CSCI 2200 — Foundations of Computer Science (FoCS)
Homework 1 (document version 1.0)
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Overview

- This homework is due by 11:59PM on Thursday, September 15
- You may work on this homework in a group of no more than four students; unlike recitation problem sets, **your teammates may be in any section**
- You may use at most **two** late days on this assignment
- Please start this homework early and ask questions during office hours and at your September 14 recitation section; also ask (and answer) questions on the Discussion Forum
- Please be concise in your answers; even if your solution is correct, if it is not well-presented, you may still lose points
- You can type or hand-write (or both) your solutions to the required graded problems below; **all work must be organized in one PDF that lists all teammate names**
- You are strongly encouraged to use LaTeX, in particular for mathematical symbols; see references in Course Materials
- **EARNING LATE DAYS:** for each homework that you complete using LaTeX (including any tables, graphs, etc., i.e., no hand-written anything), you earn one additional late day; you can draw graphs and other diagrams in another application and include them as image files

Warm-up exercises

The problems below are good practice problems to work on. Do not submit these as part of your homework submission. **These are ungraded problems.**

- **Problem 1.26**
- **Problem 2.19**
- **Problem 3.4**
- **Problem 3.13**
- **Problem 3.14**
- **Problem 3.22**
- **Problem 3.24**
- **Problem 3.43**
- **Problem 3.47**

Graded problems

The problems below are required and will be graded.

- **Problem 2.16 (Cartesian Product).**
- **Problem 2.29**
- **Problem 3.20 (DNF). Parts (a) and (b) only.**
- **Problem 3.23**
- **Problem 3.31**
- **Problem 3.44**
- **Problem 3.56**
- **Problem 4.7. Part (a) only.**

All of the above problems (both graded an ungraded) are transcribed in the pages that follow.

Graded problems are noted with an asterisk (*).

If any typos exist below, please use the textbook description.

- **Problem 1.26.** Two players alternately pick numbers without replacement from the set $\{1, 2, 3, \ldots, 9\}$. The first player to obtain three numbers that sum to 15 wins. What is your strategy?

- ***Problem 2.16 (Cartesian Product).** Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$. The Cartesian product $A \times B$ is the set of pairs formed from elements of A and elements of B,

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\}$$

  (a) List the elements in $A \times B$. What is $|A \times B|$?
  $A \times B = \{(1, a), (1, b), (1, c), (1, d), (2, a), (2, b), (2, c), (2, d), (3, a), (3, b), (3, c), (3, d)\}$
  $|A \times B| = 12$

  (b) List the elements in $B \times A$. What is $|B \times A|$?
  $B \times A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3), (d, 1), (d, 2), (d, 3)\}$
  $|B \times A| = 12$

  (c) List the elements in $A \times A = A^2$. What is $|A \times A|$?
  $A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
  $|A \times A| = 9$

  (d) List the elements in $B \times B = B^2$. What is $|B \times B|$?
  $B \times B = \{(a, a), (a, b), (a, c), (a, d), (b, a), (b, b), (b, c), (b, d), (c, a), (c, b), (c, c), (c, d), (d, a), (d, b), (d, c), (d, d)\}$
  $|B \times B| = 16$

  Generalize the definition of $A \times B$ to a Cartesian product of three sets $A \times B \times C$.
  $A \times B \times C = \{(a, b, c) \mid a \in A, b \in B, c \in C\}$

- **Problem 2.19.** How many binary sequences are of length $1, 2, 3, 4, 5$? Guess the pattern.
  $2, 4, 8, 16, 32$
  2^n

- ***Problem 2.29.** Mimic the method we used to prove $\sqrt{2}$ is irrational and prove $\sqrt{3}$ is irrational.
  Assume that $\sqrt{3}$ is rational, which means we can write it as a fraction
  $\sqrt{3} = \{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots\}$
  Each numerator is unique; and each denominator is unique
  From the Well-Ordering Principle, there must be a minimum denominator b_*
  and a corresponding minimum numerator a_* $\quad\sqrt{3} = \frac{a_*}{b_*}$
  For b_* to be the minimum possible, it must be that a_* and b_* have no common factors
  $a_*^2 = 3b_*^2$
  If n^2 is threeven, n must be threeven. A threeven number squared is threeven because $(3k)^2 = 3(3k^2)$. A non-threeven number squared is not threeven because $(3k+1)^2 = 3(3k^2)+3(2k)+1$ and $(3k+2)^2 = 3(3k^2) + 3(4k) + 3 + 1$.
  a_*^2 is threeven since it's a multiple of 3, so a_* is threeven and we can say $a_* = 3k \qquad k \in \mathbb{N}$
  $(3k)^2 = 3b_*^2$ so $b_*^2 = 3k^2$ so b_*^2 is threeven so b_* is threeven
  since a_* and b_* are both threeven, they have a common facotor of three
  for the minimum a_* and b_* to be possible, they musst have no common factors, but they have a common factor of 3.

