Stat 134 lec 41 (final review pt 1)

warmup

Let the joint (X, Y) be standard bivariate normal distribution with correlation $\rho = 1/2$. Which of the following statements is true? Mark all that apply.

i. X and Y are independent.

ii. $P(X < 0) = 1/2$.

iii. $P(X > 0, Y > 0) = 1/4$. — we saw last time

$P(X>0, Y>0) = P(X>0, \rho X + \sqrt{1-\rho^2}\, Z > 0)$

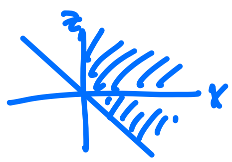
$= P(X>0, Z > -\frac{\rho}{\sqrt{1-\rho^2}} X)$

$> 1/4$

iv. X and Y are identically distributed.

v. $2X + 3Y \sim N(0, 19)$

vi. None of the above.



ii & iv & v are correct. Since the correlation is nonzero, X and Y cannot be independent. Since X is a standard normal it is symmetric. the probability is 1/4 only if $\rho = 0$ and X and Y are both standard normal so have the same distribution. 2X+3Y is normal with expectation 0 and variance= $4Var(X) + 9Var(Y) + 2Cov(2X, 3Y) = 4 + 9 + 2*2*3*\frac{1}{2} = 19$

Announcement:

Tutor Wille will run a post MT2 review session Wed.

There will be another in-class review session Fri.

Def^n (Standard Bivariate Normal Distribution)

let X, Z iid $N(0,1)$, $-1 \le \rho \le 1$

$Y = \rho X + \sqrt{1-\rho^2}\, Z \sim N(0,1)$

$\text{Corr}(X,Y) = \rho$

We call the joint distribution (X,Y) the standard bivariate normal with $\text{corr}(X,Y) = \rho$

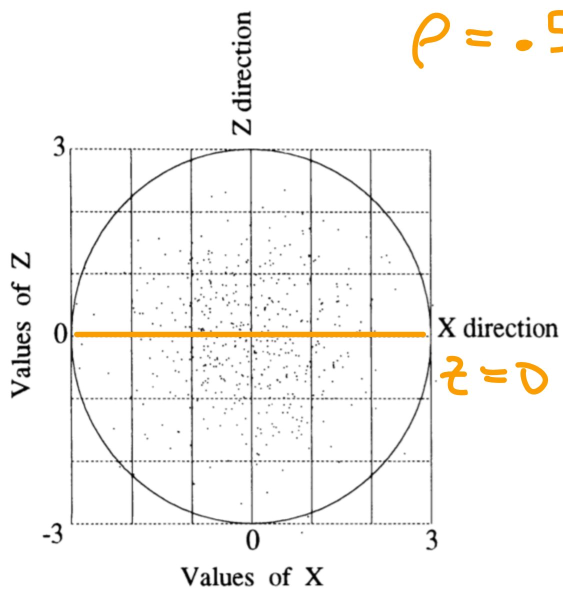
written $(X,Y) \sim BVN(0, 0, 1, 1, \rho)$

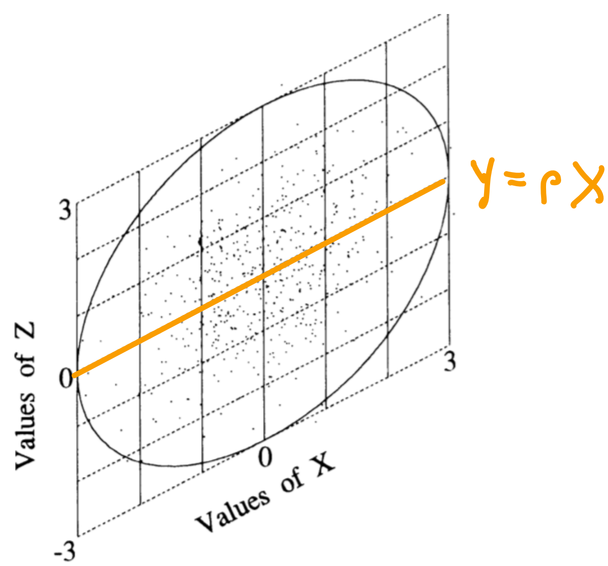
$\mu_X \quad \mu_Y \quad \sigma^2_X \quad \sigma^2_Y$

