LUND UNIVERSITY

**Written Examination**
**Discrete Mathematics**
**Monday, December 19, 2011**

Centre for Mathematical Sciences
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**1.** We use the exclusion/inclusion principle to solve the problem. We denote by $c_i$ the property of being a perfect $i$-th power. Then, we need to calculate

$$N(\bar{c_2}\bar{c_3}\bar{c_5}) = 10000 - (N(c_2) + N(c_3) + N(c_5)) + \sum_{i \neq j} N(c_i c_j) - N(c_2 c_3 c_5).$$

We know that a number is both a perfect square and a perfect cube if and only if it is a perfect 6-th power. In the same way, a number is both a perfect square and a perfect 5-th if and only if it is a perfect 10-th power; and a number is both a perfect cube and a perfect 5-th if and only if it is a perfect 15-th power. Then,

$$N(c_2) = \lfloor \sqrt{1000} \rfloor = 31,\ N(c_3) = \lfloor \sqrt[3]{1000} \rfloor = 10,\ N(c_5) = \lfloor \sqrt[5]{1000} \rfloor = 3,$$
$$N(c_2c_3) = \lfloor \sqrt[6]{1000} \rfloor = 3,\ N(c_3c_5) = \lfloor \sqrt[15]{10000} \rfloor = 1,\ N(c_2c_5) = \lfloor \sqrt[10]{10000} \rfloor = 1$$

and

$$N(c_2c_3c_5) = \lfloor \sqrt[30]{10000} \rfloor = 1.$$

Then,

$$N(\bar{c_2}\bar{c_3}\bar{c_5}) = 1000 - 44 + 5 - 1 = 960.$$

**2.** We need to calculate the coefficient of $x^{13}$ in the following generating function

$$(x^5 + x^6 + \cdots + x^9)(1 + x + \cdots + x^9)^2$$

We notice that such coefficient coincides with the coefficient of $x^{13}$ in the function

$$\begin{aligned} f(x) &= (x^5 + x^6 + \cdots + x^9)(1 + x + \cdots + x^9 + \cdots)^2 \\ &= x^5(1 + x + \cdots + x^4)(1 + x + \cdots + x^9 + \cdots)^2 \\ &= x^5(1 + x + \cdots + x^4)\frac{1}{(1-x)^2} \end{aligned}$$

Then, we need to find out the coefficient of $x^8$ in the function

$$(1 + x + \cdots + x^4)\sum_{k=0}^{\infty}(-1)^k\binom{-2}{k}x^k$$

which we decompose as

$$\begin{aligned} &\binom{-2}{8} - \binom{-2}{7} + \binom{-2}{6} - \binom{-2}{5} + \binom{-2}{4} \\ &= \binom{9}{8} + \binom{8}{7} + \binom{7}{6} + \binom{6}{5} + \binom{5}{4} \\ &= 9 + 8 + 7 + 6 + 5 = 35 \end{aligned}$$

*Please, turn over!*

**3.** The described recurence relation is

$$a_{n+2} = \frac{a_{n+1} + a_n}{2} + 1.$$

The characteristic polynomial associated with the homogeneous equation $a_{n+2} - a_{n+1}/2 - a_n/2$ is $P(r) = r^2 - r/2 - 1/2$, with roots $r_1 = 1$ and $r_2 = -1/2$. Therefore, the general solution of the homogeneous equation is

$$a_n^{(h)} = \alpha + \beta\Big(\frac{-1}{2}\Big)^n.$$

In order to find a particular solution of the non-homogeneous equation, we try with a function of the formula

$$a_n^p = An.$$

After some computations, we obtain $A = 2/3$ which offers the solution

$$a_n = \alpha + \beta\Big(\frac{-1}{2}\Big)^n + \frac{2}{3}n.$$

Finally, the solution with the given initial conditions is

$$a_n = \frac{2}{9}(1 - \Big(\frac{-1}{2}\Big)^n) + \frac{2}{3}n.$$

**4.** **a)** If the balls are all different and the containers are all different, then $VR_{5,6} = 5^6 = 15625$.

**b)** If the balls are all identical and the containers are all different, then $CR_{5,6} = \binom{10}{6} = 210$.

**c)** If the balls are all different and the containers are all identical, then $\sum_{i=1}^{5} S(6,i) = 1 + 31 + 90 + 65 + 15 = 202$.

**d)** If the balls are all identical and the containers are all identical, then $1 + 2 + 3 + 2 + 1 + 1 = 10$.

**5.** We simplify the system to

$$\begin{aligned} x &\equiv 1 \pmod 5 \\ x &\equiv 5 \pmod 6 \\ x &\equiv 2 \pmod 7 \end{aligned}$$

By the Chinese remainder theorem, we have that the solution is given by

$$x = \sum_i a_i x_i N_i$$

with $N_1 = 42, N_2 = 35, N_3 = 30, a_1 = 1, a_2 = 5, a_3 = 2$. Now, we need to calculate the factors $x_i$ which satisfy $x_i N_i \equiv 1 \pmod{n}_i$. By the euclidean algorithm, we have $x_1 = 3, x_2 = 5$ and $x_3 = 4$ wich finally offers the solution

$$x = 3 \cdot 42 + 5 \cdot 5 \cdot 35 + 2 \cdot 4 \cdot 30 = 1241 = 191.$$

**6.** The code words defined by the generator matrix are

$$C = E(W) = \{000000, 001011, 010101, 011110, 100010, 101001, 110111, 111100\}.$$

We notice that $C$ is a group code and that the minimum weight of the non-zero elements of $C$ is 2. Then, we deduce that the separation of this linear code is 2 and then, we can detect up to single errors. This also means that we cannot correct all single errors, but we can correct some.

None of the received word are code words. We calculate the syndrome of the first received word and obtain 011, which coincides with the third column of $H$. Therefore, an error in transmission happened in the third coordinate and so, so, it's associated with the code word $c = 110100 + 001000 = 111100$. Then, finally, the received word can be decoded as $w = 111$.

On the other hand, the syndrome of the second received word is 010, which coincides with the first and fifth columns of $H$. Therefore, it can't be properly decoded, even if we knew that it was the consequence of a single error.