Fields and Waves I

Lecture 19

Maxwell's Equations, Displacement Current,
EM Waves

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Exam 2

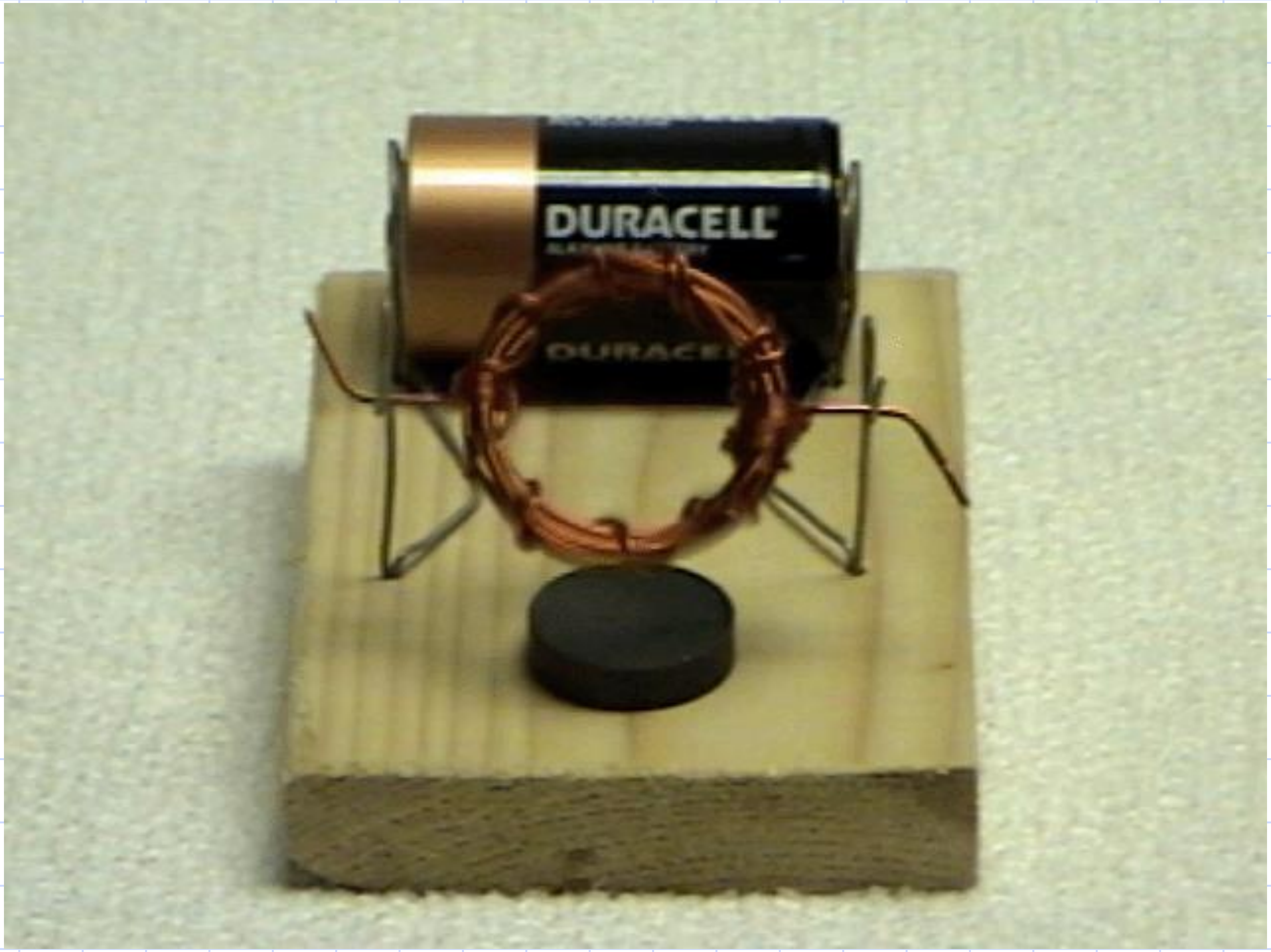
Skill	Description	Score
21	Calculate a single-stub match for an open-circuit or short-circuit load using a Smith chart.	2.81
2b	Be able to successfully calculate the voltage between two points based on the electric field between them.	2.72
2c	Use Gauss's Law to calculate an electric field from the geometries of a region's materials and charge distributions, or vice versa.	2.98
2d	Demonstrate an understanding of the geometry of electric fields and be able to draw a diagram of a given electric field distribution.	3.07
2e	Calculate electric force from electric charge using Coulomb's Law.	4.35
2f	Evaluate a static electric field at a boundary between two materials with different permittivities.	3.57

Exam 2

2g	Demonstrate an understanding of the effect of perfect conductors on the electric field both inside of them and outside their surfaces.	2.32
2h	Calculate the capacitance of a given distribution of conductors and/or dielectrics with simple geometries.	3.1
2i	Calculate the energy density in an electric field.	2.7
2j	Calculate the dielectric breakdown of a given dielectric medium.	2.35
2k	Use Laplace and/or Poisson's equations via the Finite Difference Method to solve a simple voltage field.	3.67
2l	Know the relationship between conductivity, current density, and electric field. Given appropriate information, be able to calculate these or related quantities (such as resistance or current).	3.52

Magnetic Force

How to account for forces on currents?



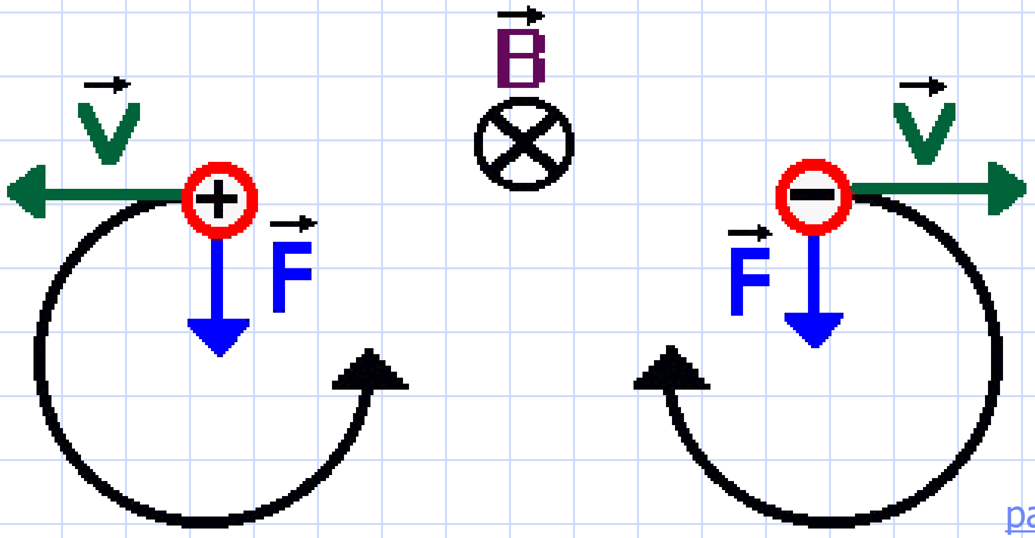
Giphy

Magnetic Force

Force on Currents

Force on a point charge: $\vec{F} = q \cdot (\vec{v} \times \vec{B})$

- For a constant B-field like the one below, F causes a change in v which in turn causes a change in F, creating a circular path.



