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OPTIMERING
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1. a) See the book.

b) See Fig. 1. $H(S) = \{(x, y) \,|\, x + y \geq 0, x - y \geq 0, y - 3x + 4 \geq 0\}$.

c) See the book.

d) For example, $y = -x - 1/2$.

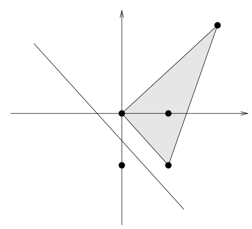

Figure 1: Drawing for Problem 1

2. Calculate the gradient and the Hessian of $f(x, y) = (x + 1)^2 + (x + y)e^{x+y}$

$$\nabla f = \begin{bmatrix} 2(x+1) \\ 0 \end{bmatrix} + (1 + x + y)e^{x+y} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \nabla^2 f = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + (2 + x + y)e^{x+y} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

a) At $(0, 0)$ we get

$$\nabla f = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \nabla^2 f = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}, \quad (\nabla^2 f)^{-1} = \frac{1}{4} \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix},$$

and one Newton step is

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}.$$

b) At $(-1, \mu)$ we get

$$\nabla f = \mu e^{\mu - 1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \nabla^2 f = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + (\mu + 1)e^{\mu - 1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

To calculate the inverse, denote $A = (\mu + 1)e^{\mu - 1}$. Then

$$\nabla^2 f = \begin{bmatrix} 2 + A & A \\ A & A \end{bmatrix}, \quad (\nabla^2 f)^{-1} = \frac{1}{2A} \begin{bmatrix} A & -A \\ -A & 2 + A \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{1}{A} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Straightforward calculation gives

$$d = -(\nabla^2 f)^{-1} \nabla f = -\frac{\mu}{1 + \mu} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} -1 \\ \mu \end{bmatrix} - \frac{\mu}{1 + \mu} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{\mu^2}{1+\mu} \end{bmatrix}.$$

c) The relation from 2ab)

$$y_{k+1} = \frac{y_k^2}{1+y_k}, \qquad y_1 = \frac{1}{2}$$

implies that for all $k \geq 1$

1. $y_k > 0$,
2. $y_{k+1} \leq y_k^2$.

Thus $0 \leq y_{k+1} \leq y_k^2$, $\forall k \geq 1$. Since $y_1 = 1/2$ it gives $0 \leq y_{k+1} \leq 1/2^{2^k} \to 0$. Then $(x_k, y_k) \to (-1, 0)$. Calculation of the minimum: note that (take $t = x+y$)

$$(x+1)^2 + (x+y)e^{x+y} \geq \min_t te^t = -1/e$$

(The last minimum exists (Endim!), the stationary point is $t = -1$). Furthermore, for $(x, y) = (-1, 0)$ we get the equality. Thus, it is the minimum.

3. a) Convert to the canonical form (add slack variables s_2, s_3 to the second and third inequalities and replace max with $-\min(-)$) to get the Simplex tableau

2	4	5	0	0	0
$\boxed{1}$	1	1	0	0	5
-1	2	1	$\boxed{1}$	0	2
1	3	2	0	$\boxed{1}$	8

An initial basis: slack variables gives two obvious candidates. To complete with the third vector we need to pick the pivot element from the first row. The only possibility is to take x_1 since x_2 and x_3 do not have the smallest ratio among all positive elements columnwise (which would result in negative values in b-column and, hence, would not be a feasible point).
After elimination we get

0	2	3	0	0	-10
1	1	1	0	0	5
0	3	2	1	0	7
0	2	$\boxed{1}$	0	1	3

Thus, $x_{start} = (5, 0, 0)$, slack $s_2 = 7$, $s_3 = 3$.

b) Picking the third column (the largest value in the top row) and the third pivot element in the column (the smalles ratio 3) we do one elimination step

0	-4	0	0	-3	-19
1	-1	0	0	-1	2
0	-1	0	1	-2	1
0	2	1	0	1	3

The top row elements are not positive, hence, we have come to the minimum. The minimum is -19, thus, the maximum is 19 and attains at the optimal point $x_{min} = (2, 0, 3)$. Slack value $s_2 = 1 \neq 0$ (will be used in CSP later).

