Stat 134 lec 23

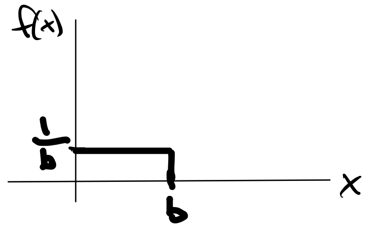
Warmup

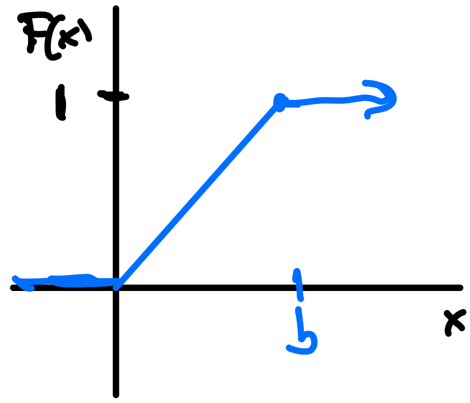
Recall that the cummulative distribution function (CDF) for a RV X is $F(x) = P(X \le x)$

Draw and give the equation of the CDF for each distributions below:

$$F(x) = \int_{-\infty}^{x} f(s)\,ds = \int_{0}^{x} \frac{1}{b}\,ds = \frac{x}{b}$$

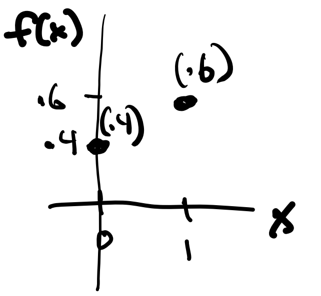
ex $X \sim \text{Unif}(0, b)$

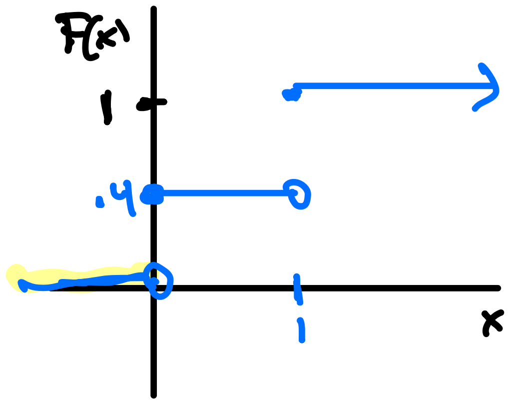




$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{b}x & 0 \le x < b \\ 1 & x \ge b \end{cases}$$

ex $X \sim \text{Bernoulli}(p = .6)$

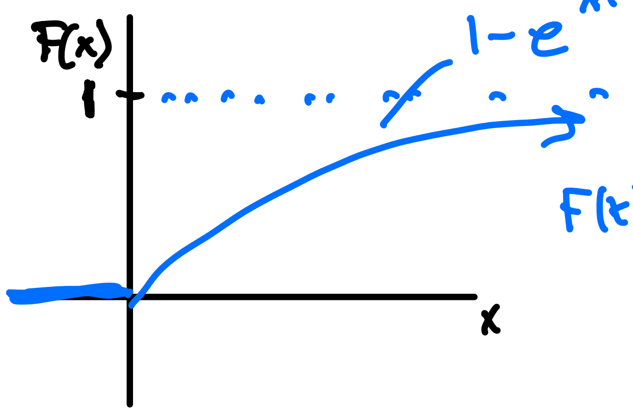




$$F(x) = \begin{cases} 0 & x < 0 \\ .4 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

ex $X \sim \text{Exp}(\lambda)$

$F(x) = 1 - e^{-\lambda x}$



$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \ge 0 \end{cases}$$

Today

1. sec 4.5 Find CDF of a mixed distribution
2. sec 4.5 Using CDF to find $E(X)$

① Sec 4.5

Let X be a continuous RV

$F(x) = P(X \le x)$ — a number between 0 and 1

If $f(x)$ is a density of X,

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

By FTC, $F'(x) = f(x)$

Consequently a density function and cdf are equivalent descriptions of a RV.

