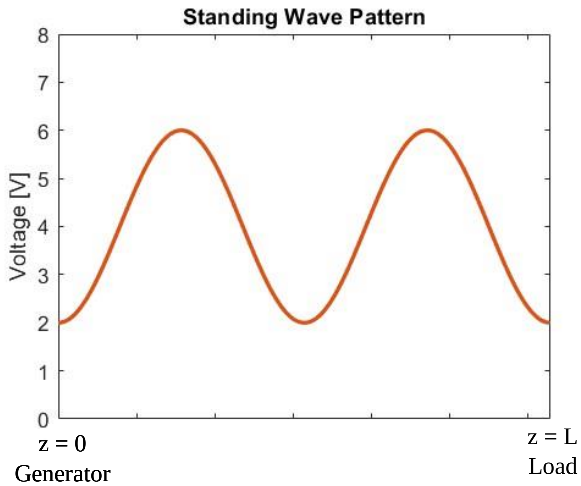
Homework 2

Released: January 25th

Due: 11:59pm February 2nd

1. Standing Wave Patterns on Lossless Transmission Lines

All parts of Problem 1 refer to the standing wave pattern below. The characteristic impedance of the transmission line is $Z_0 = 50\Omega$.



a) Simply by inspecting the features of the standing wave pattern, determine if Z_L is purely real (resistive), purely imaginary (reactive), or a combination of both. How can you tell?

Since the voltage at the load is either a maximum or minimum of the standing wave pattern, Z_L is purely real (resistive). The presence of any reactive components would result in a voltage at the load that is somewhere between the maximum and minimum voltages of the standing wave pattern.

[2 pts: 1 pt for correct answer for Z_L; 1 pt for valid justification]

b) Again, simply by inspecting features of the standing wave pattern, determine the sign of Γ_L, the reflection coefficient at the load.

Since the voltage standing wave pattern is a minimum at the load, it tells us that the reflection coefficient at the load is negative (and that $Z_L < Z_0$).

[2 pts: 1 pt for correct answer for sign of **Γ**; 1 pt for valid justification]

c) Calculate the standing wave ratio (SWR) for this standing wave pattern.

$$SWR = \frac{V_{max}}{V_{min}} = \frac{6V}{2V} = 3$$

[1 pt: 0.5 pts for valid approach/equation; 0.5 pts for correct calculation]

d) Keeping in mind your result from b), calculate Γ_L, the reflection coefficient at the load.

$$SWR = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} \rightarrow |\Gamma_L| = \frac{1}{2}$$

Since we determined in part a) that the reflection coefficient is negative, $\Gamma_L = -\frac{1}{2}$

[1 pt: 0.5 pts for valid approach/equation; 0.5 pts for correct calculation]

e) Calculate Z_L, the load impedance.

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = -\frac{1}{2} = \frac{Z_L - 50\Omega}{Z_L + 50\Omega} \rightarrow Z_L = 16.66\Omega$$

[1 pt: 0.5 pts for valid approach/equation; 0.5 pts for correct calculation]

f) What is the value of V_0^+, the amplitude of the incident voltage wave?

At the load, the minimum voltage via the standing wave pattern is 2V. In the case, we have

$$V(load) = V_0^+ + V_0^- = V_0^+(1 + \Gamma_L) \rightarrow 2V = V_0^+\left(1 - \frac{1}{2}\right) \rightarrow V_0^+ = 4V$$

[1 pt: 0.5 pts for valid approach/equation; 0.5 pts for correct calculation]

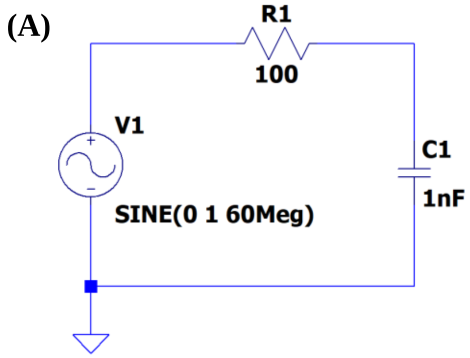
2. Input Impedance of Lossless Transmission Lines

