STA2004 Homework 8 Student ID: 121090429

(Question1): (20 points) Assume X is a random variable following the Binomial distribution $Bin(10, \theta)$ with success probability θ. Assume a $Uniform(0, 1)$ prior distribution for θ. Given the observation $X = 0$,

(a) derive the posterior distribution of θ.

(b) Then calculate the posterior mean of θ, i.e., the expectation of θ under its posterior distribution.

(a) $p(X|\theta) = (1-\theta)^{10}$. $\pi(\theta) = \frac{1}{1-0} \cdot 1_{(0,1)} = 1_{(0,1)}$

$$\int_\theta p(x|\theta) \cdot \pi(\theta) d\theta = \int_0^1 (1-\theta)^{10} \cdot 1 \, d\theta = -\frac{1}{11}(1-\theta)^{11}\Big|_0^1 = \frac{1}{11}$$

$$p(\theta|X) = \frac{p(x|\theta) \cdot \pi(\theta)}{\int_\theta p(x|\theta) \cdot \pi(\theta) d\theta} = 11(1-\theta)^{10} \cdot 1_{(0,1)}$$

The posterior distribution of θ is $p(\theta|X) = 11(1-\theta)^{10} \cdot 1_{(0,1)}$

(b) From (a), $p(\theta|x) = 11(1-\theta)^{10} \cdot 1_{(0,1)}$

$$E(\theta|x) = \int_{-\infty}^{\infty} \theta \cdot p(\theta|x) d\theta$$

$$= \int_0^1 \theta \cdot 11(1-\theta)^{10} d\theta$$

$$= 11\int_0^1 \theta(1-\theta)^{10} d\theta$$

$$= 11\left[-\frac{1}{11}\theta(1-\theta)^{11}\right]_0^1 + 11\int_0^1 \frac{1}{11}(1-\theta)^{11} d\theta$$

$$= \frac{1}{12}(1-\theta)^{12}\Big|_0^1$$

$$= \frac{1}{12}$$

(Question2): (20 points) Assume the observations $x_i|\theta \overset{iid}{\sim} Exp(\theta)$, $i = 1, 2, \ldots, n$, following the exponential distribution with pdf $f(x|\theta) = \theta e^{-\theta x}$. Assume the prior distribution of θ to be $Gamma(\alpha, \beta)$, i.e., the prior pdf is $\pi(\theta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\theta^{\alpha-1}e^{-\theta/\beta}$. Find the posterior distribution of θ.

$$p(x_1, \cdots, x_n|\theta) = \theta^n \cdot e^{-\theta \cdot \sum_{i=1}^n x_i} = \theta^n \cdot e^{-\theta \cdot n\bar{x}}$$

$$\int_\theta p(x_1, \cdots, x_n|\theta) \cdot \pi(\theta) \cdot d\theta = \int_0^\infty \theta^n \cdot e^{-\theta \cdot n\bar{x}} \cdot \frac{1}{\Gamma(\alpha) \cdot \beta^\alpha} \cdot \theta^{\alpha-1} e^{-\theta/\beta} d\theta$$

$$= \frac{1}{\Gamma(\alpha) \cdot \beta^\alpha} \int_0^\infty \theta^{\alpha+n-1} e^{-\theta(n \cdot \bar{x} + \frac{1}{\beta})} d\theta$$

$$p(\theta|x_1, \cdots, x_n) = \frac{p(x_1, \cdots, x_n|\theta) \cdot \pi(\theta)}{\int_\theta p(x_1, \cdots, x_n|\theta)\pi(\theta) d\theta} = \frac{\frac{1}{\Gamma(\alpha) \cdot \beta^\alpha} \cdot \theta^{\alpha+n-1} e^{-\theta(n\bar{x}+\frac{1}{\beta})}}{\frac{1}{\Gamma(\alpha) \cdot \beta^\alpha} \cdot \int_0^\infty \theta^{\alpha+n-1} e^{-\theta(n\bar{x}+\frac{1}{\beta})} d\theta}$$

