MAT3007 Homework 9 Student ID: 121090429

Problem 1 (30pts).
Use the branch-and-bound method to solve the following integer program.

$$
\begin{array}{lll}
\text{maximize} & 2x + y & \\
\text{subject to} & -3x + 2y & \leq 5 \\
& -x - 2y & \leq -2 \\
& 5x + 2y & \leq 17 \\
& x, y \in \mathbb{Z}. &
\end{array}
$$

You are allowed to use an LP solver to solve each of the relaxed linear program. Please specify the branch-and-bound tree and what you did at each node .

First we solve the LP relaxation.

$$
\begin{array}{lll}
\text{minimize} & -2x-y & \\
\text{Subject to} & -3x+2y & \leq 5 \\
& -x-2y & \leq -2 \\
& 5x+2y & \leq 17
\end{array}
$$

By simplex tableau, we attain optimal solution $(x,y) = (\frac{3}{2}, \frac{19}{4})$

The branch-and-bound tree:

- S: original problem; opt-val: $\frac{31}{4}$; $x=\frac{3}{2}$, $y=\frac{19}{4}$; fractional
  - S_1: $y \leq 4$; opt-val: $\frac{38}{5}$; $x=\frac{9}{5}$, $y=4$; fractional
    - S_3: $x \leq 1, y \leq 4$; opt-val: 6; $x=1, y=4$; integer
    - S_4: $x \geq 2, y \leq 4$; opt-val: $\frac{15}{2}$; $x=2$, $y=\frac{7}{2}$; fractional
      - S_5: $x \geq 2, y \leq 3$; opt-val: $\frac{37}{5}$; $x=\frac{11}{5}$, $y=3$; fractional
        - S_7: $x \leq 2, y \leq 3$; opt-val: 7; $x=2, y=3$; integer
        - S_8: $x \geq 3, y \leq 3$; opt-val: 7; $x=3, y=1$; integer
      - S_6: $x \geq 2, y \geq 4$; infeasible
  - S_2: $y \geq 5$; infeasible

By the branch-and-bound tree,

the optimal solution is $(2,3)$ or $(3,1)$ with optimal value 7.

Problem 2 (30pts).

Consider a seller who sells m different products. For product j, there are B_j units in inventory. There are n customers, each customer i is interested in buying a bundle of the product S_i, where $S_i \subseteq \{1, ..., m\}$ and is willing to pay a price v_i for it. For each customer, the seller can only decide to accept his entire request S_i or reject him. The objective of the seller is to maximize the revenue.

- Formulate this problem as an integer program.
- Consider the following example $B_1 = 1$, $B_2 = 2$, $B_3 = 3$, $S_1 = \{1, 2\}$, $v_1 = 2$, $S_2 = \{3\}$, $v_2 = 1$, $S_3 = \{1, 3\}$, $v_3 = 3$, $S_4 = \{2, 3\}$, $v_4 = 2$, $S_5 = \{2\}$, $v_5 = 2$. What is one of the optimal solution to the LP (Linear programming) and IP respectively? What is the integrality gap?

(1) Decision variable:

Let $X_i = \begin{cases} 1, & \text{if the seller accepts the request of customer } i \\ 0, & \text{if the seller rejects the request of customer } i \end{cases}$, $\forall i = 1, 2, \cdots, n$

Let $S_{ij} = \begin{cases} 1, & \text{if customer } i \text{ wants product } j \\ 0, & \text{if customer } i \text{ doesn't want product } j \end{cases}$, $\forall i = 1, 2, \cdots, n$

Formulation:

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n} X_i V_i \\ \text{subject to} \quad & \sum_{i=1}^{n} X_i S_{ij} \leq B_j, \ \forall j = 1, 2, \cdots, m \\ & X_i \in \{0, 1\}, \forall i = 1, 2, \cdots, n \\ & S_{ij} \in \{0, 1\}, \forall i = 1, 2, \cdots, n; \ \forall j = 1, 2, \cdots, m \end{aligned}$$

(2) In the example, $m = 3$, $n = 5$

$V = [2, 1, 3, 2, 2]^T$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

IP:

$$\begin{aligned} \text{maximize} \quad & V^T X \\ \text{s.t.} \quad & A^T X \leq b \\ & X_i \in \{0, 1\}, \forall i = 1, 2, 3, 4, 5 \end{aligned}$$

LP:

$$\begin{aligned} \text{maximize} \quad & V^T X \\ \text{s.t.} \quad & A^T X \leq b \\ & X_i \in [0, 1], \forall i = 1, 2, 3, 4, 5 \end{aligned}$$

By MATLAB, we can get the optimal solution of the LP:

Code:

```
HW9_2.m
clear;
clc;
% Homework9 Problem2 (2)
A=[1 1 0;
   0 0 1;
   1 0 1;
   0 1 1;
   0 1 0];
v=[2,1,3,2,2];
b=[1;
   2;
   3];

cvx_begin
    variable x(5,1);

    maximize v*x;
    subject to
        (A')*x<=b;
        x(:)>=0;
        x(:)<=1;
cvx_end
v*x % print outcome
x
```

Outcome:

```
------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): +8


ans =

    8.0000


x =

    0.0000
    1.0000
    1.0000
    1.0000
    1.0000
```

So the solution to LP is $(0,1,1,1,1)$ with optimal value 8.

