Stat 134 lec 14

Warmup

A tub of blueberry muffin batter has $\lambda = 2\ \text{bb}/\text{in}^3$ intensity of bb.

bb muffin 1 is $3\ \text{in}^3$

bb muffin 2 is $4\ \text{in}^3$

a) On average how many bb is in muffin 1?

$\mu_1 = \lambda \cdot 3 = \boxed{6\ \text{bb}}$

b) Find P(5 bb in each muffin)

$X_1 \sim \text{Pois}(6)$

$X_2 \sim \text{Pois}(8)$

$P(X_1=5, X_2=5) = \boxed{\frac{e^{-6}6^5}{5!} \cdot \frac{e^{-8}8^5}{5!}}$

c) Find P(10 bb total in both muffins together).

$X_1 + X_2 \sim \text{Pois}(14)$

$P(X_1+X_2=10) = \boxed{\frac{e^{-14}14^{10}}{10!}}$

Announcements

For Wednesday's review, write down questions in discussion board on b-course by Tuesday 8pm.

Last time

sec 3.5 Poisson distribution

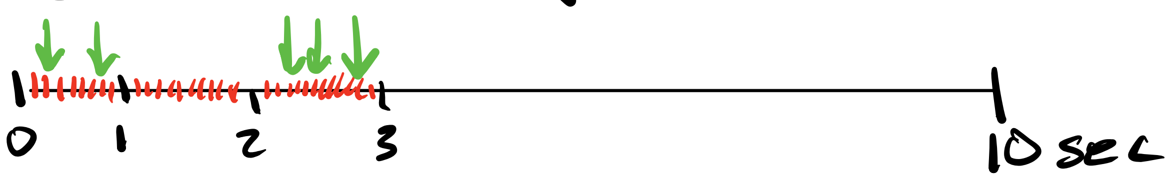
$$X \sim \text{Pois}(\mu)$$

$$P(X=k) = \frac{e^{-\mu}\mu^{k}}{k!}$$

$$E(X) = \text{Var}(X) = \mu.$$

Poisson Process or Poisson Random Scatter (PRS):

ex radioactive decay of Americium 241 in 10 seconds



Assumptions

① no two particles arrive at the same time.
(this allows us to divide 10 sec into n small time intervals each with at most one arrival.

② X is a sum of n ← large iid Bernoulli(p) trials.
← small

$\mu = np$ is avg # of arrivals in 10 sec.

$\lambda = \mu/10$ is the arrival rate per second.

X = # arrivals in 10 seconds.

Suppose $\lambda = 4$ arrivals/sec

then $\mu = \lambda \cdot 10 = 40 \Rightarrow X \sim \text{Pois}(40)$

Americium has a long half life.

Y = # arrivals in 12070 sec.

$Y \sim \text{Pois}(\lambda \cdot 12070)$

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Today

② midterm review

② midterm review

Which distributions are (exactly or approximately) a sum of a fixed number of independent Bernoulli trials?

Discrete

name and range	$P(k) = P(X = k)$ for $k \in$ range	mean	variance
uniform on $\{a, a+1, \ldots, b\}$ $\{1, 2, \ldots, n\}$	$\frac{1}{b-a+1}$	$\frac{a+b}{2}$ $\frac{n+1}{2}$	$\frac{(b-a+1)^2-1}{12}$ $\frac{n^2-1}{12}$
Bernoulli (p) on $\{0, 1\}$ ✓	$P(1) = p$; $P(0) = 1-p$	p	$p(1-p)$
binomial (n, p) on $\{0, 1, \ldots, n\}$ ✓	$\binom{n}{k} p^k (1-p)^{n-k}$	np	$np(1-p)$
Poisson (μ) on $\{0, 1, 2, \ldots\}$ ✓	$\frac{e^{-\mu}\mu^k}{k!}$	μ	μ
✗ hypergeometric (n, N, G) on $\{0, \ldots, n\}$	$\frac{\binom{G}{k}\binom{N-G}{n-k}}{\binom{N}{n}}$	$\frac{nG}{N}$	$n\left(\frac{G}{N}\right)\left(\frac{N-G}{N}\right)\left(\frac{N-n}{N-1}\right)$
✗ geometric (p) on $\{1, 2, 3 \ldots\}$	$(1-p)^{k-1}p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
✗ geometric (p) on $\{0, 1, 2 \ldots\}$	$(1-p)^k p$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$
✗ negative binomial (r, p) on $\{0, 1, 2, \ldots\}$	$\binom{k+r-1}{r-1} p^r (1-p)^k$	$\frac{r(1-p)}{p}$	$\frac{r(1-p)}{p^2}$

sum of dependent indicators

$X = I_1 + I_2 + \cdots + I_n$

G/N

$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ card good} \\ 0 & \text{else} \end{cases}$

normal $\Phi(x)$ μ σ^2

not necessarily, but the sum of iid Bernoulli trials is approx normal.

