STA2004 Homework 4 Student ID: 121090429

(**Question1):** (40 points) Suppose data $X_i, i = 1, \cdots, n$ satisfy exponential distribution with pdf $f(x, \lambda) = \lambda \exp(-\lambda x)$. We want to test the

$$H_0 : \lambda = \lambda_0, \text{ versus } H_1 : \lambda = \lambda_1$$

here λ_0, λ_1 are known numbers. Find the Uniform most powerful test of this problem. You do not need to specify the exact rejection region in this problem.

$$L(X_1, X_2, \cdots, X_n \mid \mu_0) = \lambda_0^n \cdot \exp(-\lambda_0 \cdot \sum_{i=1}^n X_i)$$

$$L(X_1, X_2, \cdots, X_n \mid \mu_1) = \lambda_1^n \cdot \exp(-\lambda_n \cdot \sum_{i=1}^n X_i)$$

Define $\Lambda(X) = \dfrac{L(X_1, \cdots, X_n \mid \mu_0)}{L(X_1, \cdots, X_n \mid \mu_1)}$

$$= \left(\frac{\lambda_0}{\lambda_1}\right)^n \cdot \exp((\lambda_1 - \lambda_0) \cdot \sum_{i=1}^n X_i)$$

By Nayman-Pearson Lemma, we apply the likelihood-ratio test.

For a given number k ($k > 0$),

$$\Lambda(X) \leq k \iff n \cdot \log\lambda_0 - n\log\lambda_1 + (\lambda_1 - \lambda_0)\sum_{i=1}^n X_i \leq \log k$$

$$\iff n \cdot \log\lambda_0 - n \cdot \log\lambda_1 + n(\lambda_1 - \lambda_0)\bar{X} \leq \log k$$

Given that λ_0、λ_1 are fixed, we have:

we have: $\begin{cases} \bar{X} \leq C, & \text{if } \lambda_0 < \lambda_1 \\ \bar{X} \geq C, & \text{if } \lambda_0 > \lambda_1 \end{cases}$

① Suppose $\lambda_0 < \lambda_1$

$$P_{\lambda_0}(\Lambda(X) \leq k) = \alpha$$

$$\Rightarrow P_{\lambda_0}(\bar{X} \leq c) = \alpha \Rightarrow P_{\lambda_0}\left(\frac{\bar{X} - \frac{1}{\lambda_0}}{\frac{1}{\lambda_0}/\sqrt{n}} \leq \frac{c - \frac{1}{\lambda_0}}{\frac{1}{\lambda_0}/\sqrt{n}}\right) = \alpha$$

$$\Rightarrow \frac{c - \frac{1}{\lambda_0}}{\frac{1}{\lambda_0}/\sqrt{n}} = -z_\alpha$$

$$c = \frac{1}{\lambda_0} - \frac{1}{\lambda_0\sqrt{n}} z_\alpha . \quad R = \{\bar{X} : \bar{X} \leq \frac{1}{\lambda_0} - \frac{1}{\lambda_0\sqrt{n}} z_\alpha\}$$

The uniform most powerful test is: $\varphi(x)$ where

$$\varphi(x) = 1_{x \in R} = \begin{cases} 1, & \text{if } x \in R \\ 0, & \text{if } x \notin R. \end{cases}$$

② Suppose $\lambda_0 > \lambda_1$

$$P_{\lambda_0}(\Lambda(X) \leq k) = \alpha$$

$$\Rightarrow P_{\lambda_0}(\bar{X} \geq c) = \alpha \Rightarrow P_{\lambda_0}\left(\frac{\bar{X} - \frac{1}{\lambda_0}}{\frac{1}{\lambda_0}/\sqrt{n}} \geq \frac{c - \frac{1}{\lambda_0}}{\frac{1}{\lambda_0}/\sqrt{n}}\right) = \alpha$$

$$\Rightarrow \frac{c-\frac{1}{\lambda_0}}{\frac{1}{\lambda_0}/\sqrt{n}} = z_\alpha$$

$$c = \frac{1}{\lambda_0} + \frac{1}{\lambda_0\sqrt{n}} z_\alpha. \quad R = \{\bar{x} : \bar{x} \geq \frac{1}{\lambda_0} + \frac{1}{\lambda_0\sqrt{n}} z_\alpha\}$$

