Stat 134 lec 25

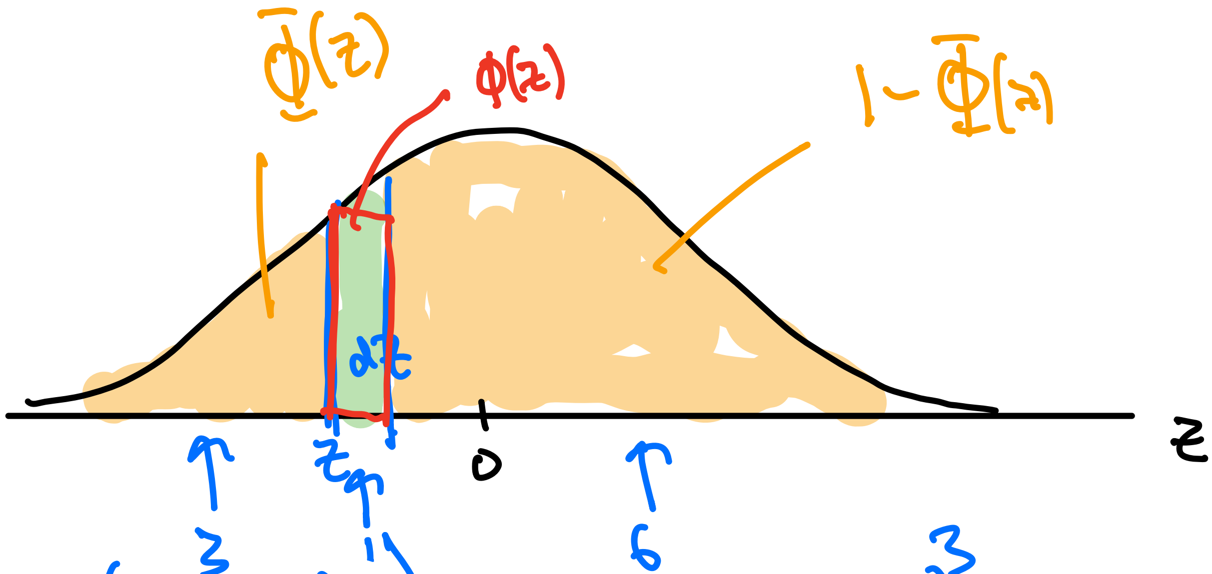
warmup

Let $Z_1, Z_2, \dots, Z_{10} \overset{iid}{\sim} N(0,1)$

$Z_{(r)}$ is the r^{th} order statistic of std normals.

$=$ r^{th} value of $Z_1, \dots, Z_n$ sorted smallest to biggest.

Find the density of $Z_{(4)}$



$$P\left(Z_{(4)} \in dz\right) = \binom{10}{3,1,6} \left(\Phi(z)\right)^3 \phi(z)\,dz \left(1-\Phi(z)\right)^6$$

$$\boxed{f(z) = \binom{10}{3,1,6} \left(\Phi(z)\right)^3 \phi(z) \left(1-\Phi(z)\right)^6}$$

Last time

Sec 4.6

Overview on standard uniform order statistic

let $U_1, \dots, U_n \overset{iid}{\sim} U(0,1)$

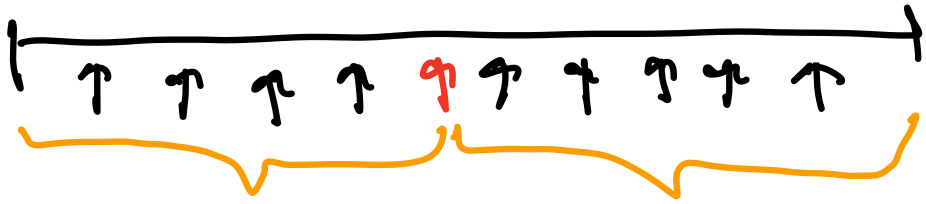
$U_{(r)}$ is the r^{th} sorted standard uniform out of n.

