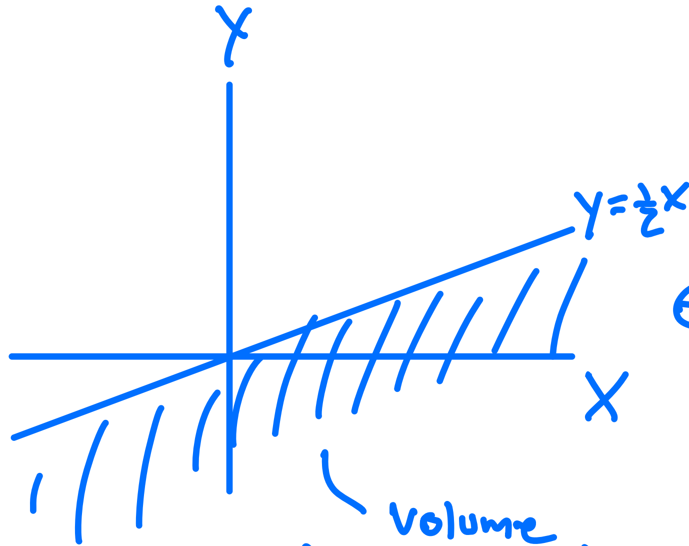
Stat 134 lec 29

Warmup

Let $X, Y \overset{iid}{\sim} N(0,1)$

Find $P(X > 2Y)$

method 1

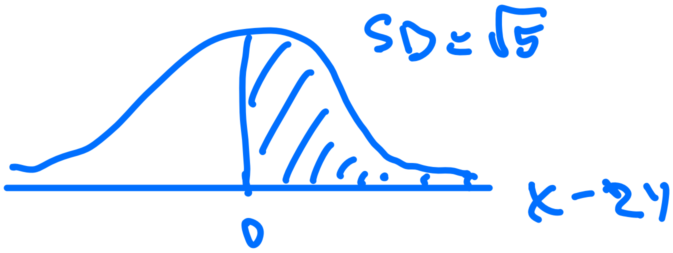


half of the entire plane

volume under the joint above shaded region is $\boxed{1/2}$ by symmetry of bell shaped joint density.

method 2

$P(\underbrace{X-2Y}_{\sim N(0,5)}>0) = \boxed{1/2}$



Announcement: MT2 Wednesday 11/8 (in class)
M6F, chap 4 (skip sec 4.3),
Chap 5,
review materials on b-courses/pages/practice quizzes and exam

Last time

Sec 5.3

A linear combination of independent normals is normal.

Thm Let $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$ $\Big\}$ indep.

then $aX_1 + bX_2 \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$

Note In chapter 6 we will generalize this result and show that $aX_1 + bX_2$ is normal iff the joint (X_1, X_2) is bivariate normal.

Today

Sec 5.4

1. Convolution formula for sum $X+Y$
2. Triangular density
3. Convolution formula for quotient $\frac{Y}{X}$.

① General convolution formula

The change of variable formula generalizes as described below:

1 dimensional change of variables

RV	transformed RV
Y	$Z(Y)$
f_Y	$f_Z = \left\lvert \frac{\partial y}{\partial z} \right\rvert f_Y$

a differentiable function

ex $Z = Y^3$

2 dimensional change of variables

RV	transformed RV
(X, Y)	$(X, Z(X,Y))$
$f_{X,Y}$	$f_{X,Z}$

a differentiable function

ex $Z = X + Y$

ex $Z = \frac{X}{Y}$, $Y \neq 0$

$$f_{X,Z} = \left\lvert \det \frac{\partial(x,y)}{\partial(x,z)} \right\rvert f_{X,Y}$$

$$= \left\lvert \det \begin{bmatrix} \underbrace{\frac{\partial x}{\partial x}}_{=1} & \frac{\partial x}{\partial z} \\ \underbrace{\frac{\partial y}{\partial x}}_{=0} & \frac{\partial y}{\partial z} \end{bmatrix} \right\rvert f_{X,Y}$$

$$= \left\lvert \frac{\partial y}{\partial z} \right\rvert f_{X,Y}$$

Convolution formula

let $Z(X,Y)$ be a differentiable function of X, Y

$$f_Z(z) = \int_{x=-\infty}^{\infty} f_{X,Z}(x,z)\, dx = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y) \left\lvert \frac{\partial y}{\partial z} \right\rvert dx$$

ex (Convultion formula for sum)

Let $Z(X,Y) = X+Y$

Find the convolution formula for Z.