The uniform most powerful test is: $\varphi(x)$ where

$$\varphi(x) = 1_{x\in R} = \begin{cases} 1, & \text{if } x \in R \\ 0, & \text{if } x \notin R. \end{cases}$$

(Question2): (60 points) Suppose data $X_1, \cdots, X_n$ satisfy normal distribution $N(\mu, \sigma^2)$ with unknown μ and known σ^2.

(a) Suppose we want to test the two-point hypothesis

$$H_0 : \mu = \mu_0 \text{ versus } H_1 : \mu = \mu_1$$

here μ_0, μ_1 are known numbers. Derive the form of the uniform most powerful test. (Hint: you need to separate $\mu_1 > \mu_0$ and $\mu_0 > \mu_1$ situation.)

(b) Determine the rejection region R_α so that the type-1 error equals α.

(c) Calculate the power of test under H_1.

(d) Determine the $p-$value of the test.

(a) $$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim Z = N(0,1).$$

$$L(X_1,\cdots,X_n \mid \mu=\mu_0) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\{(-\sum_{i=1}^n X_i^2 + 2n\mu_0\bar{X} - n\mu_0^2)/2\sigma^2\}$$

$$L(X_1,\cdots,X_n \mid \mu=\mu_1) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\{(-\sum_{i=1}^n X_i^2 + 2n\mu_1\bar{X} - n\mu_1^2)/2\sigma^2\}$$

$$\text{Define } \Lambda(X) = \frac{L(X_1,\cdots,X_n \mid \mu_0)}{L(X_1,\cdots,X_n \mid \mu_1)}$$

$$= \exp\{[2n(\mu_0-\mu_1)\bar{X} + n(\mu_1^2-\mu_0^2)]/2\sigma^2\}$$

By Nayman-Pearson Lemma, we apply the likelihood-ratio test.

For a given number k ($k>0$),

$$\Lambda(X) \leq k \iff (2n(\mu_0-\mu_1)\bar{X} + n(\mu_1^2-\mu_0^2))/2\sigma^2 \leq \log k$$

Since μ_0, μ_1 are fixed,

$$\begin{cases} \bar{X} \leq c & \text{if } \mu_0 > \mu_1 \\ \bar{X} \geq c & \text{if } \mu_0 < \mu_1 \end{cases}$$

① Suppose $\mu_0 > \mu_1$

$$P_{\mu_0}(\Lambda(X) \leq k) = \alpha$$

$$\Rightarrow P_{\mu_0}(\bar{X} \leq c) = P_{\mu_0}\left(\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} \leq \frac{c-\mu_0}{\sigma/\sqrt{n}}\right) = \alpha.$$

$$\Rightarrow \frac{c-\mu_0}{\sigma/\sqrt{n}} = -z_\alpha$$

$$c = \mu_0 - \frac{\sigma}{\sqrt{n}} z_\alpha$$

$$R = \{\bar{x} : \bar{x} \le \mu_0 - \frac{\sigma}{\sqrt{n}} z_\alpha\}.$$

The uniform most powerful test is: $\varphi(x)$ where

$$\varphi(x) = 1_{x \in R} = \begin{cases} 1, & \text{if } x \in R \\ 0, & \text{if } x \notin R. \end{cases}$$

② Suppose $\mu_0 < \mu_1$

$$P_{\mu_0}(\lambda(x) \ge k) = \alpha$$

$$\Rightarrow P_{\mu_0}(\bar{x} \ge c) = P_{\mu_0}\left(\frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \ge \frac{c - \mu_0}{\sigma/\sqrt{n}}\right) = \alpha$$

$$\frac{c - \mu_0}{\sigma/\sqrt{n}} = z_\alpha$$

$$c = \mu_0 + \frac{\sigma}{\sqrt{n}} z_\alpha$$

$$R = \{\bar{x} : \bar{x} \ge \mu_0 + \frac{\sigma}{\sqrt{n}} z_\alpha\}$$

The uniform most powerful test is: $\varphi(x)$ where

$$\varphi(x) = 1_{x \in R} = \begin{cases} 1, & \text{if } x \in R \\ 0, & \text{if } x \notin R. \end{cases}$$

