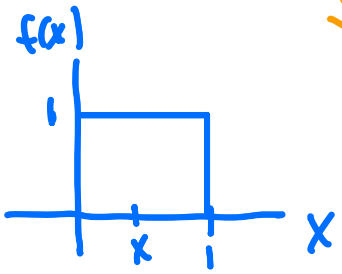
warmup

Let $X \sim Unif(0,1)$

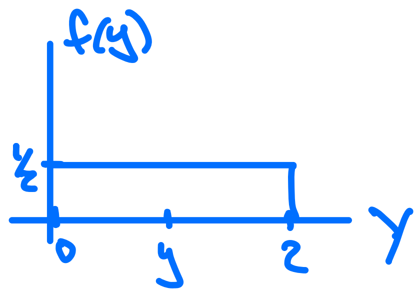
Let $Y = 2X$

What is the density of Y?

Hint: Draw a picture of the density of X and what stretching X does.



$y = 2X$



$$\Rightarrow \boxed{f_Y(y) = \begin{cases} 1/2 & \text{if } 0 \le y \le 2 \\ 0 & \text{else} \end{cases}}$$

Another way to think about this:

$$x \in dx \iff y \in dy$$

$$\Rightarrow P(x \in dx) = P(y \in dy)$$

$$\underset{f_X(x)dx}{\shortparallel} \qquad \underset{f_Y(y)dy}{\shortparallel}$$

$$\Rightarrow f_Y(y) = \frac{dx}{dy} f_X(x)$$

$$x = \frac{y}{2} \Rightarrow \frac{dx}{dy} = 1/2$$

$$\Rightarrow f_Y(y) = \frac{1}{2} f_X(x) = 1/2 \cdot (1 \text{ if } x \in (0,1]) = \boxed{\frac{1}{2} \text{ if } y \in [0,2]}$$

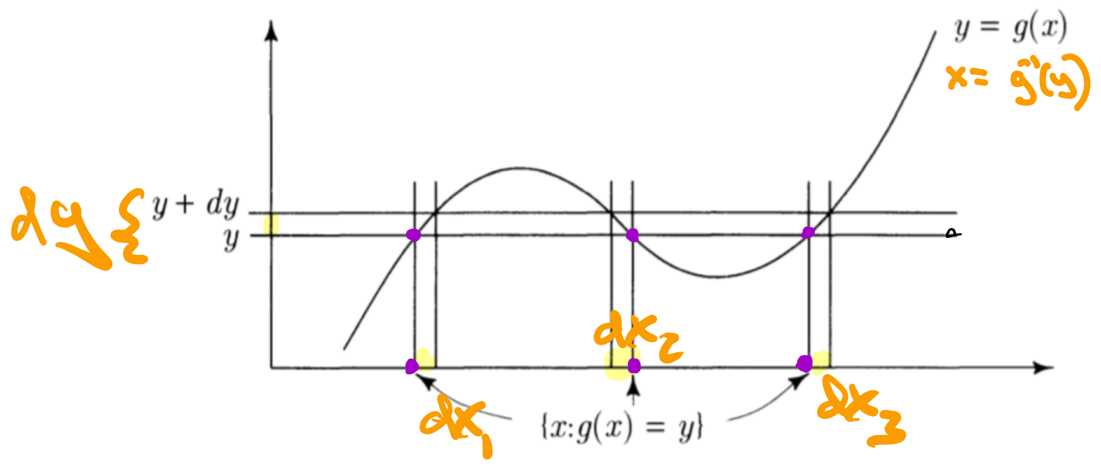
Announcement: Wednesday (Dec 21) is a special lecture on moment generating functions (not in textbook).

<u>Today</u> Sec 4.4 (skip 4.3)

① Change of variable formula for densities.

Sec 4.4 Change of Variable formula for densities

if X has density $f_X(x)$ lets find the density of $Y=g(X)$



mutually exclusive events.

$Y \in dy$ iff $X \in dx_1$ or $X \in dx_2$ or $X \in dx_3$

$$P(Y \in dy) = P(X \in dx_1) + P(X \in dx_2) + P(X \in dx_3)$$

$$f_Y(y)\,dy = f_X(x_1)dx_1 + f_X(x_2)dx_2 + f_X(x_3)dx_3$$

$$f_Y(y) = f_X(x_1)\frac{dx_1}{dy} + f_X(x_2)\left|\frac{dx_2}{dy}\right| + f_X(x_3)\frac{dx_3}{dy}$$ evaluated at $x = g^{-1}(y)$

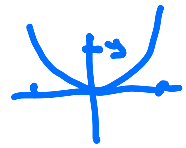
You may see it written as:

$$= \frac{f_X(x_1)}{g'(x_1)} + \frac{f_X(x_2)}{|g'(x_2)|} + \frac{f_X(x_3)}{g'(x_3)}$$ evaluated at $x = g^{-1}(y)$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{g'(x)}$$

Thm (P 307) Change of Variable Formula for densities

Let X be a continuous RV with density $f_X(x)$.



