MATH-2400 INTRODUCTION TO DIFFERENTIAL EQUATIONS FALL 2022

Problem Set 10

Due: 11pm, Tuesday, December 6, 2022

Submitted by:

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NOTES

1. Practice problems listed below and taken from the textbook are for your own practice, and are not to be turned in.
2. There are two parts of the Problem Set, an objective part consisting of multiple choice questions (with no partial credit available) and a subjective part (with partial credit possible). Please complete all questions.
3. Writing your solutions in LaTeX is preferred but not required.
4. Show all work for problems in the subjective part. Illegible or undecipherable solutions will not be graded.
5. Figures, if any, should be neatly drawn by hand, properly labelled and captioned.
6. Your completed work is to be submitted electronically to LMS as a single pdf file. Be sure that the pages are properly oriented and well lighted. (Please do not e-mail your work to Muhammad or me.)

Practice Problems from the textbook (Not to be turned in)

- Exercises from Chapter 4, pages 83–84: 1(d,e), 2(b), 3(c,d).
- Exercises from Chapter 4, page 91: 1(a,b), 2(a,b).
- Exercises from Chapter 4, page 98: 1(a,c), 2(a,c).
- Exercises from Chapter 4, page 105–107: 2(a,b), 3(a,b).

Objective part (Choose A, B, C or D; no work need be shown, no partial credit available)

1. (5 points) Let

$$A = \begin{bmatrix} 1 & 2 & \alpha \\ -4 & 1 & 2 \\ -1 & \beta & 1 \end{bmatrix}$$

where α and β are constants. Which statement is true or select "All of these choices" if all statements are true:

To be singular, the $det(A) = 0$

A A is singular if $\alpha = \beta = 1$

B A is nonsingular if $\alpha = 0$ and $\beta = 2$

C The column vectors of A are linearly dependent if $\alpha = -1$ and $\beta = -2$

[D] All of these choices.

2. (5 points) Let $\mathbf{x}(t)$ solve the constant-coefficient system

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}$$

Which statement is true or select "None of these choices" if none of the statements are true:

A The phase portrait of the system is a source.

[B] The phase portrait of the system is a saddle.

C The phase portrait of the system is a sink.

D None of these choices.

3. (5 points) Let $\mathbf{x}(t)$ solve the constant-coefficient system

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 3 & 5 \\ a & -3 \end{bmatrix}$$

The phase portrait of the system is a center if

A $a = 0$

B $a = -1$

[C] $\boldsymbol{a = -2}$

D None of these choices.

Subjective part (Show work, partial credit available)

1. (15 points) Let $\mathbf{x}(t)$ satisfy the initial-value problem

$$\mathbf{x}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

(a) Find the general solution of the constant-coefficient system.

To find the eigenvalues, use simplified formula for the roots of the characteristic polynomial:
$r = m \pm \sqrt{m^2 - p}$

Where $m = \frac{tr(\underline{\underline{A}})}{2}$ = mean of $[1,1]$ and $[2,2]$
$m = \frac{1-4}{2} = -1.5$

and $p = det(\underline{\underline{A}}) = (1 * -4) - (-2 * 3) = -4 + 6$
$p = 2$

$r = m \pm \sqrt{m^2 - p}$
$r = -1.5 \pm \sqrt{(-1.5)^2 - 2}$
$r = -1.5 \pm \sqrt{2.25 - 2} = -1.5 \pm \sqrt{0.25}$
$r = -1.5 \pm 0.5$
$\boldsymbol{r_1 = -1}$ and $\boldsymbol{r_2 = -2}$

To find the eigenvector $\underline{z}_1$, use the eigenvalue $\boldsymbol{r_1 = -1}$:
$(\underline{\underline{A}} - r_1\underline{\underline{I}})\underline{z}_1 = 0$

$$\left(\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right) \underline{z}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{Let } \underline{z}_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By the equations formed from the matrices above, $\alpha = \beta$, so:

$$\text{So } \boldsymbol{\underline{z}_1 =} \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \text{with } \alpha = 1$$