(b) By (a), if $\mu_0 > \mu_1$, $R_\alpha = \{\bar{x} : \bar{x} \le \mu_0 - \frac{\sigma}{\sqrt{n}} z_\alpha\}$;

if $\mu_0 < \mu_1$, $R_\alpha = \{\bar{x} : \bar{x} \ge \mu_0 + \frac{\sigma}{\sqrt{n}} z_\alpha\}$

(c) Under H_1, the power function $K(\mu_1) = 1 - \beta(\mu_1)$

$$\frac{\bar{x} - \mu_1}{\sigma/\sqrt{n}} = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} + \frac{\mu_0 - \mu_1}{\sigma/\sqrt{n}} \sim N(0,1)$$

① $\mu_0 > \mu_1$, $R_\alpha = \{\bar{x} : \bar{x} \le \mu_0 - \frac{\sigma}{\sqrt{n}} z_\alpha\}$

$$\begin{aligned} \beta(\mu_1) &= P(\bar{x} > \mu_0 - \frac{\sigma}{\sqrt{n}} z_\alpha) \\ &= P\left(\frac{\bar{x} - \mu_1}{\sigma/\sqrt{n}} > \frac{\sqrt{n}(\mu_0 - \mu_1)}{\sigma} - z_\alpha\right) \\ &= 1 - \Phi\left(\frac{\sqrt{n}(\mu_0 - \mu_1)}{\sigma} - z_\alpha\right) \end{aligned}$$

$$\begin{aligned} K(\mu_1) &= 1 - \beta(\mu_1) \\ &= \Phi\left(\frac{\sqrt{n}(\mu_0 - \mu_1)}{\sigma} - z_\alpha\right) \end{aligned}$$

② $\mu_0 < \mu_1$, $R_\alpha = \{\bar{x} : \bar{x} \ge \mu_0 + \frac{\sigma}{\sqrt{n}} z_\alpha\}$

$$\begin{aligned} \beta(\mu_1) &= P(\bar{x} < \mu_0 + \frac{\sigma}{\sqrt{n}} z_\alpha) \\ &= P\left(\frac{\bar{x} - \mu_1}{\sigma/\sqrt{n}} < \frac{\sqrt{n}(\mu_0 - \mu_1)}{\sigma} + z_\alpha\right) \end{aligned}$$

$$= \Phi\left(\frac{\sqrt{n}(\mu_0-\mu_1)}{\sigma} + z_\alpha\right)$$

$$K(\mu_1) = 1-\beta(\mu_1) = 1-\Phi\left(\frac{\sqrt{n}(\mu_0-\mu_1)}{\sigma} + z_\alpha\right)$$

(d) $\hat{p} := \inf\{\alpha : \bar{x} \in R_\alpha\}$. Let $z_0 = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$

① $\mu_0 > \mu_1$, $R_\alpha = \{\bar{x} : \bar{x} \leq \mu_0 - \frac{\sigma}{\sqrt{n}} z_\alpha\}$

$\Rightarrow \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \leq -z_\alpha$ i.e., $z_0 \leq -z_\alpha$

$\Rightarrow \Phi(z_0) \leq \alpha$

$\Rightarrow \hat{p} = \Phi\left(\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right)$

② $\mu_0 < \mu_1$, $R_\alpha = \{\bar{x} : \bar{x} \geq \mu_0 + \frac{\sigma}{\sqrt{n}} z_\alpha\}$

$\Rightarrow \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \geq z_\alpha$

$\Phi(z_0) \geq 1-\alpha$

$\Rightarrow \hat{p} = 1-\Phi\left(\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right)$