$(1,0) \to (1, \tfrac{1}{2})$

$(X, Z) \longrightarrow (X, Y) = (X, \rho X + \sqrt{1-\rho^2}\, Z)$

$\rho = .5$





Main properties

① (X,Y) std BVN iff $aX + bY \sim N(0, a^2 + b^2 + 2ab\rho)$

(2) X, Y inden $\Rightarrow$ $Cov(X,Y) = 0$ always

If (X,Y) BVN,

X, Y indep $\Leftrightarrow$ $Cov(X,Y) = 0$

(10 pts) (BVN - Adam) Let A and B represent the cost of a television in states A and B, respectively, with $A \sim N(\mu_A, \sigma_A^2), B \sim N(\mu_B, \sigma_B^2), \mathbb{E}[AB] = \mu_{AB}$. Furthermore, the joint distribution of A and B is bivariate normal.

(a) (4 pts) Find the correlation between A and B.

(b) (6 pts) Calculate $P(A > 2B)$. You may leave your answer in terms of Φ (the cdf of the standard normal distribution).

(a) $Corr(A,B) = \dfrac{Cov(A,B)}{SD(A)SD(B)}$

$= \dfrac{E[AB] - E[A]E[B]}{\sqrt{Var(A)Var(B)}}$

$= \dfrac{\mu_{AB} - \mu_A\mu_B}{\sigma_A\sigma_B}$

(b) $P(A > 2B) = P(A - 2B > 0)$

For convenience, define $C = A - 2B$.

By properties of bivariate normal, we know that C is normally distributed.

$\mathbb{E}[C] = E[A - 2B] = \mu_A - 2\mu_B$

$Var(C) = Var(A) + Var(-2B) + 2Cov(A, -2B) = \sigma_A^2 + 4\sigma_B^2 - 4(\mu_{AB} - \mu_A\mu_B)$

Therefore $P(A > 2B) = P(C > 0) = 1 - P(C < 0) = 1 - \Phi(\dfrac{0 - \mathbb{E}[C]}{\sqrt{Var(C)}})$

$= 1 - \Phi(\dfrac{-(\mu_A - 2\mu_B)}{\sqrt{\sigma_A^2 + 4\sigma_B^2 - 4(\mu_{AB} - \mu_A\mu_B)}})$

Conditional expectation

12. Let N have the Poisson distribution with mean μ. Let $U_1, U_2, \dots$ be independent uniform $(0,1)$ variables, independent of N.

let $M|N = \min(U_1, U_2, \dots, U_N)$. If $N = 0$, define M to be 1.

a) Find $E(M|N)$.

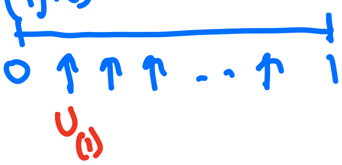
b) Find $E(M)$.

c) Find the survival function of M. $P(M>m)$

d) Sketch the c.d.f. of M. and calculate $E(M)$ using the CDF.

a)

$$M|N = U_{(1)} \sim \text{Beta}(1,N)$$



$$E(\text{Beta}(r,s)) = \frac{r}{r+s}$$

$$\boxed{E(U_{(1)}) = \frac{1}{1+N}}$$

b)

$$E(M) = E\left(E(M|N)\right) = E\left(\frac{1}{1+N}\right)$$

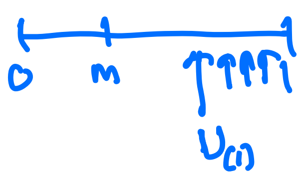
$$E(g(N)) = \sum_{n=0}^{\infty} g(n)P(N=n)$$

$$= \sum_{n=0}^{\infty} \frac{1}{1+n} \cdot \frac{e^{-\mu}\mu^{n}}{n!}$$

$$= \frac{1}{\mu}\sum_{n=0}^{\infty} \frac{e^{-\mu}\mu^{n+1}}{(n+1)!}$$

$$= \frac{1}{\mu}\sum_{n=1}^{\infty} \frac{e^{-\mu}\mu^{n}}{n!} = 1 - \frac{e^{-\mu}\mu^{0}}{0!}$$

$$= \boxed{\frac{1}{\mu}\left(1 - e^{-\mu}\right)}$$



c) $$P(M>m) = \sum_{n=0}^{\infty} \underbrace{P(M>m|N=n)}_{\substack{= P(U_1>m)P(U_2>m)\cdots P(U_n>m) \\ = P(U_1>m)^n \\ = (1-m)^n}} P(N=n)$$

rule of avg cond. prob.

