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SUBJECT; LAAG

Assignment No 1 Linear Algebra and Analytical Geometry 19SW I / II

Q1 Let $A=\begin{bmatrix}1 & 2 & -1\\ 3 & 0 & 2\\ 4 & 5 & 0\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 1 & 3\end{bmatrix}$. Verify that $(AB)^t = B^t A^t$.

Q2 Prove that the matrix $A=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1+i\\ 1-i & -1\end{bmatrix}$ is Unitary.

Solution to Qno# 1 & 2:

Assignment #01

Q:1 Let $A=\begin{bmatrix}1 & 2 & -1\\ 3 & 0 & 2\\ 4 & 5 & 0\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 1 & 3\end{bmatrix}$ verify that $(AB)^t = B^t A^t$.

Sol: (i) Taking L.H.S $(AB)^t$:

(a) $AB=\begin{bmatrix}1 & 2 & -1\\ 3 & 0 & 2\\ 4 & 5 & 0\end{bmatrix}\cdot\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 1 & 3\end{bmatrix}=\begin{bmatrix}1+4+0 & 0+2-1 & 0+0-3\\ 3+0+0 & 0+0+2 & 0+0+6\\ 4+10+0 & 0+5+0 & 0+0+0\end{bmatrix}=\begin{bmatrix}5 & 1 & -3\\ 3 & 2 & 6\\ 14 & 5 & 0\end{bmatrix}$

(b) $(AB)^t=\begin{bmatrix}5 & 3 & 14\\ 1 & 2 & 5\\ -3 & 6 & 0\end{bmatrix}$ — (i)

(ii) Now Taking R.H.S B^tA^t:

(a) $B^t=\begin{bmatrix}1 & 2 & 0\\ 0 & 1 & 1\\ 0 & 0 & 3\end{bmatrix}$ (b) $A^t=\begin{bmatrix}1 & 3 & 4\\ 2 & 0 & 5\\ -1 & 2 & 0\end{bmatrix}$

(c) $B^t.A^t=\begin{bmatrix}1 & 2 & 0\\ 0 & 1 & 1\\ 0 & 0 & 3\end{bmatrix}\cdot\begin{bmatrix}1 & 3 & 4\\ 2 & 0 & 5\\ -1 & 2 & 0\end{bmatrix}=\begin{bmatrix}1+4+0 & 3+0+0 & 4+10+0\\ 0+2-1 & 0+0+2 & 0+5+0\\ 0+0-3 & 0+0+6 & 0+0+0\end{bmatrix}$

$B^t.A^t=\begin{bmatrix}5 & 3 & 14\\ 1 & 2 & 5\\ -3 & 6 & 0\end{bmatrix}$ — (ii) $\therefore$ since (i) and (ii) are equal hence LHS = RHS, verified.

Q:2: Prove that the Matrix $A=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1+i\\ 1-i & -1\end{bmatrix}$ is Unitary.

Sol: For being Unitary it should satisfy the condition: $(\overline{A^t}).A = I$

(a) $A=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1+i\\ 1-i & -1\end{bmatrix} \Rightarrow A^t=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1-i\\ 1+i & -1\end{bmatrix} \Rightarrow (\overline{A^t})=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1+i\\ 1-i & -1\end{bmatrix}$

(b) $(\overline{A^t}).A=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1+i\\ 1-i & -1\end{bmatrix}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1+i\\ 1-i & -1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}1+1-i^2 & \cancel{1}+\cancel{i}-\cancel{1}-\cancel{i}\\ \cancel{1}-\cancel{i}-\cancel{1}+\cancel{i} & 1-i^2+1\end{bmatrix}$

$(\overline{A^t}).A=\frac{1}{3}\begin{bmatrix}1+1+1 & 0\\ 0 & 1+1+1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$ $\therefore \boxed{i^2=-1}$

$(\overline{A^t}).A=\begin{bmatrix}3/3 & 0/3\\ 0/3 & 3/3\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$

Hence $\boxed{(\overline{A^t}).A=I}$ proved, matrix is unitary.

Q3 (Revised): If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, show that (A^2)-4A-5I = O , Where O is the null matrix of order 3.

Q.3 (Revised) If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, show that $A^2 - 4A - 5I = O$

where O is the null matrix of order 3.

