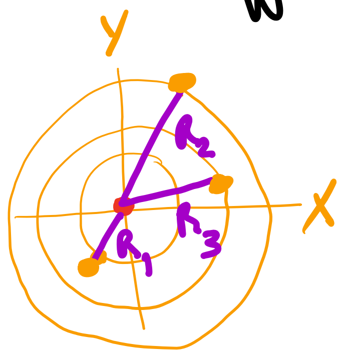
Stat 134 lec 28

warm up

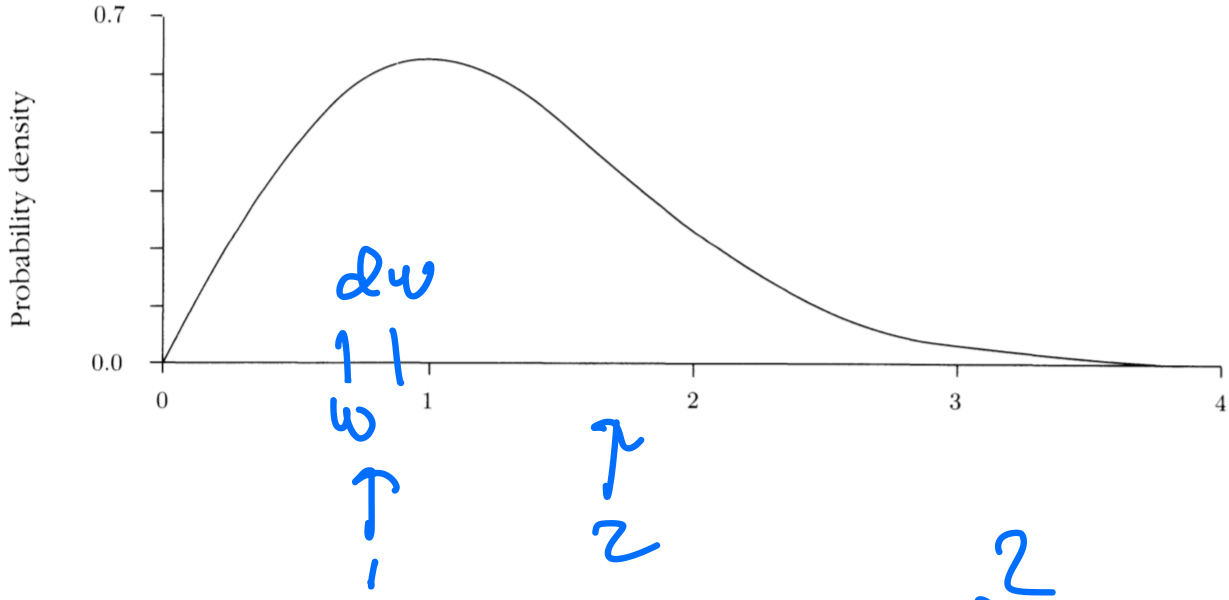
Suppose 3 shots are fired at a target. Assume for each shot, both X and Y are standard normals. Let W be the closest distance among 3 shots to the bullseye. Find $f_W(w)$

Picture



Hint: $W = \min(R_1, R_2, R_3)$ where $R_1, R_2, R_3 \overset{iid}{\sim} \text{Ray}$

FIGURE 3. Density of the Rayleigh distribution of R.



$$f_W(w) = \binom{3}{1\ 2} f(w)\,(1 - F(w))^2$$

$$= \binom{3}{1,2} w e^{-\frac{1}{2}w^2} \left(e^{-\frac{1}{2}w^2}\right)^2$$

Last time

Sec 5.3 Rayleigh distribution

$T \sim Exp(\frac{1}{2}) \Rightarrow R = \sqrt{T}$ has density $f(r) = re^{-\frac{1}{2}r^2}$, $r > 0$

and CDF $F_R(r) = 1 - e^{-\frac{1}{2}r^2}$, $r > 0$.

We saw last time that $R = \sqrt{X^2 + Y^2}$
where $X, Y \overset{iid}{\sim} N(0,1)$.

We proved that the constant of the density of standard normal is $c = \frac{1}{\sqrt{2\pi}}$ (i.e $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$ is the density of $Z \sim N(0,1)$.

Today

1. Sec 5.3 Sum of independent normals
2. Sec 5.3 Chi square distribution

① Sec 5.3 Sum of independent normals

Definition of the Normal (μ, σ^2) distribution:

We know the density of the std normal $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$ Make a change of scale $X = \mu + \sigma Z$. By definition $X \sim N(\mu, \sigma^2)$.

By the change of variable rule you can show

$$f_X(x) = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

MGF Review (lecture 24)

Recall the MGF of a RV X is $M(t) = E(e^{Xt})$.