Now use the same method to try and prove $\sqrt{9}$ is irrational. What goes wrong?
Assume that $\sqrt{9}$ is rational, which means we can write it as a fraction
$\sqrt{9} = \{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, ...\}$
Each numerator is unique; and each denominator is unique
From the Well-Ordering Principle, there must be a minimum denominator b_*
and a corresponding minimum numerator a_* $\sqrt{9} = \frac{a_*}{b_*}$
For b_* to be the minimum possible, it must be that a_* and b_* have no common factors
$a_*^2 = 9b_*^2$
a_*^2 is divisible by 9 since it's a multiple of 9,
but we can not say a_* is divisible by 9.
If n^2 is divisible by 9, n must be divisible by 9. A number divisible by 9 squared is divisible by 9 because $(9k)^2 = 9(9k^2)$. A number not divisible by 9 squared can also be divisible by 9 because $(9k+3)^2 = 9(9k^2) + 9(6k) + 9$ and $(9k+6)^2 = 9(9k^2) + 9(12k) + 9(4)$.

- **Problem 3.4.** Define the propositions p = "Kilam is a CS major" and q = "Kilam is a hockey player". Use the connectors $\vee$, $\wedge$, $\rightarrow$ to formulate these claims.

  (a) Kilam is a hockey player and CS major.
  $q \wedge p$

  (b) Kilam either plays hockey or is a CS major.
  $(qORp)ANDNOT(qANDp)$

  (c) Kilam plays hockey, but he is not a CS major.

  (d) Kilam is neither a hockey player nor a CS major.

  (e) Kilam is a CS major or a hockey player, not both.

  (f) Kilam is not a hockey player but is a CS major.

- **Problem 3.13.** If it rains on a day, it rains the next day. Today it didn't rain. On which days must there be no rain?
  (a) Tomorrow. (b) All future days. (c)X Yesterday.X (d) All previous days.

- **Problem 3.14.** For p = "You're sick", q = "You miss the final", r = "You pass FOCS", translate into English:

  (a) $q \rightarrow \neg r$. If you miss the final, you will not pass FOCS.

  (b) $(p \rightarrow \neg r) \vee (q \rightarrow \neg r)$. If you're sick, you will not pass FOCS or if you miss the final you will not pass FOCS.

  (c) $(p \wedge q) \vee (\neg q \wedge r)$. You're sick and you miss the final, or your not sick and you pass FOCS.

- ***Problem 3.20 (DNF). Parts (a) and (b) only.** Use $\neg$, $\wedge$, $\vee$ to give compound propositions with these truth-tables. [Hint: You need only consider the rows which are T and use OR of AND's.]

(a)

q	r	$q \wedge \neg r$
T	T	F
T	F	T
F	T	F
F	F	F

(b)

q	r	$\neg r$
T	T	F
T	F	T
F	T	F
F	F	T

(AND-OR-NOT formulas use only $\neg$, $\wedge$, $\vee$. Any truth-table can be realized by an AND-OR-NOT formula. Even more, one can construct an OR or AND's, the *disjunctive normal form (DNF)*.)

- **Problem 3.22.** How many rows are in the truth table of $\neg(p \vee q) \wedge \neg r$? Give the truth table.

- ***Problem 3.23.**

  (a) Give the truth-table for these compound propositions.

$$p \wedge \neg p; \quad p \vee \neg p; \quad p \rightarrow (p \vee q); \quad ((p \rightarrow q) \wedge (\neg q)) \rightarrow \neg p$$

p	q	$p \wedge \neg p$
T	T	F
T	F	F
F	T	F
F	F	F

p	q	$p \vee \neg p$
T	T	T
T	F	T
F	T	T
F	F	T

p	q	$p \rightarrow (p \vee q)$
T	T	T
T	F	T
F	T	T
F	F	T

p	q	$((p \rightarrow q) \wedge (\neg q)) \rightarrow \neg p$
T	T	T
T	F	T
F	T	T
F	F	T