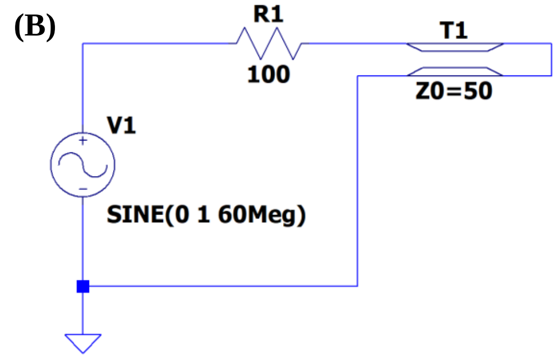
a) What is the wavelength of a 60 MHz voltage signal on a transmission line with a characteristic impedance $Z_0 = 50\Omega$ and velocity factor $v_f = 0.66$?

$$u_p = 0.66c, \text{ so } \lambda = \frac{u_p}{f} = \frac{0.66 \cdot 3\times10^8 \frac{m}{s}}{60\times10^6\ Hz} = 3.3m$$

[1 pt: 0.5 pts for correct equation/approach; 1 pt for correct calculation]

b) You are tasked with replacing the capacitor in circuit A below with a short-circuited transmission line, resulting in circuit B.



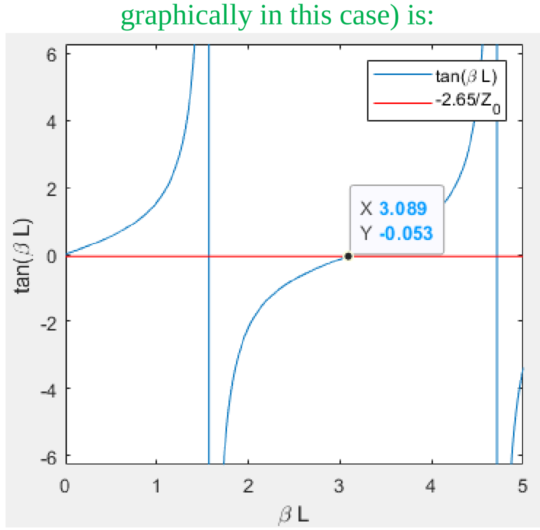


c) Using the same transmission line properties as in (a) and a source voltage frequency of 60MHz, what is the minimum length of the transmission line that would present the same input impedance to the generator circuit as the capacitor in circuit A?

The input impedance of a short-circuited transmission line of length l is $Z_{in}(l) = jZ_0 \tan(\beta l) = jZ_0 \tan\left(\frac{2\pi}{\lambda} l\right)$ and the impedance of the capacitive load is

$$Z_C = \frac{1}{j\omega C} = \frac{1}{j \cdot 2\pi \cdot 60\times10^6 Hz \cdot 1\times10^{-9} F} = -j2.65$$

The equation to solve for l is then: $jZ_0 \tan\left(\frac{2\pi}{\lambda} l\right) = j50\Omega \tan\left(\frac{2\pi}{3.3m} l\right) = -j2.65$, which gives many answers due to the periodicity of the tangent function. The smallest solution (found graphically in this case) is:



Then the condition $l = \frac{3.3m}{2\pi} 3.089 = 1.62m$ gives the smallest of these solutions.

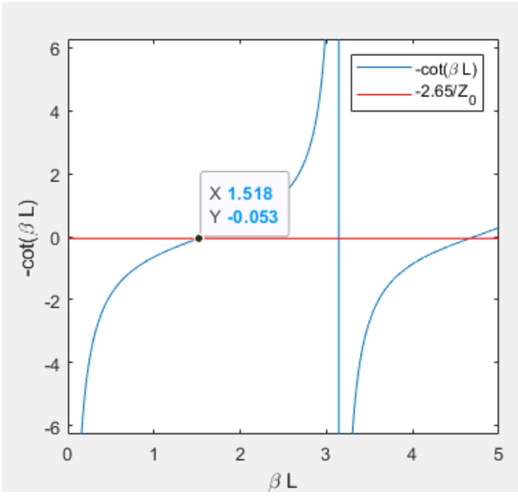
[2 pts: 1 pt for valid approach/equation; 1 pt for correct calculation]

d) Suppose instead that you were given an open-circuited transmission line with which to replace the capacitor in circuit A of part (b). What is the minimum length of the transmission line that would present the same input impedance as the capacitor in circuit A?

