CSCI 2200 — Foundations of Computer Science (FoCS)
Homework 2 (document version 1.1)

Overview

- This homework is due by 11:59PM on Thursday, September 29
- You may work on this homework in a group of no more than four students; unlike recitation problem sets, **your teammates may be in any section**
- You may use at most **two** late days on this assignment
- Please start this homework early and ask questions during office hours; also ask (and answer) questions on the Discussion Forum
- Please be concise in your answers; even if your solution is correct, if it is not well-presented, you may still lose points
- You can type or hand-write (or both) your solutions to the required graded problems below; **all work must be organized in one PDF that lists all teammate names**
- You are strongly encouraged to use LaTeX, in particular for mathematical symbols; see references in Course Materials
- **EARNING LATE DAYS:** for each homework that you complete using LaTeX (including any tables, graphs, etc., i.e., no hand-written anything), you earn one additional late day; you can draw graphs and other diagrams in another application and include them as image files
- To earn a late day, you must submit your LaTeX files (i.e., `*.tex`) along with your one required PDF file—please name the PDF file `hw2.pdf`
- Also note that the earned late day can be used retroactively, even for this first homework assignment!

Warm-up exercises

The problems below are good practice problems to work on. Do not submit these as part of your homework submission. **These are ungraded problems.**

- **Problem 3.32.**
- **Problem 3.49.**
- **Problem 4.11.**
- **Problem 4.14.**
- **Problem 4.16.**
- **Problem 4.47.**
- **Problem 5.10.**
- **Problem 6.3(a).**
- **Problem 6.15.**
- **Problem 6.32.**
- **Problem 7.3.**
- **Problem 7.4(a-b).**
(Remove the recursion in your formula for A_n.)

Graded problems

The problems below are required and will be graded.

- ***Problem 3.59 (Closure).**
- ***Problem 4.7(b).**
- ***Problem 4.10(k-l).**
- ***Problem 4.48(c).** (See Problem 4.47.)
- ***Problem 5.12(d).**
- ***Problem 5.20.**
- ***Problem 5.39.**
- ***Problem 6.8.**
- ***Problem 6.43.**
- ***Problem 7.4(c).** (Remove the recursion in your formula for A_n.)

(v1.1) Some of the above problems (graded an ungraded) are transcribed in the pages that follow.

Graded problems are noted with an asterisk (*).

If any typos exist below, please use the textbook description.

- **Problem 3.32.** Use truth tables to verify the rules for derivations in Figure 3.1 on page 29. Now use the rules in Figure 3.1 to show logical equivalence

$$\neg((p \wedge q) \vee r) \overset{\text{eqv}}{\equiv} (\neg p \wedge \neg r) \vee (\neg q \wedge \neg r).$$

- **Problem 3.49.** What is the difference between

$$\forall x : (\neg\exists y : P(x) \to Q(y)) \quad \text{and} \quad \neg\exists y : (\forall x : P(x) \to Q(y))?$$

- ***Problem 3.59 (Closure).** A set $\mathcal{S}$ is closed under an operation if performing that operation on elements of $\mathcal{S}$ returns an element in $\mathcal{S}$. Here are five examples of closure.

$$\begin{gathered}
\mathcal{S} \text{ is closed under addition} \to \forall (x,y) \in \mathcal{S}^2 : x + y \in \mathcal{S}. \\
\mathcal{S} \text{ is closed under subtraction} \to \forall (x,y) \in \mathcal{S}^2 : x - y \in \mathcal{S}. \\
\mathcal{S} \text{ is closed under multiplication} \to \forall (x,y) \in \mathcal{S}^2 : xy \in \mathcal{S}. \\
\mathcal{S} \text{ is closed under division} \to \forall (x, y \neq 0) \in \mathcal{S}^2 : x/y \in \mathcal{S}. \\
\mathcal{S} \text{ is closed under exponentiation} \to \forall (x,y) \in \mathcal{S}^2 : x^y \in \mathcal{S}.
\end{gathered}$$

Which of the five operations are the following sets closed under? (a) $\mathbb{N}$. (b) $\mathbb{Z}$. (c) $\mathbb{Q}$. (d) $\mathbb{R}$.

- ***Problem 4.7(b).** Give direct proofs:

(b) $n \in \mathbb{Z} \to n^2 + n$ is even.

- ***Problem 4.10(k-l).** You may assume n is an integer. Prove by contraposition (explicitly state the contrapositive).

(k) 3 divides $n - 2 \to n$ is not a perfect square.
(l) If $p > 2$ is prime, then $p^2 + 1$ is composite.

- **Problem 4.11.** For $x, y \in \mathbb{N}$, which statements below are contradictions (cannot possibly be true). Explain.

