Fields and Waves I

Lecture 14

Laplace + Poisson’s Equations
Numerical Methods
Current

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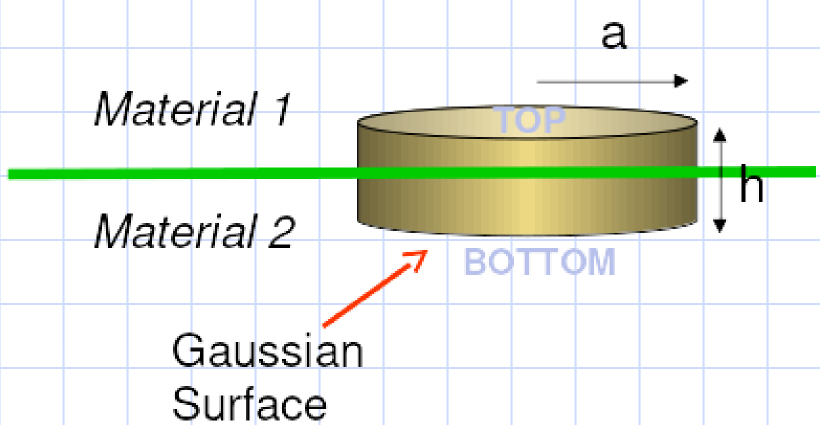
Boundary Conditions

<u>NORMAL COMPONENT</u>

$$\oint D \bullet ds = Q_{enclosed}$$

Take h << a **(a thin disc)**



$$Q_{enclosed} = \rho_s \cdot A$$

$$\oint D \bullet ds = \int_{TOP} D \bullet ds + \int_{BOTTOM} D \bullet ds$$

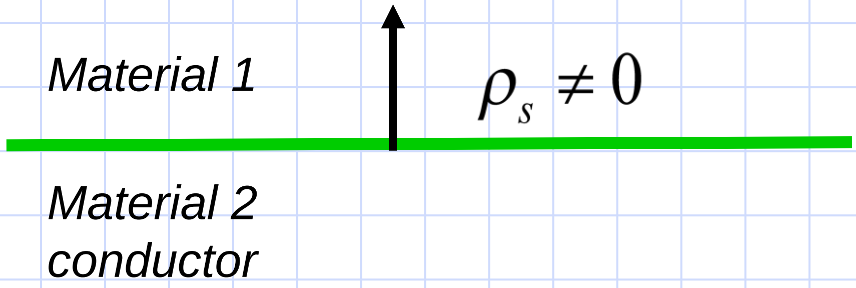
$$= (D_{1n} - D_{2n}) \cdot A$$

$$\therefore D_{1n} - D_{2n} = \rho_s$$

Boundary Conditions

Case 1: REGION 2 is a CONDUCTOR, $D_2 = E_2 = 0$

$$\therefore D_{1n} = \rho_s$$

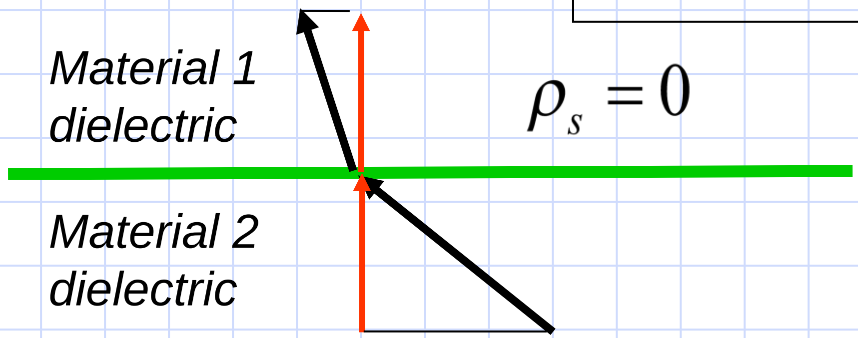


Can only really get ρ_s with conductors

Case 2: REGIONS 1 & 2 are DIELECTRICS with $\rho_s = 0$

$$\therefore D_{1n} = D_{2n}$$

$$\therefore \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

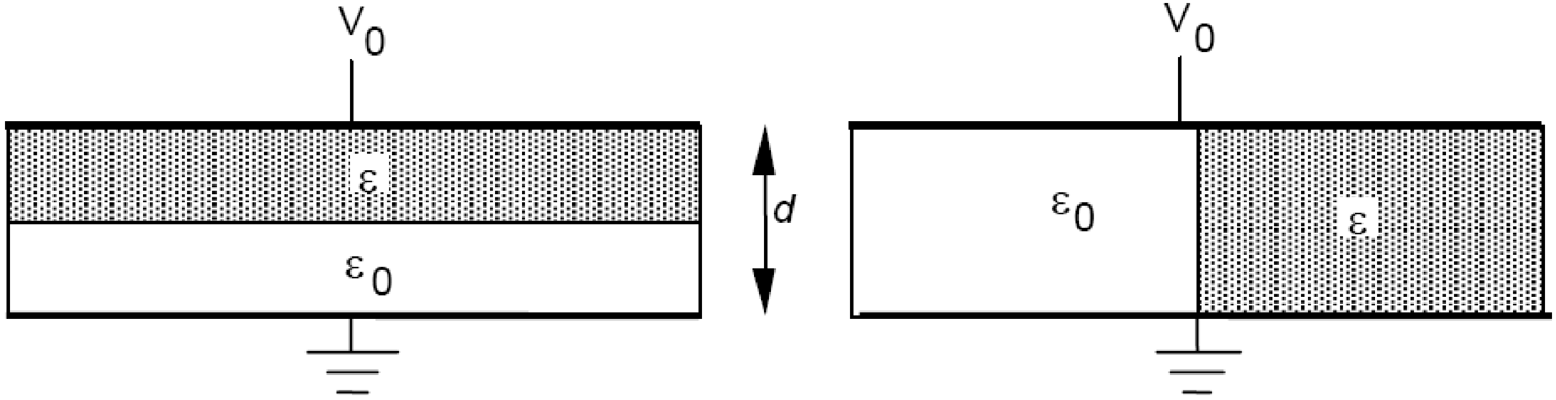


Boundary Conditions

Consider the two parallel plate geometries below. Assume that the plate dimensions are large compared to the separation *d* and ignore fringe effects. For the two figures, the electric field in the air region, (specified by ε_o) is given by:

$\mathbf{E} = -(V_0/d) * (2\varepsilon_r/(1+\varepsilon_r))\, \mathbf{a}_z$ figure on left

$\mathbf{E} = -(V_0/d)\, \mathbf{a}_z$ figure on right



Boundary Conditions

- For both cases, find **E** in the dielectric region. Find **D** in both regions. Within a given region, **D** and **E** do not vary with position.

- Find the charge density on the plates at all locations.

Boundary Conditions

a. Boundary conditions $E_{1t} = E_{2t}$; $D_{1n} = D_{2n} \Rightarrow E_{2n} = \frac{\varepsilon_1}{\varepsilon_2} E_{1n}$

Left : E is normal $\therefore$ $\vec{E}_{diel} = \frac{\varepsilon_0}{\varepsilon_r \varepsilon_0} E_{air} = \boxed{-\frac{V_0}{d} \frac{2}{1+\varepsilon_r} \hat{a}_z}$

Right : E is tangential $\therefore$ $\vec{E}_{diel} = \vec{E}_{air} = \boxed{-\frac{V_0}{d} a_z}$

$\vec{D} = \epsilon E$

Left : $\boxed{\vec{D}_{diel} = -\frac{V_0}{d} \frac{2\varepsilon_r \varepsilon_0}{1+\varepsilon_r} \hat{a}_z}$

same

$\boxed{\vec{D}_{air} = -\frac{V_0}{d} \frac{2\varepsilon_r \varepsilon_0}{1+\varepsilon_r} \hat{a}_z}$

$\vec{D}_{air} = \varepsilon_0 \vec{E}_{air}$

Right : $\boxed{\vec{D}_{diel} = -\frac{\varepsilon_r \varepsilon_0 V_0}{d} \hat{a}_z}$

$\boxed{\vec{D}_{air} = -\frac{\varepsilon_0 V_0}{d} \hat{a}_z}$

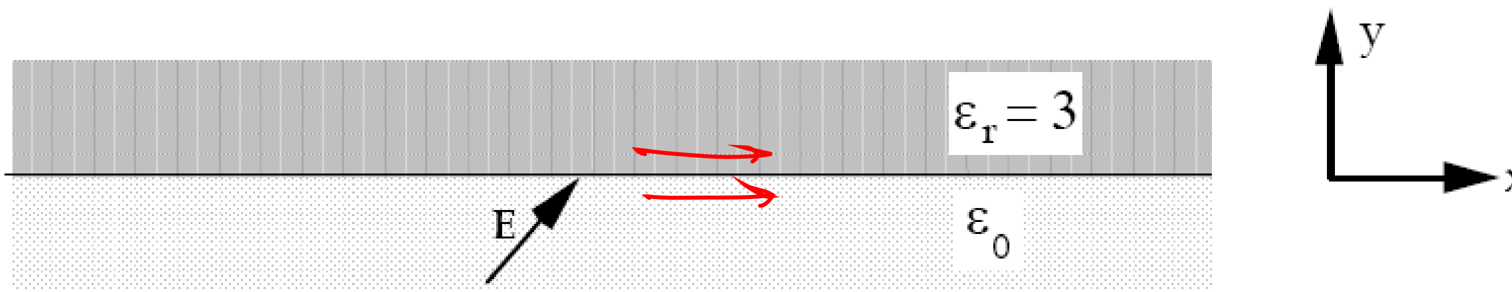
b. Boundary conditions at conductor-dielectric $D_n = \rho_s$

Left : $\boxed{\rho_s = \pm \frac{2\varepsilon_r \varepsilon_0}{1+\varepsilon_r} \frac{V_0}{d}}$ + on top
− on bottom

Right : $\boxed{\rho_{s,diel} = \pm \frac{\varepsilon_r \varepsilon_0 V_0}{d}}$ $\boxed{\rho_{s,air} = \pm \frac{\varepsilon_0 V_0}{d}}$ + on top
− on bottom

Boundary Conditions

The **E** field on the air side of a dielectric-dielectric boundary is **E** = 100 $\mathbf{a}_x$ + 100 $\mathbf{a}_y$. What is **E** on the dielectric side?



Boundary Conditions

$E_{1t} = E_{2t} \Rightarrow E_{1x} = E_{2x} \Rightarrow \therefore E_{2x} = 100$

Air = Region 1
Diel = Region 2

$D_{1n} = D_{2n} \Rightarrow \varepsilon_o E_{1y} = 3\varepsilon_o E_{2y} \Rightarrow E_{2y} = \frac{E_{1y}}{3} = \frac{100}{3} = 33\tfrac{1}{3}$

$$\boxed{\vec{E}_2 = 100\,\hat{a}_x + 33\tfrac{1}{3}\,\hat{a}_y}$$

Laplace and Poisson Equations

- Integral Form

$$\oint \vec{D} \cdot \vec{ds} = \int \rho \cdot dv$$

$$\oint \vec{E} \cdot \vec{dl} = 0$$

- Differential Form

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \times \vec{E} = 0$$

What if we try to rewrite Maxwell's Equations in terms of voltage?

$$\vec{E} = -\nabla V$$

Laplace and Poisson Equations

- First, the curl equation

$$-\nabla V$$

$$\vec{E} = -\nabla V \Rightarrow \nabla \times \vec{E} = 0$$

since $\nabla \times \left(\nabla f\right) = 0$

- Next, the divergence equation

$$\nabla \cdot \vec{D} = \nabla \cdot (\epsilon \vec{E}) = -\epsilon(\nabla \cdot (\nabla V)) = \rho$$

$$\Rightarrow \nabla^2 V = -\frac{\rho}{\varepsilon} \quad \& \quad \nabla^2 V = 0$$

Laplacian of V = divergence of gradient of V

Laplace and Poisson Equations

- Laplace's Equation:

$$\nabla^2 V = 0$$

- Poisson's Equation

$$\nabla^2 V = -\frac{\rho}{\varepsilon}$$

$$\nabla^2 = \begin{bmatrix} \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{bmatrix} \bullet \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

(in cartesian coordinates)

Laplace and Poisson Equations

- Cylindrical Laplacian Operator:

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial V}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

- Spherical Laplacian Operator:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}(sin\theta\frac{\partial V}{\partial \theta}) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

Laplace and Poisson Equations

- Laplace's Equation and Poisson's Equation are general mathematical expressions that allow us to solve scalar fields if we know the boundary conditions. They are used for solving for:
  - Voltage
  - Heat
  - Gravity
  - Aspects of fluid flow
  - Various abstract mathematical fields
  - etc.

Laplace and Poisson Equations

Boundary Conditions

- In General

$$D_{n1} - D_{n2} = \rho_s \qquad E_{t1} = E_{t2}$$

- Dielectric-Dielectric

$$D_{n1} = D_{n2} \qquad E_{t1} = E_{t2}$$

- Conductor-Dielectric

$$D_{n1} = \rho_s \qquad E_{t1} = 0$$

Laplace and Poisson Equations

Boundary Conditions

- Dielectric-Dielectric

$$D_{n1} = D_{n2} \qquad E_{t1} = E_{t2}$$

- Writing in terms of voltage:

$$\varepsilon_1 \frac{\partial V_1}{\partial n} = \varepsilon_2 \frac{\partial V_2}{\partial n} \qquad V_1 = V_2$$

here, n represents direction of boundary normal

Laplace and Poisson Equations

Boundary Conditions

- Dielectric-Dielectric

$$D_{n1} = D_{n2} \qquad E_{t1} = E_{t2}$$

- Writing in terms of voltage:

$$\varepsilon_1 \frac{\partial V_1}{\partial n} = \varepsilon_2 \frac{\partial V_2}{\partial n} \qquad V_1 = V_2$$

this limit ensures voltage continuity

Laplace and Poisson Equations

Boundary Conditions

- Conductor-Dielectric

$$D_{n1} = \rho_s \qquad E_{t1} = 0$$

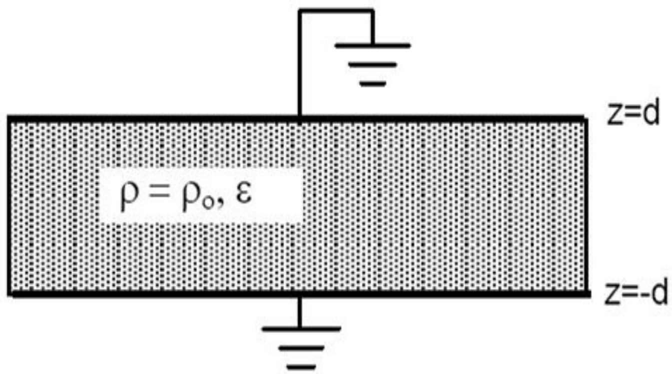
$$\varepsilon_1 \frac{\partial V_1}{\partial n} = \rho_s \qquad V_1 = const$$

Laplace and Poisson Equations

- Coulomb’s Law is already a solution
- All other voltage expressions can be checked with one of these equations
- This is the most common way of finding electric fields

Laplace and Poisson Equations

A charged region of a semiconductor is sandwiched between two grounded conductors as shown below.



Solve for V(z) directly using Poisson's Equation

Find **E** and **D**

Find the charge density on the conductors

Laplace and Poisson Equations

a. $\nabla^2 V = -\rho/\varepsilon \Rightarrow V = V(z), \therefore \nabla^2 V = \dfrac{d^2V}{dz^2}$

$$\frac{dV}{dz} = -\frac{\rho z}{\varepsilon} + C_1 \Rightarrow \therefore -\frac{\rho z^2}{2\varepsilon} + C_1 z + C_2$$

$$V(d) = 0 = -\frac{\rho d^2}{2\varepsilon} + C_1 d + C_2$$

$$V(-d) = 0 = -\frac{\rho d^2}{2\varepsilon} - C_1 d + C_2$$

Add eq. $-\dfrac{\rho d^2}{\varepsilon} + 2C_2 = 0$

$\hookrightarrow C_2 = \dfrac{\rho d^2}{2\varepsilon}$

subtract eq. $2C_1 d = 0 \Rightarrow C_1 = 0$

$$V = -\frac{\rho z^2}{2\varepsilon} + \frac{\rho d^2}{2\varepsilon} = \boxed{\frac{\rho}{2\varepsilon}(d^2 - z^2)}$$

b. $\vec{E} = -\nabla V = -\dfrac{\partial V}{\partial z}\hat{a}_z = -\dfrac{\rho}{2\varepsilon}(-2z)\hat{a}_z = \boxed{\dfrac{\rho z}{\varepsilon}\hat{a}_z}$

$\vec{D} = \varepsilon\vec{E} = \boxed{\rho z\,\hat{a}_z}$

c. Boundary condition $D_n = \rho_s$

if $\rho_s > 0$ D points into surface at both $\pm d \Rightarrow \therefore \rho_s < 0$

$\boxed{\rho_s = -\rho d}$ on both

Laplace and Poisson Equations

A coaxial cable has an inner conductor (at r = a) held at voltage V_0 and an outer conductor (at r = b) that is grounded. There is no charge other than the surface charge on the conductors.

Solve for V(r) directly using Laplace's Equation

Solve for **E** and **D**

What is the charge density on the two conductors?

What is the capacitance per unit length?

