Stat 134 Lec 8

warmup

One day, the Stanford TAs are feeling lazy, so they decide to implement an alternative method for grading quizzes. For each student, the TA will roll 10 dice. For every "good" die that shows a number ≥ 3, the student gets 10 points on their quiz. There are 30 students in the class.

(a) What distribution is the random variable $X=$ the number of "good dice" for a student?

(b) Let N be the number of students who get at least 90 points on the quiz. What is $\mathbb{E}[N]$?

(a) If X is the number of "good dice" for a student, X will have Binomial$(10, 2/3)$ distribution.

(b) Let $I_1, \ldots, I_{30}$ be the indicators of each student's score being ≥ 90; that is,

$$I_k = \begin{cases} 1 & \text{if student } k \text{ gets} \geq 90 \text{ points} \\ 0 & \text{if student } k \text{ gets} < 90 \text{ points.} \end{cases}$$

Then $N = I_1 + \cdots + I_{30}$, and

$$\begin{aligned} \mathbb{E}[N] &= \mathbb{E}[I_1] + \cdots + \mathbb{E}[I_{30}] \\ &= 30\mathbb{E}[I_1] \\ &= 30\mathbb{P}(\text{student } k \text{ gets} \geq 90 \text{ points}) \\ &= 30\mathbb{P}(9 \text{ or } 10 \text{ dice are "good"}) \\ &= 30\left(\binom{10}{9}(2/3)^9(1/3) + \binom{10}{10}(2/3)^{10}\right). \end{aligned}$$

Last time sec 3.2 Expectation

$$E(X) = \sum_{x \in X} x P(X=x)$$

If X is a count, X can be written as a sum of indicators

$$X = I_1 + I_2 + \cdots + I_n \,, \quad I_j = \begin{cases} 1 & \text{prob } p \\ 0 & \text{prob } 1-p \end{cases} \quad 1 \le j \le n$$

$$E(I_j) = 1 \cdot p + 0 \cdot (1-p) = p.$$

Idea Even if indicators are dependent the expectation of each indicator is an unconditional probability.

Try choosing indicators such that all indicators have the same expectation p.

Then $E(X) = n \cdot p$

we proved if $X \sim Bin(n,p) \Rightarrow E(X) = np$

if $X \sim HG(n,N,G) \Rightarrow E(X) = n\frac{G}{N}$

ex X = # aces in a poker hand from a deck of cards

$X \sim HG(n,N,G)$ $N = 52$, $G = 4$, $n = 5$

$$X = I_1 + I_2 + I_3 + I_4 + I_5$$

$$I_2 = \begin{cases} 1 & \text{if 2nd card is an ace} \\ 0 & \text{else} \end{cases} \quad p = 4/52$$

$$E(X) = 5 \cdot E(I_1) = \boxed{5 \cdot \left(\frac{4}{52}\right)}$$

Today

(1) sec 3.2 More expectation with indicator examples

(2) sec 3.2 tail sum formula

Sec 3.2

More expectation / indicator examples

ex Consider a 5 card deck consisting of 2, 2, 3, 4, 5.
shuffle the cards.
Let X = number of cards before the first 2.

a) What are the range of values of X?

$0, 1, 2, 3$

b) Write X as a sum of indicators

$X = I_3 + I_4 + I_5$

c) How is an indicator defined.

P

$$I_3 = \begin{cases} 1 & \text{if 3 before first 2} \\ 0 & \text{else} \end{cases}$$

d) Find $E(I_3)$

Notice the position of 4 and 5 is irrelevant.

_ 2 _ 2 _

slot can be empty or have a 3 in it.

3 places you can put 3

so $\boxed{P = 1/3}$

e) Find $E(X)$

$$E(X) = \underbrace{E(I_3)}_{1/3} + \underbrace{E(I_4)}_{1/3} + \underbrace{E(I_5)}_{1/3} = \boxed{1}$$

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Stat 134

1. Consider a well shuffled deck of cards. The expected number of cards before the first ace is?

   a 52/5

   b 48/5

   c 48/4

   d none of the above

$X = I_1 + I_2 + \dots + I_{48}$

P

$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ nonace before } 1^{st} \text{ ace} \\ 0 & \text{else} \end{cases}$