To find the eigenvector $\underline{z}_2$, use the eigenvalue $\boldsymbol{r_2 = -2}$:
$(\underline{\underline{A}} - r_2\underline{\underline{I}})\underline{z}_2 = 0$

$$\left(\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \right) \underline{z}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{Let } \underline{z}_2 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By the equations formed from the matrices above, $\beta = \frac{3}{2}\alpha$, so:

$$\boldsymbol{\underline{z}_2 =} \begin{bmatrix} \alpha \\ \frac{3}{2}\alpha \end{bmatrix} = \begin{bmatrix} \mathbf{2} \\ \mathbf{3} \end{bmatrix} \text{with } \alpha = 1$$

The eigenvalues are:
$\boldsymbol{r_1 = -1}$, and $\boldsymbol{r_2 = -2}$

The eigenvectors are:

$\boldsymbol{\underline{z}_1 =} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$ and $\boldsymbol{\underline{z}_2 =} \begin{bmatrix} \mathbf{2} \\ \mathbf{3} \end{bmatrix}$

The general solution is of the form:
$\underline{x}(t) = C_1\underline{z}_1 e^{r_1 t} + C_2\underline{z}_2 e^{r_2 t}$

Here, the general solution is:

$$\underline{x}(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-2t} \tag{1}$$

(b) Find the solution of the initial-value problem.
Plug in the initial condition:

$$\underline{x}(0) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^0 + C_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^0 = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \tag{2}$$

$C_1 + 2C_2 = -2$ (eq. 1)
$C_1 + 3C_2 = -1$ (eq. 2)

Subtract (eq. 2) from (eq. 1) to get:
$-C_2 = -1$
$\boldsymbol{C_2 = 1}$, so $\boldsymbol{C_1 = -4}$

So, the solution of the IVP is:

$$\underline{x}(t) = \begin{bmatrix} -4 \\ -4 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-2t} \tag{3}$$

2. (15 points) Consider the two systems of first-order ODEs:

$$\text{(a)} \quad \begin{aligned} x_1' &= 2x_1 + 2x_2 \\ x_2' &= x_1 + 3x_2 \end{aligned} \qquad\qquad \text{(b)} \quad \begin{aligned} x_1' &= x_1 + 5x_2 \\ x_2' &= x_1 - 3x_2 \end{aligned}$$

Determine the general solution for $\mathbf{x}(t) = [x_1(t)\,,\, x_2(t)]^T$ for each system and plot their phase portraits. Classify the solution behavior as to its type (e.g. saddle, source, etc.). Note: be sure to plot lines parallel to the eigenvectors for each phase portrait and sketch representative trajectories for increasing t in the four regions separated by the lines parallel to the eigenvectors.

(a) We can convert the system above into matrix form, as follows:

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

Find the eigenvalues and eigenvectors of A.
Using simplified formula for the roots/eigenvalues of the characteristic polynomial:
$r = m \pm \sqrt{m^2 - p}$

Where $m = \frac{tr(A)}{2} =$ mean of $[1,1]$ and $[2,2]$
$m = \frac{2+3}{2} = 2.5$

and $p = det(A) = (2*3) - (2*1) = 6 - 2$
$p = 4$

$r = m \pm \sqrt{m^2 - p}$
$r = 2.5 \pm \sqrt{2.5^2 - 4} = 2.5 \pm \sqrt{6.25 - 4} = 2.5 \pm \sqrt{2.25}$
$r = 2.5 \pm 1.5$
$\boldsymbol{r_1 = 1}$ and $\boldsymbol{r_2 = 4}$

Solve for both eigenvectors using the determined roots/eigenvalues.
For $\boldsymbol{r_1 = 1}$:
$(\underline{\underline{A}} - r_1\underline{\underline{I}})\underline{z}_1 = 0$

$$\left(\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \underline{z}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{Let } \underline{z}_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\alpha = -2\beta$, so:

$$\underline{z}_1 = \begin{bmatrix} -2\beta \\ \beta \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \tag{4}$$

For $\boldsymbol{r_2 = 4}$:
$(\underline{\underline{A}} - r_2\underline{\underline{I}})\underline{z}_2 = 0$

$$\left(\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right)\underline{z}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{Let } \underline{z}_2 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