We proved that

$$f_{U_{(r)}}(x) = \binom{n}{r-1,\ 1,\ n-r} x^{r-1}(1-x)^{n-r} \quad \text{for } 0<x<1$$

$$= \frac{n!}{(r-1)!\,(n-r-1)!}$$

Picture U_5 of $n=10$



$r=5$ gaps $\qquad s=n-r+1=6$ gaps

Define $\text{Beta}(r,s)$ to be the distribution of $U_{(r)}$ out of $n=s+r-1$

Note $s=n-r+1$

equivalently

$$U_{(r)} \text{ out of } n \sim \text{Beta}(r, \overbrace{n-r+1}^{s})$$

$s = n-r+1 \iff n = s+r-1$

$$f_{U_{(r)}}(x) = \binom{n}{r-1,1,n-r} x^{r-1}(1-x)^{\overset{s-1}{\overset{\parallel}{n-r}}} \quad \text{for } 0<x<1$$

$$\parallel$$

$$\frac{(s+r-1)!}{(r-1)!(s-1)!} = \frac{\Gamma(s+r)}{\Gamma(r)\Gamma(s)}$$

we can generalise beta to $r>0$, $s>0$:

<u>Defn</u> Let $r,s>0$

$X \sim \text{Beta}(r,s)$ if

$$f(x) = \frac{\Gamma(s+r)}{\Gamma(s)\Gamma(r)} x^{r-1}(1-x)^{s-1} \quad \text{for } 0<x<1.$$

where $\Gamma(r) = \int_0^\infty t^{r-1}e^{-t}\,dt$ Gamma function for $r>0$

or $\Gamma(r) = (r-1)!$ if $r \in \mathbb{Z}^+$

see appendix of notes

Thm $X \sim \text{Beta}(r, s)$

$$E(X) = \frac{r}{r+s}$$

Hence if $X \sim U_{(r)}$

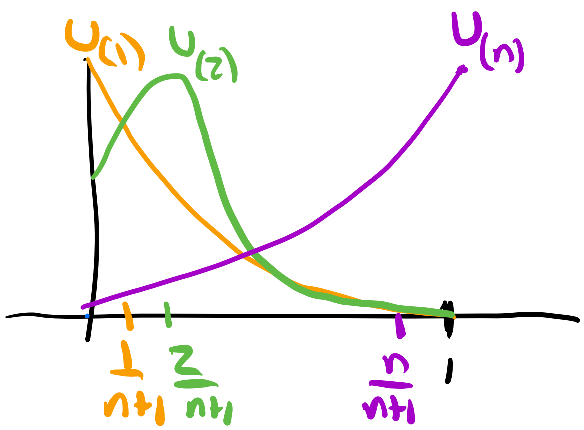
$$E(X) = \frac{r}{r+n-r+1} = \boxed{\frac{k}{n+1}}$$

$$E(U_{(1)}) = \frac{1}{n+1}$$

$$E(U_{(2)}) = \frac{2}{n+1}$$

$$\vdots$$

$$E(U_{(n)}) = \frac{n}{n+1}$$



today

1. Sec 4.6 Practice with Beta.
2. <u>Sec 5.1, 5.2</u> Continuous Joint Distribution
3. Sec <u>5.1, 5.2</u> Calculate probabilities with $f(x, y)$.

① Sec 4.6 Practice with Beta.

If $X \sim \text{Beta}(r,s)$

$$f(x) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1}(1-x)^{s-1}$$

Since $\int_0^1 f(x)\,dx = 1 \Rightarrow \int_0^1 x^{r-1}(1-x)^{s-1}dx = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$

ex Let $X \sim \text{Beta}(3,4)$

$\left(\text{so } f(x) = \frac{\Gamma(7)}{\Gamma(3)\Gamma(4)} x^2(1-x)^3 \right)$

Compute $E(7X - 5X^6)$

$$= 7\underset{\substack{\| \\ 3/7}}{E(X)} - 5E(X^6)$$

$$E(X^6) = \int_0^1 x^6 \frac{\Gamma(7)}{\Gamma(3)\Gamma(4)} x^2(1-x)^3 dx$$

$$= \frac{\Gamma(7)}{\Gamma(3)\Gamma(4)} \int_0^1 \underbrace{x^8(1-x)^3}_{\text{Var Beta}(9,4)}dx = \frac{\Gamma(9)\Gamma(7)}{\Gamma(3)\Gamma(13)}$$

$$\int_0^1 x^8(1-x)^3dx = \frac{\Gamma(9)\Gamma(4)}{\Gamma(13)}$$

$$\Rightarrow E(7x - 5x^6) = 3 - 5\frac{\Gamma(9)\Gamma(7)}{\Gamma(3)\Gamma(13)}$$

