Suggested solutions. Discrete mathematics. Exam 2009-01-16

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*Problem 1.* The solution of the recursion problem is:

$$a_n = -2^n + (-3)^n + n2^n, \quad n = 0, 1, 2, \ldots.$$

$\square$

*Problem 2.* Encoding function $E$, code words and weight function $w$ for our code $C$ are given by the following table:

| $\mathbf{x}$ | $\mathbf{y}$=E($\mathbf{x}$) | w($\mathbf{y}$) |
|---|---|---|
| 000 | 00000 | 0 |
| 001 | 00101 | 2 |
| 010 | 01111 | 4 |
| 011 | 01010 | 2 |
| 100 | 11010 | 3 |
| 101 | 11111 | 5 |
| 110 | 10101 | 3 |
| 111 | 10000 | 1 |

The separation of a linear code equals the minimum weight of its nonzero code words. Our code $C$ has separation 1. The received word 10101 is a code word in $C$. Indeed, we have that $E(110) = 10101$. The received word 00111 is not a code word in $C$. $\square$

*Problem 3.* We shall use the so-called principle of inclusion and exclusion. Denote by $X$ the set of all integers $n$ such that $1 \leq n \leq 143$. Denote by $X_2$ the subset of $X$ consisting of the even integers in $X$, denote by $X_3$ the subset of integers in $X$ divisible by 3, and denote by $X_7$ the set of integers in $X$ that are divisible by 7. Phrased in these terms we want to calculate the number of integers in the set $X \setminus (X_2 \bigcup X_3 \bigcup X_7)$.

By the above mentioned principle we have that

$$\begin{aligned}|X \setminus (X_2 \bigcup X_3 \bigcup X_7)| &= |X| - (|X_2| + |X_3| + |X_7|)\\ &+ (|X_2 \bigcap X_3| + |X_2 \bigcap X_7| + |X_3 \bigcap X_7|) - |X_2 \bigcap X_3 \bigcap X_7|,\end{aligned}$$

where the symbol $|\cdot|$ is used to indicate the number of elements in a set. A calculation gives that $|X_2| = 71$, $|X_3| = 47$ and $|X_7| = 20$. Next observe that the set $X_2 \bigcap X_3$ consists of all integers in $X$ divisible by 6. Arguing this way we see that $|X_2 \bigcap X_3| = 23$, $|X_2 \bigcap X_7| = 10$, $|X_3 \bigcap X_7| = 6$, and $|X_2 \bigcap X_3 \bigcap X_7| = 3$. We now have that the number of integers $1 \leq n \leq 143$ not divisible by 2, 3 or 7 equals

$$|X \setminus (X_2 \bigcup X_3 \bigcup X_7)| = 143 - (71 + 47 + 20) + (23 + 10 + 6) - 3 = 41.$$

$\square$

*Problem 4.* Recall that equivalence relations correspond to partitions. There is exactly one partition of the set $S = \{1, 2, 3, 4\}$ having a block with 4 or more elements. The number of equivalence relations on $S$ such that every equivalence class has at most 3 elements is

$$\sum_{k=1}^{4} S(4,k) - 1 = 1 + 7 + 6 + 1 - 1 = 14,$$

where $S(n,k)$ are Stirling numbers of the second kind. $\square$

*Problem 5.* The solutions of the congruences are the polynomials $f(x)$ in $\mathbb{Z}_3[x]$ of the form

$$f(x) = 2x^3 + 2x + 1 + (x^2+1)(x^3+2x+2)g(x), \quad g(x) \in \mathbb{Z}_3[x].$$

$\square$

*Problem 6.* We first notice that the generating function

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

for the sequence $\{c_n\}_{n=0}^{\infty}$ is given by

$$f(x) = \frac{1}{1-x}\frac{1}{1-x}\frac{1}{1-x^2} = \frac{1}{(1-x)^2(1-x^2)} = \frac{1}{(1-x)^3(1+x)}.$$

A calculation gives the partial fraction decomposition that

$$f(x) = \frac{1}{2}\frac{1}{(1-x)^3} + \frac{1}{4}\frac{1}{(1-x)^2} + \frac{1}{8}\frac{1}{1-x} + \frac{1}{8}\frac{1}{1+x}.$$

Using the standard power series expansion

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n$$

we have the expansion

$$f(x) = \sum_{n=0}^{\infty} \Big( \frac{1}{2}\binom{n+2}{n} + \frac{1}{4}\binom{n+1}{n} + \frac{1}{8} + \frac{1}{8}(-1)^n \Big) x^n$$

so that

$$c_n = \frac{1}{2}\binom{n+2}{n} + \frac{1}{4}\binom{n+1}{n} + \frac{1}{8} + \frac{1}{8}(-1)^n$$

for $n \geq 0$. A straightforward calculation now gives that

$$c_n = \frac{1}{4}n^2 + n + \frac{7}{8} + \frac{1}{8}(-1)^n$$

for $n \geq 0$. $\square$