Problem 1(a):

$$\begin{aligned} l(\theta) &= log(f(Xn|\theta)) \\ &= log(\pi^{-n}\Pi(1+(xi-\theta)^2)^{-1}) \\ &= -nlog(\pi) - \sum log(1+(\theta-xi)^2) \end{aligned}$$

$$\begin{aligned} l'(\theta) &= 0 - \sum \frac{2*(\theta-xi)}{1+(\theta-xi)^2} \\ &= -2\sum \frac{(\theta-xi)}{1+(\theta-xi)^2} \end{aligned}$$

$$\begin{aligned} l''(\theta) &= -2\sum \frac{1+(\theta-xi)^2-(\theta-xi)*2(\theta-xi)}{(1+(\theta-xi)^2)^2} \\ &= -2\sum \frac{1-(\theta-xi)^2}{(1+(\theta-xi)^2)^2} \end{aligned}$$

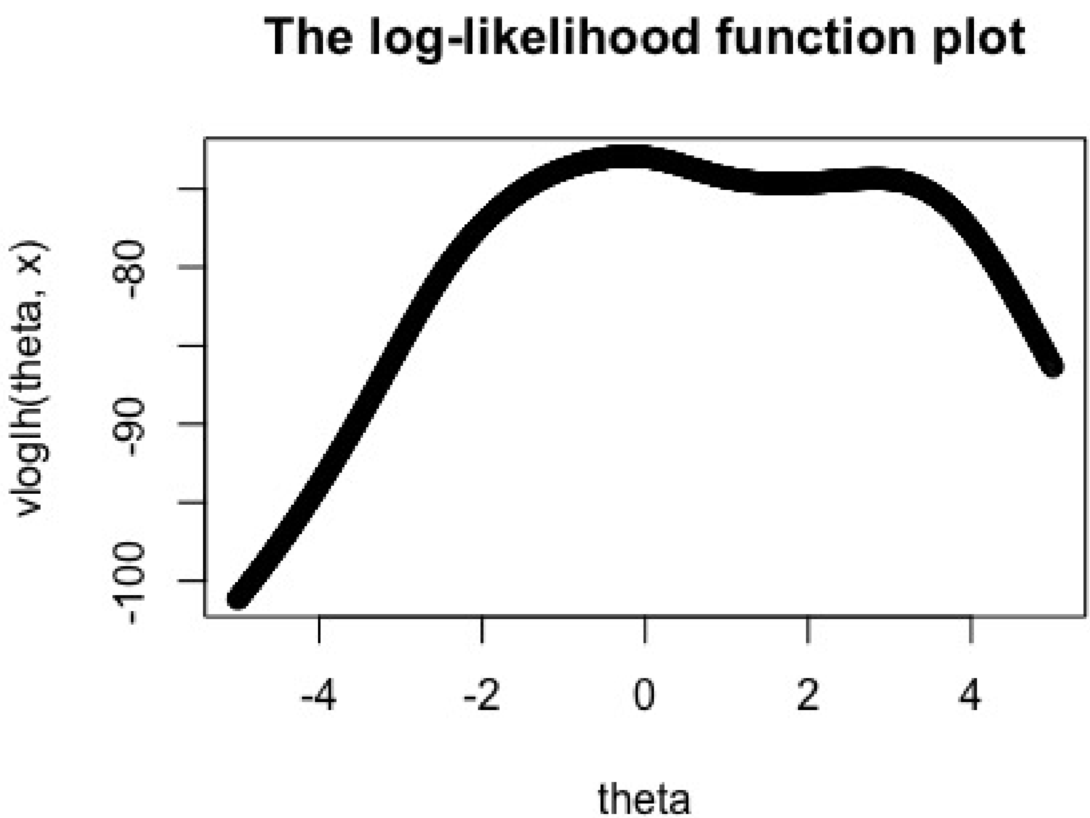
Problem 1(b):

