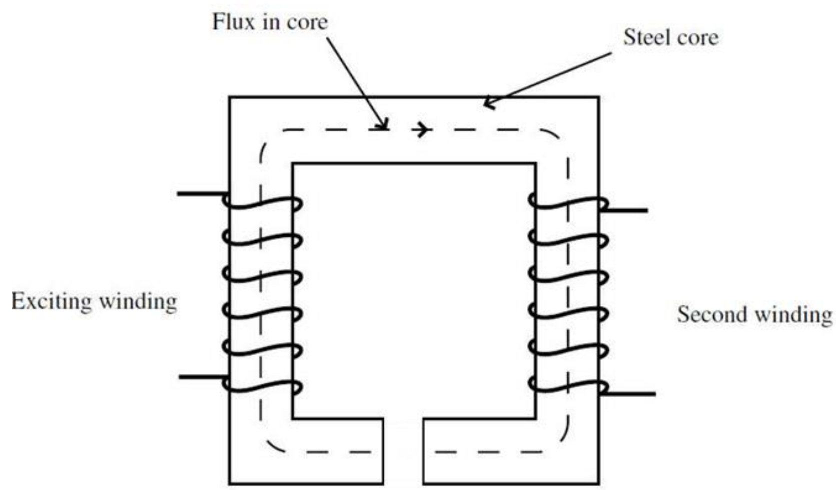
Homework 7 Solutions

1. Transformer on an Iron Core with an Air Gap [12 Points]

The transformer below consists of two coils on an iron core with an air gap. The iron core portion has a perimeter of 475cm and the air gap is 25cm wide, giving a total perimeter through the iron core region and air gap of 500cm. Additionally, the iron core has a square cross-sectional profile with side lengths of 25cm. The permeability of the iron core region is $3000\mu_0$ and the permeability of the air gap region is μ_0. The exciting winding has 1000 turns and the second winding has 3000 turns.



a) Draw a magnetic circuit to represent the structure when the current through the coil is 400mA DC and calculate and label the following:
   i. the MMF
   ii. the reluctance of the iron core and air gap region
   iii. the total magnetic flux through the core. (For this part of the problem, ignore the second winding.)

i) $m.m.f. = N \cdot I = 1000 \cdot .400A = \boxed{400\,A}$ (+1)

ii) $\mathcal{R} = \frac{l}{\mu A} \rightarrow \mathcal{R}_{iron} = \frac{0.475m}{3000\mu_0 \cdot (.25m)^2} = \underline{2016\,H^{-1}}$ (+1)

$$\mathcal{R}_{air} = \frac{0.25m}{\mu_0 \cdot (0.25m)^2} = \underline{3.18\times10^{6}\,H^{-1}} \text{ (+1)}$$

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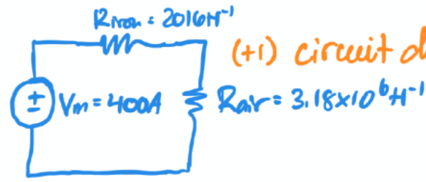
iii) $\Psi_m = \dfrac{V_m}{R_{iron} + R_{air}} = \dfrac{400A}{2016\,H^{-1} + 3.18\times10^{6}\,H^{-1}} = \underline{1.26\times10^{-4}\,Wb}$ (+1)



(+1) circuit drawn and labeled

b) Calculate the magnitude of the H field in the air gap.

$$H = \frac{B}{\mu_0} = \frac{\Psi_m/A}{\mu_0} = \frac{\Psi_m}{\mu_0 A} = \frac{1.26\times10^{-4}}{4\pi\times10^{-7}\frac{H}{m}\cdot(0.25m)^2} = 1604\ A/m$$ (+1)

c) Calculate the magnetic force felt by the two pieces of the core on either side of the air gap. Is this an attractive or repulsive force?

$$F_m = \frac{B_{air}^2}{2\mu_0} = \frac{(\mu_0\cdot 1604\ A/m)^2}{2\mu_0} = \underline{25.9\ N}$$ (+1)

attractive force, as F points in the direction of the region with higher energy density, which is in the air gap.