Laplace and Poisson Equations

a. $\nabla^2 V = 0 \qquad V = V(r) \Rightarrow \therefore \nabla^2 V = \frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0$

$\frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0 \Rightarrow r\frac{dV}{dr} = c_1 \ ; \quad \frac{dV}{dr} = \frac{c_1}{r} \ ; \quad \underline{V = c_1 \ln r + c_2}$

$V(b) = 0 = c_1 \ln b + c_2 \Rightarrow c_2 = -c_1 \ln b \Rightarrow V = c_1 \ln\frac{r}{b}$

$V(a) = V_0 = c_1 \ln\frac{a}{b} \Rightarrow c_1 = \frac{V_0}{\ln a/b} \Rightarrow V = \frac{V_0}{\ln\frac{a}{b}}\ln\frac{r}{b} = \boxed{\frac{V_0}{\ln\frac{b}{a}}\ln\frac{b}{r}}$

Laplace and Poisson Equations

b. $\vec{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{a}_r = \frac{-V_0}{\ln b/a}\frac{1}{b/r}\left(-\frac{b}{r^2}\right)\hat{a}_r = \boxed{\frac{V_0}{r\ln b/a}\hat{a}_r}$

$\vec{D} = \varepsilon\vec{E} = \boxed{\frac{\varepsilon V_0}{r\ln b/a}\hat{a}_r}$

c. Boundary conditions $D_n = \rho_s$ if $V_0 > 0$ $\vec{E}$ points from $a \to b$

$\therefore \rho_{sa} > 0 \quad \rho_{sb} < 0$

$\boxed{\rho_{sa} = \frac{\varepsilon V_0}{a\ln b/a} \qquad \rho_{sb} = \frac{-\varepsilon V_0}{b\ln b/a}}$

d. $C = \frac{Q}{V} = \frac{\rho_{sa}\,2\pi a l}{V_0} = \frac{\varepsilon \cancel{V_0}}{\cancel{a}\ln b/a}\frac{2\pi\cancel{a}l}{\cancel{V_0}} = \frac{2\pi\varepsilon l}{\ln b/a}$; $\boxed{\frac{C}{l} = \frac{2\pi\varepsilon}{\ln b/a}}$

Laplace and Poisson Equations

- ...but as engineers, you all know that integrals can be very difficult to evaluate for all but very simple geometries.
- So how do we solve for V(r) when the geometry is more complex?
- We rely on numerical methods
  - Finite Difference
  - Finite Elements
  - Method of Moments
  - Etc.

Review

- Laplace's Equation: $$\nabla^2 V = 0$$
- Poisson's Equation $$\nabla^2 V = -\frac{\rho}{\varepsilon}$$

$$\nabla^2 = \begin{bmatrix} \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{bmatrix} \bullet \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

(in cartesian coordinates)

Laplace and Poisson Equations

Boundary Conditions

- Dielectric-Dielectric

$$D_{n1} = D_{n2} \qquad\qquad E_{t1} = E_{t2}$$

- Writing in terms of voltage:

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this limit ensures voltage continuity

Laplace and Poisson Equations

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$$D_{n1} = \rho_s \qquad E_{t1} = 0$$

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Laplace and Poisson Equations

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Solve for **E** and **D**

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$\frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0 \Rightarrow r\frac{dV}{dr} = c_1 \;;\; \frac{dV}{dr} = \frac{c_1}{r} \;;\; \underline{V = c_1 \ln r + c_2}$

$V(b) = 0 = c_1 \ln b + c_2 \Rightarrow c_2 = -c_1 \ln b \Rightarrow V = c_1 \ln \frac{r}{b}$

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Laplace and Poisson Equations

b. $\vec{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{a}_r = \frac{-V_o}{\ln b/a}\frac{1}{b/r}\left(-\frac{b}{r^2}\right)\hat{a}_r = \boxed{\frac{V_o}{r \ln b/a}\hat{a}_r}$

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if $V_o > 0$ $\vec{E}$ points from $a \to b$

$\therefore \rho_{sa} > 0 \quad \rho_{sb} < 0$

$$\boxed{\rho_{sa} = \frac{\varepsilon V_o}{a \ln b/a} \qquad \rho_{sb} = \frac{-\varepsilon V_o}{b \ln b/a}}$$

d. $C = \frac{Q}{V} = \frac{\rho_{sa}\, 2\pi a l}{V_o} = \frac{\varepsilon \cancel{V_o}}{\cancel{a} \ln b/a}\frac{2\pi \cancel{a} l}{\cancel{V_o}} = \frac{2\pi\varepsilon l}{\ln b/a}$; $\boxed{\frac{C}{l} = \frac{2\pi\varepsilon}{\ln b/a}}$

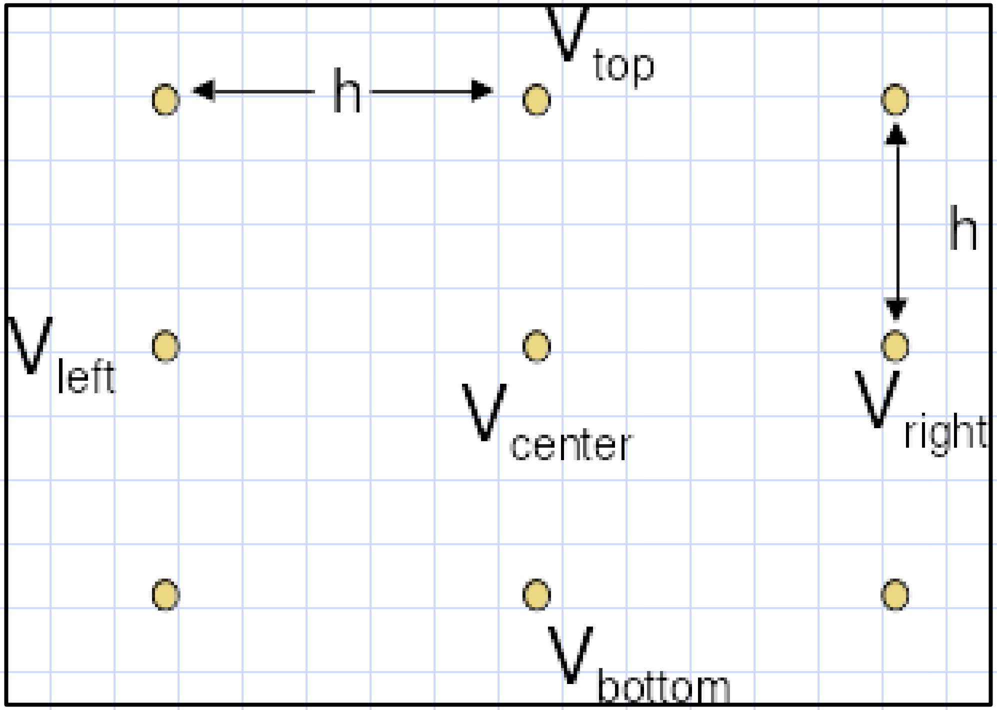
Laplace and Poisson Equations

- ...but as engineers, you all know that integrals can be very difficult to evaluate for all but very simple geometries.
- So how do we solve for V(r) when the geometry is more complex?
- We rely on numerical methods
  - Finite Difference
  - Finite Elements
  - Method of Moments
  - Etc.
- Generally, these methods break the problem into discrete pieces and linearize it while utilizing boundary conditions.

Laplace and Poisson Equations

Finite Difference Method

Consider a grid of points in a voltage field with distance h between them.



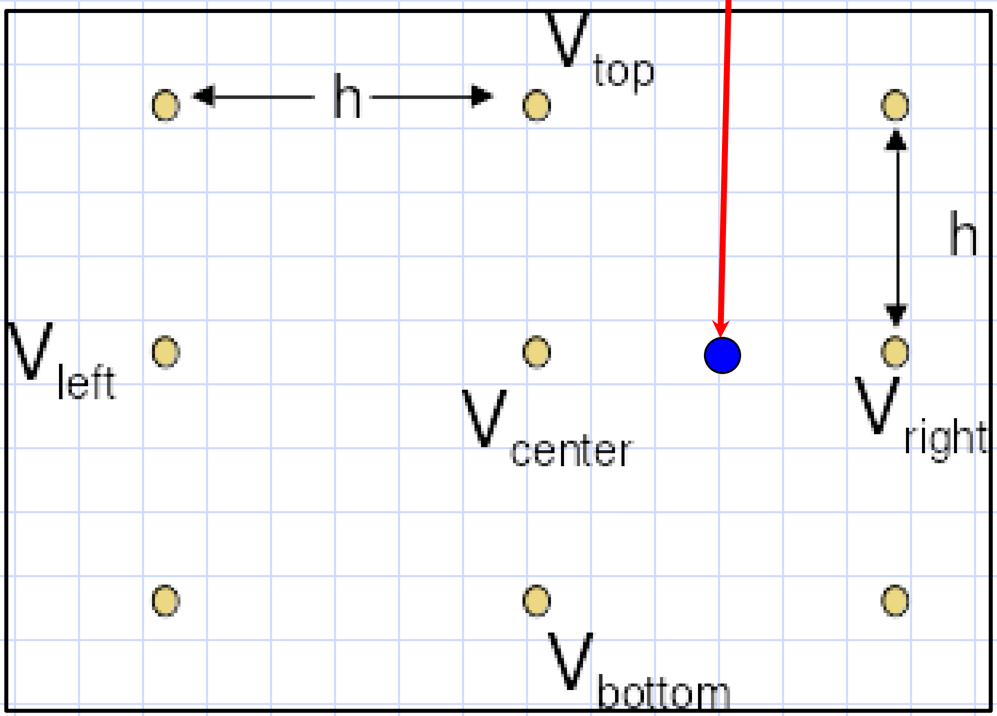
V_{center} is at the origin (0,0).

Laplace and Poisson Equations

Finite Difference Method

At (x,y) = (h/2,0)

$$E_x = -\frac{\partial V}{\partial x} \approx -\frac{\Delta V}{\Delta x} = -\frac{(V_{right} - V_{center})}{h}$$



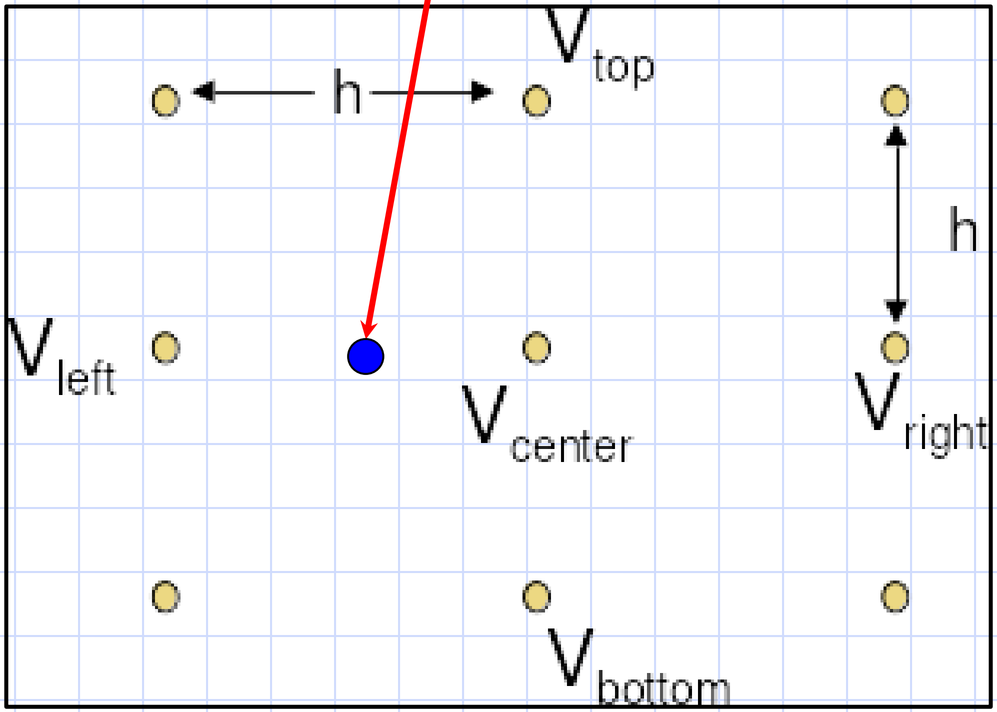
The E-field can be approximated by taking the difference of the two adjacent voltages.

Laplace and Poisson Equations

Finite Difference Method

At (x,y) = (-h/2,0)

$$E_x = -\frac{\partial V}{\partial x} \approx -\frac{\Delta V}{\Delta x} = -\frac{(V_{center} - V_{left})}{h}$$



The E-field can be approximated by taking the difference of the two adjacent voltages.

Laplace and Poisson Equations

Finite Difference Method

$$\nabla^2 V = \nabla \bullet \nabla V = -\nabla \bullet \vec{E} = -\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} - \cancel{\frac{\partial E_z}{\partial z}}$$

(no z contribution in this case)

To find the x-direction contribution to the E-field at V_{center}:

$$\frac{\partial E_x}{\partial x} \approx \frac{\Delta E_x}{\Delta x} = \frac{E_x\left(\frac{h}{2},0\right) - E_x\left(-\frac{h}{2},0\right)}{h} = \frac{2 \cdot V_{center} - V_{right} - V_{left}}{h^2}$$

We can find a similar expression for the y-direction contribution:

$$\frac{\partial \vec{E}_y}{\partial y} \approx \frac{\Delta \vec{E}_y}{\Delta y} = \frac{2 \cdot V_{center} - V_{top} - V_{bottom}}{h^2}$$

Laplace and Poisson Equations

Finite Difference Method

Finally we obtain the following expression:

$$\nabla^2 V = \frac{4 \cdot V_{center} - (V_{right} + V_{left} + V_{top} + V_{bottom})}{h^2} = -\frac{\rho}{\varepsilon}$$

Rearrange the equation to solve for V_{center} :

$$V_{center} = \frac{1}{4} \cdot \left(\sum V_{neighbors} - \frac{\rho \cdot h^2}{\varepsilon} \right)$$

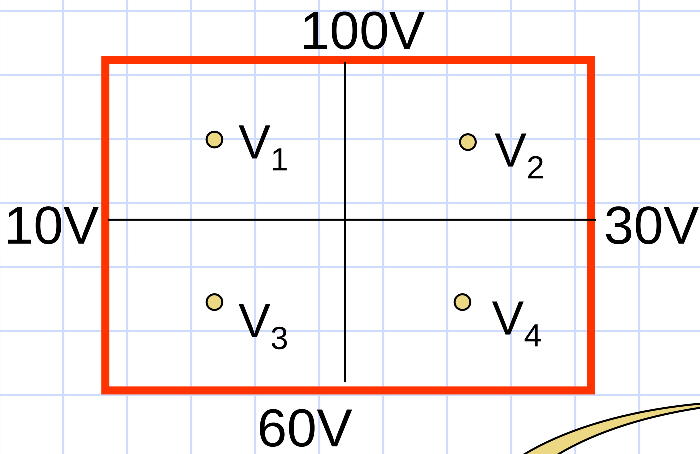
Poisson Equation Solver

$$V_{center} = \frac{1}{4} \cdot \left(\sum V_{neighbors} \right)$$

Laplace Equation Solver

Laplace and Poisson Equations

Finite Difference Method



Solution Technique - by Iteration

Guess a solution : V=0 everywhere except boundaries

$V_1 = V_2 = V_3 = V_4 = 0$

Put new values back

Start:

$$V_1 = \tfrac{1}{4}\cdot(100+10+V_2+V_3) \longrightarrow V_1 = \tfrac{1}{4}\cdot(110+0) = 27.5$$

$$V_2 = \tfrac{1}{4}\cdot(100+30+V_1+V_4) \longrightarrow V_2 = \tfrac{1}{4}\cdot(130+27.5+0) = 39.375$$

$$V_3 = \tfrac{1}{4}\cdot(10+60+V_1+V_4) \longrightarrow V_3 = \tfrac{1}{4}\cdot(70+27.5+0) = 24.375$$

$$V_4 = \tfrac{1}{4}\cdot(30+60+V_2+V_3) \longrightarrow V_4 = \tfrac{1}{4}\cdot(90+39.375+24.375) = 38.4375$$

Laplace and Poisson Equations

Finite Difference Method

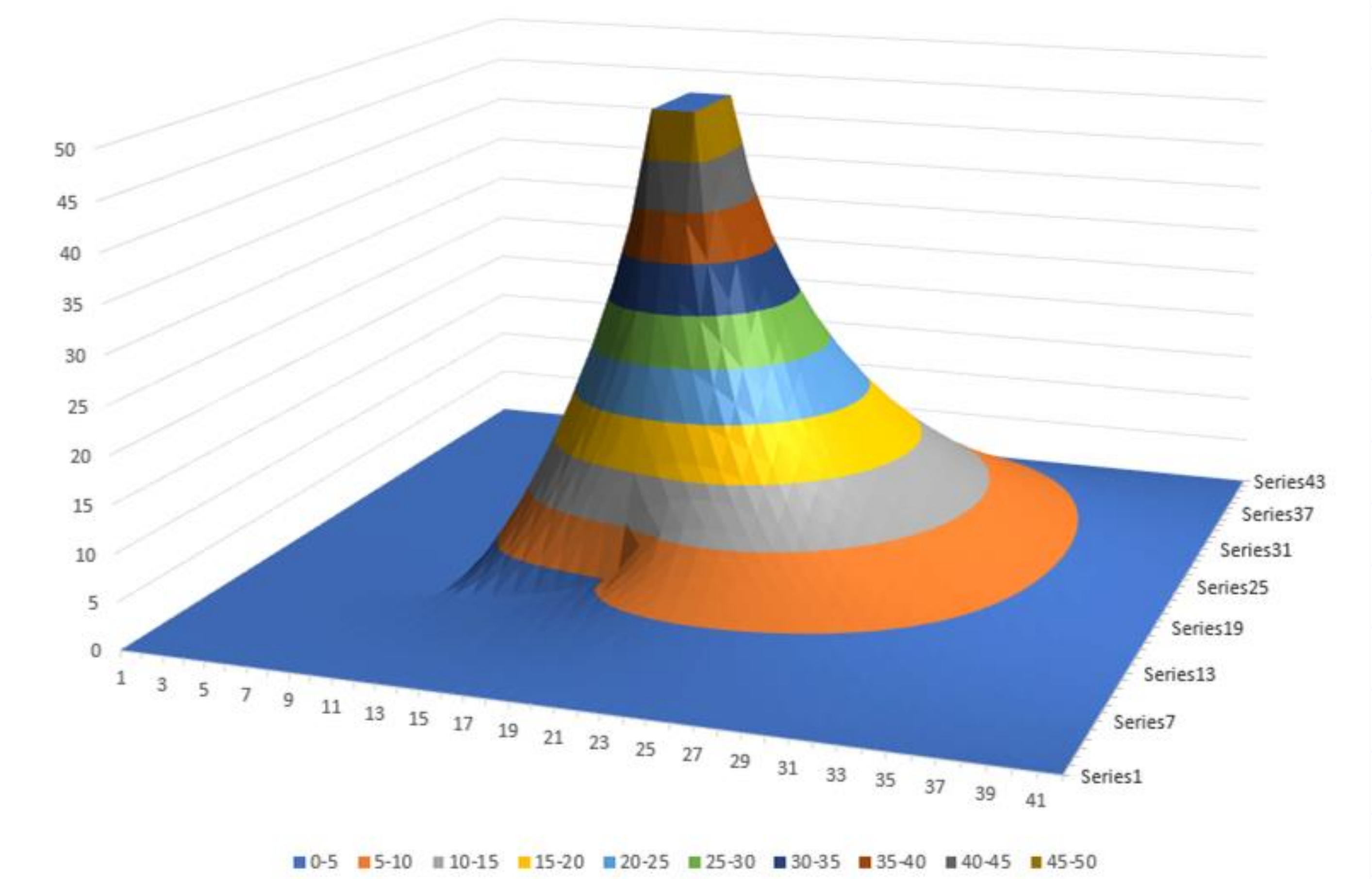
Do Lecture 13, Exercise 1 in groups of up to 4.

