Rensselaer Polytechnic Institute
Department of Electrical, Computer, and Systems Engineering
ECSE 2500: Engineering Probability, Spring 2023

Homework #7: due **Friday**, April 14, at 11:59PM.
Show all work for full credit!
Submit your work as a single PDF on Gradescope, labeling each problem number with a page.

1. (25 points.) A farmer visits their henhouse with an empty 6-egg carton to collect eggs for breakfast. They visit a group of hens whose size is a random number uniformly chosen between 5 and 7. The probability that a given hen has produced an egg overnight is $\frac{1}{2}$. Let X be the number of hens they visit, and Y be the number of eggs in the carton after their visit. Note that once the carton is full, the farmer does not continue to collect eggs (i.e., the maximum value of Y is 6), so there may be extra eggs left in the henhouse.

   (a) (10 points.) Compute the joint PMF $p_{XY}(x,y)$. Represent your answer as a table. (Hint: it may be easier for the rest of the problem to bring all the probabilities to a common denominator of 384.)

   (b) (10 points.) Compute the joint CDF $F_{XY}(x,y)$. Represent your answer as a 2D piecewise constant function (i.e., divide the XY-plane into grid squares and indicate the height of the CDF at each point).

   (c) (5 points.) Compute the marginal PMF $p_Y(y)$. Represent your answer as a table.

2. (30 points) Consider the function

$$g(x,y) = \begin{cases} cx(1-x)y & x \in [0,1], y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

   (a) (5 points) Determine the value of c so that $g(x,y)$ is a valid joint PDF $f_{XY}(x,y)$ for two random variables X and Y.

   (b) (5 points) Compute the joint CDF $F_{XY}(x,y)$ corresponding to the PDF in part (a). You only need to compute the CDF for values (x,y) such that $x \in [0,1], y \in [0,1]$.

   (c) (5 points) Compute the marginal PDF $f_X(x)$. Remember to specify the PDF at all values of x.

   (d) (5 points) Compute the marginal PDF $f_Y(y)$. Remember to specify the PDF at all values of y.

   (e) (5 points) Are X and Y independent? Justify your answer.

   (f) (5 points) Compute $P(Y \le \sqrt{X})$.

More fun on the next page ⟶

3. (20 points.) The Orville investigates a Xelayan distress signal located at the uncertain location (X, Y). We model the joint random variable as a two-dimensional Gaussian whose PDF is given by

$$f_{XY}(x,y) = c\exp\left(-\frac{3}{64}(12x^2 - 80x + 3y^2 + 24y - 4xy)\right)$$

where c is an unknown constant. We are also told that the correlation coefficient $\rho = \frac{1}{3}$.

   (a) (10 points.) Determine $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$. Hint: compare the coefficients on x^2 and y^2 for the above function and the PDF for the joint Gaussian.

   (b) (10 points.) Determine $E(X)$ and $E(Y)$. Hint: compare the coefficients on x and y for the above function and the PDF for the joint Gaussian, and solve a linear system.

4. (25 points.) Consider the following PDF:

$$f_{XY}(x,y) = \begin{cases} \frac{6}{19}(x^2 + y^3) & 0 \le x \le 2, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) (5 points) Compute the correlation of X and Y.

   (b) (10 points) Compute the covariance of X and Y.

   (c) (5 points) Compute $E(95(X(1+Y) + 2Y(1-X)))$ from your work in part (b) (i.e., don't do any new integration).

   (d) (5 points) Are X and Y uncorrelated? Justify your answer.