Stat 134 lec 11

warmup

Let X = number of sixes in 7 tosses of a fair die. $X \sim Bin(7, \frac{1}{6})$

a) write X as a sum of indicators

$X = I_1 + I_2 + \cdots + I_7$ $\quad p = 1/6$

$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ toss is a 6} \\ 0 & \text{else} \end{cases}$

b) Find $Var(X)$

$$Var(X) = Var(I_1 + I_2 + \cdots + I_7)$$
$$= Var(I_1) + Var(I_2) + \cdots Var(I_7)$$
$$= pq$$
$$= \boxed{7pq}$$

$X \sim Bin(n, p)$

$Var(X) = npq$

If n is large p small and $np \to \mu$

then $X \sim Pois(\mu)$

$Var(X) = npq \to \mu \cdot 1 = \mu$

Last time

Sec 3.3 $\mathrm{Var}(X) = E\left((X - E(X))^2\right)$

or $\mathrm{Var}(X) = E(X^2) - (E(X))^2$

ex $I = \begin{cases} 1 & \text{with prob } p \\ 0 & \text{with prob } q \end{cases}$

$\mathrm{var}(I) = pq$

see p 193 Pitman for the proof.

Thm $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$ if X, Y are independent.

ex $X \sim \mathrm{Bin}(n, p)$

$\mathrm{Var}(X) = npq$

$\mathrm{SD}(X) = \sqrt{npq}$

Stat 134

1. X is nonnegative random variable with $E(X) = 3$ and $SD(X) = 2$. True, False or Maybe:

$$P(X^2 \geq 40) \leq \frac{1}{3}$$

a True

b False

c Maybe

maybe

b	Both Markov's and Chebyshev's give values bigger than 1/3 when we think about P(X>= sqrt(40))

$P(X^2 \geq 40) = P(X \geq \sqrt{40}) \leq \frac{3}{\sqrt{40}} > 1/3$

X nonneg

a	We can solve for E[x^2] and using Markov Inequality, we go 1/4 and it is true for both that it is less than 1/4 and 1/3.

Today

(1) Sec 3.6 (next three sec 3.4) Calculating the variance of a sum of dependent indicators.

(2) Sec 3.6 Hypergeometric Distribution

(1) Sec 3.6 Var of Sum of dependent indicators.

Ex

14. A building has 10 floors above the basement. If 12 people get into an elevator at the basement, and each chooses a floor at random to get out, independently of the others, at how many floors do you expect the elevator to make a stop to let out one or more of these 12 people?

X = number of elevator stops.

$p_1 = 1 - (\frac{9}{10})^{12}$

$I_2 = \begin{cases} 1 \text{ if at least 1 person gets off at 2nd floor} \\ 0 \text{ else.} \end{cases}$

a) Find $E(X)$

$X = I_1 + \cdots + I_{10}$

$E(X) = 10 \cdot p_1$

b) Find $\text{Var}(X)$.

$X = I_1 + \cdots + I_{10}$ sum of dependent indicators

$\text{Var}(X) = E(X^2) - (E(X))^2$

$E(X^2) = E((I_1 + \cdots + I_{10})^2) = E\left(\sum_{i,j=1}^{10} I_i I_j\right)$

$I_1 = \begin{cases} 1 \text{ if stop 1st floor} \\ 0 \text{ else} \end{cases}$

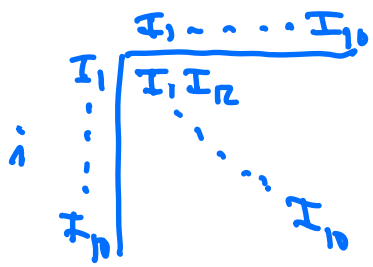
$I_2 = \begin{cases} 1 \text{ if stop 2nd floor} \\ 0 \text{ else} \end{cases}$

$I_1 \cdot I_2 = \begin{cases} 1 \text{ if stop 1st and 2nd floor} \\ 0 \text{ else} \end{cases}$