MGF of std normal

$Z \sim N(0,1)$, $M_Z(t) = e^{\frac{t^2}{2}}$ for all t

Properties of MGF

If X, Y are independent then $M_{X+Y}(t) = M_X(t) M_Y(t)$

for t in an interval containing zero

If $M_X(t) = M_Y(t)$ then $X \overset{d}{=} Y$ ← same distribution

$$M_{\mu+\sigma Z}(t) = e^{\mu t} M_Z(\sigma t)$$

ex Use the properties of MGF to find the MGF of $X \sim N(\mu, \sigma^2)$

Hint let $Z \sim N(0,1)$, $\mu \in \mathbb{R}$, $\sigma > 0$

and $X = \mu + \sigma Z$

$$M_X(t) = M_{\mu+\sigma Z}(t) = e^{\mu t} M_Z(\sigma t)$$

$$= e^{\mu t} e^{\frac{1}{2}(\sigma t)^2}$$

$$= e^{\mu t} e^{\frac{\sigma^2 t^2}{2}} \text{ for all } t \in \mathbb{R}$$

σ^2

if $X \sim N(0, 2)$

$$M_X(t) = e^{t^2}$$

ex Use MGF to prove that the sum of two independent std normals is $N(0,2)$.

hint

$$X \sim N(\mu, \sigma^2) \Rightarrow M_X(t) = e^{\mu t} e^{\frac{\sigma^2 t^2}{2}}$$

$$M_{Z_1+Z_2}(t) = M_{Z_1}(t)\, M_{Z_2}(t)$$

$$= e^{t^2/2} \cdot e^{t^2/2}$$

$$= e^{t^2} \text{ for all } t$$

MGF of $N(0,2)$

Hint $X \sim N(0, 2)$ σ^2

$$M_X(t) = e^{t^2}$$

Thm Let $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ indep.

then $aX_1 + bX_2 \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$

Pf/ $M_{X_1}(t) = e^{\mu_1 t} e^{\sigma_1^2 \frac{t^2}{2}}$

$M_{aX_1}(t) = M_{X_1}(at) = e^{\mu_1 a t} e^{\sigma_1^2 a^2 \frac{t^2}{2}}$

Then $M_{aX_1 + bX_2}(t) = M_{aX_1}(t) M_{bX_2}(t)$

$= e^{\mu_1 a t} e^{\sigma_1^2 a^2 \frac{t^2}{2}} \cdot e^{\mu_2 b t} e^{\sigma_2^2 b^2 \frac{t^2}{2}}$

$= e^{(\mu_1 a + \mu_2 b)t} e^{(\sigma_1^2 a^2 + \sigma_2^2 b^2)\frac{t^2}{2}}$ for all t.

by uniqueness of MGF

$aX_1 + bX_2 \sim N(\mu_1 a + \mu_2 b, \sigma_1^2 a^2 + \sigma^2 b^2)$

$\square$

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2. Let $X \sim N(68, 3^2)$ and $Y \sim N(66, 2^2)$ be independent. $P(X > Y)$ equals