c) The dual problem is

$$\min(5y_1+2y_2+8y_3) \quad \text{subject to} \quad \left\{ \begin{array}{rcl} y_1-y_2+y_3 & \geq & 2, \\ y_1+2y_2+3y_3 & \geq & 4, \\ y_1+y_2+2y_3 & \geq & 5, \\ y_2, y_3 & \geq & 0. \end{array} \right.$$

CSP gives (we take only nontrivial equalities):

$$\left\{ \begin{array}{rcl} x_1(2-y_1+y_2-y_3) & = & 0, \\ x_3(5-y_1-y_2-2y_3) & = & 0, \\ y_2 s_2 & = & 0 \end{array} \right.$$

Since x_1, x_3 and s_2 are not zeros we get

$$\left\{ \begin{array}{rcl} y_2 & = & 0, \\ y_1+y_3 & = & 2, \\ y_1+2y_3 & = & 5. \end{array} \right.$$

that has the solution $y_{opt}=(-1,0,3)$. It is feasible and gives the dual objective function value 19 (which complies with the optimal primal value).

4. a) See the book.

b) The function $h_k(x)=\max\{0,g_k(x)\}$ is convex (as max of convex) and takes non-negative values only, i.e. $h_k(x)\geq 0$. The function $q(t)=t^2$ is convex and growing for $t\geq 0$, thus $q(h_k(x))=\max\{0,g_k(x)\}^2$ is convex. Finally, the sum of convex functions (times the positive μ) is convex.

c) Calculate the Hessian matrix

$$\nabla^2 f(x,y)=\begin{pmatrix} \alpha(\alpha-1)x^{\alpha-2}y^{\alpha} & \alpha^2x^{\alpha-1}y^{\alpha-1} \\ \alpha^2x^{\alpha-1}y^{\alpha-1} & \alpha(\alpha-1)x^{\alpha}y^{\alpha-2} \end{pmatrix}$$

Sylvester criterion for a negative-definite matrix (minus positive-definite!) gives $\alpha(\alpha-1)<0$ and $\alpha^2(\alpha-1)^2-\alpha^4=-\alpha^2(2\alpha-1)>0$ that corresponds to $\alpha\in(0,1/2)$. Finally, the function is concave for $\alpha\in[0,1/2]$ as the limit of negative-definite matrices is a negative-semidefinite matrix.

5. a) The point $(0,1)$ is feasible and gives $f(0,1)=1$, thus, it is sufficient to look at those (x,y) only that satisfy $x^2+2xy+2y^2=(x+y)^2+y^2\leq 1$. Clearly, the last inequality makes the set bounded, hence, compact. The minimum exists by Weierstrass theorem.

b) Calculate first $\nabla g_1=(-2x,-2)^T\neq 0$ and $\nabla g_2=(-1,0)^T\neq 0$. Sketch the set and observe that at all boundary points the gradients of active constraints (both constraints are active at one point only!) are linearly independent (then, of course, they are *positively* linearly independent, which is the CQ condition).

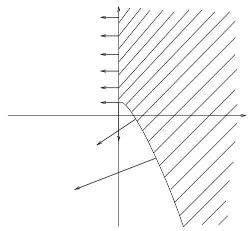

Figure 2: Drawing for Problem 5b

c) State the KKT condition

$$\begin{cases} 2x + 2y - 2xu_1 - u_2 &= 0, & (1)\\ 2x + 4y - 2u_1 &= 0, & (2)\\ u_1(1 - 2y - x^2) &= 0, & (3)\\ u_2 x &= 0, & (4)\\ 2y + x^2 &\geq 1, & (5)\\ x &\geq 0. & (6) \end{cases}$$

Case $u_1 = u_2 = 0$: (1) and (2) give $x = y = 0$. Not feasible (contradicts (5)).