panomics.pnnl.gov

Magnetic Force

Force on Currents

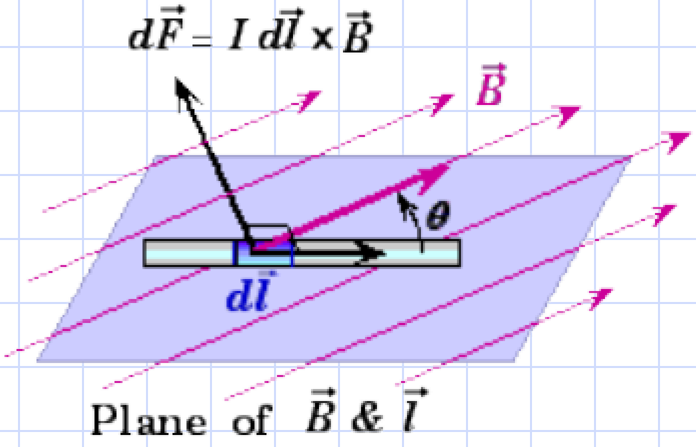
First approach - similar to that for individual particles

For one particle: $F = qE = q \cdot (v \times B)$

For many particles: $\dfrac{F}{volume} = \rho \cdot (\vec{v} \times \vec{B}) = \vec{j} \times \vec{B}$

For a wire in a magnetic field.

$$F = \int j \times B dv = \int I dl \times B$$

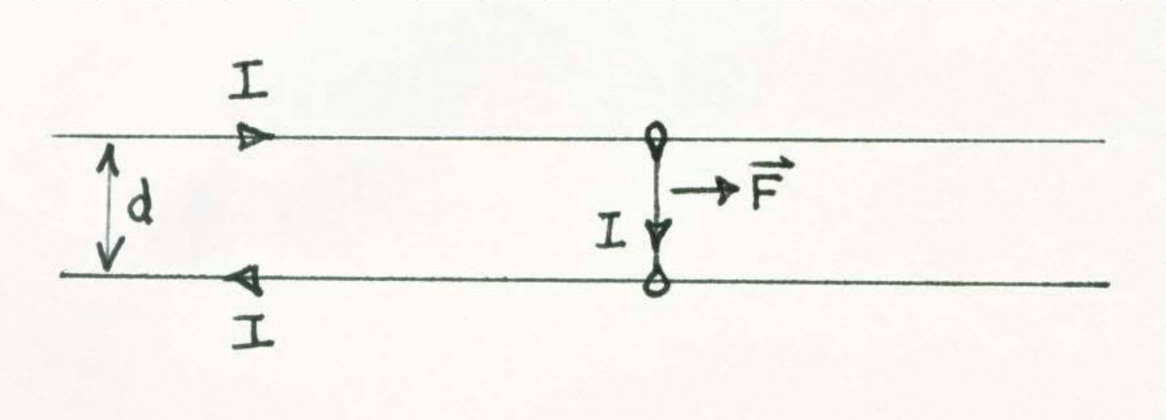


http://www.ac.wwu.edu/~vawter/PhysicsNet/Topics/MagneticField/MFOnWire.html

Magnetic Force

Rail Gun

If a sliding contact is placed across a two wire transmission line carrying a large current, a very large force can result on the contact. Assume that all the wires (including the slider) have a radius = *a* and that the transmission line wires are separated by a distance *d*.



Magnetic Force

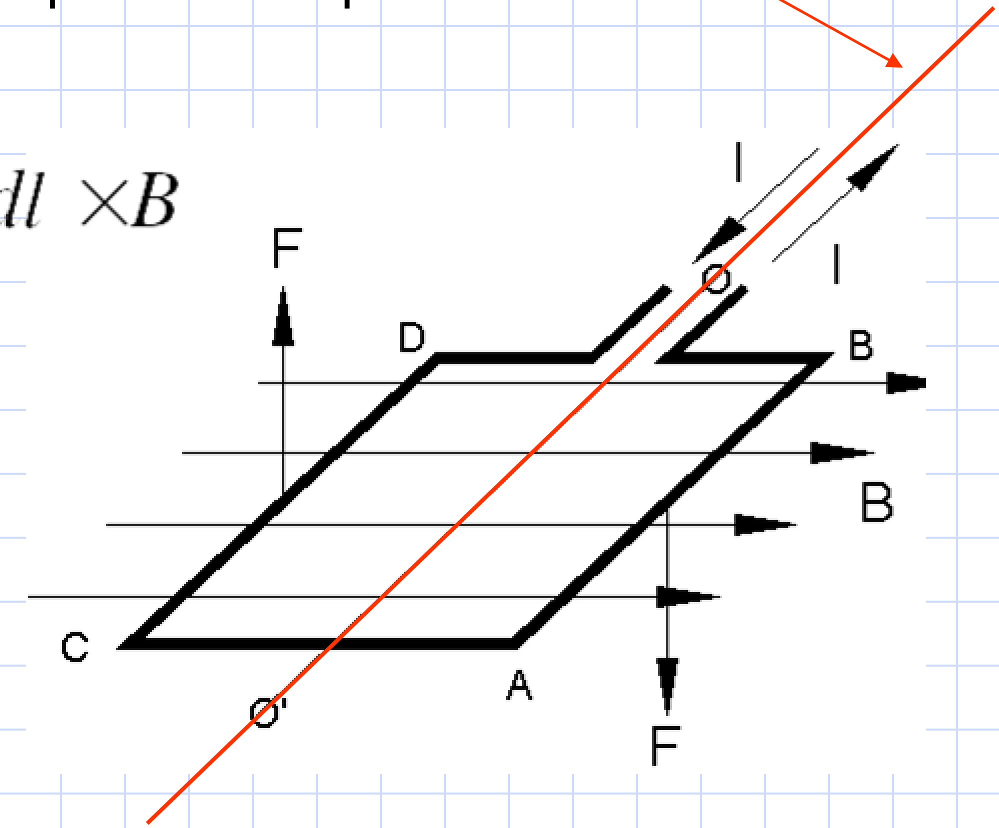
Rail Gun

Magnetic Force

Current Loop

The force on a current loop in a magnetic field can result in rotational torque if the loop has a fixed axis as shown.

$$F = \int j \times B dv = \int I dl \times B$$

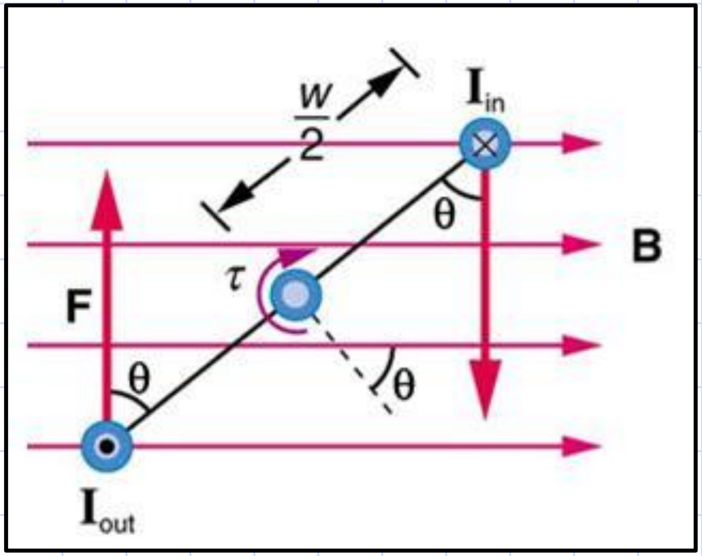


Magnetic Force

Current Loop

$$F = \int j \times B dv = \int I dl \times B$$

$$\tau = IAB sin\theta$$



libretexts.org

B = field strength
θ = angle between the loop surface normal and direction of B field
A = area of loop