$= I_{12}$

$p_{12} = 1 - P(\text{no one gets off at 1st or 2nd floor})$

$= 1 - \left[(\frac{9}{10})^{12} + (\frac{9}{10})^{12} - (\frac{8}{10})^{12}\right]$



$E(X^2) = 10\,E(I_1) + 90\,E(I_{12}) = 10p_1 + 90p_{12}$

$\text{Var}(X) = E(X^2) - (E(X))^2 = 10p_1 + 90p_{12} - (10p_1)^2$

Summary

Identically Distributed

Variance of sum of dependent i.d. indicators

$X = I_1 + \cdots + I_n$

$p_1 = E(I_1)$

$p_{12} = E(I_{12}) = E(I_1 I_2)$

$E(X) = np_1$

$\text{var}(X) = \underbrace{np_1 + n(n-1)p_{12}}_{E(X^2)} - \underbrace{(np_1)^2}_{E(X)^2}$

Variance of sum of i.d. independent indicators

$X = I_1 + \cdots + I_n$

$p_1 = E(I_1)$

$p_{12} = p_1 \cdot p_2 = p_1^2$

$\text{var}(X) = \underbrace{np_1 + n(n-1)p_1^2}_{E(X^2)} - \underbrace{(np_1)^2}_{E(X)^2} = np_1 - np_1^2 = np_1(1-p_1)$

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Stat 134

1. A fair die is rolled 14 times. Let X be the number of faces that appear exactly twice. Which of the following expressions appear in the calculation of $Var(X)$

should be 6.5

a $14 * 13 * \binom{14}{2,2,10}(1/6)^2(1/6)^2(4/6)^{10}$

b $\binom{14}{2}(1/6)^2(5/6)^{12}$

c more than one of the above

d none of the above

X = # faces that appear twice.

$X = I_1 + \cdots + I_6$

$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ face appears twice} \\ 0 & \text{else} \end{cases}$

$p_1 = \binom{14}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{12}$

$I_{12} = \begin{cases} 1 & \text{if } 1^{st} \text{ and } 2^{nd} \text{ face appear twice} \\ 0 & \text{else} \end{cases}$

$p_{12} = \binom{14}{2,2,10}\left(\frac{1}{6}\right)^2\left(\frac{1}{6}\right)^2\left(\frac{4}{6}\right)^{10}$

$$Var(X) = \underbrace{np_1 + n(n-1)p_{12}}_{E(X^2)} - \underbrace{(np_1)^2}_{E(X)^2}$$

Extra Practice

6. A drawer contains s black socks and s white socks, where s is a positive integer. I pull two socks out at random without replacement and call that my first pair. Then I pull two socks out at random without replacement from the remaining socks in the drawer, and call that my second pair. I proceed in this way till I have s pairs and the drawer is empty.

Let D be the number of pairs in which the two socks are of different colors.

a) Find $E(D)$.

b) Find $Var(D)$.

$P_1 = \frac{\binom{s}{1}\binom{s}{1}}{\binom{2s}{2}}$ (W B)

Soln

a) $D = I_1 + \cdots + I_s$ where $I_i = \begin{cases} 1 & \text{if } 2^{nd} \text{ pair different} \\ 0 \end{cases}$

$\Rightarrow E(D) = s \cdot \frac{\binom{s}{1}\binom{s}{1}}{\binom{2s}{2}}$

b) $I_{12} = \begin{cases} 1 & \text{if } 1^{st} \text{ and } 2^{nd} \text{ pair different} \\ 0 & \text{else} \end{cases}$

$P_{12} = \frac{\binom{s}{1}\binom{s}{1}}{\binom{2s}{2}} \cdot \frac{\binom{s-1}{1}\binom{s-1}{1}}{\binom{2s-2}{2}}$

then

$Var(D) = sP_1 + s(s-1)P_{12} - (sP_1)^2$