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Stat 134

1. Let P be the chance a coin lands head. Suppose the prior distribution of P is $f_P(p) = c(1-p)^4$ for $0 \leq p \leq 1$ for some constant c. Which of the following is true:

   a $P \sim Beta(1,4)$ ✗

   b $c = 5$ ✓

   c $E(P) = \frac{1}{5}$ ✗

   d more than one of the above

compare $(1-p)^4$ and $p^{r-1}(1-p)^{s-1}$ $\Rightarrow$ $r=1$, $s=5$

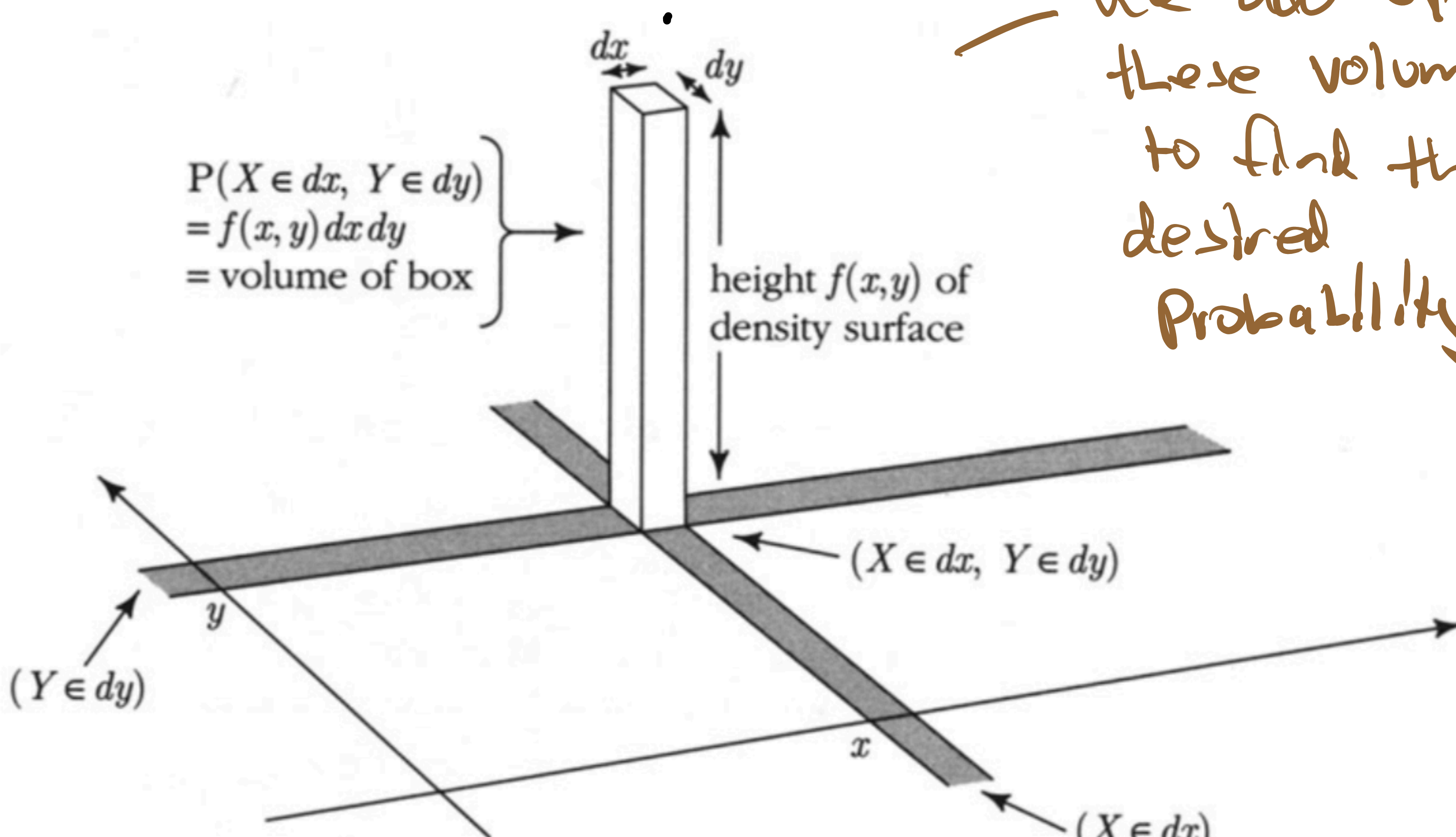
$\Rightarrow P \sim Beta(1,5)$

$$c = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} = \frac{\Gamma(6)}{\Gamma(1)\Gamma(5)} = \frac{5!}{0!\,4!} = 5$$

