MAT3007 Homework 4 Student ID: 121090429

Problem 1 (20pts). Consider the following linear program:

$$\begin{array}{ll} \text{maximize} & 3x_1 + x_2 + 4x_3 \\ \text{subject to} & x_1 + 3x_2 + x_3 \leq 5 \\ & x_1 + 2x_2 + 2x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

(a). What is the corresponding dual problem?

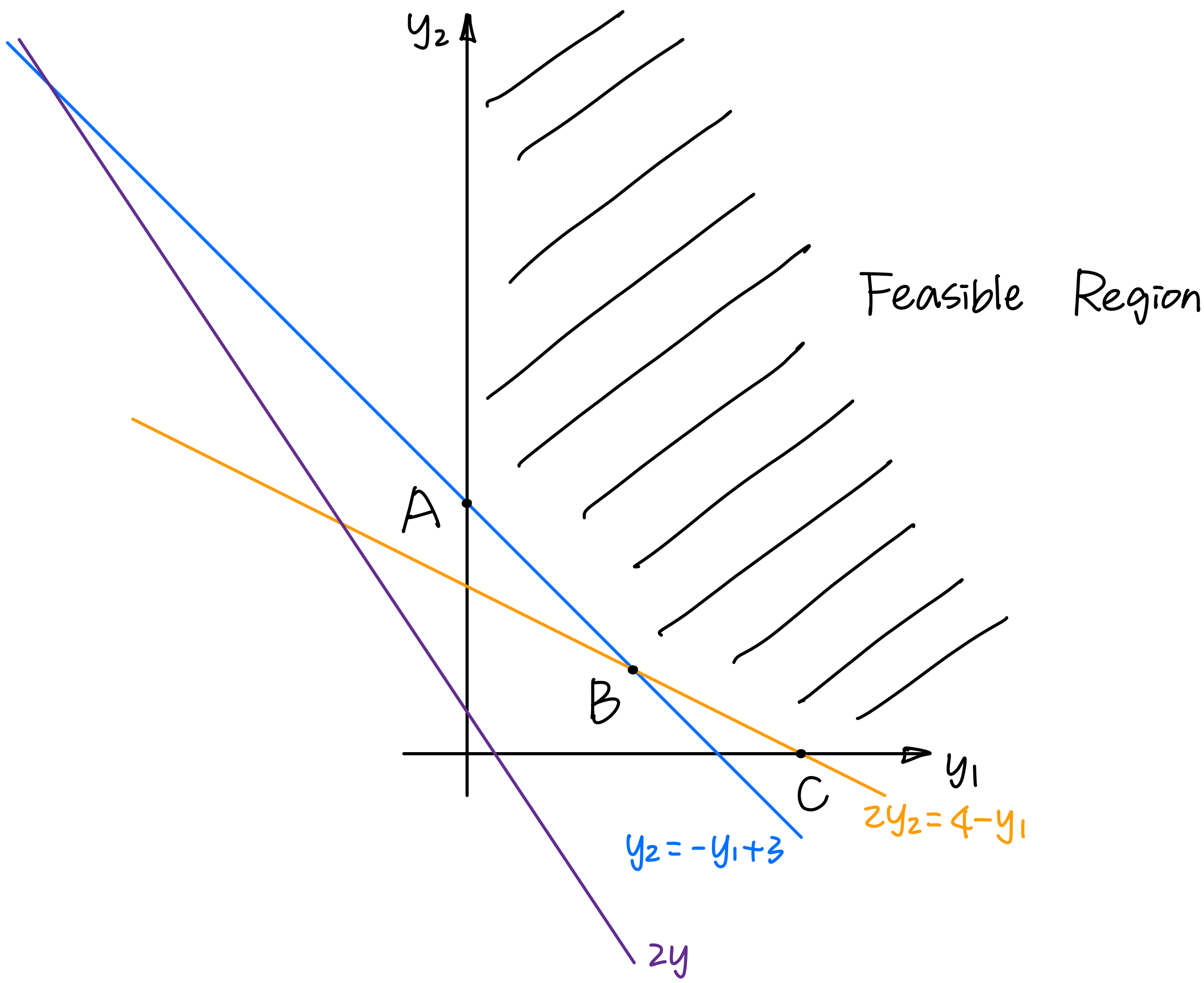
(b). Solve the dual problem graphically.

(c). Use complementarity conditions for the primal-dual pair to solve the primal problem.