The input impedance of an open-circuited transmission line of length l is

$$Z_{in}(l) = -jZ_0 \cot(\beta l) = -jZ_0 \cot\left(\frac{2\pi}{\lambda} l\right).$$

The equation to solve for l is then: $-jZ_0 \cot\left(\frac{2\pi}{\lambda} l\right) = -j50\Omega \cot\left(\frac{2\pi}{3.3m} l\right) = -j2.65$, which also gives many answers due to the periodicity of the cotangent function, the smallest of which is found here:

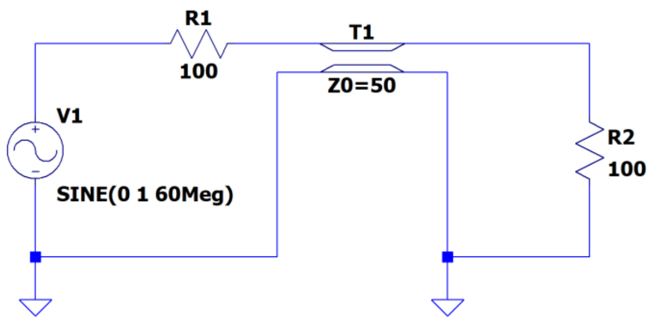


The condition

$$l = \frac{3.3m}{2\pi} 1.5178 = 0.797m \text{ gives the smallest of these solutions.}$$

[2 pts: 1 pt for valid approach/equation; 1 pt for correct calculation]

e) Given the circuit below, what is the minimum length of the transmission line T1 (same properties as in part a) that can be used to prevent reflection from occurring between the generator resistance R1 and the input of the transmission line? Will reflections still occur on the transmission line at the load?



If we use a transmission line with length $\lambda/2$=1.65m, the input impedance looks to the generator like $Z_{in} = Z_L$, which will prevent any reflections at the transmission line input since R1 and R2 have the same impedance. However, since there is a mismatch between the characteristic impedance and the

load R2, there will still be reflections on the transmission line originating from the load end.
[2 pts: 1 pt for correct line length; 1 pt for correct answer regarding reflections on the t-line]

f) If R2 in the circuit above is instead 150Ω and transmission line T1 is a quarter wave transformer, what characteristic impedance Z_0 of T1 will ensure that no reflection occurs between the generator impedance R1 and the transmission line? Will reflections still occur on the transmission line at the load?

If we use a transmission line with length $\boldsymbol{\lambda}/4$ (quarter wave transformer), the input impedance looks to the generator like

$Z_{in} = \frac{Z_0^2}{Z_L}$, so if we want to match Z_{in} to R1=100Ω, we need the characteristic impedance

$$Z_0 = \sqrt{Z_{in}Z_L} = \sqrt{100\Omega * 150\Omega} = 122.47\Omega$$

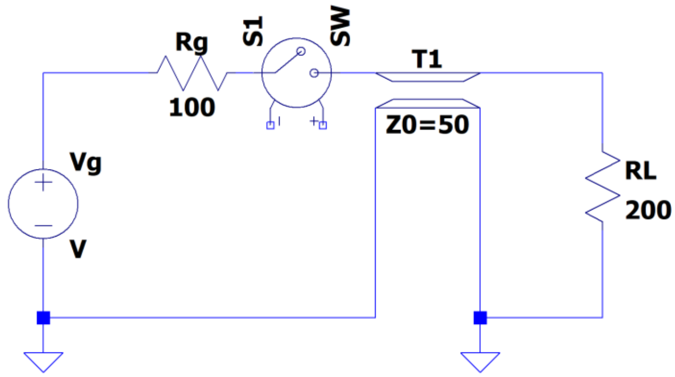
[2 pts: 1 pt for correct approach; 1 pt for correct calculation]

g) If the frequency of the source voltage in the circuit you designed in part f is changed, will the quarter wave transformer still ensure that no reflections occur between R1 and the input to the transmission line? Why or why not?

If the frequency of the source changes, the quarter wave transformer will no longer function as intended because the wavelength will also change ($\boldsymbol{\lambda}=u_p/f$) and the transmission line will no longer be exactly ¼ the wavelength.
[1 pt: 0.5 pts for correct answer; 0.5 pts for correct justification]

3. Transient Signals on Lossless Transmission Lines

All parts of Problem 3 refer to the circuit below. At time t=0, the switch S1 closes and a 3V DC source supplies voltage to the circuit. The transmission line has a characteristic impedance of Z_0 = 50Ω.



a) What is the amplitude of the forward-traveling voltage wave V_0^+ that enters the transmission line at t = 0?