CDF of mixed distributions

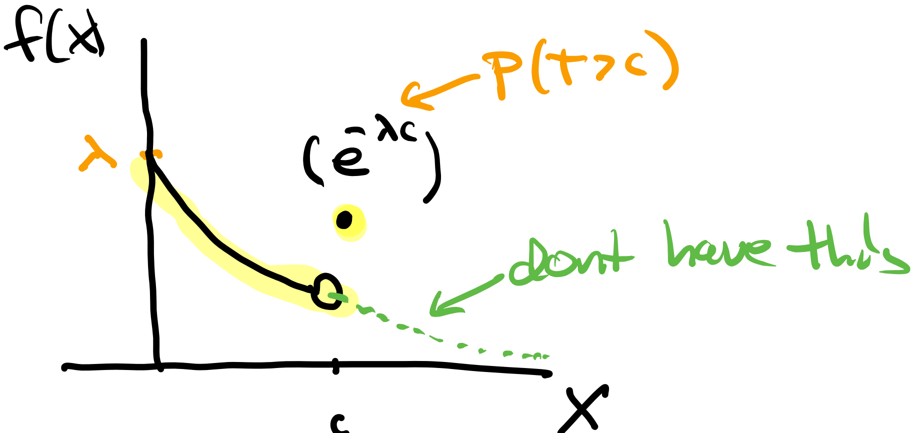
eg (Mixed distribution)

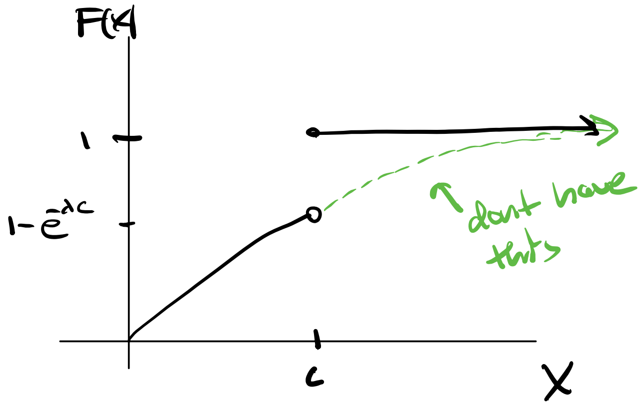
$T \sim Exp(\lambda)$

$c > 0$

$$X = \min(T, c) = \begin{cases} T & \text{if } x < c \\ c & \text{if } x = c \end{cases}$$

"T killed at c"



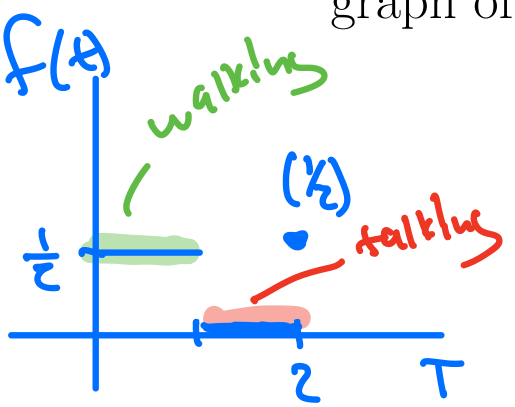


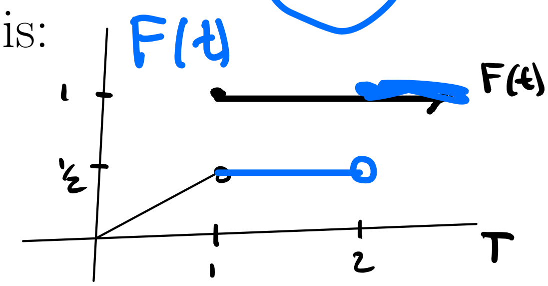
Find the CDF of $X = \min(T, c)$

$$F(x) = \begin{cases} 0 & x \le 0 \\ 1-e^{-\lambda x} & 0 < x < c \\ 1 & x \ge c \end{cases}$$

ex

Suppose you are trying to discretly leave a party. Your time to leave is uniform from 0 to 2 minutes. However, if your walk to the exit takes more than 1 minute, you run into a friend at the door and and must spend the full 2 minutes to leave. Let T represent the time it takes you to leave. True or false, the graph of the cdf of T is:



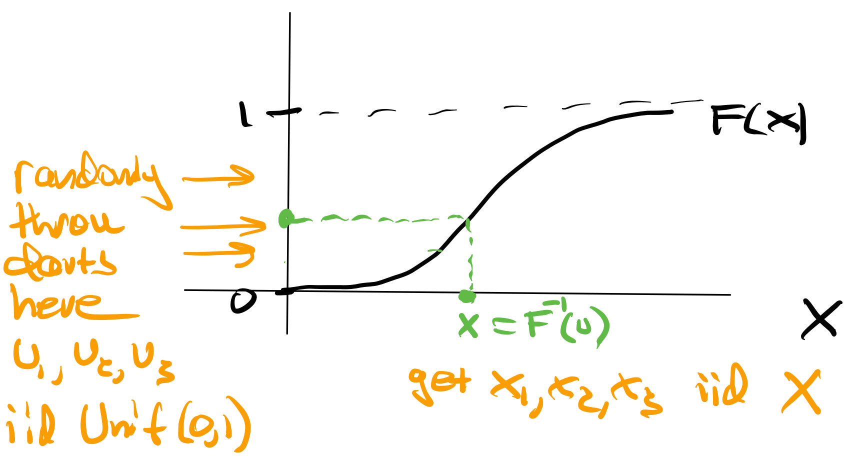


see #9 p324

(2) Sec 4.5 Using CDF to find $E(X)$ for $X \geq 0$

Inverse distribution function, $F^{-1}(u)$

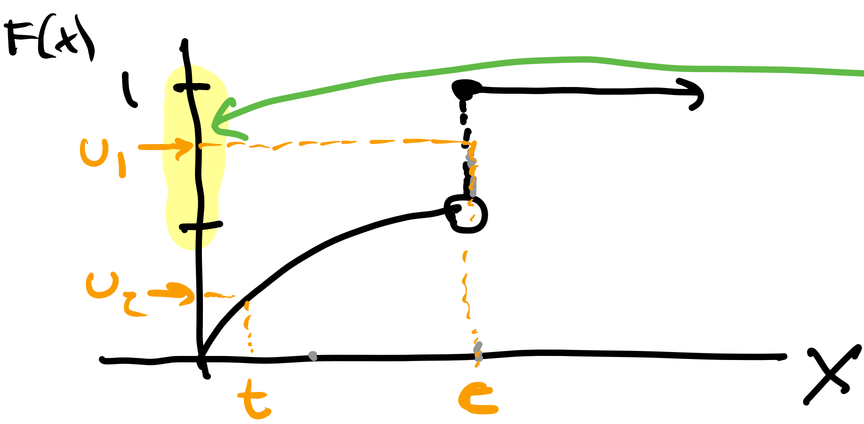
Let X have CDF $F(x)$.



$\boxed{F^{-1}(u) = x}$

Note: doesn't have to be continuous RV.

ex $X = \min(T, c)$, $T \sim Exp(\lambda)$

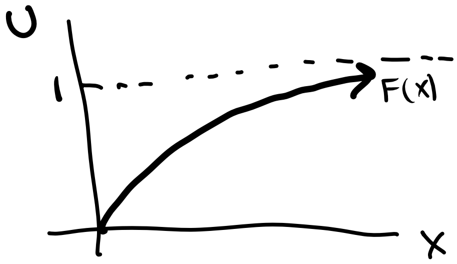


This yellow region gets assigned the single value c.

Thm (P322) — Proof at end of lecture.

Let X have CDF F.

Then the RV $F^{-1}(U) = X$

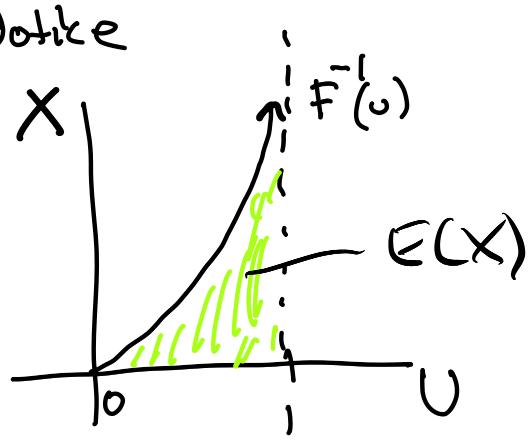


How is this useful to us finding $E(X)$?

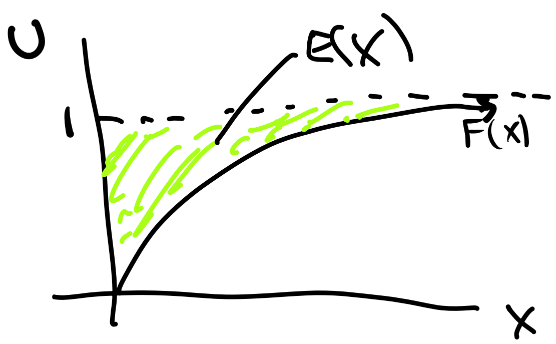
1 since $U \sim \text{Unif}(0,1)$

$$E(X) = E(F^{-1}(U)) = \int_0^1 F^{-1}(u) f_U(u)\, du$$

Notice



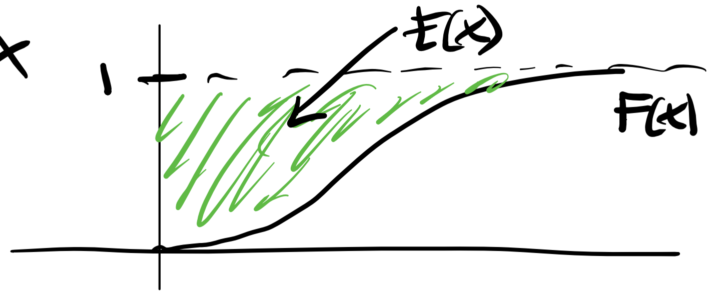
Now reflect the above graph about the diagonal $y=x$



