Cadi Ayyad University
Faculty of Sciences Semlalia – Marrakech
Department : Computer Science
Program : Master ISI

Mini-Project Report in Quantum Computing

Design and Simulation of Three-Qubit Entangled Quantum States

Conducted by :

LAMSSANE Fatima

Conducted by :

DADDA Hiba

Supervised by : Pr. JARIR Zahi

Academic Year : 2025/2026

Acknowledgement

Table des matières

Table des figures

Inroduction

1 Opening : From Classical Bits to Quantum Possibilities

“Imagine a coin spinning in the air ; not yet heads, not yet tails, but somehow both at once. Now imagine three such coins, spinning together in perfect synchrony, their fates so intertwined that observing one instantly reveals the state of all three, regardless of the distance separating them. This is not magic ; this is quantum mechanics."

The story of computation is fundamentally a story about information. For decades, the classical bit : a definitive 0 or 1, on or off, true or false—has been the atomic unit of all digital information. Like the binary choices we make daily (yes or no, left or right), classical bits are reassuringly concrete : they exist in one state at a time, and measuring them simply reveals what was already there.

Quantum computing shatters this paradigm. At its heart lies the **qubit**—a quantum bit that can exist in a superposition of both 0 and 1 simultaneously. But the true power emerges not from individual qubits, but from their ability to become **entangled** : a uniquely quantum phenomenon where particles become so deeply correlated that the whole becomes fundamentally more than the sum of its parts.

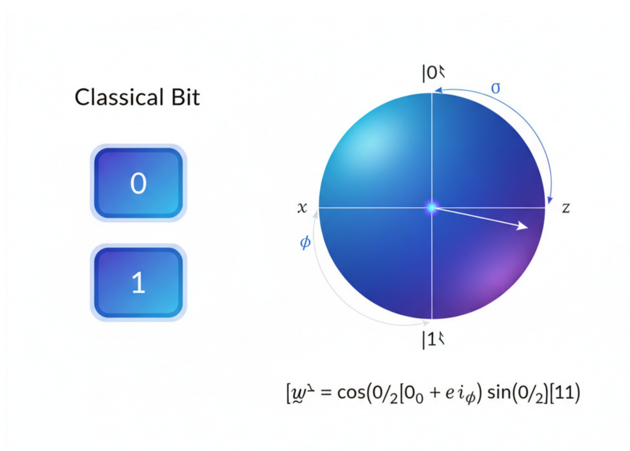

FIGURE 1 – The Computational Paradigm Shift

Consider three classical bits : at any moment, they represent exactly one of eight possible configurations (000, 001, 010, ..., 111). Three qubits, however, can exist in a *superposition* of all eight states simultaneously. More remarkably, when entangled, these qubits exhibit correlations that have no classical counterpart—correlations that Einstein famously dismissed as “spooky action at a distance," yet which lie at the heart of quantum computation’s potential.

2 The Emergence of Quantum Information Science

2.1 Historical Context : From Philosophical Paradox to Technological Reality

The journey from quantum mechanics to quantum computing spans nearly a century of scientific revolution. In 1935, Einstein, Podolsky, and Rosen (EPR) proposed their famous paradox **?**, attempting to demonstrate that quantum mechanics was incomplete. They could not accept that measuring one particle could instantaneously affect another, distant particle. Yet in 1964, John Bell formalized this intuition into testable inequalities **?**, and subsequent experiments by Aspect, Clauser, and others confirmed the unsettling truth : quantum entanglement is real, and nature is fundamentally non-local.

What began as a philosophical crisis has become a technological opportunity. In 1982, Richard Feynman proposed that quantum systems could be efficiently simulated only by other quantum systems **?**, planting the seed of quantum computation. David Deutsch formalized the quantum Turing machine in 1985 **?**, and Peter Shor's 1994 algorithm for factoring large numbers **?** demonstrated that quantum computers could solve certain problems exponentially faster than classical machines—threatening modern cryptography and igniting worldwide research.

2.2 Current Landscape : The NISQ Era

Today, we stand in the *Noisy Intermediate-Scale Quantum* (NISQ) era. Quantum processors with 50–1000 qubits are accessible through cloud platforms such as IBM Quantum, Google Quantum AI, and IonQ. However, these devices suffer from noise, decoherence, and limited connectivity. Despite these imperfections, they enable genuine quantum experiments and algorithm testing, thereby bridging the gap between theoretical models and scalable, fault-tolerant quantum computing.