$$E(P) = \frac{r}{r+s} = \frac{1}{1+5} = 1/6$$

② Sec 5.1, 5.2 Joint Density

$P(X \in dx, Y \in dy) \approx f(x,y)\,dx\,dy$



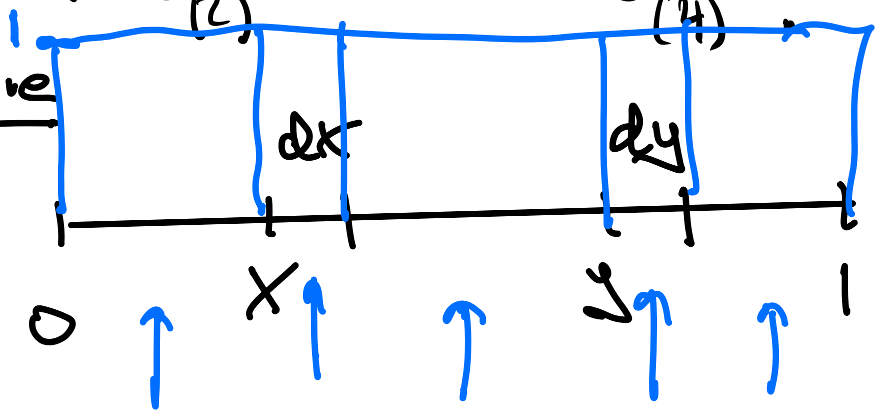
we add up these volumes to find the desired probability.

$$\int_y \int_x f(x,y)\,dx\,dy = \int_x \int_y f(x,y)\,dy\,dx = 1$$

ex Throw down 5 darts on $(0,1)$.

Find the joint density, $f(x,y)$, of $X = U_{(2)}$ and $Y = U_{(4)}$

Picture



Hint: Find $P(X \in dx, Y \in dy) \approx f(x,y)\,dx\,dy$

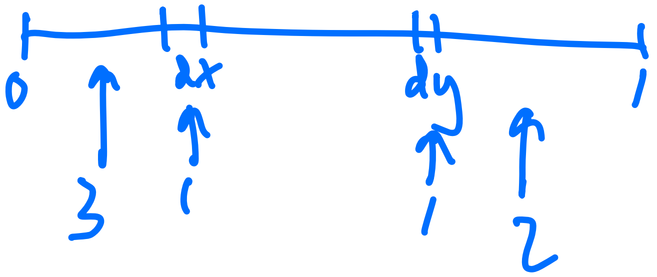
$$P(X \in dx, Y \in dy) \approx \binom{5}{1} x \binom{4}{1} dx \binom{3}{1} (y-x) \binom{2}{1} dy \binom{1}{1} (1-y)$$

$$= \underbrace{\binom{5}{1,1,1,1,1} x (y-x)(1-y)}_{= f(x,y)} \, dx\,dy \quad \text{for } 0 < x < y < 1$$

Let (X,Y) have joint density $f_{X,Y}(x,y) = 420x^3(1-y)^2$ for $0<x<y<1$.

Fill in the blanks: X and Y represent the 4 smallest and 5 smallest of 7 i.i.d. Unif (0,1) random variables, respectively.

