}(\mu_1) \\ X_2 \sim \text{Pois}(\mu_2) \end{array}\right\} \text{ independent.}$$

show that $X_1 + X_2 \sim \text{Pois}(\mu_1 + \mu_2)$.

Fact The MGF of $X \sim \text{Pois}(\mu)$ is

$$M_X(t) = e^{\mu(e^t - 1)}$$

$$M_{X_1}(t) = e^{\mu_1(e^t-1)} \quad \text{for all } t$$

$$M_{X_2}(t) = e^{\mu_2(e^t-1)} \quad \text{for all } t$$

$$M_{X_1+X_2}(t) = M_{X_1}(t)M_{X_2}(t) = \boxed{e^{(\mu_1+\mu_2)(e^t-1)}}$$

by ②

MGF of Pois $(\mu_1+\mu_2)$ for all t.

$$\Rightarrow X_1+X_2 \sim \text{Pois}(\mu_1+\mu_2)$$

by ③

Appendix:

Let X be a discrete RV with probability mass function

$$p(x) = \begin{cases} \frac{6}{\pi^2 x^2} & x = 1, 2, \dots \\ 0 & \text{else} \end{cases}$$

The MGF, $M_X(t)$, only exists at $t \leq 0$, and hence doesn't exist on an interval around zero.

pf/
It is known that the series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$ converges to $\pi^2/6$.

Then $$p(x) = \begin{cases} \frac{6}{\pi^2 x^2} & x = 1, 2, 3, \dots \\ 0 & \text{else.} \end{cases}$$

is the prob function of a RV X,

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} \frac{6e^{tx}}{\pi^2 x^2}$$

The ratio test can be used to show this diverges if $t > 0$. Hence this RV only has an MGF at $t \leq 0$ and is not differentiable at zero.

$\square$