Stat 134 lec 36

Warmup

We have two random variables X and Y. Which statement is true:

i more than one of the choices is true

ii $E(Y|X)$ is a function of X.

iii $E(Y|X)$ is a function of Y.

iv $E(E(Y|X))$ is a non constant random variable.

v $E(E(E(Y|X))) = E(Y)$.

last time

Sec 6.3 Conditional densities.

Bayes' rule:

$f(x,y)$

$$f_X(x \mid Y = y) = \frac{f_Y(y \mid X = x) f_X(x)}{f_Y(y)}$$

Average conditional probability:

$$P(B) = \int P(B \mid X = x) f_X(x)\, dx$$

$$f_Y(y) = \int f_Y(y \mid X = x) f_X(x)\, dx$$

For example,

let $X \sim \text{Unif}(0,1)$ and $I_1 \mid X=x,\ I_2 \mid X=x \overset{iid}{\sim} \text{Ber}(x)$

$$\underbrace{P(I_2=1, I_1=1)}_{B} = \int_{x=0}^{x=1} \underbrace{P(I_2=1, I_1=1 \mid X=x)}_{= P(I_2=1 \mid X=x)\, P(I_1=1 \mid X=x)} f_X(x)\, dx$$

$P(I_2=1 \mid X=x) = x$, $P(I_1=1 \mid X=x) = x$

$$= \int_0^1 x^2\, dx = \boxed{1/3}$$

Today

Sec 6.3

1. Bayesian stats
2. Conjugate Pairs

Sec 6.3

① Bayesian Statistics

In frequentist statistics we interpret probability as a long run average constant known only to Tyche, the goddess of fortune.

In Bayesian statistics we interpret probability as a RV

ex When probability a coin lands head is a RV X rather than an unknown constant we are doing Bayesian statistics,

i.e

$$X \sim \text{Unif}(0,1)$$

$$I_1 | X=x,\ I_2 | X=x \overset{iid}{\sim} \text{Ber}(x)$$

Multiplication rule: The joint density is the product of the marginal and the conditional

$$f(x,y) = f_X(x) f_Y(y \mid X = x)$$

$$= f_Y(y) f_X(x \mid Y=y)$$

Mixed Joint X cont, Y discrete

$$\text{joint}(x,y) = f_X(x) P(Y=y \mid X=x) = P_Y(Y=y) f_X(x \mid Y=y)$$

$$\Rightarrow f_X(x \mid Y=y) = \frac{f_X(x) P(Y=y \mid X=x)}{P(Y=y)}$$

posterior

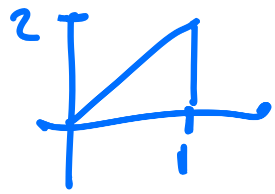
likelihood prior

$$f_X(x|I_1=1) = \frac{P(I_1=1|X=x)\cdot f_X(x)}{P(I_1=1)}$$

constant.

Posterior $\propto$ likelihood $\cdot$ Prior

ex Find $f_X(x|I_1=1) = \dfrac{x\cdot 1}{\frac{1}{2}} = 2x$



Review Beta Distribution

variable part.

$X \sim \text{Beta}(r,s)$

$$f_X(x) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\, x^{r-1}(1-x)^{s-1}, \quad 0<x<1$$

where $r \in \mathbb{Z}^+ \Rightarrow \Gamma(r) = (r-1)!$

ex If $0<x<1$,

$f_X(x) \propto 1 \Rightarrow X \sim \text{Beta}(1,1)$

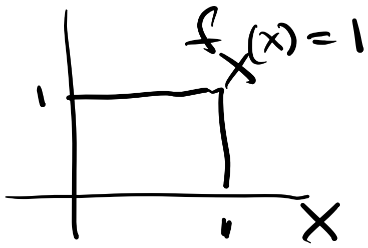
$f_X(x) \propto x \Rightarrow X \sim \text{Beta}(2,1)$

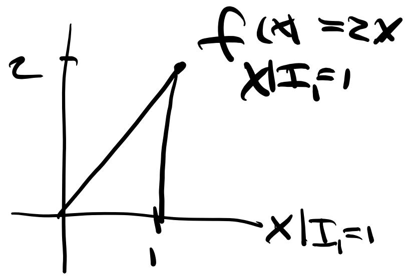
$f_X(x) \propto x(1-x) \Rightarrow X \sim \text{Beta}(2,2)$

Prior density $\quad f_X(x) = 1 \Rightarrow X \sim \text{Beta}(1,1)$

Posterior density $\quad f_X(x \mid I_1=1) = 2x \Rightarrow X \mid I_1=1 \sim \text{Beta}(2,1)$

<u>Prior</u> $X \sim \text{Unif}(0,1)$ <u>Posterior</u>





<u>ex</u>

Let A be an event with complement A^c.

