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OPTIMERING
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1. a) False. For example, take $f(x) = -x^2$. Actually, convergence in one step $\Leftrightarrow$ H is invertible.

b) True. See, for example, the last paragraph on page 317.

c) True. The fundamental idea behind the Dichotomous search is to reduce intervals of uncertainty by cutting off (almost) half of the intervals with no minimum. It is designed to work *precisely* for unimodal functions.

d) False. For example, $f(x) = x$ on $\mathbf{R}$.

e) False. The Steepest Descent method converges very slow (zigzagging) for quadratic functions with a large condition number (ill-conditioned problem). See Example 4 on page 49.

2. Consider the function $f(x,y) = x^4 - 12xy + y^4$ on $S = \{(x,y)\colon x \geq 0,\ y \geq 0\}$.

a) We need to find the largest set in S where the Hessian is positive-semidefinite. Let's calculate the Hessian

$$\nabla f(x,y) = \begin{bmatrix} 4x^3 - 12y \\ 4y^3 - 12x \end{bmatrix} \quad \Rightarrow \quad \nabla^2 f(x,y) = \begin{bmatrix} 12x^2 & -12 \\ -12 & 12y^2 \end{bmatrix} = 12 \cdot \underbrace{\begin{bmatrix} x^2 & -1 \\ -1 & y^2 \end{bmatrix}}_{=A}.$$

It is *necessary* for positive-semidefiniteness (under $x \geq 0$ and $y \geq 0$) that

(1) $\det A_1 = x^2 \geq 0$ (OK),

(2) $\det A = x^2y^2 - 1 = (xy-1)\underbrace{(xy+1)}_{\geq 1} \geq 0 \quad \Leftrightarrow \quad xy \geq 1.$

However, the condition (2) implies that $x \neq 0 \Leftrightarrow \det A_1 = x^2 > 0$, which makes (2) even the *sufficient* condition by "Modified" Sylvester criterion (Th. 8, p. 360). Thus,

$$D_{max} = \{(x,y) \in S\colon xy \geq 1\}.$$

b) Let's try first to look for a stationary point in D_{max} (we use here that $x > 0$ and $y > 0$). Solve $\nabla f(x,y) = 0 \ \Rightarrow$

$$\begin{cases} x^3 - 3y = 0 \\ y^3 - 3x = 0 \end{cases} \Rightarrow \begin{cases} x \cdot (x^3 - 3y) = x^4 - 3xy = 0 \\ y \cdot (y^3 - 3x) = y^4 - 3xy = 0 \end{cases} \Rightarrow \begin{cases} x = y \\ x^2 = 3 \end{cases} \Rightarrow \begin{cases} x = \sqrt{3} \\ y = \sqrt{3}. \end{cases}$$

The stationary point $(\sqrt{3}, \sqrt{3})$ satisfies $xy = 3 \geq 1$, hence, belongs to D_{max}. The function is convex on D_{max} $\Rightarrow$ the stationary point is the global minimum. (Therefore, the global minimum exists too.)

3. a) We change first *max* with *min* as

$$\max(4x_1 + 2x_2 + 3x_3) = -\min(-4x_1 - 2x_2 - 3x_3).$$

Then we add slack variables to the problem and build the Simplex tableau

$$
\begin{array}{|cccccc|c|}
\hline
4 & 2 & 3 & 0 & 0 & 0 & 0 \\ \hline
5 & 1 & 4 & \boxed{1} & 0 & 0 & 2 \\
1 & 2 & -1 & 0 & \boxed{1} & 0 & 6 \\
2 & \boxed{1} & 1 & 0 & 0 & -1 & 1 \\ \hline
\end{array}
\to
\begin{array}{|cccccc|c|}
\hline
0 & 0 & 1 & 0 & 0 & 2 & -2 \\ \hline
3 & 0 & 3 & 1 & 0 & 1 & 1 \\
-3 & 0 & -3 & 0 & 1 & 2 & 4 \\
2 & 1 & 1 & 0 & 0 & -1 & 1 \\ \hline
\end{array}
$$

