Stat 134 lec 26

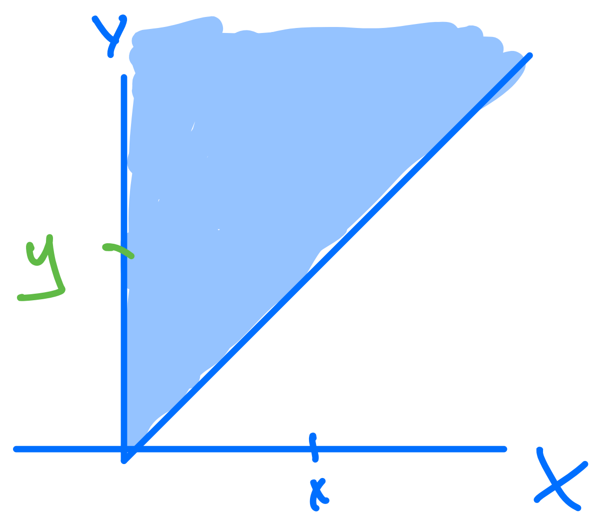
warmup

Let $X \sim Exp(\lambda)$, $Y \sim Exp(\alpha)$

(recall, $f_X(x) = \lambda e^{-\lambda x}$)

be independent lifetimes of two bulbs.

Find $P(X<Y)$.



Hint: use $f(x,y) = f_X(x) f_Y(y)$

$$f(x,y) = f_X(x) f_Y(y) = \lambda e^{-\lambda x} \alpha e^{-\alpha y}$$

$$\int_{y=0}^{\infty} \int_{x=0}^{x=y} f_X(x) f_Y(y)\,dx\,dy$$

$$\int_{x=0}^{\infty} \int_{y=x}^{\infty} f_X(x) f_Y(y)\,dy\,dx = \cancel{\alpha}\lambda \int_{x=0}^{\infty} e^{-\lambda x} \underbrace{\int_{y=x}^{\infty} e^{-\alpha y}\,dy}_{} \,dx$$

$$= \frac{e^{-\alpha y}}{-\alpha}\Big|_{y=x}^{y=\infty} = \frac{e^{-\alpha x}}{\cancel{\alpha}}$$

$$= \lambda \int_{x=0}^{x=\infty} e^{-(\alpha+\lambda)x}\,dx = \boxed{\frac{\lambda}{\alpha+\lambda}}$$

Notice: we showed this result about competing exponentials in lecture 19 using Poisson thinning.

Last time:

