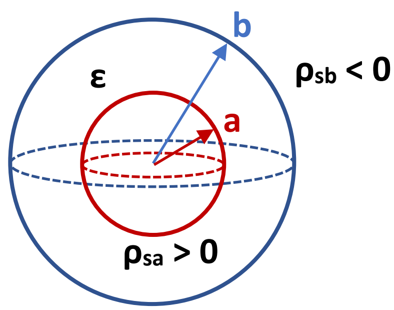
Homework 5

Released: February 29th

Due: March 20th

1. Spherical Capacitor

Consider the system below, which consists of a conducting spherical shell of radius *a* inside of a conducting spherical shell of radius *b*. The inner spherical shell has a surface charge density of $\rho_{sa} > 0$ on its surface, while the outer spherical has a surface charge density of $\rho_{sb} < 0$, such that the outer shell is at V = 0. The space between the conducting spherical shells is filled with a dielectric of permittivity ε.



a) Find the electric field in the region a < r < b.

b) Find the potential V_{ab} between the two conducting spherical shells.

c) Calculate the capacitance of the system $C = Q/V_{ab}$.

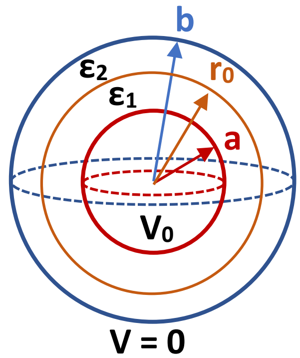
d) Calculate the total energy W_e stored in the electric field in the region a < r < b.

e) From your expression in part d, find C.

f) Do the two conducting spherical shells feel a net electrical force when a voltage is applied between them such that V(a) > V(b)? If so, in which direction does the force point?

2. Spherical Capacitor with Two Dielectrics

Consider a spherical capacitor similar to the one in Question #1, except that the space between the two conducting spherical shells is now filled with two different dielectrics, as shown below.



The inner spherical shell (still of radius *a*) now has a potential V_0, while the outer spherical shell (still of radius *b*) is grounded. From r = a to r = r_0, the space is filled with a dielectric of permittivity $\varepsilon = \varepsilon_1$, while from r = r_0 to r = b, the space is filled with a dielectric of permittivity $\varepsilon = \varepsilon_2$. The charge density in the region a < r < b is zero.

a) Using Laplace's Equation, find the general solution for V(r) in the region $a < r < r_0$ (denoted V_1) and in the region $r_0 < r < b$ (denoted V_2). The "general solution" is the functional form of the solution without specific values for the constants that arise after integration. You will solve for these constants in later steps using boundary conditions.

b) Write down all of the boundary conditions you will need to find the unique solution for V_1 and V_2. Hint: since you should have 4 unknown constants to solve for in your answer from part a, you will need 4 boundary conditions: one at r = a, one at r = b, and two at r = r_0.

c) Using the boundary conditions from part b and your general solutions from part a, find the unique solution for $V_1(r)$ for $a < r < r_0$ and $V_2(r)$ for $r_0 < r < b$.

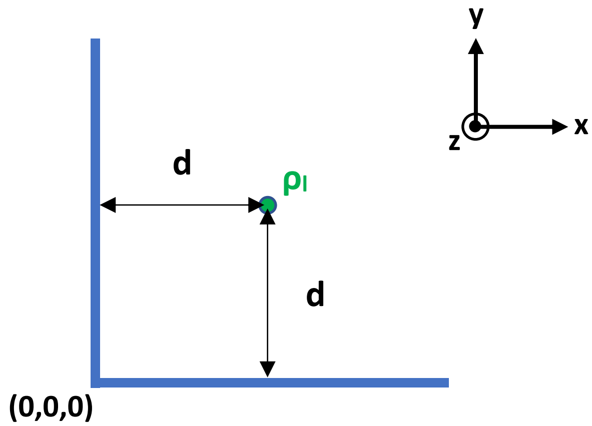
d) Calculate the electric field and displacement field for the region a < r < b. Are the relevant boundary conditions for E and D in part b satisfied by this solution?

e) Calculate the total charge on each of the conducting spherical shells in terms of V_0.

f) Calculate the capacitance between the two conducting spherical shells.