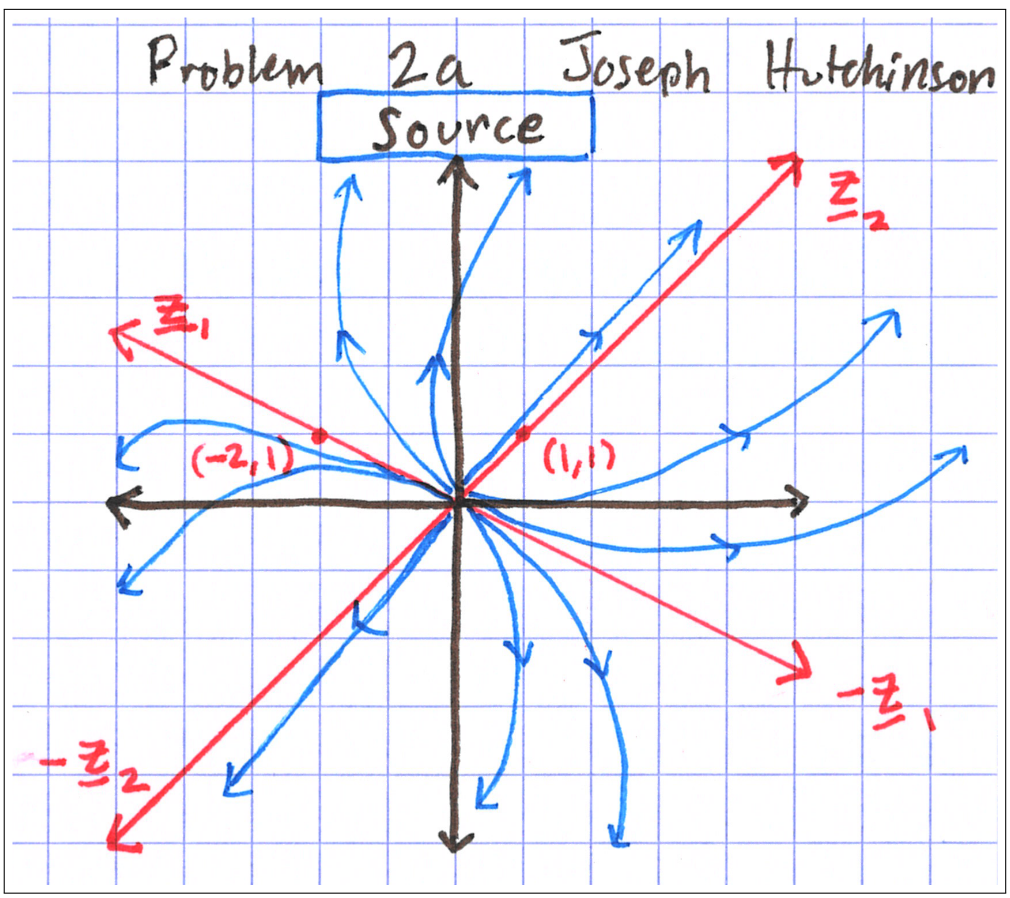
$\alpha = \beta$, so:

$$\underline{z}_2 = \begin{bmatrix} \beta \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{5}$$

The general solution is of the form:
$\underline{x}(t) = C_1\ \underline{z}_1\ e^{r_1 t} + C_2\ \underline{z}_2\ e^{r_2 t}$

$$\underline{x}(t) = C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} \tag{6}$$

The phase portrait of this system's solution is a **source**, because both roots are greater than zero ($r_1 = 1$ and $r_2 = 4$).



(b) We can convert the system above into matrix form, as follows:

$$\mathbf{x}' = B\mathbf{x}, \qquad B = \begin{bmatrix} 1 & 5 \\ 1 & -3 \end{bmatrix}$$

Find the eigenvalues and eigenvectors of B .
Using simplified formula for the roots/eigenvalues of the characteristic polynomial:
$r = m \pm \sqrt{m^2 - p}$

Where $m = \frac{tr(B)}{2} =$ mean of $[1,1]$ and $[2,2]$
$m = \frac{1-3}{2} = -1$

and $p = det(B) = (1 * -3) - (5 * 1) = -3 - 5$
$p = -8$

$r = m \pm \sqrt{m^2 - p}$
$r = -1 \pm \sqrt{1+8} = -1 \pm \sqrt{9}$
$r = -1 \pm 3$
$\boldsymbol{r_1 = -4}$ and $\boldsymbol{r_2 = 2}$