Magnetic Force

Current Loop

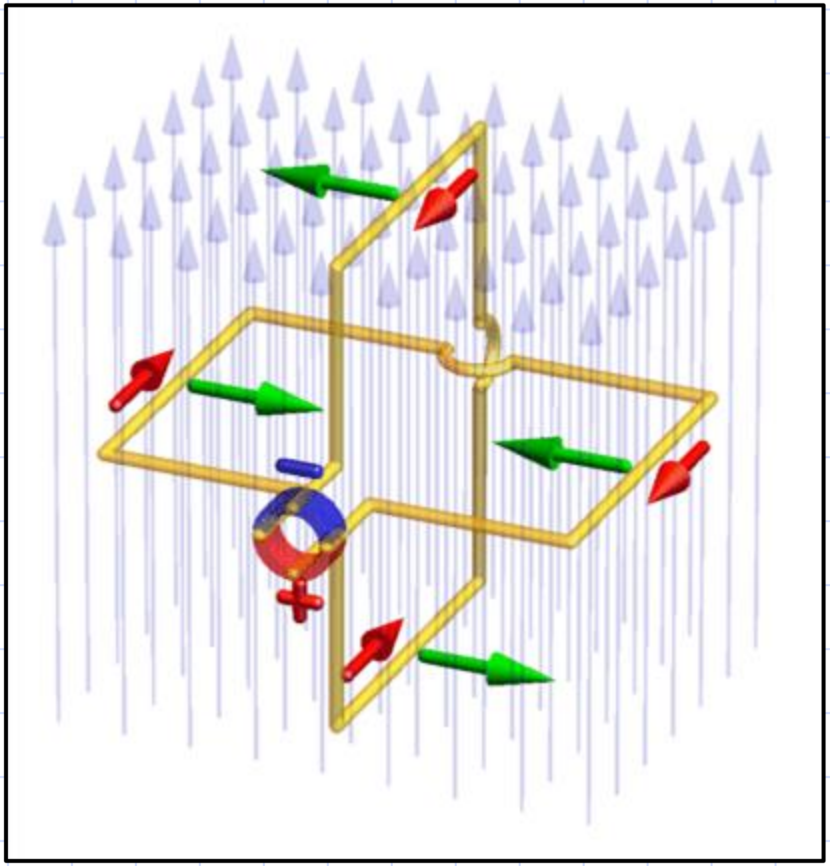
The presence of the rotational torque suggests that this loop could be used to make a DC motor. What happens when the is placed at an angle relative to the field, then allowed to rotate freely?

http://physics.bu.edu/~duffy/semester2/c13_torque.html

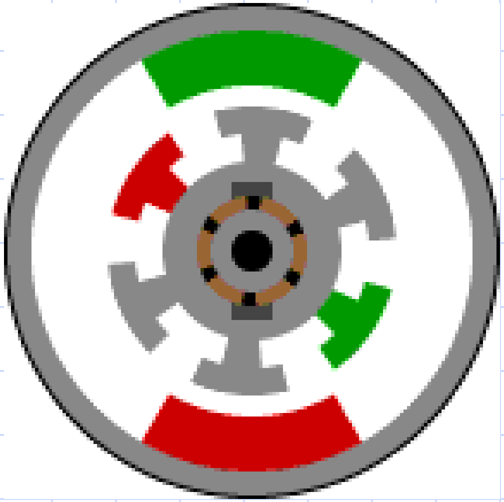
DC Motors

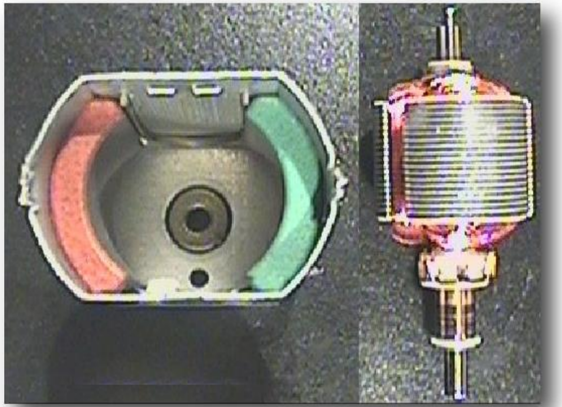
The Commutator

- A commutator allows a current loop to switch current direction at different stages of its rotation.
- As a result, the loop can achieve an average positive net torque through its rotation, and can rotate continuously while current is applied.



DC Motors

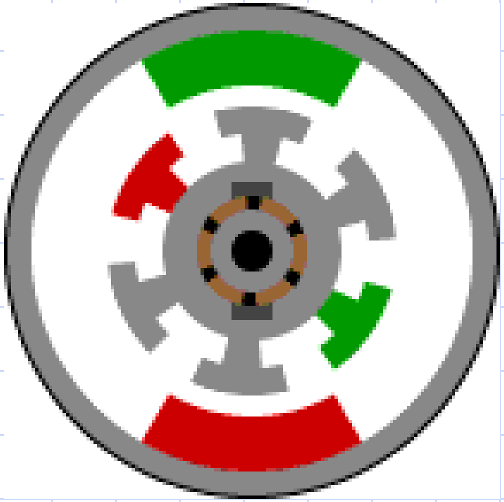




- The stator is the stationary outside part of a motor. The rotor is the inner part which rotates.
- In the motor animations, red represents a magnet or winding with a north polarization, while green represents a magnet or winding with a south polarization. Opposite, red and green, polarities attract.

http://www.freescale.com/files/microcontrollers/doc/train_ref_material/MOTORDCTUT.html

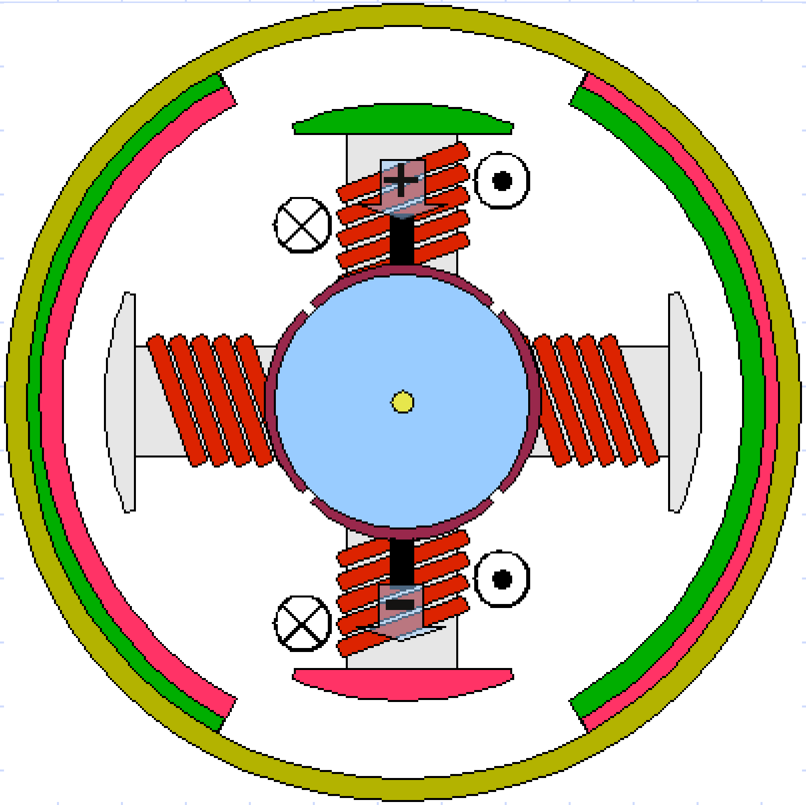
DC Motors



- Just as the rotor reaches alignment, the brushes move across the commutator contacts and energize the next winding.
- Above, the commutator contacts are brown and the brushes are dark grey.

http://www.freescale.com/files/microcontrollers/doc/train_ref_material/MOTORDCTUT.html

DC Motors



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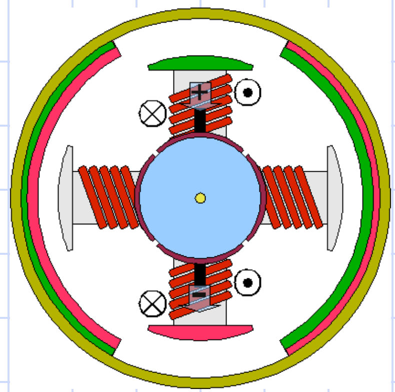
Another animated example