Sol.:

(a) $A^2 = A \cdot A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$

(b) $4A = 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$

(c) $5I = 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

=> Putting values in $A^2 - 4A - 5I = 0$ we get;

$$\begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 9-4-5 & 8-8-0 & 8-8-0 \\ 8-8-0 & 9-4-5 & 8-8-0 \\ 8-8-0 & 8-8-0 & 9-4-5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O_3$$

Hence $A^2 - 4A - 5I = O$, proved.

Q4 Show that the matrix $A = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is Involutory.

Solution to Qno# 4:

Q:4 :. Show that the matrix $A = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is involutory.

Sol:.

→ For being involutory it should satify the condition: $\boxed{A^2 = I}$

$$A^2 = A \cdot A = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0+0 & \cancel{1}-\cancel{1}+0 & \cancel{1}-\cancel{1}-0 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Since $\boxed{A^2 = I}$, matrix is involutory.

Q5 For what values of a, b and c, the matrix $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is Orthogonal.

Q:5 :. For what values of a, b and c, the matrix $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal.

Sol.: For being orthogonal $A^t \cdot A = I$

Taking L.H.S.:

(i) $A^t \cdot A$:.

$A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$, $A^t = \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix}$

(a) $A^t \cdot A = \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} \cdot \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix} \Rightarrow \begin{bmatrix} 0+a^2+a^2 & 0+\cancel{ab}-\cancel{ab} & 0-\cancel{ac}+\cancel{ac} \\ 0+\cancel{ab}-\cancel{ab} & 4b^2+b^2+b^2 & 2bc-bc-bc \\ 0-\cancel{ac}+\cancel{ac} & 2bc-bc-bc & c^2+c^2+c^2 \end{bmatrix}$

$A^t \cdot A = \begin{bmatrix} 2a^2 & 0 & 0 \\ 0 & 6b^2 & 0 \\ 0 & 0 & 3c^2 \end{bmatrix}$

→ For orthogonal $A^t \cdot A = I$

$\begin{bmatrix} 2a^2 & 0 & 0 \\ 0 & 6b^2 & 0 \\ 0 & 0 & 3c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we get,

$\Rightarrow 2a^2 = 1$ $\quad \Rightarrow 6b^2 = 1$ $\quad \Rightarrow 3c^2 = 1$

$\rightarrow a^2 = \frac{1}{2}$ $\quad \rightarrow b^2 = \frac{1}{6}$ $\quad \rightarrow c^2 = \frac{1}{3}$

$\rightarrow \boxed{a = \pm \frac{1}{\sqrt{2}}}$ $\quad \rightarrow \boxed{b = \pm \frac{1}{\sqrt{6}}}$ $\quad \rightarrow \boxed{c = \pm \frac{1}{\sqrt{3}}}$ Ans.

Q6 Show that the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is Nilpotent of index 3.

Q7 Prove that the matrix $A = \begin{bmatrix} 1 & 1+i & 2 \\ 1-i & 3 & i \\ 2 & -i & 0 \end{bmatrix}$ is Hermitian.

Q8 Show that $A = \begin{bmatrix} -i & 3+2i & -2+i \\ -3+2i & 0 & 3-4i \\ 2+i & -3-4i & -2i \end{bmatrix}$ skew – Hermitian.

Solution to Qno# 6, 7 & 8:

Q:06: Show that the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is Nilpotent of index 3.

Sol.

$\Rightarrow$ For Being Nilpotent of index 3, A should satisfy the condition.

$\rightarrow \boxed{A^3 = 0}$

(a) $A \cdot A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} \cancel{1}+\cancel{5}-\cancel{6} & \cancel{1}+\cancel{2}-\cancel{3} & \cancel{3}+\cancel{6}-\cancel{9} \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$

(b) $A \cdot (A \cdot A) = A \cdot A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0+\cancel{3}-\cancel{3} & 0+\cancel{3}-\cancel{3} & 0+\cancel{9}-\cancel{9} \\ 0+\cancel{6}-\cancel{6} & 0+\cancel{6}-\cancel{6} & 0+\cancel{18}-\cancel{18} \\ 0-\cancel{3}+\cancel{3} & 0-\cancel{3}+\cancel{3} & 0-\cancel{9}+\cancel{9} \end{bmatrix}$

$\Rightarrow A \cdot A^2 = A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$

Since, $\boxed{A^3 = 0}$, Hence A is nilpotent of index 3.