(+1) attractive / repulsive

$$\vec{F} = -\vec{\nabla}\left(\frac{B^2}{2\mu_0}\right)$$

d) Suppose that the 400mA DC current is replaced with a 400mA 100Hz AC current. What is the resulting emf across the secondary coil?

$$emf = -\frac{\partial \lambda}{\partial t} = -\frac{\partial}{\partial t}\{N_{secondary}\,\Psi_m(t)\}$$ (+1) equation

$$= -3000\cdot 1.26\times10^{-4}\,Wb\,\frac{\partial}{\partial t}\{\sin(2\pi f t)\}$$

$$= -0.378\,(2\pi\cdot 100\,Hz)\cos(200\pi t)$$

$$= \underline{-237.5\cos(200\pi t)\ V}$$ (+1)

e) Is it preferable to use a hard or a soft ferromagnetic material for the core of this transformer? Why?

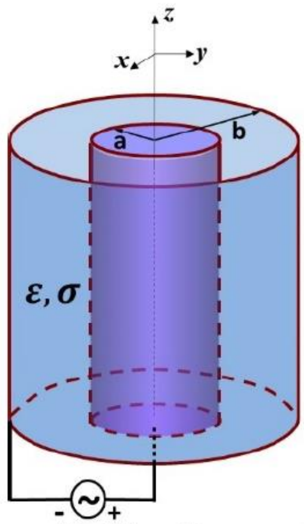
It is preferable to use a soft magnetic material for this core because it'll suffer fewer losses due to hysteresis effects than a hard core would.

(+1) soft magnetic material

(+1) justification

2. Displacement Current and the Quasi-static Approximation [16 Points]

For a coax-cable capacitor, the volume between the cylindrical copper conductors is filled with a lossy dielectric with permittivity ε_r and conductivity σ. The radius of the inner conductor is a and the effective inner radius for the outer conductor is b. The cable length $\boldsymbol{l}$ is much shorter than the wavelength, but $l \gg b$. The voltage applied across the coax-cable capacitor is $V(t) = V_0 \cos(\omega t)$ V.



$$V_g(t) = V_o \cos \omega t$$

a) Determine the displacement field $\vec{D}$ between the conductors, the displacement current I_d and the conduction current I_c passing through the capacitor. What is the phase angle between I_c and I_d? Which current leads (or which one lags)? Hint: you can use either time domain or phasor domain; you need to integrate the current passing through a cylindrical surface.

displacement field $\vec{D} = \varepsilon \vec{E}$

Via Gauss's Law: Note: can also be solved using Poisson's Equation

$$\oiint \vec{D} \cdot d\vec{s} = Q_{inner}$$

$$\varepsilon \cdot E \cdot l \cdot 2\pi r = Q_{inner}$$

$$\rightarrow \vec{E} = \frac{Q_{inner}}{2\pi \varepsilon l r} \hat{r}$$ (+1) E-field expression

$$V(r) - \cancelto{0}{V(b)} = -\int_b^r \frac{Q_{inner}}{2\pi \varepsilon l r} dr = \frac{-Q_{inner}}{2\pi \varepsilon l} \ln(r/b)$$

(+1) ✓ expression

$$V(r=a) = V_0 = -\frac{Q_{inner}}{2\pi \varepsilon l} \ln(a/b), \text{ so}$$

$$Q_{inner} = -\frac{2\pi \varepsilon l V_0}{\ln(a/b)} \cos(\omega t)$$ (+1) solve for Q_{inner}

$$\cdot\ \vec{D} = \varepsilon\vec{E} = \frac{\varepsilon}{2\pi\varepsilon l r}\ \frac{2\pi\varepsilon l V_0 \cos(\omega t)}{\ln(a/b)} = \boxed{\frac{\varepsilon V_0 \cos(\omega t)}{\ln(a/b)}\frac{1}{r}\hat{r}}\ \text{(+1) Dfield}$$

$$\cdot\ I_d = \frac{\partial}{\partial t}\iint \vec{D}\cdot d\vec{s} = \frac{\partial}{\partial t}\left\{\frac{\varepsilon\tilde{V}_0}{\ln(a/b)}\int_0^{l}\int_0^{2\pi}\frac{1}{a}\,a\,d\phi\,dz\right\}$$