Three critical challenges define current research :

1. **Quantum Supremacy/Advantage** : Demonstrating computational tasks where quantum systems outperform the best classical supercomputers (achieved by Google in 2019 **?**).
2. **Error Correction** : Developing quantum error-correcting codes to protect fragile quantum information from decoherence (surface codes, topological codes).
3. **Practical Algorithms** : Discovering quantum algorithms with real-world applications beyond Shor's and Grover's foundational results.

Entanglement stands at the nexus of all three challenges. It is simultaneously quantum computing's greatest resource and its most vulnerable aspect—easily destroyed by environmental noise, yet essential for computational advantage.

3 Scientific Problem Statement

3.1 Central Research Question

This work addresses a fundamental question at the intersection of quantum information theory and experimental quantum computing :

Research Question

How do the two inequivalent classes of three-qubit entanglement—GHZ and W states—differ in their mathematical structure, experimental implementation, and robustness to quantum noise ?

3.2 Specific Objectives

This central question decomposes into four specific research objectives :

1. **Theoretical Characterization** : Rigorously define and distinguish the mathematical properties of GHZ and W states, including their entanglement structure, symmetries, and measurement statistics.
2. **Algorithmic Synthesis** : Decompose these states into sequences of universal quantum gates implementable on current quantum hardware, optimizing for circuit depth and gate fidelity.
3. **Experimental Validation** : Implement and execute the circuits on both ideal simulators and real IBM Quantum processors, systematically characterizing the impact of noise.
4. **Comparative Analysis** : Quantify the relative robustness of GHZ versus W states under qubit loss and quantum noise, establishing their practical advantages for different quantum information tasks.

3.3 Why Three Qubits ? The Threshold of Quantum Complexity

The choice of three qubits is both pedagogically strategic and scientifically fundamental :

Pedagogical Simplicity : With 8 basis states (vs. 2 for one qubit, 4 for two), three-qubit systems remain tractable for hand calculation and visualization, making them ideal for developing intuition about multipartite entanglement.

Fundamental Richness : Remarkably, three qubits are the *minimum* required to exhibit inequivalent entanglement classes. As proven by Dür, Vidal, and Cirac **?**, three qubits support exactly two LOCC-inequivalent entanglement types : the GHZ class (maximal tripartite correlations) and the W class (distributed bipartite entanglement). Two qubits have only one entanglement class (Bell states), while four or more qubits have infinitely many.

Practical Relevance : Three-party entanglement protocols—quantum secret sharing, distributed quantum sensing, anonymous transmission—all rely fundamentally on tripartite entangled states **?**.

4 Project Architecture : From Theory to Hardware

This project follows a systematic five-phase methodology, progressing from mathematical foundations to experimental validation on real quantum hardware. Figure 1.2 illustrates the complete workflow.

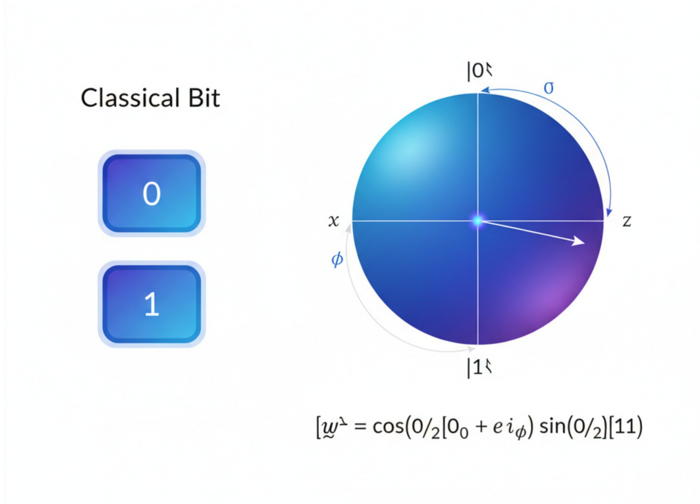

FIGURE 2 – Five-Phase Research Pipeline

4.1 Phase 1 : Theoretical Foundation

Objective : Establish rigorous mathematical formalism for three-qubit entangled states.