Finite Difference

The Finite Difference method can be done in a spreadsheet (the next Studio Session will be based on this)

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0.1	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.3	0.3	0.4	0.4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0.5	0.4	0.3	0.3	0.2	0.1	0.1	0
0	0	0	0.1	0.1	0.1	0.1	0.2	0.2	0.3	0.3	0.4	0.5	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	0.4	0.3	0.1	0
0	0	0.1	0.1	0.1	0.2	0.2	0.3	0.3	0.4	0.5	1	1	1	1	1	1	1	2	2	2	2	2	3	3	3	3	2	2	2	2	2	2	2	1	1	1	1	1	0.4	0.2	0
0	0	0.1	0.1	0.2	0.2	0.3	0.3	0.4	0.5	1	1	1	1	1	1	2	2	2	3	3	3	3	3	3	4	3	3	3	3	3	3	2	2	2	2	1	1	1	1	0.3	0
0	0	0.1	0.1	0.2	0.2	0.3	0.4	0.5	1	1	1	1	1	2	2	2	2	3	3	4	4	4	4	4	4	4	4	4	4	4	3	3	3	2	2	2	1	1	1	0.3	0
0	0	0.1	0.1	0.2	0.3	0.3	0.4	1	1	1	1	1	1	2	2	2	3	3	4	4	5	5	5	6	6	5	5	5	5	4	4	4	3	3	2	2	2	1	1	0.4	0
0	0	0.1	0.2	0.2	0.3	0.4	0.4	1	1	1	1	1	2	2	2	3	3	4	5	5	6	6	7	7	7	7	6	6	6	5	5	4	4	3	3	2	2	1	1	0.5	0
0	0	0.1	0.2	0.2	0.3	0.4	0.4	1	1	1	1	1	2	2	2	3	4	4	5	6	7	8	8	8	8	8	7	7	7	6	6	5	4	4	3	3	2	2	1	1	0
0	0	0.1	0.2	0.2	0.3	0.3	0.4	1	1	1	1	1	2	2	3	3	4	5	6	7	9	9	10	10	9	9	9	8	8	7	6	6	5	4	4	3	2	2	1	1	0
0	0	0.1	0.1	0.2	0.3	0.3	0.4	0.5	1	1	1	1	2	2	3	3	4	5	6	9	10	11	11	11	11	10	10	9	9	8	7	6	6	5	4	3	3	2	1	1	0
0	0	0.1	0.1	0.2	0.2	0.3	0.3	0.4	0.5	0.5	2	3	4	5	6	7	8	9	11	12	13	14	14	13	13	12	11	10	10	9	8	7	6	5	5	4	3	2	1	1	0
0	0	0.1	0.1	0.2	0.2	0.3	0.3	0.3	0.4	0.4	2	4	5	7	9	11	12	14	15	16	16	16	16	15	15	14	13	12	11	10	9	8	7	6	5	4	3	2	2	1	0
0	0	0.1	0.1	0.1	0.2	0.2	0.3	0.3	0.3	0.3	3	5	7	10	12	14	17	18	20	20	20	20	19	18	17	15	14	13	12	10	9	8	7	6	5	4	3	3	2	1	0
0	0	0.1	0.1	0.1	0.2	0.2	0.2	0.2	0.2	0.2	3	6	9	12	15	18	21	24	25	25	25	23	22	20	19	17	15	14	13	11	10	9	8	7	6	5	4	3	2	1	0
0	0	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.2	0.2	4	7	11	14	18	22	27	30	32	31	30	27	25	23	21	19	17	15	13	12	11	9	8	7	6	5	4	3	2	1	0
0	0	0.1	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.2	4	8	12	17	21	27	32	38	40	39	35	31	28	25	22	20	18	16	14	13	11	10	9	7	6	5	4	3	2	1	0
0	0	0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	4	9	14	19	24	30	38	50	50	50	41	35	30	27	24	21	19	17	15	13	12	10	9	8	6	5	4	3	2	1	0
0	0	0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	5	9	14	20	26	33	41	50	50	50	43	37	32	28	25	22	20	17	15	14	12	11	9	8	7	5	4	3	2	1	0
0	0	0	0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	5	10	15	21	27	34	41	50	50	50	44	38	33	29	26	23	20	18	16	14	12	11	9	8	7	6	4	3	2	1	0
0	0	0	0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	5	10	15	21	27	34	42	50	50	50	44	38	33	29	26	23	20	18	16	14	12	11	9	8	7	6	4	3	2	1	0
0	0	0	0	0	0.1	0.1	0.1	0.1	0.1	0.1	5	10	15	20	26	33	41	50	50	50	43	37	33	29	25	23	20	18	16	14	12	11	9	8	7	6	4	3	2	1	0
0	0	0	0	0	0	0.1	0.1	0.1	0.1	0.1	5	9	14	19	25	31	39	50	50	50	41	36	31	28	25	22	20	17	15	14	12	11	9	8	7	5	4	3	2	1	0
0	0	0	0	0	0	0	0	0.1	0.1	0.1	4	9	13	18	23	28	34	39	41	40	36	33	29	26	23	21	19	17	15	13	12	10	9	8	7	5	4	3	2	1	0
0	0	0	0	0	0	0	0	0	0	0	4	8	12	16	20	25	29	32	34	33	32	29	27	24	22	20	18	16	14	13	11	10	9	8	6	5	4	3	2	1	0
0	0	0	0	0	0	0	0	0	0	0	4	7	11	14	18	21	25	27	28	28	27	26	24	22	20	19	17	15	14	12	11	10	8	7	6	5	4	3	2	1	0
0	0	0	0	0	0	0	0	0	0	0	3	6	10	13	16	19	21	23	24	24	24	23	22	20	19	17	16	14	13	12	10	9	8	7	6	5	4	3	2	1	0
0	0	0	0	0	0	0	0	0	0	0	3	6	9	11	14	16	18	20	21	21	21	20	19	18	17	16	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
0	0	0	0	0	0	0	0	0	0	0	3	5	7	10	12	14	16	17	18	18	18	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	3	2	1	0
0	0	0	0	0	0	0	0	0	0	0	2	4	6	8	10	12	13	14	15	16	16	16	15	15	14	13	12	11	10	9	8	7	7	6	5	4	3	2	2	1	0
0	0	0	0	0	0	0	0	0	0	0	2	4	6	7	9	10	12	12	13	14	14	14	13	13	12	12	11	10	9	9	8	7	6	5	5	4	3	2	1	1	0
0	0	0	0	0	0	0	0	0	0	0	2	3	5	6	8	9	10	11	11	12	12	12	12	11	11	10	10	9	8	8	7	6	6	5	4	3	3	2	1	1	0
0	0	0	0	0	0	0	0	0	0	0	1	3	4	5	7	8	9	9	10	10	10	10	10	10	10	9	9	8	8	7	6	6	5	4	4	3	2	2	1	1	0
0	0	0	0	0	0	0	0	0	0	0	1	2	4	5	6	7	7	8	8	9	9	9	9	9	8	8	8	7	7	6	6	5	5	4	3	3	2	2	1	1	0
0	0	0	0	0	0	0	0	0	0	0	1	2	3	4	5	6	6	7	7	7	8	8	8	8	7	7	7	6	6	5	5	4	4	3	3	2	2	1	1	0.5	0
0	0	0	0	0	0	0	0	0	0	0	1	2	3	3	4	5	5	6	6	6	6	7	6	6	6	6	6	5	5	5	4	4	3	3	3	2	2	1	1	0.4	0
0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	3	4	4	5	5	5	5	5	5	5	5	5	5	5	4	4	4	3	3	3	2	2	1	1	1	0.4	0
0	0	0	0	0	0	0	0	0	0	0	1	1	2	2	3	3	3	4	4	4	4	4	4	4	4	4	4	4	4	3	3	3	2	2	2	2	1	1	1	0.3	0
0	0	0	0	0	0	0	0	0	0	0	0.5	1	1	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	2	2	2	2	1	1	1	1	0.5	0.2	0
0	0	0	0	0	0	0	0	0	0	0	0.3	1	1	1	2	2	2	2	2	2	3	3	3	3	3	2	2	2	2	2	2	2	1	1	1	1	1	1	0.4	0.2	0
0	0	0	0	0	0	0	0	0	0	0	0.2	0.4	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	0.5	0.4	0.2	0.1	0
0	0	0	0	0	0	0	0	0	0	0	0.1	0.2	0.3	0.4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0.5	0.4	0.4	0.3	0.2	0.2	0.1	0.1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Finite Difference



Surface chart made using Microsoft Excel

Finite Difference

Suppose that this represents a conductive rectangle
held at 50V.

$$E_x = -\frac{\partial V}{\partial x} \approx -\frac{\Delta V}{\Delta x}$$

1	1	1	1	2	2	3	3	4	5	6	7	9	9	10	10	9	9	[illegible]	8	8	7
1	1	1	1	2	2	3	3	4	5	6	9	10	11	11	11	11	10	10	9	9	8
0.5	0.5	2	3	4	5	6	7	8	9	11	12	13	14	14	13	13	12	11	10	10	9
0.4	0.4	2	4	5	7	9	11	12	14	15	16	16	16	16	15	15	14	13	12	11	10
0.3	0.3	3	5	7	10	12	14	17	18	20	20	20	20	19	18	17	15	14	13	12	10
0.2	0.2	3	6	9	12	15	18	21	24	25	25	25	23	22	20	19	17	15	14	13	11
0.2	0.2	4	7	11	14	18	22	27	30	32	31	30	27	25	23	21	19	17	15	13	12
0.2	0.2	4	8	12	17	21	27	32	38	40	39	35	31	28	25	22	20	18	16	14	13
0.1	0.1	4	9	14	19	24	30	38	50	50	50	41	35	30	27	24	21	19	17	15	13
0.1	0.1	5	9	14	20	26	33	41	50	50	50	43	37	32	28	25	22	20	17	15	14
0.1	0.1	5	10	15	21	27	34	41	50	50	50	44	38	33	29	26	23	20	18	16	14
0.1	0.1	5	10	15	21	27	34	42	50	50	50	44	38	33	29	26	23	20	18	16	14
0.1	0.1	5	10	15	20	26	33	41	50	50	50	43	37	33	29	25	23	20	18	16	14
0.1	0.1	5	9	14	19	25	31	39	50	50	50	41	36	31	28	25	22	20	17	15	14
0.1	0.1	4	9	13	18	23	28	34	39	41	40	36	33	29	26	23	21	19	17	15	13
0	0	4	8	12	16	20	25	29	32	34	33	32	29	27	24	22	20	18	16	14	13
0	0	4	7	11	14	18	21	25	27	28	28	27	26	24	22	20	19	17	15	14	12
0	0	3	6	10	13	16	19	21	23	24	24	24	23	22	20	19	17	16	14	13	12
0	0	3	6	9	11	14	16	18	20	21	21	21	20	19	18	17	16	14	13	12	11
0	0	3	5	7	10	12	14	16	17	18	18	18	18	17	16	15	14	13	12	11	10
0	0	2	4	6	8	10	12	13	14	15	16	16	16	15	15	14	13	12	11	10	9
0	0	2	4	6	7	9	10	12	12	13	14	14	14	13	13	12	12	11	10	9	9

Finite Difference

Suppose that this represents a conductive rectangle
held at 50V.

$$D_x = \epsilon_0 E_x$$

1	1	1	1	2	2	3	3	4	5	6	7	9	9	10	10	9	9	[illegible]	8	8	7
1	1	1	1	2	2	3	3	4	5	6	9	10	11	11	11	11	10	10	9	9	8
0.5	0.5	2	3	4	5	6	7	8	9	11	12	13	14	14	13	13	12	11	10	10	9
0.4	0.4	2	4	5	7	9	11	12	14	15	16	16	16	16	15	[illegible]	14	13	12	11	10
0.3	0.3	3	5	7	10	12	14	17	18	20	20	20	20	19	18	17	15	14	13	12	10
0.2	0.2	3	6	9	12	15	18	21	24	25	25	25	23	22	20	19	17	15	14	13	11
0.2	0.2	4	7	11	14	18	22	27	30	32	31	30	27	[illegible]	23	21	19	17	15	13	12
0.2	0.2	4	8	12	17	21	27	32	38	40	39	35	31	28	25	22	20	18	16	14	13
0.1	0.1	4	9	14	19	24	30	38	50	50	50	41	35	30	27	24	21	19	17	15	13
0.1	0.1	5	9	14	20	26	33	41	50	50	50	43	37	32	28	25	22	20	17	15	14
0.1	0.1	5	10	15	21	27	34	41	50	50	50	44	38	33	29	26	23	20	18	16	14
0.1	0.1	5	10	15	21	27	34	42	50	50	50	44	38	33	29	26	23	20	18	16	14
0.1	0.1	5	10	15	20	26	33	41	50	50	50	43	37	33	29	25	23	20	18	16	14
0.1	0.1	5	9	14	19	25	31	39	50	50	50	41	36	31	28	25	22	20	17	15	14
0.1	0.1	4	9	13	18	23	28	34	39	41	40	36	33	29	26	23	21	19	17	15	13
0	0	4	8	12	16	20	25	29	32	34	33	32	29	27	24	22	20	18	16	14	13
0	0	4	7	11	14	18	21	25	27	28	28	27	26	24	22	20	19	17	15	14	12
0	0	3	6	10	13	16	19	21	23	24	24	24	23	22	20	19	17	16	14	13	12
0	0	3	6	9	11	14	16	18	20	21	21	21	20	19	18	17	16	14	13	12	11
0	0	3	5	7	10	12	14	16	17	18	18	18	18	17	16	15	14	13	12	11	10
0	0	2	4	6	8	10	12	13	14	15	16	16	16	15	15	14	13	12	11	10	9
0	0	2	4	6	7	9	10	12	12	13	14	14	14	13	13	12	12	11	10	9	9

Finite Difference

Suppose that this represents a conductive rectangle
held at 50V.

$$D_x = \rho$$

1	1	1	1	2	2	3	3	4	5	6	7	9	9	10	10	9	9	[illegible]	8	8	7
1	1	1	1	2	2	3	3	4	5	6	9	10	11	11	11	11	10	10	9	9	8
0.5	0.5	2	3	4	5	6	7	8	9	11	12	13	14	14	13	13	12	11	10	10	9
0.4	0.4	2	4	5	7	9	11	12	14	15	16	16	16	16	15	[illegible]	14	13	12	11	10
0.3	0.3	3	5	7	10	12	14	17	18	20	20	20	20	19	18	17	15	14	13	12	10
0.2	0.2	3	6	9	12	15	18	21	24	25	25	25	23	22	20	19	17	15	14	13	11
0.2	0.2	4	7	11	14	18	22	27	30	32	31	30	27	[illegible]	23	21	19	17	15	13	12
0.2	0.2	4	8	12	17	21	27	32	38	40	39	35	[illegible]	28	25	22	20	18	16	14	13
0.1	0.1	4	9	14	19	24	30	38	50	50	50	41	35	30	27	24	21	19	17	15	13
0.1	0.1	5	9	14	20	26	33	41	50	50	50	43	37	32	28	25	22	20	17	15	14
0.1	0.1	5	10	15	21	27	34	41	50	50	50	44	38	33	29	26	23	20	18	16	14
0.1	0.1	5	10	15	21	27	34	42	50	50	50	44	38	33	29	26	23	20	18	16	14
0.1	0.1	5	10	15	20	26	33	41	50	50	50	43	37	33	29	25	23	20	18	16	14
0.1	0.1	5	9	14	19	25	31	39	50	50	50	41	36	31	28	25	22	20	17	15	14
0.1	0.1	4	9	13	18	23	28	34	39	41	40	36	33	29	26	23	21	19	17	15	13
0	0	4	8	12	16	20	25	29	32	34	33	32	29	27	24	22	20	18	16	14	13
0	0	4	7	11	14	18	21	25	27	28	28	27	26	24	22	20	19	17	15	14	12
0	0	3	6	10	13	16	19	21	23	24	24	24	23	22	20	19	17	16	14	13	12
0	0	3	6	9	11	14	16	18	20	21	21	21	20	19	18	17	16	14	13	12	11
0	0	3	5	7	10	12	14	16	17	18	18	18	18	17	16	15	14	13	12	11	10
0	0	2	4	6	8	10	12	13	14	15	16	16	16	15	15	14	13	12	11	10	9
0	0	2	4	6	7	9	10	12	12	13	14	14	14	13	13	12	12	11	10	9	9

Finite Difference

You now know the charge density on this surface of this cell of the conductor. You can do the same for adjacent cells.

