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*Mini-Project Report in Quantum Computing*

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# Design and Simulation of Three-Qubit Entangled Quantum States

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# Acknowledgement

First of all, when we started studying quantum computing, honestly, we didn't get a single thing about it; we felt like it was from another planet! The concepts seemed alien, the math intimidating, and the qubits mischievous by nature. But then, with consistent effort and patience, we began to realize what Mr. Zahid JARIR was truly aiming for in this course. The way he made complex ideas approachable, the creative explanations, the struggle he endured just to make us "get it" — all of that inspired us deeply. We want to sincerely acknowledge him and thank him immensely for his guidance, dedication, and unwavering support throughout this course and semester, for making complex concepts understandable and inspiring us to truly engage with quantum computing.

We also thank ourselves for exploring this mini-project, deepening our understanding of quantum concepts, and discovering more about the fascinating world of qubits.

This project reflects both the knowledge we gained and the growth in our appreciation for quantum computing.



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# Introduction

## 1 Opening: From Classical Bits to Quantum Possibilities

*“Imagine a coin spinning in the air; not yet heads, not yet tails, but somehow both at once. Now imagine three such coins, spinning together in perfect synchrony, their fates so intertwined that observing one instantly reveals the state of all three, regardless of the distance separating them. This is not magic; this is quantum mechanics.”*

The story of computation is fundamentally a story about information. For decades, the classical bit: a definitive 0 or 1, on or off, true or false—has been the atomic unit of all digital information. Like the binary choices we make daily (yes or no, left or right), classical bits are reassuringly concrete: they exist in one state at a time, and measuring them simply reveals what was already there.

Quantum computing shatters this paradigm. At its heart lies the **qubit**—a quantum bit that can exist in a superposition of both 0 and 1 simultaneously. But the true power emerges not from individual qubits, but from their ability to become **entangled**: a uniquely quantum phenomenon where particles become so deeply correlated that the whole becomes fundamentally more than the sum of its parts.

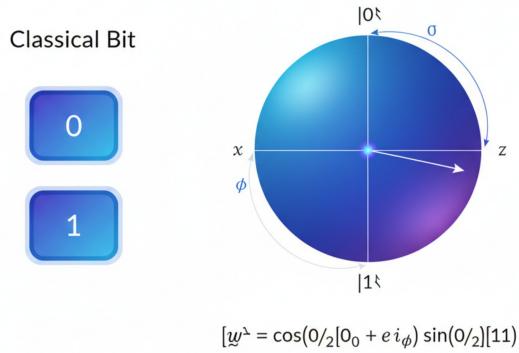


Figure 1: The Computational Paradigm Shift

Consider three classical bits: at any moment, they represent exactly one of eight possible configurations (000, 001, 010, ..., 111). Three qubits, however, can exist in a *superposition* of all eight states simultaneously. More remarkably, when entangled, these qubits exhibit correlations that have no classical counterpart—correlations that Einstein famously dismissed as “spooky action at a distance,” yet which lie at the heart of quantum computation’s potential.

## 2 The Emergence of Quantum Information Science

### 2.1 Historical Context: From Philosophical Paradox to Technological Reality

The journey from quantum mechanics to quantum computing spans nearly a century of scientific revolution. In 1935, Einstein, Podolsky, and Rosen (EPR) proposed their famous paradox ?, attempting to demonstrate that quantum mechanics was incomplete. They could not accept that measuring one particle could instantaneously affect another, distant particle. Yet in 1964, John Bell formalized this intuition into testable inequalities ?, and subsequent experiments by Aspect, Clauser, and others confirmed the unsettling truth: quantum entanglement is real, and nature is fundamentally non-local.

What began as a philosophical crisis has become a technological opportunity. In 1982, Richard Feynman proposed that quantum systems could be efficiently simulated only by other quantum systems ?, planting the seed of quantum computation. David Deutsch formalized the quantum Turing machine in 1985 ?, and Peter Shor's 1994 algorithm for factoring large numbers ? demonstrated that quantum computers could solve certain problems exponentially faster than classical machines—threatening modern cryptography and igniting worldwide research.

