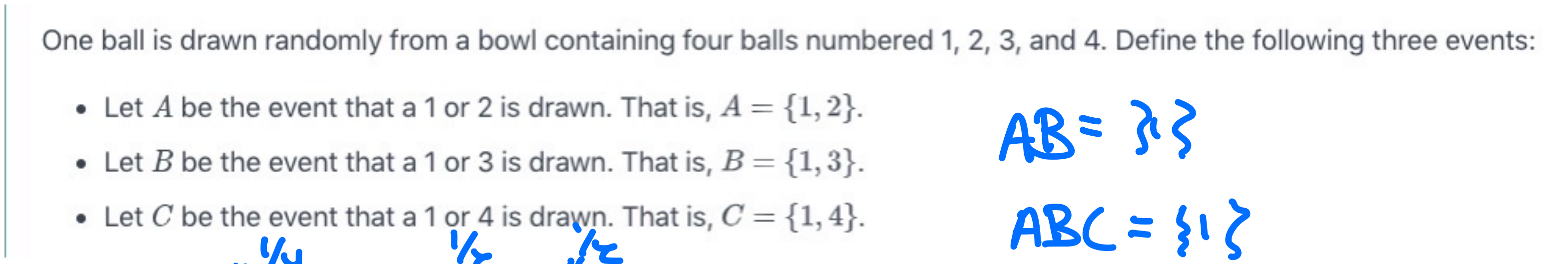
Stat 134 lec3

warmup



One ball is drawn randomly from a bowl containing four balls numbered 1, 2, 3, and 4. Define the following three events:

- Let A be the event that a 1 or 2 is drawn. That is, $A = \{1, 2\}$.
- Let B be the event that a 1 or 3 is drawn. That is, $B = \{1, 3\}$.
- Let C be the event that a 1 or 4 is drawn. That is, $C = \{1, 4\}$.

$AB = \{1\}$

$ABC = \{1\}$

Is $P(\overset{=1/4}{AB}) = P(\overset{=1/2}{A})P(\overset{=1/2}{B})$? ✓

Is A, C independent ? ✓

Is B, C independent ? ✓

Is $P(\underset{=1/4}{ABC}) \neq P(\underset{=1/2}{A})P(\underset{=1/2}{B})P(\underset{=1/2}{C})$? ✗

Last time

Bayes's rule $P(A|B) = \frac{P(AB)}{P(B)}$

multiplication rule $P(AB) = P(A|B)P(B)$

A, B mutually exclusive $\Rightarrow$ $P(AB) = 0$

A, B independent $\Rightarrow$ $P(AB) = P(A)P(B)$

If A and B are indep then so is A, B^c, and A^c, B and A^c, B^c.

Today

① sec 1.6 independence of 3 or more events

② sec 2.1 Binomial Distribution

Sec 1.6 Independence of 3 events

Def^n (pairwise independence of 3 events)
A, B, C are pairwise independent if

$P(AB) = P(A)P(B)$ and $P(AC) = P(A)P(C)$ and $P(BC) = P(B)P(C)$

Def^n (mutual independence of 3 events)
A, B, C are mutually independent if
$P(ABC) = P(A)P(B)P(C)$, (and the same for any of the events replaced by its complement)

ex In warmup we saw an example of 3 events that are pairwise indep but not mutually indep.

we require showing 8 equations is true for mutual independence. This is a strong condition.

Thm Suppose A, B, C are mutually independent. Then they are also pairwise independent.

disjoint union

$$AB = ABC \cup ABC^c$$

Pf/

we can write

$$P(AB) = P(ABC) + P(ABC^c)$$

add^n rule

$$= P(A)P(B)P(C) + P(A)P(B)P(C^c)$$

$$= P(A)P(B)\left[P(C) + P(C^c)\right]$$

$$= P(A)P(B).$$

similar for other cases

$P(AC) = P(A)P(C)$ etc.

$\square$

Note that $\boxed{P(ABC) = P(A)P(B)P(C)}$ by itself doesnt imply pairwise independence:

ex A fair eight sided die is rolled.

let $A = B = \{1, 2, 3, 4\}$ be the event you get a 1, 2, 3 or 4

let $C = \{1, 5, 6, 7\}$ $\qquad ABC = \{1\}$

Is $\underset{\overset{\shortparallel}{1/8}}{P(ABC)} = \underset{\overset{\shortparallel}{1/2}}{P(A)}\,\underset{\overset{\shortparallel}{1/2}}{P(B)}\,\underset{\overset{\shortparallel}{1/2}}{P(C)}$? ✓

Is A, B, C pairwise indep?

$$\underset{\overset{\shortparallel}{1/2}}{P(AB)} \neq \underset{\overset{\shortparallel}{1/2}}{P(A)}\,\underset{\overset{\shortparallel}{1/2}}{P(B)}$$

Thm A, B, C are mutually independent iff

1) A, B, C are pairwise indep
2) $\boxed{P(ABC) = P(A)P(B)P(C)}$.

