Exam 4

<u>Instructions</u>

1.) Unless otherwise specified, you have <u>one class period</u> to complete the questions below.

2.) Read all directions carefully.

3.) Show your work in enough detail to allow the graders to completely follow your thought process.

4.) Make sure your calculator is set to perform trigonometric functions in radians & not degrees & use at least 2 significant digits.

5.) Make sure to write your answers legibly. You can write on the back of the exam pages or ask for scratch paper.

Solutions

1. Wave Polarization

For this problem, consider the time-varying wave equation for an EM wave propagating through free space in the +z direction. (This means E with respect to z and t.) In each subproblem, you may choose all the relevant constants as long as your answer meets the stated criteria.

a.) Write the equation for a wave with linear polarization and state its inclination angle.

Solutions will generally take the form:

$$\vec{E}(z,t) = \hat{x} a_x \cos(\omega t - \beta z) + \hat{x} a_y \cos(\omega t - \beta z)$$

inclination angle: $\psi = \tan^{-1}\left(\frac{a_y}{a_x}\right)$

note that $\delta = 0$. δ could also be $\pm\pi$ or a multiple of it.

b.) Write the equation for a wave with circular polarization.

Solutions will take the form:

$$\vec{E}(z,t) = \hat{x} a \cos(\omega t - \beta z) + \hat{y} a \cos(\omega t - \beta z + \delta)$$

Note that the $\hat{x}$ and $\hat{y}$ magnitudes are the same.

$\delta = \pm \frac{\pi}{2}$ (or an odd multiple of these) c.) Write the equation for a wave with elliptical polarization and state its rotation angle and ellipticity angle.

Solutions will take the form:

$$\vec{E}(z,t) = \hat{x}\, a_x \cos(\omega t - \beta z) + \hat{y}\, a_y \cos(\omega t - \beta z + \delta)$$

At least of the following must hold:

1.) δ is not one of the following: $0, \pm\frac{\pi}{2}, \pm\pi$ (or a multiple)
$a_x \neq 0,\ a_y \neq 0$

2.) $\delta = \pm\frac{\pi}{2}$ (or an odd multiple)

and $a_x \neq a_y \neq 0$

aux angle: $\Psi_0 = \tan^{-1}\left(\frac{a_y}{a_x}\right)$ rotation angle: $\gamma = \dfrac{\tan^{-1}(\tan(2\Psi_0)\sin\delta)}{2}$

ellipticity angle: $\chi = \dfrac{\sin^{-1}(\sin(2\Psi_0)\sin\delta)}{2}$

d.) Choose one of the three waves you specified in parts a-c and calculate the average power density it contains.

$S_{av} = \hat{z}\,\dfrac{|E|^2}{2\eta}$ you may assume $\eta = \eta_0 = 377\ \Omega$

If there is no phase difference between x and y:

$|E| = \sqrt{a_x^2 + a_y^2}$

For Circular polarization, $|E| = a_x = a_y$
(when the x component is at maximum, the y component will be minimum and vice versa)

For elliptical polarization you would need to find the average $|E|^2$

$S_{av} = \hat{z}\,\dfrac{|E|^2_{av}}{2\eta}$ $\quad |E|^2_{av} = \dfrac{\int_0^{2\pi/\beta}\left[(a_x\cos(\beta z))^2 + (a_y\cos(\beta z + \delta))^2\right]dz}{2\pi/\beta}$

2. Public WiFi

Suppose that the city of Troy builds a public 5 GHz WiFi network downtown. The WiFi signal is broadcast from many sources and scatters off of buildings and other objects. A component of the scattered broadcast wave travels directly downward into the Hudson river, hitting its surface at normal incidence with an electric field magnitude of 0.05 V/m. The water of the Hudson has $\varepsilon_r = 80$ and $\sigma = 0.1$ S/m.

a.) Does the water act like a good conductor, a low-loss dielectric, or neither for the 5 GHz WiFi signal?

$$\varepsilon' = 80\varepsilon_0 \qquad \varepsilon'' = \frac{\sigma}{\omega} = \frac{0.1}{2\pi \cdot 5 \cdot 10^9}$$

$$\frac{\varepsilon''}{\varepsilon'} = \frac{0.1}{2\pi \cdot 80 \cdot \varepsilon_0 \cdot 5 \cdot 10^9} = 0.0045$$

The water is a low-loss dielectric at this frequency.

b.) Calculate α, β, η, and λ in the Hudson River. Does the wavelength of the signal increase, decrease, or stay the same when it enters the water?

