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2013-12-16 kl 08-13

1. a) $x^T Hx = (x_1 - x_2)^2 + x_3^2 \geq 0$, so the matrix H is positive semi-definite by definition. (Not positive definite, since $x = (1, 1, 0) \neq 0$ gives $x^T Hx = 0$.)

b) The minimum of $f(x) = ((x_1 - x_2)^2 + x_3^2)/2$ is trivially zero, therefore, exists.

c) Since H^{-1} does not exist the Newton method diverges.

d) Calculate $\nabla f(x) = Hx$, $\nabla^2 f(x) = H$ and

$$x_{k+1} = x_k - (H+I)^{-1}Hx_k = (I-(H+I)^{-1}H)x_k = (H+I)^{-1}(H+I-H)x_k = (H+I)^{-1}x_k.$$

For given $x_0 = (1, 1, 1)^T$ we get

$$x_1 = \begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/3 & 2/3 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1/2 \end{pmatrix}$$

from where it is easily concluded that $x_k = (1, 1, 1/2^k)^T \to (1, 1, 0)^T$. Therefore, the method converges to the limit $(1, 1, 0)^T$. The functional value $f(1, 1, 0) = 0$ is the smallest possible one, thus, the limit is a global minimum.

2. a) Assume S_1 and S_2 are convex and take two arbitrary points $x_0, x_1 \in S_1 \cap S_2 \Leftrightarrow x_0, x_1 \in S_1$ and $x_0, x_1 \in S_2$. Let $\lambda \in [0, 1]$ and consider the convex combination $x_\lambda = \lambda x_1 + (1 - \lambda)x_0$. Since S_1 is convex and $x_0, x_1 \in S_1 \Rightarrow x_\lambda \in S_1$. Similarly, $x_\lambda \in S_2$. Therefore, $x_\lambda \in S_1 \cap S_2$.

b) The set S is an intersection of $S_k = \{x_k \leq x_{k+1}\} = \{x_k - x_{k+1} \leq 0\}$ for $k = 1, \ldots, n-1$. Each set S_k is in the form $\{f(x) \leq 0\}$ where $f(x) = x_k - x_{k+1}$ is a convex function (in fact, linear!). Thus, S_k is convex [Th.7,p.210]. The convexity of S is proven by applying $n - 2$ times the result in 2a).

c) Set up the belonging condition algebraically as

$$\begin{aligned} \lambda_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 4 \end{pmatrix} &= \begin{pmatrix} -a \\ a \end{pmatrix} \\ \lambda_1 + \lambda_2 + \lambda_3 &= 1, \\ \text{all } \lambda_k &\geq 0. \end{aligned}$$

The equalities constitute the linear system

$$\begin{pmatrix} -3 & 2 & 0 \\ 1 & 0 & 4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} -a \\ a \\ 1 \end{pmatrix} \quad \overset{\text{e.g. Gauss}}{\Rightarrow} \quad \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} a+4 \\ -3a+6 \\ 2a-1 \end{pmatrix}$$

which together with the positivity condition $\lambda_k \geq 0$ gives that $a \in [1/2, 2]$.

3. a) We add slack variables to the problem and build the Simplex tableau

0	-1	-4	-2	0	0	0
-2	-3	1	5	$\boxed{1}$	0	4
1	1	0	-2	0	$\boxed{1}$	1
-2	-1	$\boxed{1}$	1	0	0	2

$\rightarrow$

-8	-5	0	2	0	0	8
0	-2	0	4	1	0	2
1	1	0	-2	0	1	1
-2	-1	1	1	0	0	2

The slack variable columns look already like basic columns. Let's complete them with a third one. There are two possibilities: x_3 and x_4, but x_4 does not give a feasible point after elimination. Trying x_3 gives it right (right hand side $(2, 1, 2) \geq 0$.) Continue with the Simplex

-8	-5	0	2	0	0	8
0	-2	0	$\boxed{4}$	1	0	2
1	1	0	-2	0	1	1
-2	-1	1	1	0	0	2

$\rightarrow$

-8	-4	0	0	-1/2	0	7
0	-1/2	0	1	1/4	0	1/2
1	0	0	0	1/2	1	2
-2	-1/2	1	0	-1/4	0	3/2

The minimum is 7 at the point $(0, 0, \frac{3}{2}, \frac{1}{2})$ with the slack variables being $(0, 2)$.

b) Multiply the first two inequalities by -1 and use the Formula Sheet to construct the dual problem

$$\max\left(-4y_1 - y_2 + 2y_3\right) \qquad \begin{cases} 2y_1 - y_2 - 2y_3 & \leq \; 0, \\ 3y_1 - y_2 - y_3 & \leq \; 1, \\ -y_1 + y_3 & \leq \; 4, \\ -5y_1 + 2y_2 + y_3 & \leq \; 2, \\ y_1, y_2 & \geq \; 0, \quad (y_3 \text{ free}) \end{cases}$$

CSP gives:

- the primal slack $s_2 = 2 \neq 0 \Rightarrow y_2 = 0$,
- the primal $x_3 = 3/2 \neq 0 \Rightarrow -y_1 + y_3 = 4$,
- the primal $x_4 = 1/2 \neq 0 \Rightarrow -5y_1 + 2y_2 + y_3 = 2$.

All together

$$\begin{cases} -y_1 + y_3 & = \; 4, \\ -5y_1 + y_3 & = \; 2 \end{cases} \quad \Leftrightarrow \quad \begin{cases} y_1 & = \; 1/2, \\ y_3 & = \; 9/2 \end{cases}$$

so the solution of the dual is $(1/2, 0, 9/2)$. (N.B. max $= 7 =$ min.)