The first two slack variable columns look already like basic columns. Let's complete them with a third basic vector. We need to pick a pivot from the last row. There is only one possibility that gives a feasible point (the right column ≥ 0) — x_2. After elimination we get the BFS $x = (0, 1, 0)$. Continue with the Simplex algorithm

$$
\begin{array}{|cccccc|c|}
\hline
0 & 0 & 1 & 0 & 0 & 2 & -2 \\ \hline
3 & 0 & 3 & 1 & 0 & \boxed{1} & 1 \\
-3 & 0 & -3 & 0 & 1 & 2 & 4 \\
2 & 1 & 1 & 0 & 0 & -1 & 1 \\ \hline
\end{array}
\to
\begin{array}{|cccccc|c|}
\hline
-6 & 0 & -5 & -2 & 0 & 0 & -4 \\ \hline
3 & 0 & 3 & 1 & 0 & 1 & 1 \\
-9 & 0 & -9 & -2 & 1 & 0 & 2 \\
5 & 1 & 4 & 1 & 0 & 0 & 2 \\ \hline
\end{array}
$$

The minimum is -4, that is, the maximum is 4 at the point $x = (0, 2, 0)$ (with the slack variables being $s = (0, 2, 1)$).

b) Multiply the third inequality by -1 and use the Formula Sheet to get the dual problem as

$$\min(2y_1 + 6y_2 - y_3) \qquad \text{subject to} \qquad \begin{cases} 5y_1 + y_2 - 2y_3 & \geq 4, \\ y_1 + 2y_2 - y_3 & \geq 2, \\ 4y_1 - y_2 - y_3 & \geq 3, \\ y_1, y_2, y_3 & \geq 0. \end{cases}$$

CSP gives:

- the primal slack $s_2 = 2 \neq 0 \Rightarrow y_2 = 0$,
- the primal slack $s_3 = 1 \neq 0 \Rightarrow y_3 = 0$,
- the primal $x_2 = 2 \neq 0 \Rightarrow y_1 + 2y_2 - y_3 = 2 \Rightarrow y_1 = 2$.

It gives the dual optimal solution $y = (2, 0, 0)$ and min $= 4$ ($=$ max from 3a).

4. a) We prove convexity of $h(x) = Ax + b$ by definition

$$\begin{aligned} h(\lambda x_1 + (1-\lambda)x_2) &= A(\lambda x_1 + (1-\lambda)x_2) + b = \lambda A x_1 + (1-\lambda)Ax_2 + \lambda b + (1-\lambda)b = \\ &= \lambda(Ax_1 + b) + (1-\lambda)(Ax_2 + b) = \lambda h(x-1) + (1-\lambda)h(x_2). \end{aligned}$$

So the function h is convex.

b) • $S = \{x \geq y^2 + z^2,\ z > 0\} = \underbrace{\{-x + y^2 + z^2 \leq 0\}}_{S_1} \cap \underbrace{\{z > 0\}}_{S_2} = S_1 \cap S_2.$

(i) $f(x,y,z) = -x + y^2 + z^2$ convex as a sum of convex $\Rightarrow$ S_1 convex.

(ii) S_2 convex (the open half-space).

Hence, $S = S_1 \cap S_2$ is convex.

• $S = \{x^2 \geq y^2 + z^2,\ y > 0\}$. Take $P_1 = (1,1,0) \in S$ and $P_2 = (-1,1,0) \in S$ and calculate $P = \frac{1}{2}P_1 + \frac{1}{2}P_2 = (0,1,0) \notin S$. Then S is not convex.

Another solution: Draw the section of S with the plane $z = 0$ and see that it is not convex $\Rightarrow$ S is not convex.

• $S = \{x^2 \geq y^2 + z^2,\ x > 0\} = \{x \geq \sqrt{y^2 + z^2},\ x > 0\} =$

$= \underbrace{\{-x + \sqrt{y^2 + z^2} \leq 0\}}_{S_1} \cap \underbrace{\{x > 0\}}_{S_2} = S_1 \cap S_2.$

(i) $f(x,y,z) = -x + \sqrt{y^2 + z^2}$ convex as a sum of convex $\Rightarrow$ S_1 convex.

(ii) S_2 convex (the open half-space).

Hence, $S = S_1 \cap S_2$ is convex.