$$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} = \frac{0.1}{2}\sqrt{\frac{\mu_0}{80\varepsilon_0}} = 2.11 \text{ m}^{-1}$$

$$\beta = \omega\sqrt{\mu\varepsilon} = 2\pi \cdot 5 \times 10^9 \cdot \sqrt{\mu_0 \cdot 80\varepsilon_0} = 937.3 \text{ m}^{-1}$$

$$\lambda = \frac{2\pi}{\beta} = 6.70 \text{ mm}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{80\varepsilon_0}} = 42.1\ \Omega$$

In the air, $\varepsilon' \approx \varepsilon_0$. This will cause β to be significantly lower in the air, making λ longer.

So λ decreases as the signal enters the water.

c.) Calculate the power density (i.e. Poynting vector magnitude) of the signal before it hits the Hudson. Then calculate the reflection coefficient of the wave as it hits the Hudson, and calculate the power densities of the reflected and transmitted waves.

initial: $S_{av} = \frac{|E|^2}{2\eta} = \frac{(0.05\ V/m)^2}{2(377\ \Omega)} = 3.32\times10^{-6}\ W/m^2$

$\Gamma = \frac{\sqrt{\varepsilon_{r_1}} - \sqrt{\varepsilon_{r_2}}}{\sqrt{\varepsilon_{r_1}} + \sqrt{\varepsilon_{r_2}}} = \frac{1-\sqrt{80}}{1+\sqrt{80}} = -0.8 \qquad \tau = 1+\Gamma = 0.2$

reflected: $S_{av} = \frac{(0.8\cdot0.05\ V/m)^2}{2(377\ \Omega)} = 2.12\times10^{-6}\ W/m^2$

transmitted: $S_{av} = \frac{(0.2\cdot0.05\ V/m)^2}{2(42.1\ \Omega)} = 1.18\times10^{-6}\ W/m^2$

d.) What is the magnitude of the signal's H-field before it hits the water?

$\eta = \frac{|E|}{|H|} \qquad |H| = \frac{|E|}{\eta} = \frac{0.05\ V/m}{377\ \Omega} = 1.33\times10^{-4}\ A/m$

e.) Calculate expressions for the conduction and displacement current for the signal transmitted into the water.

$E(z,t) = (0.2)(0.05)\, e^{-(\alpha+j\beta)z} e^{-j(2\pi\cdot5\cdot10^9)t} = 0.01 e^{-(2.11+j937.3)} e^{-j(10\pi\cdot10^9)t}\ V/m$
(in water)

Conduction current: $\vec{J} = \sigma\vec{E}$

$J(z,t) = (10^{-3})\, e^{-(2.11+j937.3)z} e^{-j(10\pi\cdot10^9)t}\ A/m^2$

displacement current:

$\frac{\partial}{\partial t}D(z,t) = 80\varepsilon_0 \frac{d}{dt}E(z,t)$

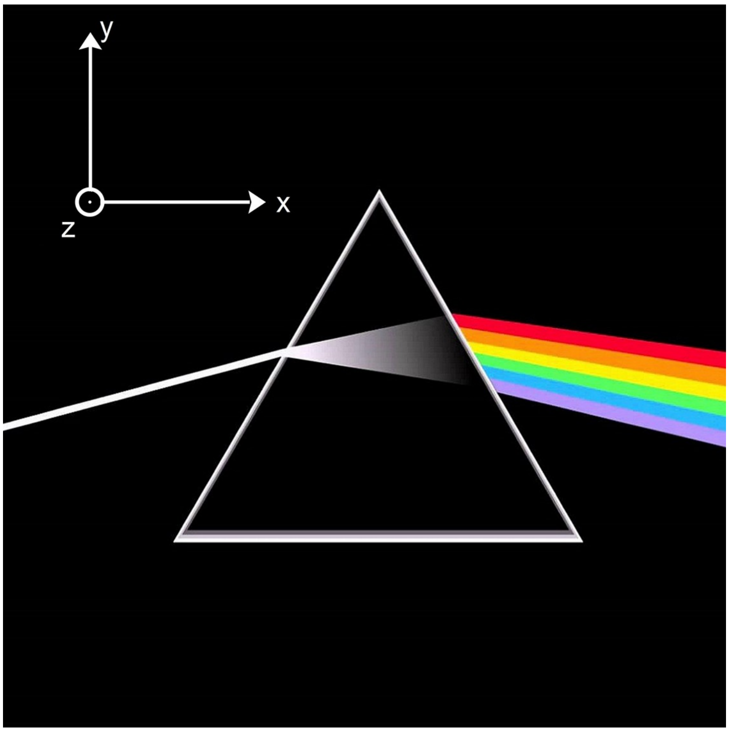
$= 80\varepsilon_0(-j\cdot10\pi\cdot10^9)E(z,t) = -0.223\, e^{-(2.11+j937.3)} e^{-j(10\pi\cdot10^9+\frac{\pi}{2})t}\ A/m^2$

f.) What is the skin depth of the signal in the waters of the Hudson? If the Hudson River is 9m deep near Troy, how many skin depths deep is it?