$$= \frac{\theta^{\alpha+n-1} e^{-\theta(n\bar{x}+\frac{1}{\beta})}}{\frac{\int_0^\infty \theta(n\bar{x}+\frac{1}{\beta})^{\alpha+n-1} e^{-\theta(n\bar{x}+\frac{1}{\beta})} d((n\bar{x}+\frac{1}{\beta})\theta)}{(n\bar{x}+\frac{1}{\beta})^{\alpha+n-1} \cdot (n\bar{x}+\frac{1}{\beta})}}$$

$$= \frac{\theta^{\alpha+n-1} e^{-\theta(n\bar{x}+\frac{1}{\beta})}}{\Gamma(\alpha+n)} \cdot (n\bar{x}+\frac{1}{\beta})^{\alpha+n}$$

$$= \frac{\theta^{\alpha+n-1} e^{-\theta(\sum_{i=1}^{n} x_i+\frac{1}{\beta})}}{\Gamma(\alpha+n)} \cdot (\sum_{i=1}^{n} x_i+\frac{1}{\beta})^{\alpha+n} \quad (\theta>0,\ \text{otherwise } p(\theta|x_1,\cdots,x_n)=0).$$

(Question3): (20 points) Assume the observations $x_i|\theta \overset{iid}{\sim} Uniform(0,\theta)$, for $i = 1, 2, \ldots, n$, following the uniform distribution. Here $\theta > 0$. Assume the prior distribution of θ is the Pareto distribution $Pareto(a, b)$ with pdf being $\pi(\theta) = \frac{ba^b}{\theta^{b+1}}$ for $\theta > a$ and 0 otherwise. Here $a > 0, b > 0$.

(a) Find the posterior distribution (pdf) of θ.

(b) Is Pareto distribution a conjugate prior in this example?

(a) Since $x_i|\theta \overset{iid}{\sim} Uniform(0,\theta)$,

$$f(x_i|\theta) = \frac{1}{\theta-0} \cdot 1_{(0,\theta)} = \frac{1}{\theta} \cdot 1_{(0,\theta)}.$$

$$p(x_1,\cdots,x_n|\theta) = (\frac{1}{\theta})^n \cdot 1_{x_1,\cdots,x_n \in (0,\theta)}$$

$$p(x_1,\cdots,x_n|\theta)\cdot\pi(\theta) = (\frac{1}{\theta})^n \cdot 1_{x_1,\cdots,x_n\in(0,\theta)} \cdot \frac{ba^b}{\theta^{b+1}} \cdot 1_{\theta>a}$$

$$\int_\theta p(x_1,\cdots,x_n|\theta)\cdot\pi(\theta)\,d\theta$$

$$= \int_a^\infty (\frac{1}{\theta})^n \cdot \frac{b\cdot a^b}{\theta^{b+1}}\,d\theta$$

$$= \int_a^\infty b\cdot a^b \cdot \theta^{-(b+n+1)}\,d\theta$$

$$= -\frac{1}{b+n}\cdot b\cdot a^b \cdot \theta^{-(b+n)} \Big|_a^\infty$$

$$= \frac{1}{b+n}\, b\cdot a^b \cdot a^{-(b+n)} = \frac{b}{(b+n)\cdot a^n}$$

$$p(\theta|x_1,\cdots,x_n) = \frac{p(x_1,\cdots,x_n|\theta)\cdot\pi(\theta)}{\int_\theta p(x_1,\cdots,x_n|\theta)\pi(\theta)d\theta} = \frac{(\frac{1}{\theta})^n \cdot 1_{x_1,\cdots,x_n\in(0,\theta)} \cdot \frac{ba^b}{\theta^{b+1}} \cdot 1_{\theta>a}}{\frac{b}{(b+n)\cdot a^n}}$$

$$= (\frac{1}{\theta})^n \cdot 1_{x_1,\cdots,x_n\in(0,\theta)} \cdot \frac{b\cdot a^b}{\theta^{b+1}} \cdot 1_{\theta>\alpha} \cdot \frac{(b+n)a^n}{b}$$

$$= \begin{cases} \frac{(b+n)\cdot a^{b+n}}{\theta^{b+n+1}} & \text{if } \theta > a \\ 0, & \text{otherwise} \end{cases}$$

(b) Yes. The posteriod distribution $\sim$ Pareto$(a, b+n)$, which is also a Pareto distribution.

