Stat 134 lec 7

Warmup

The joint distribution of X and Y is drawn below:

	3/8	1/2	1/8	P(x) / P(y)
1	1/4	1/3	1/12	2/3
0	1/8	1/6	1/24	1/3
Y / X	0	1	2	

check $P(x,y) = P(x)P(y)$ for all cells

ex $1/4 = 3/8 \cdot \frac{2}{3}$

a) X and Y are independent ✓

b) If we divide both rows by their marginal probability we get the same answer.

||

c) $P(X = x|Y = 0) = P(X = x|Y = 1)$ ✓

d) All of the above

note that

$$P(X|y=2/3) = \frac{P(X, Y=2/3)}{P(Y=2/3)}$$

by Bayes' rule

so b) and c) are equivalent

Last time

Sec 3.1 Random Variables

The event $(X=x, Y=y)$ is the intersection of events $X=x$ and $Y=y$. sometimes written (x,y)

$P(X+Y=s)$ is the sum of $P(X=x, Y=y)$ for all x,y such that $x+y=s$

i.e $$P(X+Y=s) = \sum_{(x,y):\, x+y=s} P(x,y) = \sum_{\text{all } x} P(x, s-x)$$

Independence of (X,Y,Z) means

$P(X=x, Y=y, Z=z) = P(X=x)P(Y=y)P(Z=z)$ for all $x \in X$, $y \in Y$, $z \in Z$.

Today

① Sec 3.1 Sums of independent Poissons is Poisson

② Sec 3.2 Expectation of a RV.

① Sum of independent Poisson is Poisson

informal arguement:

$X_1 \sim Bin(1000, \frac{1}{1000}) \approx Pois(1)$

$X_2 \sim Bin(2000, \frac{1}{1000}) \approx Pois(2)$

indep

$X_1 + X_2 \sim \ ?$ $Bin(3000, 1/1000) \approx Pois(3)$

heads in 3000 coin tosses of a $p = 1/1000$ coin

Proven in appendix to these notes

Claim If $X \sim Pois(\mu)$ and $Y \sim Pois(\lambda)$ are independent then

$S = X + Y \sim Pois(\mu + \lambda)$.

