ECSE 2500
Lec 6
Jan. 30

Topic: Conditional Probability
Total Probability Theorem
Bayes Rule

☐ Example on Rolling two 6-sided Dice (Con't)

$$P(\underbrace{\text{Sum} \geq 8}_{A} \mid \underbrace{\text{One die} \geq 4}_{B})$$

Definition

$$\overset{\downarrow}{=} \frac{P(A \cap B)}{P(B)}$$

$$P(B) = P(\text{One die} \geq 4) = 1 - P(B^c)$$

$$= 1 - P(\text{no dice} \geq 4)$$

$$= 1 - P(\text{one die} < 4) \cdot P(\text{2nd die} < 4)$$

$$= 1 - \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$$

$$P(A \cap B) = P(A) = P(\text{sum}=8) + P(\text{sum}=9) + P(\text{sum}=10)$$
$$+ P(\text{sum}=11) + P(\text{sum}=12)$$
$$= \frac{15}{36} = \frac{5}{12}$$

$$P(\underbrace{\text{Sum is} \geq 8}_{A} \mid \underbrace{\text{one die} = 1}_{B})$$

$$= \frac{P(A \cap B)}{P(B)} = \frac{0}{P(B)} = 0$$

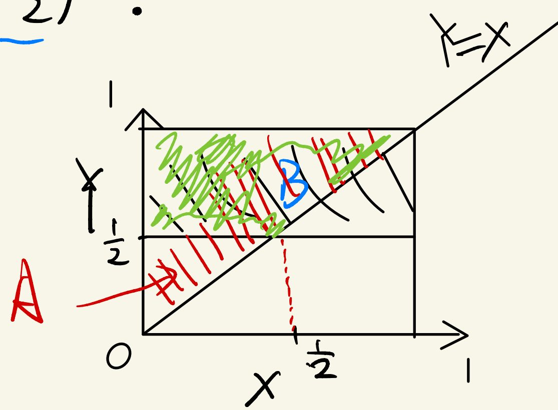
This example is for conditional probability in discrete sample space.

Similar ideas also hold for continuous sample space.

☐ Example

Two number (X, Y) uniformly randomly drawn from $[0, 1]$.

$$P(\underbrace{Y > X}_{A} \mid \underbrace{Y > \tfrac{1}{2}}_{B}) \ ?$$



$$= \frac{P(A \cap B)}{P(B)}$$

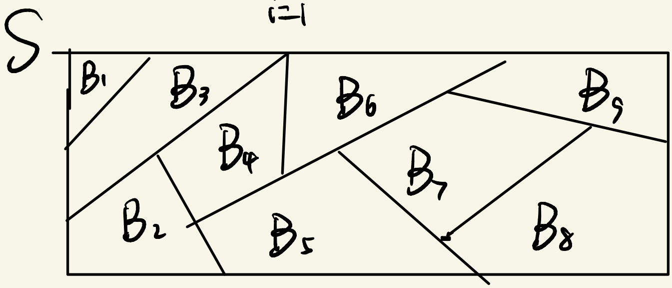
$$= \frac{\frac{1}{2} - \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}}{\frac{1}{2}}$$

$$= \frac{3}{4}$$

□ Now we introduce the idea of a partition of the sample space S:

- A set of Events $\{B_1, B_2, \cdots, B_n\}$ such that

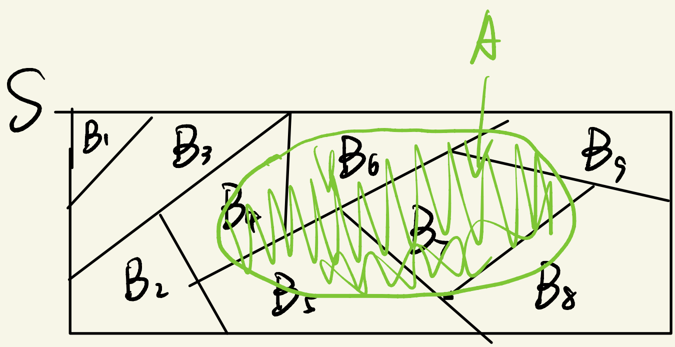
1) $B_i \cap B_j = \phi, \quad \forall i \neq j = 1, 2, \cdots, n$
2) $\bigcup_{i=1}^{n} B_i = S$



Based on Axiom 3.

$$\sum_{i=1}^{n} P(B_i) = P\left(\bigcup_{i=1}^{n} B_i\right) = P(S) \overset{\text{Axiom 2}}{=} 1$$

