Tolman–Oppenheimer–Volkoff (TOV) Stars

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(Dated: December 4, 2012)

We present a set of lecture notes for modeling stellar structure in regimes where general relativistic effects become important, such as for neutron stars. The framework draws directly from solving the Einstein equation for a spherically symmetric star in static equilibrium in terms of energy density, pressure, and a term like the gravitational potential. The equation is presented as a somewhat intuitive extension of what was covered regarding stars in hydrostatic equilibrium. Furthermore, we numerically solve the TOV equation in the case of a piecewise polytropic equation of state to find a theoretical upper limit on the mass of neutron stars. For the model considered we find a maximum mass of 2.122 $M_\odot$, in agreement with the currently accepted range of 1.44 to 3 $M_\odot$ for neutron stars.

I. INTRODUCTION

The background material for this topic is presented in Chapter II of the class notes regarding hydrostatic equilibrium. Naturally this means the solution remains constant in time, a phenomenon made possible by the balance of internal pressure support against the star's gravitational field. For stars like our sun the gas pressure is fueled by the hot nuclear reactions in its core. However, many types of stars are known to exist with varying mass and temperature as characterized by an HR diagram. Though the lives of stars are complicated by their unique dynamics, their end states are signaled by the exhaustion of the thermal fuel. Again, depending on the mass of the star it may either collapse to form a black hole or (perhaps after a collapse type explosion) find other means to oppose gravity. At this point we focus on relativistic stars for which the relativistic internal energy significantly contributes to the stellar equation of state.

II. RELATIVISTIC FRAMEWORK

We review the most relevant aspects of general relativity for stellar applications and refer the reader to Refs. [1, 2] for more detailed treatments. The *metric* is a geometric tool that relates distances in spacetime, a kind of generalized pythagorean theorem where the time coordinate is included as well. The underlying physics is more important than the relative coordinates, so all equations are written in the invariant language of tensors, or multi-indexed objects. The Einstein summation convention shortens the notation by assuming an implied sum over repeated indices. With this in mind, the Schwarschild metric for a spherically

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symmetric vacuum spacetime (valid outside a star or black hole) in coordinates (t, r, θ, ϕ) is

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{2GM}{rc^2}\right)c^2 & 0 & 0 & 0 \\ 0 & \left(1-\frac{2GM}{rc^2}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \tag{1}$$

which induces the following line element for measuring infinitesimal distances:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(1-\frac{2GM}{rc^2}\right)c^2dt^2 + \left(1-\frac{2GM}{rc^2}\right)^{-1}dr^2 + r^2\left(d\theta^2+\sin^2\theta d\phi^2\right). \tag{2}$$

Note that in spacetime, the components $dx^\mu = (dt, dr, d\theta, d\phi)$ are treated on equal footing except for a relative minus sign in front of the squared time component. However, this allows for the propagation of light along null vectors ($i.e.$ $ds^2 = 0$).

In fact, the Lorentzian form of the metric, or the $(-+++)$ signature asymmetry of time with space, helps to explain the presence of a gravitational force in curved spacetime. The geometry is essential so we define vector fields at each point and parallel transport vectors from nearby points to establish a generalized notion of the derivative. For the covariant derivative ∇ this is done by defining the Christoffel symbols Γ as a way to connect different points in spacetime:

$$\partial \to \nabla \quad \text{where} \quad \nabla_\mu T^\alpha_\beta = \partial_\mu T^\alpha_\beta + \Gamma^\alpha_{\sigma\mu}T^\sigma_\beta - \Gamma^\sigma_{\mu\beta}T^\alpha_\sigma \tag{3}$$

$$\text{and} \quad \Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left[g_{\alpha\nu,\beta}+g_{\beta\nu,\alpha}-g_{\alpha\beta,\nu}\right]. \tag{4}$$

The comma denotes differentiation with respect to the coordinate x^μ so that $f_{,\mu} \equiv \partial f/\partial x^\mu$. The Christoffels are used to construct a measure of curvature called the Ricci tensor:

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\mu}\Gamma^\beta_{\alpha\nu}, \tag{5}$$

and its contraction or trace, $R = g^{\mu\nu}R_{\mu\nu}$, known as the Ricci scalar.

This construction communicates the presence of spacetime geometry whereas the stress-energy tensor T describes the matter and energy content of the Universe. In practice, a symmetric Einstein tensor G satisfying special properties ($i.e.$ the Bianchi identities given by $\nabla_\mu G^{\mu\nu} = 0$) is hand-picked to mirror the physical properties of the stress-energy tensor T ($i.e.$ conservation laws derived from $\nabla_\mu T^{\mu\nu} = 0$). The final results needed for our relativistic stars is the deduced Einstein equation from the requirement given above:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{6}$$

Note that relativists tend to use units where the speed of light and the gravitational constant simplify the equations so that $c = G = 1$. Equation 6 simply relates the fact that 'matter tells space how to curve, and space tells matter how to move.'