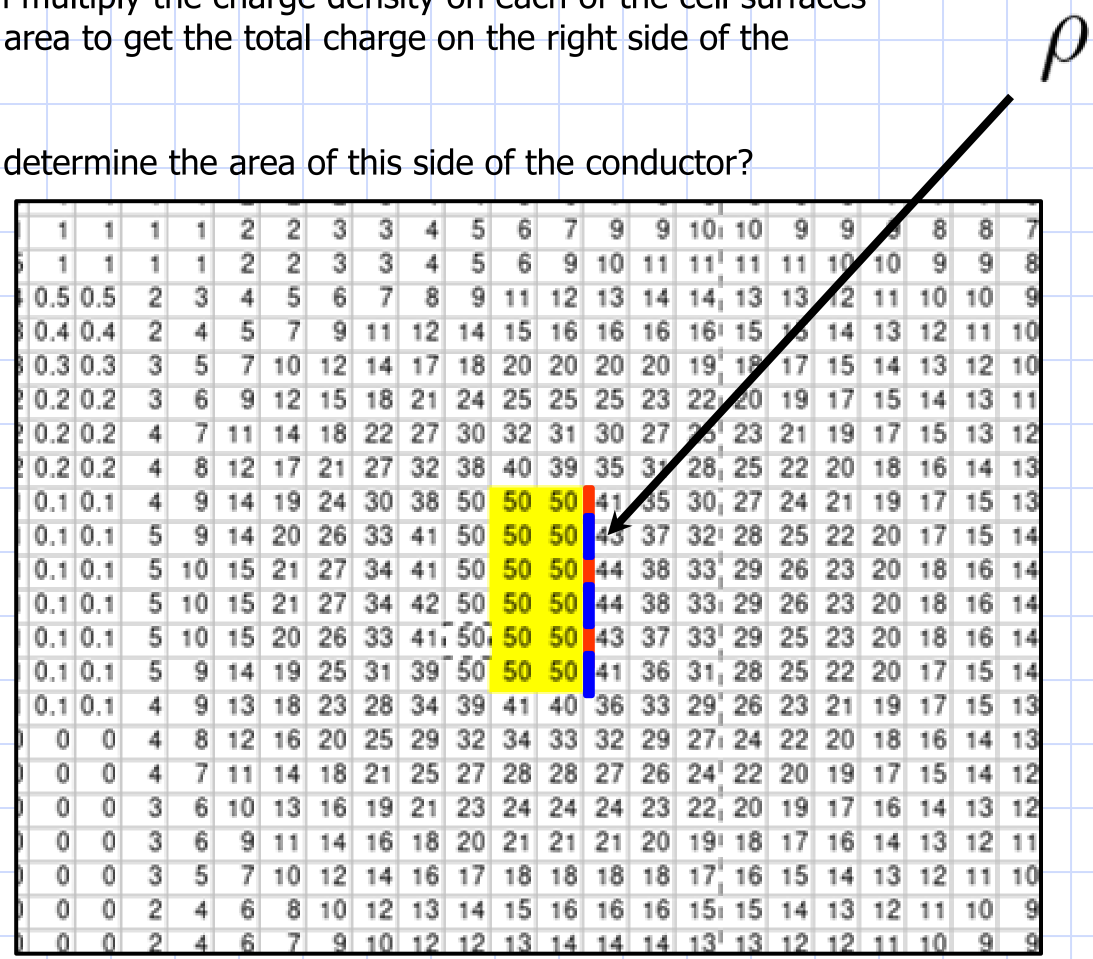
ρ

1	1	1	1	2	2	3	3	4	5	6	7	9	9	10	10	9	9	[illegible]	8	8	7
1	1	1	1	2	2	3	3	4	5	6	9	10	11	11	11	11	10	10	9	9	8
0.5	0.5	2	3	4	5	6	7	8	9	11	12	13	14	14	13	13	12	11	10	10	9
0.4	0.4	2	4	5	7	9	11	12	14	15	16	16	16	16	15	[illegible]	14	13	12	11	10
0.3	0.3	3	5	7	10	12	14	17	18	20	20	20	20	19	18	17	15	14	13	12	10
0.2	0.2	3	6	9	12	15	18	21	24	25	25	25	23	22	20	19	17	15	14	13	11
0.2	0.2	4	7	11	14	18	22	27	30	32	31	30	27	[illegible]	23	21	19	17	15	13	12
0.2	0.2	4	8	12	17	21	27	32	38	40	39	35	31	28	25	22	20	18	16	14	13
0.1	0.1	4	9	14	19	24	30	38	50	50	50	41	35	30	27	24	21	19	17	15	13
0.1	0.1	5	9	14	20	26	33	41	50	50	50	43	37	32	28	25	22	20	17	15	14
0.1	0.1	5	10	15	21	27	34	41	50	50	50	44	38	33	29	26	23	20	18	16	14
0.1	0.1	5	10	15	21	27	34	42	50	50	50	44	38	33	29	26	23	20	18	16	14
0.1	0.1	5	10	15	20	26	33	41	50	50	50	43	37	33	29	25	23	20	18	16	14
0.1	0.1	5	9	14	19	25	31	39	50	50	50	41	36	31	28	25	22	20	17	15	14
0.1	0.1	4	9	13	18	23	28	34	39	41	40	36	33	29	26	23	21	19	17	15	13
0	0	4	8	12	16	20	25	29	32	34	33	32	29	27	24	22	20	18	16	14	13
0	0	4	7	11	14	18	21	25	27	28	28	27	26	24	22	20	19	17	15	14	12
0	0	3	6	10	13	16	19	21	23	24	24	24	23	22	20	19	17	16	14	13	12
0	0	3	6	9	11	14	16	18	20	21	21	21	20	19	18	17	16	14	13	12	11
0	0	3	5	7	10	12	14	16	17	18	18	18	18	17	16	15	14	13	12	11	10
0	0	2	4	6	8	10	12	13	14	15	16	16	16	15	15	14	13	12	11	10	9
0	0	2	4	6	7	9	10	12	12	13	14	14	14	13	13	12	12	11	10	9	9

Finite Difference

You can then multiply the charge density on each of the cell surfaces below by its area to get the total charge on the right side of the conductor.

How do you determine the area of this side of the conductor?



1	1	1	1	2	2	3	3	4	5	6	7	9	9	10	10	9	9	[illegible]	8	8	7
1	1	1	1	2	2	3	3	4	5	6	9	10	11	11	11	11	10	10	9	9	8
0.5	0.5	2	3	4	5	6	7	8	9	11	12	13	14	14	13	13	12	11	10	10	9
0.4	0.4	2	4	5	7	9	11	12	14	15	16	16	16	16	15	[illegible]	14	13	12	11	10
0.3	0.3	3	5	7	10	12	14	17	18	20	20	20	20	19	18	17	15	14	13	12	10
0.2	0.2	3	6	9	12	15	18	21	24	25	25	25	23	22	20	19	17	15	14	13	11
0.2	0.2	4	7	11	14	18	22	27	30	32	31	30	27	[illegible]	23	21	19	17	15	13	12
0.2	0.2	4	8	12	17	21	27	32	38	40	39	35	[illegible]	28	25	22	20	18	16	14	13
0.1	0.1	4	9	14	19	24	30	38	50	50	50	41	35	30	27	24	21	19	17	15	13
0.1	0.1	5	9	14	20	26	33	41	50	50	50	43	37	32	28	25	22	20	17	15	14
0.1	0.1	5	10	15	21	27	34	41	50	50	50	44	38	33	29	26	23	20	18	16	14
0.1	0.1	5	10	15	21	27	34	42	50	50	50	44	38	33	29	26	23	20	18	16	14
0.1	0.1	5	10	15	20	26	33	41	50	50	50	43	37	33	29	25	23	20	18	16	14
0.1	0.1	5	9	14	19	25	31	39	50	50	50	41	36	31	28	25	22	20	17	15	14
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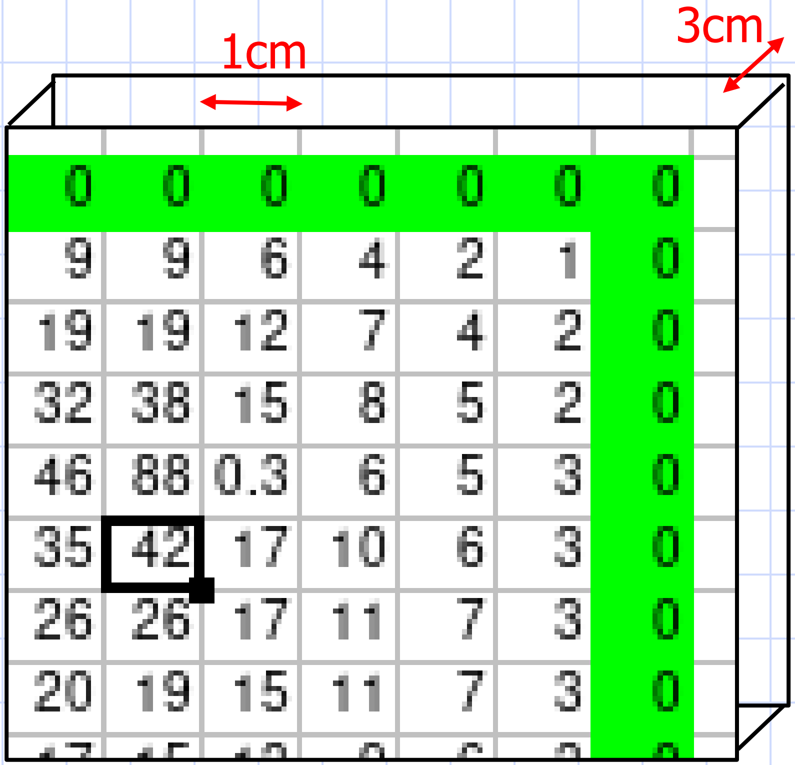
0	0	4	7	11	14	18	21	25	27	28	28	27	26	24	22	20	19	17	15	14	12
0	0	3	6	10	13	16	19	21	23	24	24	24	23	22	20	19	17	16	14	13	12
0	0	3	6	9	11	14	16	18	20	21	21	21	20	19	18	17	16	14	13	12	11
0	0	3	5	7	10	12	14	16	17	18	18	18	18	17	16	15	14	13	12	11	10
0	0	2	4	6	8	10	12	13	14	15	16	16	16	15	15	14	13	12	11	10	9
0	0	2	4	6	7	9	10	12	12	13	14	14	14	13	13	12	12	11	10	9	9

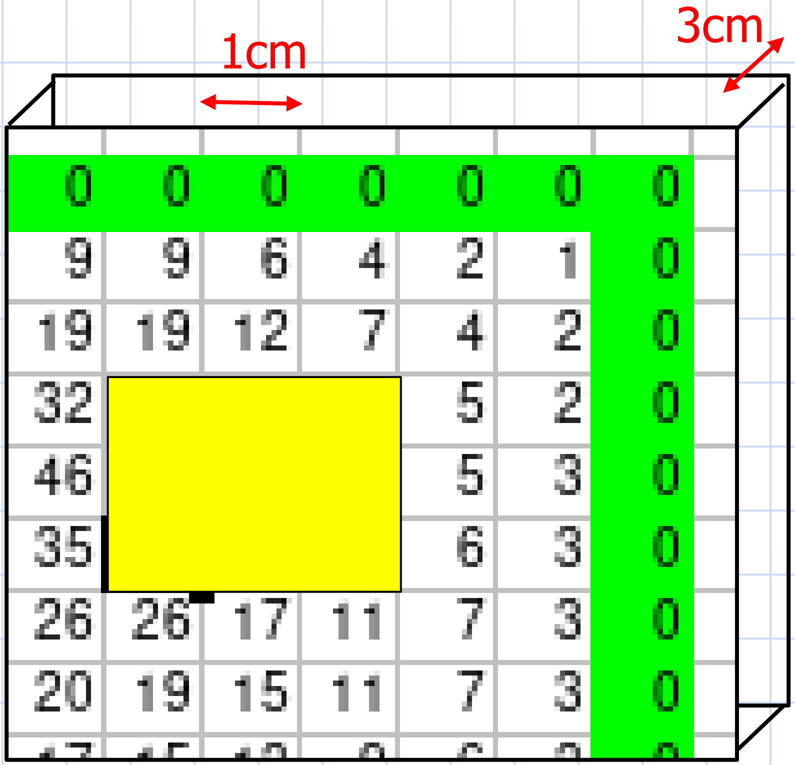
Finite Difference

- To calculate real quantities other than voltage, you must assign some dimension to each cell
- To represent true three-dimensional problem, each cell must also represent some depth (or represent per-unit length or depth)



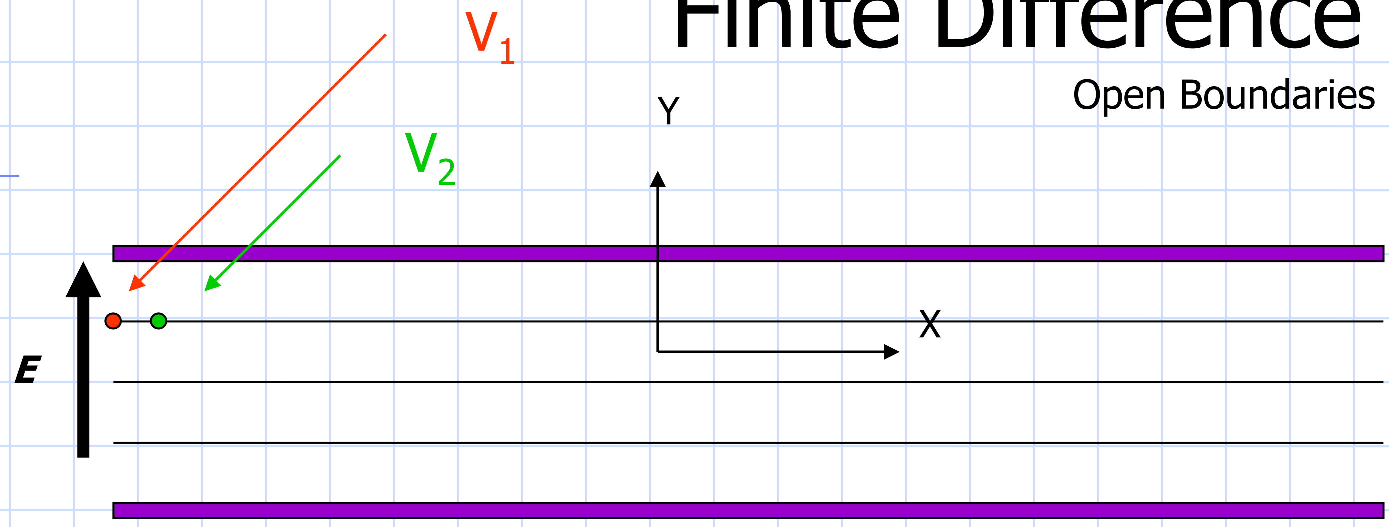
Finite Difference

- What is the area of the surfaces of this region parallel to the spreadsheet plane?



Finite Difference

Open Boundaries



- For a parallel plate capacitor the side boundaries are open and the equipotentials are horizontal.
- For such voltage lines, the boundary voltage will equal the value of its immediate inside neighbor.

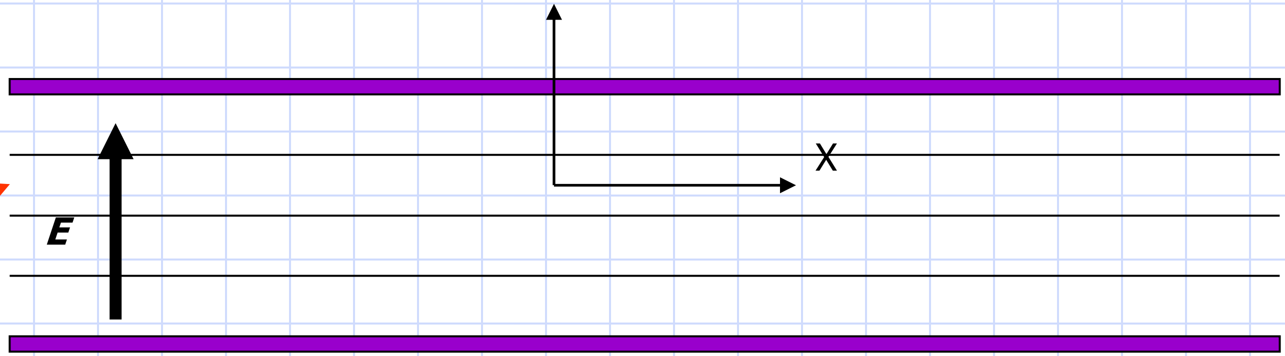
$$E_x = -\frac{\partial V}{\partial x} = 0$$

$$\therefore V_1 = V_2$$

Finite Difference

Open Boundaries

$$\frac{\partial V}{\partial n} = 0$$



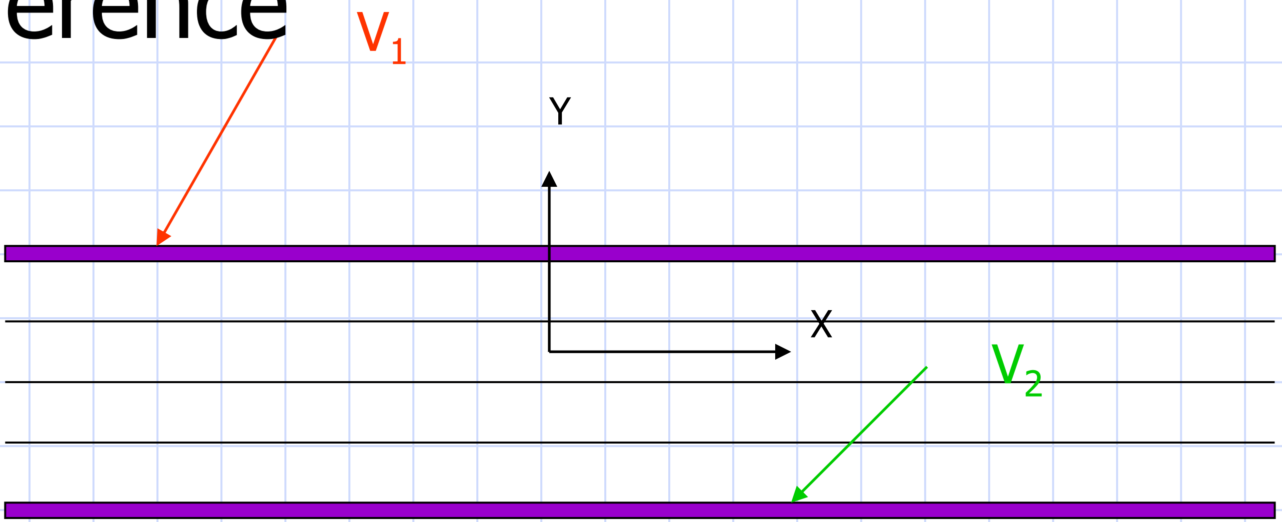
- Note that the condition that two points are on the same equipotential is the same as

$$D \bullet n = 0 \Leftrightarrow \hat{a}_x \bullet \varepsilon \nabla V = 0 \quad \text{written as} \quad \frac{\partial V}{\partial n} = 0$$

- This type of BC is called a Uniform Neumann Boundary Condition (from Mathematics).