Motors vs Generators

Mechanical Work ⟷ Electrical Work



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- In general, motors and generators can perform a two-way conversion of electrical and mechanical work (a motor can act as a generator and vice versa)
- You can think of motors and generators as a <u>single class</u> of electromechanical device with variations for specific applications

Maxwell's Equations

Full Version

Added term in curl H equation for time varying electric field that gives a magnetic field.

$$\oint \vec{H} \cdot d\vec{l} = \int \vec{J} \cdot d\vec{S} + \int \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\oint \vec{B} \cdot d\vec{S} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{S}$$

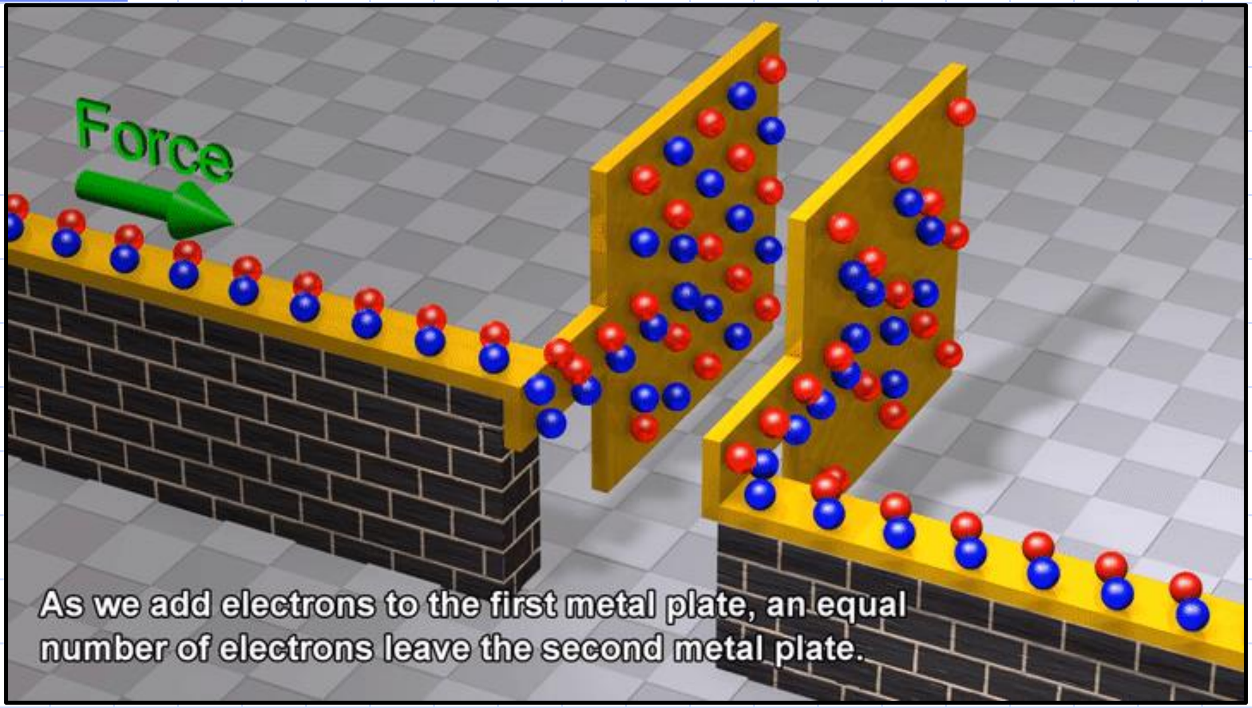
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\oint \vec{D} \cdot dS = \oint \rho dv = Q_{encl}$$

$$\nabla \cdot \vec{D} = \rho$$

First introduced by Maxwell in 1873

Displacement Current



Displacement current is the current that flows into and out of a capacitor, not due any connection between the two plates but due to the electric field between them.

Displacement Current

Ampere's Law – Curl H Equation

(quasi) Static field

$$\nabla \times H = j$$

Time varying field

$$\nabla \times \vec{H} = \vec{j} + \frac{\partial D}{\partial t}$$

Displacement current density

Integral Form of Ampere's Law for time varying fields

$$\oint_C \vec{H} \bullet d\vec{l} = I_c + \int_S \frac{\partial D}{\partial t} \bullet d\vec{s}$$

Displacement current

I_c – Conduction Current [A] linked to a conductivity property

D – Electric Flux Density (Electric Displacement) [in C/unit area]

j_c – Conduction Current Density (in A/unit area)

Displacement Current

$$\oint_C H \bullet dl = I_c + I_d = I \longrightarrow \textit{Total current}$$

Conduction current density

$$I_c = \int j_c \bullet ds = \int \sigma E \bullet ds \qquad (j_c = \sigma E)$$

$$I_d = \int_S \vec{j}_d \bullet \vec{ds} = \int_S \frac{\partial D}{\partial t} \bullet \vec{ds}$$

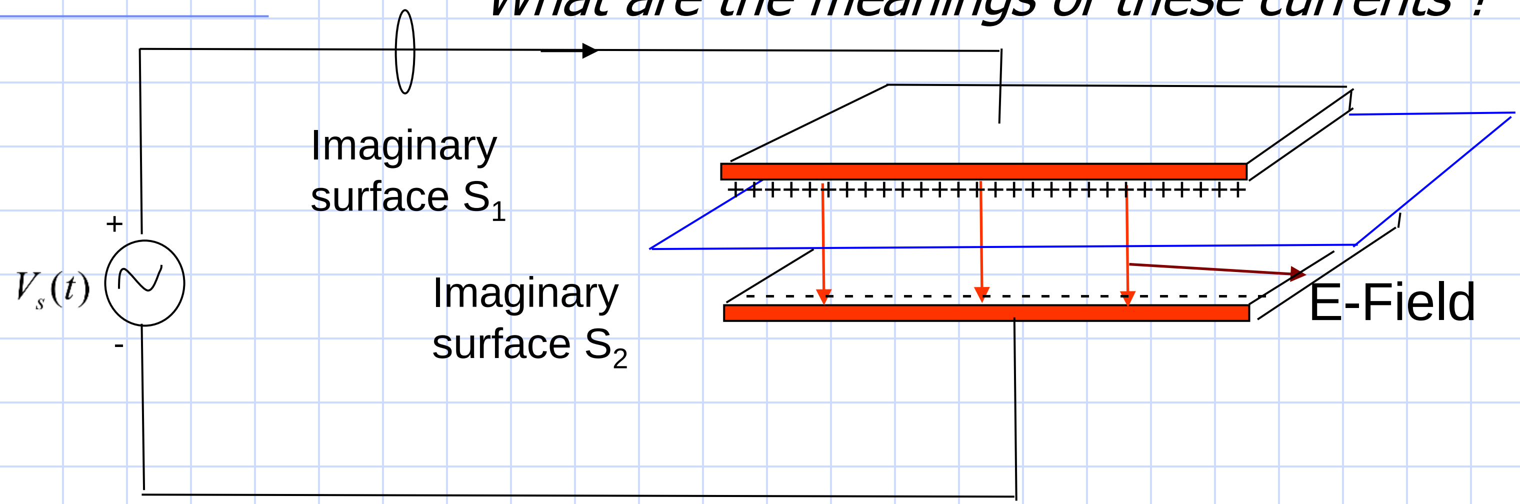
Displacement current density

Connection between electric and magnetic fields under time varying conditions

Displacement Current

Parallel Plate Capacitor

What are the meanings of these currents ?



$$V_s(t) = V_0 \cos \omega t$$

S_1=cross section of the wire

S_2=cross section of the capacitor

I_{1c}, I_{1d} : conduction and displacement currents in the wire

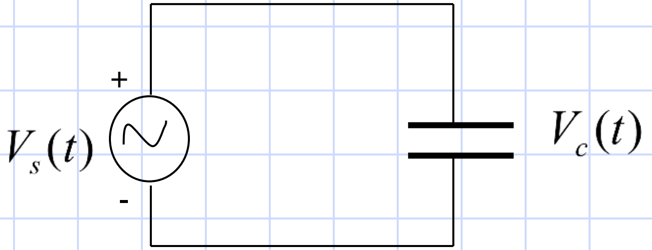
I_{2c}, I_{2d} : conduction and displacement currents through the capacitor