III. DERIVATION OF THE TOV EQUATION

According to Birkhoff's theorem the Schwarzschild solution is the most general description outside a nonrotating, spherically symmetric star. However, inside the star we must consider a more general metric to describe a static density and pressure profile. Once again using Birkhoff's theorem we are free to write the general metric for the stellar interior in the time-independent form (we have set $c = G = 1$ but can restore the dependence later)

$$ds^2 = -e^{2\Phi(r)}dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) . \tag{7}$$

The functions in front of dt^2 and dr^2 need to be independent functions of the radial coordinate r, however, we are free to choose their form which we do in anticipation of a physical interpretation of a mass function down the road. It is important to see the relationship between a general spherically symmetric spacetime and the further constraint of a vacuum solution. The empty space outside the star is precisely the one given by the Schwarzschild metric. In fact, in the limit that the initial distribution is given by a point source then the solution represents the physics of a static black hole.

III.1. Perfect Fluid

Consider a perfect fluid with total energy density ϵ, isotropic pressure P, metric $g^{\mu\nu}$, and four-velocity $u^\mu = (\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\phi}{d\tau}) = \gamma(1, \vec{v})$ as are familiar from special relativistic calculations. The stress-energy tensor for is built from these components according to

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + Pg^{\mu\nu} . \tag{8}$$

The total energy density consists of both the rest mass density of the fluid ρ and the internal energy ε, which in this case represents the thermal motion of the constituent fluid particles. By this we mean the total energy density is given by

$$\epsilon = \rho c^2 + \varepsilon . \tag{9}$$

For simplicity we may indiscriminately drop the factors of c knowing we may recover them if needed. Finally, we define the specific enthalpy h, or enthalpy per unit mass, as

$$h = \frac{\epsilon + P}{\rho} . \tag{10}$$

III.2. The TOV Equation

In the spirit of completion we now present the Tolman–Oppenheimer–Volkoff (TOV) equation. From the metric of Equation 7 and the method of Section II we can calculate the Christoffel symbols $\Gamma^\alpha_{\mu\nu}$, the curvature tensor $R_{\mu\nu}$, the Einstein tensor $G_{\mu\nu}$, and the stress-energy tensor $T_{\mu\nu}$ for a perfect fluid. We begin with the nonzero, independent Christoffels, which can be calculated from Equation 3, where we use the notation $dm(r)/dr = m'$, etc.:

$$\begin{aligned}
&\Gamma^t_{tr} = \Phi' \qquad \Gamma^r_{tt} = \Phi' e^{2\Phi}\left(1 - \frac{2m}{r}\right) \qquad \Gamma^r_{rr} = \frac{rm' - m}{r^2 - 2rm} \qquad \Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = \frac{1}{r} \\
&\Gamma^r_{\theta\theta} = \csc^2\theta\, \Gamma^r_{\phi\phi} = 2m - r \qquad \Gamma^\theta_{\phi\phi} = -\csc^2\theta\, \Gamma^\phi_{\theta\phi} = -\sin\theta\cos\theta .
\end{aligned} \tag{11}$$

From Equation 5 the nonzero, independent components of the Ricci tensor $R_{\mu\nu}$ are

$$\begin{aligned}
R_{tt} &= e^{2\Phi}\left[\left(\Phi'' + \Phi'^2\right)\left(1 - \frac{2m}{r}\right) + \Phi'\left(\frac{2r - 3m - rm'}{r^2}\right)\right] \\
R_{rr} &= \left(1 - \frac{2m}{r}\right)^{-1}\left[\frac{(rm' - m)(2 + r\Phi')}{r^3}\right] - \Phi'' - \Phi'^2 \\
R_{\theta\theta} &= \csc^2\theta\, R_{\phi\phi} = (2m - r)\Phi' + m' + \frac{m}{r}\,.
\end{aligned} \tag{12}$$

Therefore the Ricci scalar is

$$R = g^{\mu\nu}R_{\mu\nu} = 2\left[\frac{2m'}{r^2} + \Phi'\left(3m - 2r + rm'\right) - \left(1 - \frac{2m}{r}\right)\left(\Phi'' + \Phi'^2\right)\right]. \tag{13}$$

