Homework 8 Solutions

1. Electromagnetic Power Density [7 Points]

A wave traveling in a non-magnetic medium with $\epsilon_r = 16$ has an electric field:

$$\vec{E} = [2\cos(\pi 10^6 t + kx)\,\hat{y} - 3\cos(\pi 10^6 t + kx)\,\hat{z}]\ [\text{V/m}]$$

a) Determine the propagation direction of the wave.

propagation direction : $-x$ (+1)

b) Determine the magnetic field $\vec{H}$.

$$\vec{H} = \hat{k} \times \frac{\vec{E}}{\eta} \qquad \eta = \sqrt{\frac{\mu}{\varepsilon}} = \frac{1}{4}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{377\Omega}{4} = 94.25\Omega$$

$$\vec{H} = (-\hat{x}) \times \frac{[2\cos(\pi 10^6 t + kx)\hat{y} - 3\cos(\pi 10^6 t + kz)\hat{z}]\ \text{V/m}}{94.25\ \Omega}$$

$$= -0.021\cos(\pi 10^6 t + kx)\hat{z} - 0.032\cos(\pi 10^6 t + kx)\hat{y}\ \text{A/m}$$

(+1) correct magnitudes; (+1) correct propagation direction
(+1) correct H direction vectors

c) Calculate the average power density of the wave.

$$S_{av} = \frac{1}{2}\text{Re}\{\vec{E} \times \vec{H}^*\} = \frac{1}{2}\frac{|\tilde{E}|^2}{\eta} = \frac{1}{2}\left(\frac{2^2 + 3^2}{94.25\Omega}\right) = \boxed{0.069\ [\text{W/m}^2]}$$

(+1) formula (+1)

d) If we are dealing with a lossy medium, $\vec{E}$ and $\vec{H}$ acquire an extra exponential term $e^{-\alpha z}$. What additional term does the Poynting vector $\vec{S}$ then acquire in a lossy medium?

$e^{-2\alpha z}$, a factor of $e^{-\alpha z}$ from each $\vec{E}$ and $\vec{H}$ since $\vec{S} = \vec{E} \times \vec{H}$ (+1)

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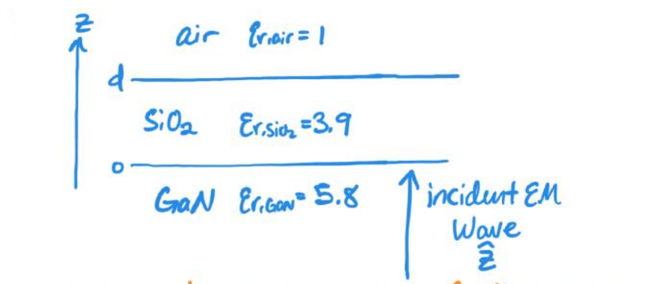
2. Electromagnetic Waves at Material Interfaces (Normal Incidence) [17 Points]

The light from an LED originates from holes and electrons recombining in a semiconductor material and emitting a photon, which then passes through another dielectric material before it is emitted into air. Consider the following system of material interfaces:

- for $z < 0$ the material is GaN with $\epsilon_{r,GaN} = 5.8$
- for $0 < z < d$, the material is SiO_2 with $\epsilon_{r,SiO2} = 3.9$
- for $d < z < \infty$, the material is air

An electromagnetic wave with $\tilde{E}(r) = 2e^{-jk_{GaN}z}\hat{x}$ [V/m] and a wavelength in GaN equal to 450nm is normally incident on the GaN/SiO_2 interface.

a) Draw a diagram of this system of materials, including axes, material properties and the direction of travel of the wave.