Methods :
— Explicit construction of GHZ and W states in the computational basis
— Calculation of density matrices : $\rho_{\text{GHZ}} = \text{GHZGHZ}$, $\rho_{\text{W}} = \text{WW}$
— Derivation of theoretical measurement probabilities
— Entanglement quantification : von Neumann entropy, concurrence
Deliverables : Complete theoretical predictions for comparison with experimental results.

4.2 Phase 2 : Quantum Circuit Design (Qiskit Implementation)

Objective : Translate abstract quantum states into executable gate sequences.

Methods :
— Decomposition into universal gate set : $\{H, \text{CNOT}, R_Y, R_Z\}$
— State verification using statevector simulation
— Circuit optimization for minimal depth and gate count
Tools : Qiskit SDK (Python), including Terra (circuit construction), Aer (simulation backends).

Validation : Fidelity check : $F = |\psi_{\text{target}}\psi_{\text{circuit}}|^2 \geq 0.9999$

4.3 Phase 3 : Ideal Simulation

Objective : Establish noise-free baseline performance.

Methods :
— Execute circuits on `qasm_simulator` (Qiskit Aer)
— Vary shot count : $N \in \{1024, 4096, 16384\}$
— Statistical analysis : χ^2 goodness-of-fit tests
Expected Results : Near-perfect agreement with theory (limited only by finite sampling).

4.4 Phase 4 : Real Hardware Execution (IBM Quantum)

Objective : Validate states on noisy quantum processors.

Methods :
- Backend selection based on calibration data (T_1, T_2, gate errors)
- Adaptive transpilation to native gate set and qubit topology
- Error mitigation : readout error correction, measurement error filtering
- Noise modeling : import backend noise model for local simulation

Key Metrics :

$$F_{\text{Hellinger}} = \left(\sum_i \sqrt{p_i^{\text{ideal}} \cdot p_i^{\text{exp}}} \right)^2 \tag{1}$$

where p_i are probability distributions over measurement outcomes.

4.5 Phase 5 : Comparative Analysis

Objective : Quantify relative performance of GHZ vs. W states.

Analyses :
- *Robustness to Qubit Loss* : Partial trace over one qubit, measure residual entanglement
- *Noise Susceptibility* : Fidelity degradation as function of simulated noise levels
- *Implementation Complexity* : Circuit depth, two-qubit gate count
- *Statistical Testing* : Student's t-test for significant differences in fidelity

5 Experimental Methodology

5.1 Computational Infrastructure

Simulation Environment :
- *Software* : Qiskit 1.0+, Python 3.9+, NumPy, SciPy, Matplotlib
- *Simulator* : Qiskit Aer (C++ backend with OpenMP parallelization)
- *Hardware* : Local workstation (16 GB RAM sufficient for 3-qubit simulations)

Quantum Hardware Access :
- *Platform* : IBM Quantum Experience (cloud-based)
- *Backends* : 5-127 qubit superconducting processors (ibmq_lima, ibmq_jakarta, ibm_brisbane)
- *Authentication* : API token-based access

5.2 Circuit Construction Protocol

All quantum circuits follow a standardized structure :

```python
from qiskit import QuantumCircuit

def create_state_circuit(n_qubits, measure=True):
    """
    Template for three-qubit state preparation.

    Args:
        n_qubits: Number of qubits (3 for this work)
        measure: Whether to include measurement gates

    Returns:
        QuantumCircuit: Prepared quantum circuit
    """
    qc = QuantumCircuit(n_qubits, n_qubits)

    # State preparation gates (GHZ or W specific)
    # ...

    qc.barrier()  # Visual separator

    if measure:
        qc.measure(range(n_qubits), range(n_qubits))

    return qc
```

Listing 1 – Quantum Circuit Template

2. Reproducible Methodology

The complete pipeline—from mathematical formalism to executable code to data analysis—establishes a template for studying multipartite entanglement. All code, circuits, and analysis scripts are documented and reusable.

3. Hardware Characterization Insights

By benchmarking known quantum states, we indirectly characterize the noise profile of IBM Quantum processors, contributing to the broader understanding of NISQ device behavior.