### 2.2 Current Landscape: The NISQ Era

Today, we stand in the *Noisy Intermediate-Scale Quantum* (NISQ) era. Quantum processors with 50–1000 qubits are accessible through cloud platforms such as IBM Quantum, Google Quantum AI, and IonQ. However, these devices suffer from noise, decoherence, and limited connectivity. Despite these imperfections, they enable genuine quantum experiments and algorithm testing, thereby bridging the gap between theoretical models and scalable, fault-tolerant quantum computing.

Three critical challenges define current research:

1. **Quantum Supremacy/Advantage:** Demonstrating computational tasks where quantum systems outperform the best classical supercomputers (achieved by Google in 2019 ?).
2. **Error Correction:** Developing quantum error-correcting codes to protect fragile quantum information from decoherence (surface codes, topological codes).
3. **Practical Algorithms:** Discovering quantum algorithms with real-world applications beyond Shor's and Grover's foundational results.

Entanglement stands at the nexus of all three challenges. It is simultaneously quantum computing's greatest resource and its most vulnerable aspect—easily destroyed by environmental noise, yet essential for computational advantage.

## 3 Scientific Problem Statement

### 3.1 Central Research Question

This work addresses a fundamental question at the intersection of quantum information theory and experimental quantum computing:

### Research Question

*How do the two inequivalent classes of three-qubit entanglement—GHZ and W states—differ in their mathematical structure, experimental implementation, and robustness to quantum noise?*

## 3.2 Specific Objectives

This central question decomposes into four specific research objectives:

1. **Theoretical Characterization:** Rigorously define and distinguish the mathematical properties of GHZ and W states, including their entanglement structure, symmetries, and measurement statistics.
2. **Algorithmic Synthesis:** Decompose these states into sequences of universal quantum gates implementable on current quantum hardware, optimizing for circuit depth and gate fidelity.
3. **Experimental Validation:** Implement and execute the circuits on both ideal simulators and real IBM Quantum processors, systematically characterizing the impact of noise.
4. **Comparative Analysis:** Quantify the relative robustness of GHZ versus W states under qubit loss and quantum noise, establishing their practical advantages for different quantum information tasks.

## 3.3 Why Three Qubits? The Threshold of Quantum Complexity

The choice of three qubits is both pedagogically strategic and scientifically fundamental:

**Pedagogical Simplicity:** With 8 basis states (vs. 2 for one qubit, 4 for two), three-qubit systems remain tractable for hand calculation and visualization, making them ideal for developing intuition about multipartite entanglement.

**Fundamental Richness:** Remarkably, three qubits are the *minimum* required to exhibit inequivalent entanglement classes. As proven by Dür, Vidal, and Cirac ?, three qubits support exactly two LOCC-inequivalent entanglement types: the GHZ class (maximal tripartite correlations) and the W class (distributed bipartite entanglement). Two qubits have only one entanglement class (Bell states), while four or more qubits have infinitely many.

**Practical Relevance:** Three-party entanglement protocols—quantum secret sharing, distributed quantum sensing, anonymous transmission—all rely fundamentally on tripartite entangled states ?.

## 4 Project Architecture: From Theory to Hardware

This project follows a systematic five-phase methodology, progressing from mathematical foundations to experimental validation on real quantum hardware.

### 4.1 Phase 1: Theoretical Foundation

**Objective:** Establish rigorous mathematical formalism for three-qubit entangled states.

**Methods:**

- Explicit construction of GHZ and W states in the computational basis
- Calculation of density matrices:  $\rho_{\text{GHZ}} = \text{GHZGHZ}$ ,  $\rho_{\text{W}} = \text{WW}$
- Derivation of theoretical measurement probabilities
- Entanglement quantification: von Neumann entropy, concurrence

**Deliverables:** Complete theoretical predictions for comparison with experimental results.

## 4.2 Phase 2: Quantum Circuit Design (Qiskit Implementation)

**Objective:** Translate abstract quantum states into executable gate sequences.

**Methods:**

- Decomposition into universal gate set:  $\{H, \text{CNOT}, R_Y, R_Z\}$
- State verification using statevector simulation
- Circuit optimization for minimal depth and gate count

