MAT3007 Homework 5 Student ID: 121090429

Problem 1 (50pts). Consider the following linear program:

$$\begin{array}{lll} \text{maximize} & 3x_1 + 4x_2 + 3x_3 + 6x_4 & \\ \text{subject to} & 2x_1 + x_2 - x_3 + x_4 & \geq 12 \\ & x_1 + x_2 + x_3 + x_4 & = 8 \\ & -x_2 + 2x_3 + x_4 & \leq 10 \\ & x_1, x_2, x_3, x_4 & \geq 0. \end{array} \quad (1)$$

After transforming the problem into standard form and apply Simplex method, we obtain the final tableau as follow:

B	0	2	9	0	3	0	36
1	1	0	−2	0	−1	0	4
4	0	1	3	1	1	0	4
6	0	−2	−1	0	−1	1	6

a) Derive the dual problem of the linear program (1) and calculate a dual solution based on complementarity conditions. Given that the optimal solution to the primal solution is unique, investigate whether the dual solution is unique.

b) Do the optimal solution and the objective function value change if we

- decrease the objective function coefficient for x_3 to 0?
- increase the objective function coefficient for x_3 to 9?
- decrease the objective function coefficient for x_4 to 5?
- increase the objective function coefficient for x_1 to 7?

e) Find the possible range for adjusting the coefficient 8 of the second constraint such that the current basis is kept optimal.

(a) Dual Problem:

$$\begin{array}{ll} \text{minimize} & 12y_1 + 8y_2 + 10y_3 \\ \text{s.t.} & 2y_1 + y_2 \geqslant 3 \\ & y_1 + y_2 - y_3 \geqslant 4 \\ & -y_1 + y_2 + 2y_3 \geqslant 3 \\ & y_1 + y_2 + y_3 \geqslant 6 \\ & y_1 \leqslant 0,\ y_2 \text{ free},\ y_3 \leqslant 0 \end{array}$$

By the tableau, the optimal solution for primal problem is $(4, 0, 0, 4)$

i.e. $x_1 = x_4 = 4,\ x_2 = x_3 = 0$

By the complementarity conditions,

$$x_1(2y_1 + y_2 - 3) = 0$$
$$x_4(y_1 + y_2 + y_3 - 6) = 0$$
$$y_1(2x_1 + x_2 - x_3 + x_4 - 12) = 0$$
$$y_2(x_1 + x_2 + x_3 + x_4 - 8) = 0$$
$$y_3(-x_2 + 2x_3 + x_4 - 10) = 0$$

$$\Rightarrow \begin{cases} y_1 = -3 \\ y_2 = 9 \\ y_3 = 0 \end{cases}$$

The dual solution is $(-3, 9, 0)$

The dual solution is unique.

(b) (i) $c = [3, 4, 3, 6, 0, 0]$

$\tilde{c} = [3, 4, 0, 6, 0, 0]$

The new reduced cost is:

$$\tilde{c}^T - \tilde{c}_B^T A_B^{-1} A$$

The reduced cost of basic parts is still zero.

For non-basic part:

$$\tilde{r}_N^T = c_N^T + (-3) e_3^T - c_B^T A_B^{-1} A_N = r_N^T + (-3) e_3^T$$

Know from the tableau,

$$r_N^T = [2, 9, 3]$$

$$\tilde{r}_N^T = [2, 9, 3] + (-3)[0, 1, 0]$$

$$= [2, 6, 3] > 0$$

The the optimal solution and the objective function value don't change.

(ii) $c = [3, 4, 3, 6, 0, 0]$

$\tilde{c} = [3, 4, 9, 6, 0, 0]$

The new reduced cost is:

$$\tilde{c}^T - \tilde{c}_B^T A_B^{-1} A$$

The reduced cost of basic parts is still zero.

For non-basic part:

$$\tilde{r}_N^T = c_N^T + 9 e_3^T - c_B^T A_B^{-1} A_N = r_N^T + 9 e_3^T$$

Know from the tableau,

$$r_N^T = [2, 9, 3]$$

$$\tilde{r}_N^T = [2, 9, 3] + 9[0, 1, 0]$$

$$= [2, 18, 3] > 0$$

The the optimal solution and the objective function value don't change.

