LUNDS TEKNISKA HÖGSKOLA
MATEMATIK

TENTAMENSSKRIVNING
OPTIMERING
2016-01-12 kl 14-19

Students may use the formulae sheet and a pocket calculator. All solutions should be properly justified.

1. Consider the function $f(x,y,z) = \frac{1}{2}(x^2 + (y^2 - z)^2)$ and the starting point $P\colon (0,1,3)$.

a) Calculate the Hessian of the function f at the point P. Is it positive-definite? Is the function convex on $\mathbb{R}^3$? (0.5)

b) Make one step of Newton's method from P. What kind of point have you found (stationary point, local/global minimum or something else)? (0.5)

2. Consider the optimization problem

$$\min(x - y) \quad \text{subject to } (x,y) \in S = \{(x,y)\colon 0 \le y \le x^3\}.$$

a) Draw the set S and find all CQ and KKT points graphically. Confirm your result by plotting the corresponding gradients for the objective and the constraint functions. (0.6)

b) Is any of the points found in 2a) a local minimum? A global minimum? (0.4)

3. a) State the dual problem to the following LP problem

$$\max\,(x_2 + 2x_3) \qquad \text{subject to} \quad \begin{cases} x_2 + x_3 & \le & 2, \\ x_1 - 3x_2 + 2x_3 & \ge & 1, \\ x_1 + x_2 - x_3 & = & -1, \\ \text{all } x_k & \ge & 0, \end{cases}$$

and show by CSP that $x = (1,0,2)$ is the optimal solution. (0.5)

b) Let a vector c and matrices A and B (of suitable dimensions) be given. Show that one and only one of the following two systems has a solution (0.5)

- $Ax \le 0$, $Bx = 0$, $c^T x > 0$,
- $A^T y + B^T z = c$, $y \ge 0$.

Please, turn over

4. Consider

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

a) Solve the optimization problem $\min \|Ax - b\|^2$, $x \in \mathbb{R}^2$. (0.4)

b) Solve the optimization problem

$$\min \|Ax - b\|^2 \qquad \text{subject to } x \in \mathbb{R}^2,\ x \geq 0$$

using the KKT method. (0.6)

5. a) Under certain circumstances the optimization problem $\min_{x \in S} f(x)$ has more than one solution. Prove that if the function f is convex and the set S is convex then the set of all optimal solutions is convex too. (0.3)

b) Is the function $\phi(x, y, z) = \max(x, 0) + |y^2 - z|$ convex on $\mathbb{R}^3$? (0.3)

c) Is the following set convex? (0.4)

$$\Omega = \{(x, y, z) \colon \frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{x+z} \leq 1,\ x > 0,\ y > 0,\ z > 0\}.$$

6. a) Solve the problem in 4b) by duality with $X = \{(x_1, x_2) \colon x_2 \geq 0\}$. (0.5)

b) State and prove the sufficient KKT condition for a convex constraint optimization problem. (0.5)

GOOD LUCK!