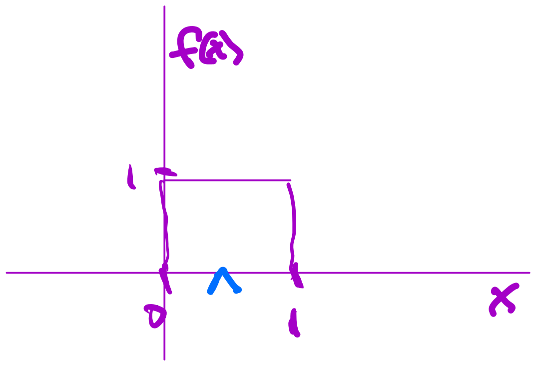
Stat 134 lec 17 (no lec 16)

Warmup

Let $X \sim Unif(0,1)$ be the standard uniform distribution with histogram (density)

$$f(x) = \begin{cases} 1 \text{ if } 0<x\le 1 \\ 0 \text{ else} \end{cases}$$

Picture



Define

$$E(X) = \int_{x=-\infty}^{\infty} x f(x)\,dx$$

Find $E(x)$, $E(x^2)$, and $Var(X)$.

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x f(x)\,dx = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \boxed{1/2}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \boxed{1/3}$$

$$Var(x) = E(x^2) - (E(x))^2 = \frac{1}{3} - 1/4 = \boxed{1/12}$$

today

Sec 4.1 Continuous Distributions

① probability density
② change of scale
③ Sec 4.2 Exponential dist

① sec 4.1 Probability density.

let X be a continuous RV

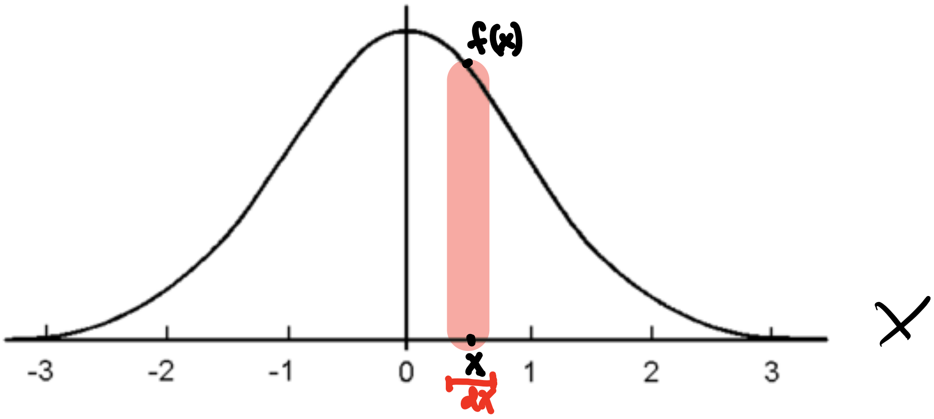
The probability density (histogram) of X is described by a prob densisity function

$$f(x) \geq 0 \text{ for } x \in X$$

and $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$

ex The standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)$$



The area of the red strip above is $\overset{\text{height}}{f(x)}\,\overset{\text{width}}{dx}$.

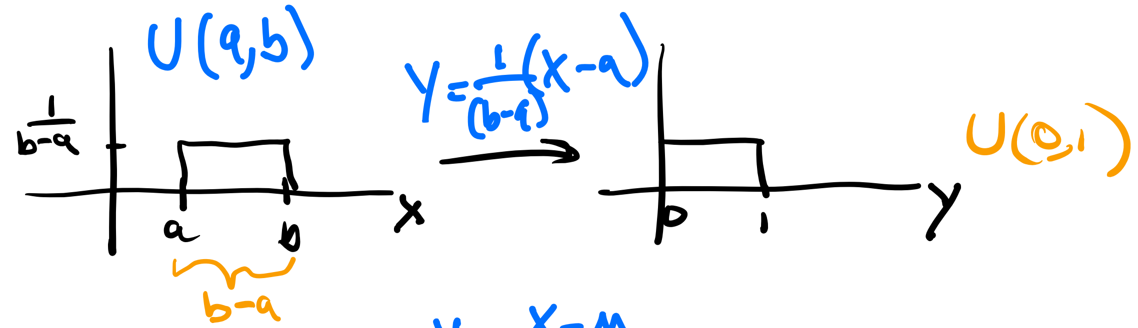
The probability of choosing a point x in the little interval dx is $P(x \in dx) = f(x)dx$