The Einstein equation is $G_{\mu\nu} \equiv R_{\mu\nu} - g_{\mu\nu}R/2 = 8\pi T_{\mu\nu}$, where in the Eularian rest frame $T_{tt} = \epsilon e^{2\Phi}$ so the 'time-time' component of the equation gives

$$G_{tt} = \frac{2m' e^{2\Phi}}{r^2} = 8\pi\epsilon e^{2\Phi} \qquad \text{or} \qquad m' = 4\pi r^2\epsilon\,. \tag{14}$$

The radial equation with $T_{rr} = (1 - \frac{2m}{r})^{-1}P$ is

$$G_{rr} = \frac{2}{r}\left(\Phi' - \frac{m}{1 - 2m/r}\right) = \frac{8\pi P}{1 - 2m/r} \qquad \text{or} \qquad \Phi' = \frac{m + 4\pi r^3 P}{r(r - 2m)}\,. \tag{15}$$

Finally, we need a differential equation for the pressure. An easy way to do this is to use conservation of energy to say the divergence of the stress-energy tensor vanishes. The radial component is all we need. If $T^{\mu\nu} = \text{diag}(\epsilon e^{-2\Phi}, (1 - \frac{2m}{r})P, r^{-2}P, r^{-2}csc^2\theta P)$ then we have

$$\begin{aligned}
0 = \nabla_\nu T^{r\nu} &= \frac{\partial T^{r\nu}}{\partial x^\nu} + T^{\sigma\nu}\Gamma^r_{\sigma\nu} + T^{r\sigma}\Gamma^\nu_{\sigma\nu} \\
&= \frac{\partial T^{rr}}{\partial r} + T^{tt}\Gamma^r_{tt} + T^{rr}\Gamma^r_{rr} + T^{\theta\theta}\Gamma^r_{\theta\theta} + T^{\phi\phi}\Gamma^r_{\phi\phi} + T^{rr}\Gamma^\nu_{r\nu} \\
&= \left(1 - \frac{2m}{r}\right)\left[P' + (P + \epsilon)\Phi'\right] \qquad \text{or} \qquad P' = -(\epsilon + P)\Phi'\,.
\end{aligned} \tag{16}$$

The TOV equation is summarized as

$$\begin{aligned}
\frac{dm}{dr} &= 4\pi r^2\epsilon \\
\frac{dP}{dr} &= -(\epsilon + P)\frac{m + 4\pi r^3 P}{r(r - 2m)} \\
\frac{d\Phi}{dr} &= -\frac{1}{\epsilon + P}\frac{dP}{dr}\,.
\end{aligned} \tag{17}$$

Finally, we express this equation in a form suggestive of the post-Newtonian corrections [3]

$$\frac{dP}{dr} = -\frac{G\epsilon(r)m(r)}{c^2r^2}\left[1 + \frac{P(r)}{\epsilon(r)}\right]\left[1 + \frac{4\pi r^3 P(r)}{m(r)c^2}\right]\left[1 - \frac{2Gm(r)}{c^2 r}\right]^{-1}. \tag{18}$$

It is apparent that the factor in front is the classical equation for hydrostatic equilibrium. The first two factors in square brackets represent special relativistic corrections of order v^2/c^2

that arise from the mass–energy relation so that the denominators, ϵ and mc^2, vary relativistically in connection with Eistein's famous equation $E = Mc^2$. The last term in brackets is a general relativistic correction based on the physical significance of the Schwarzschild form of the metric and the meaning of $m(r)$ as the total integrated mass out to a radial distance r. These corrections each act to strengthen the gravitational interaction. Indeed, we note that Φ in Equation 17 is a kind of gravitational potential.

III.3. Polytropic Equation of State

To progress toward a solution of the Equation 17 we assume a polytropic equation of state (EOS) for a relation between the isotropic pressure and the rest mass density

$$P = K\rho^{\Gamma}\,, \tag{19}$$

where Γ is the adiabatic index and K is a normalization constant. For adiabatic processes we may neglect heat transfer (*i.e.* $dQ = 0$) so the first law of thermodynamics is simply

$$dU = -PdV\,, \tag{20}$$

where $U = \epsilon V$ is the total energy of the fluid in a volume V, including both the rest energy and internal energy. However, we may write the rest mass density as $\rho = mN/V$, where N is the number of particles of mass m in the same volume V. In other words, the first law of thermodynamics can be written

$$d\left(\frac{\epsilon}{\rho}\right) = -Pd\left(\frac{1}{\rho}\right) = \frac{P}{\rho^2}d\rho = K\rho^{\Gamma-2}d\rho\,. \tag{21}$$

The last equality was obtained by applying the polytropic EOS $P = K\rho^{\Gamma}$ to the star. The integrated equation is written suggestively as

$$\frac{\epsilon}{\rho} = (1+a) + \frac{K}{\Gamma-1}\rho^{\Gamma-1}\,. \tag{22}$$

The constant of integration is placed in the equation to ensure continuity of the piecewise polytropic EOS which we will adopt later on. In fact, our primary concern is that in the low density limit all energy originates from the rest mass. Specifically, we require $\lim_{\rho\to 0}\epsilon/\rho = 1$ and set $a = 0$ for the stellar boundary.

