Fields and Waves I

Lecture 3

Input Impedance on Transmission Lines

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These slides were prepared through the work of the following people:

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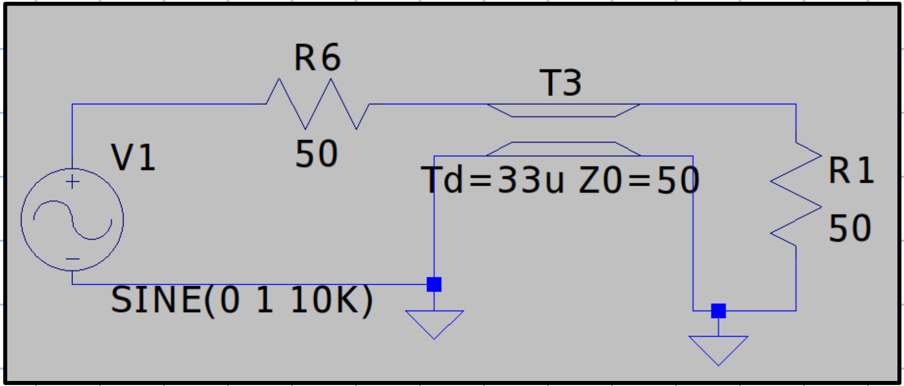
Materials from other sources are referenced where they are used.
Those listed as Ulaby are figures from Ulaby's textbook.

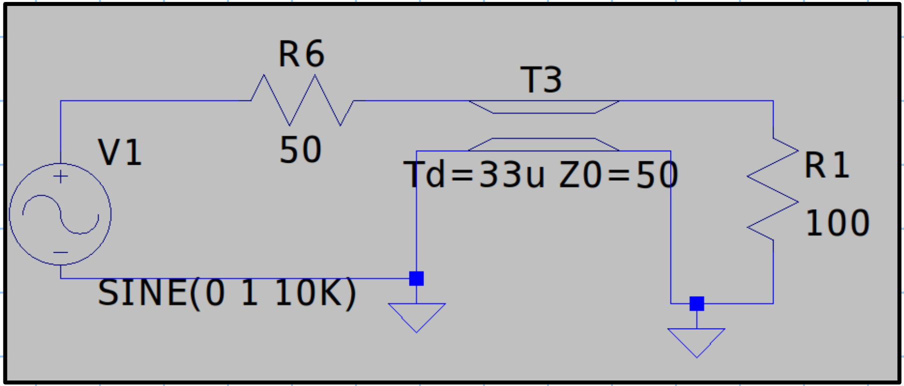
Outline

- Review
- Reflection and Standing Waves
- Generalized Impedance
- Input Impedance
- Wrap-Up

Review

What happens at the end of the transmission line in these two cases?

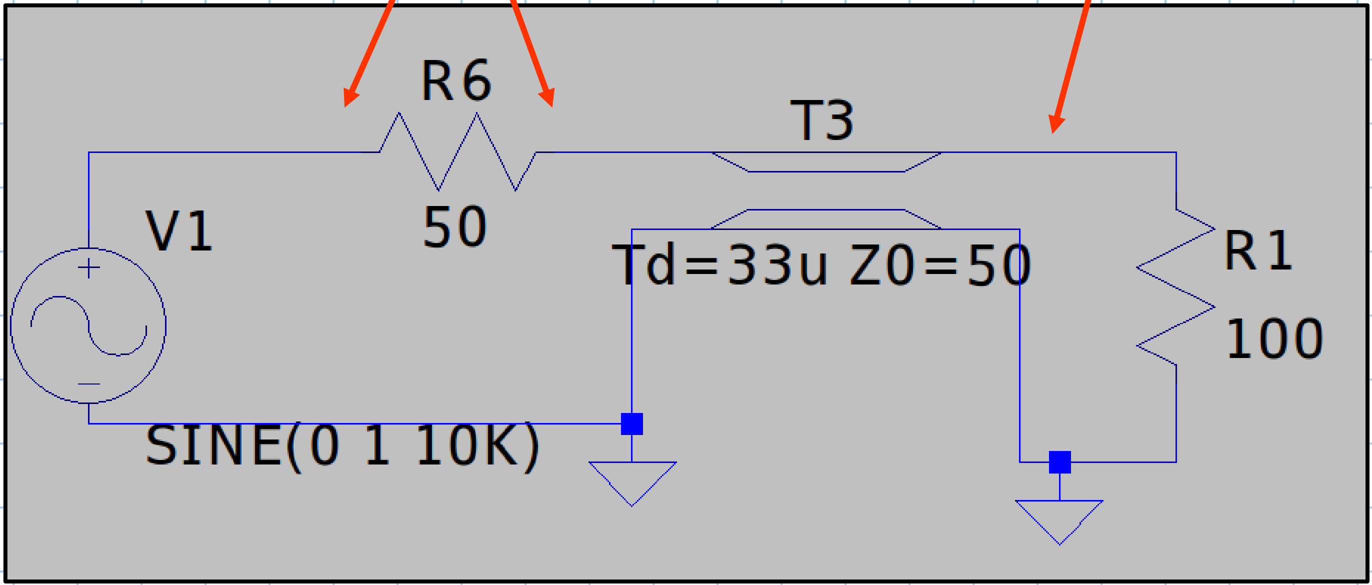




Review

Voltage drop across this resistor

Voltage measured here is that of the initial wave PLUS the load-reflected wave



You can't measure the forward and the backward traveling waves individually at the load. You measure both at once.

Review

Reflection Coefficient

When z=0 (i.e. we are at the load), we get:

$$\boxed{\tilde{V}(0) = V_o^+ + V_o^-}$$

$$\boxed{\tilde{I}(0) = I_o^+ - I_o^- = \frac{V_o^+}{Z_o} - \frac{V_o^-}{Z_o}}$$

Doing some algebra...

$$\boxed{V_o^- = \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) V_o^+} \qquad \Gamma$$

Review

Properties of Waves

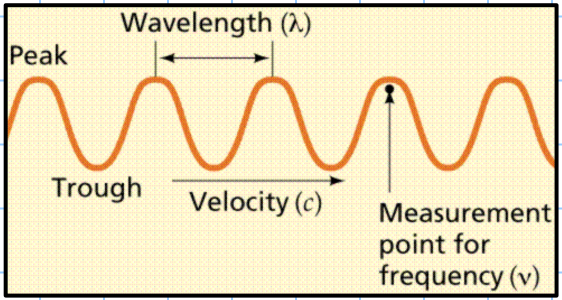


Figure from http://www.emc.maricopa.edu/

- Sinusoidal traveling waves take this form:

$$A \cos(\omega t \mp \boldsymbol{\beta} z)$$

- Another way to write this equation is:

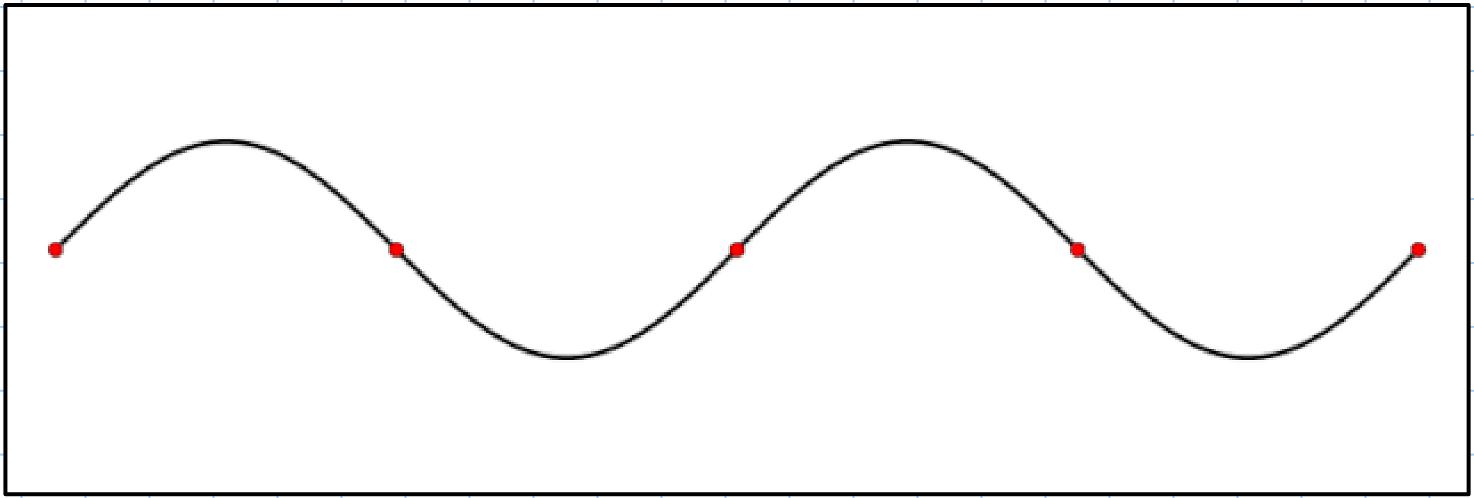
$$A \cos((2\boldsymbol{\pi}/T)t \mp (2\boldsymbol{\pi}/\lambda)x)$$

- T is the time period, λ is the wavelength (i.e. "space period"
- Therefore the first term in the argument tells us how rapidly the wave changes in *time*. The second tells us how rapidly it changes in *space*.