Sketch

Suppose (1) and (2) hold,

lets show $P(ABC^c) = P(A)P(B)P(C^c)$

write

$$P(ABC^c) = \underbrace{P(ABC^c) + P(ABC)}_{\substack{= \\ P(AB) \\ = \\ P(A)P(B)}} - \underset{\substack{= \\ P(A)P(B)P(C)}}{P(ABC)}$$

$$= P(A)P(B)\Big[\underset{\substack{= \\ P(C^c)}}{1 - P(C)}\Big]$$ ✓

Similar for the other cases,

mutual indep $\Rightarrow$ ①, ② ✓

□

② Sec 2.1 Binomial distribution.

Bernoulli(p) trial (distribution)

two outcomes
- success — p
- failure — 1-p

ex roll a die.
success is getting a 6 — $1/6$
failure is not getting a 6 — $5/6$

Binomial (n,p) distribution (Bin(n,p))

we have n independent Bernoulli(p) trials

n: fixed

p: fixed (unconditional probability)

ex we roll a die n times.

What are the possible number of successes?

$0, 1, 2, \ldots, n$

In Bin(n,p) the chance of having k successes $(0 \le k \le n)$ is given by the Binomial formula:

trials

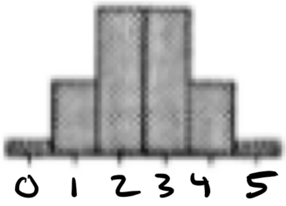
$$P(k) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

number of successes

chance of success.

ex You roll a die 5 times. What is the chance of getting 2 sixes?

$n = ?$ 5
$k = ?$ 2
$p = ?$ $1/6$



$$P(2) = \frac{5!}{2!3!}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 = \frac{5\cdot4\cdot\cancel{3\cdot2\cdot1}}{2\cdot1\cdot\cancel{3\cdot2\cdot1}}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3$$

$$= \boxed{10\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3}$$

What is chance of getting

success (6)

failure (not 6)

$1\ 1\ 0\ 0\ 0$? $\leftarrow \left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3$

$0\ 1\ 1\ 0\ 0$? $\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2 = \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)$

$\vdots$

How many of these are there?

$$\frac{5!}{2!3!} = \binom{5}{2} = \binom{5}{3}$$

We write $\frac{5!}{2!3!}$ as $\binom{5}{2}$ or $\binom{5}{3}$ or $\binom{5}{2,3}$

$$\binom{5}{2} = \frac{5!}{2!3!} = \frac{5!}{3!2!} = \binom{5}{3}$$

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ex

1. Ten cards are dealt off the top of a well shuffled deck. The binominal formula doesn't apply to find the chance of getting exactly three diamonds because:

   a The probability of a trial being successful changes

   [✓ **b**] The trials aren't independent

   c There isn't a fixed number of trials

   d more than one of the above

A trial is whether a card in the deck is a diamond or not. There are $n=10$ trials, each with prob $1/4$ because p is an unconditional prob so we don't look at the other cards first.

The trials are dependent

$$\underset{\substack{\shortparallel \\ 12/51}}{P(2^{nd}\text{ card diamond} \mid 1^{st}\text{ diamond})} \neq \underset{\substack{\shortparallel \\ 1/4}}{P(2^{nd}\text{ card diamond})}$$

Example of Bernoulli trials with different unconditional probability p:

Imagine the 1st five cards are from a normal deck (with chance of a diamond $1/4$), and the last 5 cards are from a smaller deck of 39 cards with no diamonds, then the trials would either have $p=0$ or $p=1/4$, (not always the same p).

extra

2. Suppose A and B are two events with

$$P(A) = 0.8 \text{ and } P(A \cup \overset{B=\phi}{B}) = 0.8.$$

Is it possible for A and B to be both **mutually exclusive** and **independent**?

[✓ **a** yes]

b no

c there isn't enough information to decide

we saw on Friday that you can have two events be both ME and Ind if one of the sets is empty,