MAT3007 · Homework 8

Due: 11:59 pm, Dec 9, 2022

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- You must submit your assignment in Blackboard. Please upload a pdf file. The file name should be in the format **last name-first name-hw8**.
- The homework must be written in English.
- Late submission will not be graded.
- Each student **must not copy** homework solutions from another student or from any other source.

Problem 1 Gradient Descent Method. (50 pts)
Implement the gradient descent method to solve the optimization problem:

$$\min_{x\in\mathbb{R}^2}\ f(x) = e^{1-x_1-x_2} + e^{x_1+x_2-1} + x_1^2 + x_1x_2 + x_2^2 + x_1 - 3x_2.$$

The following input parameters should be considered:

- $x^0 = (0,0)^\top$ – the initial point.
- $\texttt{tol} = 10^{-5}$ – the tolerance parameter. The method should stop whenever the current iterate x^k satisfies the criterion $\|\nabla f(x^k)\| \le \texttt{tol}$.
- $\sigma, \gamma = \frac{1}{2}$ – parameters for backtracking and the Armijo condition.

Implement different methods using the given parameter.

a) Implement the gradient descent method using constant stepsize. Choose $\alpha_1 = 1$ and $\alpha_2 = 0.1$, report the number of iterations, the final objective function value, and the point to which the methods converged. (15 pts)

b) Implement the gradient descent method using backtracking (Armijo) line search method, report the number of iterations, the final objective function value, and the point to which the methods converged. (15 pts)

c) Plot figures of the solution path for each of the different step size strategies. (20 pts)

Please include **(1) essential parts of code (2) output results and figures** in your answer. Your plots should look similar to Figure 1.

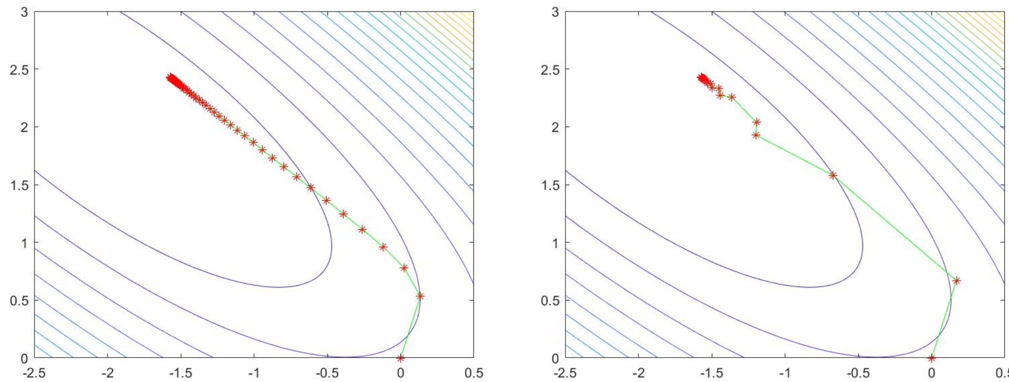

Figure 1: Solution path using constant step for $\alpha_2 = 0.1$ and backtracking, respectively.

Problem 2 Newton's Method. (30 pts)
Implement the Newton's method to solve the same problem in **Problem 1**.
Use only Armijo backtracking line search. Report the number of iterations, the final objective function value, the point to which the methods converged. (10 pts)
Plot figures of the solution path. (10 pts)
Hint: You could use some implemented functions to calculate matrix inversion. And your plot should look similar to Figure 2.

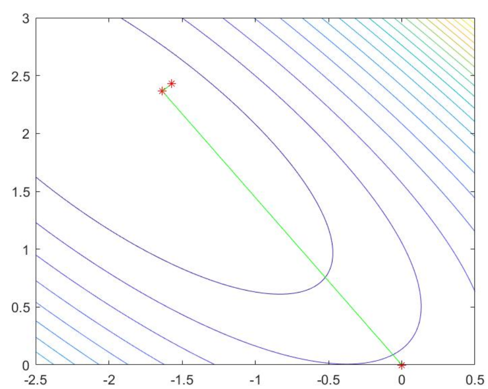

Figure 2: Solution path using Newton's Method.

Problem 3 Projection onto Convex Sets. (20 pts)
In the class, we have seen interesting examples of projection onto some convex sets, described by linear constraints, box constraints, and ball constraints (Page 8, Lec 23 of Prof.Pu or Page 17, Lec 24 of Prof.Wu).

Another important convex set, Δ_d, is standard simplex (or so-called probability simplex), where

$$\Delta_d := \left\{ \mathbf{x} \in \mathbb{R}^d : \sum_{i=1}^{d} x_i = 1, x_i \geq 0 \quad \forall i \right\}$$

Obviously, it is a convex set.
Now, you are supposed to prove the following statement.
Let $\mathbf{x}^* := \operatorname{argmin}_{\mathbf{x} \in \Delta_d} \|\mathbf{x} - \mathbf{v}\|_2^2$, which means we project a point $\mathbf{v}$ in $\mathbb{R}^d$ onto a standard simplex. Under the assumption that $v_1 \geq v_2 \geq v_3 \geq \cdots \geq v_d$, there exists $p \in \{1, 2, \cdots, d\}$, which is also unique, such that

$$\begin{aligned} x_i^* &> 0 \qquad i \leq p \\ x_i^* &= 0 \qquad i > p \end{aligned}$$