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Answers and comments only for re-exams. No complete solutions.

1. a) See the book, page 348.

b) Since $f(x) = x^T Hx = (x_1 + x_2)^2 + x_3^2 \geq 0$ and $f(1, -1, 0) = 0$ the matrix is positive semidefinite and not positive definite.

c) Farkas $\Rightarrow$ $Hy = c$, $y \geq 0$ has a solution $\Leftrightarrow$ $c_1 = c_2 \geq 0$, $c_3 \geq 0$.

2. a) $(0, 0)$ feasible $\Rightarrow$ add $f(x, y) = (x + y)^2 + (y - 1)^2 - 1 \leq f(0, 0) = 0$ to the constraints. The new set is compact $\Rightarrow$ min exists by Weierstrass.

b) $(x_1, y_1) = (-1, 1)$. Not feasible. (The global minimum without constraints.)

c) Adding the penalty gives $g(x, y) = f(x, y) + \alpha(\max(-x, 0)^2 + \max(-y, 0)^2)$. At $(-1, 1)$ the second penalty term is zero. One Newton's step gives for $\alpha \to +\infty$

$$(x_2, y_2) = (-1/(1 + 2\alpha), (1 + \alpha)/(1 + 2\alpha)) \to (0, 1/2)$$

which is the optimal point in the constrained optimization.

3. a) The optimal point is $(0, -1)$.

b)
$$(P): \ \min(2x_1 + x_2) \quad \begin{cases} -x_1 & \geq -4, \\ \quad\quad x_2 & \geq -2, \\ x_1 + x_2 & \geq -1, \\ x_1 \geq 0, \ x_2 \text{ fri.} \end{cases} \quad \Rightarrow \quad (D): \ \max(-4y_1 - 2y_2 - y_3) \quad \begin{cases} -y_1 + y_3 & \leq 2, \\ y_2 + y_3 & = 1, \\ y_1, \ y_2, \ y_3 \geq 0. \end{cases}$$

CSP gives $y_1 = 0$ and $y_2 = 0$, thus $y_3 = 1$.

c) Canonical form: write all inequalities as "$\leq$ positive", add three slack variables and change $x_1 = z_1$, $x_2 = z_2 - z_3$ where all $z_k \geq 0$. The initial BFS = the slack variables, then one step gives the minimum.

4. a) $\det(\nabla^2 f(x, y)) = -(x^{-2} - y^{-2})^2 \Rightarrow \nabla^2 f$ not pos.-semidef. $\Rightarrow$ f not convex.

b) $\frac{x}{y} + \frac{y}{x} \leq 4 \Leftrightarrow x^2 + y^2 \leq 4xy \Leftrightarrow (y - 2x)^2 \leq 3x^2 \Leftrightarrow |y - 2x| \leq |\sqrt{3}x| \Leftrightarrow -\sqrt{3}x \leq y - 2x \leq \sqrt{3}x \Rightarrow$ convex.

c) Rewrite $G(x) = e^{F(x)\ln(1+F(x))} = e^{H(F(x))}$ where $H(t) = t\ln(1 + t)$. $F(x)$ convex and $H(t)$ convex and growing for $t \geq 0$ (derivate to see it) $\Rightarrow$ $J(x) = H(F(x))$ convex. Since also e^t convex and growing $\Rightarrow$ $e^{J(x)} = G(x)$ convex.

5. a) $(1, 0)$ feasible $\Rightarrow$ add $f(x, y) = 2(y - x)^2 + x^2 \leq f(1, 0)$ to the constraints. The new set is compact $\Rightarrow$ min exists by Weierstrass. No CQ points. KKT system linear $\Rightarrow$ $(x, y) = (0, 0)$ (impossible) or $\det = 0 \Rightarrow u = 2$ ($u = -1$ impossible). KKT points $(x, y) = \pm(2/\sqrt{3}, 1/\sqrt{3})$ are the minimum points, min $= 2$.

b) $\Theta(u) = u$ for $0 \leq u \leq 2$, otherwise $-\infty$. Maximum is $\Theta(2) = 2$. No duality gap.

6. See the book, pages 135–137 and pages 73–74 respectively.