Finite Difference

Closed Boundaries



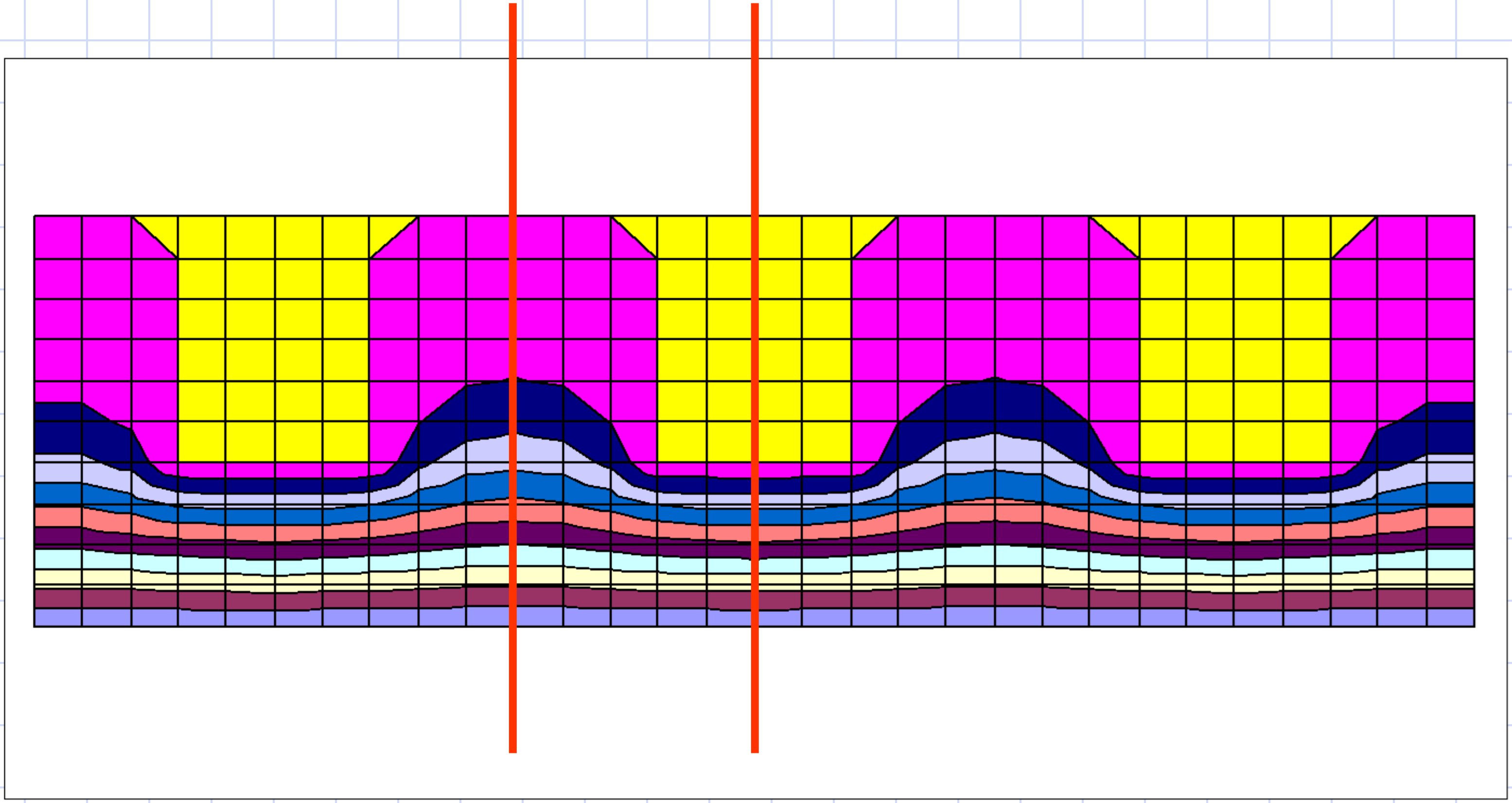
- For completeness, we should also note that any part of the boundary that is given a fixed voltage has what is called a Dirichlet Boundary Condition

- For a unique solution
  - At least one Dirichlet needs to be specified

Finite Difference

Symmetry

Lines of symmetry



- We can greatly reduce the work to find numerical solutions by using symmetry.

Finite Difference

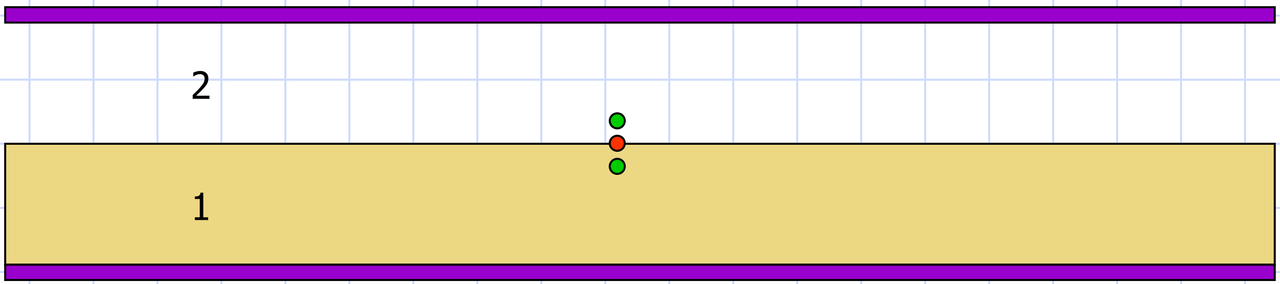
- At a dielectric-dielectric boundary, the voltage is continuous, but the normal derivative is not. (Rate of change of voltage can be different.)

$$D_{n1} = D_{n2}$$

- What does this means in terms of potential ?

Finite Difference

Dielectric Boundary



If there are two dielectrics, then the boundary condition at the interface must satisfy the diel-diel BC $D_{n1} = D_{n2}$

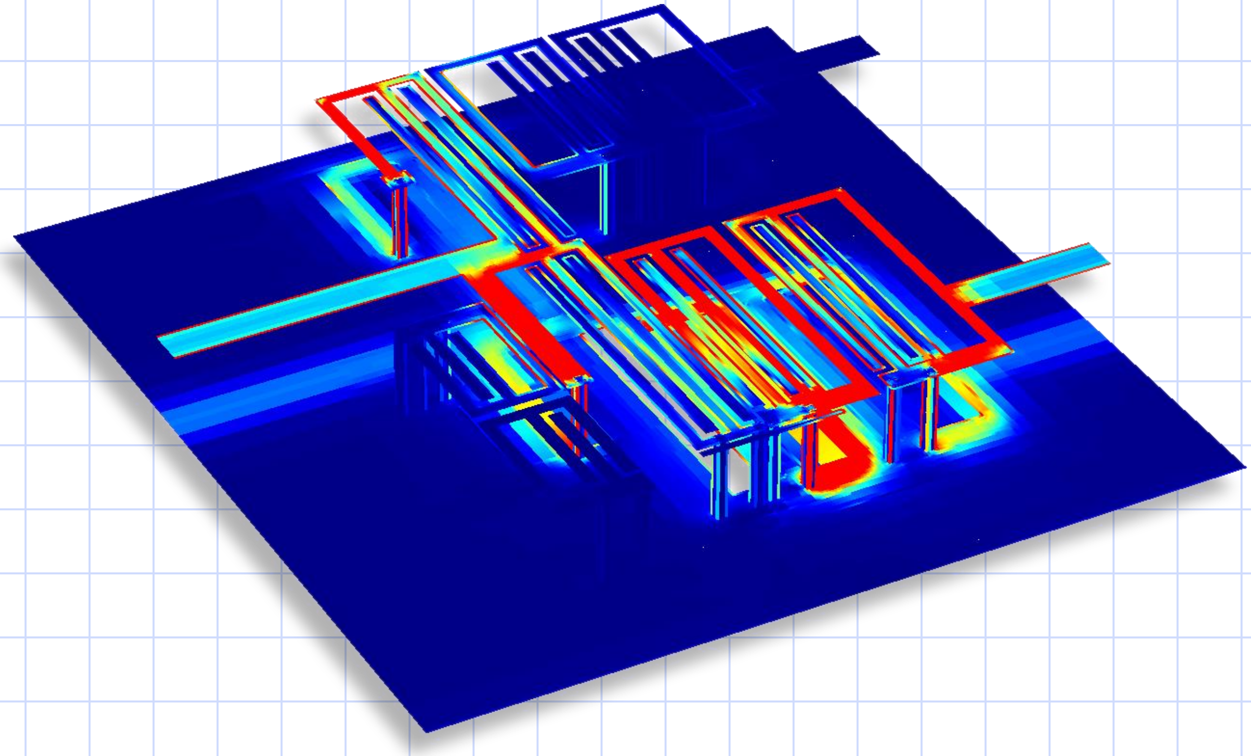
$$\varepsilon_1 \frac{V - V_{down}}{h} = \varepsilon_2 \frac{V_{up} - V}{h} \Rightarrow V = \frac{\varepsilon_1 V_{down} + \varepsilon_2 V_{up}}{\varepsilon_1 + \varepsilon_2}$$

Finite Element

- Finite Difference: generally breaks a field region into a square grid
- Finite Element: can break a field region into subdivisions of any geometry

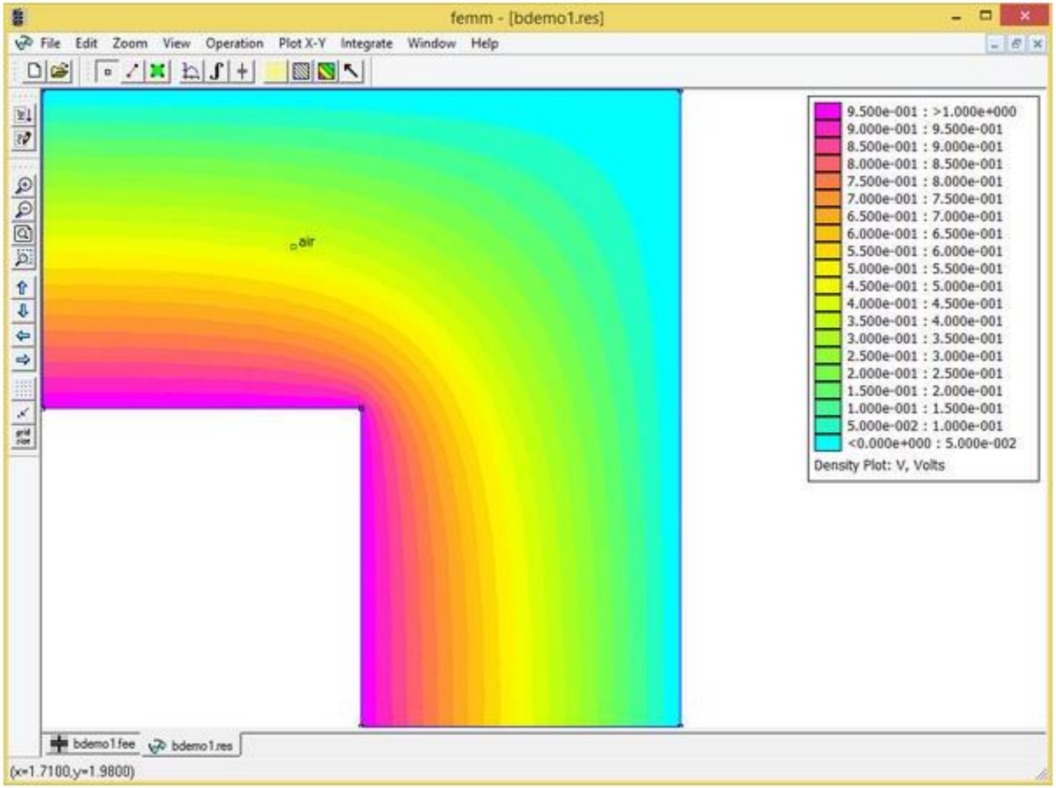
Finite Element

Sonnet (used for analog and digital circuit design applications)



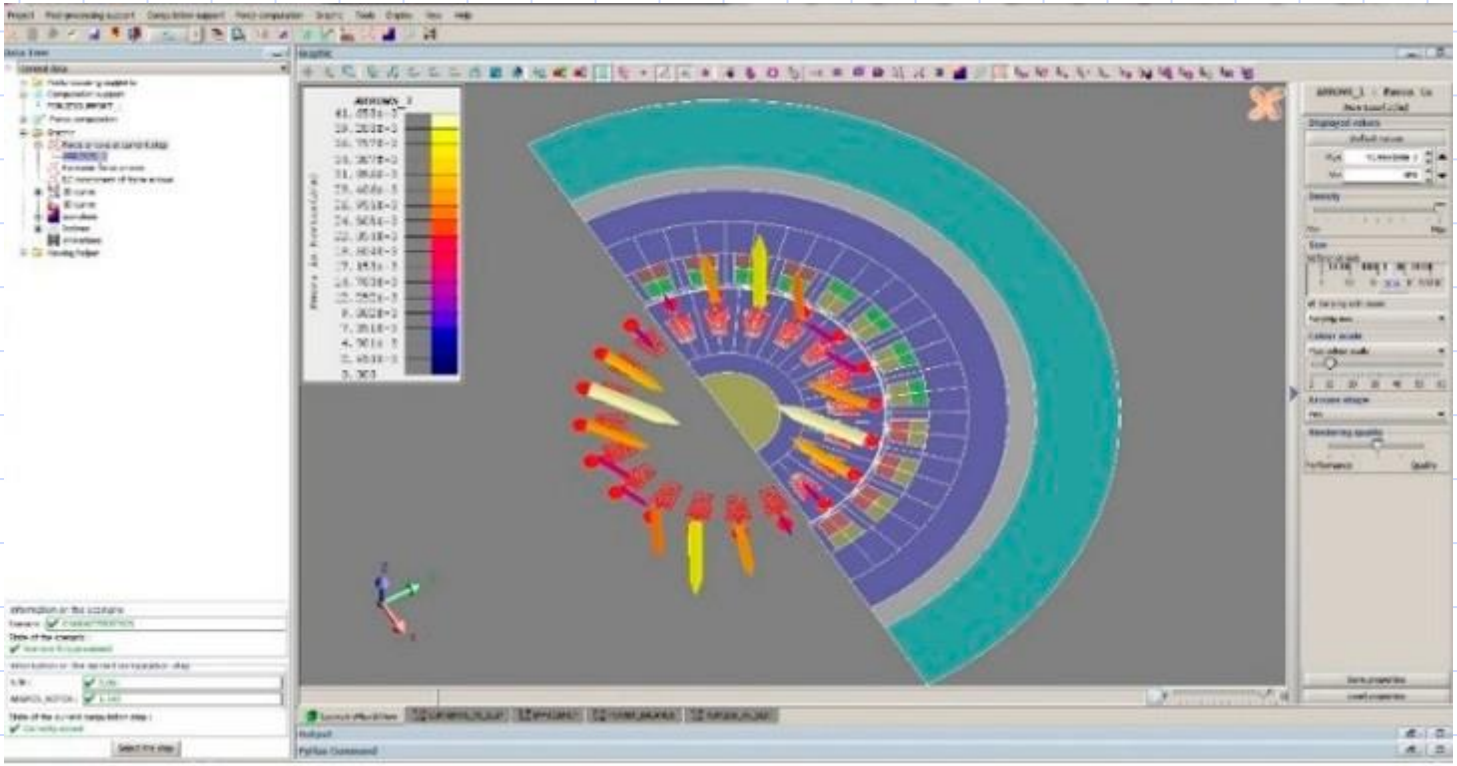
Finite Element

FEMM (open source, designed for magnetics but can be used for electrostatics)