4. a) See the book, Lemma 2 (item 4), page 211, and the solution to the Exercise 6.7, page 394.

b) Completing the squares gives

$$f(x) = (x_1 + 2x_2 - x_3)^2 + (x_2 + x_3)^2 + (a - 5)x_2^2$$

which is convex if and only if $a \geq 5$.

Alternative solution: Calculate the Hessian (times 1/2 does not affect positive-semidefiniteness)

$$A = \frac{1}{2}H = \begin{pmatrix} 1 & 2 & -1 \\ 2 & a & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

A is positive-semidefinite $\Leftrightarrow$ $B = A + \epsilon I$ is positive-definite for all $\epsilon > 0$. Use Syvester criterion for B:

- $\forall \epsilon > 0$: $\det B_1 = 1 + \epsilon > 0 \Leftrightarrow$ OK,
- $\forall \epsilon > 0$: $\det B_2 = a - 4 + \epsilon(a + 1 + \epsilon) > 0 \Leftrightarrow a \geq 4$,
- $\forall \epsilon > 0$: $\det B = a - 5 + \epsilon(\epsilon^2 + (a+3)\epsilon + 3a - 4) > 0 \Leftrightarrow a \geq 5$.

Thus, $a \geq 5$.

5. a) The function is continuous, the set is closed. Since $x^2 \leq 4 - y^3 \leq 4$ the set is bounded in x ($-2 \leq x \leq 2$). Moreover, $0 \leq y^3 \leq 4 - x^2 \leq 4$, and the set is bounded in y as well, thus, compact. By Weierstrass theorem, **the minimum exists**. For $f = x^2(y-1) + y^4$, $g_1 = x^2 + y^3 - 4$ and $g_2 = -y$ calculate

$$\nabla f = \begin{pmatrix} 2x(y-1) \\ x^2 + 4y^3 \end{pmatrix}, \quad \nabla g_1 = \begin{pmatrix} 2x \\ 3y^2 \end{pmatrix}, \quad \nabla g_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

CQ points: Case 1: only g_1 is active $\Rightarrow$ $(x, y) \neq (0, 0)$ $\Rightarrow$ $\nabla g_1 \neq 0$ $\Rightarrow$ no CQ points.
Case 2: only g_2 is active. $\nabla g_2 \neq 0 \Rightarrow$ no CQ points.
Case 3: both g_1, g_2 are active $\Rightarrow$ only two points $(x, y) = (\pm 2, 0)$. At both points the gradients ∇g_1 and ∇g_2 are linearly independent $\Rightarrow$ no CQ points.
KKT points:

$$\begin{aligned} 2x(y-1) + 2u_1 x &= 0, && (1) \\ x^2 + 4y^3 + 3u_1 y^2 - u_2 &= 0, && (2) \\ u_1(x^2 + y^3 - 4) &= 0, && (3) \\ u_2 y &= 0, && (4) \\ u_1, u_2 \geq 0&, && (5) \\ x^2 + y^3 \leq 4, y \geq 0&. && (6) \end{aligned}$$

Case 1: $u_2 = 0$. Then (2)+(5) and $y \geq 0 \Rightarrow x = y = 0$. All other conditions are satisfied ($u_1 = 0$). $(0, 0)$ is a KKT point.
Case 2: $u_2 > 0$. Then (4) $\Rightarrow y = 0 \Rightarrow$ by (2) we get $u_2 = x^2 \neq 0 \Rightarrow$ in (1) $2x(y - 1 + u_1) = 0$ gives $u_1 = 1 \Rightarrow$ by (3) $x^2 = 4 \Rightarrow x = \pm 2$. All conditions are satisfied. $(\pm 2, 0)$ are KKT points.
Testing the candidates: $f(0, 0) = 0$, $f((\pm 2, 0)) = -4$. The global minimum is -4 at $(\pm 2, 0)$.

b) The set is no longer bounded. Does the minimum exist? Take $x^2 = 4 - y^3 \Rightarrow f = (4 - y^3)(y - 1) + y^4 = y^3 + 4y - 4$. Clearly the function is unbounded from below: $f \to -\infty$ when $y \to -\infty$. Thus the minimum does not exist.

6. a) Consider the Lagrangian

$$L(x, y, u) = x^2(y - 1) + y^4 + u(x^2 + y^3 - 4) = y^4 + uy^3 + x^2y + (u - 1)x^2 - 4u.$$

Clearly for $y \geq 0$, $y^4 + uy^3 + x^2y \geq 0$ and $= 0$ at $y = 0$, hence, the minimization gives $\min_{y \geq 0} L(x, y, u) = (u - 1)x^2 - 4u$. Minimization over x is now trivial: if $u < 1$ then the minimum is $-\infty$, otherwise (for $u \geq 1$) the minimum is $-4u$ (when $(u - 1)x^2 = 0$). Therefore, the dual function is

$$\Theta(u) = \begin{cases} -4u & \text{if } u \geq 1, \\ -\infty & \text{if } 0 \leq u < 1. \end{cases}$$

Maximization of Θ gives $\max_{u \geq 1}(-4u)$. The maximum is attained at the smallest u, i.e. at $\bar{u} = 1$, and is equal to $\Theta(\bar{u}) = -4 = f((\pm 2, 0))$ from 5a). Hence, no duality gap.

b) See the book, Theorem 5 and Corollary 1, pp.264–265.