Math 3100 Zachary Hampton
Problem Set 11 Due Date: 11-29-2024

Problem 1

Problem 1. *Let A, B, C, and D be subsets of a universal set U.*

(a) *Complete the following sentence (write the full sentence in your work):*

We say that $(A - B) \subseteq (C \cap D)$ provided that for each $x \in U$, if $x \in (A - B)$, then __________________.

(b) *Write a useful negation of the statement from part (a).*

(c) *Write the contrapositive of the statement from part (a).*

Solution:

(a) We say that $(A - B) \subseteq (C \cap D)$ provided that for each $x \in U$, if $x \in (A - B)$, then $x \in C$ **and** $x \in D$.

(b) The negation of the statement is:

There exists an $x \in U$ such that $x \in (A - B)$ and **(**$x \notin C$ **or** $x \notin D$**)**.

(c) The contrapositive of the statement is:

For each $x \in U$, if $x \notin C$ **or** $x \notin D$, then $x \notin (A - B)$.

Since $x \notin (A - B)$ means that $x \notin A$ or $x \in B$, we can also express it as:

For each $x \in U$, if $x \notin C$ or $x \notin D$, then $x \notin A$ or $x \in B$.

Problem 2

Problem 2. *Let* $S = \{ x \in \mathbb{R} \mid x^2 < 9 \}$ *and* $T = \{ x \in \mathbb{R} \mid x < 3 \}$.

(a) Is $S \subseteq T$*? Justify your conclusion with a formal proof or a formal counterexample argument.*

(b) Is $T \subseteq S$*? Justify your conclusion with a formal proof or a formal counterexample argument.*

Solution:

(a) **Yes,** $S \subseteq T$**.**

Proof:

Let $x \in S$. Then by definition, $x^2 < 9$.

This implies that $-3 < x < 3$.

Therefore, $x < 3$.

Hence, $x \in T$.

Since every element of S is also in T, we conclude that $S \subseteq T$.

(b) **No,** $T \nsubseteq S$**.**

Counterexample:

Consider $x = -4$.

We have $x = -4 < 3$, so $x \in T$.

However, $x^2 = (-4)^2 = 16 \geq 9$, so $x \notin S$.

Therefore, $T \nsubseteq S$.

Problem 3

Problem 3. *Let* $A = \{ x \in \mathbb{Z} \mid x \equiv 1 \pmod 5 \}$ *and* $B = \{ y \in \mathbb{Z} \mid y \equiv 7 \pmod{10} \}$*.*

(a) *List at least five different elements of the set* A *and at least five different elements of the set* B*. Each list should contain at least one negative integer. Use proper set notation to write your lists.*

(b) *Write a formal proof to show that* A *and* B *are disjoint (i.e.,* $A \cap B = \emptyset$*).*

Solution:

(a) Elements of A:

Since $x \equiv 1 \pmod 5$, $x = 5k + 1$ for some $k \in \mathbb{Z}$.

Examples:

$$
\begin{aligned}
k = -3 &: \quad x = 5(-3) + 1 = -14 \\
k = -2 &: \quad x = 5(-2) + 1 = -9 \\
k = -1 &: \quad x = 5(-1) + 1 = -4 \\
k = 0 &: \quad x = 5(0) + 1 = 1 \\
k = 1 &: \quad x = 5(1) + 1 = 6
\end{aligned}
$$

Therefore, a subset of A is:

$$A = \{-14,\ -9,\ -4,\ 1,\ 6,\ \ldots\}$$

Elements of B:

Since $y \equiv 7 \pmod{10}$, $y = 10m + 7$ for some $m \in \mathbb{Z}$.

Examples:

$$
\begin{aligned}
m = -2 &: \quad y = 10(-2) + 7 = -13 \\
m = -1 &: \quad y = 10(-1) + 7 = -3 \\
m = 0 &: \quad y = 10(0) + 7 = 7 \\
m = 1 &: \quad y = 10(1) + 7 = 17 \\
m = 2 &: \quad y = 10(2) + 7 = 27
\end{aligned}
$$

Therefore, a subset of B is:

$$B = \{-13,\ -3,\ 7,\ 17,\ 27,\ \ldots\}$$

(b) *Proof that* $A \cap B = \emptyset$*:*

Suppose, for contradiction, that there exists an integer x such that $x \in A \cap B$.

Then:

$$\begin{cases} x \equiv 1 \pmod{5} \\ x \equiv 7 \pmod{10} \end{cases}$$

Since $x \equiv 1 \pmod 5$, it means $x \mod 5 = 1$.

Since $x \equiv 7 \pmod{10}$, we have $x \mod 10 = 7$.

Observe that:

$$x \mod 5 = (x \mod 10) \mod 5.$$

Compute $(x \mod 10) \mod 5$:

$$(7 \mod 5) = 2.$$

Therefore:

$$x \mod 5 = 2.$$

But this contradicts the earlier conclusion that $x \mod 5 = 1$.

Therefore, no such x exists, and $A \cap B = \emptyset$.

Problem 4

Problem 4. *Read the proof below. For each algebraic step, provide justification by citing the appropriate parts of Theorem 5.18 and Theorem 5.20.*
Proposition: *If A, B, and C are subsets of some universal set U, then:*

$$A - (B \cup C) = (A - B) \cap (A - C).$$

Proof: *Let A, B, and C be subsets of some universal set U. Beginning with the left side, we have:*

$$\begin{aligned} A - (B \cup C) &= A \cap (B \cup C)^c \\ &= A \cap (B^c \cap C^c) \\ &= (A \cap B^c) \cap C^c \\ &= (A \cap B^c) \cap (A \cap C^c) \\ &= (A - B) \cap (A - C) \end{aligned}$$

Therefore,

$$A - (B \cup C) = (A - B) \cap (A - C)$$

for any sets A, B, and C.

Solution:

Step 1: $A - (B \cup C) = A \cap (B \cup C)^c$ [Definition of Set Difference]
Step 2: $= A \cap (B^c \cap C^c)$ [De Morgan's Law]
Step 3: $= (A \cap B^c) \cap C^c$ [Associative Law of Intersection]
Step 4: $= (A \cap B^c) \cap (A \cap C^c)$ [Distributive Law]
Step 5: $= (A - B) \cap (A - C)$ [Definition of Set Difference]