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Problem Set 10 Due Date: 11-29-2024

Problem 1

Problem 1. *1. Use roster notation to explicitly list the elements in each of the following sets. When applicable, include any supporting work.*

(a) $\{\, a \in \mathbb{N} \mid 3a - 4 \leq 17 \,\}$

(b) $\{\, x \in \mathbb{R} \mid 2x^2 + 3x - 2 = 0 \,\}$

(c) $\{\, p \in \mathbb{Z} \mid -4 < \sqrt{p} < 4 \,\}$

(d) $\{\, n \in \mathbb{Z} \mid n \equiv 1 \pmod 4 \,\}$

(e) $\{\, y \in \mathbb{Q} \mid \left| y - \frac{1}{3} \right| = \frac{8}{3} \,\}$

Solution:

(a) $\{\, a \in \mathbb{N} \mid 3a - 4 \leq 17 \,\}$

Work: Solve the inequality for a:

$$3a - 4 \leq 17 \implies 3a \leq 21 \implies a \leq 7.$$

Since $a \in \mathbb{N}$, the set is:

$$\{1, 2, 3, 4, 5, 6, 7\}.$$

(b) $\{\, x \in \mathbb{R} \mid 2x^2 + 3x - 2 = 0 \,\}$

Work: Factor the quadratic equation:

$$2x^2 + 3x - 2 = 0.$$

Look for factors of $2x^2$ and -2 that sum to $3x$:

$$(2x - 1)(x + 2) = 2x^2 + 4x - x - 2 = 2x^2 + 3x - 2.$$

Set each factor to zero:

$$2x - 1 = 0 \implies x = \frac{1}{2}, \quad x + 2 = 0 \implies x = -2.$$

Therefore, the set is:

$$\left\{-2,\ \frac{1}{2}\right\}.$$

(c) $\{\, p \in \mathbb{Z} \mid -4 < \sqrt{p} < 4 \,\}$

Work: Since $\sqrt{p}$ is real, $p \geq 0$. Then:

$$0 \leq \sqrt{p} < 4 \implies 0 \leq p < 16.$$

Therefore, p can be any integer from 0 to 15. The set is:

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}.$$

(d) $\{\, n \in \mathbb{Z} \mid n \equiv 1 \pmod{4} \,\}$

Work: This set includes all integers that leave a remainder of 1 when divided by 4:

$$\{\ldots, -7, -3, 1, 5, 9, 13, 17, \ldots\}.$$

(e) $\{\, y \in \mathbb{Q} \mid \left|y - \frac{1}{3}\right| = \frac{8}{3} \,\}$

Work: Solve for y:

$$\left|y - \frac{1}{3}\right| = \frac{8}{3} \implies y - \frac{1}{3} = \pm\frac{8}{3}.$$

Thus:

$$y = \frac{1}{3} + \frac{8}{3} = \frac{9}{3} = 3, \quad y = \frac{1}{3} - \frac{8}{3} = -\frac{7}{3}.$$

Therefore, the set is:

$$\left\{-\frac{7}{3},\ 3\right\}.$$

Problem 2

Problem 2. *2. Let A and B be subsets of a universal set U, and let $x \in U$. Write useful negations of the definitions of set intersection, set union, and set difference by completing the sentences below:*

(a) $x \notin A \cap B$ provided that __________________.

(b) $x \notin A \cup B$ provided that __________________.

(c) $x \notin A - B$ provided that __________________.

Solution:

(a) $x \notin A \cap B$ provided that $x \notin A$ **or** $x \notin B$.

(b) $x \notin A \cup B$ provided that $x \notin A$ **and** $x \notin B$.

(c) $x \notin A - B$ provided that $x \notin A$ **or** $x \in B$.