Displacement Current

Parallel Plate Capacitor

The wire is considered as a perfect conductor

$I_{1d} = 0$



From circuit theory:

$$V_c = V_s(t)$$

$$I_{1c} = C\frac{dV_C}{dt} = C\frac{d}{dt}(V_0 \cos \omega t) = -CV_0 \omega \sin \omega t$$

Total current in the wire:

$$\boxed{I_1 = I_{1c} = -CV_0 \omega \sin \omega t}$$

Displacement Current

Parallel Plate Capacitor

The dielectric is considered as perfect (zero conductivity)

Electrical charges can't move physically through a perfect dielectric medium

$I_{2c} = 0$ no conduction between the plates

The electric field between the capacitors

$$\vec{E} = \frac{V_c}{d}\hat{a}_y = \frac{V_0}{d}\cos\omega t\,\hat{a}_y$$

d :spacing between the plates

Displacement Current

Parallel Plate Capacitor

The displacement current I_{2d}

$$I_{2d} = \int_S \frac{\partial D}{\partial t} \bullet \vec{ds}$$

$$= \int_A \left[\frac{\partial}{\partial t} \left(\frac{\varepsilon V_0}{d} \cos \omega t \hat{a}_y \right) \right] \bullet \left(\hat{a}_y ds \right)$$

$$= - \frac{\varepsilon A}{d} V_0 \omega \sin \omega t = - C V_0 \omega \sin \omega t \quad \Rightarrow \quad \boxed{I_{2d} = I_{1c}}$$

Displacement current doesn't carry real charge, but behaves like a real current

If wire has a finite conductivity σ then both wire and dielectric have conduction AND displacement currents

Displacement Current

Do Lecture 18 Exercise 1 in groups of up to 4.

Maxwell's Equations

Full Version

$$\oint \vec{H} \cdot d\vec{l} = \int \vec{J} \cdot d\vec{S} + \int \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\oint \vec{B} \cdot d\vec{S} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{S}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

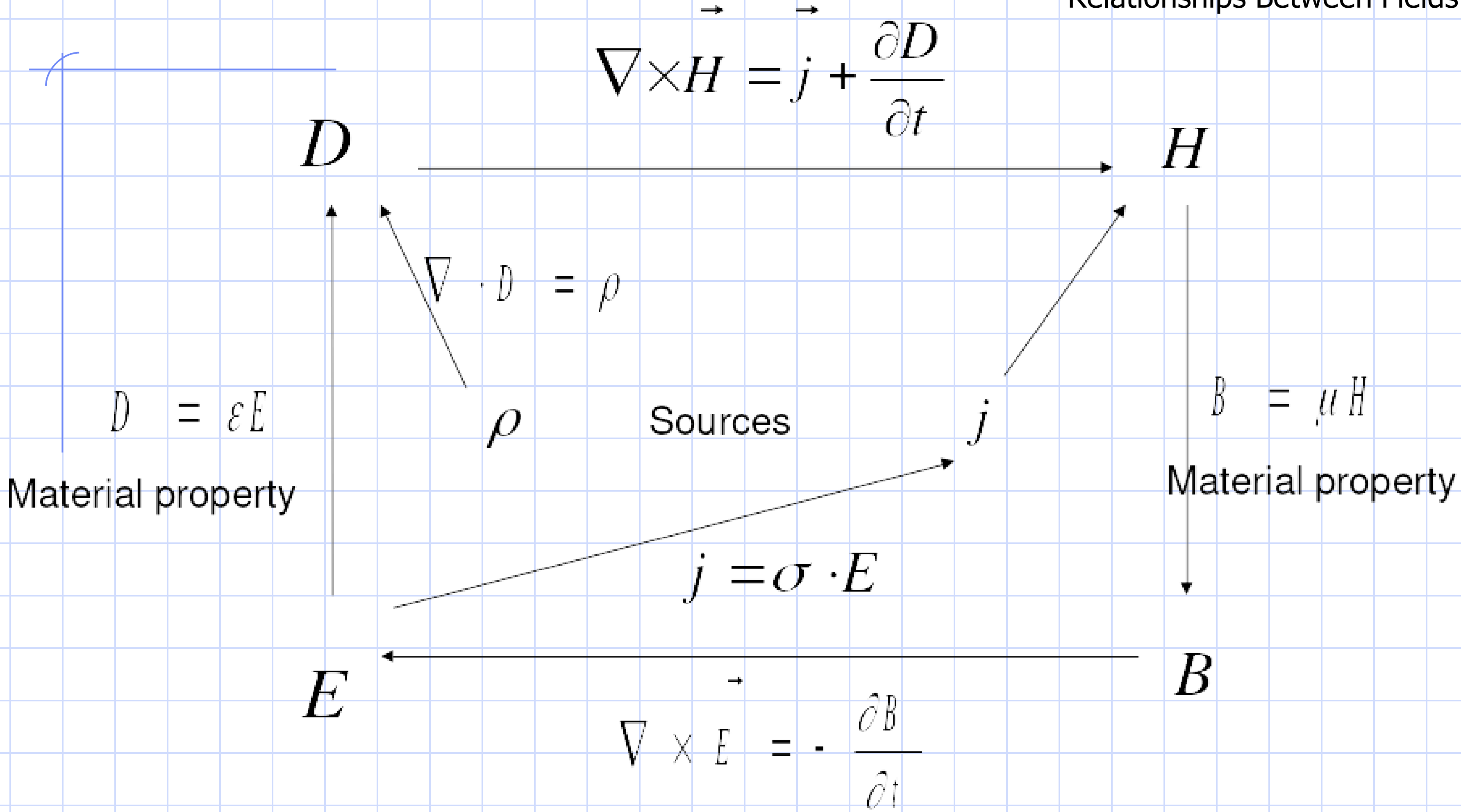
$$\oint \vec{D} \cdot dS = \oint \rho dv = Q_{encl}$$

$$\nabla \cdot \vec{D} = \rho$$

Note that the time-varying terms couple electric and magnetic fields in both directions. Thus, in general, we cannot have one without the other.

Maxwell’s Equations

Relationships Between Fields



Maxwell’s equations are fully coupled.

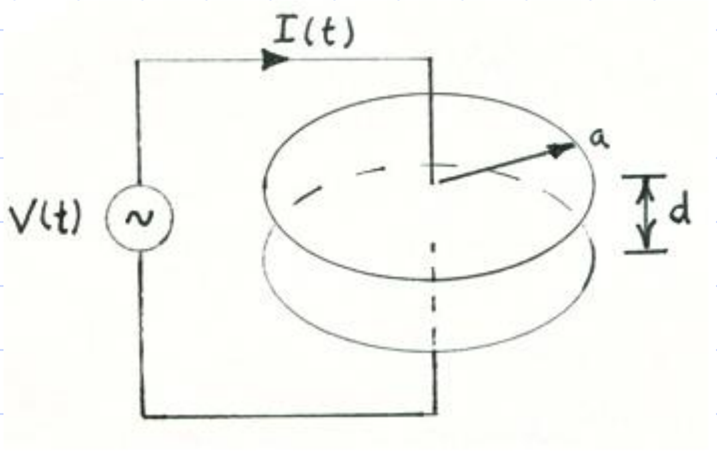
Capacitor Example

A parallel plate capacitor with circular plates and an air dielectric has a plate radius of 5 mm and a plate separation of d=10 μm. The voltage across the plates is $V = 5\cos\omega t$ where $\omega = 2\pi 100 kHz$

a. Find **D** between the plates.

a. Determine the displacement current density, ∂**D**/∂t.

c. Compute the total displacement current, ∫ ∂**D**/∂t · **ds** , and compare it with the capacitor current, I = C dV/dt.



d. What is **H** between the plates?

e. What is the induced emf ?