Step 1 Solve for Y treating X as a fixed constant

$$Y = Z - X$$

Step 2 Find $\frac{dY}{dZ}$

$$\frac{dy}{dZ} = 1 - \underbrace{\frac{dx}{dZ}}_{=0} = 1$$

Step 3 substitute y, $\frac{\partial y}{\partial z}$ in $f_Z(z) = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y) \left|\frac{\partial y}{\partial z}\right| dx$

$$f_Z(z) = \int_{x=-\infty}^{x=\infty} f_{X,Y}(x, z-x)\, dx$$

convolution formula for sum.

If X and Y are indep, $f_{X,Y}(x, z-x) = f_X(x) f_Y(z-x)$

tinyurl.com/Apl7-2023

Stat 134

Friday October 21 2022

1. Let X and Y be iid $Exp(\lambda)$ (recall $f_X(x) = \lambda e^{-\lambda x}$). Find the density of $Z = X + Y$ using the convolution formula for sum

$$f_Z(z) = \int_{x=-\infty}^{x=\infty} f_{(X,Y)}(x, z-x)dx$$

a $f_Z(z) = \lambda^2 e^{-\lambda(z-2x)}$

b $f_Z(z) = \lambda^2 z e^{-\lambda z}$

c $f_Z(z) = \lambda^2 z^2 e^{-2\lambda z}$

d none of the above

$$\underset{\substack{\uparrow \\ \text{fixed}}}{f_Z(z)} = \int_{x=-\infty}^{x=\infty} f_{x,y}(x, z-x)dx = \int_{x=0}^{x=z} f_x(x) f_y(z-x)dx$$

$$= \int_{x=0}^{x=z} \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)} dx$$

$$= \lambda^2 e^{-\lambda z}\int_0^z dx = \lambda^2 e^{-\lambda z} \cdot x\Big|_0^z = \boxed{\lambda^2 z e^{-\lambda z},\ 0 < z < \infty}$$

$Z \sim Gamma(2, \lambda)$ ✓

② Sec 5.4 Triangular density

Let $\left.\begin{array}{l} X \sim \text{Unif}\{0,1,2,\dots,6\} \\ Y \sim \text{Unif}\{0,1,2,\dots,6\} \end{array}\right\}$ indep.

Find probability mass function of $Z = X+Y$

<u>$0 \le Z \le 6$</u>

$$P(Z=4) = \overbrace{P(0,4)}^{\left(\frac{1}{7}\right)^2} + \overbrace{P(1,3)}^{\left(\frac{1}{7}\right)^2} + P(2,2) + P(3,1) + P(4,0) = \frac{5}{49}$$

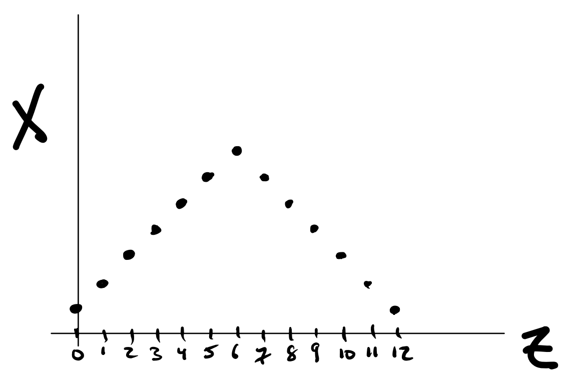
$$P(Z=z) = \sum_{x=0}^{z} P(X=x, Y=z-x)$$

<u>$7 \le Z \le 12$</u>

$$P(Z=8) = P(2,6) + P(3,5) + \cdots + P(6,2) = \frac{5}{49}$$

$$P(Z=z) = \sum_{x=z-6}^{6} P(x, z-x)$$

The distribution of $Z = X+Y$ looks like



Continuous case:

$$\left.\begin{array}{l} X \sim U(0,1) \\ Y \sim U(0,1) \end{array}\right\} \text{indep}$$

Find density of $Z = X + Y$

$$f_Z(z) = \int_{x=-\infty}^{x=\infty} f_{X,Y}(x, z-x)\, dx = \int_{x=-\infty}^{x=\infty} 1_{x\in(0,1)}\, 1_{z-x\in(0,1)}\, dx$$

range of values of Z? — 0-2

For $0 < z < 1$ $\quad$ x cant be too large since $z = x+y < 1$

$$f_Z(z) = \int_0^z 1\, dx = z$$

For $1 < z < 2$

x cant be too small since $x + y = z > 1$

ex $z = 1.2$ $\quad$ x cant be smaller than $.2$

$z = 1.7$ $\quad$ x " " " " $.7$

$1 < z < 2$ $\quad$ x " " " " $z - 1$

$$f(z) = \int_{z-1}^{1} 1\,dt = 1 - (z-1) = 2 - z$$

So $\boxed{f_Z(z) = \begin{cases} z & 0 \le z \le 1 \\ 2-z & 1 < z < 2 \end{cases}}$