CSCI 2200 — Foundations of Computer Science (FoCS)
Homework 2

Hayden Fuller, Ben Dennison, Alex Litchfield, Kaylin Rackley

- ***Problem 3.59 (Closure).**

  (a) $\mathbb{N}$
  addition, multiplication, exponentiation

  (b) $\mathbb{Z}$
  addition, subtraction, multiplication, exponentiation

  (c) $\mathbb{Q}$
  addition, subtraction, multiplication, division, exponentiation

  (d) $\mathbb{R}$
  addition, subtraction, multiplication, division, exponentiation

- ***Problem 4.7(b).**
  $n \in \mathbb{Z} \to n^2 + n$ is even.
  We prove the implication using a direct proof.
  assume $n \in \mathbb{Z}$
  if n is even, $n = 2k$
  $(2k)^2 + 2k = 4k^2 + 2k = 2(2k^2 + k)$
  therefore $n^2 + n$ is even for even n
  if n is odd, $n = 2k + 1$
  $(2k+1)^2 + (2k+1) = 4k^2 + 4k + 1 + 2k + 1 = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1)$
  therefore $n^2 + n$ is even for odd n
  so $n^2 + n$ is even for $n \in \mathbb{N}$
  therefore $n \in \mathbb{Z} \to n^2 + n$ is even. ■

- ***Problem 4.10(k-l).**
  (k)
  3 divides $n - 2 \to n$ is not a perfect square.
  We will prove this by contra position:
  Contraposition:
  n is a perfect square $\to$ 3 does not divide $n - 2$
  assume n is a perfect square k^2
  $k^2 \to$3 does not divide $k^2 - 2$
  Because n is an integer, we can prove this for both even and odd cases
  Even Case:
  Assume k is an even number 4m + 2
  $k^2 = (4m + 2)^2 = 16m^2 + 8m + 4 = 2(8m^2 + 4m + 2)$
  This means that k^2 is even when k is even which implies that $k^2 - 2$ will also be even which means it will not be divisible by 3
  Odd Case:
  Assume k is an odd number 2m+1
  $k^2 = (2m + 1)^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m + 1)$
  This means that k^2 is even which also implies that $k^2 - 2$ will always be even. This means

$k^2 - 2$ will not be divisible by 3. ■

(**L**) If $p > 2$ is prime, then $p^2 + 1$ is composite
We prove the implication using contraposition.
Contraposition:
If $p^2 + 1$ is prime $\rightarrow p > 2$ is composite
A prime is always odd. This means that $p^2 + 1$ will always be odd if we assume that our statement **P** is true.
therefore, p^2 is always even. Even numbers are always composite because they can be divided by 1, 2, and itself.
This means that if $p^2 + 1$ is prime, $p > 2$ must be even which makes it a composite number.■

- ***Problem 4.48(c).**
  Use the concept of "without loss of generality" to prove these claims.
  For any non-zero real number x, $x^2 + \frac{1}{x^2} \geq 2$.
  This is a direct proof. Suppose x is positive or negative. Then, x^2 will always be positive.
  Without loss of generality, $x = \sqrt{2}$.
  $\sqrt{2}^2 + 1/\sqrt{2}^2 \geq 2$
  $2 + 1/2 \geq 2$
  $2 + 1/2 \geq 2$ means this must be true for all cases. ■

- ***Problem 5.12(d).**
  For $n \geq 1$, prove by induction:
  $3^n > n^2$.
  Base Step P(1):
  $3^1 > 1^2$.
  $3 > 1$ so $3^n > n^2$ is true for $n = 1$
  Induction Step P(n+1):
  $3^{n+1} > (n+1)^2$
  $\frac{3^{n+1}}{n+1} > n + 1$
  $\frac{3^{n+1}}{n+1} - n > 1$
  With this we can plug in n = 0
  $\frac{3^1}{1} > 1$
  $3 > 1$ means this must be true of all cases. ■

- ***Problem 5.20.**
  Prove, by induction, that every $n \geq 1$ is a sum of distinct powers of 2.
  Base Step P(1):
  $n = 1 = \sum_{i=0}^{k=1} 2^k = 2^1 - 2^0 = 2 - 1 = 1$
  $\sum_{i=0}^{k=1} 2^k = 1$ so $n \geq 1$ being a distinct power of 2 is true.
  Induction Step P(n+1):
  $\sum_{i=0}^{k=n+1} 2^k = n + 1$
  $2^{n+1} - 2^0 = n + 1$
  $2^{n+1} - 1 = n + 1$
  $2^{n+1} = n + 2$ so $n \geq 1$ being a distinct power of 2 is true. ■

- ***Problem 5.39.**
  Prove you can make any postage greater than 12c using only 4c and 5c stamps.