**Tools:** Qiskit SDK (Python), including Terra (circuit construction), Aer (simulation backends).

**Validation:** Fidelity check:  $F = |\psi_{\text{target}}\psi_{\text{circuit}}|^2 \geq 0.9999$

## 4.3 Phase 3: Ideal Simulation

**Objective:** Establish noise-free baseline performance.

**Methods:**

- Execute circuits on `qasm_simulator` (Qiskit Aer)
- Vary shot count:  $N \in \{1024, 4096, 16384\}$
- Statistical analysis:  $\chi^2$  goodness-of-fit tests

**Expected Results:** Near-perfect agreement with theory (limited only by finite sampling).

## 4.4 Phase 4: Real Hardware Execution (IBM Quantum)

**Objective:** Validate states on noisy quantum processors.

**Methods:**

- Backend selection based on calibration data ( $T_1, T_2$ , gate errors)
- Adaptive transpilation to native gate set and qubit topology
- Error mitigation: readout error correction, measurement error filtering
- Noise modeling: import backend noise model for local simulation

**Key Metrics:**

$$F_{\text{Hellinger}} = \left( \sum_i \sqrt{p_i^{\text{ideal}} \cdot p_i^{\text{exp}}} \right)^2 \quad (1)$$

where  $p_i$  are probability distributions over measurement outcomes.

## 4.5 Phase 5: Comparative Analysis

**Objective:** Quantify relative performance of GHZ vs. W states.

**Analyses:**

- *Robustness to Qubit Loss*: Partial trace over one qubit, measure residual entanglement
- *Noise Susceptibility*: Fidelity degradation as function of simulated noise levels
- *Implementation Complexity*: Circuit depth, two-qubit gate count
- *Statistical Testing*: Student's t-test for significant differences in fidelity

## 5 Experimental Methodology

### 5.1 Computational Infrastructure

**Simulation Environment:**

- *Software*: Qiskit 1.0+, Python 3.9+, NumPy, SciPy, Matplotlib
- *Simulator*: Qiskit Aer (C++ backend with OpenMP parallelization)
- *Hardware*: Local workstation (16 GB RAM sufficient for 3-qubit simulations)

**Quantum Hardware Access:**

- *Platform*: IBM Quantum Experience (cloud-based)
- *Backends*: 5-127 qubit superconducting processors (ibmq\_lima, ibmq\_jakarta, ibm\_brisbane)
- *Authentication*: API token-based access

### 5.2 Circuit Construction Protocol

All quantum circuits follow a standardized structure:

```

1 from qiskit import QuantumCircuit
2
3 def create_state_circuit(n_qubits, measure=True):
4     """
5         Template for three-qubit state preparation.
6
7     Args:
8         n_qubits: Number of qubits (3 for this work)
9         measure: Whether to include measurement gates
10

```

```

11     Returns:
12         QuantumCircuit: Prepared quantum circuit
13     """
14     qc = QuantumCircuit(n_qubits, n_qubits)
15
16     # State preparation gates (GHZ or W specific)
17     # ...
18
19     qc.barrier()    # Visual separator
20
21     if measure:
22         qc.measure(range(n_qubits), range(n_qubits))
23
24     return qc

```

Listing 1: Quantum Circuit Template

## 2. Reproducible Methodology

The complete pipeline—from mathematical formalism to executable code to data analysis—establishes a template for studying multipartite entanglement. All code, circuits, and analysis scripts are documented and reusable.

## 3. Hardware Characterization Insights

By benchmarking known quantum states, we indirectly characterize the noise profile of IBM Quantum processors, contributing to the broader understanding of NISQ device behavior.

### 5.3 Educational Value

**Pedagogical Bridge:** Three-qubit systems occupy a “Goldilocks zone”—complex enough to exhibit rich quantum phenomena, simple enough for detailed hand calculation. This work serves as an educational resource for:

- Undergraduate/graduate quantum computing courses
- Quantum algorithm development tutorials
- Benchmarking studies for new quantum hardware

**Accessible Quantum Theory:** By providing explicit calculations alongside code, we demystify quantum computing, showing that rigorous quantum experiments are achievable with modest resources.

