MAT3007 Homework 2 Student ID: 121090429

Problem 1 (20pts). Reformulate NLP as LP

Reformulate the following problems as linear programming

$$\begin{aligned} \text{minimize} \quad & 2x_2 + |x_1 - x_3| \\ \text{s.t.} \quad & |x_1 + 2| + |x_2| \le 5 \\ & x_3^2 \le 1 \end{aligned}$$

Please also write down its standard form.

Reformulate:

$$\begin{aligned} \min \quad & 2X_2 + z \\ \text{s.t.} \quad & m + n \le 5 \\ & m \ge X_1 + 2 \\ & m \ge -X_1 - 2 \\ & n \ge X_2 \\ & n \ge -X_2 \\ & z \ge X_1 - X_3 \\ & z \ge -X_1 + X_3 \\ & X_3 \ge -1 \\ & X_3 \le 1 \end{aligned}$$

Standard form:

$$\begin{aligned} \min \quad & 2X_2^+ - 2X_2^- + z \\ \text{s.t.} \quad & m + n + S_1 = 5 \\ & m - S_2 = X_1^+ - X_1^- + 2 \\ & m - S_3 = -X_1^+ + X_1^- - 2 \\ & n - S_4 = X_2^+ - X_2^- \\ & n - S_5 = -X_2^+ + X_2^- \\ & z - S_6 = X_1^+ - X_1^- - X_3 \\ & z - S_7 = -X_1^+ + X_1^- + X_3 \\ & X_3 - S_8 = -1 \\ & X_3 + S_9 = 1 \\ & S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_9 \ge 0 \end{aligned}$$

Problem 2 (35pts). Basic solutions and basic feasible solutions
Consider the following linear optimization problem:

$$\begin{array}{rl} \text{maximize} & x_1 + 2x_2 + 4x_3 \\ \text{s.t.} & x_1 + x_3 \leq 8 \\ & x_2 + 2x_3 \leq 15 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

(a) Transform it into standard form;

(b) Argue without solving this LP that there must exist an optimal solution with no more than 2 positive variables;

(c) List all the basic solutions and basic feasible solutions (of the standard form);

(d) Find the optimal solution by using the results in step (c).

(a) $\min\ -X_1 - 2X_2 - 4X_3$

s.t. $X_1 + X_3 + S_1 = 8$

$X_2 + 2X_3 + S_2 = 15$

$X_1, X_2, X_3, S_1, S_2 \geq 0$

(b) With standard form from (a), we can denote the feasible set by $\{x : Ax = b, x \geq 0\}$, where:

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 8 \\ 15 \end{bmatrix}$$

So $\text{rank}(A) = 2$.

There are five variables with only 2 constraints, so 3 non-basic variables will be set equal to zero.

Hence the optimal solution will have no more than 2 positive variables.

(c)

Choose 2 independent columns of A, we have:

Basic Solution:

Indices	{1, 2}	{1, 3}	{1, 5}	{2, 3}
Solution	(8, 15, 0, 0, 0)	(0.5, 0, 7.5, 0, 0)	(8, 0, 0, 0, 15)	(0, -1, 8, 0, 0)
Outcome	38	30.5	8	30

Indices	{2, 4}	{3, 4}	{3, 5}	{4, 5}
Solution	(0, 15, 0, 8, 0)	(0, 0, 7.5, 0.5, 0)	(0, 0, 8, 0, -1)	(0, 0, 0, 8, 15)
Outcome	30	30	32	0

Since $X_1, X_2, X_3, S_1, S_2 \geq 0$,

we have Basic Feasible Solution:

Indices	{1,2}	{1,3}	{1,5}
Solution	(8,15,0,0,0)	(0.5,0,7.5,0,0)	(8,0,0,0,15)
Outcome	38	30.5	8

Indices	{2,4}	{3,4}	{4,5}
Solution	(0,15,0,8,0)	(0,0,7.5,0.5,0)	(0,0,0,8,15)
Outcome	30	30	0

(d) By comparing all the outcome of $X_1+2X_2+4X_3$ of the results in (c), we have the optimal solution is $(8,15,0,0,0)$