(+1) equation

$$= \frac{2\pi l\varepsilon}{\ln(a/b)}\frac{\partial\tilde{V}_0}{\partial t} = \boxed{j\,\frac{2\pi\omega l\varepsilon V_0}{\ln(a/b)}\sin(\omega t)}\ (+1)$$

$$I_c = \iint \vec{J}\cdot d\vec{s} = \sigma\int_0^{l}\int_0^{2\pi}\frac{V_0\cos(\omega t)}{\ln(a/b)}\frac{1}{r}\,r\,d\phi\,dz \quad \text{(+1) equation}$$

$$= \boxed{\frac{2\pi\sigma l V_0\cos(\omega t)}{\ln(a/b)}}\ (+1)$$

b) Evaluate I_d, I_c and the ratio of their amplitudes when $\varepsilon_r = 27$, $\sigma = 2\times10^{-8}$ S/m, $a = 0.45$ mm, $b = 1.57$ mm, $l = 100$ m, $V_0 = 10$V and $f = 1$MHz.

$$|I_d| = \frac{2\pi\omega l\varepsilon V_0}{\ln(a/b)}\sin(\omega t) = -7.54\sin(2\pi\times10^6 t)\,[A]$$

$$= \underline{7.54\cos(2\pi\times10^6 t + \tfrac{\pi}{2})\,[A]}\ (+1)$$

$$|I_c| = \frac{2\pi\sigma l V_0}{\ln(a/b)}\cos(\omega t) = 0.0001\cos(\omega t)\ A$$

$$= \underline{100\cos(\omega t)\ \mu A}\ (+1)$$

$$\frac{|I_d|}{|I_c|} \approx \frac{7.54\,A}{100\times10^{-6}A} = \boxed{7.54\times10^4}\ (+1) \rightarrow \text{Displacement Current Dominates}$$

c) Using your answer from part b, determine at which frequency the amplitudes of I_d and I_c are equal.

when does $I_D = I_C$?

$$\frac{\cancel{2\pi}\omega\cancel{l}\varepsilon\cancel{V_0}}{\cancel{\ln(a/b)}} = \frac{\cancel{2\pi}\sigma\cancel{l}\cancel{V_0}}{\cancel{\ln(a/b)}} \rightarrow \omega\varepsilon = \sigma \rightarrow f = \frac{\sigma}{2\pi\varepsilon}$$

$$f = \frac{2\times10^{-8}\ \text{S/m}}{2\pi\cdot27\cdot8.85\times10^{-12}\ \text{F/m}} = \boxed{13.3\text{Hz}}\ (+1)$$

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d) Using your knowledge from circuit theory, calculate I_d and I_c, then compare them with your results from part a. *Hint*: you will need the expressions for capacitance and conductance per unit length of a coaxial cable to calculate these quantities.

$$I_d = C\frac{dV}{dt} = \frac{2\pi\varepsilon l}{\ln(b/a)} \cdot V_0 \frac{d}{dt}(\sin(\omega t))$$

(+1) equation

$$= \frac{2\pi\omega\varepsilon l V_0}{\ln(b/a)} \sin(\omega t) \quad (+1)$$

* This differs by a "−" sign due to the fact that we defined the direction of displacement current flow to go from low voltage to high voltage ($-\hat{r}$).