Let $Y = g(X)$ have a derivative that is zero at only finitely many pts,

$y = x^2$

$x = \pm\sqrt{y}$

then $$f_Y(y) = \sum_{\{x \mid g(x)=y\}} \left|\frac{dx}{dy}\right| f_X(x)\Big|_{x = g^{-1}(y)}$$

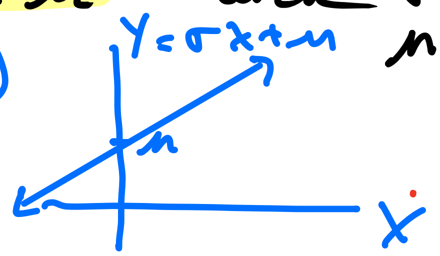
replace x with $g^{-1}(y)$

ex

let $X = N(0,1)$, $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$*$ — we will show later in the semester that this is a density

Find the density of $Y = \sigma X + \mu$ where $\sigma > 0$, $\mu \in \mathbb{R}$



Steps

1) Range of Y $(-\infty, \infty)$

2) Find $X = \frac{Y-\mu}{\sigma}$

3) Find $\frac{dx}{dy} = \frac{1}{\sigma}$

4) Find $f_Y(y) = \frac{dx}{dy} \cdot f_X(x)\Big|_{x = \frac{y-\mu}{\sigma}}$

$$= \boxed{\frac{1}{\sigma} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\frac{y-\mu}{\sigma}\right)^2}{2}}}$$

Note this is the density of $N(\mu, \sigma^2)$.

tinyurl.com/mar10-2023

$f_X(x) = 1_{x \in [0,1]}$

Stat 134

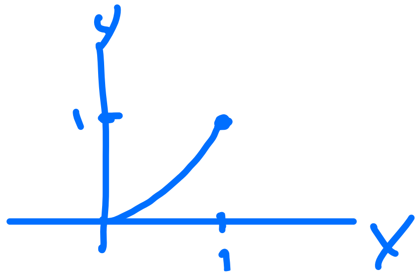
1. Let $X \sim Unif(0,1)$. The density of $Y = X^2$ is:

   a $f(y) = \frac{1}{\sqrt{y}}$ for $y \in [0,1]$, zero else.

   b $f(y) = \frac{1}{2\sqrt{y}}$ for $y \in [0,1]$, zero else.

   c $f(y) = 1$ for $y \in [0,1]$, zero else.

   d none of the above



range of Y is [0,1]

$x = \sqrt{y}$

$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$

$f_Y(y) = \frac{1}{2\sqrt{y}} \cdot 1_{x \in [0,1]}$

$x = \sqrt{y}$

$= \frac{1}{2\sqrt{y}} \cdot 1_{\sqrt{y} \in [0,1]}$

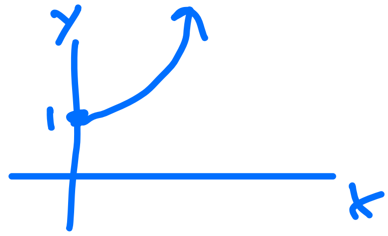
$= \boxed{\frac{1}{2\sqrt{y}} \cdot 1_{y \in [0,1]}}$

ex

(3 pts) Suppose the random variable X, which measures the magnitude of an earthquake (on the Richter scale) in the Bay Area, follows the Exponential (λ) distribution. Since the Richter scale is logarithmic, we want to study the distribution of the total energy of earthquakes. Find the distribution of $Y = e^X$.

