Problem Set 7

Problem 1. *Let $n \in \mathbb{N}$ and let $n \geq 2$. Consider the following proposition:*
Proposition. *For all integers a and b, if $a + b \not\equiv 0 \pmod{n}$, then $a \not\equiv 0 \pmod{n}$ or $b \not\equiv 0 \pmod{n}$.*

(a) *Write the contrapositive of the proposition.*

 (a) *For all integers a and b, if $a \equiv 0 \pmod{n}$ and $b \equiv 0 \pmod{n}$, then $a + b \equiv 0 \pmod{n}$.*

(b) *Construct a know-show table to outline the proof of the proposition (using the contrapositive). You do not need to write a formal proof.*

Step	***Know***	***Reason***
P1	$a \equiv 0 \pmod{n}$ *and* $b \equiv 0 \pmod{n}$	*Hypothesis*
P2	*For some integers k and l, $a = kn$ and $b = ln$*	*Definition of Congruence*
P3	$a + b = kn + ln = n(k + l)$	*Substitution*
P4	$k + l \in \mathbb{Z}$	*$\mathbb{Z}$ is closed under multiplication*
P5	*$\exists m \in \mathbb{Z}$ such that $m = k + l$ and $a + b = m \cdot n$*	*Substitute $m = k + l$*
Q1	$a + b \equiv 0 \pmod{n}$	*Definition of Congruence*

Problem 2. *Is the following conjecture true or false? If the conjecture is true, write a formal proof. If the conjecture is false, write a formal counterexample argument. (Be sure to aim for "practitioner" on the writing rubrics).*
Conjecture. *For all integers a and b, $a + b \not\equiv 0 \pmod{n}$ if and only if $a \not\equiv 0 \pmod{n}$ or $b \not\equiv 0 \pmod{n}$.*

(a) *The statement, "For all integers a and b, $a + b \not\equiv 0 \pmod{n}$ if and only if $a \not\equiv 0 \pmod{n}$ or $b \not\equiv 0 \pmod{n}$," is false. One counterexample is when $n = 5$, $a = 1$, and $b = 4$. In this case, we see the hypothesis is false because $a + b = 1 + 4 = 5 \equiv 0 \pmod{5}$. However, the conclusion is true because $a \not\equiv 0 \pmod{5}$ and $b \not\equiv 0 \pmod{5}$. Since the hypothesis is false and the conclusion is true, the statement is false.*

Problem 3. *Practice using previous results.*

(a) *Write the contrapositive of the lemma below:*

Lemma. *If n is an odd integer, then 4 does not divide n.*

(a) *If 4 divides n, then n is an even integer.*

(b) *Write a formal proof of the lemma.*

(a) *To prove the lemma by contrapositive, we assume that 4 divides n. By the definition of divisibility, there exists an integer k such that $n = 4k$. Using algebra, we get $n = 2(2k)$. Since $2k$ is an integer, there exists an integer m such that $m = 2k$. Then we can express n as:*

$$n = 2 \times (2k) = 2m.$$

This shows that n is divisible by 2, which means n is an even integer.

(c) *Consider the following proposition:*

Proposition. *For all integers a and b, if a is even and b is odd, then 4 does not divide $a^2 + b^2$.*

Use previously proved results to construct a know-show table that outlines a proof of the proposition.

Step	**Know**	**Reason**
P1	a *is even and* b *is odd*	*Hypothesis*
P2	*There exists* $k \in \mathbb{Z}$ *such that* $a = 2k$	*Definition of even*
P3	*There exists* $m \in \mathbb{Z}$ *such that* $b = 2m + 1$	*Definition of odd*
P4	*Compute* $a^2 = (2k)^2 = 4k^2$	*Algebra*
P5	*Compute* $b^2 = (2m+1)^2 = 4m^2 + 4m + 1$	*Algebra*
P6	*Sum* $a^2 + b^2 = 4k^2 + 4m^2 + 4m + 1 = 4(k^2 + m^2 + m) + 1$	*Substitution*
P7	*Let* $x = k^2 + m^2 + m$*, which is an integer since* $k, m \in \mathbb{Z}$	$\mathbb{Z}$ *is closed under multiplication*
P8	$a^2 + b^2 = 4x + 1$	*Substitution*
P9	$a^2 + b^2 \equiv 1 \pmod 4$	*Definition of congruence*
P10	*Since* $a^2 + b^2 \equiv 1 \pmod 4$*,* $a^2 + b^2$ *is an odd integer*	
P11	*Previously proven lemma: If* n *is odd, then 4 does not divide* n	
P12	*Since* $a^2 + b^2$ *is odd, 4 does not divide* $a^2 + b^2$	*By the lemma*
Step	**Show**	**Reason**

Problem 4. *Use the proposition and proof on the next page to do the following:*

(a) *Clearly identify the assumptions in the proof.*

(a) *The assumptions of the proof are:*

- x *is a rational number (*$x \in \mathbb{Q}$*)*
- $x \neq 0$
- y *is an irrational number (*$y \notin \mathbb{Q}$*)*
- *The product* $x \cdot y$ *is a rational number (*$x \cdot y \in \mathbb{Q}$*)*

(b) *Clearly identify the contradictory statement.*

(a) *The contradictory statement is that* y *is both rational and irrational. Specifically, the proof concludes that* y *must be rational based on the assumptions, which contradicts the initial assumption that* y *is irrational.*

(c) *In 2 - 3 sentences, summarize the main idea of the proof in a way that makes sense to you.*

(a) *The main idea of the proof is to assume that multiplying a nonzero rational number x by an irrational number y yields a rational result. By manipulating this assumption, it deduces that y must be rational, contradicting the original assumption that y is irrational. This contradiction implies that the product $x \cdot y$ cannot be rational, thus proving it must be irrational.*

Proposition and formal proof for Problem 4

Proposition: For all real numbers x and y, if x is rational and $x \neq 0$ and y is irrational, then $x \cdot y$ is irrational.

Proof. We will use a proof by contradiction. So we assume that there exist real numbers x and y such that x is rational, $x \neq 0$, y is irrational, and $x \cdot y$ is rational. Since $x \neq 0$, we can divide by x. Since the rational numbers are closed under division by nonzero rational numbers, we know that $\frac{1}{x} \in \mathbb{Q}$. We now know that $x \cdot y$ and $\frac{1}{x}$ are rational numbers, and since the rational numbers are closed under multiplication, we conclude that

$$\frac{1}{x} \cdot (x \cdot y) \in \mathbb{Q}.$$

However, $\frac{1}{x} \cdot (x \cdot y) = y$ and hence, y must be a rational number. Since a real number cannot be both rational and irrational, this is a contradiction to the assumption that y is irrational. We have therefore proved that for all real numbers x and y, if x is rational and $x \neq 0$ and y is irrational, then $x \cdot y$ is irrational.