# Problem Set 7

**Problem 1.** *Let $n \in \mathbb{N}$ and let $n \geq 2$. Consider the following proposition:*
***Proposition.*** *For all integers $a$ and $b$, if $a + b \not\equiv 0 \pmod{n}$, then $a \not\equiv 0 \pmod{n}$ or $b \not\equiv 0 \pmod{n}$.*

(a) *Write the contrapositive of the proposition.*

  (a) *For all integers $a$ and $b$, if $a \equiv 0 \pmod{n}$ and $b \equiv 0 \pmod{n}$, then $a + b \equiv 0 \pmod{n}$.*

(b) *Construct a know-show table to outline the proof of the proposition (using the contrapositive). You do not need to write a formal proof.*

| ***Step*** | ***Know*** | ***Reason*** |
|---|---|---|
| *P1* | $a \equiv 0 \pmod{n}$ *and* $b \equiv 0 \pmod{n}$ | *Hypothesis* |
| *P2* | *For some integers $k$ and $l$, $a = kn$ and $b = ln$* | *Definition of Congruence* |
| *P3* | $a + b = kn + ln = n(k + l)$ | *Substitution* |
| *P4* | $k + l \in \mathbb{Z}$ | *$\mathbb{Z}$ is closed under multiplication* |
| *P5* | *$\exists m \in \mathbb{Z}$ such that $m = k + l$ and $a + b = m \cdot n$* | *Substitute $m = k + l$* |
| *Q1* | $a + b \equiv 0 \pmod{n}$ | *Definition of Congruence* |

**Problem 2.** *Is the following conjecture true or false? If the conjecture is true, write a formal proof. If the conjecture is false, write a formal counterexample argument. (Be sure to aim for "practitioner" on the writing rubrics).*
***Conjecture.*** *For all integers $a$ and $b$, $a + b \not\equiv 0 \pmod{n}$ if and only if $a \not\equiv 0 \pmod{n}$ or $b \not\equiv 0 \pmod{n}$.*

(a) *The statement, "For all integers $a$ and $b$, $a + b \not\equiv 0 \pmod{n}$ if and only if $a \not\equiv 0 \pmod{n}$ or $b \not\equiv 0 \pmod{n}$," is false. One counterexample is when $n = 5$, $a = 1$, and $b = 4$. In this case, we see the hypothesis is false because $a + b = 1 + 4 = 5 \equiv 0 \pmod{5}$. However, the conclusion is true because $a \not\equiv 0 \pmod{5}$ and $b \not\equiv 0 \pmod{5}$. Since the hypothesis is false and the conclusion is true, the statement is false.*

**Problem 3.** *Practice using previous results.*

(a) *Write the contrapositive of the lemma below:*

***Lemma.*** *If $n$ is an odd integer, then 4 does not divide $n$.*

(a) *If 4 divides $n$, then $n$ is an even integer.*

(b) *Write a formal proof of the lemma.*

(a) *To prove the lemma by contrapositive, we assume that 4 divides $n$. By the definition of divisibility, there exists an integer $k$ such that $n = 4k$. Using algebra, we get $n = 2(2k)$. Since $2k$ is an integer, there exists an integer $m$ such that $m = 2k$. Then we can express $n$ as:*

$$n = 2 \times (2k) = 2m.$$

*This shows that $n$ is divisible by 2, which means $n$ is an even integer.*

(c) *Consider the following proposition:*

***Proposition.*** *For all integers $a$ and $b$, if $a$ is even and $b$ is odd, then 4 does not divide $a^2 + b^2$.*

*Use previously proved results to construct a know-show table that outlines a proof of the proposition.*

| **Step** | **Know** | **Reason** |
|---|---|---|
| *P1* | $a$ *is even and* $b$ *is odd* | *Hypothesis* |
| *P2* | *There exists* $k \in \mathbb{Z}$ *such that* $a = 2k$ | *Definition of even* |
| *P3* | *There exists* $m \in \mathbb{Z}$ *such that* $b = 2m + 1$ | *Definition of odd* |
| *P4* | *Compute* $a^2 = (2k)^2 = 4k^2$ | *Algebra* |
| *P5* | *Compute* $b^2 = (2m+1)^2 = 4m^2 + 4m + 1$ | *Algebra* |
| *P6* | *Sum* $a^2 + b^2 = 4k^2 + 4m^2 + 4m + 1 = 4(k^2 + m^2 + m) + 1$ | *Substitution* |
| *P7* | *Let* $x = k^2 + m^2 + m$*, which is an integer since* $k, m \in \mathbb{Z}$ | $\mathbb{Z}$ *is closed under multiplication* |
| *P8* | $a^2 + b^2 = 4x + 1$ | *Substitution* |
| *P9* | $a^2 + b^2 \equiv 1 \pmod 4$ | *Definition of congruence* |
| *P10* | *Since* $a^2 + b^2 \equiv 1 \pmod 4$*,* $a^2 + b^2$ *is an odd integer* | |
| *P11* | *Previously proved lemma: If* $n$ *is odd, then 4 does not divide* $n$ | |
| *P12* | *Since* $a^2 + b^2$ *is odd, 4 does not divide* $a^2 + b^2$ | *By the lemma* |
| **Step** | **Show** | **Reason** |

**Problem 4.** *Use the proposition and proof on the next page to do the following:*

(a) *Clearly identify the assumptions in the proof.*

   (a) *The assumptions of the proof are:*

   - $x$ *is a rational number (*$x \in \mathbb{Q}$*)*
   - $x \neq 0$
   - $y$ *is an irrational number (*$y \notin \mathbb{Q}$*)*
   - *The product* $x \cdot y$ *is a rational number (*$x \cdot y \in \mathbb{Q}$*)*

(b) *Clearly identify the contradictory statement.*

   (a) *The contradictory statement is that* $y$ *is both rational and irrational. Specifically, the proof concludes that* $y$ *must be rational based on the assumptions, which contradicts the initial assumption that* $y$ *is irrational.*

(c) *In 2 - 3 sentences, summarize the main idea of the proof in a way that makes sense to you.*

(a) *The main idea of the proof is to assume that multiplying a nonzero rational number $x$ by an irrational number $y$ yields a rational result. By manipulating this assumption, it deduces that $y$ must be rational, contradicting the original assumption that $y$ is irrational. This contradiction implies that the product $x \cdot y$ cannot be rational, thus proving it must be irrational.*

---

**Proposition and formal proof for Problem 4**
**Proposition:** For all real numbers $x$ and $y$, if $x$ is rational and $x \neq 0$ and $y$ is irrational, then $x \cdot y$ is irrational.
**Proof.** We will use a proof by contradiction. So we assume that there exist real numbers $x$ and $y$ such that $x$ is rational, $x \neq 0$, $y$ is irrational, and $x \cdot y$ is rational. Since $x \neq 0$, we can divide by $x$. Since the rational numbers are closed under division by nonzero rational numbers, we know that $\frac{1}{x} \in \mathbb{Q}$. We now know that $x \cdot y$ and $\frac{1}{x}$ are rational numbers, and since the rational numbers are closed under multiplication, we conclude that

$$\frac{1}{x} \cdot (x \cdot y) \in \mathbb{Q}.$$

However, $\frac{1}{x} \cdot (x \cdot y) = y$ and hence, $y$ must be a rational number. Since a real number cannot be both rational and irrational, this is a contradiction to the assumption that $y$ is irrational. We have therefore proved that for all real numbers $x$ and $y$, if $x$ is rational and $x \neq 0$ and $y$ is irrational, then $x \cdot y$ is irrational.