Stat 135 Lec 37

Warm up

Conditional expectation of $g(Y)$ given $X = x$: Integrate g against the conditional density:

$$E(g(Y) \mid X = x) = \int g(y) f_Y(y \mid X = x) dy$$

ex

Suppose

$f_Y(y \mid X = x) = 2x + 2y - 4xy$ for $0 \le x \le 1$ and $0 \le y \le 1$

Find $E(Y \mid X = x)$

$$\int_{y=0}^{y=1} y(2x + 2y - 4xy) dy = \int_0^1 (2xy + 2y^2 - 4xy^2) dy$$

$$= \frac{2xy^2}{2} + \frac{2y^3}{3} - \frac{4xy^3}{3} \Big|_{y=0}^{y=1} = x + \frac{2}{3} - \frac{4x}{3}$$

$$= \boxed{\frac{2}{3} - \frac{x}{3}, \quad 0 \le x \le 1}$$

Last time

Average conditional expectation: (rule of iterated expect.)

$E(Y) = E(E(Y|X))$

$$E(Y) = \int E(Y \mid X = x) f_X(x)\, dx$$

In above problem, given $f_X(x) = 1$,

find $E(Y)$.

$$E(Y) = E\left(\frac{2}{3} - \frac{x}{3}\right) = \int_{x=0}^{x=1} \left(\frac{2}{3} - \frac{x}{3}\right) \cdot 1\, dx$$

$$= \frac{2}{3}x - \frac{x^2}{6}\Big|_0^1 = \frac{2}{3} - \frac{1}{6} = \frac{3}{6} = \boxed{1/2}$$

Multiplication rule: The joint density is the product of the marginal and the conditional

$$f(x, y) = f_X(x) f_Y(y \mid X = x)$$
$$= f_Y(y) f_X(x \mid Y = y)$$

<u>mixed joint</u> X cont, Y discrete

$$\text{joint}(x, y) = f_X(x) P(Y = y \mid X = x) = P_Y(Y = y) f_X(x \mid Y = y)$$

$$\Rightarrow f_X(x \mid Y = y) = \frac{f_X(x) P(Y = y \mid X = x)}{P(Y = y)}$$

Bayesian Statistics treats unknown parameters of distributions (ex Ber(p)) as a RV.

i.e

$X \sim \text{Unif}(0,1)$ ← RV

$I_1, I_2 \sim \text{Ber}(X)$

$I_1 | X=x,\ I_2 | X=x \overset{iid}{\sim} \text{Ber}(x)$

posterior

likelihood, prior

$$f_X(x|I_1=1) = \frac{P(I_1=1|X=x) \cdot f_X(x)}{P(I_1=1)}$$

constant.

Posterior $\propto$ likelihood $\cdot$ Prior

A likelihood which is binomial and a prior that is beta always has a beta posterior (i.e lik = binomial, prior = beta are conjugate pairs.

The old posterior can now become the new prior if you get more data (say you know $I_1 = 1$, $I_2 = 1$).

Today

1. Sec 6.4 covariance and the variance of sum

① Sec 6.4 Covariance and variance of a sum

$X, Y, \quad S = X+Y$

mean $\mu_X, \mu_Y, \mu_S = \mu_X + \mu_Y$

$S - \mu_S = X + Y - (\mu_X + \mu_Y)$ D_S deviation from mean

$= (X - \mu_X) + (Y - \mu_Y) = D_X + D_Y$

$$\mathrm{Var}(S) = E\left((S-\mu_S)^2\right) = E\left((D_X + D_Y)^2\right)$$

$$= E\left(D_X^2 + D_Y^2 + 2D_XD_Y\right)$$

$$= \underset{\mathrm{Var}(X)}{\underset{\shortparallel}{E(D_X^2)}} + \underset{\mathrm{Var}(Y)}{\underset{\shortparallel}{E(D_Y^2)}} + 2\,\underbrace{E(D_XD_Y)}_{\substack{\shortparallel \\ \mathrm{Cov}(X,Y)}}$$

Def^n The Covariance of X and Y is

$$\mathrm{Cov}(X,Y) = E\left((X-\mu_X)(Y-\mu_Y)\right)$$

Bilinearity Properties

Proved end of lecture.

