1 Portfolio Problem 1 - Divisibility by 4

₂ 1.1 Examples and Definitions

- 3 Problem to prove: Construct both a know-show table and a first attempt at a formal proof of
- 4 the following biconditional statement.
- **Proposition.** For each integer n, n is even if and only if 4 divides n^2 .

6 Exploratory work/Examples:

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- Find four even integers a. At least one integer must be negative. Square each integer a and determine if 4 divides a^2 .
 - If we set a = -2, then $(-2)^2 = 4$, and $4 \mid 4$.
- If we set a = 0, then $0^2 = 0$, and $4 \mid 0$.
 - If we set a = 2, then $2^2 = 4$, and $4 \mid 4$.
- If we set a = 4, then $4^2 = 16$, and $4 \mid 16$.
 - Find four perfect squares, b, such that $4 \mid b$. Then find the integer c such that $b = c^2$. Determine if c is even or odd.
 - If we set b=0, then $4\mid 0$. Using $b=c^2$, we get $0=0^2$, so c=0, which is even.
 - If we set b=4, then $4\mid 4$. Using $b=c^2$, we get $4=2^2$, so c=2, which is even.
 - If we set b=16, then $4\mid 16$. Using $b=c^2$, we get $16=4^2$, so c=4, which is even.
- If we set b=144, then $4\mid 144$. Using $b=c^2$, we get $144=12^2$, so c=12, which is even.

$_{ ilde{\mathsf{u}}}$ 1.2 Know-Show Table: If n is even, then $4\mid n^2$ (Forward)

Step	Know	Reason
P1	n is even	Hypothesis
P2	$\exists k \in \mathbb{Z} \text{ s.t. } n=2k$	Definition of even
P3	$n^2 = (2k)^2 = 4k^2$	Substitution
P4	$4k^2 = 2(2k^2)$	Factoring out 2
P5	$2k^2 \in \mathbb{Z}$	${\mathbb Z}$ closed under multiplication
P6	$n^2=2q$ where $q\in\mathbb{Z}$	Set $q = k^2$
Q1	4 divides n^2	Defn divides
Step	Show	Reason

Step	Know	Reason
P1	$4 \mid n^2$	Hypothesis
P2	$n^2=4m$ for some $m\in\mathbb{Z}$	Definition of divisibility
P3	$n^2 \equiv 0 \mod 4$	Definition of congruence
P4	Assume n is odd	For contradiction
P5	$n=2k+1$ for some $k\in\mathbb{Z}$	Definition of odd
P6	$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$	Expand and simplify n^2
P7	$n^2 = 4(k^2 + k) + 1$	Factor out 4
P8	$k^2 + k \in \mathbb{Z}$	${\mathbb Z}$ closed under multipli-
		cation
P9	$n^2 \equiv 1 \mod 4$	$4(k^2 + k) \equiv 0 \mod 4$
P10	Contradicts $n^2 \equiv 0 \mod 4$	From P2, since $n^2 =$
		4m
Q1	n is even	Definition of even
Step	Show	Reason

1.4 First Draft

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Theorem. For each integer n, n is even if and only if 4 divides n^2 .

26 Proof. In order to prove this biconditional statement, we must prove the if/then statement in

27 both the forward and reverse directions.

²⁸ For the forward direction, we assume n is even, and we will prove that $4\mid n^2$. By the definition

of even, there exists an integer k such that n=2k. Substituting for n into n^2 , we get:

$$n^2 = (2k)^2 = 4k^2$$

Then, we factor:

$$4k^2 = 2(2k^2)$$

We find that $2k^2$ is an integer due to closure under multiplication. Now, we can write $n^2=2q$,

where $q=2k^2$ is an integer, which shows that n^2 is divisible by 4. Therefore, we have proven

that if n is even, then $4\mid n^2$.

For the reverse direction, we assume $4 \mid n^2$, and we will prove that n is even. By the definition

of divides, there exists an integer m such that $n^2=4m$. Because of the relationship with

36 congruence, this can be restated as:

$$n^2 \equiv 0 \pmod{4}$$

Due to integers not being closed under roots, we assume for contradiction that n is odd. By the definition of odd, n=2k+1 for some $k\in\mathbb{Z}$. Substituting for n into n^2 , we get:

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$$

Then, we factor:

$$4k^2 + 4k + 1 = 4(k^2 + k) + 1$$

So, $n^2=4(k^2+k)+1$, which implies that, $n^2\equiv 1\pmod 4$. However, we assumed that $4\mid n^2$, which implies $n^2\equiv 0\pmod 4$. Therefore, our assumption that n is odd must be false, and it follows that n is even.

References

[1] Sundstrom, T., *Mathematical Reasoning: Writing and Proof*, Version 3, Creative Commons, 2020.