(+1) regions drawn in correct locations + axes drawn
(+1) materials + material properties labeled
(+1) direction of travel of incident wave shown

b) What is the frequency f of this electromagnetic wave? Also, calculate k in each material. Is the wavelength different in each material?

$$f = \frac{u}{\lambda} = \frac{1}{\lambda\sqrt{\mu\varepsilon}} = \frac{3\times10^8 \text{ m/s}}{\sqrt{5.8}\cdot 450\times10^{-9}\text{m}} = \boxed{276.8 \text{ THz}} \quad (+1)$$

- in GaN: $k_{GaN} = \frac{2\pi}{\lambda_{GaN}} = \frac{2\pi}{450\times10^{-9}\text{m}} = \boxed{1.40\times10^7 \frac{\text{rad}}{\text{m}}}$ (+1)

- in SiO_2: $k_{SiO_2} = \frac{2\pi f}{u_{SiO_2}} = 2\pi f\sqrt{\mu\varepsilon_{SiO_2}} = \frac{2\pi\sqrt{3.9}f}{c}$
$= \boxed{1.14\times10^7 \text{ rad/m}}$ (+1)

- in air: $k_{air} = \frac{2\pi f}{u_{air}} = \frac{2\pi f}{c} = \boxed{5.80\times10^6 \text{ rad/m}}$ (+1)

c) Calculate the reflection coefficient Γ and transmission coefficient τ at the GaN/SiO_2 interface and write the phasor expression for wave transmitted from the GaN into the SiO_2 $\tilde{\boldsymbol{E}}_{T1}$.

$$\cdot\ \eta_{GaN} = \sqrt{\frac{\mu_0}{\varepsilon_{GaN}}} = \frac{\eta_0}{\sqrt{\varepsilon_{rGaN}}} = \frac{377\Omega}{\sqrt{5.8}} = 156.5\Omega$$

$$\cdot\ \eta_{SiO_2} = \frac{377\Omega}{\sqrt{3.9}} = 190.9\Omega$$

$$\cdot\ \eta_{air} = 377\Omega$$

$$\cdot\ \Gamma_{GaN/SiO_2} = \frac{\eta_{SiO_2} - \eta_{GaN}}{\eta_{SiO_2} + \eta_{GaN}} = \frac{190.9\Omega - 156.5\Omega}{190.9\Omega + 156.5\Omega} = \underline{0.1|}\ (+1)$$

$$\cdot\ \tau_{GaN/SiO_2} = 1 + \Gamma_{GaN/SiO_2} = \underline{1.1|}\ (+1)$$

$$\cdot\ \tilde{E}_{T1} = \tau_{GaN/SiO_2} \cdot |\tilde{E}| e^{-jk_{SiO_2} z} \hat{x}$$

$$= 1.1 \cdot 2 \cdot e^{-j \cdot 1.14\times10^{7} \cdot z} \hat{x} = \underline{2.2 e^{-j \cdot 1.14\times10^{7} z} \hat{x}\ V/m|}\ (+1)$$

d) Calculate the reflection coefficient Γ and transmission coefficient τ at the SiO_2/air interface and write the phasor expression for the portion of the incident wave $\tilde{\boldsymbol{E}}_{T1}$ that is transmitted from the SiO_2 into air $\tilde{\boldsymbol{E}}_{T2}$.

$$\cdot\ \Gamma_{SiO_2/air} = \frac{\eta_{air} - \eta_{SiO_2}}{\eta_{air} + \eta_{SiO_2}} = \frac{377\Omega - 190.9\Omega}{377\Omega + 190.9\Omega} = \underline{0.328|}\ (+1)$$

$$\cdot\ \tau_{SiO_2/air} = 1 + \Gamma_{SiO_2/air} = \underline{1.328|}\ (+1)$$

$$\cdot\ \tilde{E}_{T2} = \tau_{SiO_2/air} |\tilde{E}_{T1}| e^{-jk_{air} z} \hat{x}$$

$$= \underline{2.92 e^{-j \cdot 5.8\times10^{6} \cdot z} \hat{x}\ V/m|}\ (+1)$$

e) Is more power transmitted into air in this case than would be if there were only a GaN/air interface (i.e. no layer of SiO_2 in between GaN and air)? Ignore transmitted wave components from multiple reflections between the interfaces.