(iii) $c = [3, 4, 3, 6, 0, 0]$

$\tilde{c} = [3, 4, 3, 5, 0, 0]$

The new reduced cost is:

$\tilde{c}^T - \tilde{c}_B^T A_B^{-1} A$

The reduced cost of basic parts is still zero.

For non-basic part:

$$\tilde{r}_N^T = c_N^T - (c_B^T - e_4^T) A_B^{-1} A_N$$
$$= r_N^T - (-1) e_4^T A_B^{-1} A_N$$
$$= [2, 9, 3] - (-1)[0, 1, 0] \begin{bmatrix} 0 & -2 & -1 \\ 1 & 3 & 1 \\ -2 & -1 & -1 \end{bmatrix}$$
$$= [2, 9, 3] + [1, 3, 1]$$
$$= [3, 12, 4] > 0$$

The the optimal solution and the objective function value don't change.

(iv) $c = [3, 4, 3, 6, 0, 0]$

$\tilde{c} = [7, 4, 3, 6, 0, 0]$

The new reduced cost is:

$\tilde{c}^T - \tilde{c}_B^T A_B^{-1} A$

The reduced cost of basic parts is still zero.

For non-basic part:

$$\tilde{r}_N^T = c_N^T - (c_B^T + 4e_1^T) A_B^{-1} A_N$$
$$= r_N^T - 4 e_1^T A_B^{-1} A_N$$
$$= [2, 9, 3] - 4[1, 0, 0] \begin{bmatrix} 0 & -2 & -1 \\ 1 & 3 & 1 \\ -2 & -1 & -1 \end{bmatrix}$$
$$= [2, 9, 3] - 4[0, -2, -1]$$
$$= [2, 17, 7] > 0$$

The the optimal solution and the objective function value don't change.

(e) To keep the current basis optimal, we need to have

$$x_B^* + \lambda A_B^{-1} e_i \geq 0$$

$$\Rightarrow [4,4,6] + \lambda \begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= [4-\lambda, 4+\lambda, 6-\lambda]$$

we have $\begin{cases} 4-\lambda \geq 0 \\ 4+\lambda \geq 0 \\ 6-\lambda \geq 0 \end{cases}$

$$\Rightarrow \lambda \in [-4,4], \quad 8+\lambda \in [4,12]$$

The possible range for adjusting the coefficient 8 of the second constraint is $[4,12]$

Problem 2 (50pts). An insurance company is introducing three products: special risk insurance, mortgage insurance, and long-term care insurance. The expected profit is \$500 per unit on special risk insurance, \$250 per unit on mortgage insurance and \$600 per unit on long term care insurance. The work requirements are as follows:

Department	Working hours per unit			Working hours available
	Special risk	Mortgage	Long-term care	
Underwriting	2	1	1	240
Administration	3	1	2	150
Claims	1	2	4	180

The management team wants to establish sales quotas for each product to maximize the total expected profit.

1. Formulate this problem as a linear optimization problem. Specify the decision variables, objective function, and constraints.
2. After solving the problem, the final simplex tableau (for the standard form) is given as below (the variables are in the natural order as in the description of the problem):

B	0	50	0	0	140	80	35400
4	0	0.5	0	1	-0.7	0.1	153
1	1	0	0	0	0.4	-0.2	24
3	0	0.5	1	0	-0.1	0.3	39

   Show the dual variables corresponding to the services of the three departments. Using complementarity conditions to explain why mortgage insurance is not sold.
3. Find the range of working hours available for underwriting to keep the current basis optimal.
4. Find the range of the expected profit on long-term care insurance such that the current basis remains optimal.

