Suggested solutions. Discrete mathematics. Exam 2009-08-26

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Problem 1. The solution of the recurrence problem is:

$$a_n = \frac{45}{64}3^n + \frac{19}{64}(-5)^n + \frac{1}{8}n3^n, \quad n = 0, 1, 2, \ldots$$

$\square$

Problem 2. Encoding function E, code words and weight function w for our code C are given by the following table:

x	**y**=E(**x**)	w(**y**)
000	000000	0
001	001011	3
010	010110	3
011	011101	4
100	101100	3
101	100111	4
110	111010	4
111	110001	3

The separation of a linear code equals the minimum weight of its nonzero code words. Our code C has separation 3. The received word 011101 is a code word in C. Indeed, we have that $E(011) = 011101$. The received word 110101 is not a code word in C. The nearest code word in C to 110101 is 110001. $\square$

Problem 3. We shall use the so-called principle of inclusion and exclusion. Denote by X the set of all integer solutions (x_1, x_2, x_3) of the problem

$$\begin{cases} x_1 + x_2 + x_3 = 20, \\ x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0. \end{cases}$$

Also denote by X_j the subset of X consisting of solutions (x_1, x_2, x_3) in X with $x_j \geq 9$ $(j = 1, 2, 3)$. Phrased in these terms we want to calculate the number of elements in the set $X \setminus \bigcup_{j=1}^{3} X_j$.

By the above mentioned principle we have that

$$|X \setminus \bigcup_{j=1}^{3} X_j| = |X| - (|X_1| + |X_2| + |X_3|)$$
$$+ (|X_1 \bigcap X_2| + |X_1 \bigcap X_3| + |X_2 \bigcap X_3|) - |X_1 \bigcap X_2 \bigcap X_3|,$$

where the symbol $|\cdot|$ denotes the number of elements in a set. By standard theory we have that $|X| = \binom{20+3-1}{20} = \binom{22}{20} = 231$. Also $|X_1| = |X_2| = |X_3| = \binom{13}{11} = 78$, $|X_1 \bigcap X_2| = |X_1 \bigcap X_3| = |X_2 \bigcap X_3| = \binom{4}{2} = 6$, and $|X_1 \bigcap X_2 \bigcap X_3| = 0$. We now have that the number of solutions (x_1, x_2, x_3) of the given problem equals

$$|X \setminus \bigcup_{j=1}^{3} X_j| = 231 - 3 \cdot 78 + 3 \cdot 6 - 0 = 15.$$

$\square$

Problem 4. It is straightforward to see that the polynomial p has no zero in $\mathbb{Z}_3$. Indeed, we have that $p(0) = p(1) = p(2) = 2 \neq 0$ in $\mathbb{Z}_3$. Since p has degree 3 this gives that $p(x)$ is irreducible in $\mathbb{Z}_3[x]$. A standard result now gives that the quotient ring $R = \mathbb{Z}_3[x]/(p(x))$ is a field. The multiplicative inverse is given by

$$[x^2+1]^{-1} = [2x^2+2x+2].$$

$\square$

Problem 5. Multiplying $f_k(x)$ by y^k and summing over $k \geq 0$ using the binomial theorem we have that

$$\sum_{k=0}^{\infty} f_k(x) y^k = \sum_{n\geq 0}\Big(\sum_{k=0}^{n}\binom{n}{k} y^k\Big)x^n = \sum_{n\geq 0}(1+y)^n x^n = \frac{1}{1-(1+y)x},$$

where the last equality follows by the formula for a geometric series. Another expansion in a geometric series gives that

$$\frac{1}{1-(1+y)x} = \frac{1}{1-x}\frac{1}{1-\frac{xy}{1-x}} = \sum_{k\geq 0}\frac{x^k}{(1-x)^{k+1}}y^k.$$

Comparing coefficients we now see that

$$f_k(x) = \frac{x^k}{(1-x)^{k+1}} \tag{1}$$

for $k = 0, 1, 2, \ldots$ $\square$

Remark. The result of Problem 5 can alternatively be obtained along the following lines. Using the formula $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for binomial coefficients we can derive the recursion formula

$$f_k(x) = \frac{x}{1-x} f_{k-1}(x)$$

for $k = 1, 2, 3, \ldots$. Also $f_0(x) = 1/(1-x)$ by the formula for a geometric series. Solving this recursion problem for $f_k(x)$ leads to formula (1).

Problem 6. Recall that $S(n,k)$ is the number of partitions of a set with n elements into k parts. For this purpose let X be a set with n elements and fix an element $x \in X$. Consider a partition $A_1, \ldots, A_k$ of X into k parts.

The element x belongs to one of the blocks, say, block A_1 which then has $r+1$ elements for some integer $0 \leq r \leq n-1$. The set $A_1 \setminus \{x\}$ can be any subset of $X \setminus \{x\}$ with r elements and there are $\binom{n-1}{r}$ possible such subsets. The remaining blocks $A_2, \ldots, A_k$ form a partition of $X \setminus A_1$ into $k-1$ parts and there are $S(n-r-1, k-1)$ such partitions. By these observations we conclude that

$$S(n,k) = \sum_{r=0}^{n-1}\binom{n-1}{r} S(n-r-1, k-1).$$

A change of variables $r \mapsto n-r-1$ in this sum using the formula $\binom{m}{m-l} = \binom{m}{l}$ for binomial coefficients gives the identity stated in the problem. $\square$