Review

Properties of Waves

- How do you create a standing wave on a transmission line? (Usually you don't want to do this.)



Reflection and Standing Waves

Define the Reflection Coefficient:

$$\left|V_m^-\right| = \left|\Gamma_L\right| \cdot \left|V_m^+\right|$$

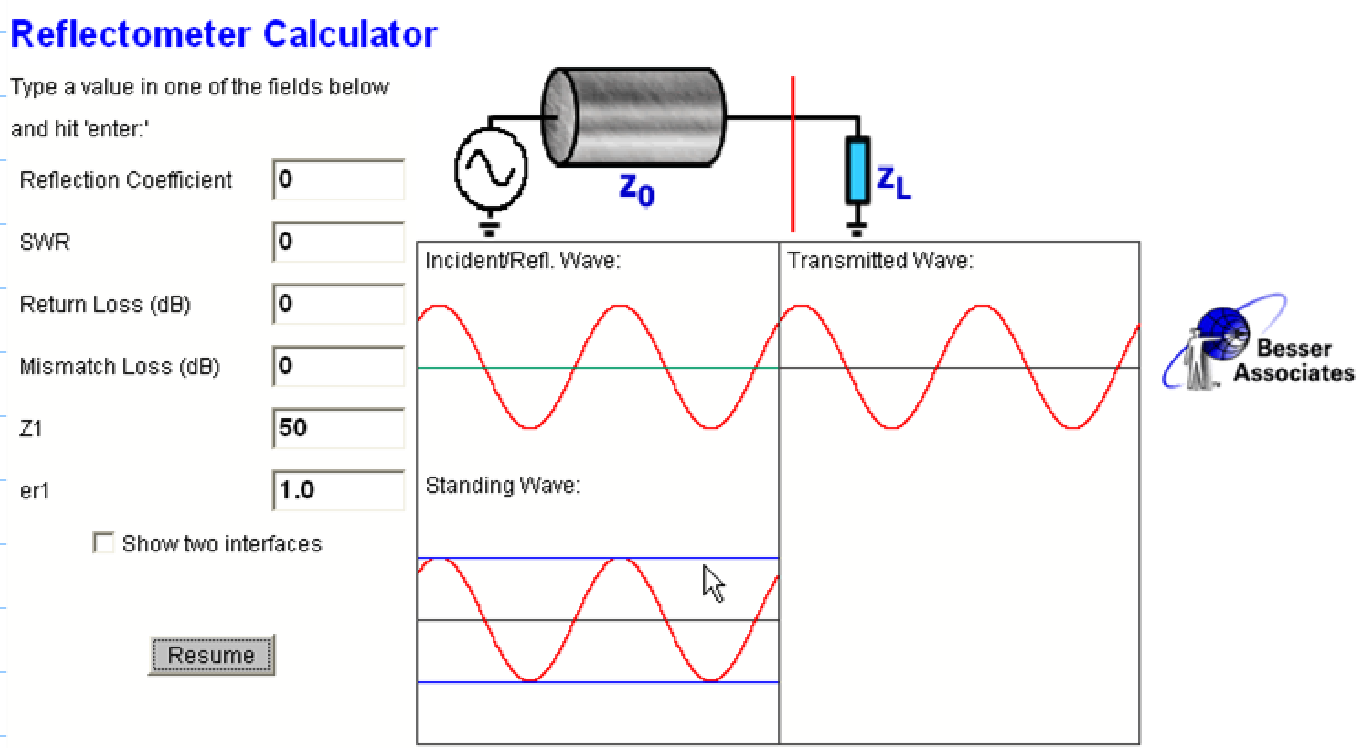
Maximum Amplitude when in Phase: $V_{max} = \left|V_m^+\right| + \left|V_m^-\right|$

$$\therefore V_{max} = \left|V_m^+\right| \cdot \left(1 + \left|\Gamma_L\right|\right)$$

Similarly: $V_{min} = \left|V_m^+\right| \cdot \left(1 - \left|\Gamma_L\right|\right)$

Standing Wave Ratio (SWR) = $\dfrac{V_{max}}{V_{min}} = \dfrac{1 + \left|\Gamma_L\right|}{1 - \left|\Gamma_L\right|}$

Reflection and Standing Waves



Reflection and Standing Waves

Deriving a Standing Wave Expression

- Voltage general wave solution (as a phasor):

$$v(z) = V^{+} e^{-j\beta z} + V^{-} e^{+j\beta z}$$

- Using the reflection coefficient:

$$v(z) = V^{+} e^{-j\beta z} + \Gamma_L V^{+} e^{+j\beta z}$$

Reflection and Standing Waves

Deriving a Standing Wave Expression

- For $Z_L=0$, we have $\Gamma_L = \dfrac{Z_L - Z_o}{Z_L + Z_o} = \dfrac{0 - Z_o}{0 + Z_o} = -1$

$$v(z) = V^+ e^{-j\beta z} + \Gamma_L V^+ e^{+j\beta z} = V^+\left(e^{-j\beta z} - e^{+j\beta z}\right)$$

$$e^{+j\beta z} = \cos\beta z + j\sin\beta z$$

$$e^{-j\beta z} = \cos\beta z - j\sin\beta z$$

$$\Rightarrow \quad v(z) = -V^+\left(j2\sin\beta z\right)$$

Reflection and Standing Waves

Short Circuit Load Case

- Convert to space-time form

$$v(z,t) = \mathrm{Re}\left(v(z)e^{j\omega t}\right) = \mathrm{Re}\left(V^{+}\left(-j2\sin\beta z\right)e^{j\omega t}\right)$$

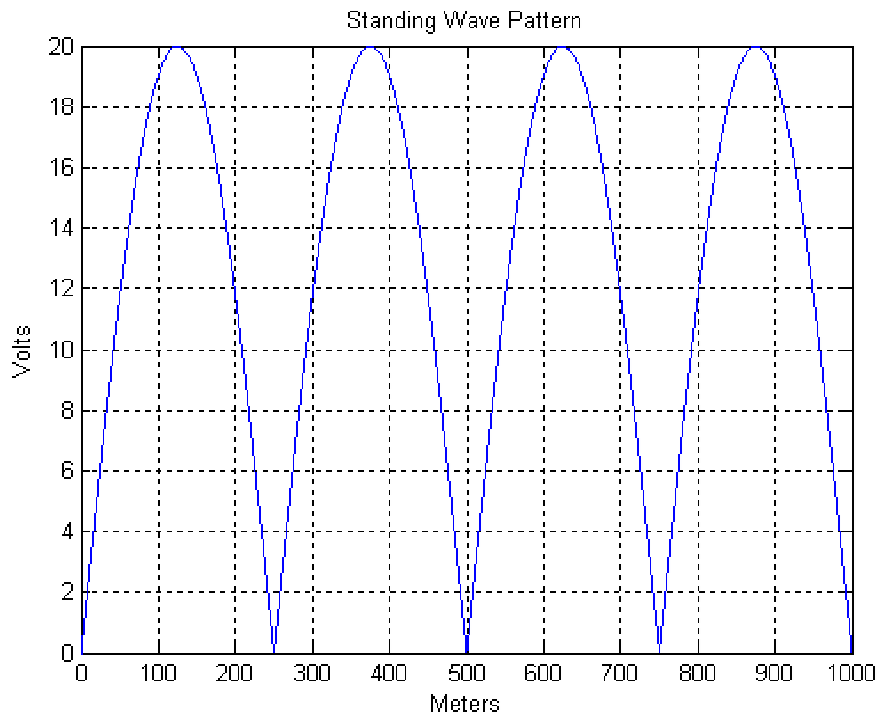
$$\mathrm{Re}\left(\left(-j2\sin\beta z\right)e^{j\omega t}\right) = \mathrm{Re}\left(-2\sin\beta z\left(j\cos\beta z - \sin\beta z\right)\right)$$

$$v(z,t) = 2V^{+}\sin\beta z\sin\omega t$$

- Is this is a standing wave? How do we know?