### 5.4 Broader Impact

Understanding tripartite entanglement has implications for:

- **Quantum Networks:** Three-party entanglement distribution protocols
- **Quantum Cryptography:** Secret sharing schemes requiring W-type states
- **Quantum Sensing:** Distributed metrology with entangled sensors
- **Quantum Error Correction:** GHZ states form the basis of many stabilizer codes

## 6 Report Organization

The remainder of this report is structured as follows:

**Chapter I: Theoretical Foundations** Rigorous mathematical treatment of three-qubit Hilbert space, quantum gates, and entangled states (GHZ and W). Includes all manual calculations of norms, density matrices, partial traces, and entanglement measures (von Neumann entropy, three-tangle).

**Chapter II: Circuit Implementations** Presents the complete computational implementation framework, including all quantum circuit designs for GHZ and W state preparation, statevector simulations, measurement protocols, and execution on both ideal simulators and real IBM Quantum hardware. All source code, circuits, and experimental data are accessible through the provided GitHub repository and Google Colab notebook. **Detailed discussion and explanation of experimental results, comparative analysis, and visualization techniques are presented in the recorded video demonstrations available in the Google Drive folder.**

**Chapter III: Perspectives and Final Gains** Discussion of future research directions (noise mitigation, N-qubit extensions, entanglement classification), reflection on educational outcomes, and technical skills acquired throughout the project.

**General Conclusion:** Synthesis of the learning journey, summary of key findings regarding GHZ vs. W state properties, and motivation for continued exploration in quantum computing.

### Key Takeaway

This research journey transforms abstract quantum mechanics into executable quantum programs, revealing how two fundamentally different forms of entanglement—GHZ and W—manifest in practice on noisy quantum hardware. Through rigorous theory, careful implementation, and systematic experimentation, we illuminate both the promise and challenges of harnessing quantum entanglement for computation.

*Let us now descend from the conceptual heights of quantum superposition into the mathematical bedrock: the formal theory of three-qubit quantum states.*

# Theoretical Foundations of Three-Qubit States

## 1 The Quantum State Space: From Geometry to Algebra

### 1.1 Single Qubits and the Bloch Sphere

A single qubit lives on the **Bloch sphere**, where every pure state corresponds to a point on a unit sphere:

$$\psi = \cos\left(\frac{\theta}{2}\right)0 + e^{i\phi} \sin\left(\frac{\theta}{2}\right)1 \quad (1.1)$$

This elegant geometry captures quantum superposition: a qubit explores a continuous manifold of possibilities, not just binary choices.

### 1.2 Three-Qubit Hilbert Space

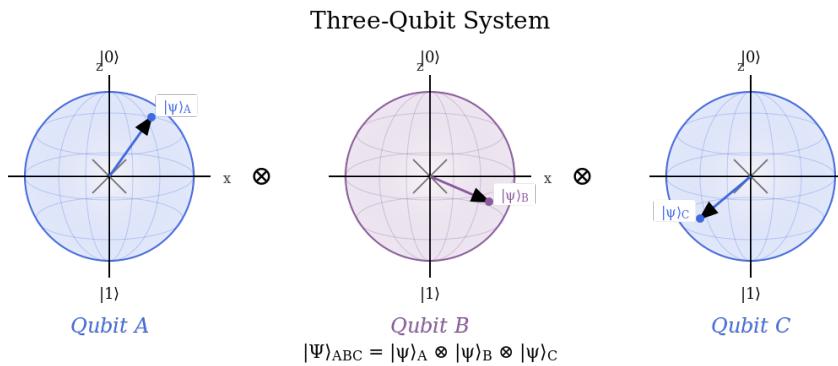


Figure 1.1: Three-Qubit Hilbert Space

For three qubits, the state space explodes into eight complex dimensions:

$$\mathcal{H} = C^2 \otimes C^2 \otimes C^2 \cong C^8 \quad (1.2)$$

The computational basis consists of eight orthonormal vectors:  $\{000, 001, 010, 011, 100, 101, 110, 111\}$ .