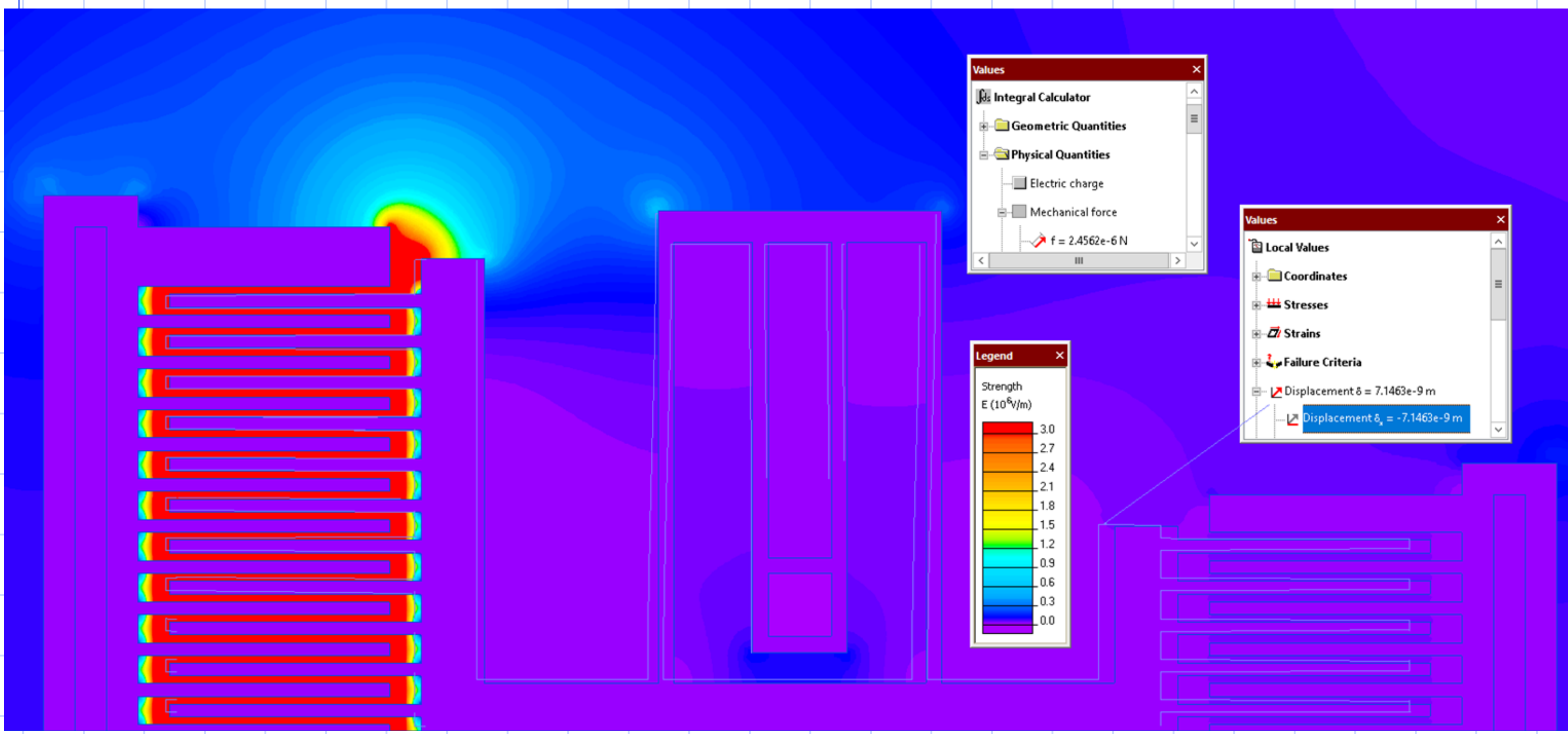
Finite Element

Altair Flux (wide range of electromagnetic capabilities)



Finite Element

[QuickField]

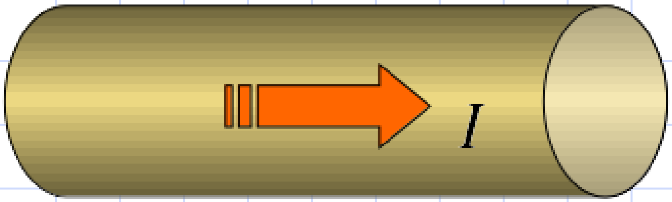


Current and Resistance

- We have been considering electrostatics - the case in which there are no moving charges.
- This obviously doesn't hold in general. When charges move, we have current.
- We will consider two parameters: current (measured in amps), and current density (measured in amps per unit area)

Current and Resistance

Wire with current, *I*



Definition: $I = \frac{\Delta Q}{\Delta t}$

Charge passing through cross-section in time Δt

Define: $\vec{j} = \frac{I}{Area} \cdot \hat{a}_z$

cross-section

points in direction of current flow

In general,

$$I = \int j \bullet ds$$

Flux Integral

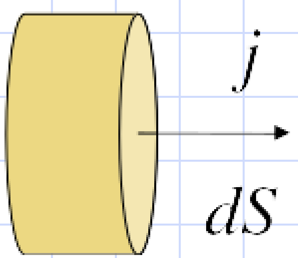
Current and Resistance

$$I = \int j \bullet ds$$

Example: wire with constant current density $j = j_0 \cdot \hat{a}_z$

$$\Rightarrow \quad I = \int_0^{2\pi}\int_0^{a} j_0 \cdot \hat{a}_z \bullet r \cdot dr \cdot d\varphi \cdot \hat{a}_z = j_0 \cdot \pi \cdot a^2$$

ds , in cylindrical geometry



Current and Resistance

a - Find I in terms of J_0 for $\vec{j} = j_0 \cdot \left(\frac{r}{a}\right)^8 \cdot \hat{a}_z$

Peak Density

a. $I = \int \vec{J} \cdot \vec{ds} = \int_0^{2\pi} \int_0^a J_0 \left(\frac{r}{a}\right)^8 r\,dr\,d\phi = \frac{2\pi J_0}{a^8} \left[\frac{r^{10}}{10}\right]_0^a$

integrate over wire cross-section

$= \frac{2\pi J_0}{a^8} \frac{a^{10}}{10} = \frac{\pi a^2}{5} J_0$

$\therefore \boxed{J_0 = \frac{5I}{\pi a^2}}$

For $I = 0.01\,A$ and $a = 10^{-3}\,m$

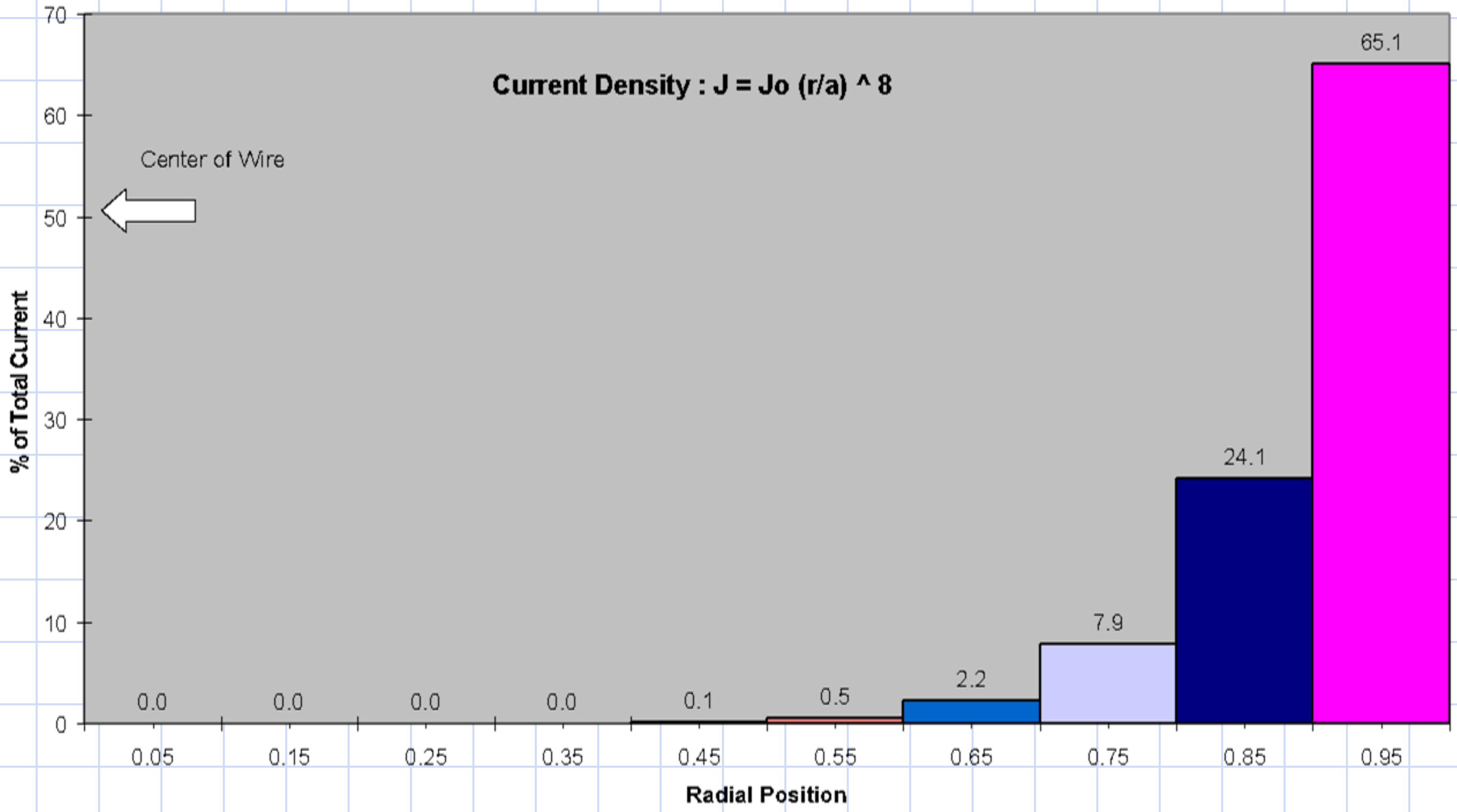
$\boxed{J_0 = 1.59 \times 10^4\ A/m^2 \text{ or } 1.59\ A/cm^2}$

NOTE: IF CURRENT WAS DISTRIBUTED UNIFORMLY, THEN $J = \frac{I}{\pi a^2}$

Since it is concentrated at edge, the peak value (J_0) is larger.

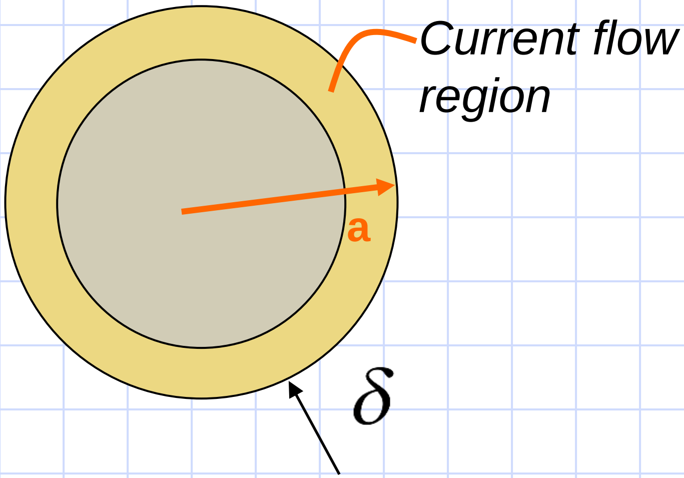
Current and Resistance

Note that for this current distribution, there is much more current in the outer shell. This is a combination of the increased density and the larger area of each shell.



Current and Resistance

- In conductors, current tends to flow in increasingly thin surface sheets at higher frequencies (the "skin effect")
- Skin effect in a wire:



$$j = \frac{I}{Area} \cong \frac{I}{2 \cdot \pi \cdot a \cdot \delta}$$

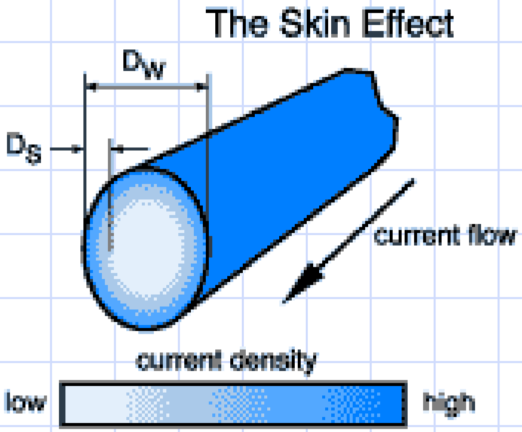
But, $\delta \xrightarrow{\lim it} 0 \Rightarrow j \rightarrow \infty$

Current and Resistance

For high frequency, the area for resistance for a circular wire is

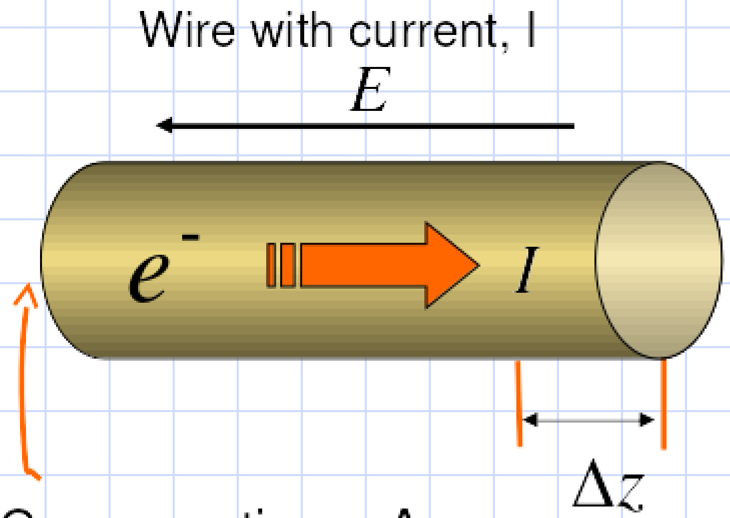
$$A = 2\pi a\delta \qquad A = 2\pi b\delta$$

$$\delta = \frac{1}{\sqrt{\pi f \mu_c \sigma_c}}$$



Current and Resistance

We now obtain an alternate expression for current density



Cross-section = A

$$I = \frac{\Delta Q}{\Delta t} \Rightarrow j = \frac{\Delta Q}{A \cdot \Delta t}$$

Assume all particles move at same v

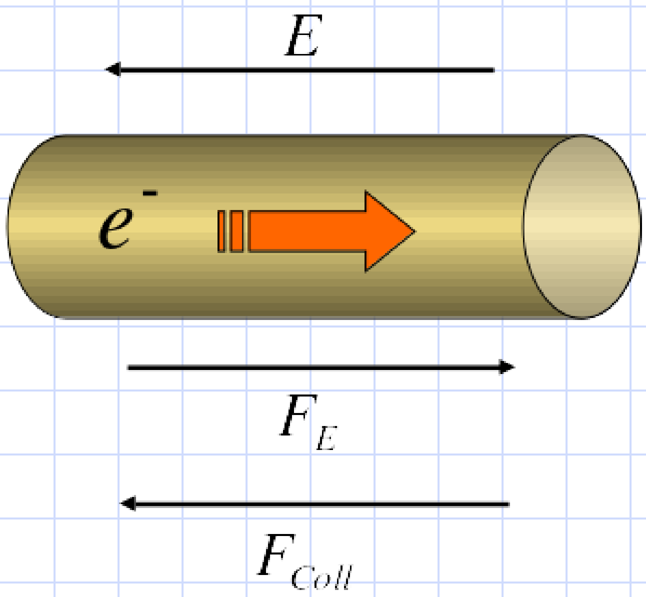
- In Δt, all particles within $\Delta z = v.\Delta t$ will pass through the right face

$$\therefore \Delta Q = \rho \cdot \Delta z \cdot A = \rho \cdot v \cdot A \cdot \Delta t$$

$$\Rightarrow j = \frac{\Delta Q}{A \cdot \Delta t} = \frac{\rho \cdot v \cdot A \cdot \Delta t}{A \cdot \Delta t} = \rho \cdot v \Rightarrow j = \rho \cdot v$$

Can we define this speed ?

Current and Resistance



E , in wire pushed electrons to the right

F_E , force (from E-field) driving electrons

e^- s, collide with lattice ions

F_{Coll} , balances F_E at some $v_{average}$

$$v_{average} = \mu \cdot E$$

(μ= electron mobility, not permeability)

Since $j = \rho \cdot v_{average}$ ⇨ $j = \rho \cdot v_{average} = \rho \cdot \mu \cdot E = \sigma \cdot E$

Current and Resistance

$$j = \sigma \cdot E$$ = Ohm's Law

Conductivity - units of S/m or 1/ohm-m

- varies from 10^7 to 10^{-15}

Good conductor eg. Cu

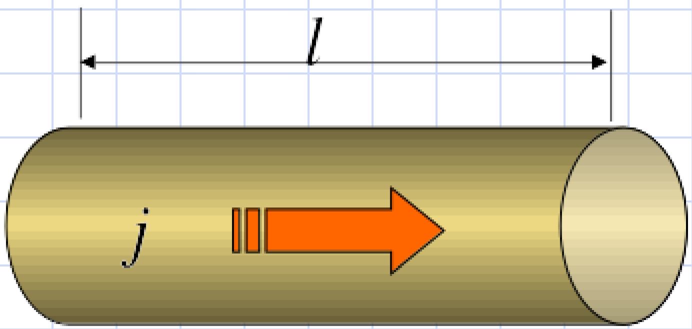
Good insulator

$j = \sigma \cdot E$, is Fields and Waves version of Ohm's Law

Note that this violates one of our electrostatic assumptions: that the e-field will have no curl.