(Question4): (20 points) Assume the observations $x_i|\theta \overset{iid}{\sim} Poisson(\theta)$, for $i = 1, 2, \ldots, n$, following the Poisson distribution with parameter θ. Assume the prior distribution of θ is the exponential distribution $Exp(\lambda)$ with pdf being $\pi(\theta) = \lambda e^{-\lambda\theta}$ for $\theta > 0$ and 0 otherwise. Find the posterior distribution of θ and its posterior mean.

Since $X_i|\theta \overset{i.i.d}{\sim} \text{Poisson}(\theta)$, $p(x_i|\theta) = \dfrac{\theta^{x_i} \cdot e^{-\theta}}{x_i!}$

$$\Rightarrow p(x_1,\cdots,x_n|\theta) = \frac{\theta^{n\bar{x}} \cdot e^{-n\theta}}{\prod_{i=1}^n x_i!}, \quad \text{where } \bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$$

$$p(x_1,\cdots,x_n|\theta)\cdot\pi(\theta) = \lambda \cdot \frac{\theta^{n\bar{x}} \cdot e^{-\theta(\lambda+n)}}{\prod_{i=1}^n x_i!} \cdot 1_{\theta>0}$$

$$\int_\theta p(x_1,\cdots,x_n|\theta)\cdot\pi(\theta)\,d\theta = \int_0^\infty \lambda \cdot \frac{\theta^{n\bar{x}} \cdot e^{-\theta(\lambda+n)}}{\prod_{i=1}^n x_i!}\,d\theta$$

$$= \frac{\lambda}{\prod_{i=1}^n x_i!}\int_0^\infty \theta^{n\bar{x}} \cdot e^{-\theta(\lambda+n)}\,d\theta$$

$$= \frac{\lambda}{\prod_{i=1}^n x_i!}\int_0^\infty [(\lambda+n)\theta]^{n\bar{x}} \cdot e^{-(\lambda+n)\theta} \cdot \frac{1}{(\lambda+n)^{n\bar{x}}\cdot(\lambda+n)} \cdot d((\lambda+n)\theta)$$

$$= \frac{\lambda}{\prod_{i=1}^n x_i! \cdot (\lambda+n)^{n\bar{x}+1}}\int_0^\infty [(\lambda+n)\theta]^{n\bar{x}} \cdot e^{-(\lambda+n)\theta}\,d((\lambda+n)\theta)$$

$$= \frac{\lambda}{\prod_{i=1}^n x_i! \cdot (\lambda+n)^{n\bar{x}+1}} \cdot \Gamma(n\bar{x}+1)$$

$$= \frac{\lambda}{\prod_{i=1}^n x_i! \cdot (\lambda+n)^{n\bar{x}+1}} \cdot \Gamma\left(\sum_{i=1}^n x_i+1\right)$$

$$p(\theta|x_1,\cdots,x_n) = \frac{p(x_1,\cdots,x_n|\theta)\cdot\pi(\theta)}{\int_\theta p(x_1,\cdots,x_n|\theta)\cdot\pi(\theta)\,d\theta}$$

$$= \lambda \cdot \frac{\theta^{n\bar{x}} \cdot e^{-\theta(\lambda+n)}}{\prod_{i=1}^n x_i!} \cdot \frac{\prod_{i=1}^n x_i! \cdot (\lambda+n)^{n\bar{x}+1}}{\lambda \cdot \Gamma\left(\sum_{i=1}^n x_i+1\right)} \cdot 1_{\theta>0}$$

$$= \begin{cases} \dfrac{(\lambda+n)^{n\bar{x}+1}\,\theta^{n\bar{x}}\,e^{-\theta(\lambda+n)}}{\Gamma\left(\sum_{i=1}^n x_i+1\right)} & \text{if } \theta>0 \\ 0, & \text{otherwise} \end{cases}$$