3. Method of Images

An infinite line of charge (line charge density ρ_l) is oriented along the z-axis. It is located a distance *d* above a grounded, conducting half-plane oriented in the x-z plane and a distance *d* away from a grounded, conducting half-plane oriented in the y-z plane, as shown below. Consider the half-planes to be infinite.



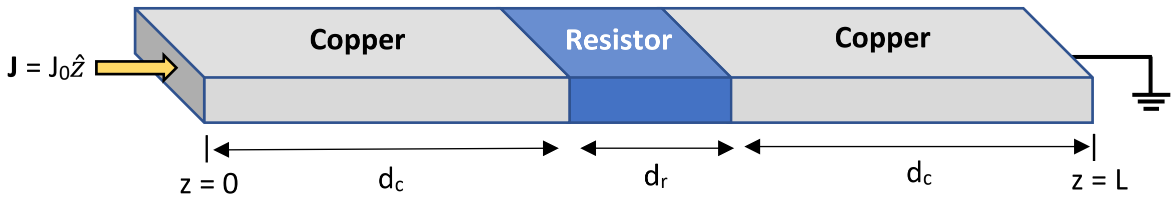
a) Draw the locations, magnitudes, and polarities of the image lines of charge with respect to each of the two grounded conductors.

b) Find the electric field at an arbitrary point P = (x, y, 0).

Current and Resistance

The diagram below shows a rectangular copper wire segment (length d_c and conductivity $\sigma = \sigma_c$) connected in series with a rectangular resistor (length d_r and resistivity $\rho = \rho_r = 1/\sigma_r$) and another segment of rectangular copper wire (length d_c and $\sigma = \sigma_c$). The cross-sectional area of the wires and resistor is A.

A uniform current density $J = J_0\hat{z}$ is injected into the wire at z = 0, while the other end of the wire/resistor/wire system at z = L is grounded.



a) Find the electric field in each region (the two copper wires and the resistor).

b) What is the ratio of the electric field in the resistor to the electric field in the copper wire segments?

c) If $\sigma_c > \sigma_r$, what can you conclude about the magnitude of the electric field in a resistor as compared to the electric field in a copper wire? Explain why this physically makes sense in terms of the current density and conductivity. Does this align with what you know about the potential drop across a resistor in a circuit, as compared with the potential drop across a copper wire?

d) Using the electric field, find the potential dropped across each of the segments (the two wires and the resistor).

e) Using the geometry of the wires and resistor, calculate R for each.

f) Using the voltage divider equation and the resistances you calculated in part e, calculate the voltage drop across the resistor. Does it match what you found in part d?

4. Ampere's Law

a) Using Ampere's law, find the H field and B field at an arbitrary point P = (x, y, z) of an infinitely long wire which is oriented along the z-axis and located in the x-y plane at a point (x_0, y_0). The wire is carrying a current **I** = I$\hat{z}$ and is surrounded by free space.

b) Find the H field and B field at an arbitrary point P = (x, y, z) of two infinitely long wires oriented along the z-axis. One is located in the x-y plane at ($-x_0$, $-y_0$) and is carrying current **I** = -I$\hat{z}$ and the other is located at (x_0, y_0) and is carrying current **I** = I$\hat{z}$.

c) Sketch the total magnetic field around each of the wires. Is the magnetic field zero at any point between the wires? If so, where?

d) According to Maxwell's Equations (magnetostatics version), what physical quantity do you obtain when you calculate the curl of the H field? If you were to integrate the curl of your H field from part c over the surface bounded by an Amperian loop containing both wires (likely a circle): $\oiint \left(\nabla \times \vec{H}\right) \cdot d\vec{S}$, what value would you get? Hint: look at Ampere's Law and Stokes' Theorem – you don't have to actually do the calculation.