Math 3100 **Zachary Hampton**
Problem Set 12 **Due Date: 12-06-2024**

Problem 1

Problem 1. *Let $A = \{a, b, c, d\}$, $B = \{a, b, c\}$, and $C = \{s, t, u, v\}$. Draw an arrow diagram of a function for each of the following descriptions. If no such function exists, briefly explain why.*

(a) A function $f : A \to C$ whose range is the set C.

(b) A function $g : B \to C$ whose range is the set C.

(c) A function $g : B \to C$ that is injective.

(d) A function $j : A \to C$ that is not bijective.

Solution:

Diagram for (a): $f : A \to C$ **(Range = C)**

$$\begin{array}{ccc} a & \longrightarrow & s \\ b & \longrightarrow & t \\ c & \longrightarrow & u \\ d & \longrightarrow & v \end{array}$$

Diagram for (b): $g : B \to C$ **(Range = C) - No such function exists**

Explanation: The domain B has only three elements, but C has four elements. Therefore, there is no way to map all elements of C (to cover its range) from B.

Diagram for (c): $g : B \to C$ **(Injective)**

$$\begin{array}{ccc} a & \longrightarrow & s \\ b & \longrightarrow & t \\ c & \longrightarrow & u \\ & & v \end{array}$$

Diagram for (d): $j : A \to C$ **(Not Bijective)**

$$\begin{tikzcd}
a \arrow[r] & s \\
b \arrow[ur] & t \\
c \arrow[ur] & u \\
d \arrow[ur] & v
\end{tikzcd}$$

Problem 2

Problem 2. *Determine whether each function is an injection **and** determine whether each is a surjection. You do not need formal proofs, but you must clearly justify your conclusion and show neat (and mathematically accurate) supporting work.*

(a) $f : \mathbb{Z}_6 \to \mathbb{Z}_6$ *defined by* $f(x) = x^2 + 4 \pmod 6$.

(b) $g : \mathbb{Z}_5 \to \mathbb{Z}_5$ *defined by* $g(x) = x^2 - 11 \pmod 5$.

(c) $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ *defined by* $h(x, y) = x + 2y$.

(d) $j : \mathbb{R} - \{3\} \to \mathbb{R}$ *defined by* $j(x) = \frac{4x}{x-3}$.

Solution:

(a) In $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$:

$$f(0) = 4,\ f(1) = 5,\ f(2) = 2,\ f(3) = 1,\ f(4) = 2,\ f(5) = 5.$$

Since $f(2) = f(4)$ and $f(1) = f(5)$, f is not injective. The range is $\{1, 2, 4, 5\}$, not all of $\mathbb{Z}_6$, so it is not surjective.

(b) In $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$, note that $-11 \equiv 4 \pmod 5$. Thus:

$$g(x) = x^2 + 4 \pmod 5.$$

Check values:

$$g(0) = 4,\ g(1) = 0,\ g(2) = 3,\ g(3) = 3,\ g(4) = 0.$$

Not injective since $g(1) = g(4)$ and $g(2) = g(3)$. The range is $\{0, 3, 4\}$, not all of $\mathbb{Z}_5$, so not surjective.

(c) Consider $h(x, y) = x + 2y$ on integers. To check injectivity:

$$h(0, 0) = 0, \quad h(2, -1) = 0.$$

Different inputs map to the same output, so not injective. For surjectivity, given any integer z, choose $x = z, y = 0$, so $h(x, y) = z$. Thus h is surjective.

(d) For $j(x) = \frac{4x}{x-3}$, if $j(x_1) = j(x_2)$:

$$\frac{4x_1}{x_1 - 3} = \frac{4x_2}{x_2 - 3}.$$

Cross-multiplying shows $x_1 = x_2$. Thus j is injective. For surjectivity, set $y = \frac{4x}{x-3}$. Solve for x:

$$y(x-3) = 4x \implies yx - 4x = 3y \implies x(y-4) = 3y \implies x = \frac{3y}{y-4}.$$

This works for all $y \neq 4$. When $y = 4$, no solution exists. Not all real numbers are attained, so not surjective.

Problem 3

Problem 3. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined by* $f(x) = x^3 + 5$*.*

(a) Determine if f *is injective. If so, write a formal proof. If not, write a counterexample.*

(b) Determine if f *is surjective. If so, write a formal proof. If not, write a counterexample.*

(c) Based upon (a) and (b), is f *bijective?*

Solution:

(a) *Injectivity:* Suppose $f(a) = f(b)$. Then:

$$a^3 + 5 = b^3 + 5 \implies a^3 = b^3 \implies a = b.$$

Thus, f is injective.

(b) *Surjectivity:* For any $y \in \mathbb{R}$, let:

$$x = \sqrt[3]{y - 5}.$$

Then:

$$f(x) = x^3 + 5 = (y - 5) + 5 = y.$$

Thus, f is surjective.

(c) Since f is both injective and surjective, f is bijective.