$$E(\theta|x_1,\cdots,x_n) = \int_0^\infty \theta \cdot p(\theta|x_1,\cdots,x_n)\,d\theta$$

$$= \int_0^\infty \frac{(\lambda+n)^{n\bar{x}+1}\,\theta^{n\bar{x}+1}\,e^{-\theta(\lambda+n)}}{\Gamma\left(\sum_{i=1}^n x_i+1\right)}\,d\theta$$

$$= \frac{1}{\Gamma\left(\sum_{i=1}^n x_i+1\right)}\int_0^\infty [(\lambda+n)\theta]^{n\bar{x}+1}\,e^{-\theta(\lambda+n)} \cdot d((\lambda+n)\theta) \cdot \frac{1}{\lambda+n}$$

$$= \frac{\Gamma(n\bar{x}+2)\cdot \frac{1}{\lambda+n}}{\Gamma(n\bar{x}+1)}$$

$$= (n\bar{x}+1)\cdot \frac{1}{\lambda+n}$$

$$= \frac{\sum_{i=1}^{n} X_i + 1}{\lambda+n}$$

(Question5): (20 points*) Assume the observations $x_i|\phi \overset{iid}{\sim} N(0,\phi)$, for $i = 1, 2, \ldots, n$, following the zero-mean Normal distribution with variance ϕ. Assume the prior distribution of ϕ is that $\frac{1}{\phi} \sim Exp(\lambda)$, and pdf of the exponential distribution $X \sim Exp(\lambda)$ is $\lambda e^{-\lambda x}$ for $x > 0$ and 0 otherwise. Prove that the posterior distribution of ϕ is Inverse-Gamma (see Lecture note 12 for the definition of Inverse-Gamma). Hint: you may first derive the pdf of the prior distribution of ϕ from $\frac{1}{\phi} \sim Exp(\lambda)$.

Since $X_i|\phi \overset{i.i.d}{\sim} \text{Normal}(0,\phi)$,

$$P(X_i|\phi) = \frac{1}{\sqrt{2\pi\phi}} e^{-\frac{X_i^2}{2\phi}}$$

$$P(x_1,\cdots,x_n|\phi) = \left(\frac{1}{\sqrt{2\pi\phi}}\right)^n \cdot e^{-\frac{1}{2\phi}\cdot\sum_{i=1}^{n} X_i^2}$$

$$\pi\left(\frac{1}{\phi}\right) = \lambda\cdot e^{-\lambda\cdot\frac{1}{\phi}}$$

Let $\frac{1}{\phi} = \theta$.

$$\pi(\theta) = \lambda\cdot e^{-\lambda\theta}$$

$$\phi = g(\theta) = \frac{1}{\theta}.$$

Since g is a one-to-one function, g^{-1} exists.

$$\theta = g^{-1}(\phi) = \frac{1}{\phi}$$

$$\pi(\phi) = \pi(\theta)\cdot\left|\frac{dg^{-1}}{d\phi}\right|$$

$$= \pi(g^{-1}(\phi))\cdot\left|\frac{dg^{-1}}{d\phi}\right|$$

$$= \lambda e^{-\lambda\cdot\frac{1}{\phi}}\cdot\left|\frac{d}{d\phi}\cdot\frac{1}{\phi}\right|$$

$$= \frac{\lambda}{\phi^2}\cdot e^{-\lambda\cdot\frac{1}{\phi}} = \frac{\lambda}{\Gamma(1)}\cdot\phi^{-1-1} e^{-\frac{\lambda}{\phi}} \sim IG(1,\lambda)$$

$$\Rightarrow P(x_1,\cdots,x_n|\phi)\pi(\phi) = \left(\frac{1}{\sqrt{2\pi\phi}}\right)^n \cdot e^{-\frac{1}{2\phi}\cdot\sum_{i=1}^{n} X_i^2} \cdot \frac{\lambda}{\phi^2}\cdot e^{-\lambda\cdot\frac{1}{\phi}}$$