Problem 3 (45pts). A Robust LP Formulation
In this exercise, we consider the following optimization problem:

$$\min_{x\in\mathbf{R}^n} c^\top x \quad \text{subject to} \quad \|Ax-b\|_\infty \le \delta, \quad x \ge 0 \tag{1}$$

where $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}m$, $c \in \mathbf{R}n$, and $\delta \ge 0$ are given and $\|y\|_\infty = \max_{1\le i\le p}|y_i|$ denotes the maximum norm of a vector $y \in \mathbf{R}^p$. In the case $\delta = 0$, problem (1) coincides with the standard form for linear programs. The choice $\delta > 0$ can be useful to model situations where A and/or b are not fully or exactly known, e.g., when A and/or b contains certain uncertainty (can be caused by noise). In this case, problem (1) belongs to the so-called robust optimization.

(a) Rewrite the optimization problem (1) as a linear problem.

(a) $\min_{X\in\mathbb{R}^n} \quad c^TX$

subject to $\quad A_iX-b_i \geq \delta,\ b_i-A_iX \geq \delta,\ \forall i\in[n]$

$X\geq 0$

(b) We now consider a specific application of problem (1).

The fruit store in Pandora is producing two different fruit salads A and B. The smaller fruit salad A consists of "1/4 mango, 1/8 pineapple, 3 strawberries"; the larger fruit salad B consists of "1/2 mango, 1/4 pineapple, 1 strawberry". The profits per fruit salad and the total number of fruits in stock are summarized in the following table:

	Mango	Pineapple	Strawberry	Net profit
Fruit salad A	1/4	1/8	3	10 RMB
Fruit salad B	1/2	1/4	1	20 RMB
Stock / Resources	25	10	120	

Suppose all fruits need to be processed and *completely used* to make the fruit salads A and B.

Given these constraints, formulate a linear program to maximize the total profits of the fruit store. Show that this program can be expressed in standard form

$$\min_{x\in\mathbf{R}^n} c^\top x \quad \text{subject to} \quad Ax = b, \quad x \ge 0,$$

with $n = 2$ and $m = 3$. In addition, is this linear programming solvable?

(b) Denote the number of A, B to X_1, X_2 respectively.

$$X = [X_1, X_2]^T, \quad C = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

The standard form of this problem is:

$$\begin{aligned} \min \quad & -10X_1 - 20X_2 \\ \text{s.t.} \quad & \tfrac{1}{4}X_1 + \tfrac{1}{2}X_2 = 25 \\ & \tfrac{1}{8}X_1 + \tfrac{1}{4}X_2 = 10 \\ & 3X_1 + X_2 = 120 \\ & X_1, X_2 \geq 0 \end{aligned}$$

$$\therefore A = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{8} & \frac{1}{4} \\ 3 & 1 \end{bmatrix} \quad \text{with } n=2, m=3, \quad b = \begin{bmatrix} 25 \\ 10 \\ 120 \end{bmatrix}$$

The augmented matrix of A & b is:

$$[A|b] = \left[\begin{array}{cc|c} \frac{1}{4} & \frac{1}{2} & 25 \\ \frac{1}{8} & \frac{1}{4} & 10 \\ 3 & 1 & 120 \end{array}\right] \xrightarrow{\text{row operation}} \left[\begin{array}{cc|c} 1 & 2 & 100 \\ 1 & 2 & 80 \\ 3 & 1 & 120 \end{array}\right]$$

$$\xrightarrow{\text{row operation}} \left[\begin{array}{cc|c} 1 & 2 & 100 \\ 0 & 1 & 36 \\ 0 & 0 & 1 \end{array}\right]$$

Since the augmented matrix contains $(0, 0, \cdots, 0, 1)$ this linear programming is not soluable.

(c) One of the employee found some additional fruits in a storage crate and the manager of the fruit shop decides to determine the production plan by using the robust formulation (1). Consider the robust variant of the problem in part (b) with $\delta = 5$.