Sec 4.6 Beta Distribution

Let $r, s > 0$

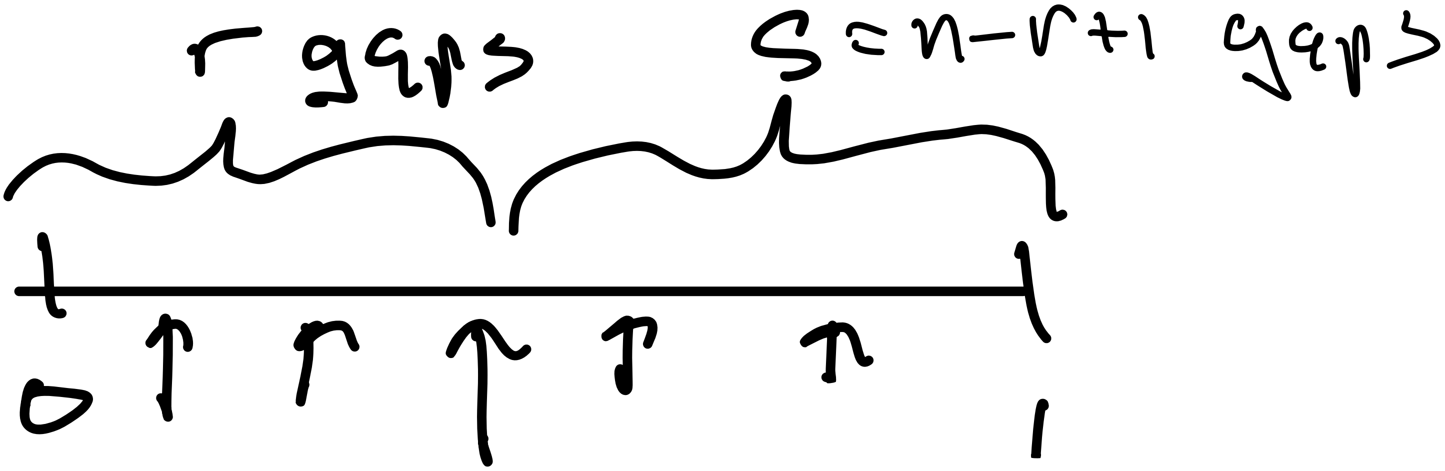
$P \sim \text{Beta}(r,s)$ if

$$f(p) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} p^{r-1}(1-p)^{s-1} \quad \text{for } 0<p<1$$

$$E(X) = \frac{r}{r+s}$$

Application $U_1, \ldots, U_n \overset{iid}{\sim} U(0,1)$

then $U_{(r)}$ of $n \sim \text{Beta}(r, n-r+1)$



$$U_{(r)} \sim \text{Beta}(r,s)$$

sec 5.1, 5.2 joint density.

ex

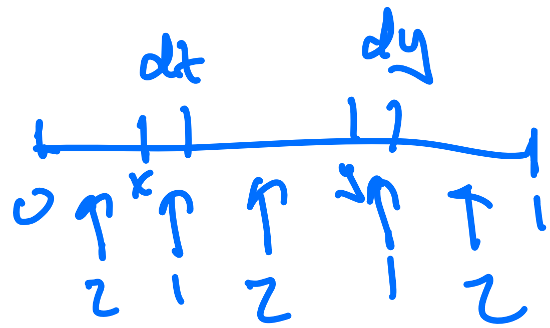
I throw down 8 darts on $(0, 1)$. The variable part of the joint density of $X = U_{(3)}$ and $Y = U_{(6)}$ is:

a $x(y-x)^5(1-y)^2$

b $x^2(y-x)^2(1-y)^2$

c $x^4(y-x)^2(1-y)^2$

d none of the above



we also saw how to use the joint density to compute probabilities (see webwork).

Today

1. sec 5.1, 5.2 Independent RVs
2. sec 5.2 Marginal densities

① Sec 5.1, 5.2

Independent RVs

defⁿ X and Y are independent if
$f(x,y) = f(x)\, f(y)$.

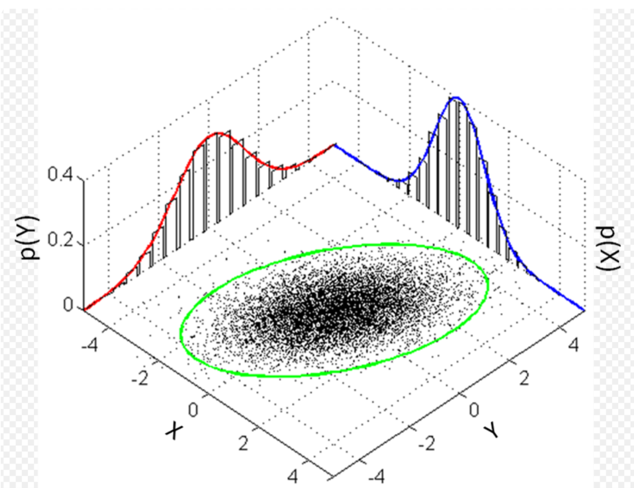
This is consistent with our defⁿ of independent events since:

$$\underbrace{P(X\in dx, Y\in dY)}_{\approx\, f(x,y)\,dx\,dy} = \underbrace{P(X\in dx)}_{\approx\, f(x)\,dx}\ \underbrace{P(Y\in dY)}_{\approx\, f(y)\,dy}$$

ex $X, Y \overset{iid}{\sim} N(0,1)$

$$f(x,y) = \phi(x)\,\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$$



Not a great picture because the oval in green should be a circle. This is the picture of a correlated bivariate normal from chapter 6 instead of an uncorrelated bivariate normal.