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3. Plane Wave in a Lossy Medium [13 Points]

Today's computer microprocessors (and many other electronic and photonic devices) are built on a silicon (Si) substrate. For pure Si, $\varepsilon_r \approx 12$. Assume the Si substrate resistivity is $\rho = 10\ \Omega\cdot\text{cm}$.

a) Determine the frequency range in which the Si substrate can be treated as a good insulator (dielectric), and the frequency range in which the Si substrate can be treated as a good conductor.

$\rho = 10\Omega\cdot cm \times \frac{1m}{100cm} = 0.1\Omega\cdot m$

for a good insulator: $\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon'} << 0.01$ (+1) condition

so $f >> \frac{\sigma}{2\pi\varepsilon'} = \frac{1}{0.01\cdot 2\pi\varepsilon'\rho} \cong 1.5\times10^{12}Hz$

· good insulator when $\boxed{f >> 1.5\ THz}$ (+1)

· for a good conductor: $\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon'} \gg 100$ (+1) condition

so $f << \frac{1}{100\cdot 2\pi\cdot\varepsilon'\cdot\rho} = 1.5\times10^{8}\ Hz \rightarrow \boxed{f << 150\ GHz}$ (+1)

b) If a plane wave is traveling in the Si substrate at 100 MHz, find the attenuation constant and phase constant (α and β), wavelength, and intrinsic impedance (η).

at f = 100 MHz, Si acts like a good conductor, so : (+1) correct regime

$\alpha = \sqrt{\pi f\mu\sigma} = \boxed{20\pi\ Np/m}$ (+1)

$\beta = \alpha = \boxed{20\pi\ rad/m}$ (+1)

$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{20\pi} = \boxed{0.1 m}$ (+1)

$\eta = (1+j)\frac{\alpha}{\sigma} = \frac{20\pi}{10}\sqrt{2}e^{j\frac{\pi}{4}} = \boxed{2\sqrt{2}\pi e^{j\pi/4}\ \Omega}$ (+1)

c) Using your results from b, if the wave is traveling in the y-direction with an **E**-field amplitude of 10 V/m measured at y = 0, find **E** and **H** for the wave in the phasor domain. $\vec{E} = E\hat{\boldsymbol{x}}$.

$$|\tilde{E}| = 10 \text{V/m}$$

$$\boxed{\tilde{E} = 10 \text{V/m}\, e^{-20\pi y} e^{-j20\pi y}\,\hat{x}}$$ (+1) correct $\vec{E}$

$$\tilde{H} = \frac{\hat{k}}{\eta} \times \tilde{E} = (+\hat{y}) \times \left(\frac{10 e^{-20\pi y} e^{-j20\pi y}\hat{x}}{2\sqrt{2}\pi e^{j\pi/4}}\right)$$

$$\boxed{\tilde{H} = -\frac{5}{\sqrt{2}\pi} e^{-20\pi y} e^{-j(20\pi y - \pi/4)}\,\hat{z}}$$

(+1) correct H magnitude
(+1) correct H direction
(+1) correct H phase

4. Wave Polarization [12 Points]

Determine the polarization of the following waves (i.e., linear, circular or elliptical) and their propagation direction. If a wave is linearly polarized, also determine the inclination angle. For non-linear polarization, determine the rotation direction (LH or RH). Draw a polarization diagram for each case with a few data points to show your work.

a) $\tilde{E}(z) = (3\hat{\boldsymbol{x}} - j3\hat{\boldsymbol{y}})e^{j25\pi z}$ [V/m]

$\tilde{E}(z) = (3\hat{x} - j3\hat{y})e^{j25\pi z}$ [V/m]

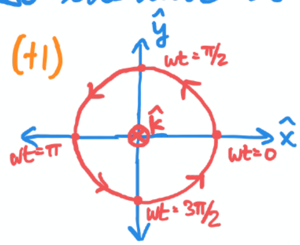
· direction of propagation : $\underline{-z}$ (+1)

· polarization:

$\tilde{E}(z) = (3\hat{x} + 3\hat{y}e^{-j\frac{\pi}{2}})e^{j25\pi z}$

→ E_x out of phase with E_y by $-\frac{\pi}{2}$, so the wave is <u>circularly polarized</u> (+1)

