LUNDS TEKNISKA HÖGSKOLA
MATEMATISKA INSTITUTIONEN

LÖSNINGAR
OPTIMERING
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1. Let's calculate the gradient and the Hessian at the point $(x, y, z) = (0, 1, 3)$

$$\nabla f = \begin{bmatrix} x \\ (y^2 - z) \cdot 2y \\ z - y^2 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 2 \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6y^2 - 2z & -2y \\ 0 & -2y & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 1 \end{bmatrix}.$$

a) It is necessary for positive-semidefiniteness that $\det(H_k) \geq 0$ (see page 360 after Example 5). Since $\det(H) = -4 < 0$, H is not positive-semidefinite. Hence, the function cannot be convex (Theorem 13, page 216).

b)

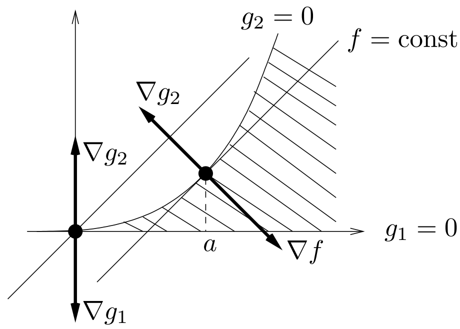
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} - H^{-1}\nabla f = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/4 & -1/2 \\ 0 & -1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Since $f \geq 0$ everywhere and $f(0, 1, 1) = 0$, we have arrived to a global minimum. It is easy to see that there are many global minima here: $(0, t, t^2)$, $t \in \mathbb{R}$.

2. Define $f(x, y) = x - y$, $g_1(x, y) = -y$, $g_2(x, y) = y - x^3$, and calculate

$$\nabla f = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \nabla g_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad \nabla g_2 = \begin{bmatrix} -3x^2 \\ 1 \end{bmatrix},$$

a) Draw the set $g_1 \leq 0$ and $g_2 \leq 0$. We get



- $(0, 0)$ is a CQ point as ∇g_1 and ∇g_2 have the opposite directions.
- (a, a^3) is a KKT point as ∇f and ∇g_2 have the opposite directions.

To find $a > 0$: solve

$$\det[\nabla f \ \nabla g_2] = \det \begin{bmatrix} 1 & -3a^2 \\ -1 & 1 \end{bmatrix} = 0 \quad \Rightarrow \quad a = \frac{1}{\sqrt{3}}.$$

b) The CQ point is a local minimum (feasible points are from one side of the level curve), the KKT point is a sadle point (feasible points are from both sides of the level curve). The global minimum does not exist (one can move the level curve against ∇f up to infinity without loosing contact with feasible points).

3. a) A form of the dual LP problem is

$$\min\,(2y_1 + y_2 - y_3) \qquad \text{subject to} \qquad \begin{cases} y_2 + y_3 & \geq \; 0, \\ y_1 - 3y_2 + y_3 & \geq \; 1, \\ y_1 + 2y_2 - y_3 & \geq \; 2, \\ y_1 \geq 0, \; y_2 \leq 0, \; y_3 \text{ free.} \end{cases}$$

CSP gives:

- the primal slack $s_2 = -4 \neq 0 \Rightarrow y_2 = 0$,
- the primal $x_1 = 1 \neq 0 \Rightarrow y_2 + y_3 = 0 \Rightarrow y_3 = 0$,
- the primal $x_3 = 2 \neq 0 \Rightarrow y_1 + 2y_2 - y_3 = 2 \Rightarrow y_1 = 2$.

It gives the dual feasible solution $y = (2, 0, 0)$ and min $= 4 =$ max from 3a, hence, both are optimal.

b) We rewrite the first system to get one of the Farkas canonical forms

$$\begin{cases} \begin{bmatrix} A \\ B \\ -B \end{bmatrix} x & \leq \; 0, \\ c^T x & > \; 0. \end{cases}$$

Then Farkas' theorem gives us the alternative system as

$$\begin{cases} \begin{bmatrix} A^T & B^T & -B^T \end{bmatrix} \begin{bmatrix} y \\ v \\ w \end{bmatrix} & = \; c, \\ y, v, w & \geq \; 0. \end{cases}$$

Now introduce the new variable $z = v - w$, which is no longer sign-definite.