$X \sim \text{Unif}(0,1)$

Suppose $P(A \mid X=x) = x$

$P(A^c \mid X=x) = 1-x$

$X \sim \text{Beta}(r,s)$

$f_X(x) = \dfrac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1}(1-x)^{s-1}, \quad 0<x<1$

variable part.

use the formula

$f_X(x \mid A^c) \propto \text{likelihood} \cdot \text{prior}$

proportional $\qquad$ 1

to find $f_X(x \mid A^c)$ is proportional to $(1-x)$

$\Rightarrow X \mid A^c \sim \text{Beta}(1,2)$

$$f_X(x \mid A^c) = \frac{\Gamma(3)}{\Gamma(1)\Gamma(2)}(1-x) = \boxed{2(1-x) \quad \text{for } 0<x<1}$$

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1. Let A, B and C be events and let X be a random variable uniformly distributed on (0,1). Suppose conditional on X=x, that A, B, and C are independent each with probability x. The conditional density of X given that A and B occurs and C doesn't is:

   a $Beta(2, 2)$

   b $Beta(3, 2)$

   c $Beta(2, 3)$

   d none of the above

$$f_{X|ABC^c} \propto \underbrace{\text{likelihood}}_{P(ABC^c|X=x)} \cdot \underbrace{\text{prior}}_{f_X(x) = 1}$$

$$P(ABC^c|X=x) = P(A|X=x)P(B|X=x)P(C^c|X=x) = x^2(1-x)$$

$$\propto x^2(1-x) \sim \boxed{B(3,2)}$$

The posterior can be difficult to calculate except when the prior and likelihood are conjugate pairs:

ex prior $X \sim \text{beta}(r,s)$

likelihood $Y|X=x \sim \text{Bin}(n,x)$

Posterior $\propto$ likelihood $\cdot$ prior

$$f_{X|Y=j}(x) \propto P(Y=j|X=x)\, f_X(x)$$

similar

$$\underbrace{x^j(1-x)^{n-j} \cdot x^{r-1}(1-x)^{s-1}}_{\shortparallel}$$

$$x^{j+r-1}(1-x)^{n-j+s-1}$$

$$\Rightarrow X|Y=j \sim \text{Beta}(j+r,\, n-j+s)$$

Def^n (conjugate pairs)

The prior and likelihood are conjugate pairs when the prior and posterior belong to the same distribution family.

ex Suppose $\Theta \sim \text{Gamma}(r, \lambda)$ with r, λ known.

Let $N_1|\Theta=\theta,\ N_2|\Theta=\theta,\ N_3|\Theta=\theta \overset{iid}{\sim} \text{Pois}(\theta)$.

Find the posterior distribution of Θ.

Gamma: $f_\Theta(\theta) \propto \theta^{r-1} e^{-\lambda\theta}$

Poisson: $P(N=n|\Theta=\theta) \propto e^{-\theta}\theta^{n}$

$$f_{\Theta|N_1=n_1,N_2=n_2,N_3=n_3}(\theta) \propto \text{likelihood} \cdot \text{prior}$$

$$\propto P(N_1=n_1, N_2=n_2, N_3=n_3 | \Theta=\theta)\, f_\Theta(\theta)$$

$$= (e^{-\theta}\theta^{n_1})(e^{-\theta}\theta^{n_2})(e^{-\theta}\theta^{n_3}) \cdot \theta^{r-1} e^{-\lambda\theta}$$

$$= \theta^{(n_1+n_2+n_3)+r-1} e^{-(3+\lambda)\theta}$$

$$\sim \text{Gamma}(n_1+n_2+n_3+r,\ 3+\lambda)$$

$\Rightarrow$ prior = Gamma and likelihood = Poisson is a conjugate pair.