

# 1 Portfolio Problem 1 - Divisibility by 4

## 1.1 Examples and Definitions

**Problem to prove:** Construct both a know-show table and a first attempt at a formal proof of the following biconditional statement.

**Proposition.** For each integer  $n$ ,  $n$  is even if and only if 4 divides  $n^2$ .

**Exploratory work/Examples:**

- Find four even integers  $a$ . At least one integer must be negative. Square each integer  $a$  and determine if 4 divides  $a^2$ .
  - If we set  $a = -2$ , then  $(-2)^2 = 4$ , and  $4 \mid 4$ .
  - If we set  $a = 0$ , then  $0^2 = 0$ , and  $4 \mid 0$ .
  - If we set  $a = 2$ , then  $2^2 = 4$ , and  $4 \mid 4$ .
  - If we set  $a = 4$ , then  $4^2 = 16$ , and  $4 \mid 16$ .
- Find four perfect squares,  $b$ , such that  $4 \mid b$ . Then find the integer  $c$  such that  $b = c^2$ . Determine if  $c$  is even or odd.
  - If we set  $b = 0$ , then  $4 \mid 0$ . Using  $b = c^2$ , we get  $0 = 0^2$ , so  $c = 0$ , which is even.
  - If we set  $b = 4$ , then  $4 \mid 4$ . Using  $b = c^2$ , we get  $4 = 2^2$ , so  $c = 2$ , which is even.
  - If we set  $b = 16$ , then  $4 \mid 16$ . Using  $b = c^2$ , we get  $16 = 4^2$ , so  $c = 4$ , which is even.
  - If we set  $b = 144$ , then  $4 \mid 144$ . Using  $b = c^2$ , we get  $144 = 12^2$ , so  $c = 12$ , which is even.

## 1.2 Know-Show Table: If $n$ is even, then $4 \mid n^2$ (Forward)

Step	Know	Reason
P1	$n$ is even	Hypothesis
P2	$\exists k \in \mathbb{Z}$ s.t. $n = 2k$	Definition of even
P3	$n^2 = (2k)^2 = 4k^2$	Substitution
P4	$4k^2 = 2(2k^2)$	Factoring out 2
P5	$2k^2 \in \mathbb{Z}$	$\mathbb{Z}$ closed under multiplication
P6	$n^2 = 2q$ where $q \in \mathbb{Z}$	Set $q = k^2$
Q1	4 divides $n^2$	Defn divides
Step	Show	Reason

### 1.3 Know-Show Table: If $4 \mid n^2$ , then $n$ is even (Reverse)

Step	Know	Reason
P1	$4 \mid n^2$	Hypothesis
P2	$n^2 = 4m$ for some $m \in \mathbb{Z}$	Definition of divisibility
P3	$n^2 \equiv 0 \pmod{4}$	Definition of congruence
P4	Assume $n$ is odd	For contradiction
P5	$n = 2k + 1$ for some $k \in \mathbb{Z}$	Definition of odd
P6	$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1$	Expand and simplify $n^2$
P7	$n^2 = 4(k^2 + k) + 1$	Factor out 4
P8	$k^2 + k \in \mathbb{Z}$	$\mathbb{Z}$ closed under multiplication
P9	$n^2 \equiv 1 \pmod{4}$	$4(k^2 + k) \equiv 0 \pmod{4}$
P10	Contradicts $n^2 \equiv 0 \pmod{4}$	From P2, since $n^2 = 4m$
Q1	$n$ is even	Definition of even
Step	Show	Reason

### 1.4 First Draft

**Theorem.** For each integer  $n$ ,  $n$  is even if and only if 4 divides  $n^2$ .

*Proof.* In order to prove this biconditional statement, we must prove the if/then statement in both the forward and reverse directions.

For the forward direction, we assume  $n$  is even, and we will prove that  $4 \mid n^2$ . By the definition of even, there exists an integer  $k$  such that  $n = 2k$ . Substituting for  $n$  into  $n^2$ , we get:

$$n^2 = (2k)^2 = 4k^2$$

Then, we factor:

$$4k^2 = 2(2k^2)$$

We find that  $2k^2$  is an integer due to closure under multiplication. Now, we can write  $n^2 = 2q$ , where  $q = 2k^2$  is an integer, which shows that  $n^2$  is divisible by 4. Therefore, we have proven that if  $n$  is even, then  $4 \mid n^2$ .

For the reverse direction, we assume  $4 \mid n^2$ , and we will prove that  $n$  is even. By the definition of divides, there exists an integer  $m$  such that  $n^2 = 4m$ . Because of the relationship with

36 congruence, this can be restated as:

$$n^2 \equiv 0 \pmod{4}$$

37 Due to integers not being closed under roots, we assume for contradiction that  $n$  is odd. By the  
38 definition of odd,  $n = 2k + 1$  for some  $k \in \mathbb{Z}$ . Substituting for  $n$  into  $n^2$ , we get:

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1$$

39 Then, we factor:

$$4k^2 + 4k + 1 = 4(k^2 + k) + 1$$

40 So,  $n^2 = 4(k^2 + k) + 1$ , which implies that,  $n^2 \equiv 1 \pmod{4}$ . However, we assumed that  $4 \mid n^2$ ,  
41 which implies  $n^2 \equiv 0 \pmod{4}$ . Therefore, our assumption that  $n$  is odd must be false, and it  
42 follows that  $n$  is even. □

## 43 References

44 [1] Sundstrom, T., *Mathematical Reasoning: Writing and Proof*, Version 3, Creative Commons,  
45 2020.