We can find the shaded region by integrating $1 - F(x)$ with respect to x:

Thm Let X be a pos. random variable, with CDF F. (continuous, discrete, mixed),

$$E(X) = \int_0^\infty (1 - F(x))\,dx$$



ex

$$T \sim \text{expon}(\lambda)$$

$$F_T(t) = 1 - e^{-\lambda t}$$

Calculate $E(T)$.

$$E(T) = \int_0^\infty \underbrace{(1 - F(t))}_{1-(1-e^{-\lambda t})}\,dt = \int_0^\infty e^{-\lambda t}\,dt = -\frac{1}{\lambda} e^{-\lambda t}\Big|_0^\infty = \boxed{1/\lambda}$$

It is sometimes easier to calculate $E(X)$ using the cdf (avoid doing integration by parts): $E(T) = \int_0^\infty t\lambda e^{-\lambda t}\,dt$

tinyurl.com/mar17-2023

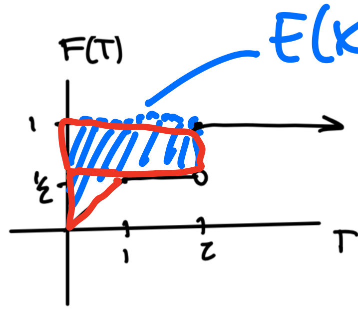
Stat 134

Friday October 21 2022

1. Suppose you are trying to discretely leave a party. Your time to leave is uniform from 0 to 2 minutes. However, if your walk to the exit takes more than 1 minute, you run into a friend at the door and must spend the full 2 minutes to leave. Let T represent the time it takes you to leave.

Find E(T)

$E(X) = 1 + 1/4 = \boxed{5/4}$



Your expected time to leave, $E(T)$, is:

a 0.5 min

b 0.75 min

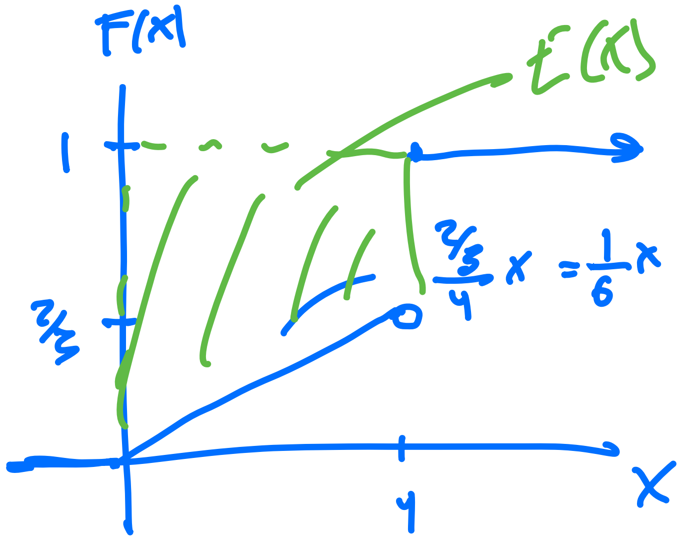
c 1.25 min

d none of the above

ex Let $T \sim \text{Unif}(0,6)$ and let $X = \min(T, 4)$.

a) Draw and find the CDF of X.

b) Find $E(X)$



$$E(X) = \frac{1}{3}\cdot 4 + \frac{1}{2}\cdot\frac{2}{3}\cdot 4$$

$$= \frac{4}{3} + \frac{4}{3} = \boxed{\frac{8}{3}}$$

$$F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{6}x & 0 < x < 4 \\ 1 & x \geq 4 \end{cases}$$

Appendix

See p 322 in book

Claim for any CDF F

$X = F^{-1}(U)$ is a RV with cdf F.

Unif(0,1)

Proof/ let $X = F^{-1}(U)$

$$F_X(x) = P(X \le x)$$

we will show $F_X = F$

$$= P(F^{-1}(U) \le x)$$

$$= P(F \cdot F^{-1}(U) \le F(x))$$

since F is increasing

$$= P(U \le F(x))$$

$$= F(x)$$

since $P(U \le u) = u$

$\square$