Stat 134 Lec 22

warmup

Show that the MGF of $X \sim Pois(\mu)$ is
$M_X(t) = e^{\mu(e^t - 1)}$ for all t.

Recall $M_X(t) = E(e^{tX})$ ← think of e^{tX} as $g(X)$.

and $P(X=k) = \dfrac{e^{-\mu}\mu^k}{k!}$

$$M_X(t) = E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk} P(X=k) = \sum_{k=0}^{\infty} e^{tk} \frac{e^{-\mu}\mu^k}{k!}$$

$$= e^{-\mu} \sum_{k=0}^{\infty} \frac{e^{tk}\mu^k}{k!} = e^{-\mu} \sum_{k=0}^{\infty} \frac{(e^t\mu)^k}{k!}$$

$e^{tk} = (e^t)^k$

Recall from Calculus

$e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!}$ Taylor for all $a \in \mathbb{R}$

$$= e^{-\mu} e^{e^t \mu}$$

$$= \boxed{e^{\mu(e^t - 1)} \text{ for all } t}$$

Note

$$E(X) = \frac{d}{dt} M_X(t) \Big|_{t=0} = e^{\mu(e^t - 1)} \cdot \mu e^t \Big|_{t=0} = \boxed{\mu}$$

Announcement: Q2 Tuesday October 24
covers Sec 4.1, 4.2, 4.4, 4.5, MGF

Last time

Key Properties of MGF

1. If an MGF exists in an interval containing zero, $M^{(k)}(t)\Big|_{t=0} = E(X^k)$

last time

2. If X and Y are independent RVs,

$M_{X+Y}(t) = M_X(t)M_Y(t)$.

Proved in MGF HW.

3. If $M_X(t) = M_Y(t)$ for all t in an interval around 0 then $F_X(z) = F_Y(z)$ (i.e X and Y have the same distribution).

Skip proof — we can invert a MGF to get the CDF.

4. $M_{aX}(t) = M_X(at)$

Pf/ $M_{aX}(t) = E(e^{aXt})$

$= E(e^{Xat})$

$= M_X(at)$ $\square$

$\underline{\underline{ex}}$ A RV X takes values 1, 2, 3
with prob $1/2, 1/3, 1/6$.

Find $M_X(t)$.

$$E(e^{tX}) = \sum_{x=1}^{3} e^{tx} p(x)$$
$$= e^{t\cdot 1}\cdot\frac{1}{2} + e^{2t}\frac{1}{3} + e^{3t}\cdot\frac{1}{6}$$

for all t

$\underline{\underline{ex}}$ If $M_X(t) \overset{E(e^{tX})}{=} \frac{1}{2}e^{1t} + \frac{1}{3}e^{2t} + \frac{1}{6}e^{3t}$,

e^{xt} tells us the value of X and the associated coefficients tell us the probabilities

($\underline{\text{i.e}}$ $X = 1, 2, 3$ w/ prob $1/2, 1/3, 1/6$.)

so MGF $\Rightarrow$ distribution of X when X has finite # values.

<u>Today</u>

1. Practice with MGF
2. Recognizing a distribution from the variable part of its density
3. Extra practice

① Practice with MGF

For $X \sim \text{Gamma}(r, \lambda)$

recall $M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^r$ for $t < \lambda$

ex Let $X \sim \text{Exp}(\lambda)$ and $a > 0$.

Show that $Y = aX$ is also exponential, and specify the new parameter.

Note $M_X(s) = \left(\frac{\lambda}{\lambda - s}\right)^1$ for $s < \lambda$

$Y = aX$

for $at < \lambda$

$$M_{aX}(t) = M_X(at) = \left(\frac{\lambda}{\lambda - at}\right)^1 = \left(\frac{\frac{\lambda}{a}}{\frac{\lambda}{a} - t}\right)^1 \quad \text{for } t < \frac{\lambda}{a}$$

$$\Rightarrow Y \sim \text{Exp}\left(\frac{\lambda}{a}\right)$$

ex

r λ

(3 pts) Let X_i follow the Gamma $(1/100,\ 2/100)$ distribution for $i = 1, 2 \ldots, 100$, independently of each other. We are interested in finding the distribution of the sample average, $Y = \frac{1}{100}\sum_{i=1}^{100} X_i$. Using properties of MGFs, identify the distribution of Y.

