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Problem Set 8 Due Date: November 2, 2024

Problem 1

Problem 1. *Consider the following proposition: If n is an integer, then $3 \mid (n^3 - n)$.*

(a) *Write a paragraph proof for the third case of this proposition without using properties of congruence.*

Solution: *To prove $3 \mid (n^3 - n)$, we examine the form $n = 3k + 2$ for some integer k, as we assume two other cases (for $n = 3k$ and $n = 3k + 1$) were covered previously.*

$$n^3 - n = (3k + 2)^3 - (3k + 2).$$

Expanding $(3k + 2)^3$:

$$(3k + 2)^3 = 27k^3 + 54k^2 + 36k + 8,$$

so

$$n^3 - n = 27k^3 + 54k^2 + 36k + 8 - (3k + 2) = 27k^3 + 54k^2 + 33k + 6.$$

Each term in this expression is divisible by 3, so we conclude $3 \mid (n^3 - n)$.

(b) *Briefly explain why this proposition is equivalent to the statement $n^3 \equiv n \pmod 3$.*

Solution: *The statement $3 \mid (n^3 - n)$ implies that $n^3 - n$ is divisible by 3. This is equivalent to saying $n^3 \equiv n \pmod 3$ because both expressions ensure that n^3 and n yield the same remainder modulo 3.*

(c) *Using congruences, what are the three cases to consider for n?*

Solution: *The three cases for n modulo 3 are:*

(a) $n \equiv 0 \pmod 3$

(b) $n \equiv 1 \pmod 3$

(c) $n \equiv 2 \pmod 3$

(d) *Construct a know-show table for the case where $n \equiv 2 \pmod 3$ using congruences.*

Step	Know	Reason
Step 1	*Assume* $n \equiv 2 \pmod 3$	*Given case*
Step 2	$n^3 \equiv 8 \equiv 2 \pmod 3$	*Calculation of* n^3 *in terms of mod 3*
Step 3	*Conclude* $n^3 \equiv n \pmod 3$	*Matches* $n \equiv 2 \pmod 3$

Problem 2

Problem 2. *Consider the proposition: Let* $n \in \mathbb{Z}$*. If* $n \not\equiv 0 \pmod 5$*, then* $n^2 \equiv 1 \pmod 5$ *or* $n^2 \equiv 4 \pmod 5$*.*

(a) *Explore the theorem by completing the following chart.*

n	n^2	*Does* $n^2 \equiv 1$ *or* $4 \pmod 5$*?*
± 1	*1*	*Yes,* $1 \equiv 1 \pmod 5$
± 2	*4*	*Yes,* $4 \equiv 4 \pmod 5$
± 3	*9*	*Yes,* $9 \equiv 4 \pmod 5$
± 4	*16*	*Yes,* $16 \equiv 1 \pmod 5$

(b) *Patterns observed:*

- n^2 *for values not divisible by 5 are either congruent to 1 or 4 modulo 5.*

(c) *The cases to consider based on the Division Algorithm are:*

  (a) $n \equiv 1 \pmod 5$

  (b) $n \equiv 2 \pmod 5$

  (c) $n \equiv 3 \pmod 5$

  (d) $n \equiv 4 \pmod 5$

(d) *Know-show table for* $n \equiv 3 \pmod 5$*.*

Step	Know	Reason
Step 1	*Assume* $n \equiv 3 \pmod 5$	*Given*
Step 2	$n^2 \equiv 9 \equiv 4 \pmod 5$	*Calculation*

Problem 3

Problem 3. *Consider the following proposition:*
Proposition: *If x and y are real numbers, then*

$$\max(x, y) = \frac{|x-y| + x + y}{2}.$$

Note: For $x, y \in \mathbb{R}$, the function $\max(x, y)$ is defined to output the larger of the two values if $x \neq y$, and the value x if $x = y$.

(a) *Explore the proposition by filling in the chart below:*

(x, y)	$\max(x, y)$	$\frac{\lvert x-y\rvert+x+y}{2}$
$(-3, 10)$	*10*	$\frac{\lvert -3-10\rvert+(-3)+10}{2} = \frac{13+7}{2} = 10$
$(-3, -10)$	*-3*	$\frac{\lvert -3+10\rvert+(-3)+(-10)}{2} = \frac{7-13}{2} = -3$
$(0, 5)$	*5*	$\frac{\lvert 0-5\rvert+0+5}{2} = \frac{5+5}{2} = 5$
$\left(\frac{2}{3}, \frac{12}{18}\right)$	$\frac{2}{3}$	$\frac{\left\lvert\frac{2}{3}-\frac{2}{3}\right\rvert+\frac{2}{3}+\frac{2}{3}}{2} = \frac{0+2/3}{2} = \frac{2}{3}$
$(\sqrt{2}, \sqrt{3})$	$\sqrt{3}$	$\frac{\lvert\sqrt{2}-\sqrt{3}\rvert+\sqrt{2}+\sqrt{3}}{2} = \sqrt{3}$

(b) *Using the definition of absolute value, there are two cases to consider when proving this equation algebraically. These cases are:*

- *Case 1: When $x \geq y$, which implies $|x - y| = x - y$.*
- *Case 2: When $x < y$, which implies $|x - y| = y - x$.*

(c) *Show the thinking work for a proof by cases of the proposition above.*

Case	***Steps***
Case 1	*Assume $x \geq y$. Then $\lvert x - y\rvert = x - y$. We have:* $$\frac{\lvert x-y\rvert + x + y}{2} = \frac{(x-y) + x + y}{2} = \frac{2x}{2} = x.$$ *Since $x \geq y$, $\max(x, y) = x$, and the proposition holds.*
Case 2	*Assume $x < y$. Then $\lvert x - y\rvert = y - x$. We have:* $$\frac{\lvert x-y\rvert + x + y}{2} = \frac{(y-x) + x + y}{2} = \frac{2y}{2} = y.$$ *Since $y > x$, $\max(x, y) = y$, and the proposition holds.*

(d) *Write a formal proof of the proposition using a proof by cases.*

Proof: *Let $x, y \in \mathbb{R}$. We will prove the proposition by considering two cases based on the relationship between x and y.*

Case 1: *Assume $x \geq y$. Then $|x - y| = x - y$, so we calculate:*

$$\begin{aligned} \frac{|x-y| + x + y}{2} &= \frac{(x-y) + x + y}{2} \\ &= \frac{2x}{2} \\ &= x. \end{aligned}$$

Since $x \geq y$, it follows that $\max(x, y) = x$. Thus, the expression $\frac{|x-y|+x+y}{2}$ is equal to $\max(x, y)$ for this case.

Case 2: *Assume $x < y$. Then $|x - y| = y - x$, so we calculate:*

$$\begin{aligned} \frac{|x-y| + x + y}{2} &= \frac{(y-x) + x + y}{2} \\ &= \frac{2y}{2} \\ &= y. \end{aligned}$$

Since $y > x$, it follows that $\max(x, y) = y$. Thus, the expression $\frac{|x-y|+x+y}{2}$ is equal to $\max(x, y)$ for this case.

In both cases, we have shown that $\max(x, y) = \frac{|x-y|+x+y}{2}$. Therefore, the proposition holds for all $x, y \in \mathbb{R}$.