Reflection and Standing Waves

Short Circuit Load Case



This is a voltage standing wave plot - a plot of the **magnitude** of the standing wave versus distance on the line.

Reflection and Standing Waves

Short Circuit Load Case

- What are the voltage maxima and minima?

$$v(z,t) = 2V^{+} \sin \beta z \sin \omega t$$

- Where are they?

  They exist when $\beta z = 0$, $\boldsymbol{\pi}$, $2\boldsymbol{\pi}$, etc.

  Since $\beta = 2\boldsymbol{\pi}/\lambda$,

  $z = 0, \lambda/2, \lambda, 3\lambda/2$, etc.

In general, standing wave patterns will repeat every half wavelength of the input signal, and every quarter wavelength they will alternate between maxima and minima.

Reflection and Standing Waves

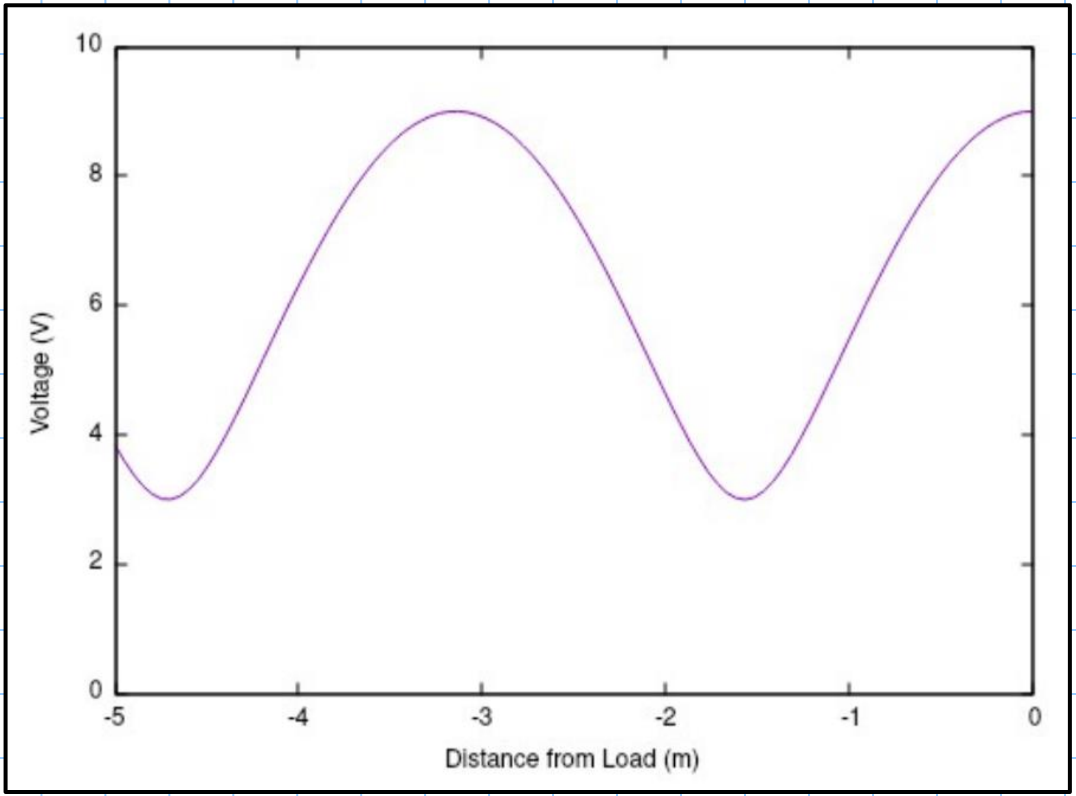
Short Circuit Load Case

- What are the voltage maxima and minima?

$$v(z,t) = 2V^{+} \sin \beta z \sin \omega t$$

- The standing wave pattern is the envelope of this function - a graph of its magnitude on the z axis.

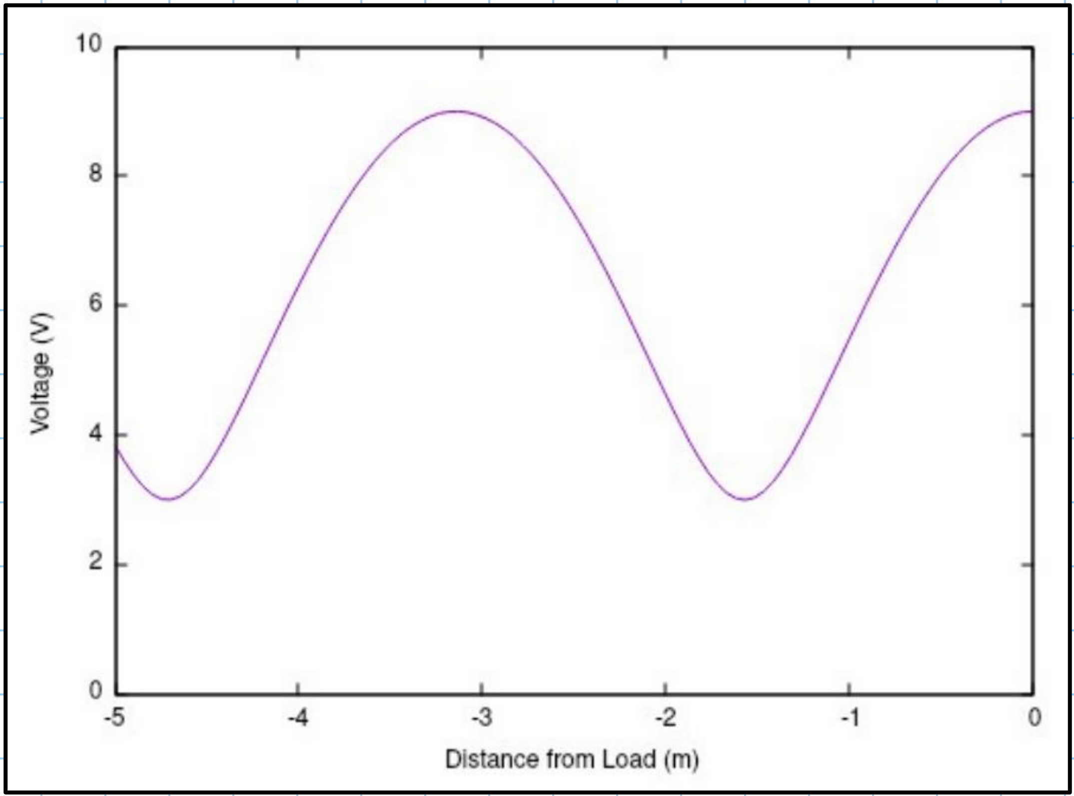
Reflection and Standing Waves



Z_0 =50

Let's say that you're given a standing wave graph like the one above **and Z_0 of the line**, but no other information. How do you extract key wave / line parameters?