Recall that for $X \sim$ Gamma (r, λ), $M_X(t) = (\frac{\lambda}{\lambda - t})^r$, $t < \lambda$.

let $S = \sum_{i=1}^{100} X_i$

$$M_S(t) = M_{X_1}(t) \cdots M_{X_{100}}(t) = \left(\left(\frac{.02}{.02-t}\right)^{.01}\right)^{100} = \left(\frac{.02}{.02-t}\right)^{1}$$

$$M_Y(t) = M_{\frac{1}{100}S}(t) = M_S(.01t) = \frac{.02}{.02 - .01t} = \frac{2}{2-t}$$

Since this MGF exists in a neighborhood of zero,

By the uniqueness of MGF, $\boxed{Y \sim Exp(2)}$

(2) Recognizing a distribution from the variable part of its density.

A density can be written as

$$f(t) = c\,h(t)$$

c ← constant

$h(t)$ ← variable part.

$T_r \sim \text{Gamma}(r, \lambda)$, $r, \lambda > 0$

$$f(t) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t} & t > 0 \\ 0 & \text{else} \end{cases}$$

$X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$1 = \int_{-\infty}^{\infty} f(t)\,dt = c \int_{-\infty}^{\infty} h(t)\,dt \implies c = \frac{1}{\int_{-\infty}^{\infty} h(t)\,dt}$$

So you can figure out the density from its variable part.

ex Name the distribution with the following variable part **ex** Gamma($r=1/2$, $\lambda=3$)

a) $h(t) = t^3 e^{-\frac{1}{2}t}$ Gamma$\left(4, \frac{1}{2}\right)$

b) $h(t) = e^{-\frac{1}{2}t^2}$ Normal$(0,1)$

c) $h(t) = e^{-3t}$ Exp(3)

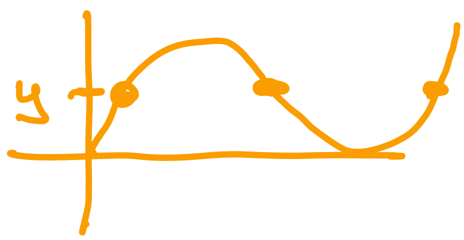
d) $h(t) = t^{-1/2} e^{-t}$ Gamma$\left(\frac{1}{2}, 1\right)$

e) $h(t) = 1_{(t \in (0,2))}$ Unif$(0,2)$

③ Extra practice

Sec 4.4 Change of variable rule.

Find density of $Y = g(X)$.



steps

① range of y

② solve for X

③ Find $\left|\frac{dx}{dy}\right|$

④ $f_Y(y) = n \left|\frac{dx}{dy}\right| f_X(x) \Big|_{x = g^{-1}(y)}$

$n = \#\{x : g(x) = y\}$

$n = 3$ above

tinyurl.com/Mar14-23

Let X be the standard normal RV. The distribution of $Y = X^2$ is:

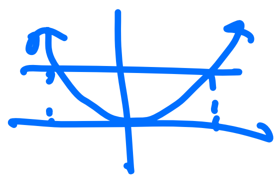
a Gamma$(\frac{1}{2}, \frac{1}{2})$

b Gamma$(\frac{3}{2}, \frac{1}{2})$

c Normal(0,1)

d none of the above

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$



every y has 2 x for y>0

range of y is $[0, \infty)$

$x = \pm\sqrt{y}$

$$\left|\frac{dx}{dy}\right| = \frac{1}{2\sqrt{y}}$$

$$f_Y(y) = 2 \cdot \frac{1}{2\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} = \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2}$$

$\#\{x : x^2 = y\}$

variable part of Gamma$(\frac{1}{2}, \frac{1}{2})$

MGF Challenge

Can you solve this problem using MGF?

<u>Hint</u> $X \sim N(0, \sigma^2)$ has density $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$

<u>Hint</u>

MGF of Gamma(r, λ) is

$$\left(\frac{\lambda}{\lambda - t}\right)^r$$

so MGF of Gamma$\left(\frac{1}{2}, \frac{1}{2}\right)$ is

$$\left(\frac{\frac{1}{2}}{\frac{1}{2} - t}\right)^{1/2}$$

$$M_{x^2}(t) = E(e^{tx^2})$$

$$= \int_{-\infty}^{\infty} e^{tx^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2\left(\frac{1}{2}-t\right)} dx$$

need this to be $\frac{1}{2\sigma^2}$

$$\Rightarrow \sigma^2 = \frac{1}{2\left(\frac{1}{2}-t\right)}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi\sigma^2} = \sigma \quad \text{for } t < 1/2$$

$$= \left(\frac{\frac{1}{2}}{\frac{1}{2}-t}\right)^{1/2}$$

for $t < 1/2$

MGF of $\text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$