DISCRETE MATHEMATICS – SOLUTIONS TO EXAM 2010-12-20

1. The characteristic polynomial is $r^2 + r - 6 = (r+3)(r-2)$. Therefore, the solutions to the homogeneous, and particular equations are of the form

$$a_n^{(h)} = \alpha(-3)^n + \beta 2^n$$
$$a_n^{(p)} = \delta 3^n + \gamma 2^n n.$$

After some computations, we should arrive at the solution

$$a_n = 3(-3)^n - 2^{n+1} + 3^{n+1} + 2^n n = 3^{n+1}(1 + (-1)^n) + 2^n(n-2).$$

2. Compute xG for $x = \in \mathbb{Z}_2^3$ to obtain the list of all code-words and their weights:

x	xG	weight
000	0000000	0
100	1000111	4
010	0101101	4
001	0011011	4
110	1101010	4
101	1011100	4
011	0110110	4
111	1110001	4

Since the smallest weight is equal to the Hamming separation $d(C)$ of the code, we get $d(C) = 4$. Also, it is clear that 1101010 is in the code, and 1010100 is not. The code-word 1011100 is the unique code-word at a Hamming distance of 1 from 1010100, and is therefore its correction.

3. Following the scheme of the Chinese Remainder theorem, we let $N_1 = 28, N_2 = 21, N_3 = 12$, and solve

$$\begin{aligned} s_1 28 &\equiv 1 \pmod 3 \\ s_1 21 &\equiv 1 \pmod 4 \\ s_1 12 &\equiv 1 \pmod 7 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} s_1 &\equiv 1 \pmod 3 \\ s_2 &\equiv 1 \pmod 4 \\ s_3 &\equiv 3 \pmod 7 \end{aligned}$$

The smallest positive x satisfying this is then

$$x = 1 \cdot 28 \cdot 2 + 1 \cdot 21 \cdot 3 + 3 \cdot 12 \cdot 3 = 227 \equiv 59 \pmod{84}.$$

4. We use the principle of inclusion and exclusion. Let $N = 6!$ denote the number of ways to arrange $1, 2, 3, 4, 5, 6$ along a line, and consider the conditions

$$c_1 = 12 \text{ appears},$$
$$c_2 = 23 \text{ appears},$$
$$c_3 = 34 \text{ appears}.$$

We calculate

$$N(c_1) = N(c_2) = N(c_3) = 5!$$
$$N(c_1c_2) = N(c_2c_3) = N(c_1c_3) = 4!$$
$$N(c_1c_2c_3) = 3!.$$

In the calculation of $N(c_1)$, say, we treat $'12'$ as being one object. In the calculation of $N(c_1c_2)$ we have to treat $'123'$ as one object, while in the calculation of $N(c_1c_3)$ we treat each of $'12'$ and $'34'$ as one object.

So, by the principle of inclusion and exclusion, our answer is

$$N(\overline{c}_1\overline{c}_2\overline{c}_3) = N - [N(c_1) + N(c_2) + N(c_3)] + [N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] - N(c_1c_2c_3)$$

$$= 6! - 3 \cdot 5! + 3 \cdot 4! - 3! = 426.$$

5.

a. We check that $p(x) = x^4 + 2x^3 + x^2 + 1$ is a prime polynomial. First, we exclude the possibility of it having a linear factor by observing that it has no zeroes: $p(0) = 1, p(1) = 2, p(2) = 1$ in $\mathbb{Z}_3$. The only possibility for it to be reducible is if it is a product of two irreducible polynomials of degree 2.

