LUNDS TEKNISKA HÖGSKOLA
MATEMATIK

TENTAMENSSKRIVNING
OPTIMERING
2011-12-20 kl 08-13

Students may use the formulae sheet and a pocket calculator. All solutions should be properly justified.

1. a) Give the definition of a convex set in $\mathbf{R}^n$. (0.2)

b) For the set $S = \{(0,0),(2,2),(1,0),(1,-1)\}$, sketch the convex hull $H(S)$ in the xy-plane and provide its analytical description by linear inequalities. The number of inequalities should be as small as possible. (0.3)

c) Give the definition of a separating hyperplane for two sets A and B in $\mathbf{R}^n$. (0.2)

d) Find the equation for a separating hyperplane for the set $H(S)$ from 1b) and the point $(0,-1)$. (0.3)

2. We apply Newton's method to the function $f(x,y) = (x+1)^2 + (x+y)e^{x+y}$ in $\mathbf{R}^2$ starting at $(x_0, y_0) = (0,0)$ and obtain the sequence $\{(x_k, y_k)\}_{k\geq 0}$.

a) Calculate the first iteration and show that $(x_1, y_1) = (-1, 0.5)$. (0.3)

b) Suppose that after k iterations we get $(x_k, y_k) = (-1, \mu)$. Calculate the next iteration and show that $(x_{k+1}, y_{k+1}) = (-1, \frac{\mu^2}{1+\mu})$. (0.4)

c) Combine the results of 2a) and 2b) and show that the sequence (x_k, y_k) converges. What is the limit? Is the limit the global minimum? Does the global minimum exist? (0.3)

3. We apply the Simplex method to the LP problem

$$\max\,(2x_1 + 4x_2 + 5x_3) \quad \text{subject to} \quad \begin{cases} x_1 + x_2 + x_3 & = & 5, \\ -x_1 + 2x_2 + x_3 & \leq & 2, \\ x_1 + 3x_2 + 2x_3 & \leq & 8, \\ x_1,\ x_2,\ x_3 & \geq & 0. \end{cases}$$

a) To initialize the Simplex method, pick an initial basic feasible solution of your choice. What is the corresponding extreme point? (0.3)

b) Proceed with the Simplex method and solve the problem. (0.3)

c) State the dual LP problem and solve it by the CSP[1]. (0.4)

Please turn over

[1]CSP stands for Complementary Slackness Principle

4. a) Prove that the sum of two convex functions is a convex function. (0.2)

b) To solve the problem $\min f(x)$ subject to $g_k(x) \le 0$, $k = 1, \ldots, m$ the penalty function method can be used. Show that the modified function appearing in the method

$$F(x) = f(x) + \mu \sum_{k=1}^{m} \max\{0, g_k(x)\}^2, \qquad \mu \ge 0,$$

is convex on $\mathbf{R}^n$ if the functions f and all g_k are convex there. (0.4)

c) Find all $\alpha \in \mathbf{R}$ such that the function $f(x, y) = (xy)^\alpha$ is concave on $\{(x, y) \in \mathbf{R}^2 \mid x > 0, y > 0\}$. (0.4)

5. We would like to solve the problem

$$\min(x^2 + 2xy + 2y^2) \quad \text{subject to } 2y + x^2 \ge 1,\ x \ge 0.$$

a) Show that the minimum exists. (0.2)

b) Show that the constraints satisfy the CQ condition (i.e. there are no CQ points). You may prove it either algebraically or geometrically (in both cases, a convincing explanation is necessary). (0.2)

c) Calculate KKT points and solve the problem. (0.4)

d) Sketch the constraints in the xy-plane and illustrate graphically that the point $(0, 0.5)$ satisfies the necessary condition for local minimum (in terms of ∇f). (0.2)

6. We would like to solve the modified problem from **5**

$$\min(x^2 + 2xy + 2y^2) \quad \text{subject to } 2y + x^2 \ge 1,\ x \ge 0,\ y \ge 0,$$

by the duality principle.

a) Taking $X = \{x \ge 0, y \ge 0\}$, calculate the dual function Θ. (Hint: minimize with respect to x-variable first.) (0.3)

b) Solve the dual problem and then solve the primal problem by showing no duality gap. (0.2)

c) Prove the theorem you used in 6b), i.e. prove that if there exist feasible $\bar{x}$ and $(\bar{u}, \bar{v})$ such that $f(\bar{x}) = \Theta(\bar{u}, \bar{v})$ then $\bar{x}$ is the global minimum for the primal problem. (0.5)

Merry Christmas, Folks!