Any pure state writes as:

$$\psi = \sum_{i,j,k \in \{0,1\}} c_{ijk} ijk, \quad \text{where} \quad \sum_{i,j,k} |c_{ijk}|^2 = 1 \quad (1.3)$$

### Hand Calculation: Tensor Product

Computing  $000 = 0 \otimes 0 \otimes 0$  explicitly:

$$0 \otimes 0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (1.4)$$

Then:

$$00 \otimes 0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 000 \quad (1.5)$$

### 1.3 Separable vs. Entangled States

A state is **separable** if it factorizes:  $\psi = \phi_1 \otimes \phi_2 \otimes \phi_3$ .

An **entangled state** cannot be factored. Example:  $\Phi^+ = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$  exhibits irreducible correlations between qubits. This is quantum entanglement: *global certainty from local uncertainty*.

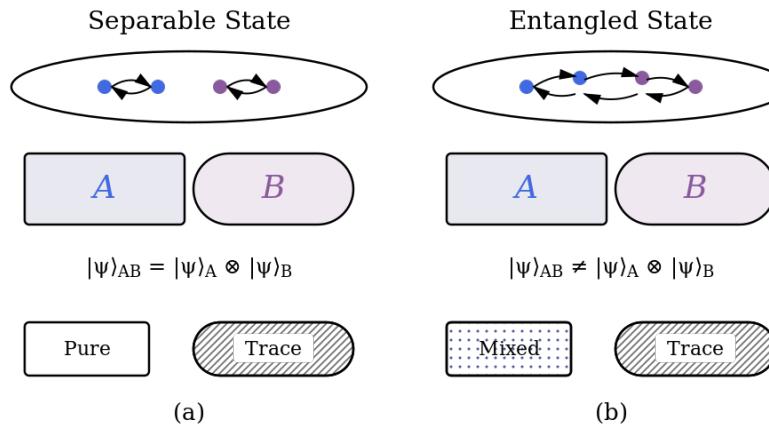


Figure 1.2: Separable vs. Entangled States

## 2 Quantum Gates: The Building Blocks

### 2.1 Hadamard Gate: Creating Superposition

Matrix form:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (1.6)$$

Action on basis states:

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{|0+1\rangle}{\sqrt{2}} \quad (1.7)$$

$$H|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{|0-1\rangle}{\sqrt{2}} \quad (1.8)$$

Property:  $H^2 = I$  (self-inverse).

### 2.2 CNOT Gate: Weaving Entanglement

Matrix form (control  $\otimes$  target):

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (1.9)$$

Action verification:

$$\text{CNOT}|00\rangle = |00\rangle \quad (1.10)$$

$$\text{CNOT}|01\rangle = |01\rangle \quad (1.11)$$

$$\text{CNOT}|10\rangle = |11\rangle \quad (\text{target flips}) \quad (1.12)$$

$$\text{CNOT}|11\rangle = |10\rangle \quad (\text{target flips}) \quad (1.13)$$

Creating entanglement:

$$\text{CNOT} \left( \frac{|00\rangle + |10\rangle}{\sqrt{2}} \right) = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \quad (\text{Bell state}) \quad (1.14)$$

## 3 Measurement: Collapsing the Wavefunction

When measuring in the computational basis, outcome  $ijk$  occurs with probability:

$$P(ijk) = |ijk|\psi|^2 \quad (1.15)$$

Post-measurement, the state collapses to  $ijk$ , destroying superposition.

## 4 The GHZ State: Maximal Tripartite Entanglement

### 4.1 Definition

$$\text{GHZ} = \frac{1}{\sqrt{2}} (000 + 111) \quad (1.16)$$

A superposition of "all zeros" and "all ones" with perfect correlations.