⇒ DeMorgan's rule: $(A \cap B)^c = A^c \cup B^c$

$$\Rightarrow \quad A \cap B = (A^c \cup B^c)^c$$

so $\boxed{P(A \cap B) = 1 - P(A^c \cup B^c)}$

Inclusion exclusion formula:

Let A_1, A_2, A_3 be dependent RVs with $P(A_i) = .9$ for $i = 1, 2, 3$.

Find a lower bound for $P\left(\bigcap_{i=1}^{3} A_i\right)$

$$P\left(\bigcap_{i=1}^{3} A_i\right) = 1 - P\left(\bigcup_{i=1}^{3} A_i^c\right)$$ DeMorgan's rule,

$$P\left(\bigcup_{i=1}^{3} A_i^c\right) = P(A_1^c) + P(A_2^c) + P(A_3^c) - P(A_1^c A_2^c) - \quad - \quad + P(A_1^c A_2^c A_3^c)$$

$$\le P(A_1^c) + P(A_2^c) + P(A_3^c)$$

$$3(.1) = .3$$

$$\Rightarrow P\left(\bigcap_{i=1}^{3} A_i\right) \ge 1 - .3 = \boxed{.7}$$

expectation question

An urn contains 90 marbles, of which there are 20 greens, 20 blacks and 50 red marbles. Tom draw marble without replacement until the 6^{th} green marble. Let $X = \#$ of marbles drawn. Example: **GGG**BRB**GG**BR**G** with $x = 11$. Find $\mathbb{E}[X]$.

Hint First find the expected number of marbles until the 1^{st} green marble.

What is the min and max of X?

X = # marbles until first green. $1 - 71$

$p = 1/21$

$X = I_1 + \dots + I_{70} + 1$ ← green marble at the end.

$E(X) = 70\left(\frac{1}{21}\right) + 1$

$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ nG before } 1^{st} \text{ green} \\ 0 & \text{else.} \end{cases}$

$1 - G_2 - G_3 - G_6 - G_{20} -$

nG here.

Y = # marbles until 6^{th} green

$$\boxed{E(Y) = 6\left(70\left(\frac{1}{21}\right) + 1\right)}$$

by symmetry.

ex Conditional distribution, Poisson

8. Let X_1 and X_2 be independent random variables such that for $i = 1, 2$, the distribution of X_i is Poisson (μ_i). Let m be a fixed positive integer. Find the distribution of X_1 given that $X_1 + X_2 = m$. Recognize this distribution as one of the famous ones, and provide its name and parameters.

$$\left.\begin{array}{l} X_1 \sim \text{Pois}(\mu_1) \\ X_2 \sim \text{Pois}(\mu_2) \end{array}\right\} \text{ indep} \qquad P(X_1 = k) = \frac{e^{-\mu_1}\mu_1^k}{k!}$$

$X_1 \mid X_1 + X_2 = m$ takes value $0, 1, 2, \ldots, m$

$$P(X_1 = k \mid X_1 + X_2 = m) = \frac{P(X_1 = k, X_2 = m-k)}{P(X_1 + X_2 = m)}$$

$$= \frac{P(X_1 = k)\,P(X_2 = m-k)}{P(X_1 + X_2 = m)}$$

$$= \frac{\dfrac{\cancel{e^{-\mu_1}}\mu_1^k}{k!} \cdot \dfrac{\cancel{e^{-\mu_2}}\mu_2^{m-k}}{(m-k)!}}{\dfrac{\cancel{e^{-(\mu_1+\mu_2)}}(\mu_1+\mu_2)^m}{m!}}$$

$$= \binom{m}{k}\left(\frac{\mu_1}{\mu_1+\mu_2}\right)^k \left(\frac{\mu_2}{\mu_1+\mu_2}\right)^{m-k}$$

$$\Rightarrow X_1 \mid X_1 + X_2 = m \sim \text{Bin}\left(m, \frac{\mu_1}{\mu_1+\mu_2}\right)$$

Problem 4 (conditional probability)

Two jars each contains r red marbles and b blue marbles. A marble is chosen at random from the first jar and placed in the second jar. A marble is then randomly chosen from the second jar. Find the probability this marble is red.

X = the first marble color

Y = the second marble color.

R = red

B = blue

$$\frac{r+1}{r+b+1}\cdot\frac{r}{r+b}$$

$$P(Y=R) = P(Y=R \mid X=R)P(X=R) + P(Y=R \mid X=B)P(X=B)$$

$$\frac{r}{r+b+1}\cdot\frac{b}{r+b}$$