Capacitor Example

The electric field for a parallel plate capacitor driven by a time-varying source is

$$\vec{E}(t) = -\frac{V(t)}{d}\hat{z} = -\frac{\rho_s(t)}{\varepsilon}\hat{z}$$

The time-varying electric field now produces a source for a magnetic field through the displacement current . We can solve for the magnetic field in the usual manner.

$$\oint \vec{H}\cdot d\vec{l} = \cancelto{0}{\int \vec{J}\cdot d\vec{S}} + \frac{d}{dt}\int \vec{D}\cdot d\vec{S}$$

Capacitor Example

The total displacement current between the capacitor plates

$$I_d = -\epsilon \int \frac{\partial}{\partial t}\left(-\frac{V(t)}{d}\right)\hat{z} \cdot d\vec{S} = \frac{\epsilon \pi a^2}{d}\frac{\partial V(t)}{\partial t}$$

Using phasor notation for the voltage and current

$$V(t) = \mathrm{Re}\left(V_o e^{j\omega t}\right) \qquad I_D = j\omega \frac{\varepsilon \pi a^2}{d} V_o$$

Capacitor Example

Applying Ampere's Law to a circular contour with radius $r < a$, the fraction of the displacement current enclosed is

$$I_D \frac{r^2}{a^2}$$

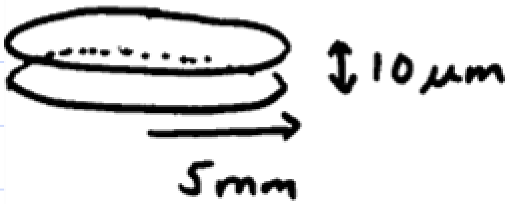
Ampere's Law then gives us

$$\oint \vec{H} \cdot d\vec{l} = H_\phi 2\pi r = I_D \frac{r^2}{a^2}$$

$$H_\phi = I_D \frac{r}{2\pi a^2} = j\omega \frac{\varepsilon \pi a^2}{d} V_o \frac{r}{2\pi a^2} = j\omega \frac{\varepsilon r}{2d} V_o$$

Thus, we now have both electric and magnetic fields between the plates

Capacitor Example



For ∥ plate capacitor

$\vec{E} \approx$ constant between plates $= -\nabla V$

$$= -\frac{\partial V}{\partial z}\hat{a}_z$$

$$\vec{E} = -\frac{\Delta V}{\Delta z}\hat{a}_z = \frac{-5\cos\omega t}{10\,\mu m}\hat{a}_z = -5\times 10^{5}\cos\omega t\,\hat{a}_z$$

$$\vec{D} = \varepsilon_0\vec{E} = \boxed{-4.43\times 10^{-6}\,\hat{a}_z \cos\omega t\ C/m^2} = -D_0\cos\omega t\ \hat{a}_z$$

$$\frac{\partial\vec{D}}{\partial t} = \omega D_0\sin\omega t\ \hat{a}_z = (2\pi\times 10^{5})(4.43\times 10^{-6})\sin\omega t\,\hat{a}_z$$

$$\boxed{\frac{\partial\vec{D}}{\partial t} = 2.78\ \sin\omega t\,\hat{a}_z}$$

Capacitor Example

NO

$I_D = \int \frac{\partial \vec{D}}{\partial t} \cdot d\vec{s} = \frac{\partial \vec{D}}{\partial t} \times \pi r^2$ since $\frac{\partial \vec{D}}{\partial t}$ is constant $\quad r = 5\times10^{-3}$

$I_D = 2.78\,\pi\,(5\times10^{-3})^2 \sin\omega t = \boxed{2.18\times10^{-4}\sin\omega t}$

$I = C\frac{dV}{dt}\ ;\ C = \varepsilon_0 \frac{A}{d} = \varepsilon_0 \frac{\pi r^2}{d} = \varepsilon_0 \frac{\pi(5\times10^{-3})^2}{10^{-5}} = 6.95\times10^{-11}\ \text{F}$

$\frac{dV}{dt} = -5\omega \sin\omega t\ ;\ I = -5\omega C \sin\omega t = -5(2\pi\times10^5)(6.95\times10^{-11}) \sin\omega t$

$\boxed{I = -2.18\times10^{-4}\sin\omega t}$

sign difference $\underline{\downarrow I} \uparrow v$ $\quad I = C\frac{dV}{dt}$ convention

For I_D took $d\vec{s}$ in $+\hat{a}_z$ $\therefore$ $I\uparrow$

$\oint \vec{H}\cdot d\vec{l} = \iint\left(\cancel{\vec{J}}^{\,0} + \frac{\partial \vec{D}}{\partial t}\right)\cdot d\vec{s}$

$2\cancel{\pi}\cancel{r} H_\varphi = \cancel{\pi} r^{\cancel{2}} \frac{\partial D}{\partial t} \Rightarrow \vec{H} = \frac{r}{2}\frac{\partial D}{\partial t}\hat{a}_\varphi = 1.39\, r \sin\omega t\ \hat{a}_\varphi$

Conductors vs. Dielectrics

The analysis of the capacitor under time-varying conditions assumed that the insulator had no conductivity. If we generalize our results to include both σ and ε we will have both a conduction and a displacement current.

$$I = I_C + I_D = \sigma \pi a^2 \frac{V_o}{d} + j\omega \frac{\varepsilon \pi a^2}{d} V_o = (\sigma + j\omega\varepsilon)\pi a^2 \frac{V_o}{d}$$

Note that the conduction current has a phase angle of zero degrees while the displacement current has an angle of 90 degrees.

Conductors vs. Dielectrics

The material will behave mostly like a conductor when

$$\frac{|I_C|}{|I_D|} = \frac{\sigma}{\omega\varepsilon} >> 1$$

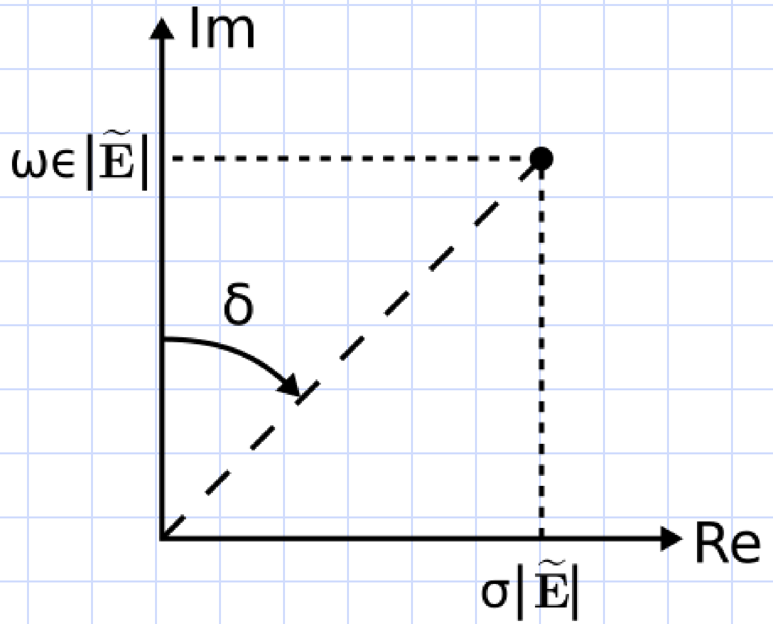
The material will behave mostly like a dielectric when

$$\frac{|I_C|}{|I_D|} = \frac{\sigma}{\omega\varepsilon} << 1$$

Conductors vs. Dielectrics

Loss tangent of the material: $$\tan\delta = \frac{\sigma}{\omega\varepsilon}$$

This tells us the phasor-domain angle of the current that results from the conduction and displacement currents combined.