Solve for both eigenvectors using the determined roots/eigenvalues.
For $\boldsymbol{r_1 = -4}$:
$(\underline{\underline{B}} - r_1\underline{\underline{I}})\underline{z}_1 = 0$

$$\left(\begin{bmatrix} 1 & 5 \\ 1 & -3 \end{bmatrix} - \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} \right) \underline{z}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{Let } \underline{z}_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\begin{bmatrix} 5 & 5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\alpha = -\beta$, so:

$$\underline{z}_1 = \begin{bmatrix} -\beta \\ \beta \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{7}$$

For $\boldsymbol{r_2 = 2}$:
$(\underline{\underline{B}} - r_2\underline{\underline{I}})\underline{z}_2 = 0$

$$\left(\begin{bmatrix} 1 & 5 \\ 1 & -3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \underline{z}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{Let } \underline{z}_2 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\begin{bmatrix} -1 & 5 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

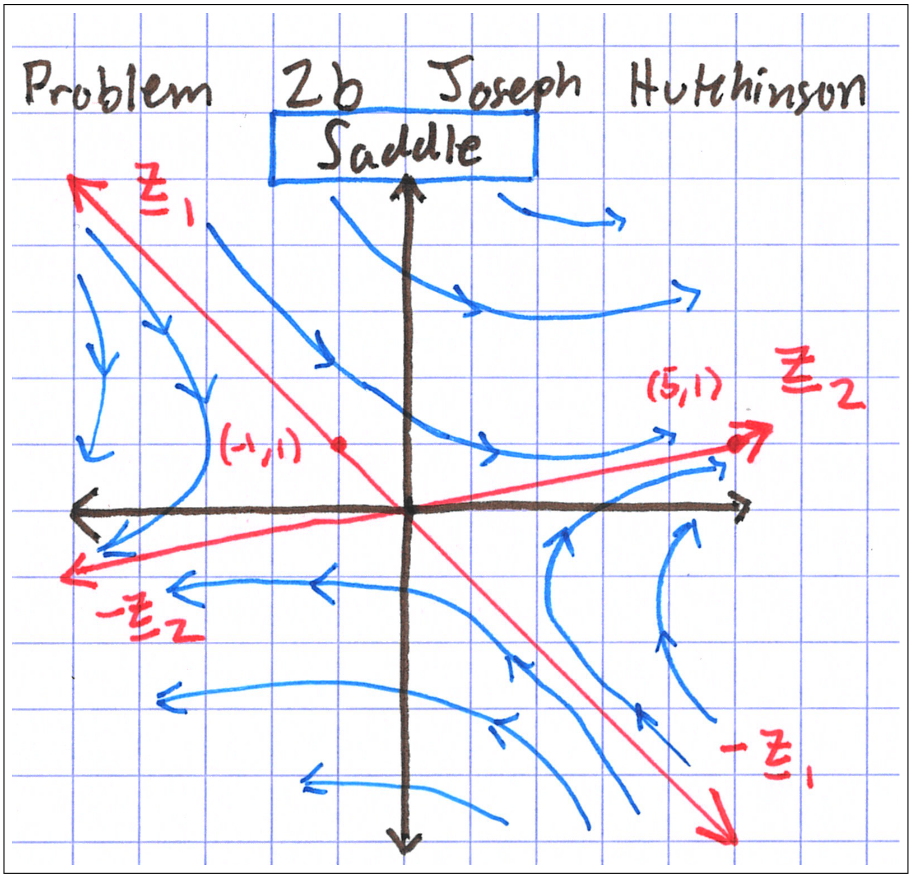
$\alpha = 5\beta$, so:

$$\underline{z}_2 = \begin{bmatrix} 5\beta \\ \beta \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \tag{8}$$

The general solution is of the form:
$\underline{x}(t) = C_1 \ \underline{z}_1 \ e^{r_1 t} + C_2 \ \underline{z}_2 \ e^{r_2 t}$

$$\underline{x}(t) = C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-4t} + C_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{2t} \tag{9}$$

The phase portrait of this system's solution is a **saddle**, because one root is less than zero ($r_1 = -4$), while the other is greater than zero ($r_2 = 2$).



3. (15 points) Consider the constant-coefficient system

$$\mathbf{x}' = A\mathbf{x}, \qquad A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

(a) Find the eigenvalues and eigenvectors of A.

