The Chinese University of Hong Kong, Shenzhen
School of Data Science

MAT3007 · Homework 3

Due: 11:59.59 pm, Oct 20, 2021

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- You must submit your assignment in Blackboard. If some problems need coding, you should paste your code in the file. Please upload only one file(pdf). The file name should be in the format **last name-first name-hw2. Any nonstandard assignment will not be graded**..
- The homework must be written in English.
- **Late submission will not be graded.**
- Each student **must not copy** homework solutions from another student or from any other source.

Problem 1 (5+5+10=20pts).
Consider the following linear program:

$$\begin{array}{rll} \text{maximize} & x_1 + 3x_2 + 3x_3 + 8x_4 & \\ \text{subject to} & x_1 - x_2 + x_3 & \leq 2 \\ & x_3 - x_4 & \leq 1 \\ & 2x_2 + 3x_3 + 4x_4 & \leq 10 \\ & x_1, x_2, x_3, x_4 & \geq 0. \end{array}$$

(a) Transform this problem to standard form and find a trivial BFS.

(b) Compute the reduced costs $\bar{\mathbf{c}}$.

(c) Choose the incoming basis according to **Bland's Rule**, compute the step size θ^* and and jth basic direction, then apply the simplex method to update the current BFS to a neighboring one.

Problem 2 (20pts). Consider the same linear program in **Problem 1**, please use simplex tableau to completely solve it. For each step, draw the simplex tableau. Clearly mark what is the current basis, the current basic solution, and the corresponding objective function value. You can start from the same trivial BFS found in **Problem 1**. Also, compare the BFS obtained in **Problem 1** and the second tableau obtained in this question. They should be consistent.

Problem 3 (20pts).
Use the two-phase simplex method (implemented by simplex tableau) to completely solve the linear optimization problem. For each step, draw the simplex tableau. Clearly mark what is the current basis, the current basic solution, and the corresponding (negative) objective function value.

$$\begin{aligned} \text{minimize} \quad & x_1 + 3x_2 + x_4 - 2x_5 \\ \text{subject to} \quad & x_1 + 2x_2 + x_4 + x_5 &&= 2 \\ & x_1 + 2x_2 - 6x_4 + x_5 &&= 2 \\ & x_1 + 4x_2 - 3x_3 + x_4 &&= -1 \\ & x_1, x_2, x_3, x_4, x_5 &&\geq 0. \end{aligned}$$

Problem 4 (20pts).
Apply the two-phase simplex method (implemented by simplex tableau) to solve the following linear program. For each step, draw the simplex tableau. Clearly mark what is the current basis, the current basic solution, and the corresponding (negative) objective function value.

$$\begin{aligned} \text{minimize} \quad & x_1 - x_2 + 3x_3 \\ \text{subject to} \quad & 2x_1 - x_2 + 4x_3 &&\leq -1 \\ & x_1 - x_2 - x_3 &&\leq 4 \\ & x_2 - x_4 &&= 0 \\ & x_1, x_2, x_3, x_4 &&\geq 0. \end{aligned}$$

Problem 5 Conditions for a Unique Optimum (10+10=20pts).
Let x^* be a basic feasible solution associated with some basic indices B. Prove the following:

a) If the reduced cost of every non-basic variable is positive, then x^* is the unique optimal solution.

   Hint: Let $y \in \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}$ be given with $y \neq x^*$. First, show that there exists $\ell \in N = B^{\mathsf{C}}$ such that $y_\ell > 0$. You can then mimic the proof of the theorem of stopping criterion based on the reduced costs.

b) If x^* is the unique optimal solution and if x^* is nondegenerate, then the reduced cost of every non-basic variable is positive.

   Hint: The construction of the simplex method via basic directions can be helpful.