$$\begin{aligned} I(\theta) &= 2\int_{-\infty}^{+\infty} \frac{1-(x-\theta)^2}{(1+(x-\theta)^2)^2}\frac{1}{\pi}\frac{1}{1+(x-\theta)^2}\,dx \\ &= \frac{2}{\pi}\int_{-\infty}^{+\infty} \frac{1-x^2}{(1+x^2)^3}\,dx(x=tant) \\ &= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{1-tan^2t}{(1+tan^2t)^3}sec^2t\,dt \\ &= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{1-tan^2t}{(sec^2t)^2}t\,dt \\ &= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} (cos^4t-sin^2tcos^2t)^2\,dt \\ &= \frac{8}{\pi}\int_{0}^{+\frac{\pi}{2}} cos^4t\,dt - \frac{4}{\pi}\int_{0}^{+\frac{\pi}{2}} cos^2t\,dt \\ &= \frac{8}{\pi}\times\frac{3\pi}{16} - \frac{4}{\pi}\times\frac{\pi}{4} \\ &= \frac{1}{2} \end{aligned}$$

so

$$In(\theta) = \frac{n}{2}$$

Problem 1(c):



Problem 1(d): With 100 iterations:

Initial	-11	-1	0	1.4	
MLE	-7.942004e+30	-0.1922866	-0.1922866	1.713587	
Initial	4.1	4.8	7	8	38
MLE	2.817472	-1.870601e+30	41.04085	-5.537776e+30	42.79538

TABLE 1. MLE with different initials

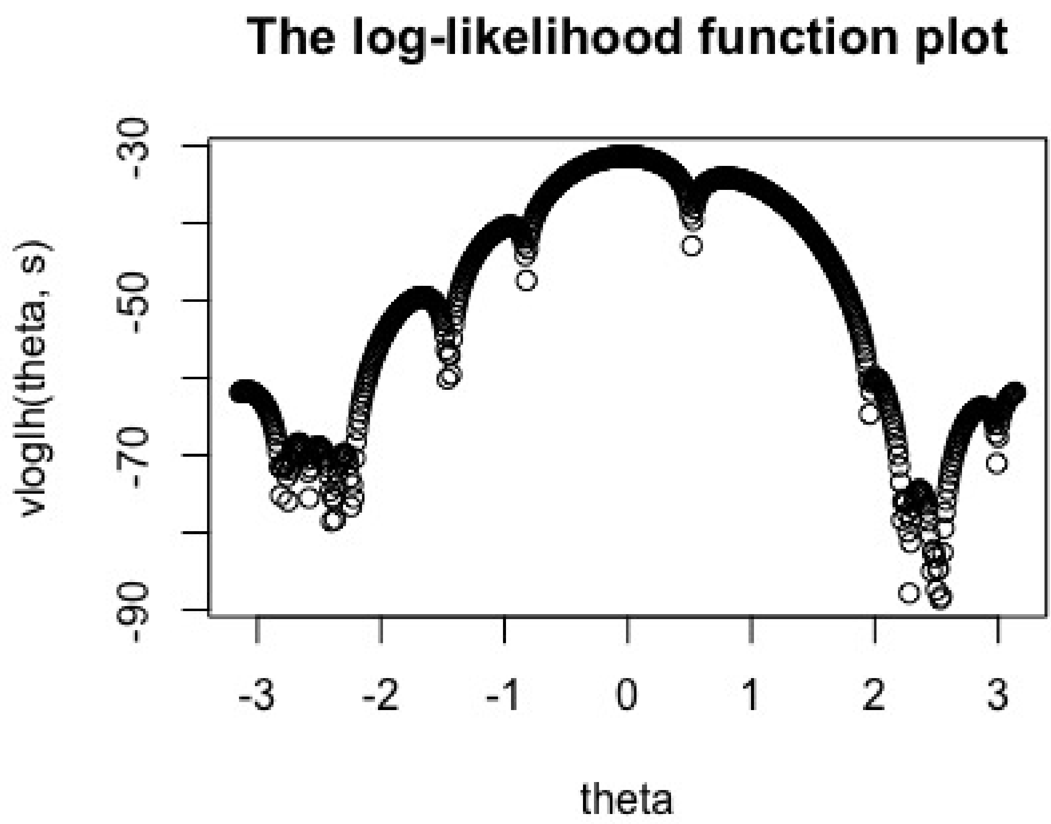
Different initials have different MLEs, two of them has the same value of -0.1922866.