$$\int_\phi P(x_1,\cdots,x_n|\phi)\pi(\phi)\,d\phi = \int_0^\infty \left(\frac{1}{\sqrt{2\pi\phi}}\right)^n \cdot e^{-\frac{1}{2\phi}\cdot\sum_{i=1}^{n} X_i^2} \cdot \frac{\lambda}{\phi^2}\cdot e^{-\lambda\cdot\frac{1}{\phi}}\,d\phi$$

$$= \frac{\lambda}{(2\pi)^{\frac{n}{2}}}\int_0^\infty \phi^{-\frac{n}{2}-2} e^{-\frac{1}{2\phi}\left(\sum_{i=1}^{n} X_i^2 + 2\lambda\right)}\,d\phi$$

$$\Rightarrow P(\phi \mid x_1, \cdots, x_n) = \frac{P(x_1, \cdots, x_n \mid \phi)\pi(\phi)}{\int_\phi P(x_1, \cdots, x_n \mid \phi)\pi(\phi)\,d\phi}$$

$$= \frac{\phi^{-\frac{n}{2}-2} e^{-\frac{1}{2\phi}(\sum_{i=1}^n x_i^2 + 2\lambda)}}{\int_0^\infty \phi^{-\frac{n}{2}-2} e^{-\frac{1}{2\phi}(\sum_{i=1}^n x_i^2 + 2\lambda)} d\phi}$$

$$= \frac{\phi^{-\frac{n}{2}-2} e^{-\frac{1}{2\phi}(\sum_{i=1}^n x_i^2 + 2\lambda)}}{\int_0^\infty \phi^{-\frac{n}{2}-2} e^{-\frac{1}{2\phi}(\sum_{i=1}^n x_i^2 + 2\lambda)} d\phi}$$

$$= \frac{\phi^{-\frac{n}{2}-2} e^{-\frac{1}{2\phi}(\sum_{i=1}^n x_i^2 + 2\lambda)} \cdot \left(\frac{1}{2}\sum_{i=1}^n x_i^2 + \lambda\right)^{\frac{n}{2}}}{\int_0^\infty \left(\frac{1}{\phi}\right)^{\frac{n}{2}+2} e^{-\frac{1}{\phi}(\frac{1}{2}\sum_{i=1}^n x_i^2 + \lambda)} d\left(\frac{1}{\phi}\left(\frac{1}{2}\sum_{i=1}^n x_i^2 + \lambda\right)\right) \cdot \frac{\phi^2}{\frac{1}{2}\sum_{i=1}^n x_i^2 + \lambda}}$$

$$= \frac{\left(\frac{1}{2}\sum_{i=1}^n x_i^2 + \lambda\right)^{\frac{n}{2}+1} \phi^{-\frac{n}{2}-2} \cdot e^{-\frac{1}{\phi}(\frac{1}{2}\sum_{i=1}^n x_i^2 + \lambda)}}{\int_0^\infty \left[\frac{1}{\phi}\left(\frac{1}{2}\sum_{i=1}^n x_i^2 + \lambda\right)\right]^{\frac{n}{2}} e^{-\frac{1}{\phi}(\frac{1}{2}\sum_{i=1}^n x_i^2 + \lambda)} d\left(\frac{1}{\phi}\left(\frac{1}{2}\sum_{i=1}^n x_i^2 + \lambda\right)\right)}$$

$$= \frac{\left(\frac{1}{2}\sum_{i=1}^n x_i^2 + \lambda\right)^{\frac{n}{2}+1} \phi^{-\frac{n}{2}-2} \cdot e^{-\frac{1}{\phi}(\frac{1}{2}\sum_{i=1}^n x_i^2 + \lambda)}}{\Gamma(\frac{n}{2}+1)} \sim IG\left(1+\frac{n}{2}, \lambda + \frac{1}{2}\sum_{i=1}^n x_i^2\right)$$

So the posteriod distribution of ϕ is Inverse-Gamma