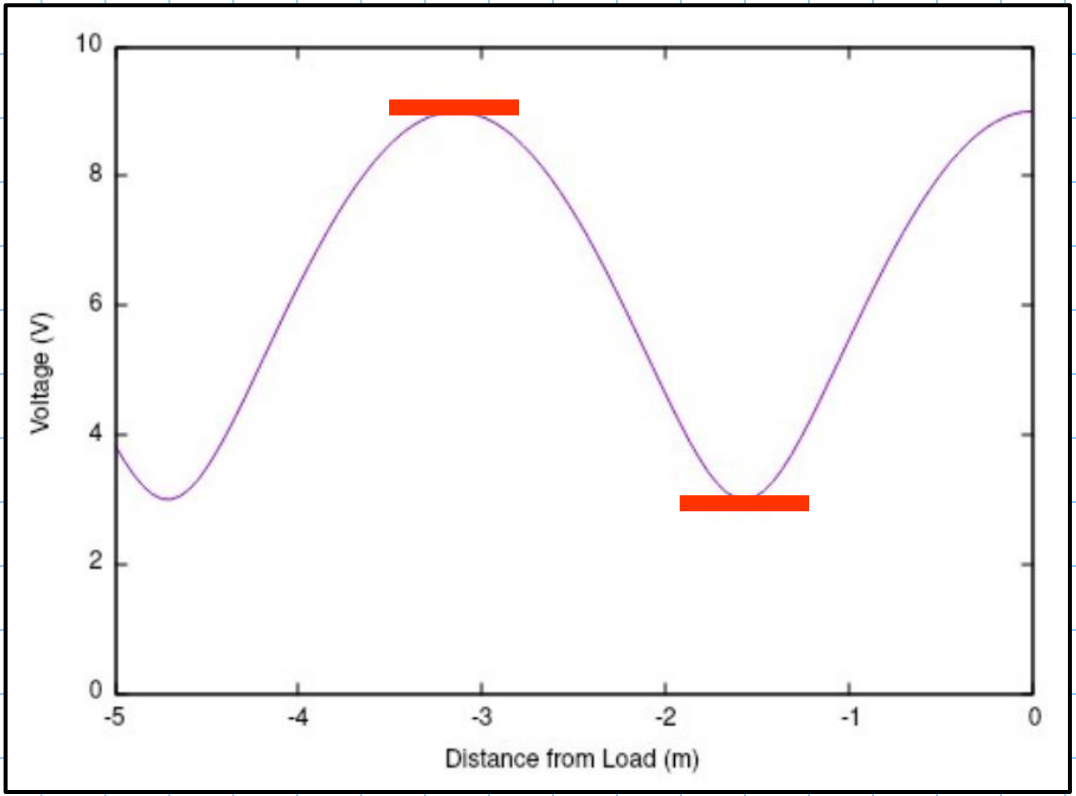
Reflection and Standing Waves



$$Z_0 = 50$$

First, what is the standing wave ratio?

Reflection and Standing Waves



Z_0 =50

First, what is the standing wave ratio?

$V_{max}/V_{min} = 9/3 = 3$

Reflection and Standing Waves

Z_0 =50

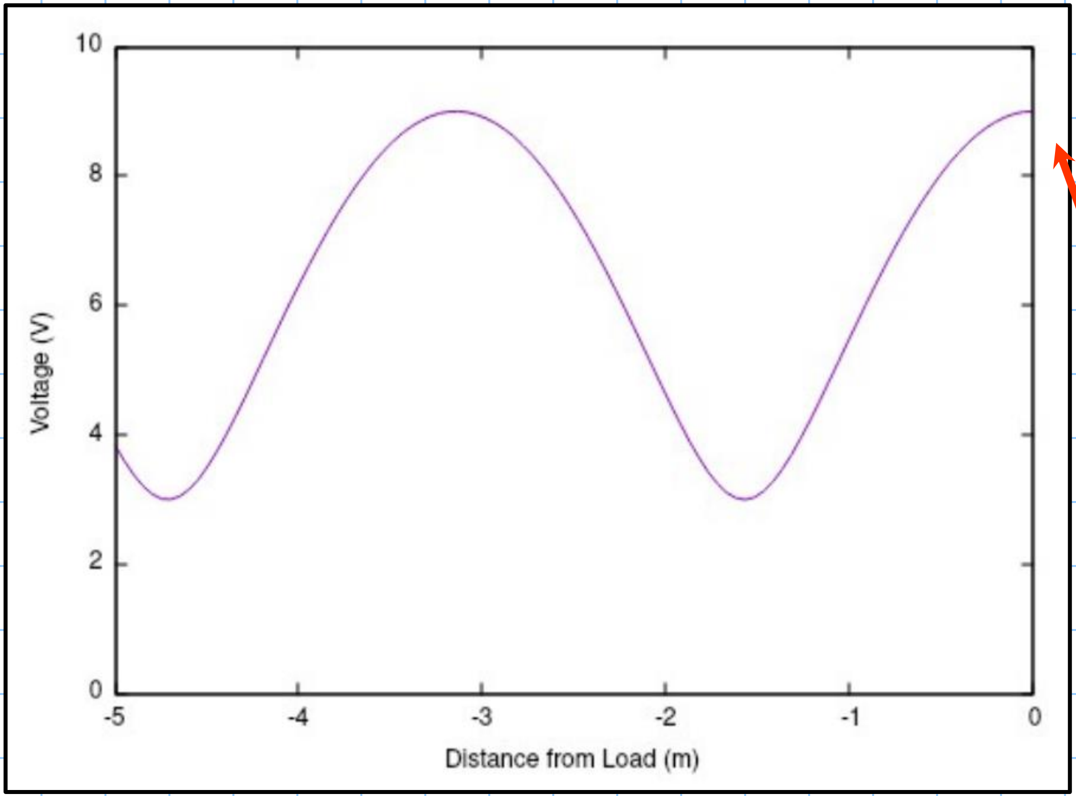
$$3 = \frac{V_{max}}{V_{min}} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|}$$

$$3 - 3\,|\Gamma_L| = 1 + |\Gamma_L|$$

$$|\Gamma_L| = \frac{1}{2} \qquad \Gamma_L = \pm\frac{1}{2}$$

What does this mean?

Reflection and Standing Waves



Z_0 =50

$$\Gamma_L = \pm\frac{1}{2}$$

To figure out the sign of Γ_L,

look at whether the load end of the graph is a **maximum or a minimum.**

Reflection and Standing Waves

$$\boxed{\frac{V_{max}}{V_{min}} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|}}$$

$$v(z) = V^{+}e^{-j\beta z} + \Gamma_L V^{+}e^{+j\beta z}$$

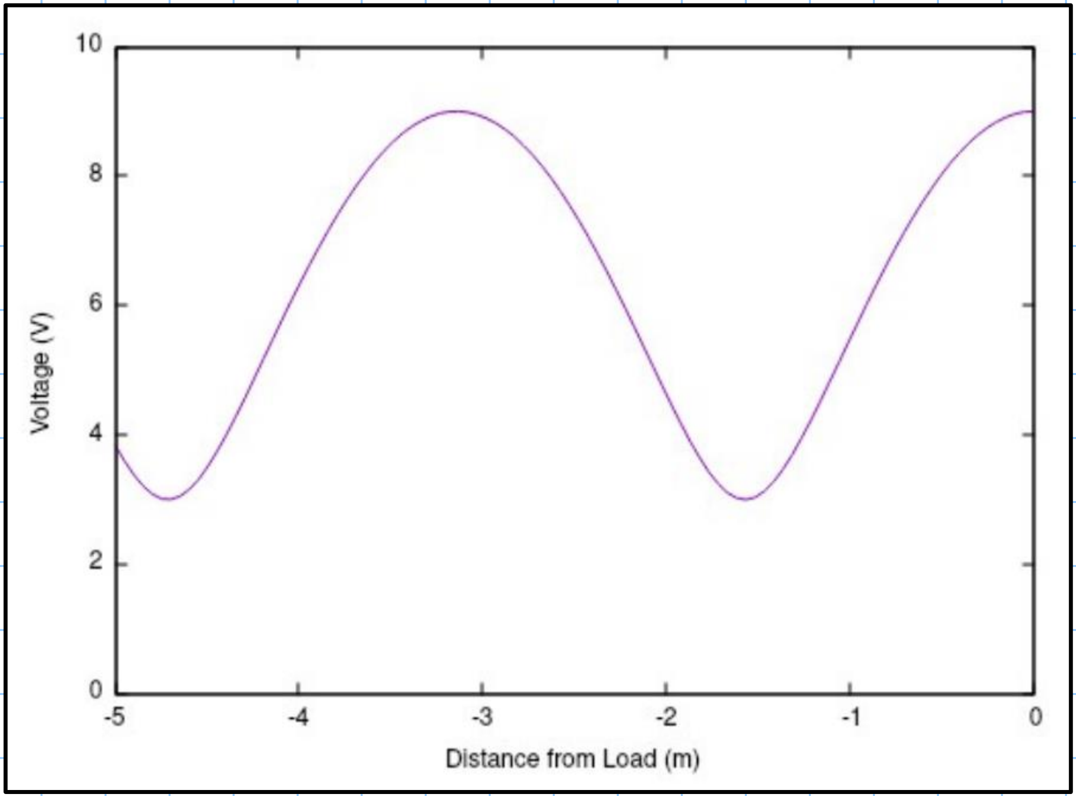
At the load,

$$v(0) = V^{+} + \Gamma_L V^{+}$$

Therefore:

- If the load is not matched to the line, **if it's a resistive load,** there will be either a voltage maximum or a voltage minimum at the load.
- If the load has a voltage maximum, **the reflection coefficient is positive**
- If the load has a voltage minimum, **the reflection coefficient is negative.**

Reflection and Standing Waves



Z_0 =50

$$\Gamma_L = \pm\frac{1}{2}$$

So what is Γ_L for this line?