Q:7: Prove that the matrix $A = \begin{bmatrix} 1 & 1+i & 2 \\ 1-i & 3 & i \\ 2 & -i & 0 \end{bmatrix}$ is Hermitian.

Sol.

$\Rightarrow$ For being Hermitian, A should satisfy the condition: $\boxed{(\bar{A})^t = A}$

(a) $A = \begin{bmatrix} 1 & 1+i & 2 \\ 1-i & 3 & i \\ 2 & -i & 0 \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} 1 & 1-i & 2 \\ 1+i & 3 & -i \\ 2 & i & 0 \end{bmatrix}$

(b) $(\bar{A})^t = \begin{bmatrix} 1 & 1+i & 2 \\ 1-i & 3 & i \\ 2 & -i & 0 \end{bmatrix} = A$, since $\boxed{(\bar{A})^t = A}$ matrix A is Hermitian.

Q:8: Show that $A = \begin{bmatrix} -i & 3+2i & -2+i \\ -3+2i & 0 & 3-4i \\ 2+i & -3-4i & -2i \end{bmatrix}$ is skew-hermitian.

$\Rightarrow$ for being skew-hermitian, A should satisfy condition: $\boxed{(\bar{A})^t = -A}$

(a) $A = \begin{bmatrix} -i & 3+2i & -2+i \\ -3+2i & 0 & 3-4i \\ 2+i & -3-4i & -2i \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} i & 3-2i & -2-i \\ -3-2i & 0 & 3+4i \\ 2-i & -3+4i & 2i \end{bmatrix}$

(b) $(\bar{A})^t = \begin{bmatrix} i & -3-2i & 2-i \\ 3-2i & 0 & -3+4i \\ -2-i & 3+4i & 2i \end{bmatrix} = -A$

Since $\boxed{(\bar{A})^t = -A}$, Hence A is skew-Hermitian.

Q9 Express $A=\begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$ as the sum of a Hermitian and a skew -Hermitian matrices.

Solution to Qno# 9:

Q:9:: Express $A=\begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$ as the sum of a Hermitan and a skew-hermitian matrices.

Soln

$$A=\begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}, \quad A^t=\begin{bmatrix} i & 6+i & -i \\ 2-3i & 0 & 2-i \\ 4+5i & 4-5i & 2+i \end{bmatrix} \Rightarrow (\overline{A^t})=\begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix}$$

$\Rightarrow$ For hermitian $(\overline{A^t}) = A$

$\therefore$ Adding b/s 'A'

$\therefore A+(\overline{A^t}) = 2A$

$\therefore \boxed{A = \frac{1}{2}\left(A+(\overline{A^t})\right)}$

$\Rightarrow$ for skew-hermitian $(\overline{A^t}) = -A$

$\therefore$ subtracting b/s 'A'

$\therefore -A+(\overline{A^t}) = -2A$

$\therefore \boxed{A = \frac{1}{2}\left(A-(\overline{A^t})\right)}$

$\rightarrow$ For expressing A as a sum of hermitian and skew-hermitian.

$$\therefore \boxed{A = \frac{1}{2}\left(A+(\overline{A^t})\right) + \frac{1}{2}\left(A-(\overline{A^t})\right)} \text{ — (i)}$$

(a) $A+(\overline{A^t}) = \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix} + \begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix} = \begin{bmatrix} 0 & 8-4i & 4+6i \\ 8+4i & 0 & 6-4i \\ 4-6i & 6+4i & 4 \end{bmatrix}$

$\Rightarrow \frac{1}{2}\left(A+(\overline{A^t})\right) = \begin{bmatrix} 0 & 4-2i & 2+3i \\ 4+2i & 0 & 3-2i \\ 2-3i & 3+2i & 2 \end{bmatrix}$

(b) $A-(\overline{A^t}) = \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix} - \begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix} = \begin{bmatrix} 2i & -4-2i & 4+4i \\ 4-2i & 0 & 2-6i \\ -4+4i & -2-6i & 2i \end{bmatrix}$