② Change of scale

A change of scale is a transformation $Y = m + nX$, of X. The purpose is that it makes it easier to calculate $E(X)$ and $Var(X)$. It maps one density to another.

constants.

ex



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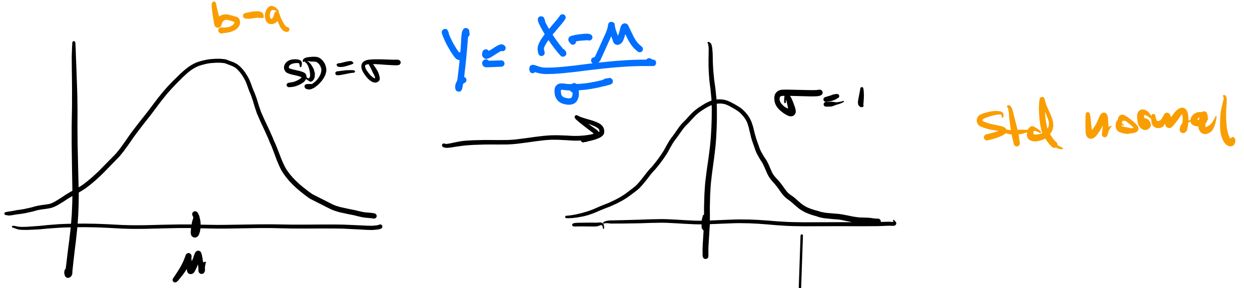

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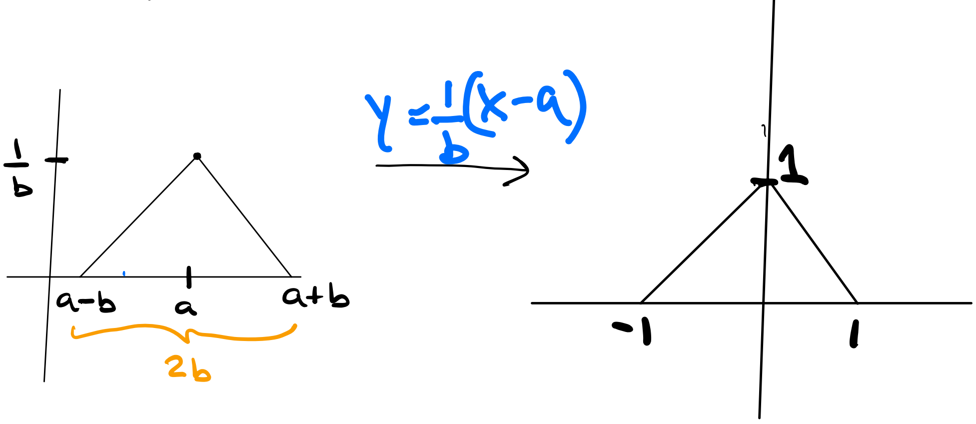
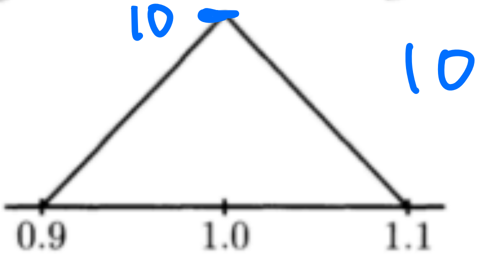


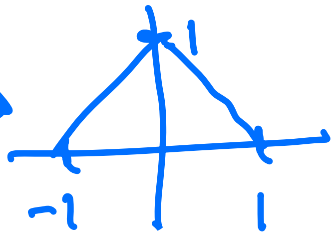
tinyurl.com/oct7-2022

this might not work !

Suppose a manufacturing process designed to produce rods of length 1 inch exactly, in fact produces rods with length distributed according to the density graphed below.