(1) Denote sale quotas of special risk, mortgage and long-term care that are sold as X_1, X_2, X_3

The profit generated is: $500X_1 + 250X_2 + 600X_3$ (objective function)

Formulation:

$$\begin{aligned} \text{maximize} \quad & 500X_1 + 250X_2 + 600X_3 \\ \text{subject to} \quad & 2X_1 + X_2 + X_3 \leq 240 \\ & 3X_1 + X_2 + 2X_3 \leq 150 \end{aligned}$$

$$x_1 + 2x_2 + 4x_3 \leq 180$$
$$x_1, x_2, x_3 \geq 0$$

(2) The standard form of primal problem:

$$\begin{aligned} \text{minimize} \quad & -500x_1 - 250x_2 - 600x_3 \\ \text{subject to} \quad & 2x_1 + x_2 + x_3 + x_4 = 240 \\ & 3x_1 + x_2 + 2x_3 + x_5 = 150 \\ & x_1 + 2x_2 + 4x_3 + x_6 = 180 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

The dual variables y_1, y_2, y_3 are the fair prices of underwriting, administration and claims 3 departments.

The dual problem is:

$$\begin{aligned} \text{minimize} \quad & 240y_1 + 150y_2 + 180y_3 \\ \text{subject to} \quad & 2y_1 + 3y_2 + y_3 \geq 500 \\ & y_1 + y_2 + 2y_3 \geq 250 \\ & y_1 + 2y_2 + 4y_3 \geq 600 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

By the tableau, $(x_1, x_2, x_3) = (24, 0, 39)$

By complementarity conditions,

$$\begin{aligned} & x_1(2y_1 + 3y_2 + y_3 - 500) = 0 \\ & x_2(y_1 + y_2 + 2y_3 - 250) = 0 \\ & x_3(y_1 + 2y_2 + 4y_3 - 600) = 0 \\ & y_1(2x_1 + x_2 + x_3 + x_4 - 240) = 0 \\ & y_2(3x_1 + x_2 + 2x_3 + x_5 - 150) = 0 \\ & y_3(x_1 + 2x_2 + 4x_3 + x_6 - 180) = 0 \end{aligned}$$

We get: $\begin{cases} y_1 = 200 \\ y_2 = 0 \\ y_3 = 100 \end{cases}$

In optimal situation, $y_1 + y_2 + 2y_3 - 250 > 0$ if $x_2 \neq 0$, this means the resource is redundant in the optimal situation.

The shadow price is zero.

So x_2 must be zero, that's why mortgage is not sold.

(3) Current basis: $\{1, 3, 4\}$

To keep the current basis optimal, we need to have

$$x_B^* + \lambda A_B^{-1} e_i \geq 0$$

$$\begin{bmatrix} 24 \\ 39 \\ 153 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0.4 & -0.2 \\ 0 & -0.1 & 0.3 \\ 1 & -0.7 & 0.1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 24 \\ 39 \\ 153+\lambda \end{bmatrix}$$

$\Rightarrow 153+\lambda \geq 0$

$\Rightarrow \lambda \geq -153$

So $240+\lambda \in [87,+\infty)$

The range of working hour available for underwriting is $[87,+\infty)$

(4) $C = [500, 250, 600, 0, 0, 0]^T$

$r_N = [50, 140, 80]^T$

$$A_B^{-1}A_N = \begin{bmatrix} 0 & 0.4 & -0.2 \\ 0.5 & -0.1 & 0.1 \\ 0.5 & -0.7 & 0.3 \end{bmatrix}$$

Change profit from 600 to $600+\lambda$ ($-600-\lambda$ in the standard form)

$r_N^T - \lambda A_B^{-1}A_N e_3^T$

$$= [50, 140, 80]^T - \lambda \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.4 & -0.1 & -0.7 \\ -0.2 & 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$= [50, 140, 80]^T + \lambda [0.5, -0.7, 0.1]^T$$

$$\begin{cases} 50+0.5\lambda \geq 0 \\ 140-0.7\lambda \geq 0 \\ 80+0.1\lambda \geq 0 \end{cases} \Rightarrow \lambda \in [-100, 200]$$

$250+\lambda \in [150, 450]$

The range of expected profit on long-term care insurance is $[150, 450]$