Stat 134 lec 24

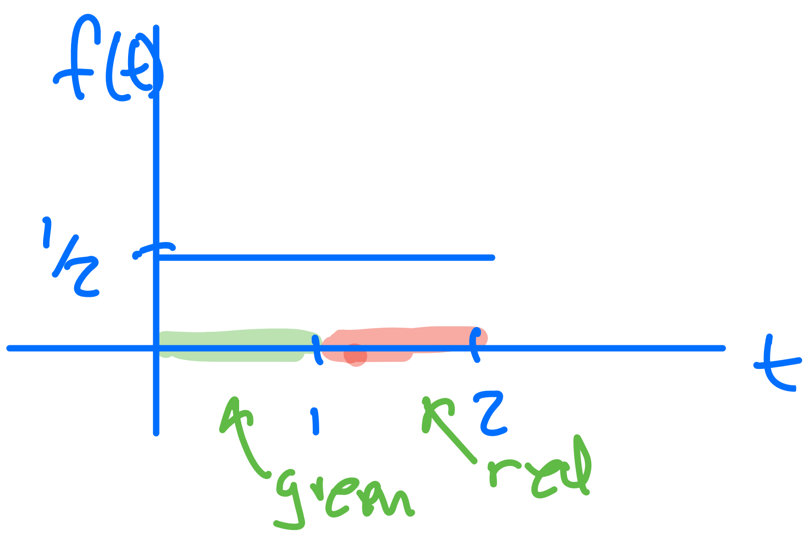
warmup:

Suppose stop lights at an intersection alternately show green for one minute, and red for one minute (no yellow). Suppose a car arrives at the lights at a time distributed uniformly from 0 to 2 minutes. Let X be the delay of the car at the lights (assuming there is only one car on the road). Graph the density and the cdf of X. Also find E(X)

Let T = arrival time of car

$T \sim Unif(0,2)$

Picture of density of T:



let X = delay time

$$X = \begin{cases} 0 & \text{if } 0 < t < 1 \\ 2-t & \text{if } 1 < t < 2 \end{cases}$$