Reflection and Standing Waves

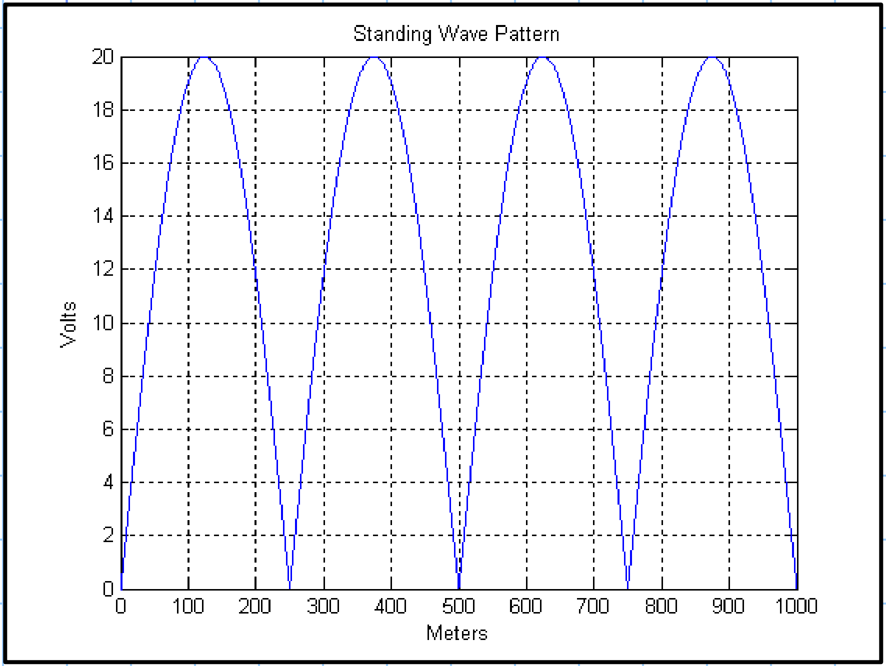
We can now calculate the load impedance:

$$\Gamma_L = \frac{1}{2} \qquad Z_0 = 50 \qquad \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

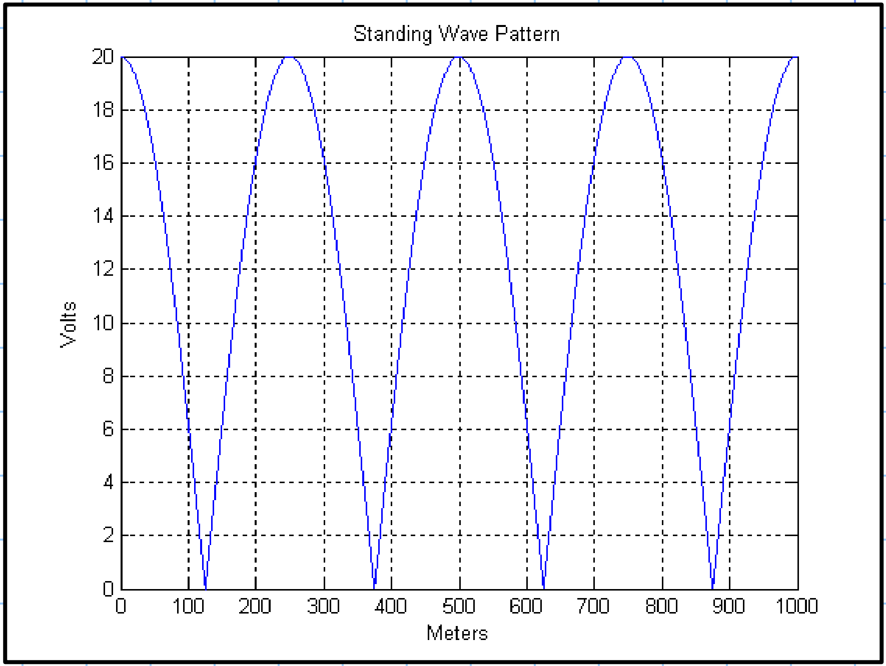
$$\boxed{Z_L = 150\Omega}$$

Reflection and Standing Waves

Short-Circuit Load Standing Wave



Open-Circuit Load Standing Wave



- Note that both standing wave patterns have an SWR of infinity since $V_{min} = 0$.
- The standing wave patterns are identical except for a **quarter-wavelength phase shift.**

Reflection and Standing Waves

What will the standing wave pattern look like on a matched transmission line $Z_0 = Z_L$?

Reflection and Standing Waves

- Can we use what we just displayed to find the current standing wave patterns?

Reflection and Standing Waves

- Can we use what we just displayed to find the current standing wave patterns?

Yes, because the reflection coefficient for current is always just the negative of the voltage reflection coefficient.

$$i(z) = \frac{V^{+}e^{-j\beta z} - V^{-}e^{+j\beta z}}{Z_0} = \frac{V^{+}}{Z_0}e^{-j\beta z} - \frac{V^{-}}{Z_0}e^{+j\beta z}$$

Reflection and Standing Waves

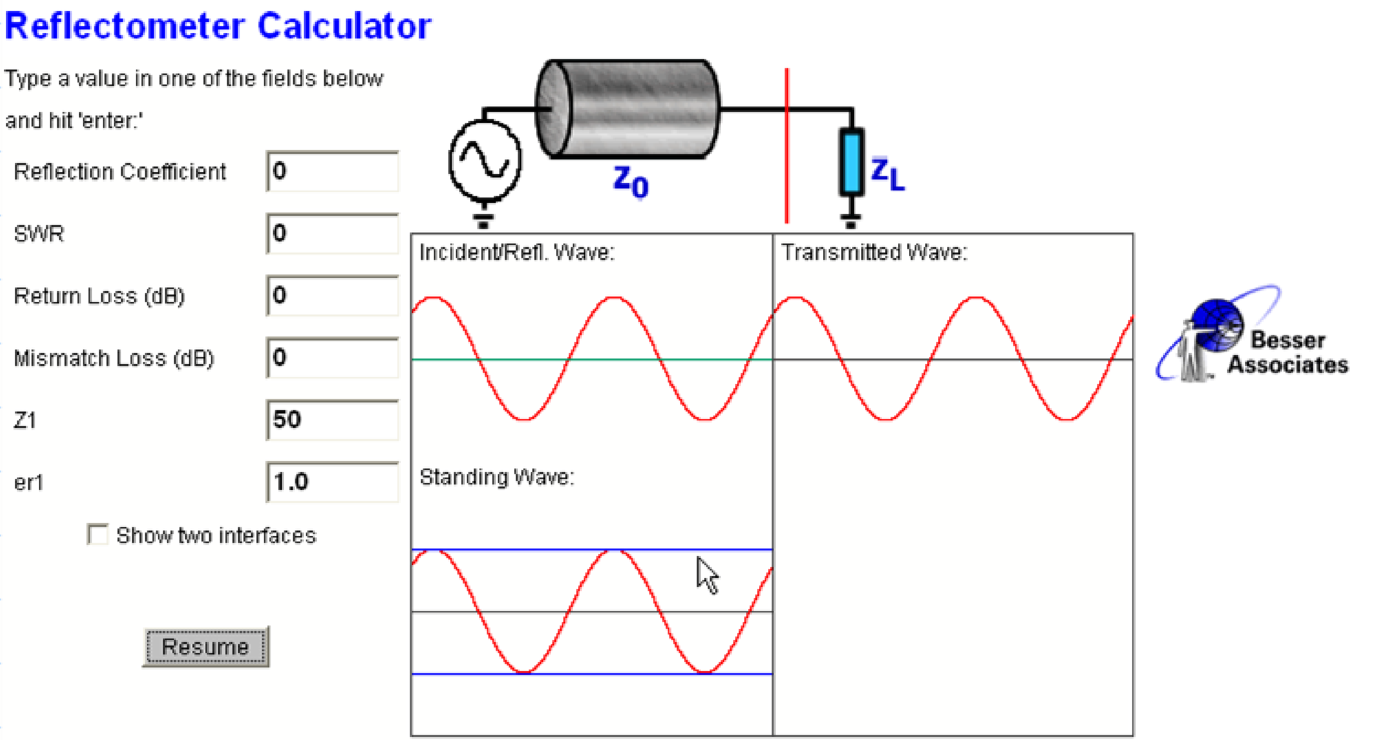
- So… are standing wave patterns sine waves?

