MAT3007 Homework 6 Student ID: 121090429

Problem 1 (25pts). Consider the following function:

$$f(x, y, z) = 3x^2 - 2x - 2xy + 3y^2 - 2y - 2zy + 3z^2 - 2z - 2xz$$

(a). Considering the 1st-order necessary condition, try to find the candidate minimizers of $f(x, y, z)$.

(b). Considering the 2nd-order sufficient condition, whether these candidates are indeed local minimizers?

(c). Is (0,0,0) a local minimizer? Why?

(a) By 1^{st}-order necessary condition, $\nabla f(x,y,z) = 0$

$$\nabla f(x,y,z) = (6x-2-2y-2z, -2x+6y-2-2z, -2+6z-2x-2y) = 0$$

we get $\begin{cases} 6x-2y-2z=2 \\ -2x+6y-2z=2 \\ -2x-2y+6z=2 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=1 \\ z=1 \end{cases}$

$f(1,1,1) = -3$

The candidate minimizer is $(x=1, y=1, z=1)$

(b)

$$\nabla^2 f(x,y,z) = \begin{pmatrix} 6 & -2 & -2 \\ -2 & 6 & -2 \\ -2 & -2 & 6 \end{pmatrix}$$

By 2^{nd}-order sufficient condition, $\nabla^2 f(x,y,z)$ should be positive definite.

Therefore all its eigenvalues are positive.

Denote $\nabla^2 f(x,y,z)$ as A

$$\det(A-\lambda I) = \begin{vmatrix} 6-\lambda & -2 & -2 \\ -2 & 6-\lambda & -2 \\ -2 & -2 & 6-\lambda \end{vmatrix}$$

$$= (6-\lambda)\cdot[(6-\lambda)^2-4] - (-2)[-2(6-\lambda)-4] + (-2)[4+2(6-\lambda)]$$

$$= (6-\lambda)^3 - 4(6-\lambda) - 4(6-\lambda) - 8 - 8 - 4(6-\lambda)$$

$$= -\lambda^3 + 18\lambda^2 - 108\lambda + 216 - 12(6-\lambda) - 16$$

$$= -\lambda^3 + 18\lambda^2 - 96\lambda + 128$$

$$= (\lambda-2)(-\lambda^2+16\lambda-64)$$

$$= (\lambda-2)(\lambda-8)^2$$

$\Rightarrow \lambda_1 = 2, \lambda_2 = \lambda_3 = 8.$

Since $\lambda_1, \lambda_2, \lambda_3 > 0$, SOSC is satisfied.

The candidate $(x=1, y=1, z=1)$ is indeed local minimizer.

(C) No. $\nabla f(0,0,0) = (-2,-2,-2) \neq 0$, doesn't satisfy FONC.

Problem 2 (25pts). Given a symmetric matrix $A \in R^{n\times n}$, consider the following problem:

$$\begin{array}{ll} \min_{x\in R^n} & x^T Ax \\ \text{subject to} & 2 - x^T x = 0 \end{array}$$

(a). Give the KKT conditions of this problem.

(b). If A is positive definite without repeated eigenvalues, how many different KKT points are there at most?

(c). If A is positive definite without repeated eigenvalues, what is the minimum value of this problem, and how many ~~local~~ global minimizers? (Hint:According to Rayleigh quotient, $min\{x^T Ax/(x^T x)\} = \lambda_{min}$, where λ_{min} is the minimum eigenvalue of A)

(a) $f(x) = x^T A x$

$g(x) = 2 - x^T x = 0$

$L(x,\nu) = x^T A x + \nu(2 - x^T x)$ with $\lambda \geq 0$

$\nabla f(x) = 2Ax$

$\nabla g(x) = -2x$

KKT conditions:

① Main Conditions:

$2Ax - 2\nu x = 0$

② Primal Feasibility:

$2 - x^T x = 0$

③ Dual Feasibility:

ν free

④ Complementarity Conditions:

$x_i \cdot \nabla_{x_i} L(x,\nu) = x_i(2Ax_i - 2\nu x_i)$

$= 2(A-\nu)x_i^T x_i$

$= 0, \ \forall i$

(b) According to KKT conditions:

$$\begin{cases} 2Ax - 2\nu x = 0 & \cdots ① \\ 2 - x^T x = 0 \end{cases}$$

So the objective function $f(x) = x^T A x = x^T \nu x = \nu x^T x = 2\nu$

Since A is positive definite without repeated eigenvalues, suppose A has n eigenvalues.

