

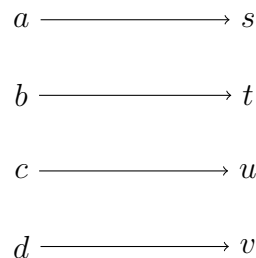
## Problem 1

**Problem 1.** Let  $A = \{a, b, c, d\}$ ,  $B = \{a, b, c\}$ , and  $C = \{s, t, u, v\}$ . Draw an arrow diagram of a function for each of the following descriptions. If no such function exists, briefly explain why.

- (a) A function  $f : A \rightarrow C$  whose range is the set  $C$ .
- (b) A function  $g : B \rightarrow C$  whose range is the set  $C$ .
- (c) A function  $g : B \rightarrow C$  that is injective.
- (d) A function  $j : A \rightarrow C$  that is not bijective.

**Solution:**

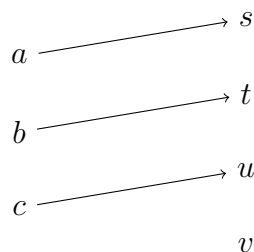
**Diagram for (a):  $f : A \rightarrow C$  (Range =  $C$ )**



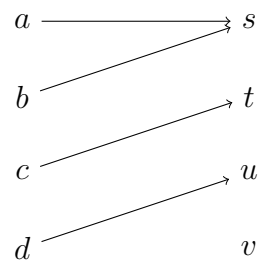
**Diagram for (b):  $g : B \rightarrow C$  (Range =  $C$ ) - No such function exists**

*Explanation: The domain  $B$  has only three elements, but  $C$  has four elements. Therefore, there is no way to map all elements of  $C$  (to cover its range) from  $B$ .*

**Diagram for (c):  $g : B \rightarrow C$  (Injective)**



**Diagram for (d):  $j : A \rightarrow C$  (Not Bijective)**



## Problem 2

**Problem 2.** Determine whether each function is an injection **and** determine whether each is a surjection. You do not need formal proofs, but you must clearly justify your conclusion and show neat (and mathematically accurate) supporting work.

(a)  $f : \mathbb{Z}_6 \rightarrow \mathbb{Z}_6$  defined by  $f(x) = x^2 + 4 \pmod{6}$ .

(b)  $g : \mathbb{Z}_5 \rightarrow \mathbb{Z}_5$  defined by  $g(x) = x^2 - 11 \pmod{5}$ .

(c)  $h : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$  defined by  $h(x, y) = x + 2y$ .

(d)  $j : \mathbb{R} - \{3\} \rightarrow \mathbb{R}$  defined by  $j(x) = \frac{4x}{x-3}$ .

**Solution:**

(a) In  $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ :

$$f(0) = 4, f(1) = 5, f(2) = 2, f(3) = 1, f(4) = 2, f(5) = 5.$$

Since  $f(2) = f(4)$  and  $f(1) = f(5)$ ,  $f$  is not injective. The range is  $\{1, 2, 4, 5\}$ , not all of  $\mathbb{Z}_6$ , so it is not surjective.

(b) In  $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ , note that  $-11 \equiv 4 \pmod{5}$ . Thus:

$$g(x) = x^2 + 4 \pmod{5}.$$

Check values:

$$g(0) = 4, g(1) = 0, g(2) = 3, g(3) = 3, g(4) = 0.$$

Not injective since  $g(1) = g(4)$  and  $g(2) = g(3)$ . The range is  $\{0, 3, 4\}$ , not all of  $\mathbb{Z}_5$ , so not surjective.

(c) Consider  $h(x, y) = x + 2y$  on integers. To check injectivity:

$$h(0, 0) = 0, \quad h(2, -1) = 0.$$

Different inputs map to the same output, so not injective. For surjectivity, given any integer  $z$ , choose  $x = z, y = 0$ , so  $h(x, y) = z$ . Thus  $h$  is surjective.

(d) For  $j(x) = \frac{4x}{x-3}$ , if  $j(x_1) = j(x_2)$ :

$$\frac{4x_1}{x_1 - 3} = \frac{4x_2}{x_2 - 3}.$$

Cross-multiplying shows  $x_1 = x_2$ . Thus  $j$  is injective. For surjectivity, set  $y = \frac{4x}{x-3}$ . Solve for  $x$ :

$$y(x - 3) = 4x \implies yx - 4x = 3y \implies x(y - 4) = 3y \implies x = \frac{3y}{y - 4}.$$

This works for all  $y \neq 4$ . When  $y = 4$ , no solution exists. Not all real numbers are attained, so not surjective.

## Problem 3

**Problem 3.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = x^3 + 5$ .

- (a) Determine if  $f$  is injective. If so, write a formal proof. If not, write a counterexample.
- (b) Determine if  $f$  is surjective. If so, write a formal proof. If not, write a counterexample.
- (c) Based upon (a) and (b), is  $f$  bijective?

**Solution:**

- (a) *Injectivity:* Suppose  $f(a) = f(b)$ . Then:

$$a^3 + 5 = b^3 + 5 \implies a^3 = b^3 \implies a = b.$$

Thus,  $f$  is injective.

- (b) *Surjectivity:* For any  $y \in \mathbb{R}$ , let:

$$x = \sqrt[3]{y - 5}.$$

Then:

$$f(x) = x^3 + 5 = (y - 5) + 5 = y.$$

Thus,  $f$  is surjective.

- (c) Since  $f$  is both injective and surjective,  $f$  is bijective.