The form of the equation is nicer when we remember the form of the internal energy $\varepsilon = \epsilon - \rho$ and the specific enthalpy $h = (\epsilon + P)/\rho$. The resulting equations are

$$\varepsilon = a\rho + \frac{P}{\Gamma-1} \qquad \text{and} \qquad h = 1 + a + \frac{\Gamma}{\Gamma-1}\frac{P}{\rho}\,. \tag{23}$$

III.4. Piecewise Polytrope

A nuclear equation of state is often given as a table or a piecewise polytrope for dividing densities $\rho_0 < \rho_1 < \rho_2 < \cdots$ based on the various envelopes and crusts [4]. Thus, we continue by expressing the above equations in a piecewise manner.

$$P = \begin{cases} K_0\rho^{\Gamma_0} & \rho < \rho_1 \\ K_1\rho^{\Gamma_1} & \rho_1 < \rho < \rho_2 \\ \vdots & \\ K_n\rho^{\Gamma_n} & \rho_n < \rho \end{cases} \tag{24}$$

and the important fluid variables are smoothly given in each section by

$$\begin{aligned} \epsilon &= (1+a_i)\rho + \frac{K_i}{\Gamma_i - 1}\rho^{\Gamma_i} \\ \varepsilon &= a_i\rho + \frac{K_i}{\Gamma_i - 1}\rho^{\Gamma_i} \\ h &= 1 + a_i + \frac{\Gamma_i}{\Gamma_i - 1} K_i \rho^{\Gamma_i - 1} \,. \end{aligned} \tag{25}$$

Recall that the integration constants a_i are chosen to ensure the energy is smooth at the transitions in the piecewise function so that

$$\begin{aligned} a_0 &= 0 \\ a_i &= a_{i-1} + \frac{K_{i-1}}{\Gamma_{i-1} - 1}\rho_i^{\Gamma_{i-1}-1} - \frac{K_i}{\Gamma_i - 1}\rho_i^{\Gamma_i - 1} \,. \end{aligned} \tag{26}$$

Now when integrating the TOV equations, it is useful to define a generalization of the Newtonian specific enthalpy

$$\eta = h - 1 \,, \tag{27}$$

which subtracts off the contribution from the rest mass of the fluid. Furthermore, the polytropic index $n_i = 1/(\Gamma_i - 1)$ is defined exactly as in the class notes. Thus we may write the fluid variables in terms of η in the following manner

$$\begin{aligned} \rho(\eta) &= \left(\frac{\eta - a_i}{K_i(n_i+1)}\right)^{n_i} \\ P(\eta) &= K_i \left(\frac{\eta - a_i}{K_i(n_i+1)}\right)^{n_i+1} \\ \epsilon(\eta) &= \rho(\eta)\left(1 + \frac{a_i + n_i\eta}{n_i+1}\right) \,. \end{aligned} \tag{28}$$

The TOV equations of Eq. 17 diverge at $r = 0$ and can be difficult to integrate numerically for $r \to 0$. A common technique in the literature to continue analytically and avoid singular equations is to change variables by defining a pseudo-enthalpy

$$y(P) = \int^P \frac{dP'}{\epsilon(P') + P'} \,. \tag{29}$$

With this definition we may relate the pseudo-enthalpy to the specific enthalpy by $y = \log(h)$. From a quick application of the first law of thermodynamics in terms of dh we have

$$\frac{dh}{h} = dy = \frac{dP}{\epsilon + P} \,. \tag{30}$$

Indeed the TOV equation for $d\Phi/dr$ can be integrated immediately to give $e^{y+\Phi} = he^{\Phi} = \sqrt{1 - 2M/R}$, where M and R are the mass and radius of the star. This follows from the relation $d\Phi = dy$ and choosing the constant of integration to match the Schwarzschild spacetime beyond the surface of the star. The meaning of this new insight is that the 'gravitational potential' Φ for the star is fully determined if we can integrate the other two

TOV equations. This becomes apparent when we complete the change of variables to the Newtonian specific enthalpy η. Equation 17 becomes

$$\begin{aligned}\frac{dr}{d\eta} &= -\frac{r(r-2m)}{m+4\pi r^3 P(\eta)}\frac{1}{\eta+1} \\ \frac{dm}{d\eta} &= 4\pi r^2 \epsilon(\eta)\frac{dr}{d\eta}\,,\end{aligned} \tag{31}$$

which are well-behaved both at the center of the star and at the surface.

