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1. a) The function can be written as $f(x,y) = (x-1)^2 + g(xy) - 1$ where $g(t) = e^t - t$. Since $g'(t) = e^t - 1 = 0$ gives $t = 0$ (a stationary point), and g is convex, the least value for g is $g(0) = 1$. Thus $f(x,y) \geq 0$ and $f(1,0) = 0$ gives the global minimum at $(1,0)$.

b) Let us calculate the gradient and the Hessian of f

$$\nabla f(x,y) = \begin{bmatrix} 2x - 2 - y + ye^{xy} \\ -x + xe^{xy} \end{bmatrix}, \qquad H = \begin{bmatrix} 2 + y^2 e^{xy} & -1 + e^{xy} + xye^{xy} \\ -1 + e^{xy} + xye^{xy} & x^2 e^{xy} \end{bmatrix}.$$

The Hessian at $(0,1)$ is not invertible and one cannot do a Newton step from this point.

c) Modify the Hessian with $\epsilon = 1$ to get the positive-definite matrix

$$H + \epsilon I = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \underbrace{- \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 0 \end{bmatrix}}_{=d} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}.$$

The direction d is descent since $d^T \nabla f(0,1) = -1 < 0$. The functional value goes down

$$f(1/2, 1) = -\frac{5}{4} + \sqrt{e} \approx 0.4 < 1 = f(0,1).$$

2. a) The implication is equivalent to (cmp. Exercises 4.14–17)

$$\begin{cases} 2x_1 - x_2 & \leq \ 0 \\ -x_1 + 3x_2 & \leq \ 0 \\ 3x_1 + 5x_2 & > \ 0 \end{cases}$$

having no solution. By Farkas theorem it means that the “dual” system

$$\begin{cases} 2y_1 - y_2 & = \ 3 \\ -y_1 + 3y_2 & = \ 5 \\ y_1, y_2 \geq 0 \end{cases}$$

must have a solution. It is trivial to see that the solution to the linear system is positive

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 14/5 \\ 13/5 \end{bmatrix} \geq 0.$$

b) We can easily see that the solution in this case is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3c_1 + c_2 \\ c_1 + 2c_2 \end{bmatrix}$$

and the cone $y \geq 0$ is shown on Figure 1.

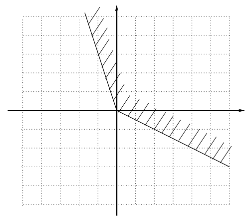

Figure 1: Drawing for Problem 2b

3. a) After converting to the canonical form (adding 3 slack variables and replacing max with $-\min(-)$) we get the following Simplex tableau

2	8	6	0	0	0	0
1	3	1	$\boxed{1}$	0	0	6
3	5	1	0	$\boxed{1}$	0	7
3	1	$\boxed{1}$	0	0	-1	2

An initial basis: slack variables gives two obvious candidates. To complete with the third vector we need to pick the pivot element from the last row. The only two possibilities is to take x_1 or x_3 since others would result in negative values in b-column and, hence, would not be a feasible point. Clearly to pick x_3 is preferable since the pivot element is already 1. After elimination we get

-16	2	0	0	0	6	-12
-2	2	0	1	0	$\boxed{1}$	4
0	4	0	0	1	1	5
3	1	1	0	0	-1	2

Thus, $x_{start} = (0, 0, 2)$, slack $s_1 = 4$, $s_2 = 5$ and $s_3 = 0$. Picking the last column (the largest value in the top row) and the first pivot element in the column (the smalles ratio 4) we do one elimination step

-4	-10	0	-6	0	0	-36
-2	2	0	1	0	1	4
2	2	0	-1	1	0	1
1	3	1	1	0	0	6

The top row elements are not positive, hence, we have come to the minimum. The minimum is -36, thus, the maximum is 36 and attains at the optimal point $x_{max} = (0, 0, 6)$. Non-zero slack values $s_2 = 1$ and $s_3 = 4$ (will be used in CSP later).

b) Let's multiply the third inequality by -1 to get all $\leq$. Then it looks like the dual problem (see the formula sheet) without equalities and free variables. Thus the dual to it is the primal problem without equalities and free variables too, that is

$$\min(6y_1 + 7y_2 - 2y_3) \quad \text{subject to} \quad \begin{cases} y_1 + 3y_2 - 3y_3 & \geq & 2, \\ 3y_1 + 5y_2 - y_3 & \geq & 8, \\ y_1 + y_2 - y_3 & \geq & 6, \\ \text{all } y_k & \geq & 0. \end{cases}$$

CSP gives (we take only nontrivial equalities):

$$\begin{cases} x_3(y_1+y_2-y_3-6) &= 0, \\ y_2 s_2 &= 0 \\ y_3 s_3 &= 0 \end{cases} \quad\Rightarrow\quad \begin{cases} y_2 &= 0, \\ y_3 &= 0, \\ y_1 &= 6, \end{cases}$$

so the solution is $y_{min} = (6,0,0)$. It is feasible and gives the dual objective function value 36 (which complies with the optimal primal value).