Notice that X takes values $[0,1]$

$P(X=0)$ = Prob arrive at green light

$= 1/2$

lets find $f_X(x)$ for $0 < x < 1$

By change of variable rule

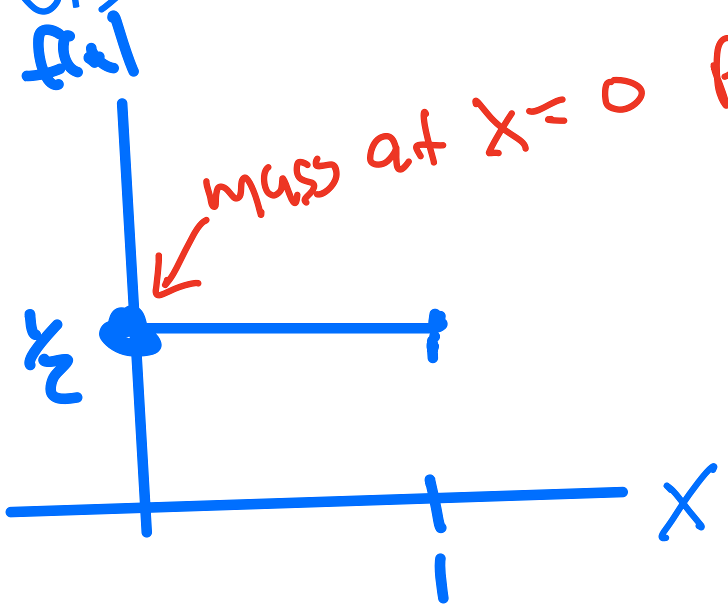
$X = 2 - t$

$t = 2 - X$

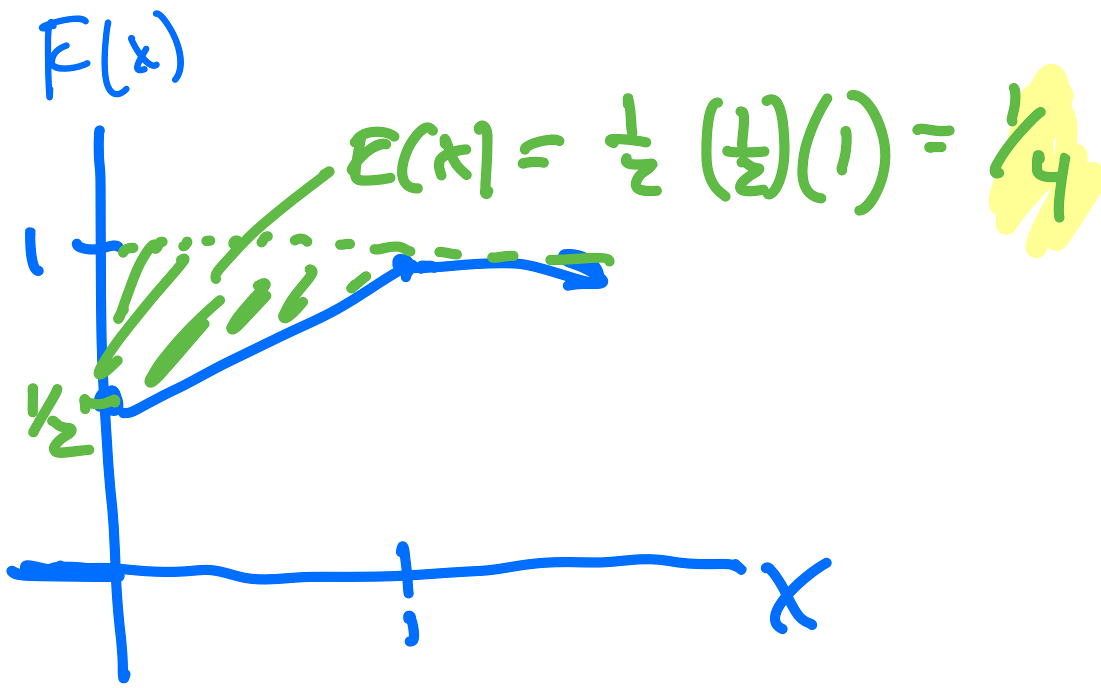
$$f_X(x) = \left|\frac{dx}{dt}\right| f_T(t)$$

$$= 1 \cdot \frac{1}{2} \text{ for } 1 < X < 2$$

So the mixed density of X looks like



Then draw $F(x)$:



Last time

sec 4.5 Expectation of a nonnegative RV using CDF

$$E(X) = \int_0^\infty (1 - F(x))\,dx$$

ex let $X \sim \text{Geom}\left(\frac{1}{2}\right)$

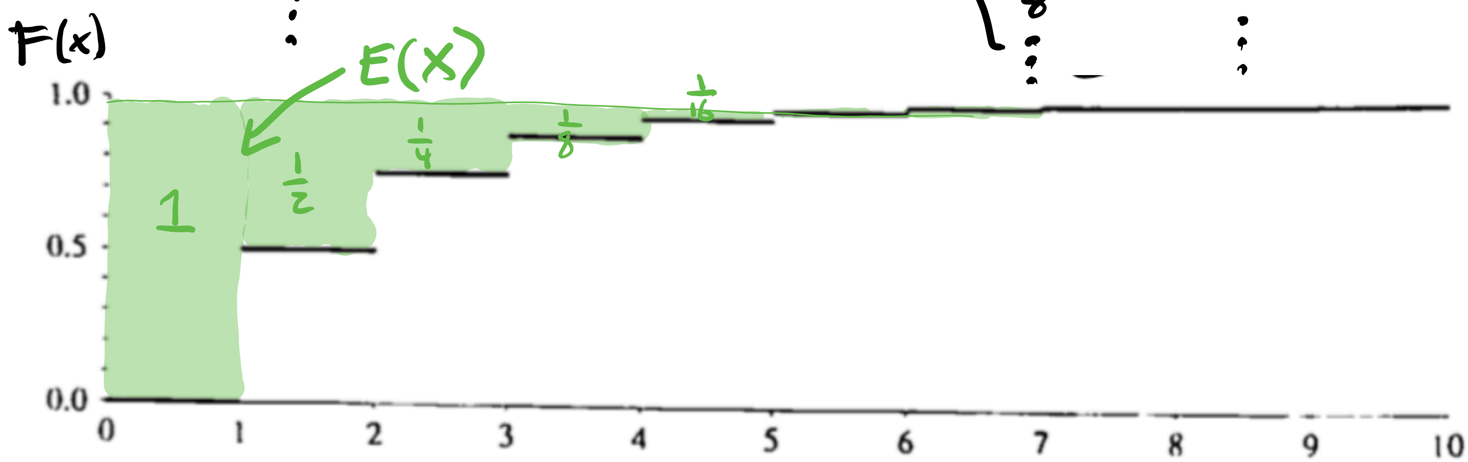
$P(X=1) = 1/2$

$P(X=2) = \frac{1}{2}\cdot\frac{1}{2} = 1/4$

Picture $P(X=3) = \left(1/2\right)^2 \cdot \frac{1}{2} = 1/8$

$\vdots$

$$\Rightarrow F(x) = \begin{cases} 0 & x < 1 \\ 1/2 & 1 \le x < 2 \\ \frac{3}{4} & 2 \le x < 3 \\ \frac{7}{8} & 3 \le x < 4 \\ \vdots & \vdots \end{cases}$$



$$E(X) = \int_0^\infty (1 - F(x))\,dx$$

$$E(X) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = \boxed{2}$$

p

$$= \sum_{j=0}^{\infty} P(X > j) = \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j \longleftarrow \text{tail sum formula.}$$

$\parallel$

$q^j = (1/2)^j$

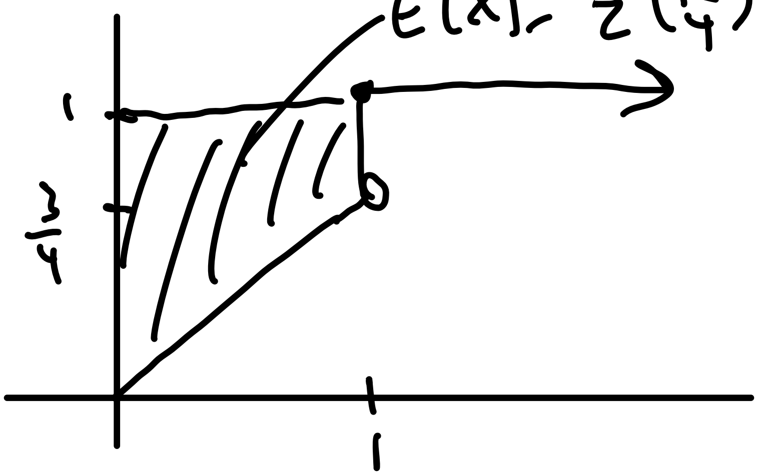
ex

A random variable X has CDF

$$F(x) = \begin{cases} \frac{3x}{4} & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

The expectation of X is:

$E(X) = \frac{1}{2}\left(\frac{3}{4}\right) + \frac{1}{4} = \frac{5}{8}$



① Overview of what we have learned since the midterm.
② sec 4.6 Order statistics
③ sec 4.6 Beta distribution

① Overview

Chap 4

single variable unconditional prob

density of distribution

change of variable formula for densities,

expectation

continuous distributions

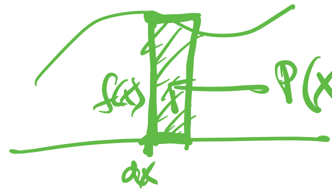
- uniform
- exponential/gamma

MGF - useful tool — calculate moments; identify a distribution by its MGF

CDF / mixed distributions

calculating expectation from cdf.

order statistics / beta distribution (today)



$P(X \in dx) \approx f(x)\,dx$

Chap 5

multiple variable unconditional prob

joint distributions

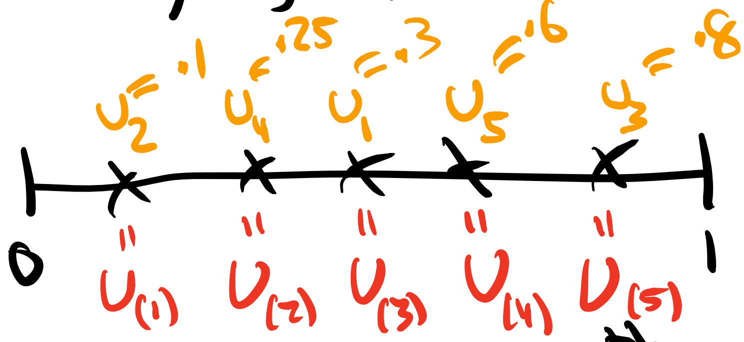
Chap 6

multiple variable conditional prob.

