LUNDS TEKNISKA HÖGSKOLA
MATEMATIK

TENTAMENSSKRIVNING
Lösningar till Linjär Algebra
2014–10–27 kl 08–13

1. We list below two alternative solutions for this problem.
*S*olution.
Note that the line ℓ has direction vector $\boldsymbol{u} = (4,0,-7)$. We first pick a point Q on the line ℓ and construct the vector $\overrightarrow{QP}$ from the line to the point P. Then we project that vector $\overrightarrow{QP}$ onto the vector $\boldsymbol{u}$. Lets call that projection vector $\boldsymbol{v}$ and note that this vector is on the line ℓ. Therefore the vector from the point P and orthogonal to the line ℓ is $\boldsymbol{w} = \overrightarrow{QP} - \boldsymbol{v}$. Thus the distance is simply found by computing the length of $|\boldsymbol{w}|$.

All of the above is now computed here. We pick, without loss of generality, $Q = (1,2,3)$. As a result $\overrightarrow{QP} = (11,4,-3), \boldsymbol{v} = (4,0,-7)$ and therefore $\boldsymbol{w} = (7,4,4)$. And therefore the shortest distance is $|\boldsymbol{w}| = \sqrt{81} = 9$.
*A*lternative Solution.
We begin by computing the distance between the point P and every other point on that line. So in effect it is a general distance formula that must be computed.

The distance formula between the point P and a general point (x,y,z) on the line ℓ is given by,

$$\sqrt{(12-x)^2 + (6-y)^2 + z^2}.$$

Every point (x,y,z) on the line ℓ is represented by $x = 1+4t, y = 2$ and $z = 3-7t$ for $t \in R$. Thus the distance is

$$\sqrt{(12-1-4t)^2 + (6-2)^2 + (3-7t)^2}.$$

This after simplification gives the distance to be $\sqrt{65t^2 - 130t + 146}$ for $t \in R$.

Now to compute the shortest distance we must find the minimum of the parabola $65t^2 - 130t + 146$. The minimum of the parabola is 81 and is achieved for $t = 1$. As a result the shortest distance is $\sqrt{81} = 9$.

2. Let's call this system $A\boldsymbol{x} = \boldsymbol{b}$. To show that $(1,0,0)$ is a solution just substitute $x = 1, y = 0, z = 0$ into the system above and show that it works.

So now we have shown that there exists at least one solution although it is possible that more could exist. One easy and fast way to know which is the case is to compute the determinant of the given matrix A. According to the fundamental theorem of algebra from our textbook if $\det A \neq 0$ then there exist only one solution. Otherwise there exists infinite solutions. Computing the determinant gives $\det A = a(a-1)$.
Case $a \neq 0$ **and** $a \neq 1$. Then the $\det A \neq 0$ and there exists a single solution $(1,0,0)$. Otherwise for $a = 0$ or $a = 1$ there exist infinite solutions which we will describe completely below by undertaking Gauss elimination in each case.
Case $a = 0$. The system becomes

$$\begin{cases} 0x + \ \ y + 4z &= 0 \\ \ \ x + \ \ y + 0z &= 1 \\ 2x + 2y + 0z &= 2 \end{cases} \text{ and after Gauss elimination} \ldots \begin{cases} \ \ x + \ \ y + 0z &= 1 \\ 0x + \ \ y + 4z &= 0 \\ 0x + 0y + 0z &= 0. \end{cases}$$

The infinite solution set is expressed through a parameter. Without loss of generality we let one of the variables, say $z = t$, for $t \in R$ and solve the above for x and y. So the solution is $(x, y, z) = (1+4t, -4t, t)$ for $t \in R$.
Case $a = 1$. The system becomes

$$\left\{\begin{array}{rcl} x+\ \ y+4z & = & 1 \\ x+\ \ y \quad & = & 1 \\ 2x+2y+z & = & 2. \end{array}\right. \text{ and after Gauss elimination} \ldots \left\{\begin{array}{rcl} x+\ \ y+4z & = & 1 \\ 0x+0y-4z & = & 0 \\ 0x+0y-7z & = & 0. \end{array}\right.$$

Here it is clear that $z = 0$ solves both the 2nd and 3rd equations. In order to solve the first equation however we need to use a parameter. Without loss of generality we let $y = t$ for $t \in R$ and therefore $x = 1-t$. Thus $(x, y, z) = (1-t, t, 0)$ for $t \in R$.

