The Chinese University of Hong Kong, Shenzhen
School of Data Science

MAT3007 · Homework 7

Due: 11:59 pm, Dec 2, 2022

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- You must submit your assignment in Blackboard. If some problems need coding, you should paste your code in the file. Please upload only **one** file(pdf). The file name should be in the format **last name-first name-hw7**.
- The homework must be written in English.
- Late submission will not be graded.
- Each student **must not copy** homework solutions from another student or from any other source.

Problem 1 (25pts). Suppose that $f : \mathbf{R} \to \mathbf{R}$ is convex, and $a, b \in \operatorname{dom} f$ with $a < b$, where dom denotes the domain of the function. More specifically, $f : \mathbf{R}^p \to \mathbf{R}^q$ means that f is an $\mathbf{R}^p$-valued function on some *subset* of $\mathbf{R}^p$, and this *subset* of $\mathbf{R}^p$ is the domain of the function f. Show that

(a)

$$f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b), \text{ for all } x \in (a, b)$$

Hint: (Jensen's Inequality) If $p_1, ..., p_n$ are positive numbers which sum to 1 and f is a real continuous function that is convex, then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i)$$

(b)

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a} \leq \frac{f(b) - f(x)}{b - x}$$

for all $x \in (a, b)$. Draw a sketch that illustrates this inequality.

(c) Suppose f is differentiable. Use the result in (b) to show that:

$$f'(a) \leq \frac{f(b) - f(a)}{b - a} \leq f'(b)$$

Note that these inequalities also follow form:

$$f(b) \geq f(a) + f'(a)(b - a), \quad f(a) \geq f(b) + f'(b)(a - b)$$

(d) Suppose f is twice differentiable. Use the result in (c) to show that $f''(a) \geq 0$ and $f''(b) \geq 0$.

Problem 2 (30pts) Show that the following functions are convex:

(a) $f(x) = -\log\left(-\log\left(\sum_{i=1}^m e^{a_i^T x + b_i}\right)\right)$ on $\mathrm{dom}\, f = \left\{x \mid \sum_{i=1}^m e^{a_i^T x + b_i} < 1\right\}$. You can use the fact that $\log\left(\sum_{i=1}^n e^{y_i}\right)$ is convex.

(b)

$$f(x,u,v) = -\log\left(uv - x^T x\right) \text{ on } \mathrm{dom}\, f = \left\{(x,u,v) \mid uv > x^T x, u, v > 0\right\}$$

(c) Let $T(x,\omega)$ denote the trigonometric polynomial

$$T(x,\omega) = x_1 + x_2 \cos\omega + x_3 \cos 2\omega + \cdots + x_n \cos(n-1)\omega$$

Show that the function

$$f(x) = -\int_0^{2\pi} \log T(x,\omega) d\omega$$

is convex on $\{x \in \mathbf{R}^n \mid T(x,\omega) > 0, 0 \le \omega \le 2\pi\}$.

Hint: Nonnegative weighted sum of convex functions is still convex. Let this property extend to infinite sums and integrals. Assume that $f(x,y)$ is convex in x for each $y \in \mathcal{A}$ and $w(y) \ge 0$ for each $y \in \mathcal{A}$ and integral exists. Then the function g defined as

$$g(x) = \int_{\mathcal{A}} w(y) f(x,y) dy$$

is convex in x.

Problem 3 (20pts). Consider the following function:

$$\begin{array}{ll} \text{minimize} & -x_1 - x_2 + \max\{x_3, x_4\} \\ \text{s.t.} & (x_1 - x_2)^2 + (x_3 + 2x_4)^4 \le 5 \\ & x_1 + 2x_2 + x_3 + 2x_4 \le 6 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{array}$$

(a) Verify this is a convex optimization problem.

(b) Use CVX to solve the problem.

Problem 4 (25pts). To model the influence of price on customer purchase probability, the following logit model is often used:

$$\lambda(p) = \frac{e^{-p}}{1 + e^{-p}}$$

where p is the price, $\lambda(p)$ is the purchase probability.

Assume the variable cost of the product is 0 (e.g., iPhone Apps). As the seller, you want to maximize the expected revenue by choosing the optimal price. That is, you want to solve:

$$\text{maximize}_p \quad p\lambda(p)$$

(a) Draw a picture of $r(p) = p\lambda(p)$ (for p from 0 to 10) and use the picture to show that $r(p)$ is not concave (thus maximize $r(p)$ is not a convex optimization problem)

(b) Write down p as a function of λ (the inverse function of $\lambda(p)$). Show that you can write the objective function as a function of $\lambda : \tilde{r}(\lambda)$, where $\tilde{r}(\lambda)$ is concave in λ.

(c) From part 2, write the KKT condition for the optimal λ. Then transform it back to an optimal condition in p.