dependence

② Sec 4.6 order statistic of $U(0,1)$

let $U_1, \dots, U_n \overset{iid}{\sim} \text{Unif}(0,1)$



throw 5 darts at $[0,1]$.

let $U_{(r)}$ = called the r^{th} order statistic

$= r^{th}$ value of $U_1, \dots, U_n$ sorted smallest to biggest.

(assuming no ties)

ex

$U_{(1)} = \min(U_1, \dots, U_n)$

$U_{(n)} = \max(U_1, \dots, U_n)$

Review counting

You have 3 red, 2 green and 5 blue marbles, How many orderings of these 10 marbles are there?

ex rrrggbbbbb

grrrgbbbbb

ggrrrbbbbb

⋮

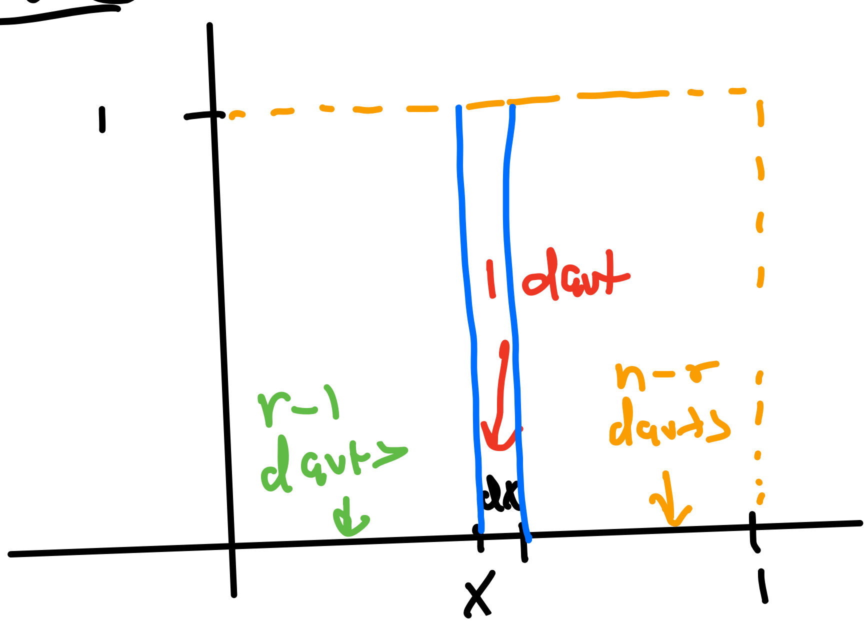
$$\frac{10!}{3!\,2!\,5!}$$

$$\binom{10}{3,2,5} = \binom{10}{3}\binom{7}{2}\binom{5}{5}$$

Next, find density of $U_{(r)}$:

Let $U_1, \ldots, U_n \overset{iid}{\sim} U(0,1)$ (i.e throw n darts at $(0,1)$)

Picture



$U_{(r)} \in dx$ means that $r-1$ darts are between 0 and x,

and one is in dx, and $n-r$ darts are between x and 1

$$\underbrace{P(U_{(r)} \in dx)}_{= f_{U_{(r)}} dx} = P\Big(\underbrace{r-1 \text{ darts} \in (0,x)}_{\binom{n}{r-1} x^{r-1}},\ 1 \text{ dart} \in dx,\ \underbrace{n-r \text{ darts} \in (x,1)}_{\binom{n-r+1}{1} 1 dx}\Big)$$

$$= \underbrace{P\big(r-1 \text{ darts} \in (0,x)\big)}_{=} \cdot \underbrace{P\big(1 \text{ dart} \in dx \mid r-1 \text{ darts} \in (0,x)\big)}_{=}$$

