Write the equation for a binary lens.

```
zeta = z + m1 / (Conjugate[z1] - Conjugate[z]) + m2 / (Conjugate[z2] - Conjugate[z])
```

$$z + \frac{m1}{-\text{Conjugate}[z] + \text{Conjugate}[z1]} + \frac{m2}{-\text{Conjugate}[z] + \text{Conjugate}[z2]}$$

Write the conditions for m1 and m2 to work in parameter space involving m and dm.

```
m1 = m - dm
m2 = m + dm
```

$$-dm + m$$

$$dm + m$$

Take the conjugate of the binary lens equation. Solve for the conjugate of z, and substitute this back into binary lens equation. Assume the lensing bodies lie on the real axis, and that the masses are real. Then ζ is equal to the following expression:

```
Refine[zeta /. Solve[Conjugate[zeta] == Conjugate[ζ], Conjugate[z]], {m, dm, z1, z2} ∈ Reals]
```

$$\left\{z + \frac{-dm + m}{z1 + \frac{-dm+m}{-z+z1} + \frac{dm+m}{-z+z2} - \text{Conjugate}[\zeta]} + \frac{dm + m}{\frac{-dm+m}{-z+z1} + z2 + \frac{dm+m}{-z+z2} - \text{Conjugate}[\zeta]}\right\}$$

Move ζ to other side and combine the expression into one fraction. Collect only the numerator (since this polynomial is equal to zero) into powers of z, and mulitply by -1 for convention. This is the form of the polynomial in a general coordinate system.

$$\textbf{pgeneral} = \textbf{Collect}\left[-\textbf{Numerator}\left[\textbf{Together}\left[-\zeta + z + \frac{-dm + m}{z1 + \frac{-dm+m}{-z+z1} + \frac{dm+m}{-z+z2} - \text{Conjugate}[\zeta]} + \frac{dm + m}{\frac{-dm+m}{-z+z1} + z2 + \frac{dm+m}{-z+z2} - \text{Conjugate}[\zeta]}\right]\right], z, \textbf{Simplify}\right]$$

$$\begin{aligned}
&-2 m^2 z1^2 z2 - 2 m^2 z1 z2^2 - m z1^3 z2^2 - m z1^2 z2^3 + m^2 z1^2 \zeta + dm^2 (z1 - z2)^2 \zeta + 2 m^2 z1 z2 \zeta + m z1^3 z2 \zeta + m^2 z2^2 \zeta + 2 m z1^2 z2^2 \zeta + m z1 z2^3 \zeta + z1^3 z2^3 \zeta - \\
&\quad dm (z1 - z2) (2 m + z1 z2) (z1 z2 - z1 \zeta - z2 \zeta) - z1 z2 ((2 dm (z1 - z2) + z1 z2 (z1 + z2)) \zeta + m (-2 z1 z2 + 2 z1 \zeta + 2 z2 \zeta)) \text{Conjugate}[\zeta] + \\
&\quad z1^2 z2^2 \zeta \text{Conjugate}[\zeta]^2 + z^5 (z1 - \text{Conjugate}[\zeta]) (-z2 + \text{Conjugate}[\zeta]) + \\
&\quad z^4 \left(m z1 + m z2 + 2 z1^2 z2 + 2 z1 z2^2 + dm (-z1 + z2) + z1 z2 \zeta - (2 m + (z1 + z2) (2 z1 + 2 z2 + \zeta)) \text{Conjugate}[\zeta] + (2 z1 + 2 z2 + \zeta) \text{Conjugate}[\zeta]^2\right) + \\
&\quad z \Big(m^2 z1^2 - dm^2 (z1 - z2)^2 + 6 m^2 z1 z2 + m z1^3 z2 + m^2 z2^2 + 2 m z1^2 z2^2 + m z1 z2^3 - z1^3 z2^3 - 4 m^2 z1 \zeta - m z1^3 \zeta - 4 m^2 z2 \zeta - \\
&\qquad 5 m z1^2 z2 \zeta - 5 m z1 z2^2 \zeta - 2 z1^3 z2^2 \zeta - m z2^3 \zeta - 2 z1^2 z2^3 \zeta + dm (z1 - z2) \left(z1 z2 (z2 - 2 \zeta) + z1^2 (z2 - \zeta) - \left(4 m + z2^2\right) \zeta\right) + \\
&\qquad \left(m \left(-2 z1 z2 (z2 - 4 \zeta) - 2 z1^2 (z2 - \zeta) + 2 z2^2 \zeta\right) + 2 dm (z1 - z2) (z2 \zeta + z1 (z2 + \zeta)) + z1 z2 (z1 + z2) (2 z2 \zeta + z1 (z2 + 2 \zeta))\right) \text{Conjugate}[\zeta] - \\
&\qquad z1 z2 (2 z2 \zeta + z1 (z2 + 2 \zeta)) \text{Conjugate}[\zeta]^2\Big) + z^3 \Big(dm z1^2 - m z1^2 - 2 m z1 z2 - z1^3 z2 - dm z2^2 - m z2^2 - 4 z1^2 z2^2 - z1 z2^3 - 2 m z1 \zeta - \\
&\qquad 2 m z2 \zeta - 2 z1^2 z2 \zeta - 2 z1 z2^2 \zeta + \left(2 dm (z1 - z2) + 2 m (z1 + z2 + 2 \zeta) + (z1 + z2) \left(z1^2 + 4 z1 z2 + z2^2 + 2 z1 \zeta + 2 z2 \zeta\right)\right) \text{Conjugate}[\zeta] - \\
&\qquad \left(z1^2 + 2 z1 (2 z2 + \zeta) + z2 (z2 + 2 \zeta)\right) \text{Conjugate}[\zeta]^2\Big) + \\
&\quad z^2 \Big(-2 m^2 z1 - 2 m^2 z2 + 2 z1^3 z2^2 + 2 z1^2 z2^3 + 4 m^2 \zeta + 3 m z1^2 \zeta + 6 m z1 z2 \zeta + z1^3 z2 \zeta + 3 m z2^2 \zeta + 4 z1^2 z2^2 \zeta + z1 z2^3 \zeta + \\
&\qquad dm (z1 - z2) (2 m - 2 z1 z2 + z1 \zeta + z2 \zeta) - \left(2 dm (z1 - z2) (z1 + z2 + \zeta) + (z1 + z2) \left(\left(6 m + z2^2\right) \zeta + z1^2 (2 z2 + \zeta) + 2 z1 z2 (z2 + 2 \zeta)\right)\right) \\
&\qquad \text{Conjugate}[\zeta] + \left(z2^2 \zeta + z1^2 (2 z2 + \zeta) + 2 z1 z2 (z2 + 2 \zeta)\right) \text{Conjugate}[\zeta]^2\Big)
\end{aligned}$$