10(x-1)



a = 1
b = .1

You should change the scale of X= the length of rods to:

○ a: X-1

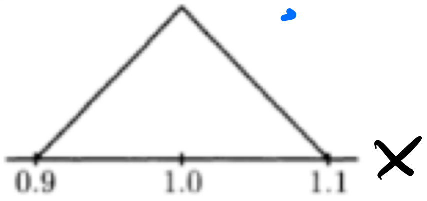
○ b: .1(X-1)

○ c: 10X-1

☒ d: none of the above — 10(x-1)

ex

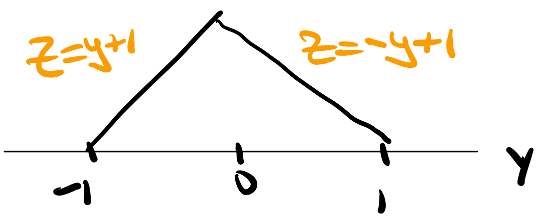
Suppose a manufacturing process designed to produce rods of length 1 inch exactly, in fact produces rods with length distributed according to the density graphed below.



Find the variance of the length of the rods.

$Y = 10(x-1)$ change of scale.

$\mathrm{Var}(Y) = 100\,\mathrm{Var}(X) \Rightarrow \mathrm{Var}(X) = \frac{\mathrm{Var}(Y)}{100}$ ← easier to find.



Find the density of Y:

$$f(y) = \begin{cases} y+1 & -1 \le y \le 0 \\ -y+1 & 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$

Find Var(X)

$$\mathrm{Var}(Y) = E(Y^2) - (E(Y))^2$$

$$E(Y^2) = \int_{-1}^{0} y^2(y+1)\,dy + \int_{0}^{1} y^2(-y+1)\,dy$$

$$= \int_{-1}^{0} y^3\,dy + \int_{-1}^{0} y^2\,dy - \int_{0}^{1} y^3\,dy + \int_{0}^{1} y^2\,dy$$

$$= \frac{y^4}{4}\Big|_{-1}^{0} + \frac{y^3}{3}\Big|_{-1}^{0} - \frac{y^4}{4}\Big|_{0}^{1} + \frac{y^3}{3}\Big|_{0}^{1} = -\frac{1}{4} + \frac{1}{3} - \frac{1}{4} + \frac{1}{3}$$

$$= 2\left(\frac{1}{12}\right) = \boxed{\frac{1}{6}}$$

$$E(Y) = 0 \Rightarrow \mathrm{Var}(Y) = \frac{1}{6}$$

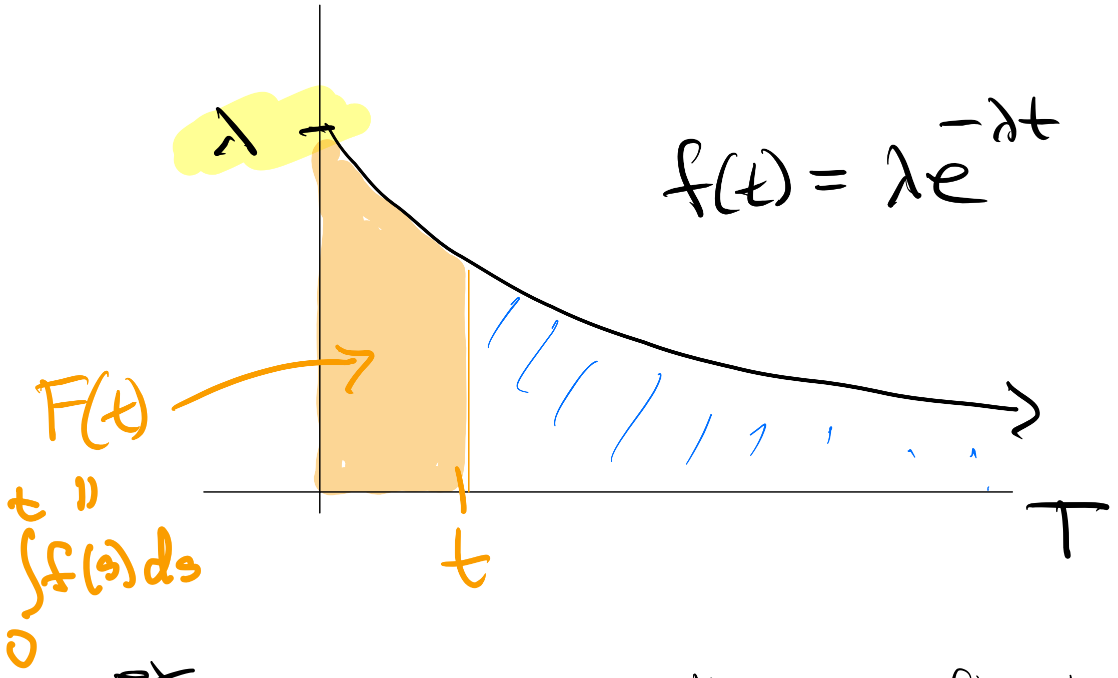
$$\mathrm{Var}(X) = \frac{1/6}{100} = \boxed{1/600}$$

③ Sec 4.2

Exponential distribution

Defn A random time T has exponential distribution with **rate** $\lambda > 0$.