Source: LibreTexts

Maxwell's Equations

Full Version

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$$\oint \vec{D} \cdot dS = \oint \rho dv = Q_{encl} \qquad \nabla \cdot \vec{D} = \rho$$

First introduced by Maxwell in 1873

Maxwell's Equations

Quasi-Static Solutions

Maxwell's Equations.

Need a simultaneous solution for the electric and magnetic fields

Lead to a wave equation identical in form to the wave equation found for transmission lines

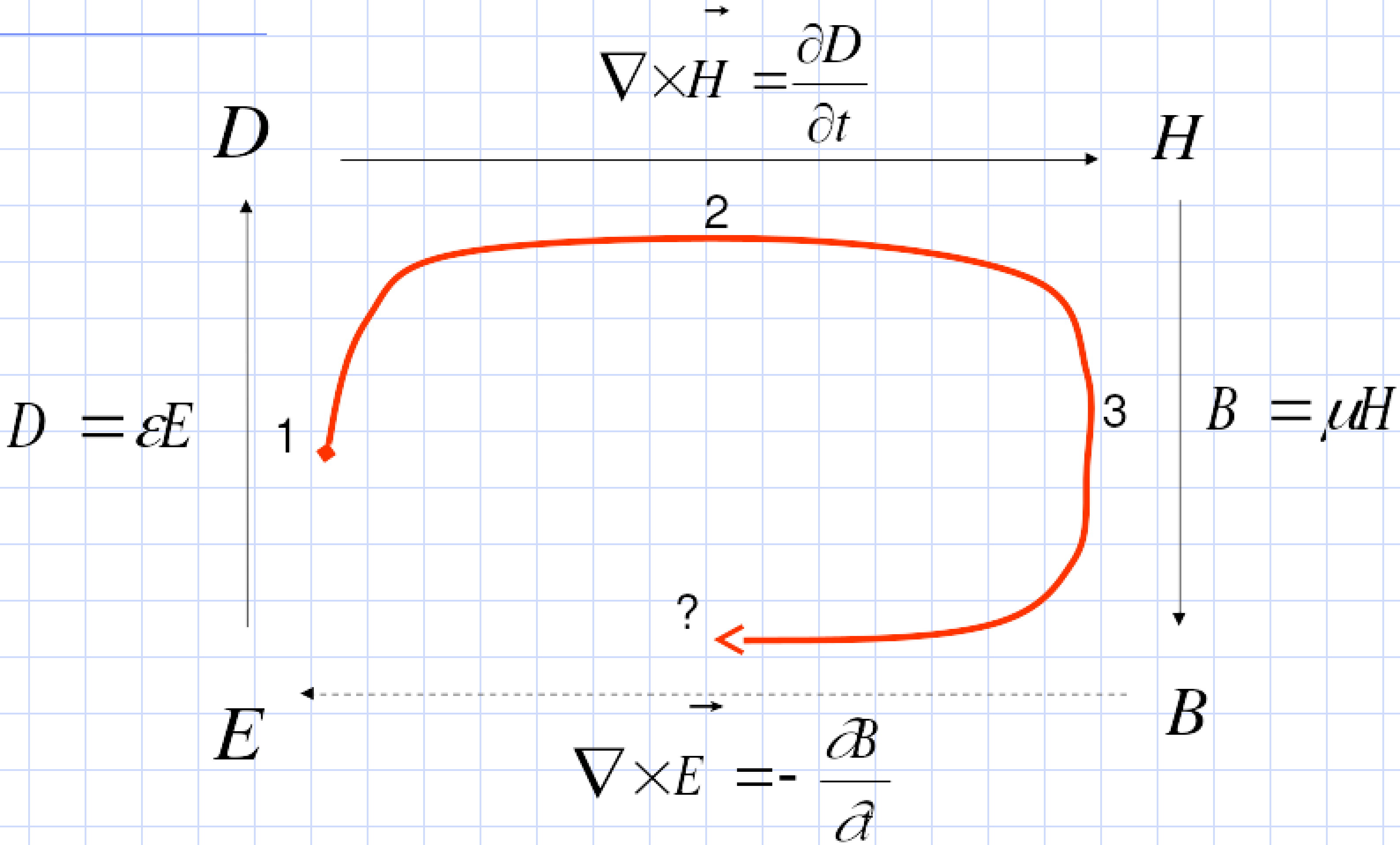
Quasi static approach

Valid if the system dimensions are small compared to a wavelength.

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\omega\sqrt{\varepsilon\mu}} = \frac{c}{f} = \frac{3\cdot 10^8}{10^5} = 300m$$

real meaning of low frequencies.

Capacitor Example

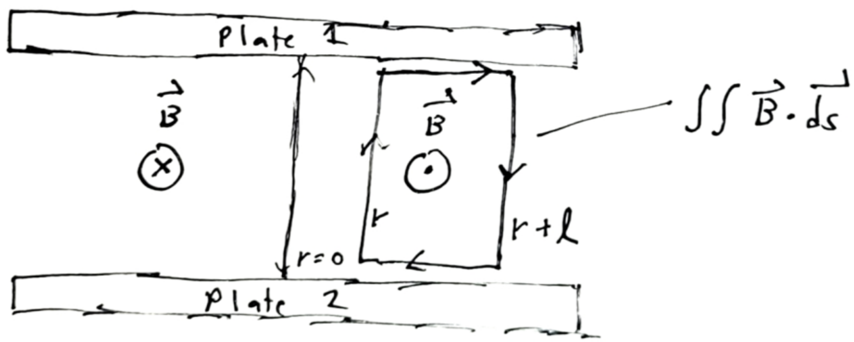


Maxwell's Equations

Full Solution

$$\vec{H} = 1.39\, r \sin \omega t\, \hat{\phi}$$

$$\vec{B} = \mu_0 1.39\, r \sin \omega t\, \hat{\phi}$$



- What if we cannot make the quasistatic assumption?
- We must then calculate the B-field between the plates, then the emf generated by the B-field

Maxwell's Equations

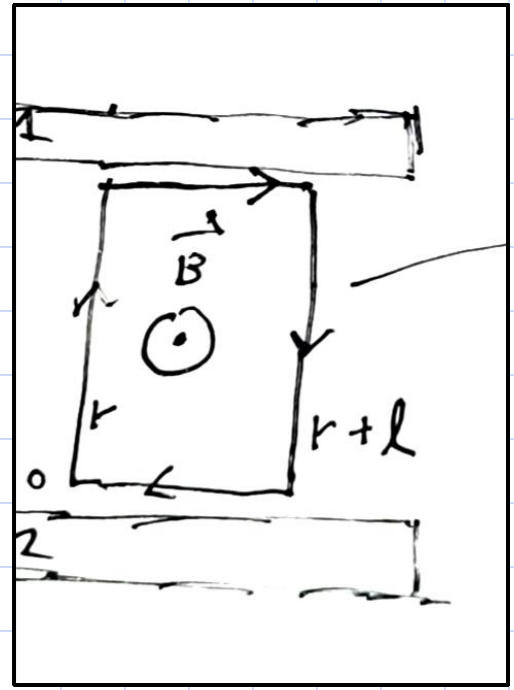
Full Solution

$$\iint \vec{B}\cdot \vec{ds} = \int_0^d \int_r^{r+\ell} 1.39\, r \sin \omega t \, dr$$