(a) The corresponding dual problem is:

$$\begin{array}{ll} \min & 5y_1 + 8y_2 \\ \text{s.t.} & y_1 + y_2 \geq 3 \\ & 3y_1 + 2y_2 \geq 1 \\ & y_1 + 2y_2 \geq 4 \\ & y_1, y_2 \geq 0 \end{array}$$

(b) The graph is as followed:



Three extreme points $A(0,3)$; $B(2,1)$; $C(4,0)$

$A(0,3)$: $5y_1 + 8y_2 = 24$

$B(2,1): 5y_1+8y_2=18$

$C(4,0): 5y_1+8y_2=20$

So the optimal value is 18.

The optimal solution of the dual problem is $(2,1)$.

(c) By the complementarity condition,

we have:

$x_1(3-y_1-y_2)=0$

$x_2(1-3y_1-2y_2)=0$

$x_3(4-y_1-2y_2)=0$

$y_1(5-x_1-3x_2-x_3)=0$

$y_2(8-x_1-2x_2-2x_3)=0$

By (b), we already know that both the primal and dual have optimal finite value, and $y_1=2, y_2=1$.

Solving these equations, we have optimal solutions:

primal problem: $(x_1,x_2,x_3)=(2,0,3)$

dual problem: $(y_1,y_2)=(2,1)$

Problem 2 (25pts). Consider the following table of food and corresponding nutritional values:

	Protein, g	Carbohydrates, g	Calories	Cost
Bread	4	7	130	3
Milk	6	10	120	4
Fish	20	0	150	8
Potato	1	30	70	2

The ideal intake for an adult is at least 30 grams of protein, 40 grams of carbohydrates, and 400 calories per day. The problem is to find the **least** costly way to achieve those amounts of nutrition by using the four types of food shown in the table.

(a). Formulate this problem as a linear optimization problem (specify the meaning of each decision variable and constraint).

(b). Solve it using MATLAB, find an optimal solution and the optimal value.

(c). Formulate the dual problem. Interpret the dual problem. (Hint: Suppose a pharmaceutical company produces each of the nutrients in pill form and sells them each for a certain price.)

(d). Use MATLAB to solve the dual problem. Find an optimal solution and the optimal value.

(a) Denote the number of bread、milk、fish and potato separately as x_1、x_2、x_3、x_4.

$$\min \quad 3x_1+4x_2+8x_3+2x_4$$

$$\text{s.t.} \quad 4x_1+6x_2+20x_3+x_4 \geq 30$$

$$7x_1+10x_2+30x_4 \geq 40$$

$$130x_1 + 120x_2 + 150x_3 + 70x_4 \geq 400$$
$$x_1, x_2, x_3, x_4 \geq 0$$

(b) The Code is as followed:

编辑器 - D:\大二上\MAT3007\assignment\homework4\HW4_2.m

HW4_2.m

```
clear;
clc;

cvx_begin
variables x1 x2 x3 x4
minimize 3*x1 + 4*x2 + 8*x3 + 2*x4
subject to
4*x1 + 6*x2 + 20*x3 + x4 >= 30;
7*x1 + 10*x2 + 30*x4 >= 40;
130*x1 + 120*x2 + 150*x3 + 70*x4 >= 400;
x1 >= 0;
x2 >= 0;
x3 >= 0;
x4 >= 0;
cvx_end

% print outcome
x1
x2
x3
x4
3*x1 + 4*x2 + 8*x3 + 2*x4
```

outcome:

命令行窗口

```
x1 =

    1.0732


x2 =

   1.1560e-08


x3 =

    1.2312


x4 =

    1.0829


ans =

   15.2351

fx >>
```

(c) Denote y_1、y_2、y_3 as the number of pills that separately contain protein、Carbonhydrates and Calories.

Each pill is sold with certain price.

The dual problem is:

$$\begin{aligned} \max \quad & 30y_1 + 40y_2 + 400y_3 \\ \text{s.t.} \quad & 4y_1 + 7y_2 + 130y_3 \leq 3 \\ & 6y_1 + 10y_2 + 120y_3 \leq 4 \\ & 20y_1 + 150y_3 \leq 8 \\ & y_1 + 30y_2 + 70y_3 \leq 2 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

(d) The Code is as followed:

```
% Homework4 Problem2 (d)
cvx_begin
variables y1 y2 y3
maximize 30*y1 + 40*y2 + 400*y3
subject to
4*y1 + 7*y2 + 130*y3 <= 3;
6*y1 + 10*y2 + 120*y3 <= 4;
20*y1 + 150*y3 <= 8;
y1 + 30*y2 + 70*y3 <= 2;
y1 >= 0;
y2 >= 0;
y3 >= 0;
cvx_end

% print outcome
y1
y2
y3
30*y1 + 40*y2 + 400*y3
```

outcome:

编辑器 - D:\大二上\MAT3007\assignment\homework4\HW4_2.m

命令行窗口

```
------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): +15.2351


y1 =

    0.3099


y2 =

    0.0283


y3 =

    0.0120


ans =

   15.2351

fx >>
```

The optimal solution: $(x_1, x_2, x_3, x_4) = (1.0732, 1.1560\times10^{-8}, 1.2312, 1.0829)$

$(y_1, y_2, y_3) = (0.3099, 0.0283, 0.0120)$

The optimal value: 15.23151

Problem 3 (25pts). Consider the max flow problem on the graph in figure 1 with the orange node being the source node and the green node being the terminal node (the number on each edge is its capacity, see the lecture slide 13). Do the following based on the lecture slides.

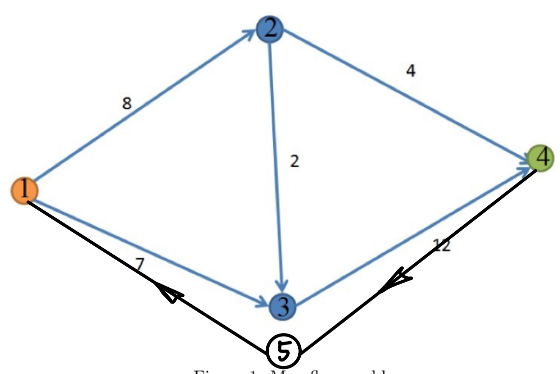


Figure 1: Max flow problem

(a). Formulate it as a linear program and solve it using MATLAB.

(b). Formulate the dual of this problem and solve it using MATLAB.

(c). Find the corresponding maximum flow and minimum cut of the graph.(Please draw the cut on figure 1).

(a) Assume there is an imaginary node 5 with edges (5,1) and (4,5)

Denote the flow from 5 to 1 as m.

The flow from $1 \to 2$ as X_1, $1 \to 3$ as X_2, $2 \to 3$ as X_3, $2 \to 4$ as X_4, $3 \to 4$ as X_5

The problem is:

$$\underset{x,\Delta}{\text{maximize}} \quad \Delta$$

$$\text{s.t.} \quad \sum_{j:(j,i)\in E} X_{ji} - \sum_{j:(j,i)\in E} X_{ij} = 0, \ \forall\, i \neq 1,5$$

$$\sum_{j:(j,1)\in E} X_{j1} - \sum_{j:(1,j)\in E} X_{1j} + \Delta = 0$$

$$\sum_{j:(j,5)\in E} X_{j5} - \sum_{j:(5,j)\in E} X_{5j} - \Delta = 0$$

$$X_{ij} \leq W_{ij} \quad \forall\, (i,j) \in E$$

$$X_{ij} \geq 0$$

The code is as followed:

```
% Homework4 Problem4 (a)
w=[0 8 7 0;
   0 0 2 4;
   0 0 0 12;
   0 0 0 0];

cvx_begin
    variable x(4,5);

    maximize sum(x(:,5));
    subject to
        sum(x(2,1:4))-sum(x(1:4,2))==0;
        sum(x(3,1:4))-sum(x(1:4,3))==0;
        sum(x(1:4,1))-sum(x(1,1:4))+sum(x(:,5))==0;
        sum(x(1:4,4))-sum(x(4,1:4))-sum(x(:,5))==0;
        x>=0;
        for i = 1:4
            for j=1:4
                x(i,j)<=w(i,j)
            end
        end
cvx_end
x,sum(x(:,5))
```

outcome:

```
命令行窗口
-------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): +13


x =

    0.0000    6.0000    7.0000    0.0000    3.2500
    0.0000    0.0000    2.0000    4.0000    3.2500
    0.0000    0.0000    0.0000    9.0000    3.2500
    0.0000    0.0000    0.0000    0.0000    3.2500


ans =

   13.0000

fx >>
```

(b) The dual problem is:

$$\text{minimize} \sum_{(i,j)\in E} W_{ij} z_{ij}$$

$$\text{s.t.} \quad z_{ij} \geq y_i - y_j$$

$$y_1 - y_5 = 1$$

$$z_{ij} \geq 0$$

The code is as followed:

```
% Homework4 Problem4 (b)
cvx_begin
    variable z(4,4);
    variable y(4);
    minimize sum(sum(w.*z));
    for i = 1:4
        for j = 1:4
            if w(i,j)~=0
                z(i,j)>=y(i)-y(j)
            end
        end
    end
    y(1)-y(4)==1
    z>=0

cvx_end
z,y
```