$X \sim U_{(4)}$ $\quad Y \sim U_{(5)}$



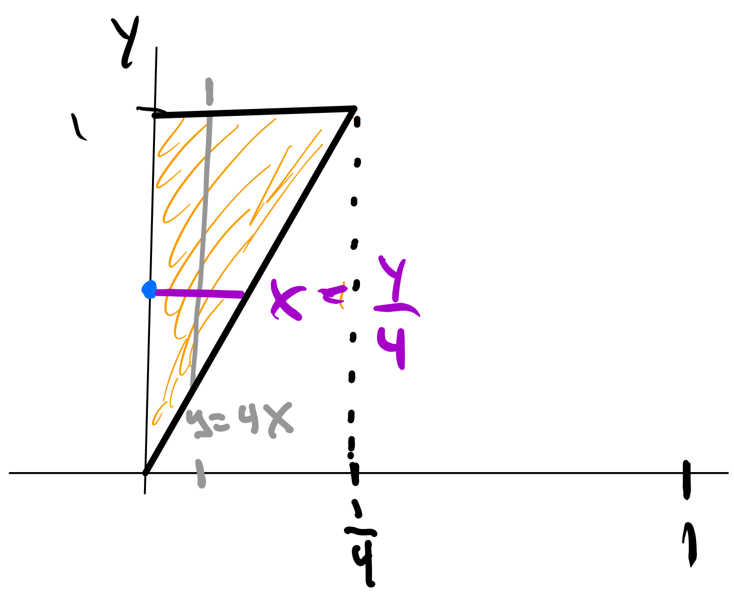
③ Sec 5.1, 5.2 Calculate probabilities with $f(x,y)$.

ex

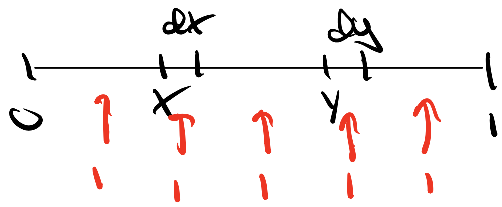
Throw down 5 darts on $(0,1)$.

$X = U_{(2)}$ $\quad Y = U_{(4)}$

Find $P(Y > 4X)$



recall,



$$f(x,y) = \binom{5}{1,1,1,1,1} x(y-x)(1-y) = \boxed{5!\,x(y-x)(1-y)}$$

What are bounds of integrals?

$$P(Y>4X) = \int_{y=0}^{y=1} \int_{x=0}^{x=y/4} 5!\,x(y-x)(1-y)\,dx\,dy$$

(or)

$$P(Y>4X) = \int_{x=0}^{x=1/4} \int_{y=4x}^{y=1} 5!\,x(y-x)(1-y)\,dy\,dx$$

details:

$$P(Y>4X) = \int_{y=0}^{y=1} \int_{x=0}^{y/4} 120\,x(y-x)(1-y)\,dx\,dy$$

$$= \int_{y=0}^{y=1} 120(1-y) \int_{x=0}^{x=y/4} (xy-x^2)\,dx\,dy$$

$$= \int_{y=0}^{y=1} 120(1-y) \left[\frac{x^2y}{2} - \frac{x^3}{3} \right] \Bigg|_{x=0}^{x=y/4} dy$$

$$= \int_{y=0}^{y=1} 120(1-y)\underbrace{\left(\frac{y^3}{32} - \frac{y^3}{3\cdot 64}\right)}_{\frac{6y^3 - y^3}{3\cdot 64} = \frac{5y^3}{192}} dy$$

$$= \frac{5}{192}\cdot 120 \int_{y=0}^{y=1} y^3 - y^4 \, dy$$

$$= \frac{5}{192}\cdot 120 \int_0^1 y^3 - y^4 \, dy = \frac{5\cdot 120}{192}\left(\frac{y^4}{4} - \frac{y^5}{5}\right)\Big|_0^1$$

$$= \frac{5\cdot 120}{192}\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{30}{192} = .156$$

Appendix

Let $X \sim \text{Beta}(r,s)$

then $E(X) = \dfrac{r}{r+s}$.

Pf/ Note that $\displaystyle\int_0^1 f(x)\,dx = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 x^{r-1}(1-x)^{s-1}\,dx = 1$

$$\Rightarrow \int_0^1 x^{r-1}(1-x)^{s-1}\,dx = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$$

$$E(X) = \int_0^1 x f(x)\,dx = \frac{\Gamma(r+s)}{\Gamma(r)\cancel{\Gamma(s)}} \underbrace{\int_0^1 \overset{x^{r+1-1}}{x\, x^{r-1}}(1-x)^{1-s}\,dx}_{\displaystyle = \frac{\cancel{\Gamma(s)}\Gamma(r+1)}{\Gamma(s+r+1)}}$$

$$= \frac{(r+s-1)!\; r!}{(s+r)!} = \boxed{\frac{r}{r+s}}$$

$\square$