

Problem Set 8

Problem 1. Consider the following proposition: If n is an integer, then $3 \mid (n^3 - n)$.

(a) Write a paragraph proof for the third case of this proposition without using properties of congruence.

Solution: To prove $3 \mid (n^3 - n)$, we examine the form $n = 3k + 2$ for some integer k , assuming the cases for $n = 3k$ and $n = 3k + 1$ were covered previously.

$$\begin{aligned} n^3 - n &= (3k + 2)^3 - (3k + 2) \\ &= 27k^3 + 54k^2 + 36k + 8 - (3k + 2) \\ &= 27k^3 + 54k^2 + 36k + 8 - 3k - 2 \\ &= 27k^3 + 54k^2 + 33k + 6. \end{aligned}$$

Each term in this expression is divisible by 3, so we conclude $3 \mid (n^3 - n)$.

(b) Briefly explain why this proposition is equivalent to the statement $n^3 \equiv n \pmod{3}$.

Solution: The statement $3 \mid (n^3 - n)$ implies that $n^3 - n$ is divisible by 3. This is equivalent to saying $n^3 \equiv n \pmod{3}$ because both expressions ensure that n^3 and n yield the same remainder modulo 3.

(c) Using congruences, what are the three cases to consider for n ?

Solution: The three cases for n modulo 3 are:

(a) $n \equiv 0 \pmod{3}$

(b) $n \equiv 1 \pmod{3}$

(c) $n \equiv 2 \pmod{3}$

(d) Construct a know-show table for the case where $n \equiv 2 \pmod{3}$ using congruences.

Step	Know	Reason
Step 1	<i>Assume</i> $n \equiv 2 \pmod{3}$	<i>Given case</i>
Step 2	$n^3 \equiv 8 \equiv 2 \pmod{3}$	<i>Calculation of n^3 in terms of mod 3</i>
Step 3	<i>Conclude</i> $n^3 \equiv n \pmod{3}$	<i>Matches $n \equiv 2 \pmod{3}$</i>

Problem 2. Consider the proposition: Let $n \in \mathbb{Z}$. If $n \not\equiv 0 \pmod{5}$, then $n^2 \equiv 1 \pmod{5}$ or $n^2 \equiv 4 \pmod{5}$.

(a) Explore the theorem by completing the following chart.

n	n^2	Does $n^2 \equiv 1$ or $4 \pmod{5}$?
± 1	1	Yes, $1 \equiv 1 \pmod{5}$
± 2	4	Yes, $4 \equiv 4 \pmod{5}$
± 3	9	Yes, $9 \equiv 4 \pmod{5}$
± 4	16	Yes, $16 \equiv 1 \pmod{5}$

(b) Patterns observed:

- n^2 for values not divisible by 5 are either congruent to 1 or 4 modulo 5.

(c) The cases to consider based on the Division Algorithm are:

- (a) $n \equiv 1 \pmod{5}$
- (b) $n \equiv 2 \pmod{5}$
- (c) $n \equiv 3 \pmod{5}$
- (d) $n \equiv 4 \pmod{5}$

(d) Know-show table for $n \equiv 3 \pmod{5}$.

Step	Know	Reason
Step 1	<i>Assume</i> $n \equiv 3 \pmod{5}$	<i>Given</i>
Step 2	$n^2 \equiv 9 \equiv 4 \pmod{5}$	<i>Calculation</i>

Problem 3. Consider the following proposition:

Proposition: If x and y are real numbers, then

$$\max(x, y) = \frac{|x - y| + x + y}{2}.$$

Note: For $x, y \in \mathbb{R}$, the function $\max(x, y)$ is defined to output the larger of the two values if $x \neq y$, and the value x if $x = y$.

(a) Explore the proposition by filling in the chart below:

(x, y)	$\max(x, y)$	$\frac{ x-y +x+y}{2}$
$(-3, 10)$	10	$\frac{ -3-10 +(-3)+10}{2} = \frac{13+7}{2} = 10$
$(-3, -10)$	-3	$\frac{ -3+10 +(-3)+(-10)}{2} = \frac{7-13}{2} = -3$
$(0, 5)$	5	$\frac{ 0-5 +0+5}{2} = \frac{5+5}{2} = 5$
$(\frac{2}{3}, \frac{12}{18})$	$\frac{2}{3}$	$\frac{ \frac{2}{3}-\frac{2}{3} +\frac{2}{3}+\frac{2}{3}}{2} = \frac{0+2/3}{2} = \frac{2}{3}$
$(\sqrt{2}, \sqrt{3})$	$\sqrt{3}$	$\frac{ \sqrt{2}-\sqrt{3} +\sqrt{2}+\sqrt{3}}{2} = \sqrt{3}$

(b) Using the definition of absolute value, there are two cases to consider when proving this equation algebraically. These cases are:

- Case 1: When $x \geq y$, which implies $|x - y| = x - y$.
- Case 2: When $x < y$, which implies $|x - y| = y - x$.

(c) Show the thinking work for a proof by cases of the proposition above.

Case	Steps
Case 1	<p>Assume $x \geq y$. Then $x - y = x - y$. We have:</p> $\frac{ x - y + x + y}{2} = \frac{(x - y) + x + y}{2} = \frac{2x}{2} = x.$ <p>Since $x \geq y$, $\max(x, y) = x$, and the proposition holds.</p>
Case 2	<p>Assume $x < y$. Then $x - y = y - x$. We have:</p> $\frac{ x - y + x + y}{2} = \frac{(y - x) + x + y}{2} = \frac{2y}{2} = y.$ <p>Since $y > x$, $\max(x, y) = y$, and the proposition holds.</p>

(d) Write a formal proof of the proposition using a proof by cases.

Proof: Let $x, y \in \mathbb{R}$. We will prove the proposition by considering two cases based on the relationship between x and y .

Case 1: Assume $x \geq y$. Then $|x - y| = x - y$, so we calculate:

$$\begin{aligned}\frac{|x - y| + x + y}{2} &= \frac{(x - y) + x + y}{2} \\ &= \frac{2x}{2} \\ &= x.\end{aligned}$$

Since $x \geq y$, it follows that $\max(x, y) = x$. Thus, the expression $\frac{|x - y| + x + y}{2}$ is equal to $\max(x, y)$ for this case.

Case 2: Assume $x < y$. Then $|x - y| = y - x$, so we calculate:

$$\begin{aligned}\frac{|x - y| + x + y}{2} &= \frac{(y - x) + x + y}{2} \\ &= \frac{2y}{2} \\ &= y.\end{aligned}$$

Since $y > x$, it follows that $\max(x, y) = y$. Thus, the expression $\frac{|x - y| + x + y}{2}$ is equal to $\max(x, y)$ for this case.

In both cases, we have shown that $\max(x, y) = \frac{|x - y| + x + y}{2}$. Therefore, the proposition holds for all $x, y \in \mathbb{R}$.