Since there is no voltage or current on the line at t = 0, generator circuit sees $Z_{in} = Z_0$ as the input impedance. As a result, $V_0^+ = V_g \frac{50\Omega}{100\Omega+50\Omega} = V_g \frac{1}{3} = 1V$

[2 pts: 1 pt for correct approach/equation; 1 pt for correct calculation]

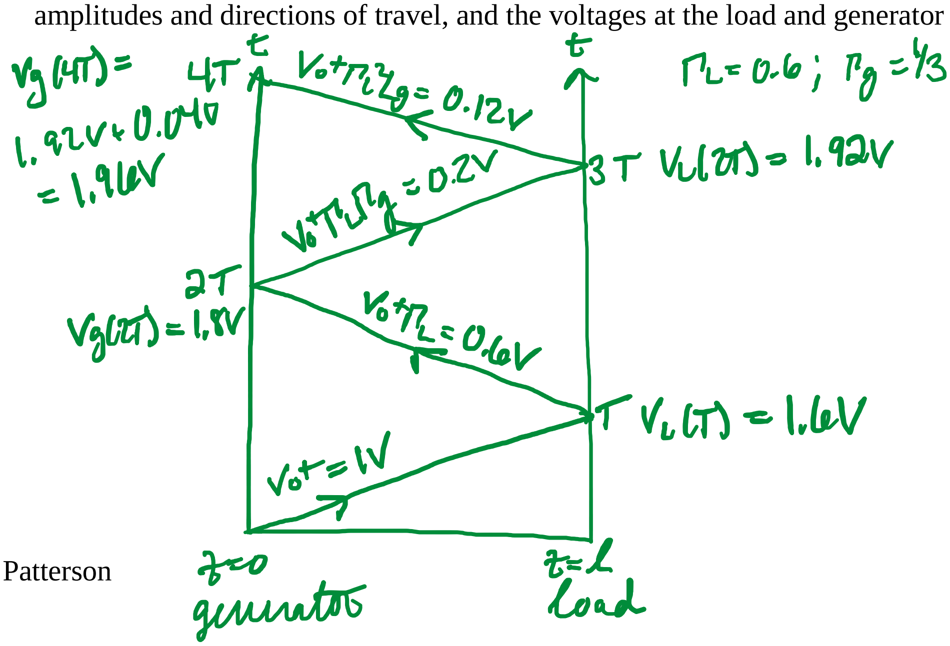
b) What are Γ_L, the reflection coefficient at the load, and Γ_g, the reflection coefficient at the input to the transmission line?

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{200\Omega - 50\Omega}{200\Omega + 50\Omega} = 0.6$$

$$\Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0} = \frac{100\Omega - 50\Omega}{100\Omega + 50\Omega} = \frac{1}{3}$$

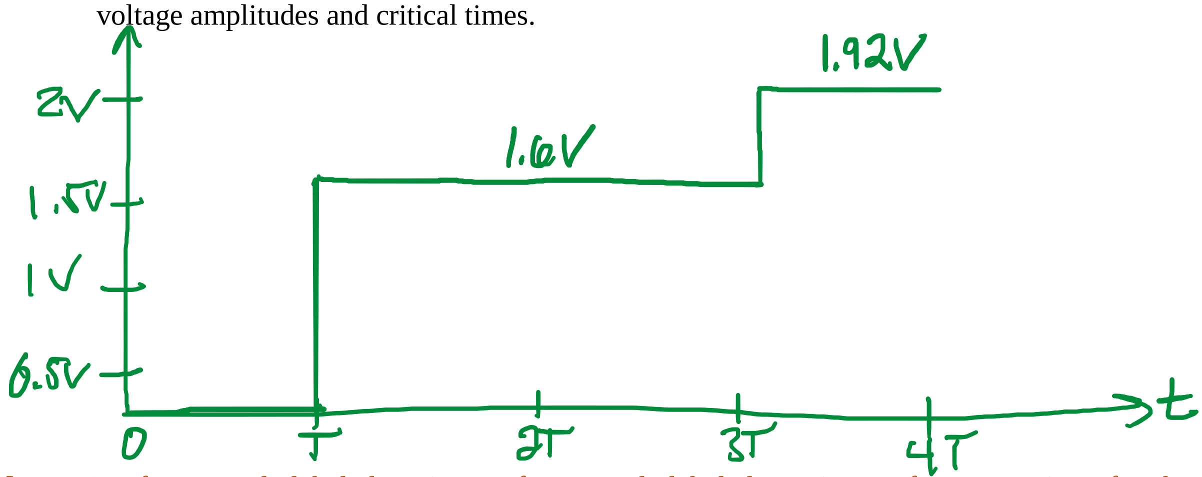
[2 pts: 0.5 pts for correct approach/equation for **Γ**$_L$; 0.5 pts for correct calculation of **Γ**$_L$; 0.5 pts for correct approach/equation for **Γ**$_g$; 0.5 pts for correct calculation of **Γ**$_g$]