$$\cdot \underbrace{P\big(n-r \text{ darts} \in (x,1) \mid 1 \text{ dart} \in dx,\ r-1 \text{ darts} \in (0,x)\big)}_{= \binom{n-r}{n-r} \cdot (1-x)^{n-r}}$$

$$= \binom{n}{r-1} x^{r-1} \binom{n-r+1}{1} 1 dx \binom{n-r}{n-r} (1-x)^{n-r}$$

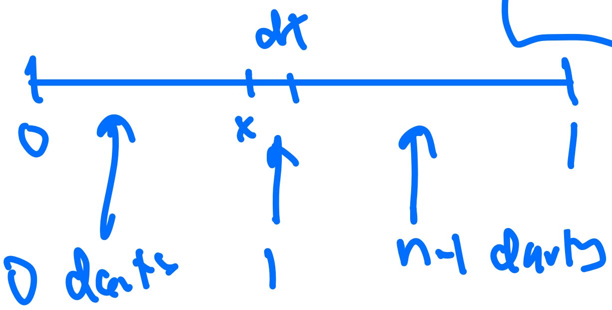
$$= \underbrace{\binom{n}{r-1,\,1,\,n-r} x^{r-1} (1-x)^{n-r}}_{f_{U_{(r)}}(x)} dx$$

for $0 < x < 1$

Ex Let $U_1, \ldots, U_n \overset{iid}{\sim} \mathrm{Unif}(0,1)$

Find the density of $U_{(1)}$

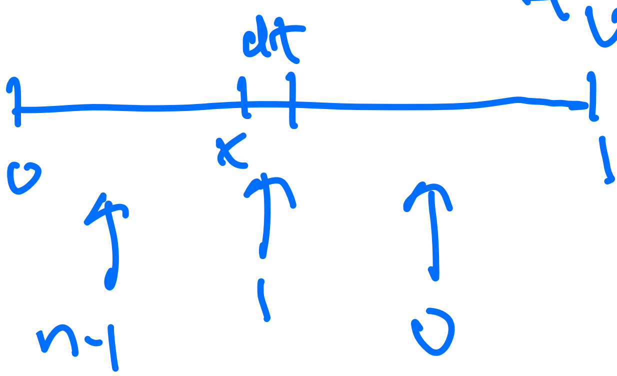
$$\boxed{f_{U_{(1)}}(x) = \binom{n}{0,1,n-1} 1 \cdot (1-x)^{n-1}}$$