Using simplified formula for the roots/eigenvalues of the characteristic polynomial:
$r = m \pm \sqrt{m^2 - p}$

Where $m = \frac{tr(A)}{2} =$ mean of $[1,1]$ and $[2,2]$
$m = \frac{3-1}{2} = 1$

and $p = det(A) = (3 * -1) - (-2 * 4) = -3 + 8$
$p = 5$

$r = m \pm \sqrt{m^2 - p}$
$r = 1 \pm \sqrt{1 - 5}$
$r = 1 \pm 2i$

$\boldsymbol{r_1 = 1 + 2i}$, and $\boldsymbol{r_2 = 1 - 2i}$

Solve for both eigenvectors using the determined roots/eigenvalues.
For $\boldsymbol{r_1 = 1 + 2i}$:
$(\underline{\underline{A}} - r_1\underline{\underline{I}})\underline{z}_1 = 0$

$$\left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - \begin{bmatrix} 1+2i & 0 \\ 0 & 1+2i \end{bmatrix} \right) \underline{z}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{Let } \underline{z}_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(2 - 2i)\alpha - 2\beta = 0$ (eq. 1)
$4\alpha + (-2 - 2i)\beta = 0$ (eq. 2)

$\beta = (1 - i)\alpha$

$$\text{So } \boldsymbol{\underline{z}_1} = \begin{bmatrix} \alpha \\ (1-i)\alpha \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \boldsymbol{1-i} \end{bmatrix} \text{with } \alpha = 1$$

Since λ_2 is the complex conjugate of λ_1 , the eigenvector $\underline{z}_2$ is the complex conjugate of $\underline{z}_1$:

$$\text{So } \boldsymbol{\underline{z}_2} = \begin{bmatrix} \mathbf{1} \\ \boldsymbol{1+i} \end{bmatrix}$$

The eigenvalues are:
$\boldsymbol{r_1 = 1 + 2i}$, and $\boldsymbol{r_2 = 1 - 2i}$
So $\boldsymbol{\lambda = 1}$ and $\boldsymbol{\mu = 2}$

The eigenvectors are:

$\boldsymbol{\underline{z}_1} = \begin{bmatrix} \mathbf{1} \\ \boldsymbol{1-i} \end{bmatrix}$ and $\boldsymbol{\underline{z}_2} = \begin{bmatrix} \mathbf{1} \\ \boldsymbol{1+i} \end{bmatrix}$

$\boldsymbol{\underline{z}_1} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} - \boldsymbol{i} \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$ and $\boldsymbol{\underline{z}_2} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} + \boldsymbol{i} \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$

So $\boldsymbol{\underline{a}} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$ and $\boldsymbol{\underline{b}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$

(b) Find the general solution of the system in terms of real-valued functions.

The general solution is of the form:
$\underline{x}(t) = C_1\ \underline{u}(t) + C_2\ \underline{v}(t)$, where:

$\underline{u}(t) = (\ \underline{a}\cos(\mu t) - \underline{b}\sin(\mu t)\ )\ e^{\lambda t}$
$\underline{v}(t) = (\ \underline{b}\cos(\mu t) + \underline{a}\sin(\mu t)\ )\ e^{\lambda t}$

Given the values of $\lambda = 1$, $\mu = 2$, $\underline{a}$, and $\underline{b}$ found above, these solutions come to be:

$$\underline{u}(t) = \left(\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \cos(2t) - \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \sin(2t) \right)\ e^t$$
$$\underline{v}(t) = \left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \cos(2t) + \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \sin(2t) \right)\ e^t$$

$$\underline{u}(t) = \begin{bmatrix} \cos(2t) \\ \cos(2t) - \sin(2t) \end{bmatrix} e^t$$

$$\underline{v}(t) = \begin{bmatrix} \sin(2t) \\ \cos(2t) + \sin(2t) \end{bmatrix} e^t$$

(c) Sketch the phase portrait for the system and classify the solution behavior as to its type (e.g. saddle, source, etc.) Note: when the eigenvalues and eigenvectors are complex conjugates, it is helpful to plot lines parallel to the real and imaginary parts of the eigenvectors.

The solution is a **spiral source**, because the eigenvalue roots are complex conjugates (case 2), and the $e^{\lambda t}$ term is growing with a positive value of $\lambda = 1$.
It will rotate **counter-clockwise**, because the row 2, column 1 element of $\underline{\underline{A}}$ is *positive*. Equivalently, it will rotate counter-clockwise because the row 1, column 2 element of $\underline{\underline{A}}$ is *negative*.