5.3 Educational Value

Pedagogical Bridge : Three-qubit systems occupy a “Goldilocks zone"—complex enough to exhibit rich quantum phenomena, simple enough for detailed hand calculation. This work serves as an educational resource for :

— Undergraduate/graduate quantum computing courses
— Quantum algorithm development tutorials
— Benchmarking studies for new quantum hardware

Accessible Quantum Theory : By providing explicit calculations alongside code, we demystify quantum computing, showing that rigorous quantum experiments are achievable with modest resources.

5.4 Broader Impact

Understanding tripartite entanglement has implications for :

- **Quantum Networks** : Three-party entanglement distribution protocols
- **Quantum Cryptography** : Secret sharing schemes requiring W-type states
- **Quantum Sensing** : Distributed metrology with entangled sensors
- **Quantum Error Correction** : GHZ states form the basis of many stabilizer codes

6 Report Organization

The remainder of this report is structured as follows :

Chapter I : Theoretical Foundations Rigorous mathematical treatment of three-qubit Hilbert space, quantum gates, and entangled states (GHZ and W). Includes all manual calculations of norms, density matrices, and entanglement measures.

Chapter II : Qiskit Infrastructure Overview of the Qiskit ecosystem, simulation backends, transpilation, and the quantum circuit model. Details computational complexity and optimization strategies.

Chapter III : Circuit Implementation Step-by-step construction of GHZ and W circuits, validation via statevector simulation, analysis of measurement statistics and visualization techniques.

Chapter IV : Hardware Validation Execution on IBM Quantum processors, noise characterization, error mitigation, and comparative analysis of simulator vs. real backend performance.

Chapter V : Conclusion Synthesis of findings, discussion of GHZ vs. W trade-offs, limitations, and extensive future research directions (quantum error correction, N-qubit extensions, algorithm applications).

Key Takeaway

This research journey transforms abstract quantum mechanics into executable quantum programs, revealing how two fundamentally different forms of entanglement—GHZ and W—manifest in practice on noisy quantum hardware. Through rigorous theory, careful implementation, and systematic experimentation, we illuminate both the promise and challenges of harnessing quantum entanglement for computation.

Let us now descend from the conceptual heights of quantum superposition into the mathematical bedrock : the formal theory of three-qubit quantum states.

Theoretical Foundations of Three-Qubit States

1 The Quantum State Space : From Geometry to Algebra

1.1 Single Qubits and the Bloch Sphere

A single qubit lives on the **Bloch sphere**, where every pure state corresponds to a point on a unit sphere :

$$\psi = \cos\left(\frac{\theta}{2}\right) 0 + e^{i\phi} \sin\left(\frac{\theta}{2}\right) 1 \tag{1.1}$$

This elegant geometry captures quantum superposition : a qubit explores a continuous manifold of possibilities, not just binary choices.

1.2 Three-Qubit Hilbert Space

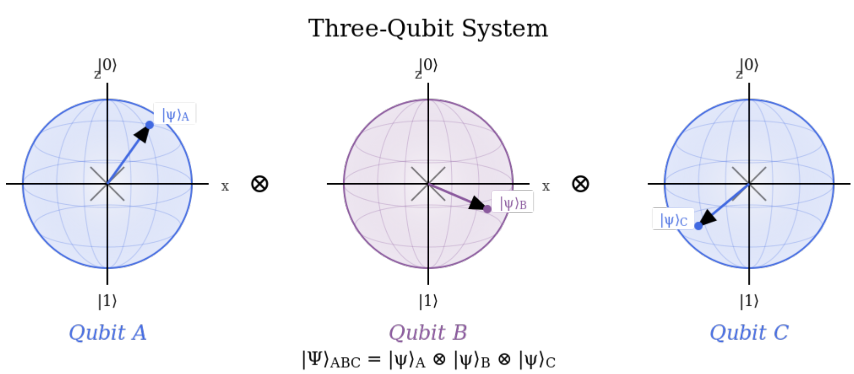

FIGURE 1.1 – Three-Qubit Hilbert Space

For three qubits, the state space explodes into eight complex dimensions :

$$\mathcal{H} = C^2 \otimes C^2 \otimes C^2 \cong C^8 \tag{1.2}$$

The computational basis consists of eight orthonormal vectors : $\{000, 001, 010, 011, 100, 101, 110, 111\}$.