Reflection and Standing Waves

- So… are standing wave patterns sine waves?

  Not quite! They are the magnitude of a forward traveling wave plus a backward traveling wave. This is a somewhat complicated expression that can be approximated as a sine wave in many cases (or as half a sine wave in the open and short circuit case).

Reflection and Standing Waves



Reflection and Standing Waves

- In most practical applications, we try to minimize standing wave ratio since this represents power reflected away from the load
- If the voltage maxima created by a standing wave pattern exceed the rated voltage of a transmission line, it can also cause the transmission line to fail.
- The SWR of real lines can be measured using several different devices and methods

Directional Coupler

krytar.com

Spectrum Analyzer

aliexpress.com

Reflection and Standing Waves

- Form into group of up to 4 and do Lecture 3 Exercise 1 on Gradescope. We'll reconvene in about 10 minutes.

Generalized Impedance

We already saw that:

$$Z_L = \frac{\tilde{V}_L}{\tilde{I}_L}$$

But we should also be able to state more generally for any z=d:

$$Z_d = \frac{\tilde{V}_d}{\tilde{I}_d}$$

It's not clear yet what an impedance like this actually represents.

Generalized Impedance

$$\frac{\tilde{V}_d}{\tilde{I}_d} = \frac{V_0^+[e^{j\beta d} + \Gamma e^{-j\beta d}]}{V_0^+[e^{j\beta d} - \Gamma e^{-j\beta d}]} Z_0$$

note: here we let z = -d (see Ulaby pg. 71)

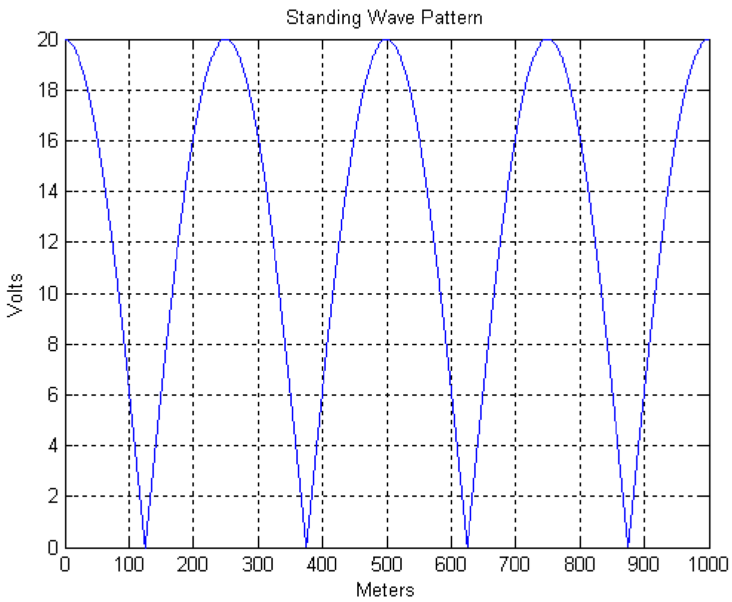
$$= Z_0 \frac{1 + \Gamma e^{-j2\beta d}}{1 - \Gamma e^{-j2\beta d}}$$

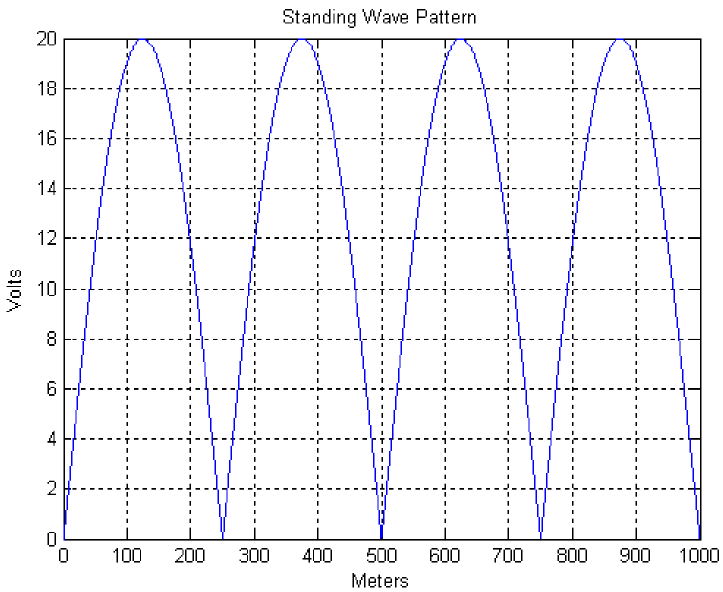
$$= Z_0 \frac{1 + \Gamma_d}{1 - \Gamma_d}$$

$$\boxed{\Gamma_d = \Gamma(z) = \Gamma e^{-2j\beta d}}$$

Generalized Impedance

$$\Gamma(z) = \Gamma_L \cdot e^{-j\cdot 2\cdot \beta\cdot (L-z)}$$





Generalized Impedance

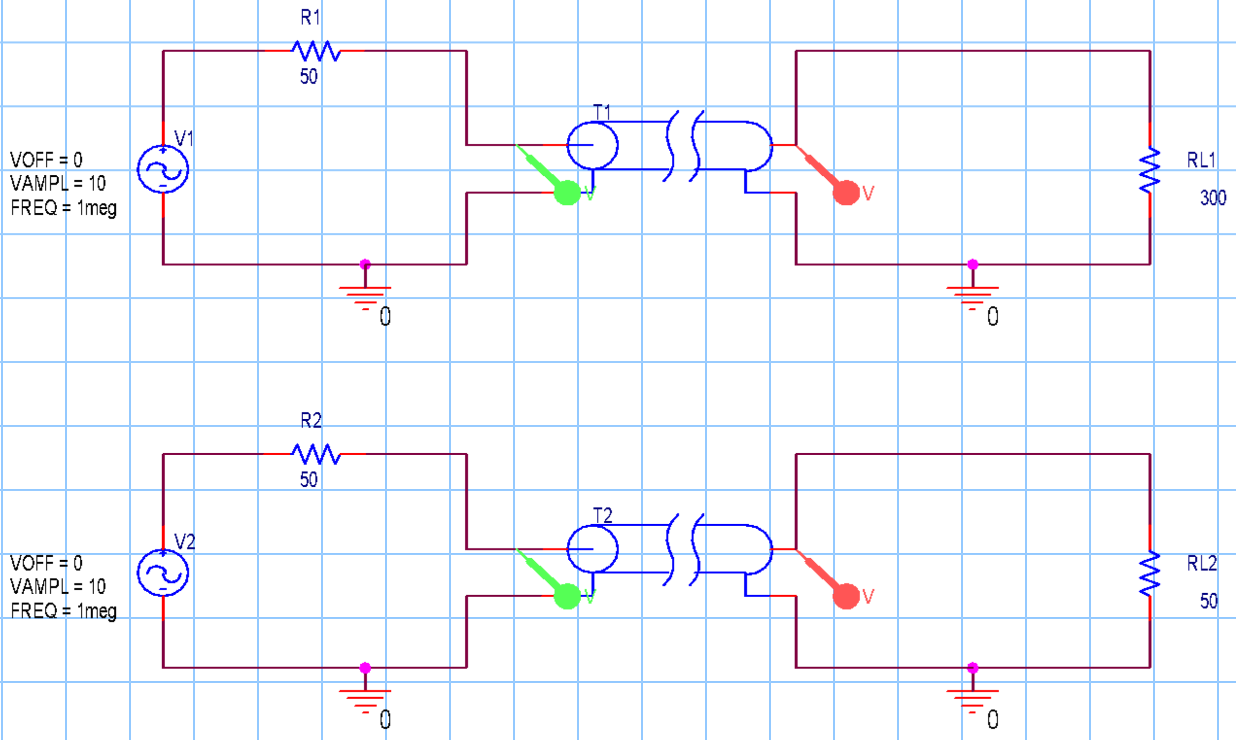
$$\Gamma(z) = \Gamma_L \cdot e^{-j \cdot 2 \cdot \beta \cdot (L - z)}$$

- We see that if you go backward a quarter wavelength from an open circuit load, you will see at that spot the <u>same voltage pattern</u> you would see if there were a short circuit load to the right of that point.