Problem 1(e): With 100 iterations for Fisher scoring and 10 iterations for Newton- Raphson:

Initial	-11	-1	0	1.4	
MLE	-0.1922866	-0.1922866	-0.1922866	-0.1922866	
Initial	4.1	4.8	7	8	38
MLE	2.817472	2.817472	2.817472	2.817472	54.87662

TABLE 2. MLE with different initials

Fisher Scoring help different initials concentrate their MLE's to almost two values: -0.1922866 and 2.817372, the outcomes are much better than in (d).



Problem 2(a):
The log likelihood function plot is:
Problem 2(b):

$$
\begin{aligned}
E(x) &= 2\int_0^{2\pi} x\frac{1-cos(x-\theta)}{2\pi}\,dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} x - xcos(x-\theta)\,dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} x\,dx - \frac{1}{2\pi}\int_{-\theta}^{2\pi-\theta}(t+\theta)cost\,dt \\
&= \pi - 0 - \frac{1}{2\pi}((2\pi-\theta)sin(-\theta) - (-\theta)sin(-\theta) + \\
(cos(2\pi-\theta) - cos(-\theta))) \\
&= \pi + sin\theta \\
\bar{x} &= \pi + sin\theta \\
\theta_{MOM} = arcsin(\bar{x}-\pi) &= 0.05844061
\end{aligned}
$$

Problem 2(c):

$$l'(\theta) = -\sum \frac{-sin(xi-\theta)}{1-cos(xi-\theta)}$$

$$l''(\theta) = -\sum \frac{1}{cos(xi-\theta)-1}$$

when initial value equals to θ_{MOM}, MLE equals to -0.011972.

Problem 2(d): when initial value equals to 2.7, MLE equals to 2.873095.
when initial value equals to -2.7, MLE equals to -2.6667.

Problem 2(e):

Start	-3.141593	-2.762707	-2.731133	-2.573264	-2.383822	2.352248	-2.225953
End	2.825855	-2.762707	-2.604838	-2.415395	-2.383822	-2.257526	-2.225953
MLE	-3.093092	-2.786167	-2.6667	-2.507613	-2.388198	-2.297256	-2.232167
Start	-2.194379	-1.436608	-1.405034	-0.8051318	0.5209676	0.5209676	1.973362
End	-1.468181	-1.436608	-0.8367056	0.4893938	0.5209676	1.941788	2.194379
MLE	-1.658283	-1.447473	-0.9533363	-0.011972	0.7891924	0.7906013	2.003645
Start	2.225953	2.2891	2.478543	2.510117	2.541791	3.015297	
End	2.257526	2.446969	2.478543	2.510117	2.983724	3.141593	
MLE	2.236219	2.360718	2.475373	2.51359	2.873096	3.190094	

TABLE 3. MLE with different initials

Problem 3(a):
First derivative:

$$f_1 = l'(\theta_1) = 2\sum yi - \frac{xi\theta_1)}{xi + \theta_2)}$$

$$f_2 = l'(\theta_2) = 2\sum(yi - \frac{xi\theta_1)}{xi + \theta_2})(\frac{xi\theta_1)}{(xi + \theta_2)^2})$$

Hessian:

$$f_{11} = 2\sum(\frac{xi}{xi + \theta_2})^2$$

$$f_{22} = 2\sum(\frac{2yixi\theta_1}{(xi + \theta_2)^3} - \frac{-3(xi\theta_1)^2}{(xi + \theta_2)^4})$$

$$f_{12} = f_{21} = 2\sum(\frac{xiyi}{(xi + \theta_2)^2} - \frac{2xi^2\theta_1}{(xi + \theta_2)^3})$$

$\theta_1 = 195.8027$ and $\theta_2 = 0.04840654$

Problem 3(b):
Newton-Raphson algorithm with line search:
$\theta_1 = 211.71934766$ and $\theta_2 = 0.06320907$

Problem 3(c):
steepest descent algorithm:
$\theta_1 = 212.13028577$ and $\theta_2 = 0.06359665$

Problem 3(d):
Gauss-Newton algorithm:
$\theta_1 = 212.6736430$ and $\theta_2 = 0.0641055$