Outcome:

```
命令行窗口
------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): +13


z =

   (1,2)        0.0000
   (1,3)        1.0000
   (2,3)        1.0000
   (2,4)        1.0000
   (3,4)        0.0000


y =

    1.0000
    1.0000
   -0.0000
   -0.0000

fx >>
```

(c)

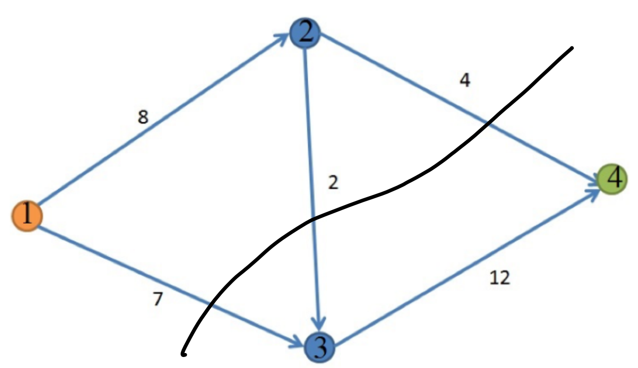


Figure 1: Max flow problem

Problem 4 (15pts). Use linear program duality to show that exactly one of the following systems has a solution

1. $A\boldsymbol{x} \leq \boldsymbol{b}$
2. $\boldsymbol{y}^T A = 0, \boldsymbol{b}^T \boldsymbol{y} < 0, \boldsymbol{y} \geq 0$

Hint: You can first show that they can't both have solutions. Then you show that if the second one is infeasible, the first one must be feasible.

Primal Problem:

$$\min \ c^T x$$
$$\text{s.t. } Ax = b, x \geq 0$$

Dual Problem:

$$\max \ b^T y$$
$$\text{s.t. } A^T y \leq c$$

In this problem, $c = 0$.

Argue by contradiction.

Suppose they can both have solutions.

Then $0 \cdot x = (y^T A) \cdot x = y^T (Ax) = y^T b = (b^T y)^T < 0$

Since $0 \cdot x = 0$, the above statement is wrong.

$\therefore$ They can't both have solutions.

If 2 is infeasible, then $y^T A \neq 0$

Suppose 1 is feasible.

$$y \cdot y^T A \cdot x \leq y \cdot y^T b = y \cdot (b^T y)^T < 0$$

$$\Rightarrow y y^T A x < 0$$

Since $y > 0$, $y \cdot y^T = \sqrt{y^2} > 0$

$$\Rightarrow Ax < 0$$

① If $b \geq 0$, then $Ax \leq b$ holds.

② If $b < 0$, then the solution to $Ax \leq b$ is a subset of the solution to $Ax \leq 0$

So $\exists x$ s.t. $Ax \leq b$ holds.

$\Rightarrow Ax \leq b$ is feasible.

So if the second one is infeasible, the first one must be feasible.

$\therefore$ Exactly one of the systems has solution.

Problem 5 (15pts).
Suppose M is a square matrix such that $M = -M^T$, for example,

$$M = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -4 \\ -2 & 4 & 0 \end{pmatrix}$$

Consider the following optimization problem:

$$\begin{array}{rl} \text{minimize}\,\boldsymbol{x} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & M\boldsymbol{x} \geq -\boldsymbol{c} \\ & \boldsymbol{x} \geq 0 \end{array}$$

(a). Show that the dual problem of it is equivalent to the primal problem.

(b). Argue that the problem has optimal solution if and only if there is a feasible solution.

(a) In this problem, $b = -c$

The dual problem is:

$$\begin{array}{rl} \text{maximize} & b^T x \\ \text{s.t.} & M^T y \leq c \\ & y \geq 0 \end{array}$$

Since $b = -c$, $M = -M^T$

The dual problem becomes:

$$\begin{array}{rl} \text{minimize} & c^T y \\ \text{s.t.} & My \geq -c \\ & y \geq 0 \end{array}$$

Which is equivalent to the primal problem.

(b) Know that the only possible cases for LP are:

P | D	Finite Optimum	Unbounded	Infeasible
Finite Optimum	✓		
Unbounded			✓
Infeasible		✓	✓

Since the primal problem and dual problem are equivalent, either they are both feasible or they are both infeasible.

When they are both infeasible, the problem will not have optimal solution.

So the problem has optimal solution if and only if there is a feasible solution.