Rensselaer Polytechnic Institute
Department of Electrical, Computer, and Systems Engineering
ECSE 2500: Engineering Probability, Spring 2023
Homework #6 Solutions

1. (a)

$$
\begin{aligned}
E(X) &= \int_{x=-\infty}^{\infty} x f_X(x)\, dx \\
&= \int_{x=-2}^{3} \frac{3}{35} x^3\, dx \\
&= \frac{3}{140} x^4 \Big]_{x=-2}^{x=3} \\
&= \frac{243}{140} - \frac{48}{140} \\
&= \frac{195}{140} \\
&= \frac{39}{28} \\
&= 1.393
\end{aligned}
$$

> Grading criteria: 8 points in total
> -0 point: correct
> -3 point: the formula is wrong or missing
> -2 point: wrong or missing in the process
> -2 point: the final answer is wrong or missing
> -8 point: completely incorrect or blank

(b) We can compute $\mathrm{Var}(X)$ either as $E\left(\left(X - \frac{39}{28}\right)^2\right)$ or $E(X^2) - \left(\frac{39}{28}\right)^2$. The integration for the latter is a little easier.

$$
\begin{aligned}
E(X^2) &= \int_{x=-\infty}^{\infty} x^2 f_X(x)\, dx \\
&= \int_{x=-2}^{3} \frac{3}{35} x^4\, dx \\
&= \frac{3}{175} x^5 \Big]_{x=-2}^{x=3} \\
&= \frac{729}{175} + \frac{96}{175} \\
&= \frac{825}{175} \\
&= \frac{33}{7} \\
&= 3.617
\end{aligned}
$$

So $\mathrm{Var}(X) = \frac{33}{7} - \left(\frac{39}{28}\right)^2 = 2.774$.

Grading criteria: 8 points in total
-0 point: correct
-3 point: the formula is wrong or missing
-2 point: wrong or missing in the process
-2 point: the final answer is wrong or missing
-8 point: completely incorrect or blank

(c)

$$\begin{aligned}
E\left(e^{X^3}\right) &= \int_{x=-\infty}^{\infty} \left(e^{x^3}\right) f_X(x)\, dx \\
&= \int_{x=-2}^{3} \frac{3}{35} x^2\, e^{x^3}\, dx \\
&= \frac{1}{35} e^{x^3} \Big]_{x=-2}^{x=3} \\
&= \frac{1}{35}(e^{27} - e^{-8}) \\
&= 1.52 \times 10^{10}
\end{aligned}$$

Grading criteria: 9 points in total
-0 point: correct
-4 point: the formula is wrong or missing
-2 point: wrong or missing in the process
-2 point: the final answer is wrong or missing
-9 point: completely incorrect or blank

2. (a) This is easy using the properties of expected value:

$$\begin{aligned}
E(Y) &= E(7X^2 - 1) \\
&= 7E(X^2) - 1 \\
&= 33 - 1 \\
&= 32
\end{aligned}$$

Grading criteria: 10 points in total
-0 point: correct
-4 point: the formula is wrong or missing
-3 point: wrong or missing in the process
-3 point: the final answer is wrong or missing
-10 point: completely incorrect or blank

(b) First we need the CDF of X, which from HW 4 we computed as:

$$F_X(x) = \begin{cases} 0 & x < -2 \\ \frac{1}{35}x^3 + \frac{8}{35} & x \in [-2,3] \\ 1 & x > 3 \end{cases}$$

Now we can compute the CDF of $Y = 7X^2 - 1$. First, since X can take on values in $[-2,3]$, Y can take on values in $[-1,62]$. However, one thing to note is that values of Y in the range $[-1,27]$ can be produced by two values of X (positive or negative) while values of Y in the range $(27,62]$ can be produced by only one value of X (between 2 and 3). So the CDF will be piecewise. Let's take the range $[-1,27]$ first; in this range,

$$\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= P(7X^2 - 1 \le y) \\
&= P\left(X \in \left[-\sqrt{\frac{y+1}{7}}, \sqrt{\frac{y+1}{7}}\right]\right) \\
&= F_X\left(\sqrt{\frac{y+1}{7}}\right) - F_X\left(-\sqrt{\frac{y+1}{7}}\right) \\
&= \frac{1}{35}\left(\frac{y+1}{7}\right)^{\frac{3}{2}} + \frac{8}{35} + \frac{1}{35}\left(\frac{y+1}{7}\right)^{\frac{3}{2}} - \frac{8}{35} \\
&= \frac{2}{35}\left(\frac{y+1}{7}\right)^{\frac{3}{2}}
\end{aligned}$$