ex

If $X, Y \overset{iid}{\sim} U(0,1)$

Find $P(Y \geq \frac{1}{2} \mid Y \geq 1-2X)$

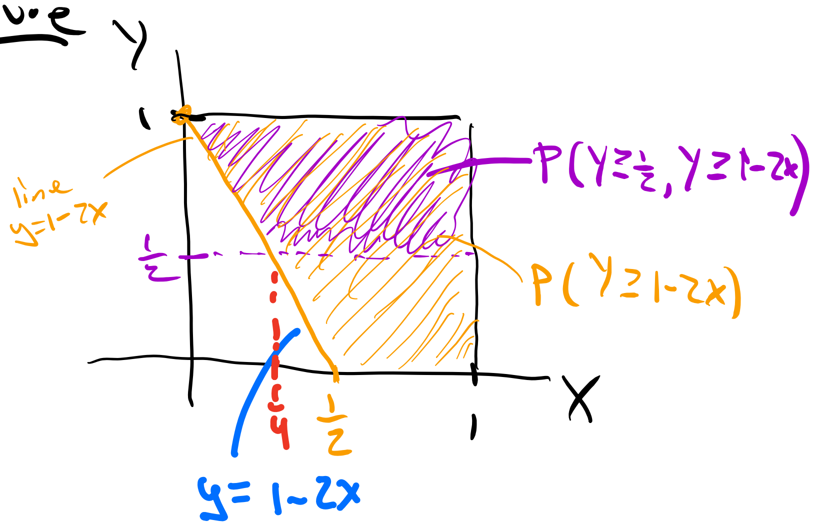
Soln

$f(x,y) = f(x)f(y) = 1$ for $0<x,y<1$, 0 else.

$f(x) = 1$, $f(y) = 1$

$$P(Y \geq \tfrac{1}{2} \mid Y \geq 1-2X) = \frac{P(Y \geq \frac{1}{2}, Y \geq 1-2X)}{P(Y \geq 1-2X)} \quad \text{Bayes' rule}$$

Picture



$$\frac{\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{4} = \frac{7}{16}}{1 - \frac{1}{4} = \frac{3}{4}} = \boxed{7/12}$$

tinyurl.com/mar25-2023

Stat 134

Friday October 21 2022

1. You are first in line to have your question answered by either of the 3 uGSI Yiming, Brian and Rowan, whose wait time to be seen, Y, B and R, are independent and exponentially distributed RVs with rates λ_Y, λ_B, and λ_R respectively. $P(Y < B < R)$ is?

   a $\frac{\lambda_Y+\lambda_B}{\lambda_Y+\lambda_B+\lambda_R}$

   b $\frac{\lambda_Y}{\lambda_Y+\lambda_B+\lambda_R} \times \frac{\lambda_B}{\lambda_B+\lambda_R}$

   c $\frac{\lambda_Y}{\lambda_Y+\lambda_B} \times \frac{\lambda_B}{\lambda_B+\lambda_R}$

   d none of the above

$= P(Y<B, B<R)$

$\frac{\lambda_B}{\lambda_B+\lambda_R}$

$\|$

$P(Y<B, B<R) = P(Y<B \mid B<R) \cdot P(B<R)$

$P(Y<B \mid B<R)$ is a forward conditional.

To write this as an unconditional probability replace B by the super GSI $\min(B,R)$ who has rate $(\lambda_B + \lambda_R)$.