Current and Resistance

- Resistance of a cylindrical wire:



$$I = j \cdot A = \sigma \cdot E \cdot A$$

$$V = -\int E \bullet dl = E \cdot l$$

$$\frac{V}{I} = \frac{E \cdot l}{\sigma \cdot E \cdot A} = \frac{l}{\sigma \cdot A} = R$$

Current and Resistance

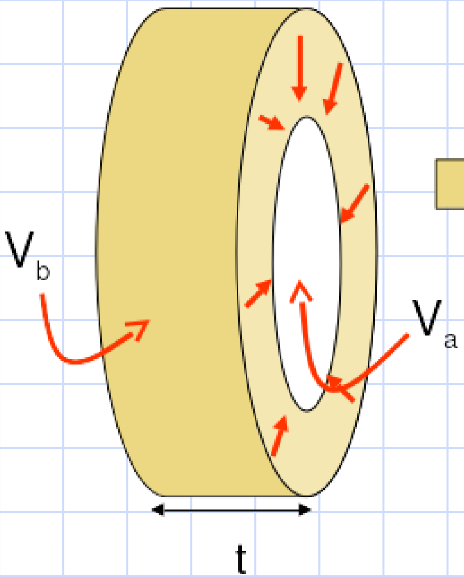
Do Lecture 13, Exercise 2 in groups of up to 4.

Current and Resistance

$$R = \frac{l}{\sigma \cdot A}$$ ⇨ Valid if only *j* and *A* are constant

What if they are not? Compute V and I separately and form V/I

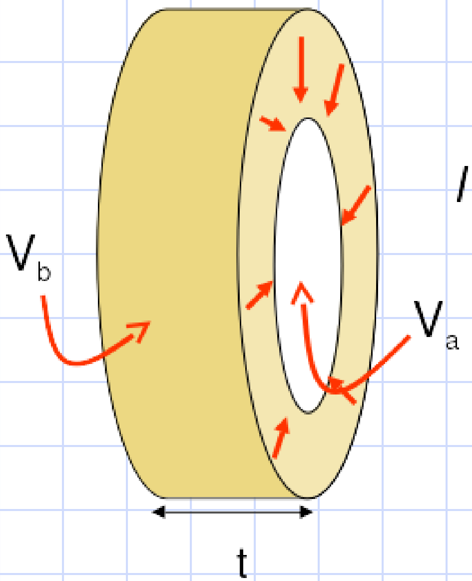
Example: Disk with Radial Current



⇨ Look at wedge (sketch)

I is the same at the inner and outer part

Current and Resistance



I is the same at the inner and outer part of disk

$$\vec{j} = \frac{I}{Area}\cdot\hat{a}_r = -\frac{I}{2\cdot\pi\cdot r\cdot t}\cdot\hat{a}_r$$

t=thickness

$$\vec{E} = \frac{\vec{j}}{\sigma} = -\frac{I}{2\cdot\pi\cdot r\cdot\sigma\cdot t}\cdot\hat{a}_r$$

Compute potential difference between inner and outer part of disk:

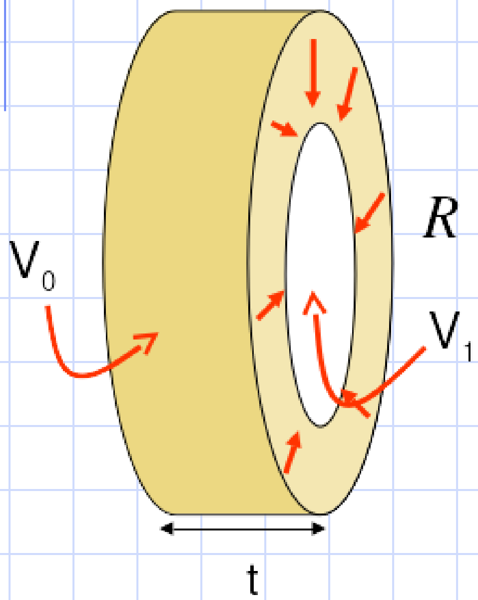
$$V_b - \underbrace{\cancel{V_a}}_{0} = -\int_a^b \vec{E}\bullet d\vec{l} = \frac{I}{2\cdot\pi\cdot\sigma\cdot t}\Big|_a^b \frac{dr}{r} = \frac{I}{2\cdot\pi\cdot\sigma\cdot t}\ln\left(\frac{b}{a}\right)$$

$$\therefore R = \frac{V}{I} = \frac{\ln(b/a)}{2\cdot\pi\cdot\sigma\cdot t}$$

Current and Resistance

- General expression of the resistance

$$R = \frac{V}{I} = \frac{\int \vec{E} \bullet d\vec{l}}{\int \vec{j} \bullet d\vec{S}} = \frac{\int \vec{j}/\sigma \bullet d\vec{l}}{\int \vec{j} \bullet d\vec{S}} = \int \frac{dl}{\sigma(l)S(l)}$$



$$R = \int_a^b \frac{dr}{\sigma \cdot 2 \cdot \pi \cdot t \cdot r} = \frac{I}{2 \cdot \pi \cdot \sigma \cdot t} \Big|_a^b \frac{dr}{r} = \frac{I}{2 \cdot \pi \cdot \sigma \cdot t} \ln\left(\frac{b}{a}\right)$$

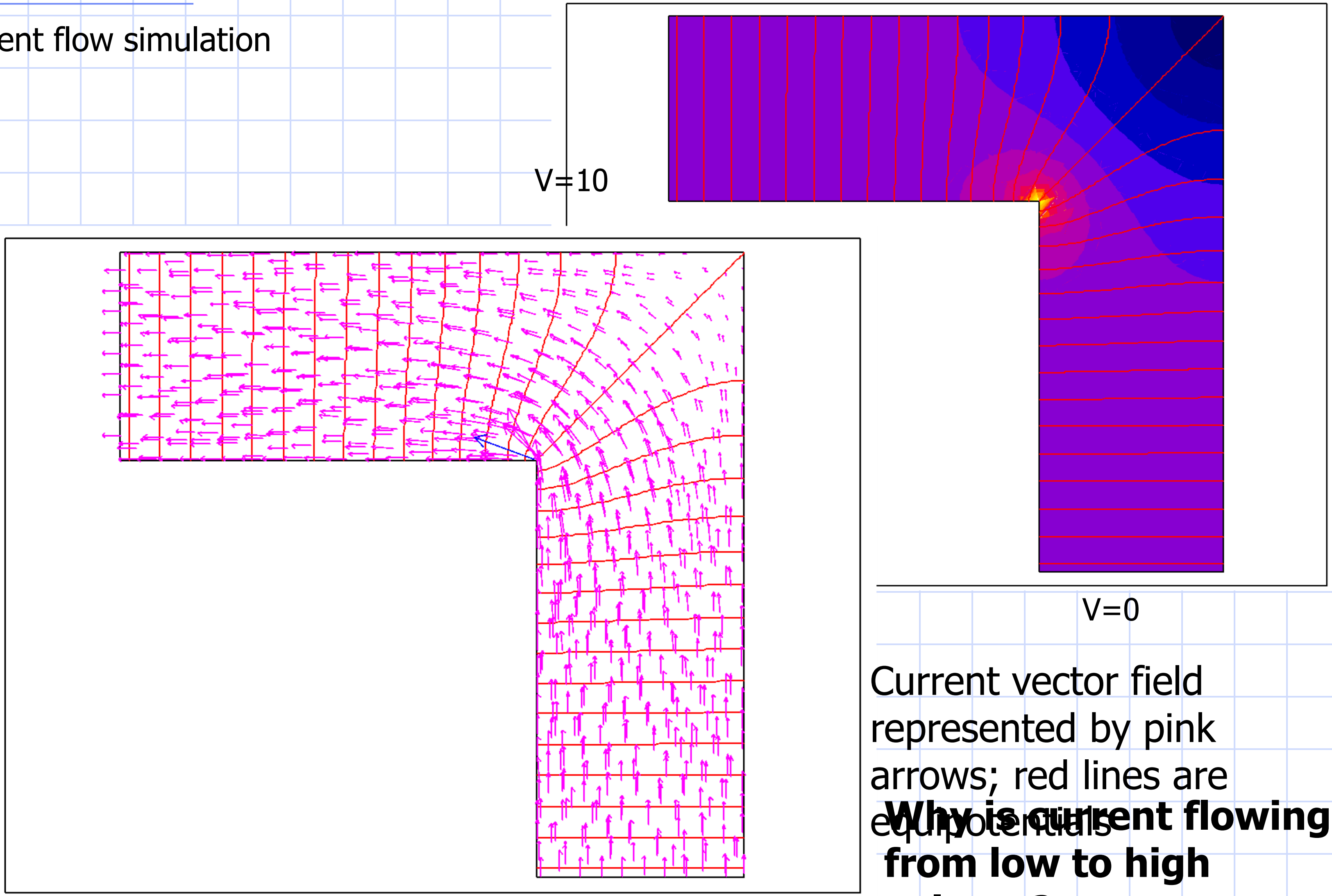
Numerical Methods for Current

- Since current density is proportional to (and parallel to) electric field, we can represent J in finite element / finite difference analysis similarly to how we would represent E
- To address resistance, note that we are addressing the flow of electrons (current) across equipotentials.

Numerical Methods for Current

Current flow simulation

V=10



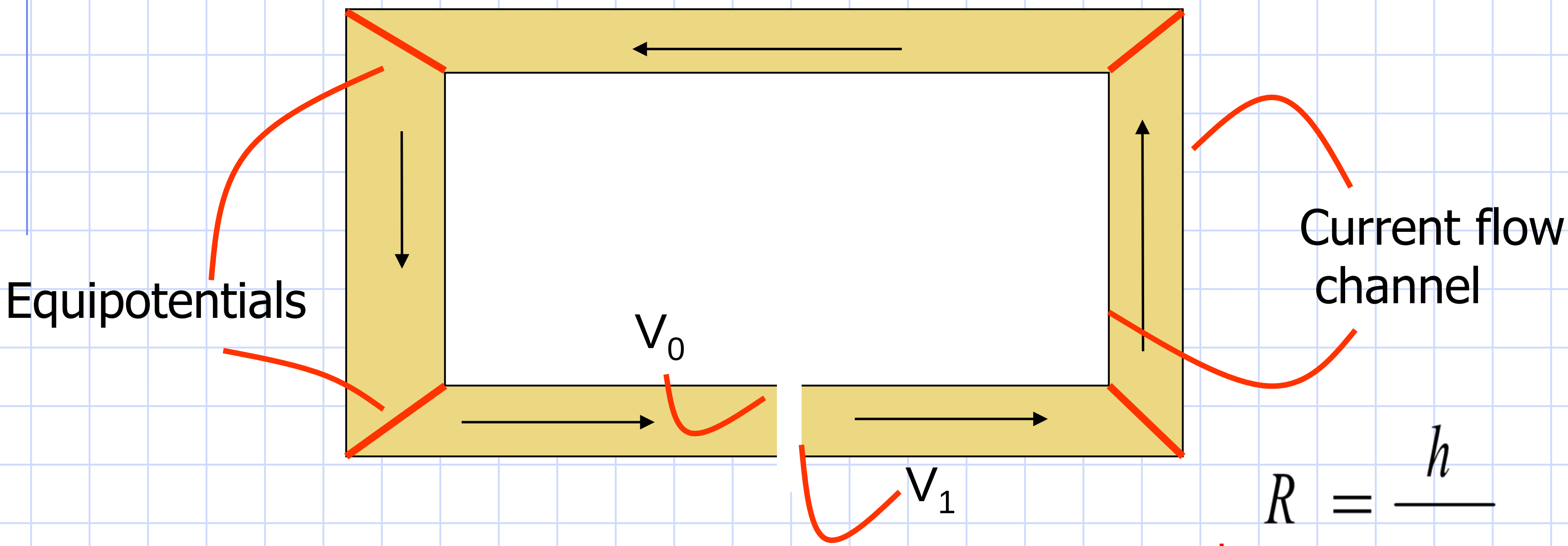
V=0

Current vector field represented by pink arrows; red lines are equipotentials

Why is current flowing from low to high voltage?

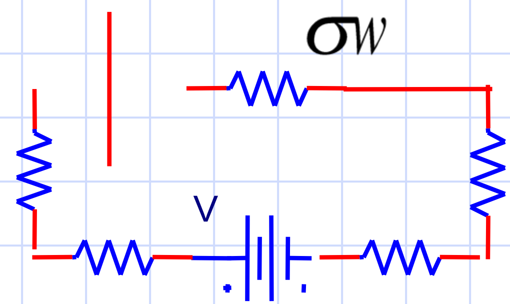
Numerical Methods for Current

- Calculation of this closed loop resistance



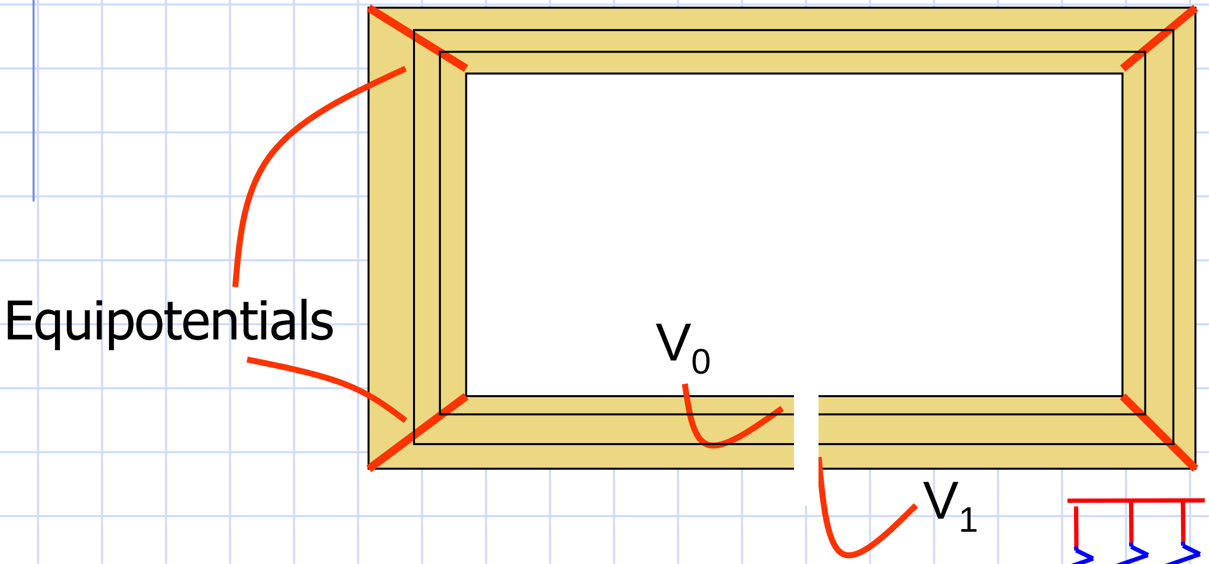
$$R = \frac{h}{\sigma w}$$

First approximation



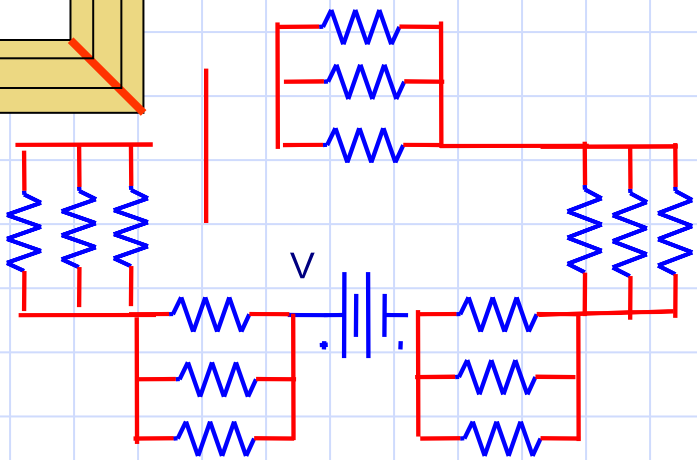
Numerical Methods for Current

- More accurate calculation
  - Subdivision according to current flow



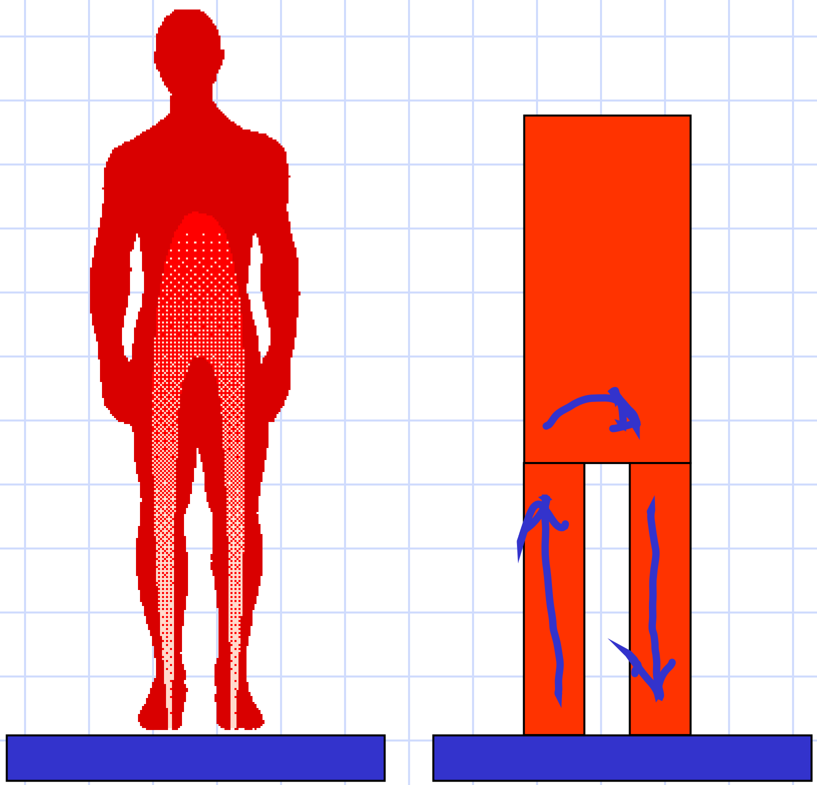
3 Current flow channels

More accurate approximation



Applications

Bioelectrical Impedance Analysis

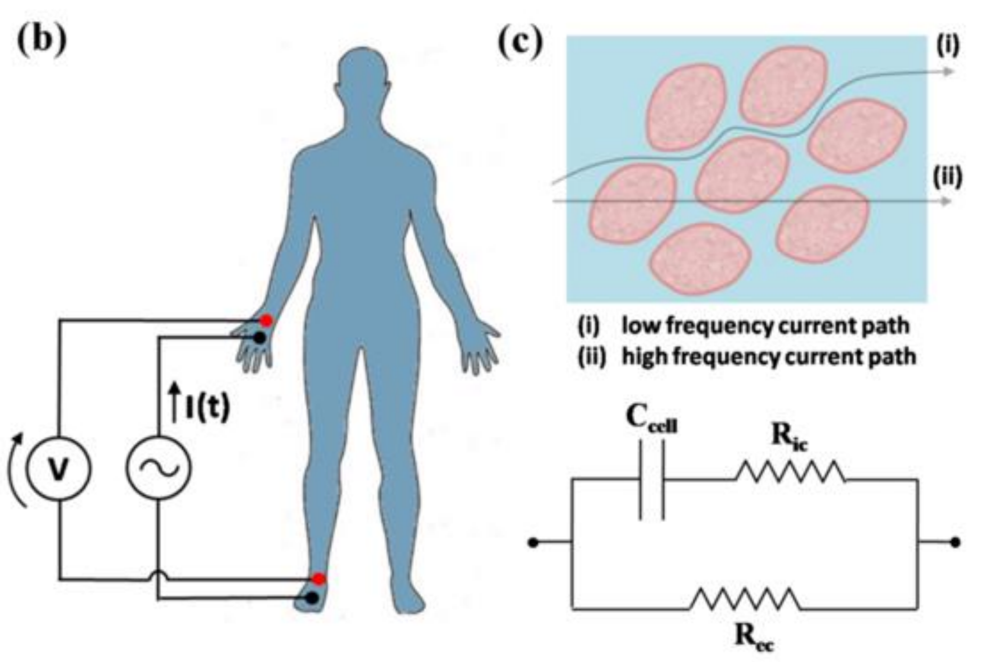


- Bioelectrical impedance analysis (BEA) is a method of electrically measuring body composition (muscle, body fat, hydration)
- Probes are used to measure the current-voltage relationship for a signal passing between to body surfaces (at left: from one foot to another)