$\_\; A_1 \;\_\; A_2 \;\_\; A_3 \;\_\; A_4 \;\_$

$p = 1/5$ since 5 equally likely slots the second nonace can go,

$$\boxed{E(X) = 48(1/5)}$$

③ Sec 3.2 Tail Sum formula for expectation

Suppose X is a count $0, 1, 2, 3, \dots$

$$E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2) + \cdots$$

$$= \underbrace{1 \cdot P(X=1)}_{P_1} + \underbrace{2 \cdot P(X=2)}_{P_2} + \cdots$$

$$= \begin{array}{cccc} \overbrace{\begin{array}{c} P_1 \\ P_2 \\ P_3 \\ P_4 \\ \vdots \end{array}}^{P(X\ge 1)} & \overbrace{\begin{array}{c} \\ P_2 \\ P_3 \\ P_4 \\ \vdots \end{array}}^{P(X\ge 2)} & \overbrace{\begin{array}{c} \\ \\ P_3 \\ P_4 \\ \vdots \end{array}}^{P(X\ge 3)} & \begin{array}{c} \\ \\ \\ P_4 \\ \vdots \end{array} \end{array}$$

$$= P(X\ge 1) + P(X\ge 2) + P(X\ge 3) + \cdots \quad \text{Tail Sum Formula}$$

This is useful when it is easy to find $P(X \ge k)$.

ex A fair die is rolled 10 times.
Let $X = \max(X_1, \dots, X_{10})$.
Find $E(X)$

$$P(X \ge k) = 1 - P(X < k)$$

$x_2 \quad x_4 \quad x_1 \quad x_3 \quad \max(x_1, x_2, \dots x_{10}) \quad k$

$$= 1 - P(X_1 < k, X_2 < k, \dots, X_{10} < k)$$

$$= 1 - P(X_1 < k)P(X_2 < k)\cdots P(X_{10} < k)$$

$$= 1 - P(X_1 < k)^{10}$$

$$= 1 - \left(\frac{k-1}{6}\right)^{10} \quad \text{for } 1 \le k \le 6$$

use $P(X \ge k) = 0$ if $k > 6$

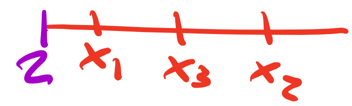
so
$$E(X) = \underset{1}{\underset{\shortparallel}{P(X\ge 1)}} + \underset{1-\left(\frac{1}{6}\right)^{10}}{\underset{\shortparallel}{P(X\ge 2)}} + \cdots + \underset{1-\left(\frac{5}{6}\right)^{10}}{\underset{\shortparallel}{P(X\ge 6)}} + \underset{0}{\underset{\shortparallel}{P(X\ge 7)}} + \cdots$$

$$= 6 - \left(\frac{1}{6}\right)^{10}\left[1^{10} + 2^{10} + 3^{10} + 4^{10} + 5^{10}\right] = 5.82$$

ex A fair die is rolled 3 times, X_1, X_2, X_3.

a) Find $P(\min(X_1,X_2,X_3) \ge 2)$

Picture



$$= P(X_1 \ge 2,\ X_2 \ge 2,\ X_3 \ge 2)$$

$$= P(X_1 \ge 2)^3 = \boxed{\left(\frac{5}{6}\right)^3}$$

(b) Find $E(\min(X_1,X_2,X_3))$

$$P(\min \ge 1) + P(\min \ge 2) + \dots + P(\min \ge 6)$$

$$\overset{\shortparallel}{P(X_1 \ge 1)^3} \qquad \overset{\shortparallel}{(X_1 \ge 2)^3} \qquad \overset{\shortparallel}{(X_1 \ge 6)^3}$$

$$= 1^3 + \left(\frac{5}{6}\right)^3 + \left(\frac{4}{6}\right)^3 + \dots + \left(\frac{1}{6}\right)^3 = \frac{1}{6^3}\left[6^3 + 5^3 + \dots + 1^3\right]$$

Let Y be the sum of the largest 2 numbers.

Notice that $Y = X_1 + X_2 + X_3 - \min(X_1,X_2,X_3)$

(c) Find $E(Y) = E(X_1) + E(X_2) + E(X_3) - E(\min)$

$$E(X_1) = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} + \dots + 6\cdot\frac{1}{6} = \frac{7}{2}$$

$$\boxed{E(Y) = 3\cdot\frac{7}{2} - \frac{1}{6^3}\left[6^3 + 5^3 + \dots + 1^3\right]}$$

extra problems

. (3 pts) On a telephone wire, n birds sit arranged in a line. A noise startles them, causing each bird to look left or right at random. Calculate the expected number of birds which are not seen by an adjacent bird.

X = # birds not seen by an adjacent bird

$X = I_1 + I_2 + \dots + I_n$

$P = 1/2$

$I_1 = \begin{cases} 1 & \text{if } 1^{st} \text{ bird not seen} \\ 0 & \text{else} \end{cases}$

$P = 1/2 \cdot 1/2 = 1/4$

$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ bird not seen} \\ 0 & \text{else} \end{cases}$

$$\boxed{E(X) = 2 \cdot 1/2 + (n-2) \cdot 1/4}$$