Any pure state writes as :

$$\psi = \sum_{i,j,k \in \{0,1\}} c_{ijk} ijk, \quad \text{where} \quad \sum_{i,j,k} |c_{ijk}|^2 = 1 \tag{1.3}$$

Hand Calculation : Tensor Product

Computing $000 = 0 \otimes 0 \otimes 0$ explicitly :

$$0 \otimes 0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.4}$$

Then :

$$00 \otimes 0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 000 \tag{1.5}$$

1.3 Separable vs. Entangled States

A state is **separable** if it factorizes : $\psi = \phi_1 \otimes \phi_2 \otimes \phi_3$.

An **entangled state** cannot be factored. Example : $\Phi^+ = \frac{00+11}{\sqrt{2}}$ exhibits irreducible correlations between qubits. This is quantum entanglement : *global certainty from local uncertainty*.

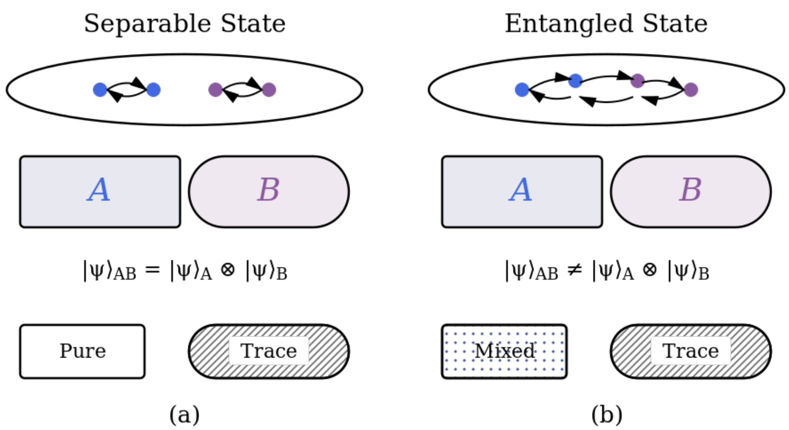

Figure 1.2 – Separable vs. Entangled States

2 Quantum Gates : The Building Blocks

2.1 Hadamard Gate : Creating Superposition

Matrix form :

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{1.6}$$

Action on basis states :

$$H0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{0+1}{\sqrt{2}} \tag{1.7}$$

$$H1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{0-1}{\sqrt{2}} \tag{1.8}$$

Property : $H^2 = I$ (self-inverse).

2.2 CNOT Gate : Weaving Entanglement

Matrix form (control $\otimes$ target) :

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{1.9}$$

Action verification :

$$\text{CNOT}00 = 00 \tag{1.10}$$
$$\text{CNOT}01 = 01 \tag{1.11}$$
$$\text{CNOT}10 = 11 \quad \text{(target flips)} \tag{1.12}$$
$$\text{CNOT}11 = 10 \quad \text{(target flips)} \tag{1.13}$$

Creating entanglement :

$$\text{CNOT} \left(\frac{00+10}{\sqrt{2}} \right) = \frac{00+11}{\sqrt{2}} \quad \text{(Bell state)} \tag{1.14}$$

3 Measurement : Collapsing the Wavefunction

When measuring in the computational basis, outcome ijk occurs with probability :

$$P(ijk) = |ijk|\psi|^2 \tag{1.15}$$

Post-measurement, the state collapses to ijk, destroying superposition.

4 The GHZ State : Maximal Tripartite Entanglement

4.1 Definition

$$\text{GHZ} = \frac{1}{\sqrt{2}}\,(000 + 111) \tag{1.16}$$

A superposition of "all zeros" and "all ones" with perfect correlations.

4.2 Normalization (Hand Calculation)

$$\begin{aligned} \text{GHZ}|\text{GHZ} &= \frac{1}{2}\,(000 + 111)\,(000 + 111) && (1.17)\\ &= \frac{1}{2}\,(000|000 + 0000|111 + 0111|000 + 111|111) && (1.18)\\ &= \frac{1}{2}(1 + 1) = 1 && (1.19) \end{aligned}$$

4.3 Density Matrix

$$\rho_{\text{GHZ}} = \frac{1}{2}\,(000000 + 000111 + 111000 + 111111) \tag{1.20}$$

In 8×8 matrix form (basis $000, \ldots, 111$) :

$$\rho_{\text{GHZ}} = \frac{1}{2}\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{1.21}$$

The off-diagonal coherences $(0, 7)$ and $(7, 0)$ signify quantum interference.