In the range $(27,63]$, the CDF will be

$$\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= P(7X^2 - 1 \le y) \\
&= P\left(X \le \sqrt{\frac{y+1}{7}}\right) \\
&= F_X\left(\sqrt{\frac{y+1}{7}}\right) \\
&= \frac{1}{35}\left(\frac{y+1}{7}\right)^{\frac{3}{2}} + \frac{8}{35}
\end{aligned}$$

Thus the complete CDF of Y is

$$F_Y(y) = \begin{cases} 0 & y < -1 \\ \frac{2}{35}\left(\frac{y+1}{7}\right)^{\frac{3}{2}} & y \in [-1,27] \\ \frac{1}{35}\left(\frac{y+1}{7}\right)^{\frac{3}{2}} + \frac{8}{35} & y \in [27,62] \\ 1 & y > 62 \end{cases}$$

Grading criteria: 20 points in total
-0 point: correct
-3 point: the case of $y < -1$ is wrong or missing
-3 point: the case of $y > 62$ is wrong or missing
-5 point: the case of $-1 < y < 27$ is wrong or missing
-5 point: the case of $27 < y < 62$ is wrong or missing
-3 point: minor wrong or missing in the process
-20 point: completely incorrect or blank

3. (a) The Markov bound says that $P(X \ge 55) \le \frac{50}{55} = 0.91$.

Grading criteria: 7 points in total
-0 point: correct
-3 point: the formula is wrong or missing
-2 point: wrong or missing in the process
-2 point: the final answer is wrong or missing
-7 point: completely incorrect or blank

(b) The Chebyshev bound says that $P(|X-50| \geq 5) \leq \left(\frac{4}{5}\right)^2 = 0.64$, which, assuming the distribution is symmetric, would bound $P(X \geq 55) \leq 0.32$ (a lot less than the previous estimate).

Grading criteria: 8 points in total
-0 point: correct
-3 point: the formula is wrong or missing
-2 point: wrong or missing in the process
-2 point: the final answer is wrong or missing
-8 point: completely incorrect or blank

(c) If the random variable is in fact Gaussian, we can compute from the Q table that $P(X \geq 55) = Q(\frac{5}{4}) = Q(1.25) = 0.106$. So while both bounds are satisfied, neither is very tight. (The bounds work better when we ask for values really far from the mean.)

Grading criteria: 5 points in total
-0 point: correct
-2 point: the formula is wrong or missing
-2 point: the final answer is wrong or missing
-5 point: completely incorrect or blank

4. (a) First we note that the PDF of R is given by

$$f_R(r) = \begin{cases} \frac{1}{10} & r \in [15, 25] \\ 0 & \text{otherwise} \end{cases}$$

If the sandworm is 400m long, the volume V and radius R are related by $V = 400\pi R^2$. Let's call this function $g(R)$. Since R is positive, this is a one-to-one function and we can also write $g^{-1}(V) = \left(\frac{V}{400\pi}\right)^{\frac{1}{2}}$. While we could go through the procedure similar to the above by determining the CDF of V and then taking the derivative, we can also apply the direct formula to get the PDF:

$$\begin{aligned} f_V(v) &= f_R(g^{-1}(v)) \cdot \frac{d}{dv} g^{-1}(v) && (1) \\ &= f_R\left(\left(\frac{V}{400\pi}\right)^{\frac{1}{2}}\right) \cdot \frac{1}{800\pi}\left(\frac{v}{400\pi}\right)^{-\frac{1}{2}} && (2) \\ &= \begin{cases} \frac{1}{8000\pi}\left(\frac{v}{400\pi}\right)^{-\frac{1}{2}} & \left(\frac{v}{400\pi}\right)^{\frac{1}{2}} \in [15, 25] \\ 0 & \text{otherwise} \end{cases} && (3) \\ &= \begin{cases} \frac{1}{8000\pi}\left(\frac{v}{400\pi}\right)^{-\frac{1}{2}} & v \in [90000\pi, 250000\pi] \\ 0 & \text{otherwise} \end{cases} && (4) \end{aligned}$$

Grading criteria: 15 points in total
-0 point: correct
-4 point: the formula is wrong or missing
-3 point: wrong or missing in the process
-3 point: the final answer is wrong or missing
-15 point: completely incorrect or blank

(b) Note that $S = \left(\frac{V}{\pi}\right)^{\left(\frac{1}{2}\right)}$, but plugging in V in terms of R, we get the easier expression $S = 20R$. We can go through the same type of one-to-one process above, or we can immediately see that S will also be a uniform random variable, just over a wider range:

$$f_S(s) = \begin{cases} \frac{1}{200} & s \in [300, 500] \\ 0 & \text{otherwise} \end{cases}$$

Grading criteria: 10 points in total
-0 point: correct
-3 point: doesn't clarify S is a uniformly distributed RV
-3 point: the final answer is wrong or missing
-10 point: completely incorrect or blank