1 Portfolio Problem 1 - Divisibility by 4

1.1 Examples and Definitions

Problem to prove: Construct both a know-show table and a first attempt at a formal proof of the following biconditional statement.

Proposition. *For each integer n, n is even if and only if 4 divides n^2.*

Exploratory work/Examples:

- Find four even integers a. At least one integer must be negative. Square each integer a and determine if 4 divides a^2.
  - If we set $a = -2$, then $(-2)^2 = 4$, and $4 \mid 4$.
  - If we set $a = 0$, then $0^2 = 0$, and $4 \mid 0$.
  - If we set $a = 2$, then $2^2 = 4$, and $4 \mid 4$.
  - If we set $a = 4$, then $4^2 = 16$, and $4 \mid 16$.
- Find four perfect squares, b, such that $4 \mid b$. Then find the integer c such that $b = c^2$. Determine if c is even or odd.
  - If we set $b = 0$, then $4 \mid 0$. Using $b = c^2$, we get $0 = 0^2$, so $c = 0$, which is even.
  - If we set $b = 4$, then $4 \mid 4$. Using $b = c^2$, we get $4 = 2^2$, so $c = 2$, which is even.
  - If we set $b = 16$, then $4 \mid 16$. Using $b = c^2$, we get $16 = 4^2$, so $c = 4$, which is even.
  - If we set $b = 144$, then $4 \mid 144$. Using $b = c^2$, we get $144 = 12^2$, so $c = 12$, which is even.

1.2 Know-Show Table: If n is even, then $4 \mid n^2$ (Forward)

Step	Know	Reason
P1	n is even	Hypothesis
P2	$\exists k \in \mathbb{Z}$ s.t. $n = 2k$	Definition of even
P3	$n^2 = (2k)^2 = 4k^2$	Substitution
P4	$4k^2 = 2(2k^2)$	Factoring out 2
P5	$2k^2 \in \mathbb{Z}$	$\mathbb{Z}$ closed under multiplication
P6	$n^2 = 2q$ where $q \in \mathbb{Z}$	Set $q = k^2$
Q1	4 divides n^2	Defn divides
Step	**Show**	**Reason**

1.3 Know-Show Table: If $4 \mid n^2$, then n is even (Reverse)

Step	Know	Reason
P1	$4 \mid n^2$	Hypothesis
P2	$n^2 = 4m$ for some $m \in \mathbb{Z}$	Definition of divisibility
P3	$n^2 \equiv 0 \mod 4$	Definition of congruence
P4	Assume n is odd	For contradiction
P5	$n = 2k + 1$ for some $k \in \mathbb{Z}$	Definition of odd
P6	$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$	Expand and simplify n^2
P7	$n^2 = 4(k^2 + k) + 1$	Factor out 4
P8	$k^2 + k \in \mathbb{Z}$	$\mathbb{Z}$ closed under multiplication
P9	$n^2 \equiv 1 \mod 4$	$4(k^2 + k) \equiv 0 \mod 4$
P10	Contradicts $n^2 \equiv 0 \mod 4$	From P2, since $n^2 = 4m$
Q1	n is even	Definition of even
Step	**Show**	**Reason**

1.4 First Draft

Theorem. *For each integer n, n is even if and only if 4 divides n^2.*

Proof. In order to prove this biconditional statement, we must prove the if/then statement in both the forward and reverse directions.

For the forward direction, we assume n is even, and we will prove that $4 \mid n^2$. By the definition of even, there exists an integer k such that $n = 2k$. Substituting for n into n^2, we get:

$$n^2 = (2k)^2 = 4k^2$$

Then, we factor:

$$4k^2 = 2(2k^2)$$

We find that $2k^2$ is an integer due to closure under multiplication. Now, we can write $n^2 = 2q$, where $q = 2k^2$ is an integer, which shows that n^2 is divisible by 4. Therefore, we have proven that if n is even, then $4 \mid n^2$.

For the reverse direction, we assume $4 \mid n^2$, and we will prove that n is even. By the definition of divides, there exists an integer m such that $n^2 = 4m$. Because of the relationship with congruence, this can be restated as:

$$n^2 \equiv 0 \pmod 4$$

Due to integers not being closed under roots, we assume for contradiction that n is odd. By the definition of odd, $n = 2k + 1$ for some $k \in \mathbb{Z}$. Substituting for n into n^2, we get:

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1$$

Then, we factor:

$$4k^2 + 4k + 1 = 4(k^2 + k) + 1$$

So, $n^2 = 4(k^2 + k) + 1$, which implies that, $n^2 \equiv 1 \pmod 4$. However, we assumed that $4 \mid n^2$, which implies $n^2 \equiv 0 \pmod 4$. Therefore, our assumption that n is odd must be false, and it follows that n is even. $\square$

1.5 Second Draft

Theorem. *For each integer n, n is even if and only if 4 divides n^2.*

Proof. Let n be an integer. To prove the biconditional statement, "n is even if and only if 4 divides n^2", we must prove the if/then statement in both the forward and reverse directions.

For the forward direction, we assume n is even, and we will prove that $4 \mid n^2$. By the definition of even, there exists an integer k such that $n = 2k$. Substituting for n into n^2, we get:

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2) \end{aligned}$$

We find that $2k^2$ is an integer due to closure under multiplication. Now, we can write $n^2 = 2q$, where $q = 2k^2$ is an integer, which shows that n^2 is divisible by 4. Therefore, we have proven that if n is even, then $4 \mid n^2$.

For the reverse direction, we assume $4 \mid n^2$, and we will prove that n is even. By the definition of divides, there exists an integer m such that $n^2 = 4m$. Because of the relationship with congruence, this can be restated as:

$$n^2 \equiv 0 \pmod 4$$

Due to integers not being closed under roots, we assume for contradiction that n is odd. By the definition of odd, $n = 2k + 1$ for some $k \in \mathbb{Z}$. Substituting for n into n^2, we get:

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 4(k^2 + k) + 1 \end{aligned}$$

So, $n^2 = 4(k^2 + k) + 1$, which implies that, $n^2 \equiv 1 \pmod 4$. However, we assumed that $n^2 \equiv 0 \pmod 4$. Thus, our assumption that n is odd must be false, and it follows that n is even.

Therefore, we have proven that $4 \mid n^2$ if and only if n is even. $\square$

1.6 Reflection

- My first draft doesn't declare n as an integer, so I added a sentence at the beginning of the first paragraph.
- I decided to state what we're proving more clearly by referencing the theorem.
- Also, it was pointed out that I was restating something on lines 35 and 36, which I trimmed down.
- I stacked the math and aligned by the equation symbol. And I also restated the conclusion a little more clearly.
- All-in-all, I think the proof is nearly dialed in, I'm excited for Dr. Johnson's critique.

References

[1] Sundstrom, T., *Mathematical Reasoning: Writing and Proof*, Version 3, Creative Commons, 2020.