- In other words, as you move backward from the load, it begins to "look like" a different load. The expression for $\Gamma(z)$, the <u>phase-shifted reflection coefficient</u>, reflects this by taking into account the load as well as the distance from it.

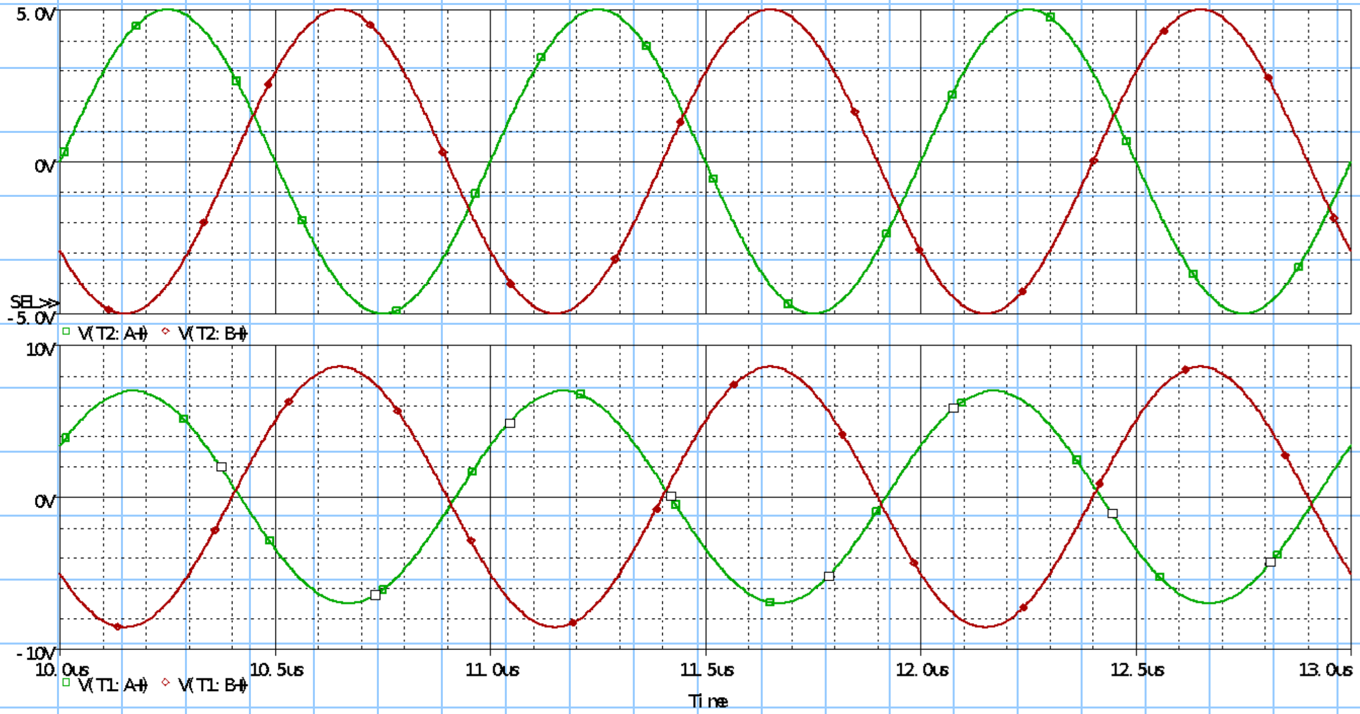
Generalized Impedance

For the same source and line, but different load:



Input Impedance

A change in the load results in a change in the voltages we measure at the input. If we define an "input impedance", this will also change.



Input Impedance

What does $Z_{in} = \dfrac{V_{in}}{I_{in}}$ look like?

To figure this out, we can use our expression for phase-shifted reflection coefficient.

$$\Gamma(z) = \frac{V^{-} \cdot e^{+j \cdot \beta \cdot z}}{V^{+} \cdot e^{-j \cdot \beta \cdot z}} = \frac{V^{-}}{V^{+}} \cdot e^{j \cdot 2 \cdot \beta \cdot z}$$

$$= \Gamma_L \cdot e^{j \cdot 2 \cdot \beta \cdot z}$$ if z =0 at LOAD

Input Impedance

Previously, we have seen:

$$\hat{V}(z) = V^{+}(z) + V^{-}(z) = V^{+} \cdot e^{-j\cdot\beta\cdot z} \cdot (1 + \Gamma(z))$$

What about I?

$$\hat{I}(z) = \frac{V^{+}(z)}{Z_o} - \frac{V^{-}(z)}{Z_o} = \frac{V^{+}}{Z_o} \cdot e^{-j\cdot\beta\cdot z} \cdot (1 - \Gamma(z))$$

Input Impedance

Form the Ratio (the generalized impedance):

$$\frac{\hat{V}(z)}{\hat{I}(z)} = Z_o \cdot \frac{1+\Gamma(z)}{1-\Gamma(z)} = Z(z)$$

We are primarily interested in the z=0 value (if we define z=0 as the source end.)

$$Z_{in}(z=0) = Z_o \cdot \frac{1+\Gamma(z=0)}{1-\Gamma(z=0)}$$

Input Impedance

Using algebra and trigonometry (Ulaby pg. 76), we can write this as:

$$Z_{in}(z=0) = Z_o \cdot \frac{Z_L + j \cdot Z_o \cdot \tan(\beta \cdot L)}{Z_o + j \cdot Z_L \cdot \tan(\beta \cdot L)}$$

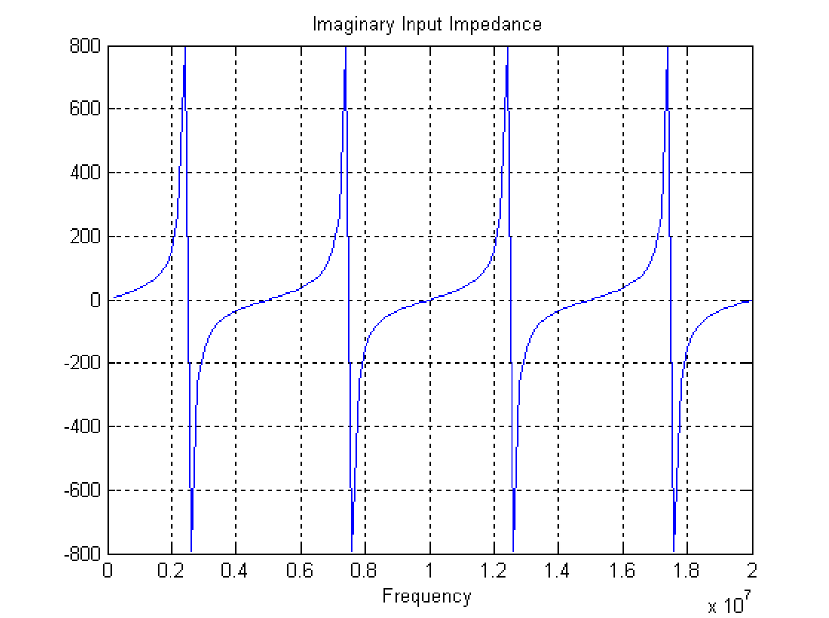
Special Case example: Z_L=0 (short circuit)

Can change Z_{in} by changing these two parameters

$$Z_{in}(z=0) = Z_o \cdot \frac{0 + j \cdot Z_o \cdot \tan(\beta \cdot L)}{Z_o + j \cdot 0 \cdot \tan(\beta \cdot L)} = j \cdot Z_o \cdot \tan(\beta \cdot L)$$