$$\delta_s = \frac{1}{\alpha} = \frac{1}{2.11 m^{-1}} = 0.473 \ m$$

$9 / 0.473 = 19$ skin depths deep

3. Dark Side of the Moon



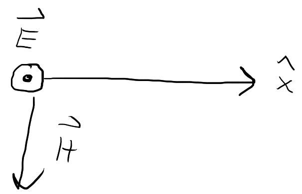
The year is 1973. The band Pink Floyd is trying to get a photograph of light through a prism so they can create some awesome cover art for their album. Unfortunately they're not very knowledgeable about optics so they ask you for help.

a.) In the image above, all the light going into and out of the prism is moving in the xy plane – there is no component moving into or out of the page. If the H-field of the light points in the +z direction, is the light parallel or perpendicularly polarized?

parallel polarized. ($\vec{E}$ is parallel to the plane of travel) b.) If some of the light is traveling in the +x direction with the E-field in the +z direction, what direction does the ~~E-field~~ H-field point?

$$\tilde{H}(x) = \hat{a}_x \times \frac{\tilde{E}(x)}{\eta}$$

$$\hat{x} \times \hat{z} = -\hat{y}$$



$\vec{H}$ points in $-\hat{y}$ direction

c.) Suppose that the light hits the outside of the prism at an incident angle of 40° relative to the normal of the prism's surface. The prism is made of glass with permittivity $4\varepsilon_0$, and the wave is propagating through free space before hitting the prism. What is the angle of transmission into the prism? What is the E-field magnitude of the transmitted wave?

$$\theta_i = 40^\circ \qquad n = \frac{\eta_0}{\eta} \qquad n_i = \frac{\eta_0}{\eta_0} = 1$$

$$\eta_t = \sqrt{\frac{\mu_0}{4\varepsilon_0}} = \frac{\eta_0}{2} \qquad n_t = 2$$

$$n_i \sin\theta_i = n_t \sin\theta_t$$

$$\theta_t = \sin^{-1}\left(\frac{1}{2}\sin 40^\circ\right) = 18.75^\circ$$

Let $|E_i| = 1$ V/m.

$$\tau_{\parallel} = \frac{2\eta_2\cos\theta_1}{\eta_2\cos\theta_2 + \eta_1\cos\theta_1} = \frac{2\left(\frac{\eta_0}{2}\right)\cos(40^\circ)}{\left(\frac{\eta_0}{2}\right)\cos(18.75^\circ) + \eta_0\cos(40^\circ)} = 0.618$$

$$|E_t| = |E_i| \cdot \tau_{\parallel} = 0.618 \text{ V/m}$$

d.) Is there any reflection off of the prism surface? If so, what is its E-field magnitude and reflection angle?

Yes, there is reflection.

$$\Gamma_{\parallel} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1} = \frac{\left(\frac{\eta_0}{2}\right)\cos(18.75^\circ) - (\eta_0)\cos 40^\circ}{\left(\frac{\eta_0}{2}\right)\cos(18.75^\circ) + (\eta_0)\cos 40^\circ}$$

$$\Gamma_{\parallel} = -0.236$$

$$|E_r| = |-0.236| \cdot |E_i| = 0.236 \text{ V/m}$$

$$\theta_r = \theta_i = 40^\circ$$

e.) Is there a critical angle for the light entering the prism? If so, what is it?

There is no critical angle because $\varepsilon_2 > \varepsilon_1$.

f.) Is there a Brewster angle for the light entering the prism? If so, what is it?

yes, there is a Brewster angle.

$$\sin\theta_{B\parallel} = \sqrt{\frac{1-(\varepsilon_1\mu_2)/(\varepsilon_2\mu_1)}{1-(\varepsilon_1/\varepsilon_2)^2}} = \sqrt{\frac{1-(1/4)}{1-(1/4)^2}} = 0.894$$

$$\theta_B = \sin^{-1}(0.894) = 63.4^\circ$$