ⓐ $\text{Cov}(X+Y, Z) = \text{Cov}(X,Z) + \text{Cov}(Y,Z)$

ⓑ $\text{Cov}(aX, bY) = ab\,\text{Cov}(X,Y)$

More generally

$$\text{Cov}\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \,\text{Cov}(X_i, Y_j)$$

Proved end of lecture.

Thm $\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$

$\text{Var}(X) = E((X-\mu_X)^2)$

$\text{Var}(X) = E(X^2) - (E(X))^2$

Easy facts

$\text{Cov}(X,X) = E(X^2) - E(X)^2 = \text{Var}(X)$

$\text{Cov}(X,X) = \text{Var}(X)$

$\text{Cov}(X,Y) = \text{Cov}(Y,X)$

$\text{Cov}(X,c) = 0$

Constant

$\underline{\underline{ex}}$

Simplify

$$\mathrm{Cov}(X-5Y,\ 3X+Y-Z+10)$$

$= 3\mathrm{Var}(X) + \mathrm{Cov}(X,Y) - \mathrm{Cov}(X,Z) + 0 - 15\,\mathrm{Cov}(X,Y)$

$- 5\mathrm{Var}(Y) + 5\,\mathrm{Cov}(Y,Z) + 0$

$$\mathrm{Cov}(X,-Z) = \mathrm{Cov}(X,(-1)Z) = -\mathrm{Cov}(X,Z)$$

$$\mathrm{Cov}(-5Y,3X) = -15\,\mathrm{Cov}(Y,X) = -15\,\mathrm{Cov}(X,Y)$$

Recall X, Y independent

$\Rightarrow \quad E(XY) = E(X)E(Y)$

$\mathrm{Cov}(X,Y) = 0$ if X, Y independent.

Hence if X, Y indep,

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\overset{=0}{\mathrm{Cov}(X,Y)}$$

$$= \mathrm{Var}(X) + \mathrm{Var}(Y)$$

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Stat 134

1. Consider a Poisson(λ) process. Let $T_r \sim$ gamma(r, λ) be the rth arrival time. Cov(T_1, T_3) equals:

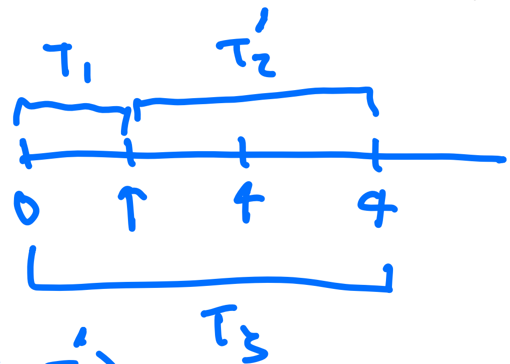
   a λ

   b λ^2

   c $1/\lambda^2$

   d none of the above

Recall $\text{Var}(T_r) = \dfrac{r}{\lambda^2}$



$$\text{Cov}(T_1, T_3) = \text{Cov}(T_1, T_1 + T_2')$$

$$= \underbrace{\text{Cov}(T_1, T_1)}_{= \text{Var}(T_1) = \boxed{\frac{1}{\lambda^2}}} + \underbrace{\text{Cov}(T_1, T_2')}_{= 0}$$

since T_1, T_2' indep

ex

Toss a fair coin 30 times.

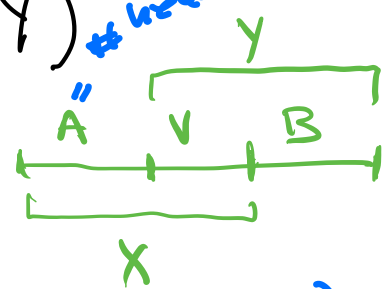
Let X = # heads in the first 20 tosses

Y = # heads in the last 20 tosses.