4. a) The sets $S_i = \{x \in \mathbf{R}^n : g_i(x) \leq \alpha_i\}$ are convex (see the proof to Th. 7, p. 210) then the set S is convex as the intersection of convex sets (Lemma 2a, p. 121).

b) On the set $\{t < 0\}$ the function $r(t) = -1/t$ is growing ($r'(t) = 1/t^2 > 0$) and convex ($r''(t) = -2/t^3 > 0$). The function g_i is convex and takes negative values only in S_i for $\alpha_i = 0$, thus by Lemma 2 (item 4), p. 211 we get that $f_i(x) = r(g_i(x))$ is convex. Finally, the sum of convex functions (times the positive ϵ) is convex too.

c) Calculate the Hessian matrix

$$\nabla^2 f(x,y) = \begin{pmatrix} 2 & 4y \\ 4y & 4x + 12ay^2 \end{pmatrix}$$

Since the first subdeterminant is $2 > 0$ the 2×2 Hessian is positive-semidefinite iff the determinant is nonnegative (Sylvester), i.e. $\det(H) = 8x + (24a-16)y^2 \geq 0$ for all $x \geq 0 \Leftrightarrow 24a - 16 \geq 0 \Leftrightarrow a \geq 2/3$.

5. Denote $f(x,y) = x^2 - 2xy + 2y$, $g_1(x,y) = y^2 - 2x$, $g_2(x,y) = -y$. The set of constraints $\{g_1 \leq 0, g_2 \leq 0\}$ is unbounded. Let's take the point $(0,0)$, which is feasible, and add the constraint $f(x,y) \leq f(0,0)$ to the set: $x^2 - 2xy + 2y \leq 0 \Rightarrow x^2 - 2xy = x(x-2y) \leq 0 \Rightarrow 0 \leq x \leq 2y$. Thus $y^2 \leq 2x \leq 4y \Rightarrow y^2 - 4y = y(y-4) \leq 0 \Rightarrow 0 \leq y \leq 4$, the new set is bounded in y. Since $0 \leq x \leq 2y \leq 4$, the set is also bounded in x and closed, hence, compact. The Weierstrass theorem gives the existence of the global minimum, and we can apply the KKT condition.
Since ∇g_1 and ∇g_2 are lineary independent everywhere, there are no CQ-points.
To calculate KKT points, we state the KKT condition:

$$\begin{cases} 2x - 2y - 2u_1 &= 0, & (1) \\ -2x + 2 + 2u_1 y - u_2 &= 0, & (2) \\ u_1(y^2 - 2x) &= 0, & (3) \\ u_2 y &= 0, & (4) \\ u_1 \geq 0, u_2 \geq 0, & \\ y^2 \leq 2x, y \geq 0. & \end{cases}$$

If $\underline{u_2 > 0}$ then $y = 0$ (from (4)), $x = u_1$ (from (1)) and $u_1 x = 0$ (from (3)) $\Rightarrow x = u_1 = 0$. One KKT point is $(0,0)$.
If $\underline{u_2 = 0,\ u_1 = 0}$ then $x = 1$ (from (2)) and $y = x = 1$ (from (1)). Another KKT point is $(1,1)$.
If $\underline{u_2 = 0,\ u_1 > 0}$ then adding (1)+(2) gives $2 - 2y + 2u_1 y - 2u_1 = 0 \Rightarrow (u_1 - 1)(y-1) = 0 \Rightarrow y = 1$ or $u_1 = 1$. If $y = 1$ then $x = 1/2$ (from (3)) and $u_1 = -1/2$ (from (1)),

which contradicts $u_1 \geq 0$. If $u_1 = 1$ then replace $2x = y^2$ (from (3)) in (1) to get $y^2 - 2y - 2 = 0 \Rightarrow y = 1 \pm \sqrt{3}$. The only feasible $y = 1 + \sqrt{3} \Rightarrow x = 2 + \sqrt{3}$ (from (3)). The last KKT point is $(2 + \sqrt{3}, 1 + \sqrt{3})$.
Calculating the functional values $f(0,0) = 0$, $f(1,1) = 1$ and $f(2+\sqrt{3}, 1+\sqrt{3}) = -1$ gives the minimum at $(2 + \sqrt{3}, 1 + \sqrt{3})$.

6. a) See Th. 3, p. 298.

b) Since $X = \{y \geq 0\}$ the Lagrange function is

$$\begin{aligned} L(x, y, u) &= x^2 - 2xy + 2y + u(y^2 - 2x) = [\text{complete square in } x] = \\ &= (x - y - u)^2 + (u - 1)(y^2 - 2y) - u^2 = [\text{complete square in } y] = \\ &= (x - y - u)^2 + (u - 1)(y - 1)^2 + 1 - u - u^2. \end{aligned}$$

The first square can be always minimized to zero (putting $x = y + u$). Now if $0 \leq u < 1$ the second term can be done $-\infty$ by letting $y \to +\infty$. If $u \geq 1$ the minimum is clearly at $y = 1$. Therefore, the dual function is

$$\Theta(u) = \begin{cases} 1 - u - u^2 & \text{if } u \geq 1, \\ -\infty & \text{if } 0 \leq u < 1. \end{cases}$$

The dual problem is $\max_{u \geq 1} 1 - u - u^2$. Since the function is decreasing (the derivative is $-1 - 2u < 0$), the largest value is $\Theta(\bar{u}) = -1$ at $\bar{u} = 1$. Taking the primal solution $\bar{x} = 2 + \sqrt{3}$ and $\bar{y} = 1 + \sqrt{3}$ we see that $\Theta(\bar{u}) = f(\bar{x}, \bar{y})$ and then we have no duality gap and the saddle point.