Solve this in our first coordinate frame: geometric center; i.e. z1=-z2

$$\textbf{pcenter} = \textbf{Collect}[\textbf{Expand}[\textbf{pgeneral} \;/.\; \textbf{z2} \to -\textbf{z1}, \textbf{z}], \textbf{z}, \textbf{Simplify}]$$

$$\begin{aligned}
&z^5 \left(z1^2 - \text{Conjugate}[\zeta]^2\right) + z^3 \left(-2 z1^4 + 4 (dm z1 + m \zeta) \text{Conjugate}[\zeta] + 2 z1^2 \text{Conjugate}[\zeta]^2\right) + \\
&\quad z z1 \left(-4 dm^2 z1 - 4 m^2 z1 + z1^5 - 8 dm m \zeta - 4 z1 (dm z1 + m \zeta) \text{Conjugate}[\zeta] - z1^3 \text{Conjugate}[\zeta]^2\right) + \\
&\quad z^4 \left(-z1 (2 dm + z1 \zeta) - 2 m \text{Conjugate}[\zeta] + \zeta \text{Conjugate}[\zeta]^2\right) + \\
&\quad z^2 \left(4 dm z1 \left(m + z1^2\right) + 2 \left(2 m^2 + z1^4\right) \zeta - 4 dm z1 \zeta \text{Conjugate}[\zeta] - 2 z1^2 \zeta \text{Conjugate}[\zeta]^2\right) + \\
&\quad z1^2 \left(4 dm m z1 - 2 dm z1^3 + 4 dm^2 \zeta - z1^4 \zeta + 2 z1 (m z1 + 2 dm \zeta) \text{Conjugate}[\zeta] + z1^2 \zeta \text{Conjugate}[\zeta]^2\right)
\end{aligned}$$