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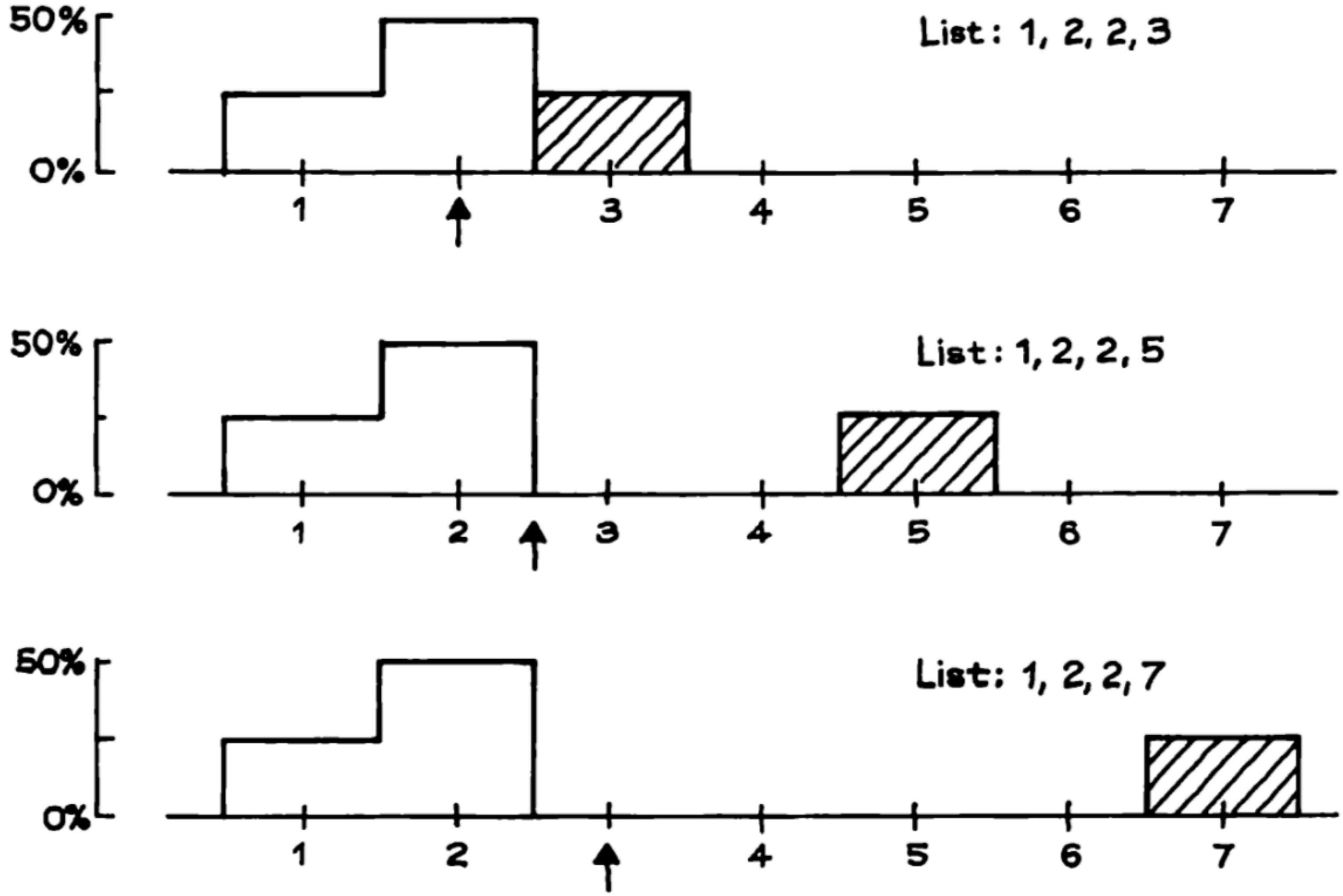
Sec 3.2 Expectation

$$E(X) = \sum_{x \in X} x \cdot P(X=x)$$

$$E(X) = 1 \cdot \tfrac{1}{4} + 2 \cdot \tfrac{1}{2} + 3 \cdot \tfrac{1}{4} = 2$$



$$E(X) = \sum_{x \in X} x \cdot P(X=x)$$

Properties of Expectation — P167 Pitman

① $E(c) = c$

② $E(X+Y) = E(X) + E(Y)$ (X, Y <u>dont</u> need to be independent)

③ $E(aX+b) = aE(X) + b$

Indicators

An indicator is a RV that has only 2 values 1 (w/ prob p) and 0 (with prob 1-p).

$$I = \begin{cases} 1 & \text{prob } p \\ 0 & \text{prob } 1-p \end{cases}$$

— same as a Bernoulli p trial.

$$E(I) = 1 \cdot p + 0 \cdot (1-p) = p$$

RV that are counts can often be written as a sum of indicators.

ex $X \sim Bin(n, p)$

successes in n Bernoulli p trials.

ex X = # heads in n flips of p coin

$X = I_1 + I_2 + \dots + I_n$

where $I_j = \begin{cases} 1 & \text{if } j^{th} \text{ trial success} \\ 0 & \text{else} \end{cases}$ — p

$$E(X) = \underset{p}{\underset{\shortparallel}{E(I_1)}} + \dots + \underset{p}{\underset{\shortparallel}{E(I_n)}} = \boxed{np}$$

indicators are independent since trials are indep.

Ex X = # aces in a poker hand from a deck of cards

$X \sim HG(n=5\ N=52\ G=4)$

a) What are the range of values of X?

0, 1, 2, 3, 4

b) Write X as a sum of indicators

$X = I_1 + I_2 + I_3 + I_4 + I_5$

c) How is I_2 defined?