Case $u_2 > 0$: (4) $\Rightarrow x = 0 \Rightarrow y = u_2/2$ (from (1)) and $y = u_1/2$ (from (2)). Thus, $u_1 = u_2 > 0 \Rightarrow 2y = 1$ (from (3) and $x = 0$) $\Rightarrow (0, 1/2)$ is a KKT point ($u_1 = u_2 = 1$).

Case $u_1 > 0, u_2 = 0$: (3) $\Rightarrow 2y = 1 - x^2 \Rightarrow u_1 = 1 + x - x^2$ (from (2)) and $2x + 1 - x^2 - 2x(1 + x - x^2) = 0$ (from (1)). Simplify the last equation $2x^3 - 3x^2 + 1 = 0$. Easy to see that $x = 1$ is a solution. Factorisation (by polynomial division) gives $(x - 1)^2(2x + 1) = 0 \Rightarrow x = 1$ (and then $y = 0$) or $x = -1/2$ (not feasible by (6)). So $(1, 0)$ is another KKT point ($u_1 = 1$).

Thus, there are two KKT points $(1, 0)$ and $(0, 1/2)$. Since $f(1, 0) = 1$ and $f(0, 1/2) = 1/2$ the latter is the minimum point. Answer: min $= 1/2$ at $(0, 1/2)$.

d) This is a geometrical interpretation of the KKT condition. Sketch the set and note that at $a = (0, 1/2)$ both constraints are active. The vector $-\nabla f(a) = (-1, -2)^T$ belongs to the positive cone generated by $\nabla g_1(a) = (0, -2)^T$ and $\nabla g_2(a) = (-1, 0)^T$ (the corresponding coordinates $u_1 = u_2 = 1$).

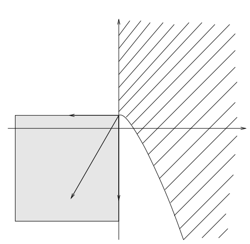

Figure 3: Drawing for Problem 5d

6. a) Writing

$$L(x,u) = x^2 + 2xy + 2y^2 + u(1-2y-x^2) = (1-u)x^2 + 2yx + 2y^2 - 2uy + u$$

makes it obvious that $\inf_{x\geq 0} L = -\infty$ if $1-u<0$ (due to the first term in L). Furthermore, if $1-u \geq 0$ then we get $(1-u)x^2 + 2xy \geq 0$ (since $x \geq 0$, $y \geq 0$) and the expression is equal to zero if $x = 0$. Therefore, the minimum attains at $x = 0$ and $\min_{x\geq 0} L = 2y^2 - 2uy + u = 2(y-u/2)^2 - u^2/2 + u$. Minimization with respect to $y \geq 0$ gives finally (for $y = u/2 \geq 0$)

$$\Theta = \begin{cases} u - u^2/2 & 0 \leq u \leq 1, \\ -\infty & \text{otherwise.} \end{cases}$$

b) The dual problem is

$$\max_{u\geq 0} \Theta(u) = \max_{0\leq u\leq 1} (u - \frac{u^2}{2}).$$

$\Theta'(u) = 1 - u = 0 \Rightarrow u = 1$ is a stationary point (NB: belongs to the interval $[0,1]$). Since Θ is concave, the maximum is attained at $\bar{u} = 1$. Furthermore, $\max_{u\geq 0}\Theta = \Theta(1) = 1/2$. The primal solution *candidate* is $\bar{x} = 0$ and $\bar{y} = \bar{u}/2 = 1/2$. Since $f(0,1/2) = 1/2 = \Theta(1)$ we have no duality gap and then $(0,1/2)$ *is* the solution to the primal problem.

c) For any *feasible* $x \in X$ (i.e. if $g(x) \leq 0$ and $h(x) = 0$) we get

$$f(\bar{x}) = \Theta(\bar{u},\bar{v}) = \inf_{x\in X} L(x,\bar{u},\bar{v}) \leq L(x,\bar{u},\bar{v}) = f(x) + \underbrace{\bar{u}^T g(x)}_{\leq 0} + \bar{v}^T \underbrace{h(x)}_{=0} \leq f(x).$$

Comparing the most left and the most right parts of the inequality, we conclude that the point $\bar{x}$ is the global minimum (by definition of minimum).