http://www.tanita.com/en/personal

Applications

Bioelectrical Impedance Analysis



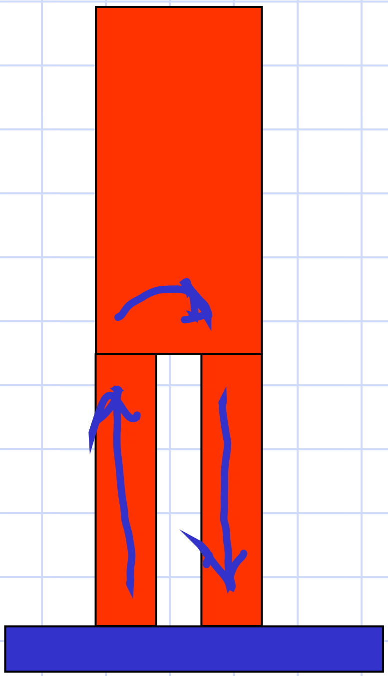
- Different cells have different impedances
- For instance, a fat cell has a much higher impedance than a muscle cell

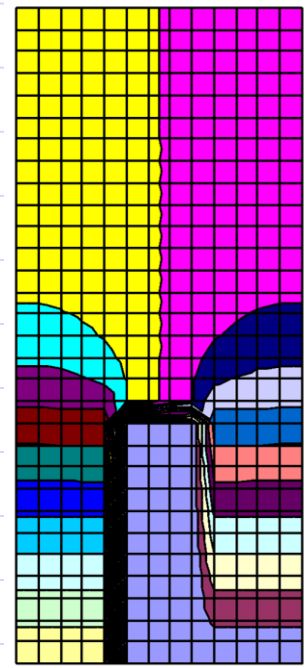
Grossi 2017

Applications

Bioelectrical Impedance Analysis

- Laplace/Poisson's Equation finite difference methods could be employed to create an equipotential map of the body
- Laplace's equation would be used since the body is presumed not to have free charge
- Dirichlet boundary conditions: electric potential is defined at either foot

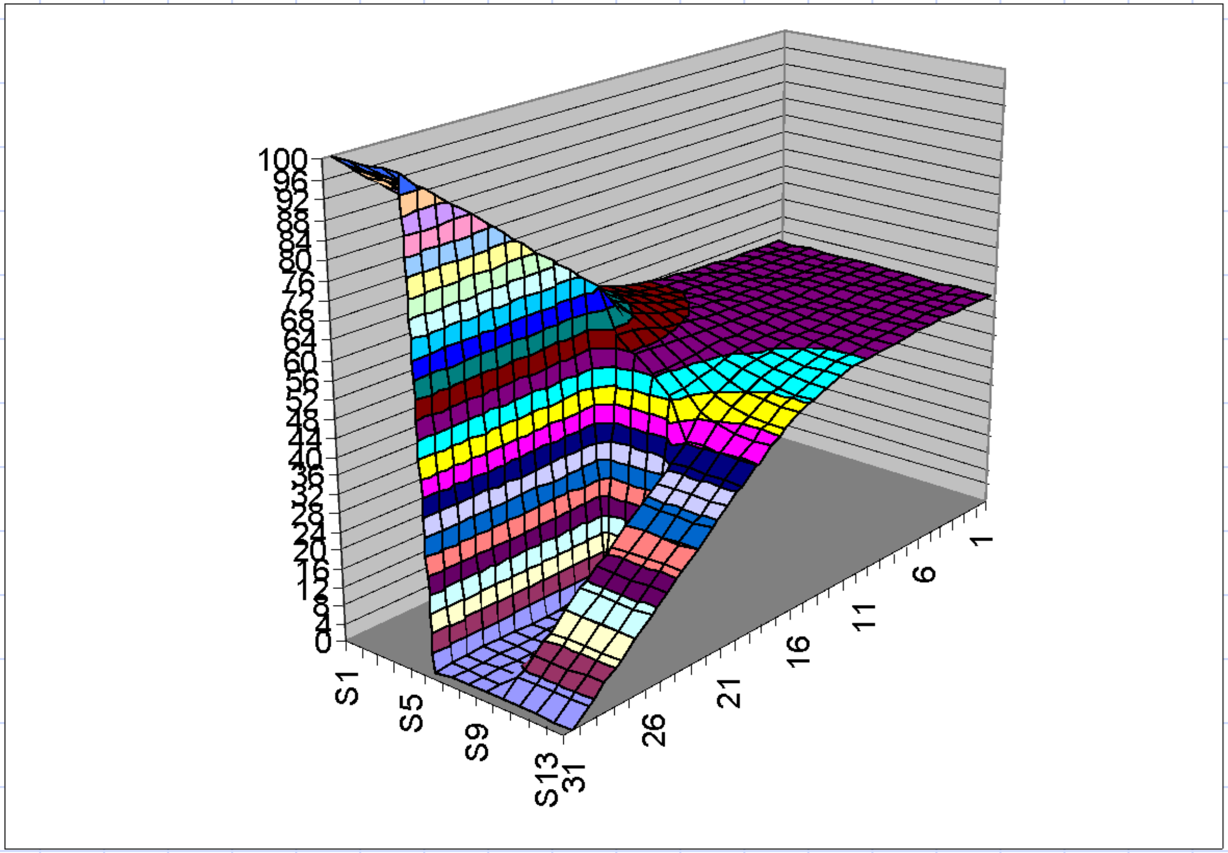




Applications

Bioelectrical Impedance Analysis

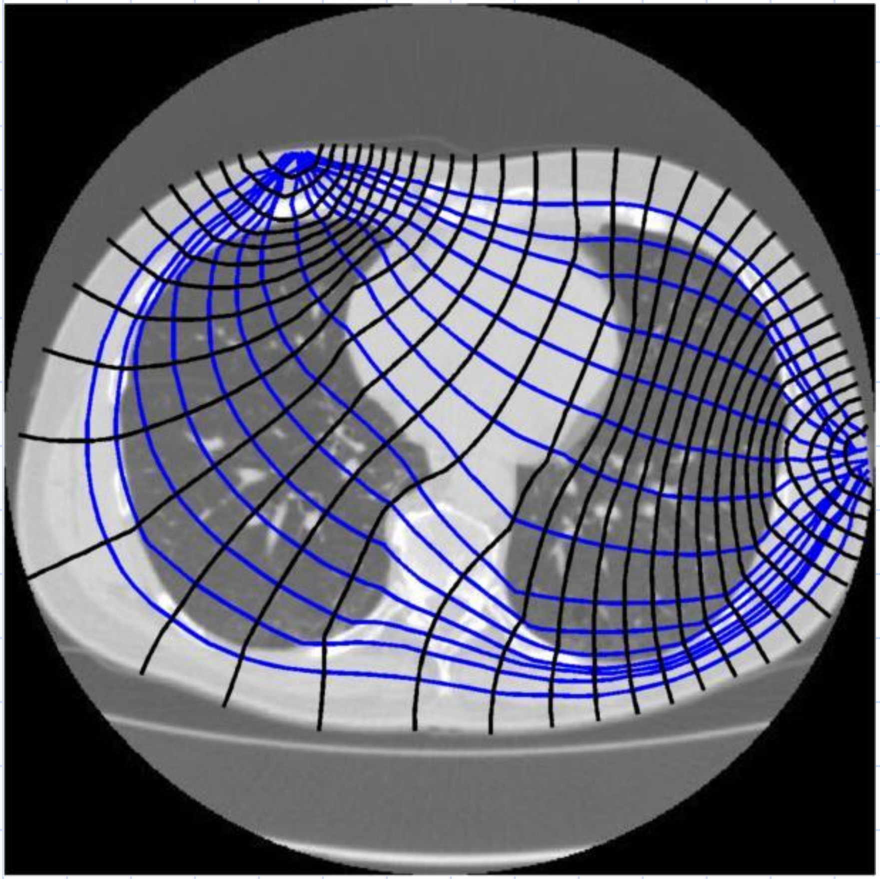
Legs dominate the resistance.



Applications

Electrical Impedance Tomography

- Small signals applied to the body's surface with electrodes will allow field and equipotenial lines to be mapped inside the body
- Using image construction algorithms, an 3D image of tissues can be created



Wikipedia

Theremins

- Who can tell me (without looking it up) what a theremin is?

Theremins

- Who can tell me (without looking it up) what a theremin is?

Theremins

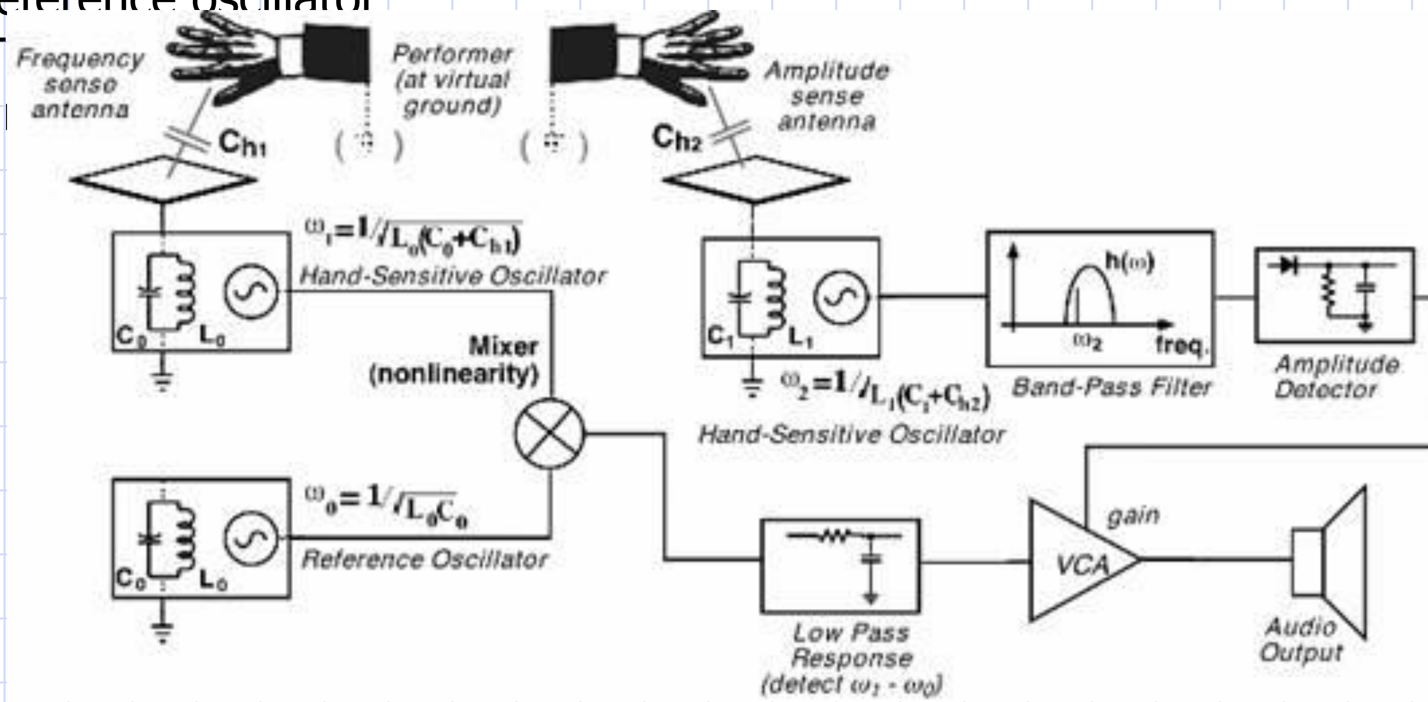
Theremins

- A theremin is an analog electronic instrument based on LRC oscillation
- Invented by Soviet inventor Leon Theremin in 1920 as an offshoot of research on proximity sensors
- First instrument that could be played without physical contact
- One of the first electronic instruments

Wikipedia

Theremins

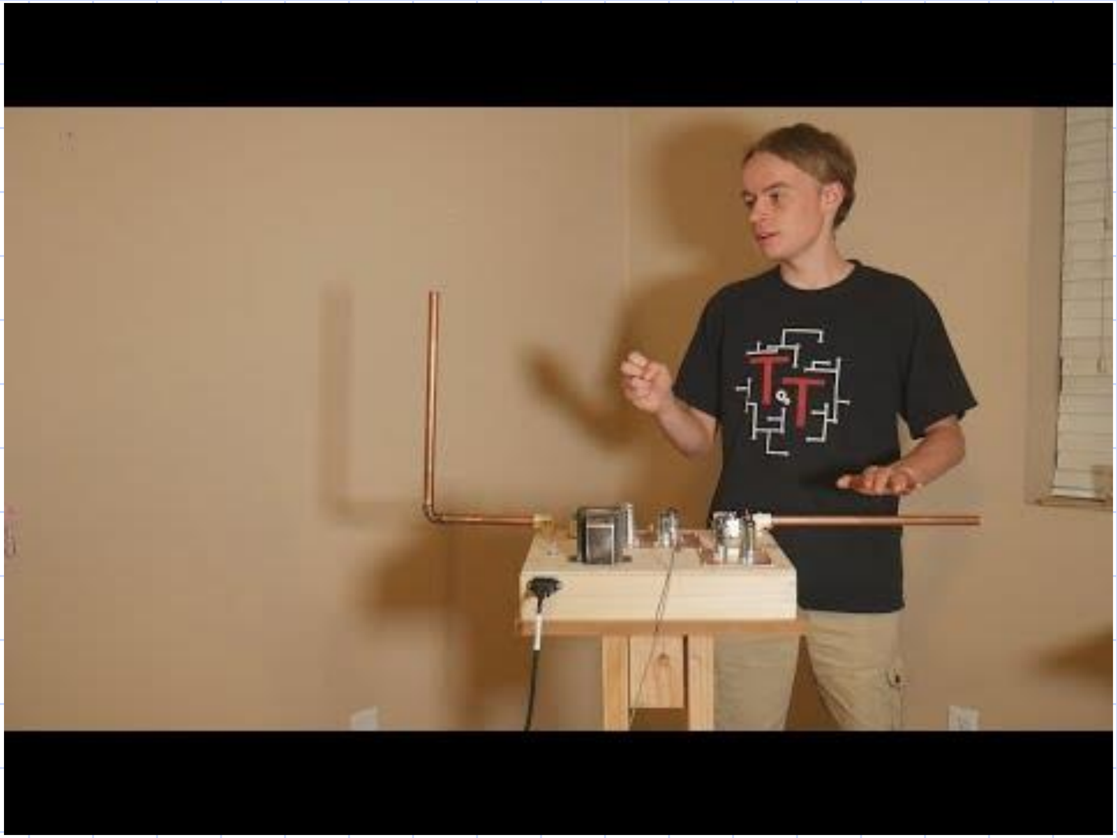
- Two high-frequency (100+ kHz LC oscillator circuits) have their output voltages added together
- One includes an antenna that interacts capacitively with the performer's hand; capacitance varies slightly as their hand moves
- Small variation in capacitance leads to a beat frequency with the reference oscillator
- T ncy a



http://asmir.info/

Theremins

- Built by many DIY hobbyists



Wrap-Up

- Quiz 2 will be next week and will cover Unit 2 skills. Today is the last lecture that will be covered under Quiz 2.
- Quiz 2 review materials will be released this weekend.
- Skill retests for this week:

1e, 1f
1h, 1i, 1j