$\Rightarrow \frac{1}{2}\left(A-(\overline{A^t})\right) = \begin{bmatrix} i & -2-i & 2+2i \\ 2-i & 0 & 1-3i \\ -2+2i & -1-3i & i \end{bmatrix}$

$\Rightarrow$ Substituting values in eq (i)

$$A = \underbrace{\begin{bmatrix} 0 & 4-2i & 2+3i \\ 4+2i & 0 & 3-2i \\ 2-3i & 3+2i & 2 \end{bmatrix}}_{\text{hermitian}} \underset{\text{Sum}}{+} \underbrace{\begin{bmatrix} i & -2-i & 2+2i \\ 2-i & 0 & 1-3i \\ -2+2i & -1-3i & i \end{bmatrix}}_{\text{skew-hermitian}}$$

Q10 Express $A=\begin{bmatrix}1 & 1 & 3\\ 5 & 2 & 6\\ -2 & -1 & -3\end{bmatrix}$ as the sum of symmetric and skew symmetric matrices.

<u>Solution to Qno# 10:</u>

Q:10:- Express $A=\begin{bmatrix}1 & 1 & 3\\ 5 & 2 & 6\\ -2 & -1 & -3\end{bmatrix}$ as the sum of symmetric and skew-symmetric matrices.

Soln

=> For expressing 'A' as a sum of symmetric and skew-symmetric

$$\boxed{A = \frac{1}{2}\left(A + A^t\right) + \frac{1}{2}\left(A - A^t\right)} \quad \text{(i)}$$

$$A = \begin{bmatrix}1 & 1 & 3\\ 5 & 2 & 6\\ -2 & -1 & -3\end{bmatrix}, \quad A^t = \begin{bmatrix}1 & 5 & -2\\ 1 & 2 & -1\\ 3 & 6 & -3\end{bmatrix}$$

(a) $$A + A^t = \begin{bmatrix}1 & 1 & 3\\ 5 & 2 & 6\\ -2 & -1 & -3\end{bmatrix} + \begin{bmatrix}1 & 5 & -2\\ 1 & 2 & -1\\ 3 & 6 & -3\end{bmatrix} = \begin{bmatrix}2 & 6 & 1\\ 6 & 4 & 5\\ 1 & 5 & -6\end{bmatrix}$$

$$\Rightarrow \frac{1}{2}\left(A + A^t\right) = \begin{bmatrix}2/2 & 6/2 & 1/2\\ 6/2 & 4/2 & 5/2\\ 1/2 & 5/2 & -6/2\end{bmatrix} = \begin{bmatrix}1 & 3 & 1/2\\ 3 & 2 & 5/2\\ 1/2 & 5/2 & -3\end{bmatrix}$$

(b) $$A - A^t = \begin{bmatrix}1 & 1 & 3\\ 5 & 2 & 6\\ -2 & -1 & -3\end{bmatrix} - \begin{bmatrix}1 & 5 & -2\\ 1 & 2 & -1\\ 3 & 6 & -3\end{bmatrix} = \begin{bmatrix}0 & -4 & 5\\ 4 & 0 & 7\\ -5 & -7 & 0\end{bmatrix}$$

$$\Rightarrow \frac{1}{2}\left(A - A^t\right) = \begin{bmatrix}0/2 & -4/2 & 5/2\\ 4/2 & 0/2 & 7/2\\ -5/2 & -7/2 & 0/2\end{bmatrix} = \begin{bmatrix}0 & -2 & 5/2\\ 2 & 0 & 7/2\\ -5/2 & -7/2 & 0\end{bmatrix}$$

=> Substituting values in eq (i)

$$A = \underbrace{\begin{bmatrix}1 & 3 & 1/2\\ 3 & 2 & 5/2\\ 1/2 & 5/2 & -3\end{bmatrix}}_{\text{symmetric}} \underset{\text{sum}}{+} \underbrace{\begin{bmatrix}0 & -2 & 5/2\\ 2 & 0 & 7/2\\ -5/2 & -7/2 & 0\end{bmatrix}}_{\text{skew-symmetric}}$$

<u>THE END</u>