· power transmitted with GaN/SiO_2/air : (+1)

$$|S_{av}| = \frac{1}{2}\frac{|E_{T2}|^2}{\eta_0} = \frac{(3.04\ V/m)^2}{2 \cdot 377\Omega} = \underline{0.123\ [W/m^2]}$$

· power transmitted in the case of only a GaN/air interface

$$\cdot\ \tau_{GaN/air} = \Gamma_{GaN/air} + 1 = \frac{\eta_{air} - \eta_{GaN}}{\eta_{air} + \eta_{GaN}} + 1 = \frac{377\Omega - 156.5\Omega}{377\Omega + 156.5\Omega} + 1$$

$$= \underline{1.41}$$

$$\cdot\ \tilde{E}_T = \tau_{GaN/air}|\tilde{E}|e^{-jk_{air}z}\hat{x}$$

$$= 1.41 \cdot 2\, e^{-j \cdot 5.8\times10^6 \cdot z}\hat{x}$$ (+1) amplitude of transmitted wave

$$\cdot\ |S_{av}| = \frac{1}{2}\frac{|E_T|^2}{\eta_0} = \frac{(1.41\ V/m)^2}{2 \cdot 377\Omega} = \underline{0.0026\ [W/m^2]}$$

(+1)

· More power is transmitted by inserting a layer of SiO_2 between the GaN and air.

(+1) conclusion consistent with mathematical findings

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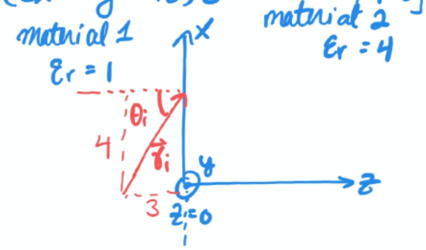
3. Electromagnetic Waves at Material Interfaces (Oblique Incidence – Snell's Law) [12 Points]

A plane wave in air has an electric field defined by $\widetilde{\boldsymbol{E}}^i = (3\hat{x} + 5\hat{y} - 4\hat{z})e^{-j(4x+3z)}$ [V/m] and is incident upon the planar surface of a dielectric material with $\epsilon_r = 4$ that occupies the half-space where $z \geq 0$. Determine:

a. The angle of incidence (θ_i), angle of refraction (θ_t), and the frequency of the wave.

$\tilde{E}^i = (3\hat{x} + 5\hat{y} - 4\hat{z})e^{-j(4x+3z)}$ [V/m]



- $\theta_i = \tan^{-1}(4/3) = \boxed{53.1°}$ (+1)

- θ_t : $n_i \sin\theta_i = n_t \sin\theta_t \rightarrow \theta_t = \sin^{-1}\left\{\frac{n_i}{n_t}\sin\theta_i\right\}$
$= \sin^{-1}\left\{\frac{1}{2}\sin(53.1°)\right\}$
$= \boxed{23.6°}$ (+1)

- $|\vec{\gamma}| = \frac{2\pi f n}{c} \rightarrow f = \frac{|\vec{\gamma}|c}{2\pi n} = \frac{\sqrt{4^2+3^2}}{2\pi \cdot 1} 3\times10^8\,\text{m/s}$
$= \boxed{238.7\text{ MHz}}$ (+1)

(+1) calculation of $|\vec{\gamma}|$

b. The reflection and transmission coefficients Γ and τ.

wave polarization:

$\tilde{E}^i = \tilde{E}^i_{\perp} + \tilde{E}^i_{\parallel} = \underbrace{5e^{-j(4x+3z)}\hat{y}}_{E_\perp} + \underbrace{(3\hat{x} - 4\hat{z})e^{-j(4x+3z)}}_{E_\parallel}$

- $\Gamma_\perp = \dfrac{\overset{\eta_0/2}{\eta_2}\cos\theta_i - \overset{\eta_0}{\eta_1}\cos\theta_t}{\underset{\eta_0/2}{\eta_2}\cos\theta_i + \underset{\eta_0}{\eta_1}\cos\theta_t} = \dfrac{\frac{1}{2}\cos(53.1°) - \cos(23.6°)}{\frac{1}{2}\cos(53.1°) + \cos(23.6°)}$
$= \boxed{-0.506}$ (+1)