The next step is to place factors of G and c back into the Equation 31 and introduce dimensionless variables to prepare for easier integration on a computer. We choose the mass scale to be $1M_{\odot}$. In order to integrate the equations we must choose a value for the enthalpy at the center of the star η_c and set up initial conditions $r(\eta_c) = 0$ and $m(\eta_c) = 0$. The equations are then integrated from the center of the star ($\eta = \eta_c$) to the surface ($\eta = 0$). Once the Newtonian specific enthaply crosses zero we have reached the surface of the star and must switch to the Schwarzschild solution if we desire to proceed further. Thus, the radius and mass of the star are respectfully $R = r(0)$ and $M = m(0)$.

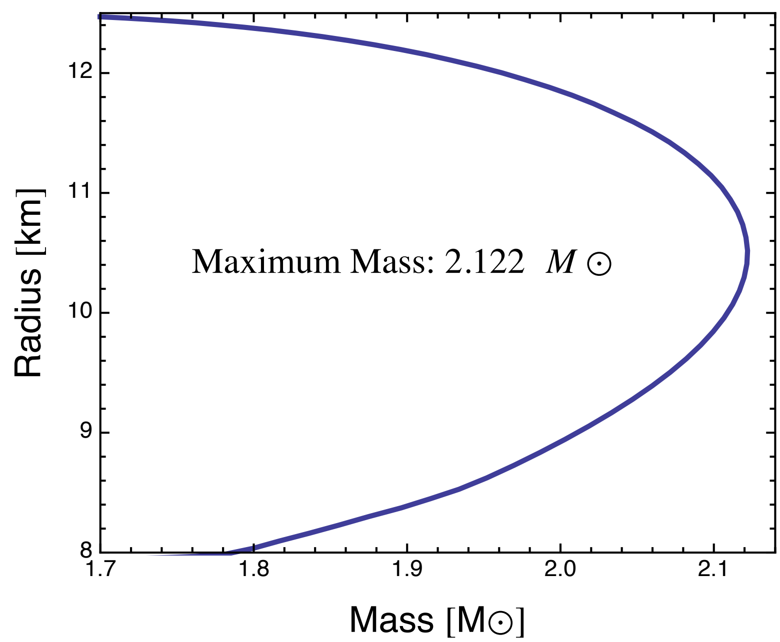

FIG. 1. The mass and radius relationship for a neutron stars with parameters found in Table I. The maximum mass of 2.122 $M_{\odot}$ is found by varying the central Newtonian specific enthalpy η_c.

K_i	Γ_i	a_i	ρ_i
$3.99873692 \times 10^{-8}$	1.35692395	0	
$2.23872092 \times 10^{-8}$	3	0.010350691	1.4172900×10^{14}

TABLE I. Summary of the parameters K_i, Γ_i, a_i, and ρ_i for a simple piecewise polytropic EOS.

IV. RESULTS

The piecewise polytropic equation of state takes into account the different envelopes of nuclear matter, and does a fairly good job with relatively few layers. One layer is simply not realistic enough because there are points where a harder or softer EOS are necessary. Realistic models may use a dozen different piecewise steps. At this point we may actually solve the TOV equation for piecewise values of K_i, Γ_i, and a_i. The transition of where the piecewise transition should be made is based on a density parameter ρ_i so that once again we have $P/c^2 = K_i\rho_i^{\Gamma_i}$. We plot points representing the mass and radius of a neutron star with the parameters given in Table I. The maximum mass is found to be about $2.122 M_\odot$, which is consistent with the currently accepted range of 1.44 to 3 $M_\odot$ for the maximum mass of a neutron star [4]. We note that there is a hard constraint from general relativity as to the maximum stationary mass within a given radius because if $R < 2GM/c^2$ then a black hole will form. There is also a causality constraint based roughly on the sound speed being less than the speed of light, which requires that $R > \ 3GM/c^2$. Finally there is a potential rotation constraint so that the spin frequency is less than the mass-shredding limit. However, the overall picture is sufficiently complicated to make observations the most authoritative constraint.

Appendix A: Code for the TOV Integration

Ask the author via email.

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