$P = 4/52 = 1/13$

$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ card is an ace} \\ 0 & \text{else} \end{cases}$

d) Find $E(I_2) = 1/13$

e) Find $E(X) = E(I_1) + E(I_2) + E(I_3) + E(I_4) + E(I_5)$

$E(I_1) = 1/13$, $E(I_2) = 1/13$, $E(I_5) = 1/13$

$= \boxed{5/13}$

Another more complicated solution:

Note

You may define $I_2 = \begin{cases} 1 & \text{if get 2 aces} \\ 0 & \text{else} \end{cases}$

so

$$X = I_1 + 2I_2 + 3I_3 + 4I_4$$

This is also correct but more complicated.

$$E(I_1) = \frac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}} \qquad E(I_3) = \frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}}$$

$$E(I_2) = \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}} \qquad E(I_4) = \frac{\binom{4}{4}\binom{48}{1}}{\binom{52}{5}}$$

so $E(X) = \frac{1}{\binom{52}{5}} \left[\binom{4}{1}\binom{48}{4} + 2 \cdot \binom{4}{2}\binom{48}{3} + 3 \cdot \binom{4}{3}\binom{48}{2} + 4\binom{4}{4}\binom{48}{1} \right]$

$= 5 \cdot \left(\frac{4}{52}\right)$ ← I checked this in R

Note $E(X) = 1 \cdot P(X=1) + 2 \cdot P(X=2) + 3 \cdot P(X=3) + 4 \cdot P(X=4)$

ex

A drawer contains s black socks and s white socks (s> 0). I pull two socks out at random without replacement and call that my first pair. Then I pull out two socks at random without replacement and call that my second pair. I proceed in this way until I have s pairs and the drawer is empty. Find the expected number of pairs in which two socks are different colors.

$X = \#$ of mismatched pairs (out of s)

$I_2 = \begin{cases} 1 & \text{if the second pair is mismatch} \\ 0 & \text{else} \end{cases}$

$P = \frac{2s}{2s} \cdot \frac{s}{2s-1}$

1st sock, 2nd sock

or $P = \frac{\binom{s}{1}\binom{s}{1}}{\binom{2s}{2}}$

$X = I_1 + \cdots + I_s$

$$\boxed{E(X) = s \cdot \frac{\binom{s}{1}\binom{s}{1}}{\binom{2s}{2}}}$$

Appendix

__Claim__ If $X \sim Pois(\mu)$ and $Y \sim Pois(\lambda)$ are independent then

$$S = X + Y \sim Pois(\mu + \lambda).$$

To prove this you need to know 2 facts:

__Recall__ binomial theorem

$$(a+b)^3 = \binom{3}{3} a^3 b^0 + \binom{3}{2} a^2 b^1 + \binom{3}{1} a^1 b^2 + \binom{3}{0} a^0 b^3$$
$$= a^3 + 3a^2 b + 3ab^2 + b^3$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

__Recall__ $X \sim Pois(\mu)$

$$P(X=k) = \frac{e^{-\mu} \mu^k}{k!}$$

Pf/

addition rule

$$P(S=s) = P(X=0, Y=s) + P(X=1, Y=s-1) + \dots P(X=s, Y=0)$$

summation notation

$$= \sum_{k=0}^{s} P(X=k, Y=s-k)$$

independence of X and Y

$$= \sum_{k=0}^{s} P(X=k) P(Y=s-k)$$

Poisson formula

$$= \sum_{k=0}^{s} \frac{e^{-\mu}\mu^{k}}{k!} \cdot \frac{e^{-\lambda}\lambda^{s-k}}{(s-k)!}$$

$\frac{s!}{s!} = 1$

$$= e^{-(\lambda+\mu)} \frac{1}{s!} \sum_{k=0}^{s} \frac{s!}{k!(s-k)!} \mu^{k} \lambda^{s-k}$$

binomial thm

$$= e^{-(\lambda+\mu)} \frac{1}{s!} (\mu+\lambda)^{s}$$

$$\Rightarrow S \sim Pois(\mu+\lambda).$$

Poisson formula.

$\square$