- $\tau_\perp = 1 + \Gamma_\perp = \boxed{0.494}$ (+1)

$$\cdot\ \Gamma_{\parallel} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} = \frac{\tfrac{1}{2}\cos(23.6^\circ) - \cos(53.1^\circ)}{\tfrac{1}{2}\cos(23.6^\circ) + \cos(53.1^\circ)} = \underline{-.134|}\ (+1)$$

$$\cdot\ \tau_{\parallel} = (1+\Gamma_{\parallel})\frac{\cos\theta_i}{\cos\theta_t} = (1-0.134)\frac{\cos(53.1^\circ)}{\cos(26.3^\circ)} = \underline{0.58|}\ (+1)$$

c. The field $\widetilde{\boldsymbol{E}}^r$ of the reflected wave, and the field $\widetilde{\boldsymbol{E}}^t$ of the wave transmitted into the dielectric medium.

$$\cdot\ \tilde{E}_r = \tilde{E}_{r\parallel} + \tilde{E}_{r\perp} = -0.134\cdot(3\hat{x}+4\hat{z})\,e^{j(4x-3z)} - 0.506\cdot(5\hat{y})\,e^{j(4x-3z)}$$

$$= (-0.402\hat{x} - 2.53\hat{y} - 0.536\hat{z})\,e^{j(4x-3z)}\ (+1)$$

$$|\vec{\gamma}_2| = n_2|\vec{\gamma}_1| = 2|\vec{\gamma}_1| = 2\sqrt{4^2+3^2} = 10$$

$$\gamma_{2x} = 10\sin(23.6^\circ) = 4 \quad (+1)\text{ each for } \gamma_{2x}, \gamma_{2z}$$

$$\gamma_{2z} = 10\cos(23.6^\circ) = 9.16$$

$$\cdot\ |E_{\parallel i}| = \sqrt{3^2+(-4)^2} = 5\ ;\ E_{\parallel tx} = 5\cos(23.6^\circ) = 4.58\ ;\ E_{\parallel tz} = 5\sin(23.6^\circ) = 2$$

$$\tilde{E}_t = \tilde{E}_{t\parallel} + \tilde{E}_{t\perp} = (2.66\hat{x} + 2.47\hat{y} - 1.16\hat{z})\,e^{-j(4x+9.16z)}$$

d. The average power density carried by the wave into the dielectric medium.

$$S_{av} = \frac{|E_0^t|^2}{2\eta_2} = \frac{2.66^2 + 2.47^2 + 1.16^2}{2\,\frac{377\Omega}{2}} = \underline{0.038\ \text{W/m}^2|}$$

4. Electromagnetic Waves at Material Interfaces (Oblique Incidence – Brewster Angle) [6 Points]

A parallel-polarized plane wave is incident from air onto a (nonmagnetic) dielectric medium with $\epsilon_r = 9$.

a. What is the Brewster angle for this interface and polarization?

parallel-polarized wave, so

$$\theta_{B\parallel} = \tan^{-1}\left(\sqrt{\frac{\epsilon_2}{\epsilon_1}}\right) = \tan^{-1}\left(\sqrt{\frac{9}{1}}\right) = \underline{71.57^\circ} \quad (+1)$$

(+1): correct equation

b. If the plane wave is incident on the interface at the Brewster angle, what is the refraction angle?

$$n_i \sin\theta_i = n_t \sin\theta_t \rightarrow \theta_t = \sin^{-1}\left\{\frac{n_i}{n_t}\sin\theta_i\right\}$$

(+1): correct equation

$$= \sin^{-1}\left\{\frac{1}{3}\sin(71.57^\circ)\right\}$$

$$= \underline{18.42^\circ} \quad (+1)$$

c. If the plane wave were perpendicular-polarized instead, what would the Brewster angle be?