Another solution: Draw the set S in $\mathbf{R}^3$. It is a cone without the origin $\Rightarrow$ convex.

c) Denote $x = (x_1, x_2, x_3)^T$ and rewrite

$$f(x_1, x_2, x_3) = \sqrt{1 + x_1^2 + x_2^2 + x_3^2} = \left\| \begin{array}{c} 1 \\ x \end{array} \right\|.$$

Let's prove convexity of $f(x)$ by definition

$$\begin{aligned} f(\lambda a + (1-\lambda)b) &= \left\| \begin{array}{c} 1 \\ \lambda a + (1-\lambda)b \end{array} \right\| = \left\| \begin{array}{c} \lambda + 1 - \lambda \\ \lambda a + (1-\lambda)b \end{array} \right\| = \left\| \lambda \begin{bmatrix} 1 \\ a \end{bmatrix} + (1-\lambda) \begin{bmatrix} 1 \\ b \end{bmatrix} \right\| \leq \\ &\leq \left\| \lambda \begin{bmatrix} 1 \\ a \end{bmatrix} \right\| + \left\| (1-\lambda) \begin{bmatrix} 1 \\ b \end{bmatrix} \right\| = \lambda \left\| \begin{array}{c} 1 \\ a \end{array} \right\| + (1-\lambda) \left\| \begin{array}{c} 1 \\ b \end{array} \right\| = \\ &= \lambda f(a) + (1-\lambda) f(b), \quad \forall \lambda \in [0,1]. \end{aligned}$$

Hence, the function f is convex.

Another solution: $f(x_1, x_2, x_3) = \sqrt{1 + x_1^2 + x_2^2 + x_3^2} = \sqrt{1 + \|x\|^2}$.

• $h(x) = \|x\| \geq 0$ is convex by definition

$$\begin{aligned} h(\lambda a + (1-\lambda)b) &= \|\lambda a + (1-\lambda)b\| \leq \|\lambda a\| + \|(1-\lambda)b\| = \\ &= \lambda\|a\| + (1-\lambda)\|b\| = \lambda h(a) + (1-\lambda)h(b), \quad \forall \lambda \in [0,1]. \end{aligned}$$

• $g(t) = \sqrt{1 + t^2}$ is convex and increasing for $t \geq 0$. Indeed

(i) $g' = \frac{t}{\sqrt{1+t^2}} \geq 0$ for $t \geq 0$ $\Rightarrow$ g increasing.

(ii) $g'' = \frac{1}{(1+t^2)\sqrt{1+t^2}}$ $\Rightarrow$ g convex.

So $f(x) = g(h(x))$ is convex.

Yet another solution (harder): Calculate the Hessian (3×3 matrix) and check by Sylvester that it is positive semi-definite (actually, it is even positive-definite).

5. We need to minimize the function $f(x, y) = 3x + 4y$ subject to 3 constraints:

$$g_1(x, y) = 2y - x^2 - y^2 \le 0, \qquad g_2(x, y) = x^2 + y^2 - 4 \le 0, \qquad g_3(x, y) = -x \le 0.$$

The functions f, g_2 and g_3 are convex. The only trouble is g_1, which is not convex. However, the KKT sufficient condition (Th. 5, p. 264 and Corollary 1, p. 265) needs convexity for active constraints only. So *if* we find a KKT point where g_1 is *inactive*, i.e. the corresponding $u_1 = 0$, then it is the global minimum.

State the KKT conditions

$$\begin{cases} 3 - 2u_1 x + 2u_2 x - u_3 & = & 0, & (1) \\ 4 + u_1(2 - 2y) + 2u_2 y & = & 0, & (2) \\ u_1(2y - x^2 - y^2) & = & 0, & (3) \\ u_2(x^2 + y^2 - 4) & = & 0, & (4) \\ u_3 x & = & 0, & (5) \\ u_1, u_2, u_3 & \ge & 0, & (6) \\ 2y \le x^2 + y^2 \le 4, \ x \ge 0 & & & (7) \end{cases}$$

and set $u_1 = 0$. Let's *try* to find a KKT point.

$x \ge 0,\ u_2 \ge 0 \overset{(1)}{\Rightarrow} u_3 = 3 + 2u_2 x \ge 3 > 0 \overset{(5)}{\Rightarrow} x = 0 \overset{(1)}{\Rightarrow} u_3 = 3.$

- $u_2 = 0 \overset{(2)}{\Rightarrow} 4 = 0$ (contradiction!).
- $u_2 > 0 \overset{(4)}{\Rightarrow} y^2 = 4$ (since $x = 0$) $\Rightarrow y = \pm 2$.

  (i) $y = 2 \overset{(2)}{\Rightarrow} 4 + 4u_2 = 0 \Rightarrow u_2 = -1$ (contradiction!).