Then $P(Y<B \mid B<R) = P(Y < \min(B,R))$

$$= \frac{\lambda_Y}{\lambda_Y + (\lambda_B + \lambda_R)}$$

It follows that ~~that~~

$$\boxed{P(Y<B<R) = \frac{\lambda_Y}{\lambda_Y + \lambda_B + \lambda_R} \cdot \frac{\lambda_B}{\lambda_B + \lambda_R}}$$

Or you can solve the integral directly

$$P(Y<B<R) = \int_{y=0}^{\infty} \lambda_y e^{-\lambda_y y} \int_{B=y}^{\infty} \lambda_B e^{-\lambda_B b} \int_{R=b}^{\infty} \lambda_R e^{-\lambda_R r} \, dr \, db \, dy$$

$$\int_{R=b}^{\infty} \lambda_R e^{-\lambda_R r} \, dr = e^{-\lambda_R b}$$

$$= \lambda_y \lambda_B \int_{y=0}^{\infty} e^{-\lambda_y y} \int_{B=y}^{\infty} e^{-(\lambda_B+\lambda_R) b} \, db \, dy \quad \text{etc}$$

$$\int_{B=y}^{\infty} e^{-(\lambda_B+\lambda_R) b} \, db = \frac{e^{-(\lambda_B+\lambda_R) y}}{\lambda_B+\lambda_R}$$

etc...

(2) Sec 5.2 Marginal densities

Recall marginal probability:

<u>discrete picture</u>

y \ x	0	1	P(y)
2	0	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
0	$\frac{1}{4}$	0	$\frac{1}{4}$
P(x)	$\frac{1}{2}$	$\frac{1}{2}$	

marginal probability of X

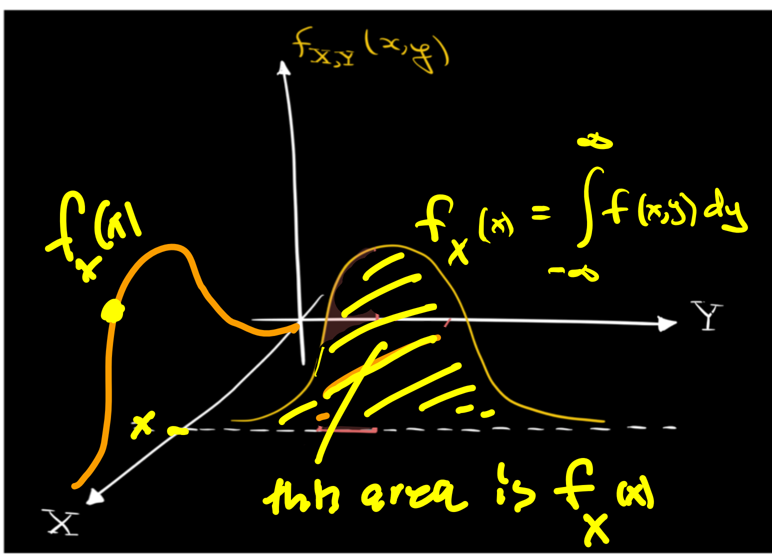
$P(x) = \sum_{y \in Y} P(x,y)$

Marginal prob of Y

$P(y) = \sum_{x \in X} P(x,y)$

2

Continuous picture: marginal density



ex S and T are iid $\text{Exp}(\lambda)$

$X = \min(S,T)$ and $Y = \max(S,T)$.

The joint density is

$f(x,y) = 2\lambda^2 e^{-\lambda(x+y)}$ for $0 < x < y$,

0 else.

Find the marginal of Y.

$$f_Y(y) = \int_{x=-\infty}^{\infty} f(x,y)\,dx$$

$$= 2\lambda^2 e^{-\lambda y} \underbrace{\int_{x=0}^{x=y} e^{-\lambda x}\,dx}_{= \frac{1-e^{-\lambda y}}{\lambda}} = \boxed{2\lambda e^{-\lambda y}\left(1-e^{-\lambda y}\right)}$$