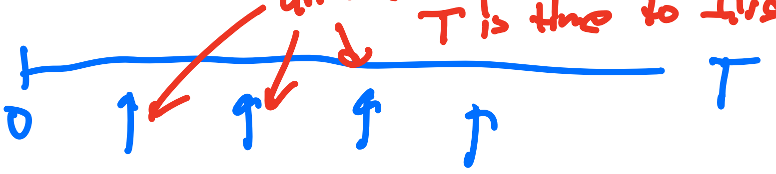
$T \sim \text{Exp}(\lambda)$, if T has density $f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{else} \end{cases}$



ex T = time until your first success where λ = rate of success.

arrivals of a Poisson process

T is time to first arrival.



~~ex~~ T = time until a lightbulb burns out

CDF and survival function

Let X be a continuous RV

$F(x) = P(X \le x)$ — a number between 0 and 1

If f(x) is a density of X,

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

$T \sim \text{Exp}(\lambda)$ $\quad f(t) = \lambda e^{-\lambda t}$

Compute the CDF of T.

$$F(t) = P(T \le t) = \int_0^t \lambda e^{-\lambda s}\,ds = \frac{\lambda e^{-\lambda s}}{-\lambda}\Bigg|_0^t$$

$$= -e^{-\lambda t} + 1 = \boxed{1 - e^{-\lambda t}}$$

$P(T > t) = e^{-\lambda t}$ is called the survival function

$T \sim \text{Exp}(\lambda)$ iff $P(T > t) = e^{-\lambda t}$

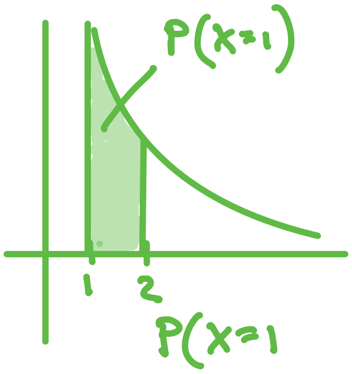
since $F(t) \overset{= 1 - P(T>t)}{}$ and $f(t)$ both define distribution.

Ex

Let $T \sim \text{Exp}(\lambda)$. Let $X = \lfloor T \rfloor$

floor of T

X takes values 0, 1, 2, 3, . . .



a) Find $P(X=x)$

b) Find $E(X)$

$P(X=1) = P(T>1) - P(T>2)$

a) $P(X=x) = P(T>x) - P(T>x+1)$

$$= \boxed{e^{-\lambda x} - e^{-\lambda(x+1)}}$$

b) $$E(X) = \sum_{x=0}^{\infty} x P(X=x)$$

$$= \sum_{x=0}^{\infty} \left(x e^{-\lambda x} - x e^{-\lambda(x+1)} \right)$$

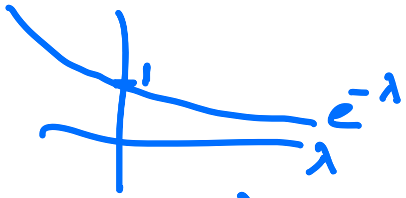
$$=(1e^{-\lambda}-1e^{-\lambda\cdot 2}+)+(2e^{-\lambda\cdot 2}-2e^{-\lambda 3})$$

$$+(3e^{-\lambda 3}-3e^{-\lambda 4})+\cdots$$

$$=e^{-\lambda}+e^{-\lambda 2}+e^{-\lambda 3}+\cdots$$

$$=e^{-\lambda}(1+e^{-\lambda}+(e^{-\lambda})^2+\cdots)$$

$$=\boxed{\frac{e^{-\lambda}}{1-e^{-\lambda}}}$$



so $0<e^{-\lambda}<1$