4. a) It is the least square optimization. To solve it we will solve the normal equation $A^T A x = A^T b$ which in this case looks like

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} x = \begin{bmatrix} 6 \\ -1 \end{bmatrix} \quad \Leftrightarrow \quad x = \begin{bmatrix} 2 \\ -\frac{1}{2} \end{bmatrix}.$$

b) We have

$$\|Ax - b\|^2 = 3x_1^2 + 2x_2^2 - 12x_1 + 2x_2 + 14.$$

Let us denote $x_1 = x$, $x_2 = y$. So we need to solve the problem

$$\min(3x^2 + 2y^2 - 12x + 2y) \qquad \text{subject to } x \geq 0, \; y \geq 0.$$

The problem is convex (prove it!), hence it is sufficient to find a KKT point. The KKT system is

$$\begin{cases} 6x - 12 - u_1 & = \; 0, \\ 4y + 2 - u_2 & = \; 0, \\ u_1 x & = \; 0, \\ u_2 y & = \; 0, \\ x \geq 0, \; y \geq 0. \end{cases}$$

One solution is, for example, $(x, y) = (2, 0)$ with $u_1 = 0$, $u_2 = 2$.

Another (longer) solution: Use the KKT/CQ necessary condition.

5. a) Denote $\gamma = \min_{x \in S} f(x)$. The set of all solutions can be described as

$$\{x \in S\colon\, f(x) \le \gamma\}$$

which is convex (prove it similar to Theorem 7, page 210).

Another solution: Prove convexity by definition.

b) Take two points $P = (0,0,0)$ and $Q = (0,1,1)$. We have $f(P) = f(Q) = 0$ and

$$f(P/2 + Q/2) = f(0, 1/2, 1/2) = \frac{1}{4} > 0 = f(P)/2 + f(Q)/2.$$

Hence, the function is not convex as it does not satisfy the definition for P, Q and $\lambda = 1/2$.

c) Define the set $S = \{x > 0,\ y > 0,\ z > 0\}$. It is convex as an intersection of three half-spaces. The function $f(t) = 1/t$ is convex for $t > 0$ (as $f'' > 0$). The function $g(x,y,z) = x + y$ is affine and takes only positive values on S. Therefore, the superposition

$$f(g(x,y,z)) = \frac{1}{x+y}$$

is convex by Lemma 2, page 211. Similarly, two other functions are convex.

Finaly, the sum of convex functions is convex, and the convex function less than or equal to 1 on S defines, thus, a convex set.

6. a) Let us denote $x_1 = x$, $x_2 = y$ again. As $y \ge 0$ is taken as an *implicit* constraint, the Lagrange function is

$$L(x,y,u) = 3x^2 + 2y^2 - 12x + y - ux = 3x^2 - (12+u)x + \underbrace{2y^2 + y}_{\ge 0}.$$

The expression $2y^2 + y$ is clearly non-negative on $X = \{y \ge 0\}$, then the minimum of L with respect to $y \ge 0$ is attained at $y = 0$. To minimize L further with respect to $x \in \mathbf{R}$ we notice that it is a convex function in x, and the minimum is attained at the stationary point

$$6x - 12 - u = 0 \qquad \Rightarrow \qquad x = \frac{12+u}{6} = 2 + \frac{u}{6}.$$

Substitution of this x and $y = 0$ to L gives the minimum as the following dual function

$$\Theta(u) = -12 - 2u - \frac{u^2}{12}, \quad u \ge 0.$$

Since $\Theta' = -2 - \frac{u}{6} < 0$ for $u \ge 0$, the dual function is decreasing, thus, the maximum is attained at $\bar{u} = 0$ where $\Theta(\bar{u}) = -12$. Taking the candidate $(\bar{x}, \bar{y})$ from the minimization of L above

$$(\bar{x}, \bar{y}) = (2 + \bar{u}/6, 0) = (2, 0)$$

we get the corresponding value of $f(\bar{x}, \bar{y}) = 12 - 24 = -12$ to be equal to $\Theta(\bar{u})$. It means that we have no duality gap, and the point $(2,0)$ is the global minimum.

b) See Theorem 5 on page 264 and Corollary 1 on page 265.