Solve this in our second coordinate frame : planet rest frame; i.e.z1 = 0

$$\textbf{pbody1} = \textbf{Collect}[\textbf{Expand}[\textbf{pgeneral} \;/.\; \textbf{z1} \to \textbf{0}, \textbf{z}], \textbf{z}, \textbf{Simplify}]$$

$$\begin{aligned}
&(dm - m)^2 z2^2 \zeta - z^5 \text{Conjugate}[\zeta] (-z2 + \text{Conjugate}[\zeta]) - (dm - m) z z2 \left(dm z2 + m z2 - 4 m \zeta - z2^2 \zeta + 2 z2 \zeta \text{Conjugate}[\zeta]\right) + \\
&\quad z^2 \left(-dm z2 (2 m + z2 \zeta) + m \left(-2 m z2 + 4 m \zeta + 3 z2^2 \zeta\right) + z2 \left(-\left(6 m + z2^2\right) \zeta + 2 dm (z2 + \zeta)\right) \text{Conjugate}[\zeta] + z2^2 \zeta \text{Conjugate}[\zeta]^2\right) + \\
&\quad z^4 \left((dm + m) z2 - (2 m + z2 (2 z2 + \zeta)) \text{Conjugate}[\zeta] + (2 z2 + \zeta) \text{Conjugate}[\zeta]^2\right) + \\
&\quad z^3 \left(-z2 (dm z2 + m (z2 + 2 \zeta)) + \left(-2 dm z2 + \left(2 m + z2^2\right) (z2 + 2 \zeta)\right) \text{Conjugate}[\zeta] - z2 (z2 + 2 \zeta) \text{Conjugate}[\zeta]^2\right)
\end{aligned}$$

Solve this in our third coordinate frame : star rest frame; i.e.z2 = 0

```
pbody2 = Collect[Expand[pgeneral /. z2 → 0, z], z, Simplify]
```

$$\begin{aligned}
&(dm+m)^2\, z1^2\, \zeta + z^5\, (z1 - \text{Conjugate}[\zeta])\, \text{Conjugate}[\zeta] - (dm+m)\, z\, z1\, \left(dm\, z1 - m\, z1 + 4\, m\, \zeta + z1^2\, \zeta - 2\, z1\, \zeta\, \text{Conjugate}[\zeta]\right) + \\
&z^2\, \left(dm\, z1\, (2\, m + z1\, \zeta) + m\, \left(-2\, m\, z1 + 4\, m\, \zeta + 3\, z1^2\, \zeta\right) - z1\, \left(\left(6\, m + z1^2\right)\, \zeta + 2\, dm\, (z1+\zeta)\right)\, \text{Conjugate}[\zeta] + z1^2\, \zeta\, \text{Conjugate}[\zeta]^2\right) + \\
&z^4\, \left((-dm+m)\, z1 - (2\, m + z1\, (2\, z1 + \zeta))\, \text{Conjugate}[\zeta] + (2\, z1 + \zeta)\, \text{Conjugate}[\zeta]^2\right) + \\
&z^3\, \left(z1\, (dm\, z1 - m\, (z1 + 2\, \zeta)) + \left(2\, dm\, z1 + \left(2\, m + z1^2\right)\, (z1 + 2\, \zeta)\right)\, \text{Conjugate}[\zeta] - z1\, (z1 + 2\, \zeta)\, \text{Conjugate}[\zeta]^2\right)
\end{aligned}$$

Solve this in our fourth and final coordinate frame : center of mass; i.e. (z1m1 + z2m2)/(m1+m2) = 0

```
pcm = Collect[Expand[pgeneral /. z2 → -z1 (m - dm) / (m + dm), z], z, Simplify]
```