3. According to the figure $\boldsymbol{u} = (2,-1), \boldsymbol{v} = (1,2), \boldsymbol{w} = (3,1), F(\boldsymbol{u}) = (-2,-2), F(\boldsymbol{v}) = (-1,4)$ and $F(\boldsymbol{w}) = (-3,1)$.

a) A mapping is linear if and only if $F(\boldsymbol{a}+\boldsymbol{b}) = F(\boldsymbol{a}) + F(\boldsymbol{b})$ and $F(\lambda\boldsymbol{a}) = \lambda F(\boldsymbol{a})$.
We begin by first examining the rule $F(\boldsymbol{u}+\boldsymbol{v}) = F(\boldsymbol{u}) + F(\boldsymbol{v})$. Note the right hand side $F(\boldsymbol{u})+F(\boldsymbol{v}) = (-2,-2)+(-1,4) = (-3,2)$. On the left hand side $F(\boldsymbol{u}+\boldsymbol{v}) = F((2,-1)+(1,2)) = F(3,1) = F(\boldsymbol{w}) = (-3,1)$ since luckily $\boldsymbol{w} = (3,1)$. But that shows that $F(\boldsymbol{u}+\boldsymbol{v}) \neq F(\boldsymbol{u})+F(\boldsymbol{v})$ and therefore the mapping F is not linear! We therefore are done and do not need to check the other rule $F(\lambda\boldsymbol{a}) = \lambda F(\boldsymbol{a})$.

b) We let the mapping F be represented by the following 2x2 matrix A: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
We know that for a linear mapping F the following must hold $A\boldsymbol{u} = F(\boldsymbol{u})$ and $A\boldsymbol{v} = F(\boldsymbol{v})$. These two equations are shown below

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \text{ and } \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

which can be solved as a system of four equations with four unknowns. The solution to this system is $a = -1, b = 0, c = 0, d = 2$ and therefore $A = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$.
Now we can figure out what is $\boldsymbol{w}$ so that $A\boldsymbol{w} = F(\boldsymbol{w})$. In other words let $\boldsymbol{w} = (w_1, w_2)$ and solve

$$\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

The solution $\boldsymbol{w} = (3, 1/2)$.

4. *C*omputing $\hat{e}_1$. Since $\hat{e}_1$ is orthogonal to the plane $x_1 + x_2 + x_4 = 0$ then $\hat{e}_1$ must have the same direction as the normal to that plane. So we pick the diretion for $\hat{e}_1$ to be $(1,1,1)$. To make this vector of length 1 we just need to divide it by its length $\sqrt{3}$ and therefore $e_1 = \frac{1}{\sqrt{3}}(1,1,1)$.
*C*omputing $\hat{e}_3$. Vector $\hat{e}_3$ is parallel with the plane $3x+2y+z = 37$. That is equivalent to $\hat{e}_3$ being orthogonal to the normal of the plane $\eta = (3,2,1)$. At the same time $\hat{e}_3$ must also be orthogonal to the vector $\hat{e}_1$ we just constructed above. We have a tool to construct such a vector that is orthogonal to two other vectors: the cross product. We therefore just compute $\eta \times (1,1,1) = (-1, 2-1)$. Then after normalizing this vector we get $\hat{e}_3 = \frac{1}{\sqrt{6}}(-1,2,-1)$.
*C*omputing $\hat{e}_2$. Finally the last vector $\hat{e}_2$ must be orthogonal to both of the previously constructed vectors $\hat{e}_1$ and $\hat{e}_3$. This vector can easily be constructed by computing the

cross-product $\hat{e}_3 \times \hat{e}_1$. For computational purposes it is easiest to compute the cross-product for the vectors before we made them length one. This will provide us with the correct orientation and then afterwards we can make the length of that vector one as well. As a result we compute the cross-product

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 3e_1 + 0e_2 - 3e_3.$$

So the direction of the third vector which will be orthognonal to $\hat{e}_1$ and $\hat{e}_2$ is $(3,0,-3)$. Clearly this vector does not have length 1. To fix this we compute its length $\sqrt{18}$. Thus $\hat{e}_2 = \frac{1}{\sqrt{18}}(3,0,-3)$.

Last, but not least, you should check whether your three vectors are in fact a *positive oriented* base. To check that you should compute the determinant of $\hat{e}_1, \hat{e}_2$ and $\hat{e}_3$. If that determinant is positive then this is a positively oriented bases. Otherwise it is not and in order to fix it we must reverse one of these vectors. If you check you will note that the determinant is 18 and therefore we do not need to change the orientation of any one of those vectors.

To express the plane $x_1 + x_2 + x_3 = 0$ with respect to the new basis vector we only need to represent its normal $\eta = (1,1,1)$ with respect to that new basis. One way to do this is to use the change of basis formula

$$\begin{pmatrix} \text{Old} \\ \text{Basis} \end{pmatrix} \begin{pmatrix} \text{Old} \\ \text{Coordinates} \end{pmatrix} = \begin{pmatrix} \text{New} \\ \text{Basis} \end{pmatrix} \begin{pmatrix} \text{New} \\ \text{Coordinates} \end{pmatrix}$$

which we denote with the following: $D\eta = G\hat{\eta}$ for $\hat{\eta} = (a,b,c)$. Computationally,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & -1 \\ 1 & 0 & 2 \\ 1 & -3 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Inverting the matrix G above (or solving by Gauss) we can obtain the solution for $\hat{\eta} = G^{-1}D\eta = (1,0,0)$. Thus the equation of the plane with respect to the basis $\hat{e}_1, \hat{e}_2, \hat{e}_3$ is $x_1 = 0$ (note that the point $(0,0,0)$ in the old basis is the same in the new basis).

