LUND UNIVERSITY

Written Examination
Discrete Mathematics
Wednesday, August 24, 2011

Centre for Mathematical Sciences
Mathematics, Faculty of Science

No aids are allowed. Use only the department's paper and write only on one side of each sheet. Fill in the cover form completely and write your initials on each sheet. Give clear and short motivations to your solutions.

1. The characteristic polynomial associated with the homogeneous equation $a_n + a_n - 2a_{n-1} = 0$ is $P(r) = r^2 + r - 2$, with roots $r_1 = 1$ and $r_2 = -2$. Therefore, the general solution of the homogeneous equation is

$$a_n^{(h)} = \alpha + \beta(-2)^n$$

In order to find a particular solution of the non-homogeneous equation, we try with a function of the formula

$$a_n^p = a2^{n-1}$$

After few computations, we obtain $a = 1$ which offers the solution

$$a_n = \alpha + \beta(-2)^n + 2^{n-1}$$

2. We need to calculate the coefficient of x^{29} in the following generating function

$$(x + x^3 + \cdots + x^9)(x^5 + x^6 + \cdots + x^{10})(x^{10} + x^{12} + \cdots + x^{28})(x^{10} + x^{11} + \cdots + x^{15})$$

$$= x^{26}(1 + x^2 + \cdots + x^8)(1 + x + \cdots + x^5)(1 + x^2 + \cdots + x^{18})(1 + x + \cdots + x^5)$$

and so, we just need to calculate the coefficient of x^3 of the last factors. Moreover, that coefficient coincides with the coefficient of x^3 in the function

$$f(x) = (1+x^2+x^4+\cdots)(1+x+x^2+x^3+\cdots)(1+x^2+x^4+\cdots)(1+x+x^2+x^3+\cdots)$$

This can be done directly and then obtain that such coefficient equals 8.

3. By the Chinese remainder theorem, we have that the solution is given by

$$x = \sum_i a_i x_i N_i$$

with $N_1 = 143$, $N_2 = 117$, $N_1 = 99$, $a_1 = -1$, $a_2 = 0$, $a_3 = 1$. Now, we need to calculate the factors x_i which satisfy $x_i N_i \equiv 1 \mod n_i$. By the euclidean algorithm, we have $x_1 = 8$, $x_2 = 8$ and $x_3 = 5$ wich finally offers the solution

$$x = -1 \cdot 8 \cdot 143 + 0 \cdot 8 \cdot 117 + 1 \cdot 5 \cdot 99 = -649 = 638$$

Please, turn over!

4. We use the exclusion/inclusion principle to solve the problem. We have the following prime decomposition $3500 = 2^2 \cdot 5^3 \cdot 7$ and then, an integer is relatively prime with 3500 if and only if is not divisible by 2, 3 or 7. Therefore, if we denote by c_i the property of begin divisible by i, we have that the number of integers relatively prime with 3500 coincides with

$$N(\bar{c_2}\bar{c_5}\bar{c_7}) = 3500 - (N(c_2) + N(c_5) + N(c_7))$$

$$+N(c_2c_5) + N(c_2c_7) + N(c_5c_7) + -N(c_2c_5c_7)$$

Now

$$N(c_2) = 2 \cdot 5^3 \cdot 7 = 1050,\ N(c_5) = 2^2 \cdot 5^2 \cdot 7 = 700,\ N(c_7) = 2^2 \cdot 5^3 = 500,$$

$$N(c_2c_5) = 2 \cdot 5^2 \cdot 7 = 350,\ N(c_2c_7) = 2 \cdot 5^3 = 150,\ N(c_5c_7) = 2^2 \cdot 5^2 = 100$$

and

$$N(c_2c_5c_7) = 2 \cdot 5^2 = 50$$

Then,

$$N(\bar{c_3}\bar{c_4}\bar{c_5}\bar{c_6}) = 3500 - 1800 + 600 - 50 = 2250$$

5. The minimum sum we can obtain by throwing 5 dice is 5, while the maximum is 30. This implies that there are 26 different possible sums. Then, by the pigeonhole principle, we need to throw the dice at least 27 times in order to ensure two equal sums in the outcome.

6. The code words defined by the generator matrix are

$$C = E(W) = \{00000, 00101, 01011, 10010, 01110, 10111, 11001, 11100\}$$

We notice that C is a group code and that the minimum weight of the non-zero elements of C is 2. Then, we deduce that the separation of this linear code is 2 and then, we can detect every single errors but we can not correct all of them.

None of the received word are a code words. We calculate the syndrome of the first received word and obtain 11. Its corresponding coset leader is then 000100 (it coincides with the fourth column of H) and so, can be associated with the code word $c = r + 00010 = 10111$. Then, finally, the received word can be correctly decoded as $w = 101$.

On the other hand, the syndrome of the second received word is 10 which does not correspond the syndromes of any coset leaders representing a single mistakes. Actually, the syndrome coincides with the first and fourth columns of H and so, there is no way to decide whether it should be associated with the code word $c = r + 10000 = 00101$ and so decoded as 001 or with the code word $c = r + 00010 = 10111$ and so decoded as 101 or it was received with multiple errors.