c) Draw a bounce diagram for the circuit above from t = 0 to t = 4T, where T is the time delay on the transmission line. Be sure to label both the time axis and distance axis, voltage wave amplitudes and directions of travel, and the voltages at the load and generator during reflection.



[4 pts: 1 pt for correctly labeled t-axis; 0.5 pts for correctly labeled z-axis; 0.5 pts for labeled voltage wave travel directions; 1 pt for correct voltage wave amplitudes between reflections; 1 pt for correct voltage wave amplitudes at generator and load]

d) Sketch the voltage amplitude at the load vs. time for the timespan t = 0 to t = 4T. Be sure to label voltage amplitudes and critical times.



[4 pts: 1 pt for correctly labeled t-axis; 1 pt for correctly labeled V-axis; 1 pt for correct times for changes in voltage (reflections); 1 pt for correct voltages at V_g]

e) Assuming enough time has elapsed, what are the steady-state voltage and current on the line? Does this agree with the result you expect from DC circuit theory?

At steady-state, $V_\infty = V_g \frac{Z_L}{Z_g+Z_L} = 3V \frac{200\Omega}{100\Omega+200\Omega} = 2V$ and $I_\infty = \frac{V_g}{Z_g+Z_L} = \frac{3V}{100\Omega+200\Omega} = 10mA$

Yes, this is exactly the expression for a voltage divider for the load voltage and Ohm's law for the current in the circuit.

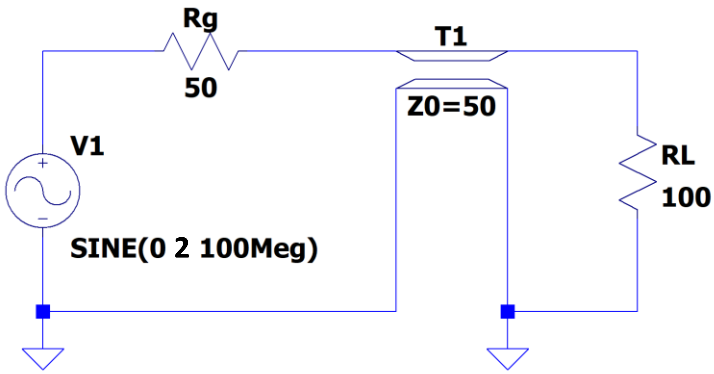
[3 pts: 0.5 pts for correct approach/equation for V_∞; 0.5 pts for correct calculation of V_∞; 0.5 pts for correct approach/equation for I_∞; 0.5 pts for correct calculation of I_∞; 1 pt for comparison to DC circuit theory]

f) If you were to replace RL with a capacitor instead, what would you expect the steady-state voltage and current to be? Why?

If RL were replaced with a capacitor, $Z_L = \frac{1}{j\omega C} \to \infty$, which looks like an open circuit, giving $\Gamma_L = \frac{\infty-50\Omega}{\infty+50\Omega} \to 1$. When $\Gamma_L = 1$, all of the incident voltage is reflected at the load and some of the reflected voltage is again reflected at the source. Eventually the voltage on the line settles to $V_\infty = V_{oc} = 3V\frac{\infty}{50\Omega+\infty} = 3V$ and the current is accordingly $I_\infty = I_{oc} = \frac{3V}{50\Omega+\infty} = 0A$.

