Suggested solutions. Discrete mathematics. Exam 2009-05-27

Anders Olofsson

Problem 1. The solution of the recurrence problem is:

$$a_n = (1-i)(1+i)^n + (1+i)(1-i)^n + n, \quad n = 0, 1, 2, \ldots$$

□

Problem 2. The solutions of the congruences are:

$$x = 59 + 154n, \quad n \in \mathbb{Z}.$$

□

Problem 3. We know that there is a one-to-one correspondence between equivalence relations on a set X and partitions of the same set X. The number of partitions of the set $\{1, 2, 3, 4\}$ is

$$\sum_{k=1}^{4} S(4, k) = 1 + 7 + 6 + 1 = 15,$$

where $S(n, k)$ denotes Stirling numbers of the second kind. □

Problem 4. We shall use the so-called principle of inclusion and exclusion. Denote by X the set of all integers n such that $1 \leq n \leq 123$. Denote by X_2 the subset of X consisting of the even integers in X, denote by X_3 the subset of integers in X divisible by 3, and denote by X_7 the set of integers in X that are divisible by 7. Phrased in these terms we want to calculate the number of integers in the set $X \setminus (X_2 \bigcup X_3 \bigcup X_7)$.

By the above mentioned principle we have that

$$\begin{aligned} |X \setminus (X_2 \bigcup X_3 \bigcup X_7)| &= |X| - (|X_2| + |X_3| + |X_7|) \\ &+ (|X_2 \bigcap X_3| + |X_2 \bigcap X_7| + |X_3 \bigcap X_7|) - |X_2 \bigcap X_3 \bigcap X_7|, \end{aligned}$$

where the symbol $|\cdot|$ is used to indicate the number of elements in a set. A calculation gives that $|X_2| = 61$, $|X_3| = 41$ and $|X_7| = 17$. Next observe that the set $X_2 \bigcap X_3$ consists of all integers in X divisible by 6. Arguing this way we see that $|X_2 \bigcap X_3| = 20$, $|X_2 \bigcap X_7| = 8$, $|X_3 \bigcap X_7| = 5$, and $|X_2 \bigcap X_3 \bigcap X_7| = 2$. We now have that the number of integers $1 \leq n \leq 123$ not divisible by 2, 3 or 7 equals

$$|X \setminus (X_2 \bigcup X_3 \bigcup X_7)| = 123 - (61 + 41 + 17) + (20 + 8 + 5) - 2 = 35.$$

□

Problem 5. Notice first the prime factorization $20000 = 2^5 5^4$, and that every positive integer n_j must have a prime factorization of the form $n_j = 2^{k_j} 5^{l_j}$ for some nonnegative integers k_j and l_j ($j = 1, 2, 3$). By the fundamental theorem of arithmetic we have that the triple (n_1, n_2, n_3) is such that $n_1 n_2 n_3 = 20000$ if and only if the triple (k_1, k_2, k_3) satisfies

$$\left\{ \begin{array}{l} k_1 + k_2 + k_3 = 5, \\ k_1 \geq 0, \ k_2 \geq 0, \ k_3 \geq 0, \end{array} \right. \tag{1}$$

and the triple (l_1, l_2, l_3) satisfies

$$\begin{cases} l_1 + l_2 + l_3 = 4, \\ l_1 \geq 0,\ l_2 \geq 0,\ l_3 \geq 0. \end{cases} \tag{2}$$

By standard theory (Section I.1.4 in Grimaldi) problem (1) has $\binom{5+3-1}{5} = \binom{7}{5}$ solutions (k_1, k_2, k_3) and problem (2) has $\binom{6}{4}$ solutions (l_1, l_2, l_3). As a result our problem has $\binom{7}{5}\binom{6}{4} = 315$ solutions (n_1, n_2, n_3). □

Problem 6. Passing from the recurrence formula for the Fibonacci numbers to the exponential generating function we see that F solves the second order constant coefficient differential equation initial value problem

$$\begin{cases} F'' = F' + F, \\ F(0) = 1, \quad F'(0) = 1, \end{cases}$$

where the prime $'$ indicates derivative. A calculation gives that

$$F(x) = \frac{1+\sqrt{5}}{2\sqrt{5}} \exp\left(\frac{1+\sqrt{5}}{2}x\right) - \frac{1-\sqrt{5}}{2\sqrt{5}} \exp\left(\frac{1-\sqrt{5}}{2}x\right),$$

where $\exp(x) = e^x$ is the usual exponential function. □