Now consider an Event A



$$\begin{aligned} P(A) &= P(A \cap S) \\ &= P\left(A \cap \left(\bigcup_{i=1}^{n} B_i\right)\right) \\ &= P\left(\bigcup_{i=1}^{n} (A \cap B_i)\right) \\ &= \sum_{i=1}^{n} P(A \cap B_i) \end{aligned}$$

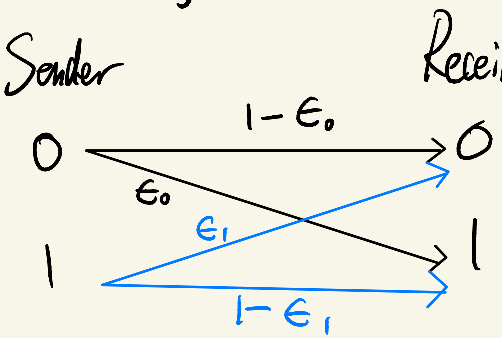
$$\Rightarrow \boxed{\begin{aligned} P(A) &= \sum_{i=1}^{n} P(A \cap B_i) \\ &= \sum_{i=1}^{n} P(A|B_i)\,P(B_i) \end{aligned}}$$

This is the so-called **Total Probability Theorem**.

This is useful since it relates probability of an event to its conditional probabilies with a set of possible events.

I will give several examples to show calculating $P(A)$ is difficult but computing $P(A|B_i)P(B_i)$ is simpler.

Example Binary communication channel



ϵ_0, ϵ_1 are predefined constants.

$$P(\text{Rec}: 0 \mid \text{Send } 0) = 1-\epsilon_0$$

$$P(\text{Rec}: 1 \mid \text{Send } 0) = \epsilon_0$$

$$P(\text{Rec}: 0 \mid \text{Send } 1) = \epsilon_1$$

$$P(\text{Rec}: 1 \mid \text{Send } 1) = 1-\epsilon_1$$

$P(\text{Rec } 0) = ?$ $\quad P(\text{Rec}=1) = ?$

Assumption: $P(\text{Send } 0) = 1-p$

(known) $P(\text{Send } 1) = p$

Now let's compute

$$P(\underbrace{\text{Error in Receiver}}_{A})$$

$$= P(\{(\text{Rec } 0, \text{Send } 1), (\text{Rec } 1, \text{Send } 0)\})$$

Total probability

$$= \underbrace{P(\text{Error} \mid \text{Send } 0)} \cdot P(\underbrace{\text{Send } 0}_{B_1}) +$$

$$P(\text{Error} \mid \text{Send } 1) \cdot P(\underbrace{\text{Send } 1}_{B_2})$$

$$= \epsilon_0 \cdot (1-p) + \epsilon_1 \cdot p$$

$$= \epsilon_0 (1-p) + \epsilon_1 p.$$

☐ An important application of conditional probability is Bayes Rule. As before, assume we have a partition $\{B_1, B_2, \ldots, B_n\}$ of the sample space S. Now we observe an event A occurs.

And we want determine the probability of $B_1, B_2 \ldots B_n$ given Event A:

$$\boxed{P(B_i|A)} = \frac{P(A \cap B_i)}{P(A)}$$

$$= \frac{\boxed{P(A|B_i)} P(B_i)}{P(A)}$$

Total Probability

$$\overset{\downarrow}{=} \frac{P(A|B_i) P(B_i)}{\sum_{k=1}^{n} P(A|B_k) P(B_k)}$$

$P(B_i), i=1,2,\ldots,n$ are called the prior or a prior distribution of the events. Early

$P(B_i|A), i=1,2,\ldots,n$ are called the posterior or a posterior distribution of the events. Late/After

e.g. We update our belife after observing an output of the state.

Q: How to use Bayes Rule?

Example: In communication systems, we often want to answer questions like

"What is Probability Symbol "1" was sent given that Symbol "1" was received?"

$$P(\underbrace{\text{Send } 1}_{A} \mid \underbrace{\text{Rec } 1}_{B})$$

This is the reverse of the conditional probability.

$$\overset{\text{Bayes rule}}{=} \frac{P(\text{Rec } 1 \mid \text{Send } 1)\, P(\text{Send } 1)}{P(\text{Rec } 1)}$$

$$= \frac{P(\text{Rec } 1 \mid \text{Send } 1)\, P(\text{Send } 1)}{P(\text{Rec } 1 \mid \text{Send } 1)\, P(\text{Send } 1) + P(\text{Rec } 1 \mid \text{Send } 0)\, P(\text{Send } 0)}$$

$$= \frac{(1-\epsilon_1)\, p}{\epsilon_0 (1-p) + (1-\epsilon_1)\, p}$$