4.4 Properties

Perfect correlations : Measuring yields "000" (50%) or "111" (50%), never intermediate states.

GHZ Paradox : Provides direct contradiction with local realism (no inequalities needed), stronger than Bell's theorem.

Fragility : Measuring one qubit destroys all entanglement—the state collapses to either 00 or 11 (separable).

5 The W State : Distributed Entanglement

5.1 Definition

$$W = \frac{1}{\sqrt{3}}(001 + 010 + 100) \tag{1.22}$$

A single excitation symmetrically shared across three qubits.

5.2 Normalization (Hand Calculation)

$$\begin{aligned} W|W &= \frac{1}{3}(001 + 010 + 100)(001 + 010 + 100) &(1.23)\\ &= \frac{1}{3}[001|001 + 0001|010 + \cdots + 100|100] &(1.24)\\ &= \frac{1}{3}(1 + 1 + 1) = 1 &(1.25)\end{aligned}$$

5.3 Robustness : Partial Trace Calculation

Tracing out qubit 3 (measuring and discarding) :

$$\rho_{12} = \mathrm{Tr}_3(WW) = \frac{1}{3}(0000 + 0101 + 1010) \tag{1.26}$$

Key result : This mixed state remains entangled ! The correlations between qubits 1 and 2 persist despite losing qubit 3.

Contrast with GHZ : For GHZ, measuring qubit 3 yields either 00 or 11 (both separable). Entanglement vanishes completely.

5.4 Physical Meaning

W state represents "quantum load balancing"—distributed, redundant entanglement ideal for :

— Quantum networks (node failure tolerance)
— Distributed sensing (particle loss robustness)
— Secret sharing protocols

6 Classification and Quantification of Entanglement

6.1 The Dür-Vidal-Cirac Theorem

DVC Theorem (2000)

Under LOCC (Local Operations + Classical Communication), three qubits have exactly two inequivalent entanglement classes :

1. **GHZ class** : Maximal tripartite correlations, fragile
2. **W class** : Distributed bipartite entanglement, robust

These are fundamentally distinct—no LOCC protocol can convert one to the other.

This reveals entanglement has distinct "flavors," optimized for different tasks.

6.2 Von Neumann Entropy

For bipartite systems, entanglement quantified by :

$$S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho) = -\sum_i \lambda_i \log_2 \lambda_i \tag{1.27}$$

GHZ entropy (hand calculation) :

Reduced state : $\rho_{12}^{\mathrm{GHZ}} = \frac{1}{2}(0000 + 1111)$

Eigenvalues : $\lambda_1 = \lambda_2 = 1/2$

$$S(\rho_{12}) = -2 \cdot \frac{1}{2} \log_2 \frac{1}{2} = -2 \cdot \frac{1}{2}(-1) = 1 \text{ bit} \tag{1.28}$$

W entropy :

Reduced state has eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 1/3$:

$$S(\rho_{12}^{W}) = -3 \cdot \frac{1}{3} \log_2 \frac{1}{3} = \log_2 3 \approx 1.585 \text{ bits} \tag{1.29}$$

W has higher entropy, reflecting its distributed character.

6.3 Three-Tangle

Coffman-Kundu-Wootters inequality :

$$C_{1(23)}^2 = C_{12}^2 + C_{13}^2 + \tau_{123} \tag{1.30}$$

— GHZ : $\tau > 0$ (genuinely tripartite, irreducible)
— W : $\tau = 0$ (decomposes into pairwise correlations)

Circuit Implementations

To enhance readability and facilitate the inspection of the implementation codes, all experiments were developed in a Google Colab notebook. The complete source code is also hosted on GitHub, while a shared Google Drive folder provides explanatory videos presenting the obtained results and capturing all essential implementation details. The links to these resources are provided below.

External Resources

— **GitHub Repository :** `https://github.com/Zohrae/QC-project`
— **Google Colab Notebook :** `https://colab.research.google.com/github/Zohrae/QC-project/blob/Entanglement_measurement/Implementation_code.ipynb`
— **Google Drive (Results and Video Explanations) :** `https://drive.google.com/drive/folders/1KLsQwdokmHipLSyHfswnhQ5AqV9E5TgE?usp=drive_link`

Conclusion générale

Références