  (b) How many rows are in the truth-table of the proposition $(p \vee q) \rightarrow (r \rightarrow s)$?
  $2^4 = 16$

  (c) Show that $(p \rightarrow q) \vee p$ is ALWAYS true. This is called a tautology.
  By implication rules, $(p \rightarrow q) \equiv \neg p \vee q$ so $(p \rightarrow q) \vee p \equiv \neg p \vee q \vee p$
  $\neg p \vee p \equiv$ True

- **Problem 3.24.** Let $q \rightarrow p$ be F and $q \rightarrow r$ be T. Answer T/F: (a) $p \vee q$ (b) $p \rightarrow q$ (c) $p \wedge q \wedge r$.

- ***Problem 3.31.** Use truth tables to determine the logical equivalence of the compound statements.

  (a) $(p \rightarrow q) \rightarrow r$ and $p \rightarrow (q \rightarrow r)$

p	q	r	a	b	
T	T	T	T	T	
T	T	F	F	F	
T	F	T	T	T	
T	F	F	T	T	not equivalent
F	T	T	T	T	
F	T	F	F	T	
F	F	T	T	T	
F	F	F	F	T	

  (b) $(p \wedge \neg q) \vee q$ and $p \vee q$

p	q	a	b	
T	T	T	T	
T	F	T	T	equivalent
F	T	T	T	
F	F	F	F	

- **Problem 3.43.** For $x \in \{1,2,3,4,5\}$ and $y \in \{1,2,3\}$, determine T/F with short justifications.

  (a) $\exists x : x + 3 = 10$

  (b) $\forall y : y + 3 \leq 7$

  (c) $\exists x : (\forall y : x^2 < y + 1)$

  (d) $\forall x : (\exists y : x^2 + y^2 < 12)$

- ***Problem 3.44.** For $x, y \in \mathbb{Z}$, determine T/F with short justifications.

  (a) $\forall x : (\exists y : x = 5/y)$ F, there is no integer solution for $0 = 5/y$.

  (b) $\forall x : (\exists y : y^4 - x < 16)$ F, y^4 can not be less than 0 since a negative real number to an even power is positive, but if x is less than -16, the left side will become greater than 16 and y^4 will not be able to subtract from that. Contradiction: $x = -16, y^4 - (-16) < 16, y^4 < 0$, which is imposible if $y \in \mathbb{R}$.

  (c) $\forall x : (\exists y : \log_2 x \neq y^3)$ T, $log_2 x$ and y^3 are not equal constants, therefore no matter the value of one, you will always be able to pick a value of the other that does not match.

- **Problem 3.47.** Use quantifiers to precisely formulate the associative laws for multiplication and addition and the distributive law for multiplication over addition.

- ***Problem 3.56.** In which (if any) of the domains $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are these claims T? (x and y can have different domains.)

  (a) $\exists x : x^2 = 4$ $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$

  (b) $\exists x : x^2 = 2$ $\mathbb{R}$

  (c) $\forall x : (\exists y : x^2 = y)$ $x\mathbb{N}, \mathbb{Z}, y\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, $x\mathbb{Q}, y\mathbb{Q}, \mathbb{R}$, $x\mathbb{R}, y\mathbb{R}$

  (d) $\forall y : (\exists x : x^2 = y)$ $y\mathbb{N}, x\mathbb{R}$, $y\mathbb{Z}, \mathbb{Q}, \mathbb{R}, x$

- ***Problem 4.7. Part (a) only.** Give direct proofs:

  (a) $x, y \in \mathbb{Q} \rightarrow xy \in \mathbb{Q}$.

  Proof. We prove the claim using a direct proof.
  Assume that $x, y \in \mathbb{Q}$.
  By the definition of $\mathbb{Q}$, x, y can be writen as $\frac{a}{b}, \frac{c}{d}$, where $a, b, c, d \in \mathbb{Z}$
  Since $\frac{a}{b}\frac{c}{d} = \frac{ac}{cd}$, and an integer multiplied by another integer always results in an integer, we have $\frac{a}{b}\frac{c}{d} = \frac{e}{f}$ where $e, f \in \mathbb{Z}$.
  By the definition of $\mathbb{Q}$, if we have $\frac{e}{f}$ where $e, f \in \mathbb{Z}$, then $\frac{e}{f} \in \mathbb{Q}$.
  Therefore, $x, y \in \mathbb{Q} \rightarrow xy \in \mathbb{Q}$ is true.