a $1 - \Phi(\frac{0-2}{\sqrt{3^2+2^2}})$

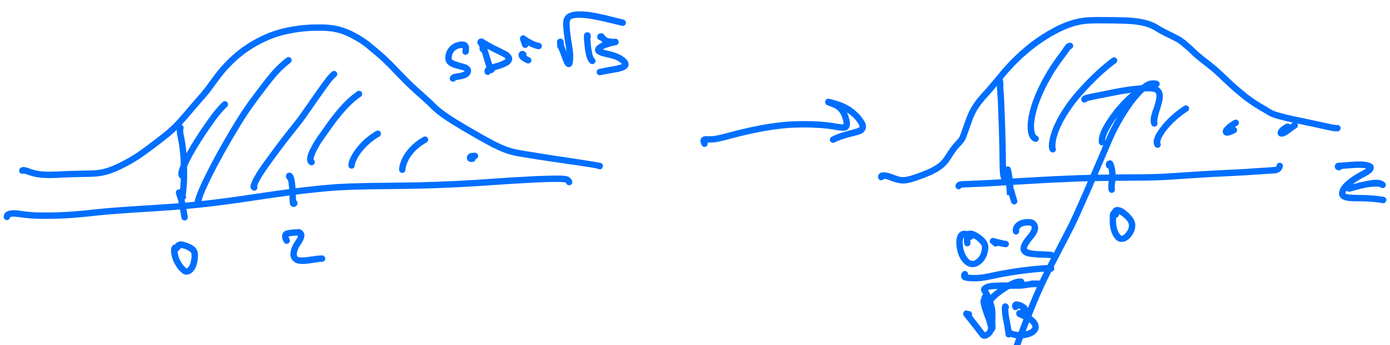
b $1 - \Phi(\frac{0-2}{3^2+2^2})$

c $1 - \Phi(\frac{68-66}{\sqrt{3^2+2^2}})$

d none of the above

$P(X>Y) = P(X-Y>0)$

$X-Y \sim N(68-66, 3^2+2^2) = N(2,13)$



area is

$1 - \Phi\left(\frac{0-2}{\sqrt{3^2+2^2}}\right)$

② Sec 5.3 Chi-square distribution

Fact see end of lecture notes

if $Z \sim N(0,1)$ then

$$Z^2 \sim \text{Gamma}\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

Recall that if $T \sim \text{Gamma}(r, \lambda)$,

$$M_T(t) = \left(\frac{\lambda}{\lambda - t}\right)^r, \quad t < \lambda$$

Hence $M_{Z^2}(t) = \left(\dfrac{\frac{1}{2}}{\frac{1}{2} - t}\right)^{1/2}, \quad t < 1/2$

Let $Z_1, \ldots, Z_n \overset{iid}{\sim} N(0,1)$

$$M_{Z_1^2 + \cdots + Z_n^2} = \left(\frac{\frac{1}{2}}{\frac{1}{2} - t}\right)^{1/2 \cdot n}, \quad t < 1/2$$

By uniqueness of MGF

$$\underbrace{Z_1^2 + \cdots + Z_n^2}_{\text{called chi squared distribution } (\chi^2) \text{ with } n \text{ degrees of freedom}} \sim \text{Gamma}\left(r = \tfrac{n}{2}, \lambda = 1/2\right)$$

ex Let $X, Y \overset{iid}{\sim} N(0,1)$

Let $R = \sqrt{X^2 + Y^2}$ be Raleigh distribution.

Then $R^2 = X^2 + Y^2 \sim \chi^2_2$ ← # degrees of freedom extra (Rayleigh)

If $R_1 \sim \text{Ray}$ $\xleftarrow{}$ Rayleigh (density $f_R(r) = re^{-\frac{1}{2}r^2}$)

find the density of $W = \frac{1}{\sqrt{3}} R_1$

$$f_W(w) = \frac{1}{\frac{1}{\sqrt{3}}} \cdot re^{-\frac{1}{2}r^2} \Big|_{r=\sqrt{3}w}$$

$$= \sqrt{3} \cdot \sqrt{3} w e^{-\frac{3}{2}w^2} = 3we^{-3/2 w^2}$$

So $W = \frac{1}{\sqrt{3}}$ Ray is also the minimum of R_1, R_2, R_3

i.e $\min(R_1, R_2, R_3) \sim \frac{1}{\sqrt{3}}$ Ray

ex Let $R_1, R_2, R_3, R_4 \overset{iid}{\sim}$ Ray

let $W = \min(R_1, R_2, R_3) \sim \frac{1}{\sqrt{3}}$ Ray

Find $P(R_4 < W)$

Hint: Find $P(R_4^2 < W^2)$

Recall

$c\,\text{Exp}(\lambda) = \text{Exp}(\frac{\lambda}{c})$

$$P(R_4 < W) = P(R_4^2 < W^2)$$

$R_4^2 \sim \text{Exp}(\lambda)$ and $W^2 \sim \frac{1}{3}\text{Ray}^2 = \frac{1}{3}\text{Exp}(\frac{1}{2}) = \text{Exp}(\frac{3}{2})$

$$P(\underbrace{\text{Exp}(\tfrac{1}{2}) < \text{Exp}(\tfrac{3}{2})}_{\text{competing exponentials}}) = \frac{1/2}{\frac{1}{2} + 3/2} = \boxed{1/4}$$

Appendix

If $Z \sim N(0,1)$, than $Z^2 \sim Gamma(\frac{1}{2}, \frac{1}{2})$

Proof/

$\lambda > 0$, pos integer r,

gamma (r, λ) density

$$f(t) = \frac{\lambda^r}{\Gamma(r)} t^{r-1} e^{-\lambda t}, \quad t > 0$$

let Z = std normal

$X = Z^2$ change of variable rule.

Find $f_X(x) = \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-\frac{1}{2}x}, \quad x > 0$

$$= \frac{(\frac{1}{2})^{1/2}}{\sqrt{\pi}} x^{\frac{1}{2}-1} e^{-\frac{1}{2}x}$$

$\sqrt{\pi} = \Gamma(\frac{1}{2})$

$\Rightarrow X \sim gamma(\frac{1}{2}, \frac{1}{2})$

$\square$