We prove this with leaping induction
$c = 4a + 5b$
bases) $12 = 4*3+5*0$, $13 = 4*2+5*1$, $14 = 4*1+5*2$, $15 = 4*0+5*3$
Case 1) c=4k
Prove $c = 4a + 5b \to c + 4 = 4c + 5d$
assume $c = 4a + 5b$
$d = 0$
$4k + 4 = 4a + 5(0)$
$4(k + 1) = 4a + 5(0)$
$a = k + 1$
therefore $c = 4a + 5b$ is true for $c = 4k$ $\blacksquare$
Case 2) c=4k+1
Prove $c = 4a + 5b \to c + 4 = 4c + 5d$
assume $c = 4a + 5b$
$d = 1$
$4k + 1 + 4 = 4a + 5(1)$
$4k + 5 = 4a + 5$
$a = k$
therefore $c = 4a + 5b$ is true for $c = 4k + 1$ $\blacksquare$
Case 3) c=4k+2
Prove $c = 4a + 5b \to c + 4 = 4c + 5d$
assume $c = 4a + 5b$
$d = 2$
$4k + 2 + 4 = 4a + 5(2)$
$4k + 6 = 4a + 10$
$4(k - 1) + 10 = 4a + 10$
$a = k - 1$
therefore $c = 4a + 5b$ is true for $c = 4k + 2$ $\blacksquare$
Case 4) c=4k+3
Prove $c = 4a + 5b \to c + 4 = 4c + 5d$
assume $c = 4a + 5b$
$d = 3$
$4k + 3 + 4 = 4a + 5(3)$
$4k + 7 = 4a + 15$
$4(k - 2) + 15 = 4a + 15$
$a = k - 2$
therefore $c = 4a + 5b$ is true for $c = 4k + 3$ $\blacksquare$
therefore $c = 4a + 5b$ is true for $c \geq 12$ $\blacksquare$

- ***Problem 6.8.**
  Prove $n^7 < 2^n$ for $n \geq 37$.
  (a) Use induction
  Base Step:
  $n = 37$
  $37^7 < 2^{37}$
  $9.4931877133 \times 10^{10} < 1.3743895347 \times 10^{11}$
  so $n^7 < 2^n$ for $n = 37$ is true.

Induction Step P(n+1)
$2^{k+1} = 2 * 2^k > 2k^7 > k^7 + 7k^6 + 21k^5 + 35k^4 + 35k^3 + 21k^2 + 7k + 1 = (k+1)^7$
$2^{k+1} > (k+1)^7$
Therefore $n^7 < 2^n$ for $n \geq 37$.

(b) Use leaping induction
Base Step 1:
$n = 37$
$37^7 < 2^{37}$
$9.4931877133 \times 10^{10} < 1.3743895347 \times 10^{11}$
so $n^7 < 2^n$ for $n = 37$ is true.
Induction step 1:
$2^{k+2} = 4 * 2^k > 4k^7 > k^7 + 14k^6 + 84k^5 + 280k^4 + 560k^3 + 672k^2 + 448k + 128 = (k+2)^7$
$2^{k+2} > (k+2)^7$
Therefore $n^7 < 2^n$ for $n = 37 + 2k$ where $k \in \mathbb{N}_0$ ■
Base Step 2:
$n = 38$
$38^7 < 2^{38}$
$1.1441558259 \times 10^{11} < 2.7487790694 \times 10^{11}$
so $n^7 < 2^n$ for $n = 38$ is true.
Induction step 2:
$2^{k+2} = 4 * 2^k > 4k^7 > k^7 + 14k^6 + 84k^5 + 280k^4 + 560k^3 + 672k^2 + 448k + 128 = (k+2)^7$
$2^{k+2} > (k+2)^7$
Therefore $n^7 < 2^n$ for $n = 38 + 2k$ where $k \in \mathbb{N}_0$ ■
Therefore $n^7 < 2^n$ for $n \geq 37$ ■

- ***Problem 6.43.**
  for a 3x3 sliding puzzle, the number of inversions always stays odd
  This is a direct proof. Suppose x represents the number of inversions and that $x = 1$.
  Suppose:
  Vertical sliding of tile in either direction can be represented as $2b$, where $b \in \mathbb{N}$.
  Horizontal sliding of tile in either does not change the number of inversions.
  $x = 1 + 2b$
  $x = 2b + 1$ is an odd number, therefore the number of inversions remains odd. ■

- ***Problem 7.4(c).**
  Guess a formula for A_n and prove it by induction.
  (a) $A_0 = 0$ and $A_n = A_{n-1} + 1$ for $n \geq 1$.
  $A_n = n$
  We prove this by induction
  $A_0 = 0$ is True.
  assume $A_n = n$
  $A_{n+1} = n + 1$
  $A_{n+1} = A_{n+1-1} + 1 = A_n + 1 = A_n + 1 = n + 1$
  $A_n + 1 = n + 1$
  $n + 1 = n + 1$
  therefore $A_n = n$ ■

(b) $A_0 = 1$; $A_1 = 2$; $A_n = 2A_{n-1} - A_{n-2} + 2$ for $n \geq 2$. [method of differences]
$A_n = n^2 + 1$
We prove this by induction
$A_0 = 0^2 + 1 = 1$ is True.
assume $A_n = n^2 + 1$
$A_{n+1} = (n+1)^2 + 1$
$A_{n+1} = 2A_{n+1-1} - A_{n+1-2} + 2 = 2A_n - A_{n-1} + 2 = (n+1)^2 + 1$
$2(n^2 + 1) - ((n-1)^2 + 1) + 2 = (n+1)^2 + 1$
$2n^2 + 2 - ((n^2 - 2n + 1) + 1) + 2 = (n+1)^2 + 1$
$2n^2 + 2 - (n^2 - 2n + 2) + 2 = (n+1)^2 + 1$
$2n^2 + 2 - n^2 + 2n - 2 + 2 = (n+1)^2 + 1$
$2n^2 + 2 - n^2 + 2n = (n+1)^2 + 1$
$n^2 + 2n + 1 + 1 = (n+1)^2 + 1$
$(n+1)^2 + 1 = (n+1)^2 + 1$
therefore $A_n = n^2 + 1$ is True $\blacksquare$