(a) $x^2 < y$.
(b) $x^2 = y/2$.
(c) $x^2 - y^2 \leq 1$.
(d) $x^2 + y^2 \leq 1$.
(e) $2x + 1 = y^2 + 5y$.
(f) $x^2 - y^2/2 = 1$.
(g) $x^2 - y^2 = 1$.

- **Problem 4.14.** Prove: If $a, b, c \in \mathbb{Z}$ are odd, then for all $x \in \mathbb{Q}$, $ax^2 + bx + c \neq 0$. (Contradiction in a direct proof.)

- **Problem 4.47 (Without Loss of Generality (wlog)).** Consider the following claim.

  If x and y have opposite parity (one is odd and one is even), then $x + y$ is odd.

  Explain why, in a direct proof, we may assume that x is odd and y is even? Prove the claim. (Such a proof starts "Without loss of generality, assume x is odd and y is even. Then, ...")

- ***Problem 4.48(c).** Use the concept of "without loss of generality" to prove these claims.

  (c) For any non-zero real number x, $x^2 + 1/x^2 \geq 2$.

- ***Problem 5.12(d).** For $n \geq 1$, prove by induction:

  (d) $3^n > n^2$.

- ***Problem 5.20.** Prove, by induction, that every $n \geq 1$ is a sum of distinct powers of 2.

- ***Problem 5.39.** Prove you can make any postage greater than 12¢ using only 4¢ and 5¢ stamps. (The USPS can set any postage above 12¢ and you don't have to buy any new stamps.)

- **Problem 6.3(a).** Strengthen the claim and prove by induction for $n \geq 1$:

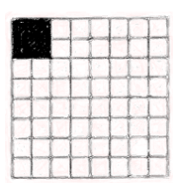
  (a) The sum of the first n odd numbers is a square. [Hint: Strengthen to a specific square.]

- ***Problem 6.8.** Prove $n^7 < 2^n$ for $n \geq 37$. (a) Use induction. (b) Use leaping induction.

- **Problem 6.15.** Prove that there are $2^{\lceil n/2 \rceil}$ distinct n-bit binary palindromes (strings that equal their reversal).

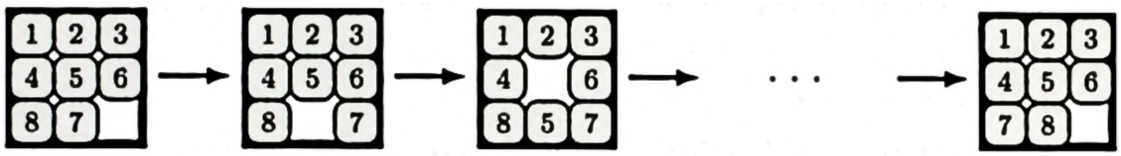
- **Problem 6.32.** We are back in L-tile land.

  (a) This time, the potted plant needs more room than just one square. For $n \geq 1$, a $2^n \times 2^n$ grid-patio is missing a (large) 2×2 square in a corner as shown in the figure. Prove that the remainder of the patio can be L-tiled, for $n \geq 1$.

  (b) We are no longer sure what the size of the potted plant is. The size may be $2^k \times 2^k$, and so a $2^k \times 2^k$ square will be missing from the corner of the $2^n \times 2^n$ grid-patio. Prove that the remainder of the patio can always be L-tiled, for $k \geq 1$ and $n \geq k$. [Hint: Tinker: try $k = 2$; $n = 3$ and $k = 2$; $n = 4$ to figure out what is going on.]

- ***Problem 6.43.** A sliding puzzle is a grid of 9 squares with 8 tiles. The goal is to get the 8 tiles into order (the target configuration). A move slides a tile into an empty square. Below, we show first a row move, then a column move.

  Prove that no sequence of moves produces the target configuration. [Hint: The tiles form a sequence going left to right, top to bottom. An inversion is a pair that is out of order. Prove by induction that the number of inversions stays odd.]

- **Problem 7.3.** Give a recursive definition of the function $f(n) = n! \times 2^n$, where $n \geq 1$.

- **Problem 7.4(a-b).** Guess a formula for A_n and prove it by induction.
  (a) $A_0 = 0$ and $A_n = A_{n-1} + 1$ for $n \geq 1$.
  (b) $A_1 = 1$, $A_2 = 2$, and $A_n = A_{n-1} + 2A_{n-2}$ for $n \geq 2$.

- ***Problem 7.4(c).** Guess a formula for A_n and prove it by induction.
  (c) $A_0 = 1$; $A_1 = 2$; $A_n = 2A_{n-1} - A_{n-2} + 2$ for $n \geq 2$. [Hint: Method of differences.]