X1.

$x^2 = x \iff x^2 - x = 0$

$\iff x(x-e) = 0$, where e is the multiplicative identity.

$\iff x = 0$ or $x = e$, since x is an element in the integral domain.

Since the multiplicative identity is unique and $e \neq 0$, there're exactly 2 solutions to $x^2 = x$, namely $x=0$ & $x=e$.

X2.

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ a & b \end{pmatrix} \xrightarrow[r_3 - ar_1]{r_2 - 3r_1} \begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 0 & b-2a \end{pmatrix} \xrightarrow[-\frac{1}{2}r_2]{r_1 + r_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & b-2a \end{pmatrix} \xrightarrow{r_3 - (b-2a)r_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 3 & a \\ 2 & 4 & b \end{pmatrix} \xrightarrow{-\frac{1}{2}(r_2 - 2r_1)} \begin{pmatrix} 1 & 3 & a \\ 0 & 1 & -\frac{1}{2}(b-2a) \end{pmatrix} \xrightarrow{r_1 - 3r_2} \begin{pmatrix} 1 & 0 & a + \frac{3}{2}(b-2a) \\ 0 & 1 & -\frac{1}{2}(b-2a) \end{pmatrix}$

(c) $M = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & a & a^2 \end{pmatrix} \xrightarrow[r_3 - r_1]{r_2 - r_1} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & a-2 & a^2-4 \end{pmatrix} \xrightarrow[r_3 - (a-2)r_2]{r_1 - 2r_2} \begin{pmatrix} 1 & 0 & -6 \\ 0 & 1 & 5 \\ 0 & 0 & (a-2)(a-3) \end{pmatrix} =: M^*$

case i. if $a=2$ or $a=3$, $\mathrm{RREF}(M) = \begin{pmatrix} 1 & 0 & -6 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$

case ii. otherwise,

$M^* \xrightarrow{r_3/(a-2)(a-3)} \begin{pmatrix} 1 & 0 & -6 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_2 - 5r_3]{r_1 - (-6)r_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

X3.1 $\left(\begin{array}{ccc|c} i & i-1 & 3 & 1 \\ i-1 & i & 2 & -1 \end{array}\right) \xrightarrow[-ir_2]{-ir_1} \left(\begin{array}{ccc|c} 1 & i+1 & -3i & -i \\ i+1 & 1 & -2i & +i \end{array}\right) \xrightarrow{r_2 - (i+1)r_1} \left(\begin{array}{ccc|c} 1 & 1+i & -3i & -i \\ 0 & 1-2i & i-3 & 2i-1 \end{array}\right)$

$\xrightarrow{r_1 - r_2 \times \frac{1+i}{1-2i}} \left(\begin{array}{ccc|c} 1 & 0 & -3i-(i-3)\times\frac{1+i}{1-2i} & 1 \\ 0 & 1-2i & i-3 & 2i-1 \end{array}\right)$

$\xrightarrow{r_2/(1-2i)} \left(\begin{array}{ccc|c} 1 & 0 & \frac{-2-i}{1-2i} & 1 \\ 0 & 1 & \frac{i-3}{1-2i} & -1 \end{array}\right)$

Thus, $\begin{cases} x_1 = 1 + \frac{2+i}{1-2i} x_3 \\ x_2 = -1 - \frac{i-3}{1-2i} x_3 \end{cases}$

$x_{3,2}$

Similarly, $\left(\begin{array}{ccc|c} i & i-1 & 3 & 2 \\ i-1 & i & 2 & -2 \end{array}\right) \xrightarrow[\substack{r_1 - r_2 \times \frac{1+i}{1-2i} \\ r_2 \times \frac{1}{1-2i}}]{\substack{-ir_1,\ -ir_2 \\ r_2 - (i+1) r_1}} \left(\begin{array}{ccc|c} 1 & 0 & \frac{-2i}{1-2i} & 2 \\ 0 & 1 & \frac{i-3}{1-2i} & -2 \end{array}\right)$

Thus, $\begin{cases} x_1 = 2 + \frac{2+i}{1-2i} x_3 \\ x_2 = -2 - \frac{i-3}{1-2i} x_3. \end{cases}$

x_4.

(a) $\begin{bmatrix} 1 & a & b & c \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & a & b & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & a & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}, \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & a & b \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & a & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & a \\ 0 & 0 & 1 & b \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 1 & a \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, where $a, b, c, d \in \mathbb{R}$

(b) Similarly, just that $a, b, c, d \in \mathbb{Z}_2$

Thus, there's $2^3 + 2^2 + 2^3 + 2^4 + 2^2 + 2 + 2^2 + 2 + 1 + 1 + 1 = 51$ number of possibilities 2×4 RREF form in $\mathbb{Z}_2$

Sec 1.3 P_7

W.L.O.G, we interchange the row i and row j in finite sequence of elementary as following:

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_i \\ \vdots \\ \alpha_j \\ \vdots \\ \alpha_n \end{pmatrix} \xrightarrow[r_j - r_i]{r_i + r_j} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_i + \alpha_j \\ \vdots \\ -\alpha_i \\ \vdots \\ \alpha_n \end{pmatrix} \xrightarrow[r_j \times (-1)]{r_i + r_j} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_j \\ \vdots \\ \alpha_i \\ \vdots \\ \alpha_n \end{pmatrix}$$

Sec 1.4 P6.

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 1 & 1 & -1 & 1 & 2 \\ 1 & 7 & -5 & -1 & 3 \end{array}\right] \xrightarrow[r_3 - r_1]{r_2 - r_1} \left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 0 & 3 & -2 & -1 & 1 \\ 0 & 9 & -6 & -3 & 2 \end{array}\right] \xrightarrow{r_3 - 3r_2} \left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 0 & 3 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

Since the last row implies $0=1$, there's no solution to the system.

Sec 1.4 P7.

$$\left[\begin{array}{ccccc|c} 2 & -3 & -7 & 5 & 2 & -2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 2 & 0 & -4 & 2 & 1 & 3 \\ 1 & -5 & -7 & 6 & 2 & -7 \end{array}\right] \xrightarrow[\substack{\frac{1}{2}r_3 \\ r_4 - r_2}]{r_1 - 2r_2} \begin{bmatrix} 0 & 1 & 1 & -1 & 0 & 2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 1 & 0 & -2 & 1 & \frac{1}{2} & 3/2 \\ 0 & -3 & -3 & 3 & 1 & -5 \end{bmatrix}$$

$$\xrightarrow[r_4 + 3r_1]{r_3 - r_2} \begin{bmatrix} 0 & 1 & 1 & -1 & 0 & 2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 0 & 2 & 2 & -2 & -1/2 & 7/2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[r_2 + 2r_1]{r_2 + r_3} \begin{bmatrix} 0 & 1 & 1 & -1 & 0 & 2 \\ 1 & 0 & -2 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[r_2 - \frac{1}{2}r_4]{r_3 + \frac{1}{2}r_4} \begin{bmatrix} 0 & 1 & 1 & -1 & 0 & 2 \\ 1 & 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $\begin{cases} x_1 = 2 + x_4 - x_3 \\ x_2 = 1 - x_4 + 2x_3 \\ x_5 = 1 \end{cases}$

Sec 1.5 - 1

$$ABC = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}$$

$$CAB = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = [3-3] = [0]$$

Sec 1.5.3.

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$