5. Based on the theory for eigenvalues and eigenvectors the matrix S consists of the eigenvectors of the matrix A written as columns for that matrix. Since there are three distinct eigenvalues (shown in the diagonal of matrix D) then there exist three distinct eigenvectors to be found. Let's call them $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$.

We do not need to compute the eigenvalues since they are already provided in the diagonal of matrix D according to the theory of diagonalization. But we do need to compute each of the three eigenvectors. We begin to do so for $\lambda_1 = 1$. We need to solve the system

$$\begin{pmatrix} -2 & 3 & 2 \\ 1 & -3 & -1 \\ -1 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Which we can solve with any one of many methods (i.e. Gaussian elimination). The solution is $\boldsymbol{v}_1 = (x_1, x_2, x_3)^T = (1,0,1)^T t$ for $t \in R$ and $t \neq 0$.

The eigenvector corresponding to the eigenvalue $\lambda_2 = -2$ can be found by solving,

$$\begin{pmatrix} 1 & 3 & 2 \\ 1 & 0 & -1 \\ -1 & 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Which we can solve with any one of many methods (i.e. Gaussian elimination). The solution is $\boldsymbol{v}_2 = (x_1, x_2, x_3)^T = (1, -1, 1)^T t$ for $t \in R$ and $t \neq 0$.

Finally the eigenvector corresponding to the eigenvalue $\lambda_3 = 0$ can be found by solving,

$$\begin{pmatrix} -1 & 3 & 2 \\ 1 & -2 & -1 \\ -1 & 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Which we can solve with any one of many methods (i.e. Gaussian elimination). The solution is $\boldsymbol{v}_3 = (x_1, x_2, x_3)^T = (-1, -1, 1)^T t$ for $t \in R$ and $t \neq 0$. Thus the matrix S is,

$$S = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

We can now also compute the nollrum and nolldimension for A but there is a much easier way. Since there are 3 distinct, real eigenvalues then there exists three linearly independent eigenvectors as found above which span the whole space $\mathbf{R}^3$. Note that only the vector $\boldsymbol{v}_3$ corresponding to eigenvalue $\lambda_3 = 0$ is send to 0 since $A\boldsymbol{v}_3 = \lambda_3 \boldsymbol{v}_3 = 0$. So the only vectors from the space $\mathbf{R}^3$ which are send to 0 are the linear combinations of the vector $\boldsymbol{v}_3$. As a result the nolldimension is 1 and therefore the rank is $3 - 1 = 2$. Furthermore nollrum is described from the corresponding eigenvector $\boldsymbol{v}_3$. No need to compute anything if you do not want to. Just argue as above directly from theory.

6. Let $\boldsymbol{a} = \frac{1}{3}(1, -2, 2), \boldsymbol{b} = \frac{1}{3}(2, 1, -2), \boldsymbol{c} = \frac{1}{9}(4, 7, -4)$. We let the unknown vector be $\boldsymbol{v} = (x, y, z)$ and solve the system of equations,

$$\cos(\theta) = \boldsymbol{a} \cdot \boldsymbol{v}/|\boldsymbol{a}||\boldsymbol{v}|, \quad \cos(\theta) = \boldsymbol{a} \cdot \boldsymbol{v}/|\boldsymbol{a}||\boldsymbol{v}|, \quad \cos(\theta) = \boldsymbol{a} \cdot \boldsymbol{v}/|\boldsymbol{a}||\boldsymbol{v}|.$$

Substitute the vectors above produces the following system of equations,

$$\begin{cases} x - 2y + 2z & = 3\cos(\theta) \\ 2x + y + 2z & = 3\cos(\theta) \\ 4x + 7y - 4z & = 9\cos(\theta) \end{cases}$$

After Gauss elimination we obtain,

$$\begin{cases} x - 2y + 2z & = 3\cos(\theta) \\ \phantom{x - {}} 5y - 6z & = -3\cos(\theta) \\ z & = \cos(\theta) \end{cases}$$

The solution is $x = \frac{11}{5}\cos(\theta), y = \frac{3}{5}\cos(\theta), z = \cos(\theta)$ or $(x, y.z) = \frac{(11,3,5)}{5}\cos(\theta)$. In general any vector of the form $(11, 3, 5)t$ for $t \in R$ is correct.