### 4.2 Normalization (Hand Calculation)

$$\text{GHZ}|\text{GHZ} = \frac{1}{2} (000 + 111) (000 + 111) \quad (1.17)$$

$$= \frac{1}{2} (000|000 + 0000|111 + 0111|000 + 111|111) \quad (1.18)$$

$$= \frac{1}{2}(1 + 1) = 1 \quad (1.19)$$

### 4.3 Density Matrix

$$\rho_{\text{GHZ}} = \frac{1}{2} (000000 + 000111 + 111000 + 111111) \quad (1.20)$$

In  $8 \times 8$  matrix form (basis 000, ..., 111):

$$\rho_{\text{GHZ}} = \frac{1}{2} \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad (1.21)$$

The off-diagonal coherences (0, 7) and (7, 0) signify quantum interference.

### 4.4 Properties

**Perfect correlations:** Measuring yields "000" (50%) or "111" (50%), never intermediate states.

**GHZ Paradox:** Provides direct contradiction with local realism (no inequalities needed), stronger than Bell's theorem.

**Fragility:** Measuring one qubit destroys all entanglement—the state collapses to either 00 or 11 (separable).

## 5 The W State: Distributed Entanglement

### 5.1 Definition

$$W = \frac{1}{\sqrt{3}} (001 + 010 + 100) \quad (1.22)$$

A single excitation symmetrically shared across three qubits.

### 5.2 Normalization (Hand Calculation)

$$W|W = \frac{1}{3}(001 + 010 + 100)(001 + 010 + 100) \quad (1.23)$$

$$= \frac{1}{3}[001|001 + 0001|010 + \dots + 100|100] \quad (1.24)$$

$$= \frac{1}{3}(1 + 1 + 1) = 1 \quad (1.25)$$

### 5.3 Robustness: Partial Trace Calculation

Tracing out qubit 3 (measuring and discarding):

$$\rho_{12} = \text{Tr}_3(WW) = \frac{1}{3}(0000 + 0101 + 1010) \quad (1.26)$$

**Key result:** This mixed state remains entangled! The correlations between qubits 1 and 2 persist despite losing qubit 3.

**Contrast with GHZ:** For GHZ, measuring qubit 3 yields either 00 or 11 (both separable). Entanglement vanishes completely.

### 5.4 Physical Meaning

W state represents "quantum load balancing"—distributed, redundant entanglement ideal for:

- Quantum networks (node failure tolerance)
- Distributed sensing (particle loss robustness)
- Secret sharing protocols

## 6 Classification and Quantification of Entanglement

### 6.1 The Dür-Vidal-Cirac Theorem

#### DVC Theorem (2000)

Under LOCC (Local Operations + Classical Communication), three qubits have exactly two inequivalent entanglement classes:

1. **GHZ class:** Maximal tripartite correlations, fragile
2. **W class:** Distributed bipartite entanglement, robust

These are fundamentally distinct—no LOCC protocol can convert one to the other.

This reveals entanglement has distinct "flavors," optimized for different tasks.

### 6.2 Von Neumann Entropy

For bipartite systems, entanglement quantified by:

$$S(\rho) = -\text{Tr}(\rho \log_2 \rho) = -\sum_i \lambda_i \log_2 \lambda_i \quad (1.27)$$

#### GHZ entropy (hand calculation):

Reduced state:  $\rho_{12}^{\text{GHZ}} = \frac{1}{2}(0000 + 1111)$

Eigenvalues:  $\lambda_1 = \lambda_2 = 1/2$

$$S(\rho_{12}) = -2 \cdot \frac{1}{2} \log_2 \frac{1}{2} = -2 \cdot \frac{1}{2}(-1) = 1 \text{ bit} \quad (1.28)$$

#### W entropy:

Reduced state has eigenvalues  $\lambda_1 = \lambda_2 = \lambda_3 = 1/3$ :

$$S(\rho_{12}^W) = -3 \cdot \frac{1}{3} \log_2 \frac{1}{3} = \log_2 3 \approx 1.585 \text{ bits} \quad (1.29)$$

W has higher entropy, reflecting its distributed character.