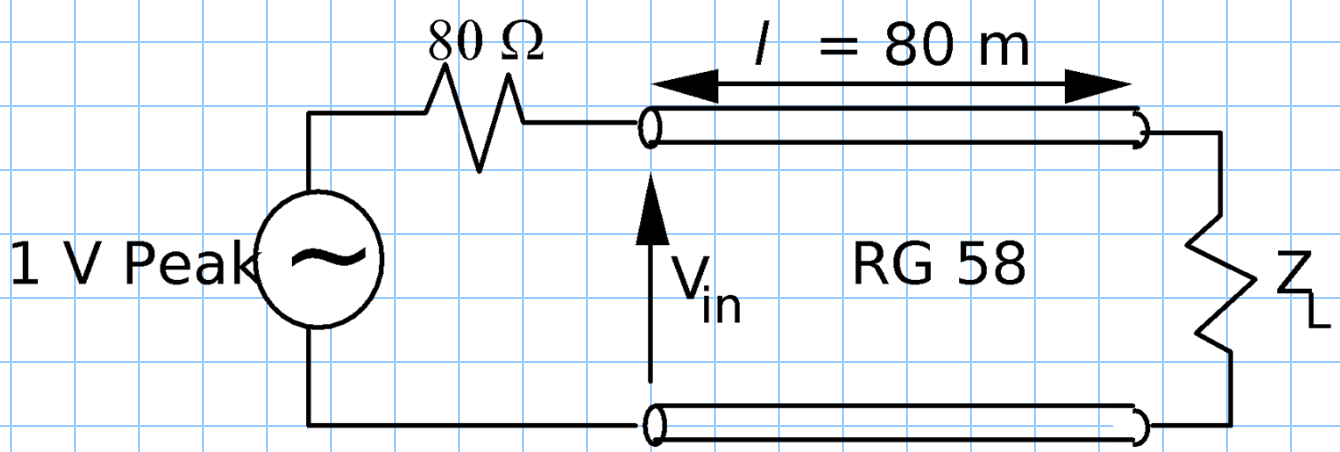
Input Impedance

For this short circuit load case, this impedance is imaginary (reactive) and you can vary frequency (and therefore β) to achieve any magnitude you want.



Input Impedance

Consider some other cases...



Input Impedance

Open Circuit Case:

$$Z_L = \infty$$

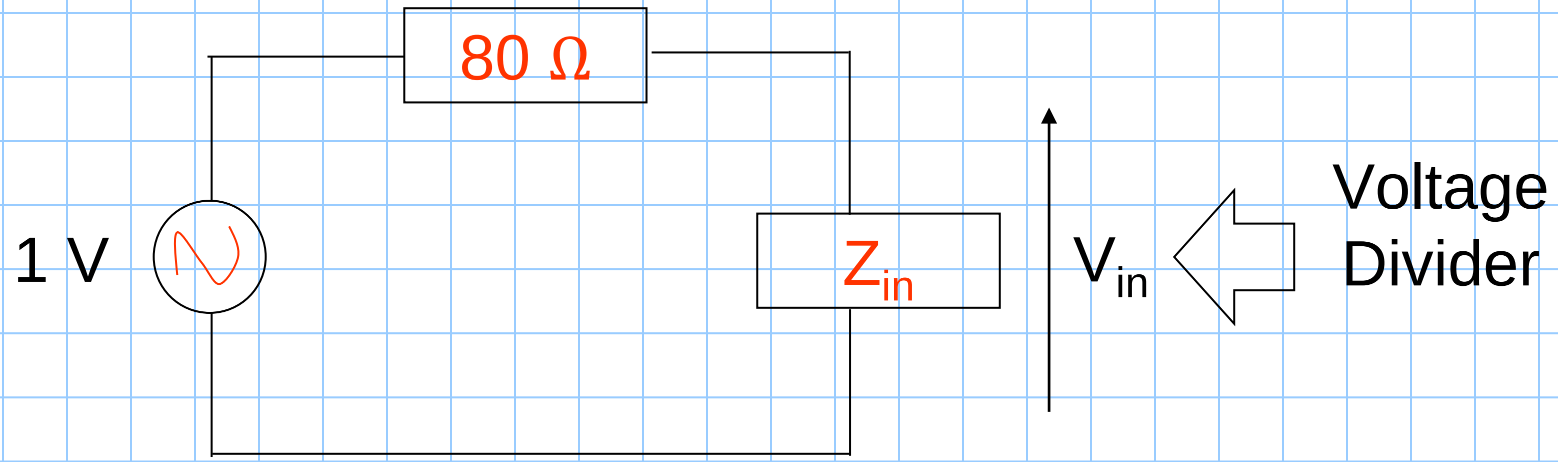
$$Z_{in} = Z_o \frac{Z_L + j \cdot Z_o \cdot \tan(\beta \cdot L)}{Z_o + j \cdot Z_L \cdot \tan(\beta \cdot L)}$$

Z_L = 93Ω:

Lots of complex algebra, but straightforward.

Input Impedance

Knowing Z_{in} (z=0), we can use it to represent an entire circuit network in calculations.



$$Power = \frac{1}{2} Re\left\{V_{in} \times I_{in}^{*}\right\} = \frac{1}{2} Re\left\{\frac{V_{in} \times V_{in}^{*}}{Z_{in}^{*}}\right\} = \frac{1}{2} Re\left\{\frac{\left|V_{in}\right|^{2}}{Z_{in}^{*}}\right\}$$

Input Impedance

In a Lossless Transmission Line, P_{in} flows into the Transmission Line and it is dissipated at the load:

$$P_{in} = \frac{1}{2} \cdot \frac{|V_L|^2}{Z_L}$$

What is the voltage at the load?

$$\hat{V}(z) = V^+ \cdot e^{-j \cdot \beta \cdot z} \cdot (1 + \Gamma(z))$$

$$\hat{V}(z = 0) = V_{in} = V^+ \cdot e^{-j \cdot \beta \cdot z} \cdot (1 + \Gamma(z = 0))$$

$$\Rightarrow V^+ = \frac{V_{in}}{1 + \Gamma(0)}$$

Can then plug back and get the full phasor expression

Input Impedance

Remember this expression for the voltage phasor:

$$V(z) = V^{+} e^{-j\beta z} + V^{-} e^{+j\beta z}$$

Armed with this expression and a transmission line's input voltage and load impedance, *we can now calculate the voltage and current everywhere on the line.*

Input Impedance

Special Cases

- Recall that the standing wave pattern repeated every half wavelength. Thus, we expect that this will also happen for $Z_{in.}$ First, consider the trivial case of $L=0$.

$$Z_{in} = Z_o \frac{Z_L + jZ_o \tan \beta L}{Z_o + jZ_L \tan \beta L} = Z_L$$

- Now let the line be a half wavelength long

$$\tan \beta L = \tan\left(\frac{2\pi}{\lambda}\frac{\lambda}{2}\right) = \tan(\pi) = 0 \qquad Z_{in} = Z_o \frac{Z_L + 0}{Z_o + 0} = Z_L$$

Input Impedance

Special Cases

- Thus, for a line that is exactly an integer number of half wavelengths long

$$Z_{in} = Z_L$$

- Thus, if you have a transmission line with the wrong characteristic impedance, you can match the load to the source by selecting a length equal to a half wavelength.

Input Impedance

Special Cases

- If the line is an odd multiple of a quarter wavelength, we also get an interesting result.

$$Z_{in} = Z_o \frac{Z_L + jZ_o \tan \beta L}{Z_o + jZ_L \tan \beta L} = Z_o \frac{jZ_o \tan \beta L}{jZ_L \tan \beta L} = \frac{Z_o^{\ 2}}{Z_L}$$

$$\tan \beta L = \tan \frac{2\pi}{\lambda} \frac{\lambda}{4} = \tan \frac{\pi}{2} \rightarrow \infty$$

Input Impedance

Special Cases

For the quarter-wave transmission line:

$$\frac{Z_{in}}{Z_0} = \frac{Z_0}{Z_L}$$

Thus, such a transmission line works like an "impedance transformer".

In applications, a quarter-wave T-line can be useful when you want to make one impedance look like another one.

Wrap-Up

- Homework 1 due **Sunday (the 21st) at 11:59pm**
- Studio Session 1 due **Wednesday (the 24th) at 11:59pm**
- Ziqiao’s office hours are 12-2pm Wednedays in **JEC 6318**
- I offer office hours by appointment. I will be setting up a booking link on the course website. In the meantime you can email/DM me.