(+1)



$\vec{E}(0,t) = 3\cos(\omega t)\hat{x} + 3\cos(\omega t - \frac{\pi}{2})\hat{y}$

$\omega t = 0: 3\hat{x} - 0\hat{y}$

$\omega t = \pi/2: 0\hat{x} + 3\hat{y}$

$\omega t = \pi: -3\hat{x} + 0\hat{y}$

$\omega t = \frac{3\pi}{2}: 0\hat{x} - 3\hat{y}$

→ <u>left-hand circular</u> (+1)

b) $\tilde{E}(z,t) = 2\cos(10^6\pi t - 0.5z + 45°)\,\hat{\boldsymbol{x}} + \sin(10^6\pi t - 0.5z - 45°)\,\hat{\boldsymbol{y}}$ [V/m]

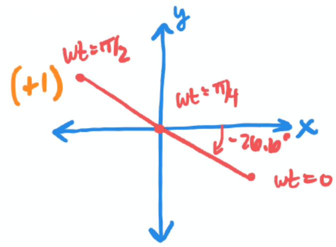
$E(z,t) = 2\cos(10^6\pi t - 0.5z + 45°)\hat{x}$
$+ \sin(10^6\pi t - 0.5z - 45°)\hat{y}$ [V/m]

· direction of propagation: $\underline{+z}$ (+1)

· polarization: $2\cos(10^6\pi t - 0.5z + \pi/4)\hat{x}$
$- \cos(10^6\pi t - 0.5z + \pi/4)\hat{y}$

→ E_x is $\pm\pi$ out of phase with E_y, so it is <u>linearly polarized</u> (+1)

→ Angle of inclination: $\tan^{-1}(1/-2) = \underline{-26.6°}$ (+1)



$\omega t = 0: 2/\sqrt{2}\,\hat{x} - 1/\sqrt{2}\,\hat{y}$
$\omega t = \pi/2: -2/\sqrt{2}\,\hat{x} + 1/\sqrt{2}\,\hat{y}$
$\omega t = \pi/4: 0\hat{x} + 0\hat{y}$

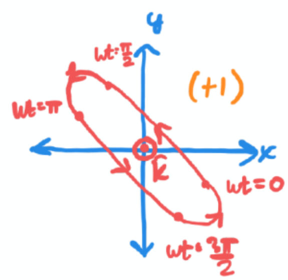
c) $\tilde{E}(z) = \left(3e^{j\frac{\pi}{6}}\hat{\boldsymbol{x}} - 3e^{j\frac{\pi}{3}}\hat{\boldsymbol{y}}\right)e^{-j3\pi z}$ [V/m]

$\tilde{E}(z) = \left(3e^{j\frac{\pi}{6}}\hat{x} - 3e^{j\frac{\pi}{3}}\hat{y}\right)e^{-j3\pi z}$

· direction of propagation: $\underline{+z}$ (+1)

· polarization: E_x is $\frac{7\pi}{6}$ out of phase with E_y
→ <u>elliptical polarization</u> (+1)

· $\omega t = 0: 3\cos(\pi/6)\hat{x} - 3\cos(\pi/3)\hat{y} = 2.6\hat{x} - 1.5\hat{y}$
$\omega t = \pi/2: 3\cos(4\pi/6)\hat{x} - 3\cos(5\pi/6)\hat{y} = -1.5\hat{x} + 2.6\hat{y}$
$\omega t = \pi: 3\cos(7\pi/6)\hat{x} - 3\cos(4\pi/3)\hat{y} = -2.6\hat{x} + 1.5\hat{y}$
$\omega t = \frac{3\pi}{2}: 3\cos(10\pi/6)\hat{x} - 3\cos(11\pi/6)\hat{y} = 1.5\hat{x} - 2.6\hat{y}$



· <u>right-hand elliptical</u>
(+1)