$X \sim Exp(\lambda)$

X is $[0, \infty)$



Y takes values $[1, \infty)$

$X = \log Y$

$$\frac{dx}{dy} = \frac{1}{y}$$

$$f_Y(y) = \frac{1}{y} \cdot \lambda e^{-\lambda x} \Big|_{x = \log y}$$

$$= \frac{1}{y} \lambda e^{-\lambda \log y}$$

$-\lambda \log y = \log y^{-\lambda}$

$$= \frac{1}{y} \lambda e^{\log y^{-\lambda}} = \frac{1}{y} \lambda y^{-\lambda} = \boxed{\lambda y^{-\lambda - 1} \text{ for } y \in [1, \infty)}$$

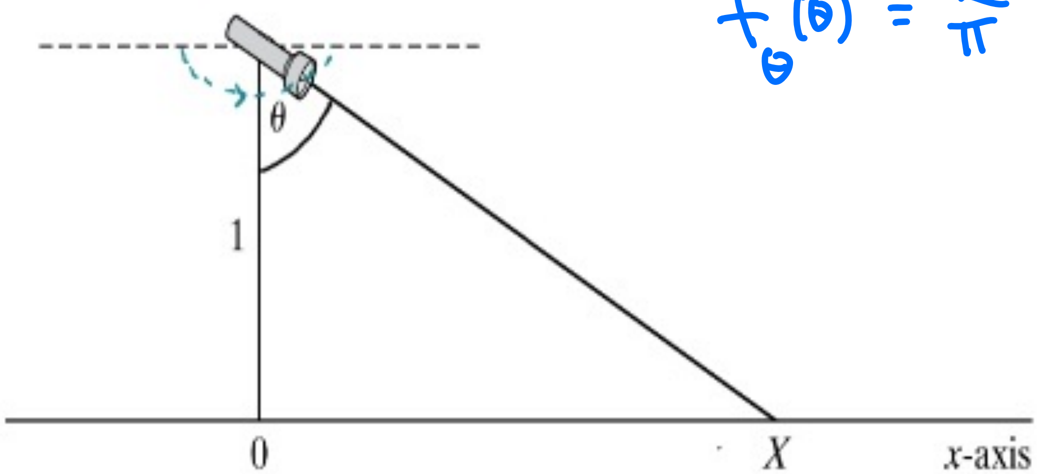
es

Suppose that a narrow-beam flashlight is spun around its center, which is located a unit distance from the x-axis. (See Figure 1) Consider the point X at which the beam intersects the x-axis when the flashlight has stopped spinning. (If the beam is not pointing toward the x-axis, repeat the experiment). As indicated in Figure 1, the point X is determined by the angle θ between the flashlight and the y-axis, which, from the physical situation, appears to be uniformly distributed between $-\pi/2$ and $\pi/2$. Show that the density of X is given by $f_X(x) = \frac{1}{\pi(1+x^2)}$, $x \in \mathbb{R}$.

Hint: Try and write X as a function of θ.

$\Theta \sim \mathrm{Unif}(-\pi/2, \pi/2)$

$f_\Theta(\theta) = \frac{1}{\pi} \mathbb{1}_{\theta \in [-\pi/2, \pi/2]}$



$X = \tan(\Theta)$

range of X is $(-\infty, \infty)$

$\Theta = \tan^{-1}(X)$

$\frac{d\theta}{dx} = \frac{1}{1+x^2}$

$f_X(x) = \frac{1}{1+x^2} \cdot \frac{1}{\pi} \mathbb{1}_{\theta \in [-\pi/2, \pi/2]} \Big|_{\theta = \tan^{-1} x}$

$= \frac{1}{1+x^2} \cdot \frac{1}{\pi} \cdot \mathbb{1}_{x \in (-\infty, \infty)}$

Extra

(5pts) Suppose a random variable U follows Uniform(0,1) distribution. Define $W = \log\left(\frac{U}{1-U}\right)$. Here log is log base e.

i. (2pt) Find the range of W.

ii. (3pts) Find the density of W.

i) $(-\infty, \infty)$ since $\frac{U}{1-U}$ takes all values between 0 and ∞

so $\log\left(\frac{U}{1-U}\right)$ takes all values between $-\infty$ and ∞.

ii) $e^{w} = \frac{U}{1-U}$

$e^{w} - e^{w}U = U$

$e^{w} = e^{w}U + U = (e^{w}+1)U$

$\Rightarrow U = \frac{e^{w}}{e^{w}+1}$

$\frac{dU}{dw} = \frac{e^{w}(e^{w}+1) - e^{w}\cdot e^{w}}{(e^{w}+1)^2}$

$= \frac{e^{w}[e^{w}+1-e^{w}]}{(e^{w}+1)^2} = \frac{e^{w}}{(e^{w}+1)^2}$

$$\boxed{\text{quotient rule} \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}}$$

$$f_W(w) = \frac{e^u}{(e^u+1)^2} \cdot 1_{u \in (0,1)}$$

$$= \boxed{\frac{e^w}{(e^w+1)^2} \cdot 1_{w \in (-\infty, \infty)}}$$