# 1 Portfolio Problem 2 - C. Squares and Cubes

## 1.1 Examples and Definitions

**Problem to prove:** Construct both a know-show table and a first attempt at a formal proof of the following proposition.

**Proposition.** *Let $a, b \in \mathbb{N}$. If $a^2 = b^3$ and $a$ is even, then $4 \mid a$.*

## 1.2 Exploratory Work/Examples

**Strategy:** To find four pairs of natural numbers $(a, b)$ such that $a^2 = b^3$ and $a$ is even, I modified my approach:

- Filtered $a$ to include only even numbers.
- Verified whether $\sqrt[3]{a^2}$ is an integer and that $b^3 = a^2$.
- Added additional columns to check if $a$ is even and ensure that both $a$ and $b$ satisfy the conditions for being natural numbers.

**Excel Formulas Used:**

- Column A: Even natural numbers $a$.
  - Formula: `=even a value`
- Column B: Squares of $a$ ($a^2$).
  - Formula: `=POWER(A2, 2)`
- Column C: Cube root of $a^2$ ($\sqrt[3]{a^2}$).
  - Formula: `=POWER(B2, 1/3)`
- Column D: Verification column for $b^3$ ($b^3 \in \mathbb{N}$).
  - Formula: `=IF(AND(C2>0, C2=INT(C2)), C2*C2*C2 & "∈ ℕ", "b∉N")`
- Column E: Even or Odd verification for $a$.
  - Formula: `=IF(A2=INT(A2), IF(MOD(A2,2)=0, "Even", "Odd"), "Not an Integer")`
- Column F: Final check for Even and Natural.

  - Formula: `=IF(AND(A2>0, A2=INT(A2), MOD(A2,2)=0, C2>0, C2=INT(C2)), "Natural and Even", "False")`

**Results:** The four pairs satisfying $a^2 = b^3$, with $a$ even, are:

| $a$ | $a^2$ | $b$ | $b^3$ |
|---|---|---|---|
| 8 | 64 | 4 | 64 |
| 64 | 4096 | 16 | 4096 |
| 216 | 46656 | 36 | 46656 |
| 512 | 262144 | 64 | 262144 |

**Spreadsheet Screenshot:**

`Portfolio Problem 2 Spreadsheet.png`

## 1.3 Know-Show Table ($a^2 = b^3$ and $a$ even $\implies 4 \mid a$)

| **Step** | **Know** | **Reason** |
|---|---|---|
| P1 | $a^2 = b^3$ | Hypothesis |
| P2 | $a$ is even $(a = 2k)$ | Hypothesis |
| P3 | $(2k)^2 = b^3$ | Substitution |
| P4 | $4k^2 = b^3$ | Algebra |
| P5 | $b$ is even $(b = 2m)$ | Cubes divisible by 4 imply base divisible by 2 |
| P6 | Substituting $b = 2m$: $4k^2 = (2m)^3$ | Substitution |
| P7 | $4k^2 = 8m^3$ | Algebra |
| P8 | $k^2 = 2m^3$ | Divide by 4 |
| P9 | $k$ is even $(k = 2n)$ | Squares divisible by 2 imply base divisible by 2 |
| Q1 | $a = 4n$ | Substituting $k = 2n$ into $a = 2k$ |
| **Step** | **Show** | **Reason** |

## 1.4 First Draft of Proof

*Proof.* Assume $a^2 = b^3$, and $a$ is even. Then there exists an integer $k$ such that $a = 2k$. Substituting into $a^2$, we get:

$$a^2 = (2k)^2 = 4k^2$$

By the hypothesis $a^2 = b^3$, we know $b^3 = 4k^2$. Since $b^3$ is divisible by 4, $b$ must be even. Let $b = 2m$ for some integer $m$. Substituting into $b^3$, we get:

$$b^3 = (2m)^3 = 8m^3$$

Thus:

$$4k^2 = 8m^3$$

Dividing both sides by 4, we find:

$$k^2 = 2m^3$$

Since $k^2$ is even, $k$ must also be even. Let $k = 2n$ for some integer $n$. Substituting into $a = 2k$, we get:

$$a = 2(2n) = 4n$$

Therefore, $a$ is divisible by 4. $\square$

## 1.5 Second Draft of Proof

We aim to prove the following proposition:

**Proposition:** Let $a, b \in \mathbb{N}$. If $a^2 = b^3$ and $a$ is even, then $4 \mid a$.

*Proof.* Assume $a^2 = b^3$, and $a$ is even. Since $a$ is even, there exists an integer $k$ such that $a = 2k$. Substituting $a = 2k$ into $a^2$, we get:

$$a^2 = (2k)^2 = 4k^2.$$

By the hypothesis $a^2 = b^3$, it follows that:

$$b^3 = 4k^2.$$

Since $b^3$ is divisible by $4$, $b$ must also be even. Let $b = 2m$ for some integer $m$. Substituting $b = 2m$ into $b^3$, we have:

$$b^3 = (2m)^3 = 8m^3.$$

Thus, the equation becomes:

$$4k^2 = 8m^3.$$

Dividing both sides by $4$, we find:

$$k^2 = 2m^3.$$

Since $k^2$ is even, $k$ must also be even. Let $k = 2n$ for some integer $n$. Recalling that $a = 2k$, we substitute $k = 2n$ into the expression for $a$, and we have:

$$a = 2(2n) = 4n.$$

Hence, $a$ is divisible by $4$, which proves the proposition that if $a^2 = b^3$ and $a$ is even, then $a$ must be divisible by $4$. $\square$

## 1.6 Final Draft of Proof

**Proposition:** Let $a, b \in \mathbb{N}$. If $a^2 = b^3$ and $a$ is even, then $4 \mid a$.

*Proof.* Assume $a$ and $b$ are natural numbers. We will prove that if $a^2 = b^3$ and $a$ is even, then $4$ divides $a$. Since $a$ is even, there exists an integer $k$ such that $a = 2k$. Substituting $a = 2k$ into $a^2$, we get:

$$a^2 = (2k)^2 = 4k^2.$$

By the hypothesis $a^2 = b^3$, it follows that:

$$b^3 = 4k^2.$$

Since $b^3$ is divisible by $4$, it follows that $b^3$ is even. By previous results, $n^3$ is even if and only if $n$ is even, so $b$ must also be even. Thus, there exists an integer $m$ such that $b = 2m$. Substituting $b = 2m$ into $b^3$, we have:

$$b^3 = (2m)^3 = 8m^3.$$

Thus, the equation becomes:

$$4k^2 = 8m^3,$$
$$k^2 = 2m^3.$$

By previous results, since $k^2$ is even, $k$ must also be even. Let $k = 2n$ for some integer $n$. Recalling that $a = 2k$, we substitute $k = 2n$ into the expression for $a$, and we have:

$$a = 2(2n) = 4n.$$

Therefore, $a$ is divisible by $4$, which proves the proposition that if $a^2 = b^3$ and $a$ is even, then $a$ must be divisible by $4$. $\square$

## 1.7 Reflection

- Initially, the proof was correct, but certain steps were less explicit. For example, it needed clearer justification for why $b^3$ being divisible by 4 implies $b$ is even.
- The second draft improved these explanations, making the reasoning more direct and ensuring each step was justified.
- The final draft streamlined the logic, referenced prior results explicitly, and presented a fully rigorous and clear argument that $a$ must be divisible by 4.

# References

[1] Sundstrom, T., *Mathematical Reasoning: Writing and Proof*, Version 3, Creative Commons, 2020.