Problem 3

Problem 3. *3. Let $U = \mathbb{N}$ and let:*

$$\begin{aligned} A &= \{\, x \in \mathbb{N} \mid x \geq 7 \,\}, \\ B &= \{\, x \in \mathbb{N} \mid x \text{ is odd} \,\}, \\ C &= \{\, x \in \mathbb{N} \mid x \equiv 0 \pmod{3} \,\}, \\ D &= \{\, x \in \mathbb{N} \mid x \text{ is even} \,\}. \end{aligned}$$

Use the roster method to list all of the elements in each of the following sets. When applicable, show your work by finding the sets in parentheses first:

(a) $(A \cup B)^{\complement}$

(b) $A^{\complement} \cap B^{\complement}$

(c) $(A \cup B) \cap C$

(d) $B \cap D$

(e) $(B \cap D)^{\complement}$

(f) $B - D$

Solution:

(a) $(A \cup B)^{\complement}$

Work: First, find $A \cup B$:

$$A = \{7, 8, 9, 10, 11, \ldots\}, \quad B = \{1, 3, 5, 7, 9, \ldots\}.$$

So:

$$A \cup B = \{1, 3, 5, 7, 8, 9, 10, 11, \ldots\}.$$

Then, the complement with respect to $U = \mathbb{N}$:

$$(A \cup B)^{\complement} = \{2, 4, 6\}.$$

(b) $A^{\complement} \cap B^{\complement}$

Work: Find the complements:

$$A^{\complement} = \{1, 2, 3, 4, 5, 6\}, \quad B^{\complement} = \{2, 4, 6, 8, 10, \ldots\}.$$

Then:

$$A^{\complement} \cap B^{\complement} = \{2, 4, 6\}.$$

(c) $(A \cup B) \cap C$

Work: We have:

$$C = \{3, 6, 9, 12, 15, \ldots\}.$$

Then:

$$(A \cup B) \cap C = \{3, 9, 12, 15, 18, \ldots\}.$$

Listing the elements:

$$\{3, 9, 12, 15, 18, 21, 24, \ldots\}.$$

(d) $B \cap D$

Work: Since B contains odd numbers and D contains even numbers:

$$B \cap D = \emptyset.$$

(e) $(B \cap D)^{\complement}$

Work: Since $B \cap D = \emptyset$, its complement is:

$$(B \cap D)^{\complement} = U = \mathbb{N}.$$

(f) $B - D$

Work: Since B contains odd numbers and D contains even numbers:

$$B - D = B.$$

So:

$$B - D = \{1, 3, 5, 7, 9, 11, \ldots\}.$$

Problem 4

Problem 4. *4. Let:*

$$A = \{1, 2\},$$
$$B = \{a, b, c, d\},$$
$$C = \{1, a, b\}.$$

Use the roster method to list all of the elements in each of the following sets. When applicable, show your work by finding the sets in parentheses first:

(a) $A \times B$

(b) $B \times A$

(c) $A \times (B \cap C)$

(d) $(A \times B) \cap (A \times C)$

Solution:

(a) $A \times B$

Work: Compute the Cartesian product:

$$A \times B = \{(1, a),\ (1, b),\ (1, c),\ (1, d),\ (2, a),\ (2, b),\ (2, c),\ (2, d)\}.$$

(b) $B \times A$

Work: Compute the Cartesian product:

$$B \times A = \{(a, 1),\ (a, 2),\ (b, 1),\ (b, 2),\ (c, 1),\ (c, 2),\ (d, 1),\ (d, 2)\}.$$

(c) $A \times (B \cap C)$

Work: First, find $B \cap C$:

$$B \cap C = \{a, b\}.$$

Then:

$$A \times (B \cap C) = \{(1, a),\ (1, b),\ (2, a),\ (2, b)\}.$$

(d) $(A \times B) \cap (A \times C)$

Work: Compute $A \times C$:

$$A \times C = \{(1,1),\ (1,a),\ (1,b),\ (2,1),\ (2,a),\ (2,b)\}.$$

Then:

$$(A \times B) \cap (A \times C) = \{(1,a),\ (1,b),\ (2,a),\ (2,b)\}.$$