By ①, there are n different ν.

Note that $x^T x$ can achieve same value by $x^T x$ and $(-x)^T(-x)$

So there are $2n$ different KKT points.

(c) By Rayleigh quotient,

$$\min\left\{\frac{x^TAx}{x^Tx}\right\} = \lambda_{min}$$

By Primal Feasibility, $x^Tx = 2$

Hence $\min\{x^TAx\} = 2\lambda_{min}$

Then we have $(x^*)^TAx^* = (-x^*)^TA(-x^*) = 2\lambda_{min}$

So there are 2 global minimizers.

Problem 3 (25pts). Construct the KKT conditions for the following linear program:

$$\begin{array}{ll} \text{maximize} & 3x_1 + x_2 + 4x_3 \\ \text{subject to} & x_1 + 3x_2 + x_3 \leq 5 \\ & x_1 + 2x_2 + 2x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

We first convert it into standard form:

$$\begin{array}{ll} \text{minimize} & -3x_1 - x_2 - 4x_3 \\ \text{Subject to} & x_1 + 3x_2 + x_3 \leq 5 \\ & x_1 + 2x_2 + 2x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

$f(x) = -3x_1 - x_2 - 4x_3$. $\nabla f(x) = (-3, -1, -4)$

$l_1(x) = x_1 + 3x_2 + x_3 - 5$. $\nabla l_1(x) = (1, 3, 1)$

$l_2(x) = x_1 + 2x_2 + 2x_3 - 8$. $\nabla l_2(x) = (1, 2, 2)$

$L(x, \eta) = -3x_1 - x_2 - 4x_3 + \eta_1(x_1 + 3x_2 + x_3 - 5) + \eta_2(x_1 + 2x_2 + 2x_3 - 8)$

① Main Conditions:

$$-3 + \eta_1 + \eta_2 \geq 0$$
$$-1 + 3\eta_1 + 2\eta_2 \geq 0$$
$$-4 + \eta_1 + 2\eta_2 \geq 0$$

② Primal Feasibility:

$$x_1 + 3x_2 + x_3 \leq 5$$
$$x_1 + 2x_2 + 2x_3 \leq 8$$
$$x_1, x_2, x_3 \geq 0$$

③ Dual Feasibility:

$$\eta_1 \geq 0, \ \eta_2 \geq 0$$

④ Complementarity Conditions:

$$\eta_1(x_1 + 3x_2 + x_3 - 5) = 0$$
$$\eta_2(x_1 + 2x_2 + 2x_3 - 8) = 0$$
$$x_1(-3 + \eta_1 + \eta_2) = 0$$

$$X_2(-1+3\eta_1+2\eta_2)=0$$
$$X_3(-4+\eta_1+2\eta_2)=0$$

Problem 4 (25pts). Construct the KKT conditions for the following nonlinear program:

$$\begin{array}{ll} \text{minimize} & x_1 ln(x_1) + (x_2-2)^2 + x_3 \\ \text{subject to} & x_1 + x_2 \leq 3 \\ & x_3 - x_2^2 \geq 3 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

$f(x) = X_1 \cdot \ln(X_1) + (X_2-2)^2 + X_3$

$\nabla f(x) = (\ln(X_1)+1,\ 2X_2-4,\ 1)$

$g(x) = -X_2^2 + X_3 - 3$. $\nabla g(x) = (0, -2X_2, 1)$

$\ell(x) = X_1 + X_2 - 3$ $\nabla \ell(x) = (1, 1, 0)$

$L(X, \lambda, \eta) = X_1 \cdot \ln(X_1) + (X_2-2)^2 + X_3 + \lambda(-X_2^2 + X_3 - 3) + \eta(X_1 + X_2 - 3)$

① Main Conditions:

$\ln(X_1) + 1 + \eta \geq 0$

$2(1-\lambda)X_2 - 4 + \eta \geq 0$

$1 + \lambda \geq 0$

② Primal Feasibility:

$X_1 + X_2 \leq 3$

$X_3 - X_2^2 \geq 3$

$X_1, X_2, X_3 \geq 0$

③ Dual Feasibility:

$\lambda \leq 0$, $\eta \geq 0$

④ Complementarity Conditions:

$\lambda(-X_2^2 + X_3 - 3) = 0$

$\eta(X_1 + X_2 - 3) = 0$

$X_1(\ln(X_1) + 1 + \eta) = 0$

$X_2(2(1-\lambda)X_2 - 4 + \eta) = 0$

$X_3(1+\lambda) = 0$