Find $Cov(X,Y)$

heads in first 10 tosses $= A$

Hint:



$$Cov(X,Y) = Cov(A+V,\ V+B)$$

$$= \underbrace{Cov(A,V)}_{=0} + \underbrace{Cov(A,B)}_{=0} + \underbrace{Cov(V,V)}_{=Var(V)} + \underbrace{Cov(V,B)}_{=0}$$

$V \sim Bin(10, \tfrac{1}{2})$

$$Var(V) = npq = 10\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = \boxed{\tfrac{10}{4}}$$

extra

Average conditional probability:

$$P(B) = \int P(B|X = x) f_X(x)\, dx$$

(6 pts) Suppose $Y \sim$ Pois (X), where $X \sim$ Exp (λ). That is, given $X = x$, $Y \sim$ Pois (x).

(a) (3 pts) Show that the unconditional distribution of Y is Geometric $(\frac{\lambda}{\lambda+1})$ on $\{0, 1, 2, \ldots\}$.

$Y \sim \text{Geom}(p)$ on $0, 1, 2, \ldots$

\# failures until first success

$P(Y=n) = q^n p$

$p = \frac{\lambda}{\lambda+1}$ $q = 1 - \frac{\lambda}{\lambda+1} = \frac{1}{\lambda+1}$

$\boxed{P(Y=n) = \left(\frac{1}{\lambda+1}\right)^n \left(\frac{\lambda}{\lambda+1}\right)}$ want this,

$$P(Y=n) = \int_{x=0}^{x=\infty} P(Y=n \mid X=x) f_X(x)\, dx$$

$$= \int_0^\infty \frac{e^{-x} x^n}{n!} \cdot \lambda e^{-\lambda x}\, dx$$

$$= \frac{\lambda}{n!} \int_0^\infty x^{(n+1)-1} e^{-(\lambda+1)x}$$

variable part of Gamma $(n+1, \lambda+1)$

$$= \frac{\lambda}{n!} \cdot \frac{n!}{(\lambda+1)^{n+1}} = \left(\frac{1}{\lambda+1}\right)^n \left(\frac{\lambda}{\lambda+1}\right)$$ as required

Appendix

Bilinearity Properties

Thm
(a) $\operatorname{Cov}(X+Y, Z) = \operatorname{Cov}(X,Z) + \operatorname{Cov}(Y,Z)$
(b) $\operatorname{Cov}(aX, bY) = ab\operatorname{Cov}(X,Y)$

Pf/
a)
$$\operatorname{Cov}(X+Y, Z) = E((X+Y-\mu_{X+Y})(Z-\mu_Z))$$
$$= E([(X-\mu_X)+(Y-\mu_Y)](Z-\mu_Z))$$
$$= E((X-\mu_X)(Z-\mu_Z) + (Y-\mu_Y)(Z-\mu_Z))$$
$$= E((X-\mu_X)(Z-\mu_Z)) + E((Y-\mu_Y)(Z-\mu_Z))$$
$$= \operatorname{Cov}(X,Z) + \operatorname{Cov}(Y,Z). \quad \square$$

b)
$$\operatorname{Cov}(aX, bY) = E((aX-\mu_{aX})(bY-\mu_{bY}))$$
$$= E((aX-a\mu_X)(bY-b\mu_Y))$$
$$= E(ab(X-\mu_X)(Y-\mu_Y))$$
$$= ab\,E((X-\mu_X)(Y-\mu_Y))$$
$$= ab\operatorname{Cov}(X,Y) \quad \square$$

Appendix

Thm $\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$

pf/ $\mathrm{Cov}(X,Y) = E(D_X D_Y) = E\big((X-\mu_X)(Y-\mu_Y)\big)$

$= E(XY - \mu_X Y - X\mu_Y + \mu_X \mu_Y)$

$= E(XY) - \mu_X\mu_Y - \mu_X\mu_Y + \mu_X\mu_Y)$

$= E(XY) - E[X]E(Y)$

$\square$