- Sketch the feasible set of this problem.
- Solve the problem graphically, i.e., Calculate the optimal value and the optimal solution set.
- Which constraints are active in the solution?
- Find one integer solution of this problem.

(c) Take $\delta = 5$ in (b),

we can rewrite the problem as:

$$\min \quad -10X_1 - 20X_2$$

$$\text{s.t.} \quad \frac{1}{4}X_1 + \frac{1}{2}X_2 - 25 \leq 5$$

$$-\frac{1}{4}X_1 - \frac{1}{2}X_2 + 25 \leq 5$$

$$\frac{1}{8}X_1 + \frac{1}{4}X_2 - 10 \leq 5$$

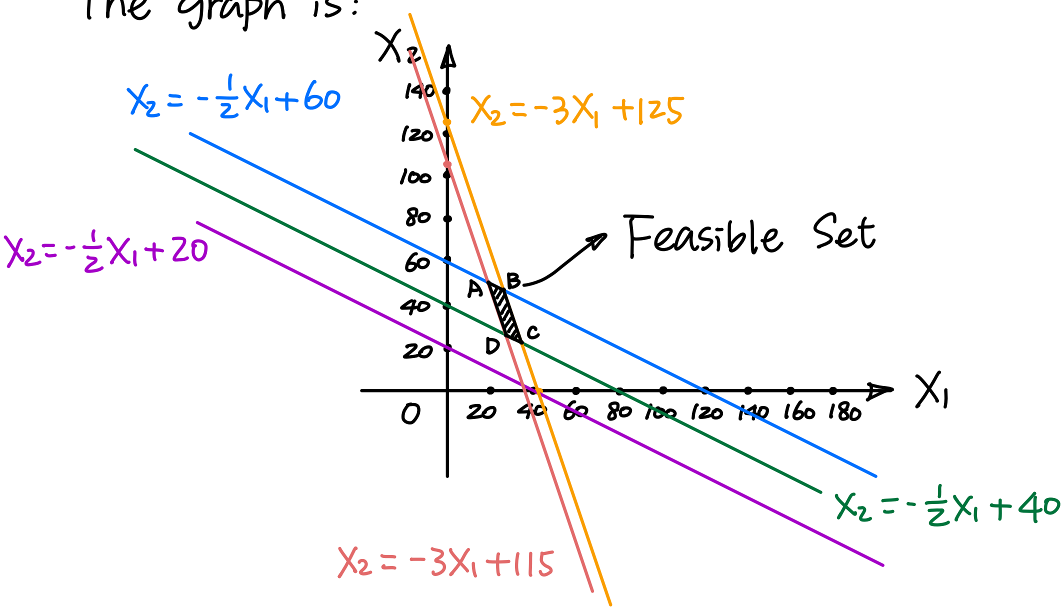
$$-\frac{1}{8}X_1 - \frac{1}{4}X_2 + 10 \leq 5$$

$$3X_1 + X_2 - 120 \leq 5$$

$$-3X_1 - X_2 + 120 \leq 5$$

$$X_1, X_2 \geq 0$$

The graph is:



Calculate different function $10X_1 + 20X_2 = C$ for different extreme points:

A: $(22, 49)$ $\quad 10X_1 + 20X_2 = 220 + 980 = 1200$

B: $(26, 47)$ $\quad 10X_1 + 20X_2 = 260 + 940 = 1200$

C: $(32, 24)$ $\quad 10X_1 + 20X_2 = 320 + 480 = 800$

D: $(30, 25)$ $\quad 10X_1 + 20X_2 = 300 + 500 = 800$

Optimal value: 1200

Optimal solution: $\{(120 - 2X_2, X_2) : 47 \leq X_2 \leq 49\}$

Active constraints: $\frac{1}{4}X_1 + \frac{1}{2}X_2 - 25 \leq 5$, $\frac{1}{8}X_1 + \frac{1}{4}X_2 - 10 \leq 5$,

$3X_1 + X_2 - 120 \leq 5$, $-3X_1 - X_2 + 120 \leq 5$

Integer solution: $\{(22, 49)\}$