$$\begin{aligned}
&z^5\, (z1 - \text{Conjugate}[\zeta])\, \left(\frac{(-dm+m)\, z1}{dm+m} + \text{Conjugate}[\zeta]\right) + \frac{1}{(dm+m)^3} \\
&\quad z1^2\, \left(-m^3\, z1^4\, \zeta + dm\, m^2\, z1\, \left(8\, m^2 - 4\, m\, z1^2 + 3\, z1^3\, \zeta\right) + dm^2\, m\, \left(8\, m\, z1^2\, (z1 - \zeta) + 16\, m^2\, \zeta - 3\, z1^4\, \zeta\right) + dm^3\, \left(-8\, m^2\, (z1 - 2\, \zeta) - 4\, m\, z1^2\, (z1 - 2\, \zeta) + z1^4\, \zeta\right) + \right. \\
&\qquad \left. 2\, (dm - m)\, z1\, \left(-m^3\, z1 + dm\, m\, \left(-4\, m + z1^2\right)\, \zeta + dm^2\, \left(m\, z1 - 4\, m\, \zeta - z1^2\, \zeta\right)\right)\, \text{Conjugate}[\zeta] + (dm - m)^2\, (dm + m)\, z1^2\, \zeta\, \text{Conjugate}[\zeta]^2\right) + \\
&\quad \frac{1}{(dm+m)^2}\, z^4\, \left(\left((dm - m)\, z1^2\, (m\, \zeta + dm\, (4\, z1 + \zeta)) - 2\, \left(m^3 + dm\, m\, (2\, m + z1\, \zeta) + dm^2\, (m + z1\, (4\, z1 + \zeta))\right)\right)\, \text{Conjugate}[\zeta] + \right. \\
&\qquad \left. (dm + m)\, (m\, \zeta + dm\, (4\, z1 + \zeta))\, \text{Conjugate}[\zeta]^2\right) + \\
&\quad \frac{1}{(dm+m)^3}\, 2\, z^3\, \left(-z1\, \left(m^3\, z1^3 + dm^2\, m\, \left(-3\, z1^3 + 4\, m\, \zeta\right) + dm\, m^2\, \left(-z1^3 + 2\, m\, \zeta - 2\, z1^2\, \zeta\right) + dm^3\, \left(3\, z1^3 + 2\, m\, \zeta + 2\, z1^2\, \zeta\right)\right) + \right. \\
&\qquad 2\, \left(m^4\, \zeta + dm\, m^2\, \left(2\, m\, z1 - z1^3 + 3\, m\, \zeta\right) + dm^3\, \left(2\, m\, z1 + 3\, z1^3 + m\, \zeta + 2\, z1^2\, \zeta\right) + dm^2\, m\, \left(4\, m\, z1 + 3\, m\, \zeta + 2\, z1^2\, \zeta\right)\right)\, \text{Conjugate}[\zeta] - \\
&\qquad \left. (dm + m)\, z1\, \left(-m^2\, z1 + 2\, dm\, m\, \zeta + dm^2\, (3\, z1 + 2\, \zeta)\right)\, \text{Conjugate}[\zeta]^2\right) + \\
&\quad \frac{1}{(dm+m)^3}\, 2\, z^2\, \left(m^3\, \left(2\, m^2 + z1^4\right)\, \zeta + dm\, m^2\, \left(2\, m\, z1^3 + z1^4\, (2\, z1 - \zeta) + 6\, m^2\, \zeta\right) + dm^2\, m\, \left(-4\, z1^5 + 6\, m^2\, \zeta + 8\, m\, z1^2\, \zeta - 3\, z1^4\, \zeta\right) + \right. \\
&\qquad dm^3\, \left(-2\, m\, z1^2\, (z1 - 4\, \zeta) + 2\, m^2\, \zeta + z1^4\, (2\, z1 + 3\, \zeta)\right) - 2\, dm\, z1\, \left(m^2\, \left(4\, m - z1^2\right)\, \zeta + 2\, dm\, m\, \left(m\, z1 - z1^3 + 4\, m\, \zeta\right) + dm^2\, \left(2\, m\, z1 + 2\, z1^3 + 4\, m\, \zeta + 3\, z1^2\, \zeta\right)\right) \\
&\qquad\quad \left. \text{Conjugate}[\zeta] + (dm + m)\, z1^2\, \left(-2\, dm\, m\, z1 - m^2\, \zeta + dm^2\, (2\, z1 + 3\, \zeta)\right)\, \text{Conjugate}[\zeta]^2\right) - \\
&\quad \frac{1}{(dm+m)^3}\, z\, z1\, \left(4\, m^5\, z1 - m^3\, z1^5 + dm\, m^2\, \left(-4\, m\, z1^2\, \zeta + 4\, m^2\, (z1 + 4\, \zeta) + z1^4\, (3\, z1 + 4\, \zeta)\right) + \right. \\
&\qquad dm^3\, \left(-4\, m^2\, (z1 - 4\, \zeta) + z1^4\, (z1 + 4\, \zeta) + 4\, m\, z1^2\, (-2\, z1 + 5\, \zeta)\right) + dm^2\, m\, \left(8\, m\, z1^3 - 4\, m^2\, (z1 - 8\, \zeta) - z1^4\, (3\, z1 + 8\, \zeta)\right) - \\
&\qquad 2\, z1\, \left(-2\, m^4\, \zeta + dm\, m^2\, \left(z1^3 - 2\, m\, \zeta\right) + 2\, dm^2\, m\, \left(-z1^3 + 5\, m\, \zeta - 2\, z1^2\, \zeta\right) + dm^3\, \left(z1^3 + 10\, m\, \zeta + 4\, z1^2\, \zeta\right)\right)\, \text{Conjugate}[\zeta] + \\
&\qquad \left. \left(dm^2 - m^2\right)\, z1^2\, (-m\, z1 + dm\, (z1 + 4\, \zeta))\, \text{Conjugate}[\zeta]^2\right)
\end{aligned}$$