### 6.3 Three-Tangle

Coffman-Kundu-Wootters inequality:

$$C_{1(23)}^2 = C_{12}^2 + C_{13}^2 + \tau_{123} \quad (1.30)$$

- GHZ:  $\tau > 0$  (genuinely tripartite, irreducible)
- W:  $\tau = 0$  (decomposes into pairwise correlations)

# Circuit Implementations

To enhance readability and facilitate the inspection of the implementation codes, all experiments were developed in a Google Colab notebook using the Qiskit framework. The implementations carried out in this mini-project are based on two complementary execution approaches:

- **Simulator-based implementations:** Quantum circuits are first executed on Qiskit simulators in order to analyze their theoretical behavior under ideal and controlled conditions. This approach allows direct access to statevectors, probability distributions, and intermediate quantum states without the influence of hardware noise.
- **Real quantum hardware implementations (IBM Quantum):** Selected circuits are then executed on real quantum processors provided by IBM Quantum. This extension enables the evaluation of circuit performance under realistic constraints such as noise, decoherence, and limited qubit connectivity.

The complete source code is hosted on GitHub, while a shared Google Drive folder provides explanatory videos presenting the obtained results and capturing all essential implementation details. The links to these resources are provided below.

## External Resources

- **GitHub Repository:** <https://github.com/Zohrae/QC-project>
- **Google Colab Notebook:** [https://colab.research.google.com/github/Zohrae/QC-project/blob/Entanglement\\_measurement/Implementation\\_code.ipynb](https://colab.research.google.com/github/Zohrae/QC-project/blob/Entanglement_measurement/Implementation_code.ipynb)
- **Google Drive (Results and Video Explanations):** <https://drive.google.com/drive/folders/1KLsQwdokmHipLSyHfswnhQ5AqV9E5TgE?usp=sharing>

# Perspectives and Final Gains

## 1 Perspectives

This mini-project opens several natural perspectives for future exploration in quantum computing, including:

- Studying the impact of noise and decoherence on multipartite entangled states by incorporating realistic noise models and applying advanced error mitigation techniques.
- Extending the current work to larger quantum systems, such as circuits with four or more qubits, in order to analyze how entanglement structure and circuit complexity scale with system size.
- Exploring other classes of quantum states and entanglement measures to gain deeper insight into potential applications in quantum communication, distributed sensing, and quantum error correction.
- Performing additional experiments on different IBM Quantum backends to enable a broader comparison of hardware performance, stability, and reliability in the NISQ era.

## 2 Final Gains

From an educational and practical perspective, this mini-project provided significant gains. It strengthened our understanding of fundamental quantum concepts such as superposition, measurement, and multipartite entanglement, while bridging the gap between abstract theory and concrete implementation. We learned how to design quantum circuits, validate them using statevector simulations, and analyze experimental results obtained from both ideal simulators and real quantum hardware. Beyond technical skills, this work enhanced our analytical thinking, scientific rigor, and confidence in approaching complex quantum systems. Overall, this project represents a meaningful step forward in our journey into quantum computing and lays a solid foundation for more advanced studies in this field.

# General Conclusion

This mini-project on three-qubit quantum states has been a fascinating journey from initial confusion to a solid understanding of quantum entanglement. At the beginning, the concepts of qubits, superposition, and entanglement felt abstract and almost unreachable. However, by implementing GHZ and W states with Qiskit, visualizing their statevectors, and simulating measurements, we gradually gained clarity and confidence in handling quantum systems.

Through step-by-step circuit construction and progressive analysis, we not only learned to prepare entangled states but also to interpret their properties and probabilities, appreciating the subtle differences between GHZ and W states. The project also offered insight into the role of simulators and the potential of executing circuits on real IBM Quantum backends, hinting at the impact of noise and experimental constraints in real quantum devices.

Beyond the technical skills, this project strengthened our understanding of quantum computing principles, our ability to translate theoretical concepts into practical circuits, and our analytical thinking when interpreting results. The suggested extensions — studying noise effects, exploring measures of entanglement, and running experiments on real quantum hardware — open exciting avenues for deeper exploration in future projects.

In summary, this mini-project has been a comprehensive introduction to quantum computation: from mastering the basics, constructing and simulating entangled states, to envisioning the next steps in quantum experimentation, leaving us motivated and better equipped to continue exploring this rapidly evolving field.

# Références

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