Since the solution to LP is already a set of integers,
the solution to IP is $(0,1,1,1,1)$ with optimal value 8.

The integrality gap is $V^{LP} - V^{IP} = 8 - 8 = 0$.

Problem 3 (40pts).
Suppose we have a set of n many items and a set of m different knapsacks. For each item i and knapsack j, the following information is given:

- The item i has value (preference) v_i.
- The weight of item i is a_i.
- The capacity of knapsack j is at most C_j.

a) Formulate an integer program to maximize the total value of items that can be packed in the different knapsack while adhering to the capacity constraint (i.e., the total weight of items in each bag j is not allowed to be larger than C_j).

Hint: You can introduce variables x_{ij} to denote whether item i is placed in knapsack j.

b) Consider the following list of items and bags:

Item	Laptop	T-Shirt	Swim. Trunks	Sunglasses	Apples	Opt. Book	Water
Value	2	1	3	2	1	4	2
Weight	2	0.5	0.5	0.1	0.5	1	1.5
Knapsack 1				Knapsack 2			
$C_1 = 3$				$C_2 = 2$			

Formulate the corresponding IP in that case. What are the optimal solutions to the IP and its LP relaxation (you can use `MATLAB` or `CVX` to solve the problems)? Is there an integrality gap in this case?

(a) Decision variable:

Let $X_{ij} = \begin{cases} 1, & \text{if item } i \text{ is placed in knapsack } j \\ 0, & \text{if item } i \text{ isn't placed in knapsack } j \end{cases}$

Formulation:

$$\text{maximize} \quad \sum_{i=1}^{n}\sum_{j=1}^{m} X_{ij} V_i$$

$$\text{subject to} \quad \sum_{i=1}^{n} X_{ij} a_i \leq C_j, \ \forall j = 1, 2, \cdots, m$$

$$\sum_{j=1}^{m} X_{ij} \leq 1, \ \forall i = 1, 2, \cdots, n$$

$$X_{ij} \in \{0, 1\}, \ \forall i = 1, 2, \cdots, n; \ \forall j = 1, 2, \cdots, m$$

(b) In the example, $n = 7$, $m = 2$

weight: $W = [2, 0.5, 0.5, 0.1, 0.5, 1, 1.5]^T$

capacity: $C = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

value: $V = [2, 1, 3, 2, 1, 4, 2]^T$

Formulation:

$$\text{maximize} \quad \sum_{i=1}^{7}\sum_{j=1}^{2} X_{ij} V_i$$

$$\text{Subject to} \quad \sum_{i=1}^{7} X_{ij} W_i \leq C_j, \ \forall j = 1, 2$$

$$\sum_{j=1}^{2} X_{ij} \leq 1, \ \forall i = 1, 2, \cdots, 7$$

$$X_{ij} \in \{0, 1\}, \ \forall i = 1, 2, \cdots, 7; \ \forall j = 1, 2$$

Code:

```
clear;
clc;
% Homework9 Problem3 (b)
v = [2, 1, 3, 2, 1, 4, 2]; % value
w = [2, 0.5, 0.5, 0.1, 0.5, 1, 1.5]; % weight
f = [v,v];
A = [w, zeros(1,7);
    zeros(1,7), w;
    diag(ones(1,7)), diag(ones(1,7))];
b = [3,2,ones(1,7)];

% LP
[x,fval] = linprog(-f,A,b,[],[],zeros(1,14),ones(1,14));
disp(x);
disp(-fval);

% IP
[x,fval] = intlinprog(-f,1:1:14,A,b,[],[],zeros(1,14),ones(1,14));
disp(x)
disp(-fval)
```

Outcome:

LP:

命令行窗口

```
Optimal solution found.

    0.4500
    1.0000
    1.0000
    1.0000
    1.0000
         0
    0.3333
         0
         0
         0
         0
         0
    1.0000
    0.6667

   13.9000

LP:                Optimal objective value is -13.900000.

Heuristics:        Found 2 solutions using ZI round.
                   Upper bound is -13.000000.
                   Relative gap is 0.00%.

Cut Generation:    Applied 1 clique cut, and 1 mir cut.
                   Lower bound is -13.000000.
                   Relative gap is 0.00%.
```

IP:

```
Optimal solution found.

Intlinprog stopped at the root node because the objective value is within a gap tolerance of the optimal value,
options.AbsoluteGapTolerance = 0 (the default value). The intcon variables are integer within tolerance,
options.IntegerTolerance = 1e-05 (the default value).

         0
         0
         0
    1.0000
    1.0000
         0
    1.0000
         0
    1.0000
    1.0000
         0
         0
    1.0000
         0

   13.0000
```

Solution to LP:

(0.45, 1, 1, 1, 1, 0, 0.3333, 0, 0, 0, 0, 0, 1, 0.6667) with

optimal value 13.9

Solution to IP:

(0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 0, 0, 1, 0)

optimal value 13

Integrality: $V^{LP} - V^{IP} = 13.9 - 13 = 0.9$