

## Problem 1

**Problem 1.** Consider the following proposition: If  $n$  is an integer, then  $3 \mid (n^3 - n)$ .

- (a) Write a paragraph proof for the third case of this proposition without using properties of congruence.

**Solution:** To prove  $3 \mid (n^3 - n)$ , we examine the form  $n = 3k + 2$  for some integer  $k$ , as we assume two other cases (for  $n = 3k$  and  $n = 3k + 1$ ) were covered previously.

$$n^3 - n = (3k + 2)^3 - (3k + 2).$$

Expanding  $(3k + 2)^3$ :

$$(3k + 2)^3 = 27k^3 + 54k^2 + 36k + 8,$$

so

$$n^3 - n = 27k^3 + 54k^2 + 36k + 8 - (3k + 2) = 27k^3 + 54k^2 + 33k + 6.$$

Each term in this expression is divisible by 3, so we conclude  $3 \mid (n^3 - n)$ .

- (b) Briefly explain why this proposition is equivalent to the statement  $n^3 \equiv n \pmod{3}$ .

**Solution:** The statement  $3 \mid (n^3 - n)$  implies that  $n^3 - n$  is divisible by 3. This is equivalent to saying  $n^3 \equiv n \pmod{3}$  because both expressions ensure that  $n^3$  and  $n$  yield the same remainder modulo 3.

- (c) Using congruences, what are the three cases to consider for  $n$ ?

**Solution:** The three cases for  $n$  modulo 3 are:

(a)  $n \equiv 0 \pmod{3}$

(b)  $n \equiv 1 \pmod{3}$

(c)  $n \equiv 2 \pmod{3}$

- (d) Construct a know-show table for the case where  $n \equiv 2 \pmod{3}$  using congruences.

<b>Step</b>	<b>Know</b>	<b>Reason</b>
<i>Step 1</i>	<i>Assume <math>n \equiv 2 \pmod{3}</math></i>	<i>Given case</i>
<i>Step 2</i>	$n^3 \equiv 8 \equiv 2 \pmod{3}$	<i>Calculation of <math>n^3</math> in terms of mod 3</i>
<i>Step 3</i>	<i>Conclude <math>n^3 \equiv n \pmod{3}</math></i>	<i>Matches <math>n \equiv 2 \pmod{3}</math></i>

## Problem 2

**Problem 2.** Consider the proposition: Let  $n \in \mathbb{Z}$ . If  $n \not\equiv 0 \pmod{5}$ , then  $n^2 \equiv 1 \pmod{5}$  or  $n^2 \equiv 4 \pmod{5}$ .

(a) Explore the theorem by completing the following chart.

$n$	$n^2$	Does $n^2 \equiv 1$ or $4 \pmod{5}$ ?
$\pm 1$	1	Yes, $1 \equiv 1 \pmod{5}$
$\pm 2$	4	Yes, $4 \equiv 4 \pmod{5}$
$\pm 3$	9	Yes, $9 \equiv 4 \pmod{5}$
$\pm 4$	16	Yes, $16 \equiv 1 \pmod{5}$

(b) Patterns observed:

- $n^2$  for values not divisible by 5 are either congruent to 1 or 4 modulo 5.

(c) The cases to consider based on the Division Algorithm are:

(a)  $n \equiv 1 \pmod{5}$

(b)  $n \equiv 2 \pmod{5}$

(c)  $n \equiv 3 \pmod{5}$

(d)  $n \equiv 4 \pmod{5}$

(d) Know-show table for  $n \equiv 3 \pmod{5}$ .

<b>Step</b>	<b>Know</b>	<b>Reason</b>
<i>Step 1</i>	<i>Assume <math>n \equiv 3 \pmod{5}</math></i>	<i>Given</i>
<i>Step 2</i>	$n^2 \equiv 9 \equiv 4 \pmod{5}$	<i>Calculation</i>

## Problem 3

**Problem 3.** Consider the following proposition:

**Proposition:** If  $x$  and  $y$  are real numbers, then

$$\max(x, y) = \frac{|x - y| + x + y}{2}.$$

*Note:* For  $x, y \in \mathbb{R}$ , the function  $\max(x, y)$  is defined to output the larger of the two values if  $x \neq y$ , and the value  $x$  if  $x = y$ .

(a) Explore the proposition by filling in the chart below:

$(x, y)$	$\max(x, y)$	$\frac{ x-y +x+y}{2}$
$(-3, 10)$	<b>10</b>	$\frac{ -3-10 +(-3)+10}{2} = \frac{13+7}{2} = 10$
$(-3, -10)$	<b>-3</b>	$\frac{ -3+10 +(-3)+(-10)}{2} = \frac{7-13}{2} = -3$
$(0, 5)$	<b>5</b>	$\frac{ 0-5 +0+5}{2} = \frac{5+5}{2} = 5$
$(\frac{2}{3}, \frac{12}{18})$	$\frac{2}{3}$	$\frac{ \frac{2}{3}-\frac{2}{3} +\frac{2}{3}+\frac{2}{3}}{2} = \frac{0+2/3}{2} = \frac{2}{3}$
$(\sqrt{2}, \sqrt{3})$	$\sqrt{3}$	$\frac{ \sqrt{2}-\sqrt{3} +\sqrt{2}+\sqrt{3}}{2} = \sqrt{3}$

(b) Using the definition of absolute value, there are two cases to consider when proving this equation algebraically. These cases are:

- Case 1: When  $x \geq y$ , which implies  $|x - y| = x - y$ .
- Case 2: When  $x < y$ , which implies  $|x - y| = y - x$ .

(c) Show the thinking work for a proof by cases of the proposition above.

Case	Steps
Case 1	<p>Assume <math>x \geq y</math>. Then <math> x - y  = x - y</math>. We have:</p> $\frac{ x - y  + x + y}{2} = \frac{(x - y) + x + y}{2} = \frac{2x}{2} = x.$ <p>Since <math>x \geq y</math>, <math>\max(x, y) = x</math>, and the proposition holds.</p>
Case 2	<p>Assume <math>x &lt; y</math>. Then <math> x - y  = y - x</math>. We have:</p> $\frac{ x - y  + x + y}{2} = \frac{(y - x) + x + y}{2} = \frac{2y}{2} = y.$ <p>Since <math>y &gt; x</math>, <math>\max(x, y) = y</math>, and the proposition holds.</p>

(d) Write a formal proof of the proposition using a proof by cases.

**Proof:** Let  $x, y \in \mathbb{R}$ . We consider two cases.

**Case 1:** Assume  $x \geq y$ . Then  $|x - y| = x - y$ , so we calculate:

$$\frac{|x - y| + x + y}{2} = \frac{(x - y) + x + y}{2} = \frac{2x}{2} = x.$$

Since  $x \geq y$ , it follows that  $\max(x, y) = x$ . Thus, the expression  $\frac{|x - y| + x + y}{2}$  is equal to  $\max(x, y)$  in this case.

**Case 2:** Assume  $x < y$ . Then  $|x - y| = y - x$ , so we calculate:

$$\frac{|x - y| + x + y}{2} = \frac{(y - x) + x + y}{2} = \frac{2y}{2} = y.$$

Since  $y > x$ , it follows that  $\max(x, y) = y$ . Thus, the expression  $\frac{|x - y| + x + y}{2}$  is equal to  $\max(x, y)$  in this case.

In both cases, we have shown that  $\max(x, y) = \frac{|x - y| + x + y}{2}$ . Therefore, the proposition holds for all  $x, y \in \mathbb{R}$ .