We consider all polynomials of the form $f(x) = x^2 + \alpha x + \beta$. (Note that a polynomial $2x^2 + \alpha x + \beta$ can be made to be of this form by multiplying with $2^{-1} = 2$.) Such a polynomial is irreducible if and only if it has no zeroes. This implies that $\beta \neq 0$. We make a table

α	β	$f(x)$	zeroes
0	1	$x^2 + 1$	
0	2	$x^2 + 2$	$x = 1, x = 2$
1	1	$x^2 + x + 1$	$x = 1$
1	2	$x^2 + x + 2$	
2	1	$x^2 + 2x + 1$	$x = 2$
2	2	$x^2 + 2x + 2$	

We now compute all products of the irreducible polynomials of degree 2:

$$\begin{aligned}
(x^2+1)^2 &= x^4 + 2x^2 + 1\\
(x^2+x+2)^2 &= x^4 + 2x^3 + 2x^2 + x + 1\\
(x^2+2x+2)^2 &= x^4 + x^3 + 2x + 1\\
(x^2+1)(x^2+x+2) &= x^4 + x^3 + x + 2\\
(x^2+1)(x^2+2x+2) &= x^4 + 2x^3 + 2x + 2\\
(x^2+x+2)(x^2+2x+2) &= x^4 + 1.
\end{aligned}$$

As our $p(x)$ is not in this list, and has no zeroes, it has to be irreducible. Hence, $\mathbb{Z}_3[x]/p(x)$ is a field.

b. By the division algorithm, if $f(x)$ is any polynomial, there exists polynomials $q(x)$ and $r(x)$ such that $f(x) = q(x)p(x) + r(x)$, and $r(x)$ has strictly lower degree than $p(x)$. This answers the first question.

To compute the equivalence class of $x^5 + 1$, we use long division to find that $x^5 + 1 = (x+1)(x^4 + 2x^3 + x^2 + 1) + (2x^2 + 2x)$. So $[x^5 + 1] = [2x^2 + 2x]$ in $\mathbb{Z}_3[x]$.

Finally, to compute the inverse of $x^5 + 1$, it is enough to compute the inverse of $2x^2 + 2$. This can be done by using the Euclidean algorithm, and we find (by just one step)

$$(x^4 + 2x^3 + x^2 + 1) = (2x^2 + 2x)(2x^2 + 2x) + 1.$$

Hence $[2x^2 + 2x]^{-1} = -[2x^2 + 2x] = [x^2 + x]$.

6.

a. For this problem, we need to find the coefficient of $x^6/6!$ of the exponential generating function

$$(1 + x + x^2/2)(1 + x)^5.$$

By the binomial theorem, this is the same as

$$\left[\sum_{k=0}^{5} \binom{5}{k} x^k\right](1 + x + x^2/2),$$

and so the answer is

$$6!\left[\binom{5}{5} + \binom{5}{4}\frac{1}{2}\right] = 7!/2 = 2520.$$

b. We first observe that with the letters 'NUNBET' we can construct $6!/2$ combinations of length six. To find the remaining combinations, where we all the time use the blank tile, consider 3 cases.

$$\begin{aligned}&\text{Case 1 :The blank tile is none of N, U, B, E, T.}\\&\text{Case 2 :The blank tile is one of U, B, E, T.}\\&\text{Case 3 :The blank tile is N.}\end{aligned}$$

Observe that in all cases we need to assume that the blank tile is used to avoid overcounting!

Case 1: There are 21 such choices for the blank tile. For each choice, the number of combinations where we use the blank tile, is equal to the coefficient of $x^6/6!$ in $x(1+x)^4(1+x+x^2/2)$, which is

$$\left[\binom{4}{4}+\frac{1}{2}\binom{4}{3}\right]6! = 3\cdot 6!.$$

Case 2: There are 4 such choices. For each, the number of combinations is equal to the coefficient of $x^6/6!$ in $(x^2/2)(1+x)^3(1+x+x^2/2)$, which is

$$\left[\frac{1}{2}\binom{3}{3}+\frac{1}{4}\frac{3}{2}\right]6! = (5/4)\cdot 6!.$$

Case 3: There is only one such choice, and the number of combinations is equal to the coefficient of $x^6/6!$ in $(x^3/3!)(1+x)^4$, which is:

$$\left[\frac{1}{3!}\binom{4}{3}\right]6! = (2/3)\cdot 6!.$$

The final answer is now (by the rule of sum):

$$6!\cdot\left[\frac{1}{2}+21\cdot 3+4\cdot\frac{5}{4}+1\cdot\frac{2}{3}\right] = 6!\cdot\frac{415}{6} = 49.800.$$