Ex Let $U_1, \ldots, U_n \overset{iid}{\sim} \mathrm{Unif}(0,1)$

Find the density of $U_{(n)}$

$$f_{U_{(n)}} = \binom{n}{n-1,1,0} x^{n-1}$$

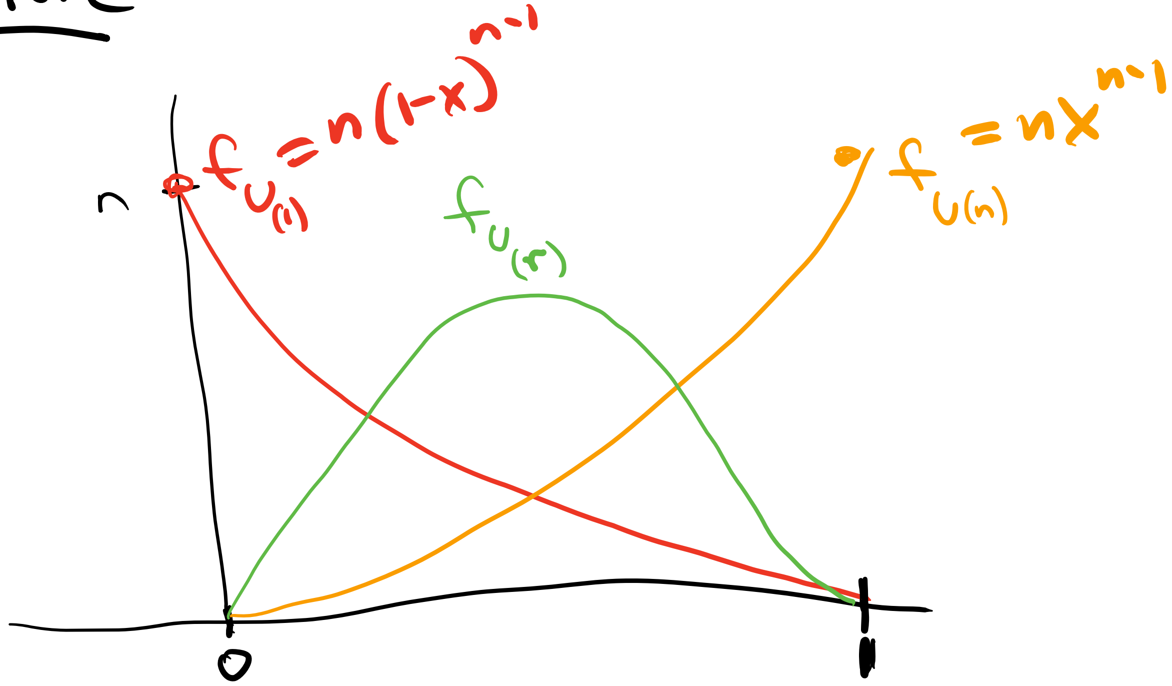


How would this change if $U_1, \ldots, U_n \overset{iid}{\sim} \mathrm{Unif}(0,b)$?

$$f_{U_{(n)}}(t) = \binom{n}{n-1,1,0} \frac{1}{b} \cdot \left(\frac{x}{b}\right)^{n-1}$$

Order statistic of $U(0,1)$ provides a family of densities on the unit interval.

Picture



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Stat 134

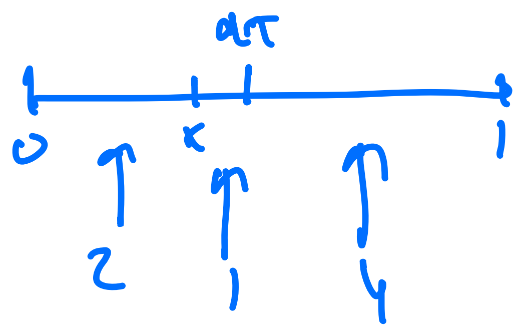
1. $x^2(1-x)^4$ for $0 < x < 1$ is the variable part of the density of what random variable?

   a $U_{(3)}$ of n=6 darts

   b $U_{(2)}$ of n=7 darts

   c $U_{(1)}$ of n=7 darts

   d none of the above



$U_{(3)}$ of 7

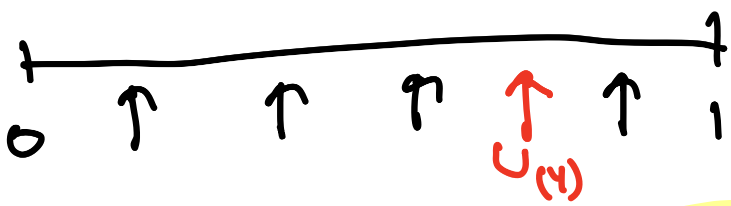
③ Sec 4.6 Beta distribution.

overview on standard uniform order statistic

let $U_1, \ldots, U_5 \overset{iid}{\sim} U(0,1)$

$U_{(r)}$ is the r^{th} sorted element.

Picture



Notice there are 4 gaps before and 2 gaps after $U_{(4)}$.

Define Beta(4,2) to be the distribution of $U_{(4)}$ out of 5.

i.e Beta(4,2) has density

$$f(x) = \binom{5}{3,1,1} x^3 (1-x) \quad \text{for} \quad 0<x<1$$

Notice that

$$n = \underbrace{r+s}_{\text{total number of gaps}} - 1$$

n: total number of darts

ex Beta(5,10) has 5 gaps before and 10 gaps after what RV? $U_{(5)}$ of 14 ($n = r+s-1$)

Let $r, s \in \mathbb{Z}^+$

Beta(r,s) has r gaps before and s gaps after?

More generally!

Def^n Let $r, s > 0$

$X \sim \text{Beta}(r,s)$ if

$$f(x) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1}(1-x)^{s-1} \quad \text{for } 0 < x < 1.$$

where $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t}\, dt$ Gamma function for $r > 0$

or $\Gamma(r) = (r-1)!$ if $r \in \mathbb{Z}^+$