$$= \underbrace{P(M>m|N=0)}_{=1}\underbrace{P(N=0)}_{=e^{-\mu}} + \sum_{n=1}^{\infty} (1-m)^n e^{-m\mu} e^{m\mu} \frac{e^{-\mu}\mu^n}{n!}$$

skip $N>0$

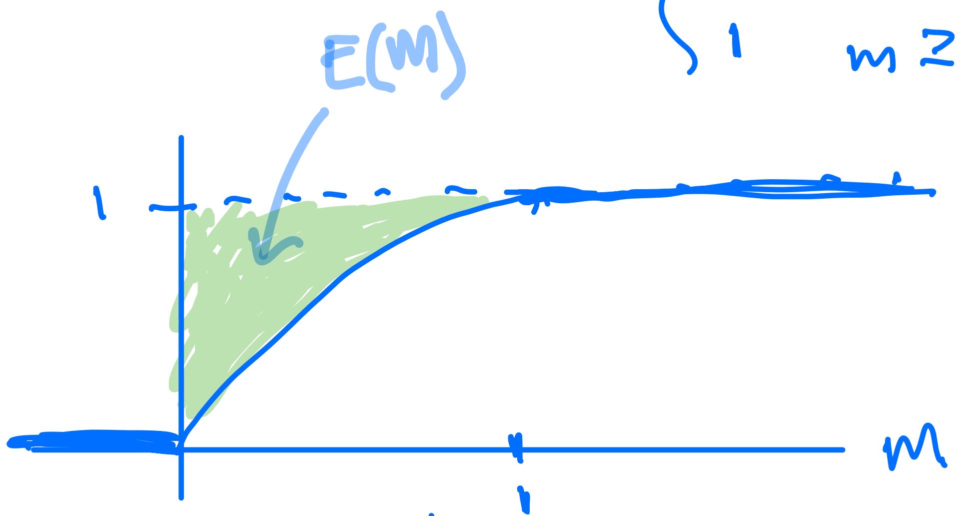
$$e^{-\mu+m\mu} = e^{-\mu(1-m)}$$

$$= e^{-\mu} + e^{-m\mu} \underbrace{\sum_{n=1}^{\infty} \frac{e^{-\mu(1-m)}\left[\mu(1-m)\right]^n}{n!}}_{=1-e^{-\mu(1-m)}}$$

$$= e^{-\mu} + e^{-m\mu}\left(1-e^{-\mu(1-m)}\right) = \boxed{e^{-\mu m}}$$

$$P(M>m) = \begin{cases} 1 & m<0 \\ e^{-\mu m} & 0 \le m < 1 \\ 0 & m \ge 1 \end{cases}$$

d) $F(m) = P(M \le m) = \begin{cases} 0 & m < 0 \\ 1 - e^{-\mu m} & 0 \le m < 1 \\ 1 & m \ge 1 \end{cases}$



$$E(M) = \int_{m=0}^{m=1} (1 - F(m))\,dm$$

$$= \int_0^1 e^{-\mu m}\,dm$$

$$= \left.\frac{e^{-\mu m}}{-\mu}\right|_{m=0}^{m=1}$$

$$= \frac{e^{-\mu}}{-\mu} - \frac{1}{-\mu} = \boxed{\frac{1 - e^{-\mu}}{\mu}}$$

Extra

Suppose that, conditionally on $N = n$ and $P = p$, B has a Binomial(n, p) distribution. Additionally, suppose P has a (continuous) uniform distribution on the interval $[0, \frac{1}{N}]$, and N is a random variable taking values in $\{1, 2, \dots\}$.

Calculate $\mathbb{E}[B]$.

$B \mid N=n, P=p \sim \text{Bin}(n, p)$

$B \mid N, P=p \sim \text{Bin}(N, p)$

$B \mid N=n, P \sim \text{Bin}(n, P)$

$$E_P E_B(B \mid N=n, P) = E_P(nP)$$
$$= n\underbrace{E(P)}_{\frac{1}{2n}}$$

Don't use $E_N E_B(B \mid N, P=p) = E(Np)$ since we don't know $E(N)$.