$$= \frac{1.39}{2} \sin \omega t \left[(r+\ell)^2 - (r)^2\right]$$

$$\Psi = \frac{1.39}{2}\left[2r\ell + \ell^2\right] \sin \omega t$$

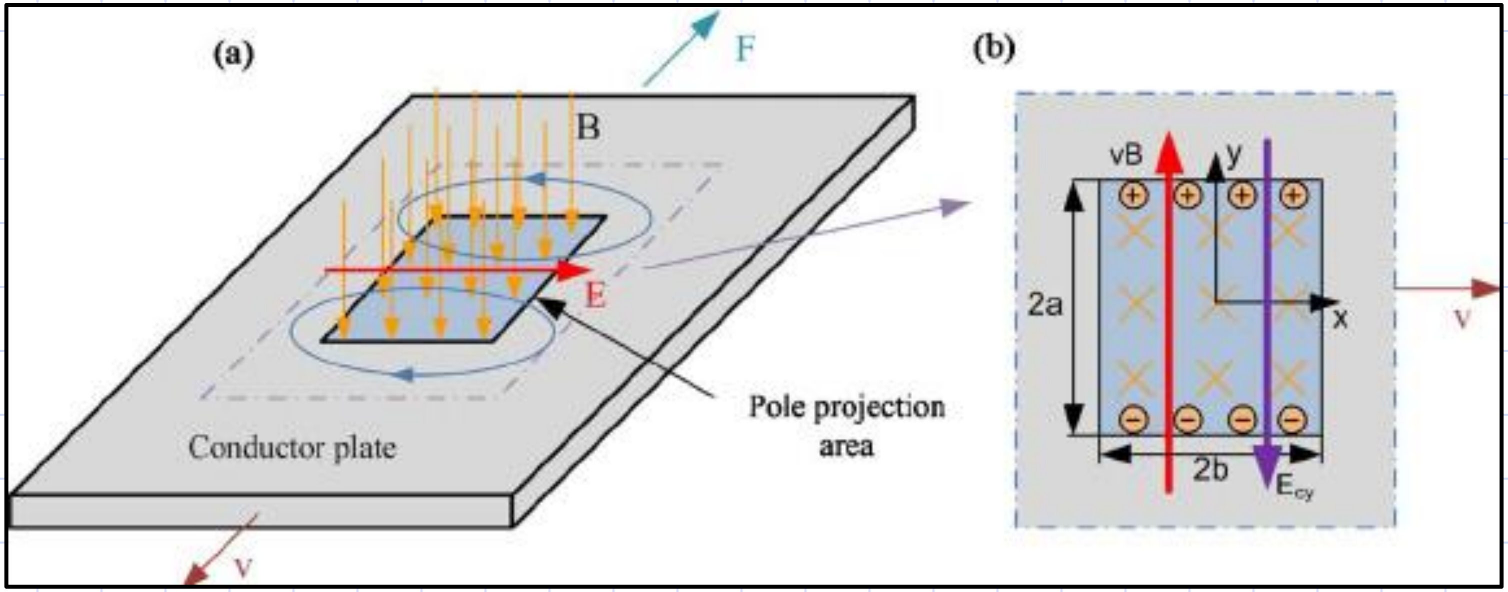
$$E_{mf} = -\frac{d\Psi}{dt} = \frac{-1.39\,\omega}{2}\left[2r\ell + \ell^2\right] \sin \omega t$$



- The emf will apply to a loop that goes up/down in z and in/out in r. What does this mean?

Maxwell's Equations

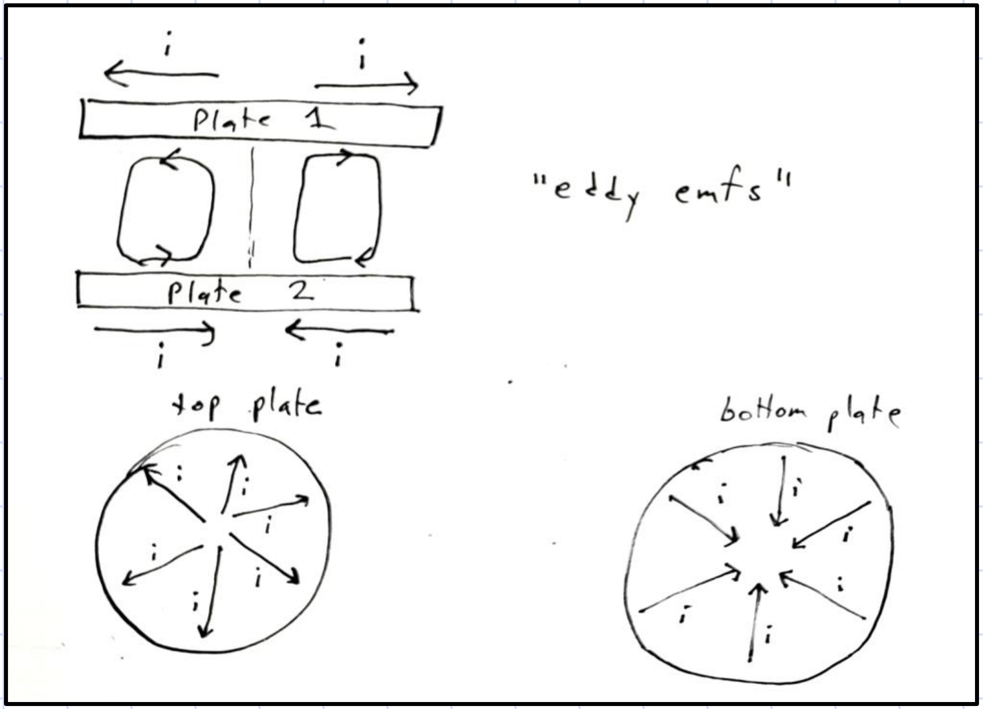
Eddy Currents



- In a conductor, a changing B-field gives rise to additional "eddy currents" that circulate within the conductor.
- Magnetic circuits can have eddy currents as well. In both cases, they lead to power losses.

Maxwell's Equations

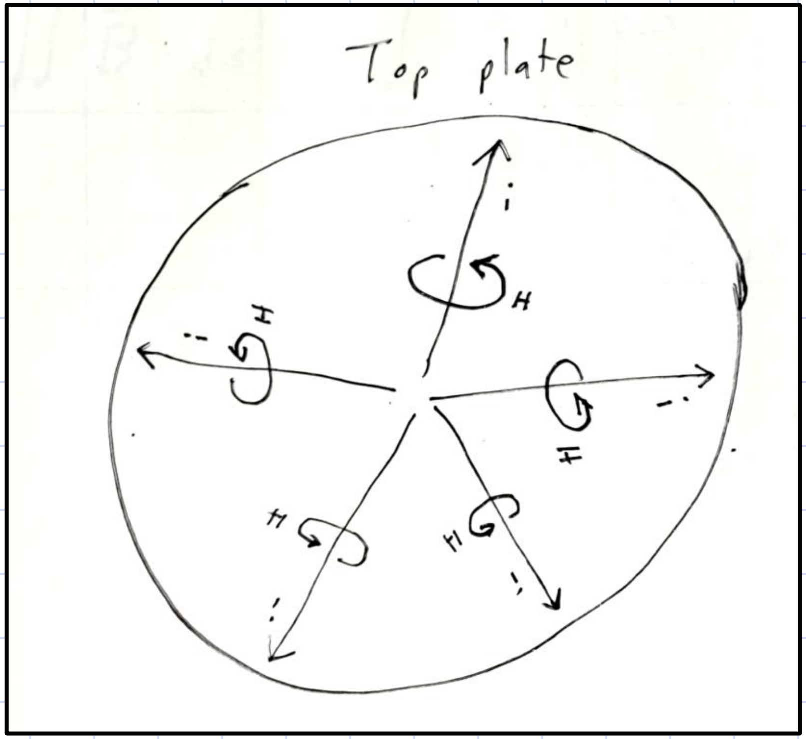
Eddy Currents



- If the capacitor dielectric does not conduct, no current will flow in it due to this emf.
- However, there WILL be radial eddy current in the capacitor plates.

Maxwell's Equations

Eddy Currents



- These eddy currents have magnetic fields of their own as governed by Ampere's Law. And these magnetic fields produce yet additional eddy currents!
- Ultimately, the total energy in this infinite series of currents/fields/emfs is finite, and the terms get progressively smaller.
- However, it is obvious why we resort to quasistatic approximations for time-varying field problems.

Maxwell’s Equations

Eddy Currents