[2 pts: 0.5 pts for correct V_∞; 0.5 pts for correct I_∞; 1 pt for valid justification]

4. Power on Lossless Transmission Lines



In the circuit above, the voltage source has an amplitude of V1 = 2V and frequency of f = 100 MHz. V_0^+ = 1V and the transmission line has a characteristic impedance Z_0 = 50Ω and velocity factor v_f = 0.66.

a) What is the instantaneous incident power on the transmission line in terms of d = -z? Express your answer in the time domain with numerical values for ω, β, V V_0^+, and Z_0.

$$\omega = 2\pi f = 200\pi \times 10^6 Hz$$

$$\lambda = \frac{u_p}{f} = \frac{0.66 \times 3 \times 10^8 \frac{m}{s}}{100 \times 10^6\ Hz} = 1.98m$$

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi}{1.98m} = 3.17\frac{1}{m}$$

$$P^i(d,t) = \frac{|V_0^+|^2}{2Z_0}\{1 + \cos(2\omega t + 2\beta d)\} = \mathbf{0.01\{1 + \cos\left(8\pi \times 10^8 t + 6.34d\right)\}\ W}$$

[2 pts: 1 pt for correct ω, β; 1 pt for correct P^i equation]

b) What is the instantaneous reflected power on the transmission line in terms of d = -z? Express your answer in the time domain with numerical values for ω, β, V V_0^+, and Z_0.

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{100\Omega - 50\Omega}{100\Omega + 50\Omega} = \frac{1}{3}$$

$$P^r(d,t) = -\frac{|V_0^+|^2}{2Z_0}|\Gamma|^2\{1 + \cos^2(2\omega t - 2\beta z)\} = \mathbf{-0.0011\{1 + \cos^2\left(8\pi \times 10^8 t - 6.34d\right)\}\ W}$$

[2 pts: 1 pt for correct $\boldsymbol{\Gamma}_L$; 1 pt for correct P^r equation]

c) What is the maximum net instantaneous power at the load (d = 0)?

The net instantaneous power at the load is the sum of P^i and P^r, evaluated at d = 0:

$$P^{net}(d = 0, t) = 0.01\{1 + \cos^2(8\pi \times 10^8 t)\} - 0.0011\{1 + \cos^2(8\pi \times 10^8 t)\}$$

$$= 0.0089\{1 + \cos^2(8\pi \times 10^8 t)\}$$

Cosine squared has a maximum value of 1, so the maximum net instantaneous power at the load is

$$P_{max}^{net}(d = 0) = \mathbf{0.0178\ W}$$

[2 pts: 1 pt for equation/approach; 1 pt for correct calculation]

d) What is the time-average power delivered to the load?

The time-average power delivered to the load is the time-average net power:

$$P_{av} = \frac{|V_0^+|^2}{2Z_0}(1 - |\Gamma|^2) = \frac{(1V)^2}{2 * 50\Omega}\left(1 - \frac{1}{9}\right) = \mathbf{0.0089\ W}$$

[1 pt: 0.5 pt for correct equation/approach; 1 pt for correct calculation]

e) Assuming two signals have the same amplitude V_0^+, but different frequencies, will a higher-frequency signal deliver more power to a resistive load than a lower-frequency signal in a fixed amount of time Δt?

No, since the time-average power is independent of frequency, both signals would deliver the same amount of power in a fixed amount of time.

[1 pt: 0.5 pts for correct answer; 0.5 pts for correct justification]

f) If both the voltage and current waves have zero DC offset, why does the instantaneous power have a non-zero DC component (which is the time-average power)?

Since P = IV, the power is the product of voltage and current waves, each of which are sinusoids. The product of two sinusoids of the same frequency results in a function that oscillates twice as fast as the original, individual waves, and also changes the function so that it is no longer centered at V = 0 (unlike the original two waves). This can be seen in the following trigonometric identity:

$$\cos\theta\cos(\theta + \phi_0) = \frac{1}{2}\{\cos(\phi_0) + \cos(2\theta + \phi_0)\}$$

If φ_0 (the phase shift between the two waves) is anything other than an odd multiple of π/2, the cos(φ_0) term will be non-zero and lead to a non-zero DC offset in power.

[1 pt: 0.5 pts for correct answer; 0.5 pts for correct justification]