LUND UNIVERSITY

Written Examination
Discrete Mathematics
Friday, May 27, 2011

Centre for Mathematical Sciences
Mathematics, Faculty of Science

No aids are allowed. Use only the department's paper and write only on one side of each sheet. Fill in the cover form completely and write your initials on each sheet. Give clear and short motivations to your solutions.

1. The characteristic polynomial associated with the homogeneous equation $a_{n+2} + 2a_{n+1} - 3a_n = 0$ is $P(r) = r^2 + 2r - 3$, with roots $r_1 = 1$ and $r_2 = -3$. Therefore, the general solution of the homogeneous equation is

$$a_n^{(h)} = \alpha + \beta(-3)^n$$

In order to find a particular solution of the non-homogeneous equation, we try with a function of the formula

$$a_n^p = (an + b)2^n$$

After some computations, we obtain $a = 3/5$ and $b = -26/25$ which offers the solution

$$a_n = \alpha + \beta(-3)^n + \Big(\frac{3}{5}n + -\frac{26}{25}\Big)2^n$$

Finally, the solution with the given initial conditions is

$$a_n = 1 + \frac{1}{25}(-3)^n + \Big(\frac{3}{5}n + -\frac{26}{25}\Big)2^n$$

2. We need to calculate the coefficient of x^{14} in the following generating function

$$(x + x^2 + \cdots + x^9)(x^5 + x^7 + \cdots + x^{13})(1 + x + x^2 + \cdots + x^8)$$

We notice that such coefficient coincides with the coefficient of x^{14} in the function

$$f(x) = (x + x^2 + \cdots)x^5(1 + x^2 + \cdots)(1 + x + x^2 + \cdots)$$

$$= (\frac{1}{1-x} - 1)\frac{x^5}{1-x^2}\frac{1}{1-x} = \frac{x}{1-x}\frac{x^5}{(1-x)(1+x)}\frac{1}{1-x}$$

$$= x^6\frac{1}{(1-x)^3(1+x)}$$

Then, we need to find out the coefficient of x^8 in the function

$$\frac{1}{(1-x)^3(1+x)}$$

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which we decompose as

$$\frac{A}{1+x}+\frac{B}{1-x}+\frac{C}{(1-x)^2}+\frac{D}{(1-x)^3}$$

with $A=1/8$, $B=-11/32$, $C=23/32$ and $D=1/2$. Then, the expression equals

$$A\sum_{k=0}^{\infty}(-1)^k x^k + B\sum_{k=0}^{\infty}x^k + C\sum_{k=0}^{\infty}\left(\begin{array}{c}-2\\k\end{array}\right)x^k + D\sum_{k=0}^{\infty}\left(\begin{array}{c}-3\\k\end{array}\right)x^k$$

and so, the coefficient of x^8 is exactly

$$A+B+C\left(\begin{array}{c}-2\\8\end{array}\right)+D\left(\begin{array}{c}-3\\8\end{array}\right)$$

with $\left(\begin{array}{c}-2\\8\end{array}\right)=\left(\begin{array}{c}9\\8\end{array}\right)$ and $\left(\begin{array}{c}-3\\8\end{array}\right)=\left(\begin{array}{c}10\\8\end{array}\right)$.

3. By the Chinese remainder theorem, we have that the solution is given by

$$x=\sum_i a_i x_i N_i$$

with $N_1=35$, $N_2=21$, $N_1=15$, $a_1=1$, $a_2=2$, $a_3=3$. Now, we need to calculate the factors x_i which satisfy $x_iN_i\equiv 1 \mod n_i$. By the euclidean algorithm, we have $x_1=2$, $x_2=1$ and $x_3=1$ wich finally offers the solution

$$x=2\cdot 35+2\cdot 21+3\cdot 15=157=52$$

4. We use the exclusion/inclusion principle to solve the problem. We denote by c_i the property of begin divisible by i. Then we have that

$$\begin{aligned}N(\bar{c_3}\bar{c_4}\bar{c_5}\bar{c_6}) &= 1000-(N(c_3)+N(c_4)+N(c_5))\\ &+\sum_{i\neq j}N(c_ic_j)-N(c_3c_4c_5)\end{aligned}$$

Now

$$N(c_3)=[1000/3]=333,\ N(c_4)=250,\ N(c_5)=200,$$
$$N(c_3c_4)=[1000/12]=83,\ N(c_3c_5)=66,\ N(c_4c_5)=50$$

and

$$N(c_3c_4c_5)=16$$

Then,

$$N(\bar{c_3}\bar{c_4}\bar{c_5}\bar{c_6})=1000-783+199-16=400$$

5. Let's assume that the system has a solution x. Then, taking classes modulo 6, the second equation implies that $2x\equiv 2 \mod 6$. Now the difference between second and first equation implies $x\equiv -3 \mod 6\equiv 3 \mod 6$ which is contradictory with first equation $x\equiv 5 \mod 6$.

The example does not contradict Chinese remainder theorem since it does not satisfy its hypotheses. In particular, the numbers 6 and 15 are not relatively prime since $\gcd(6,15)=3$.

6. The code words defined by the generator matrix are

$$C = E(W) = \{000000, 001110, 010101, 011011, 100011, 101000, 110110, 111000\}$$

We notice that C is a group code and that the minimum weight of the non-zero elements of C is 3. Then, we deduce that the separation of this linear code is 3 and then, we can detect up to double errors or correct up to single errors.

None of the received word are a code words. We calculate the syndrome of the first received word and obtain 100. Its corresponding coset leader is then 000100 (it coincides with the fourth column of H) and so, it's associated with the code word $c = r + 000100 = 110110$. Then, finally, the received word can be decoded as $w = 110$.

On the other hand, the syndrome of the second received word is 111 which does not correspond to any coset leader representing a single mistake. Then, it can't be properly decoded.