Since $\sin\theta_{B\perp} = \sqrt{\dfrac{1-(\mu_1\epsilon_2/\mu_2\epsilon_1)}{1-(\mu_1/\mu_2)^2}}$ and we have $\mu_1 = \mu_2$ for this system, $\theta_{B\perp}$ is undefined. (+1)

d. If the plane wave were unpolarized (as light typically is) and incident on the interface at the Brewster angle, what would the polarization of the transmitted wave be?

Since the parallel-polarized component would be completely transmitted and the perpendicular-polarized component would be partially reflected and transmitted, the transmitted wave would have a mixed polarization. (+1)

5. Terms, Notation, Symbols, etc. [8 Points]

On the left-hand side of the table below, write the name of the Greek letter listed in the symbol column and the units that correspond to the symbol. In the right-hand side of the table, write the correct symbol next to the name of the quantity it represents.

Symbol	Name of Greek Letter	Units
α	*alpha*	*Np/m*
β or k	*beta*	*rad/m*
γ	*gamma*	*1/m*
Γ	*gamma*	*N/A*
ϵ_0	*epsilon*	*F/m*
$\vec{D}$	N/A	C/m^2
$\vec{E}$	N/A	*V/m*
$\vec{H}$	N/A	*A/m*
$\vec{J}_s$	N/A	A/m^2
$\vec{J}_d$	N/A	A/m^2
ρ_v	*rho*	C/m^3
λ	*lambda*	*m*
Λ	*lambda*	*Wb*
n	N/A	*N/A*
σ	*sigma*	*S/m*
C	N/A	*F*
L	N/A	*H*
$\mathcal{R}$	N/A	H^{-1}
$\vec{S}_{av}$	N/A	W/m^2
τ	*tau*	*N/A*
μ_0	*mu*	*H/m*
V	N/A	*V*
η	*eta*	Ω
$\vec{B}$	N/A	Wb/m^2
$\vec{A}$	N/A	*Wb/m*
Z_0	N/A	Ω

Symbol for Quantity	Name of Quantity
$\boldsymbol{\alpha}$	Attenuation Constant
$\boldsymbol{C}$	Capacitance
$\boldsymbol{Z_0}$	Characteristic Impedance
$\boldsymbol{\sigma}$	Conductivity
$\vec{J}_d$	Displacement Current Density
$\vec{E}$	Electric Field Intensity
$\boldsymbol{V}$	Electric Scalar Potential
$\vec{D}$	Electric Flux Density (Displacement Field)
$\boldsymbol{n}$	Index of Refraction
$\boldsymbol{L}$	Inductance
$\boldsymbol{\eta}$	Intrinsic Impedance
$\vec{H}$	Magnetic Field Intensity
$\vec{B}$	Magnetic Flux Density
$\boldsymbol{\Lambda}$	Magnetic Flux Linkage
$\vec{A}$	Magnetic Vector Potential
$\boldsymbol{\epsilon_0}$	Permeability of Free Space
$\boldsymbol{\mu_0}$	Permittivity of Free Space
$\boldsymbol{k}$ ***or*** $\boldsymbol{\beta}$	Phase Constant or Wave Number
$\vec{S}_{av}$	Poynting Vector (avg. power density)
$\boldsymbol{\gamma}$	Propagation Constant
$\boldsymbol{\Gamma}$	Reflection Coefficient
$\boldsymbol{\mathcal{R}}$	Reluctance
$\vec{J}_s$	Surface Current Density
$\boldsymbol{\tau}$	Transmission Coefficient
$\boldsymbol{\rho_v}$	Volume Charge Density
$\boldsymbol{\lambda}$	Wavelength

64 total answers : +8: 64 correct
+7: 56-63 correct
+6: 48-62 correct
+5: 40-47 correct
+4 : 32 - 39 correct
+3 : 24 - 31 correct
+2 : 16 - 23 correct
+1 : 8 - 15 correct
+0 : 0 - 7 correct