  (ii) $y = -2 \overset{(2)}{\Rightarrow} 4 - 4u_2 = 0 \Rightarrow u_2 = 1 > 0$ (OK).

Thus, $(0, -2)$ is a KKT point with $u_1 = 0 \Rightarrow$ the global minimum, $\min = -8$.

Another solution: Let's just forget the non-convex constraint $2y \le x^2 + y^2$, and solve the new minimization problem over the *larger* set $\{x^2 + y^2 \le 0,\ x \ge 0\}$. The problem is convex now, so any KKT is a global minimizer (see the hint in Problem 6b). The KKT point here is the same as above $(0, -2)$. Since it belongs even to the original *smaller* set, it is the solution to the original problem too (easy to prove, though needs to be said).

Yet another solution: The origin gives $f(0, 0) = 0$, so the minimum point, if exists, should satisfy $f(x, y) = 3x + 4y \le 0$. Let's add this condition to the constraints. We get that the only non-convex constraint $2y \le x^2 + y^2$ becomes allways inactive (e.g. draw the 2D-picture to see that), so the new set is given by $x^2 + y^2 \le 4$, $x \ge 0$ and $3x + 4y \le 0$. The problem is again convex, and any KKT is a global minimizer. Existence of the minimum is granted by the sufficient condition.

Yet another solution (harder $\Rightarrow$ beter to avoid to save the time): The global minimum exists by Weierstrass (continuous + compact[= closed + bounded]).

1. Calculate all CQ points (7 cases) to get $(0, 2)$.
2. Calculate all KKT points (8 cases) to get
   - $(\frac{3}{5}, \frac{9}{5})$ for $u_1 > 0$, $u_2 = u_3 = 0$,
   - $(0, -2)$ for $u_1 = 0$, $u_2 > 0$, $u_3 > 0$,
   - $(0, 2)$ for $u_1 > 0$, $u_2 \ge 0$, $u_3 > 0$.
3. Compare $f(\frac{3}{5}, \frac{9}{5}) = \boxed{9}$, $f(0, 2) = \boxed{8}$, $f(0, -2) = \boxed{-8}$.

Remark: It is easy to draw a two-dimensional picture of the set and to do a graphical minimization since the level sets of f are straight lines. It helps a lot to understand what's going on and to see all those KKT/CQ points. Be aware though that the graphical solution is not fully accepted as a solution here since the question is to solve the problem using *KKT conditions*.

6. a) The Lagrange function is

$$\begin{aligned} L(x,y,u_1,u_2) &= 3x+4y+u_1(2y-x^2-y^2)+u_2(x^2+y^2-4) = \\ &= \underbrace{(u_2-u_1)x^2+3x}_{x-\text{terms}} + \underbrace{(u_2-u_1)y^2+(4+2u_1)y}_{y-\text{terms}} - 4u_2. \end{aligned}$$

- $u_2 \le u_1$ then for $x=0$ we get $L \le (4+2u_1)y \to -\infty$ when $y \to -\infty$.
- $u_2 > u_1$ then the x-terms are ≥ 0 and $= 0$ when $x = 0$, hence,

$$\min_{x\ge 0} L = (u_2-u_1)y^2+(4+2u_1)y-4u_2 = (u_2-u_1)y^2+2(2+u_1)y-4u_2.$$

To minimize it further with respect to $y \in \mathbf{R}$ we notice that it is a convex function in y when $u_2 > u_1$ and the minimum is attained at the stationary point

$$2(u_2-u_1)y+2(2+u_1)=0 \qquad \Rightarrow \qquad y = -\frac{2+u_1}{u_2-u_1}.$$

Substitution to $\min_{x\ge0} L$ gives

$$\Theta(u_1,u_2) = \begin{cases} -\frac{(2+u_1)^2}{u_2-u_1} - 4u_2 & \text{if} \quad u_2 > u_1 \ge 0, \\ -\infty & \quad\quad\ \text{otherwise.} \end{cases}$$

It is not hard to maximize $\Theta(u_1,u_2)$ and to see that there is no duality gap, but since we have already solved the primal problem and got $(\bar{x},\bar{y}) = (0,-2)$, with the corresponding Lagrange multipliers being $\bar{u}_1 = 0$ and $\bar{u}_2 = 1$ (see Problem 5), we will try to use these values:

$$\Theta(0,1) = -8 = f(0,-2).$$

Thus, there is no duality gap.

b) See Theorem 5 on page 264 and Corollary 1 on page 265.