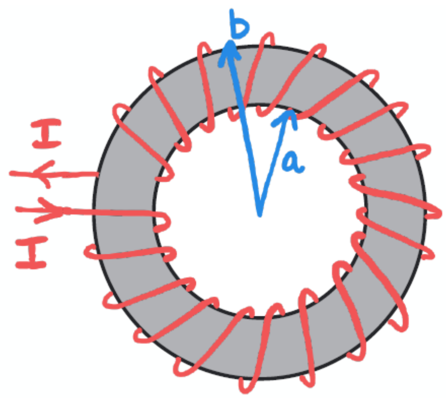
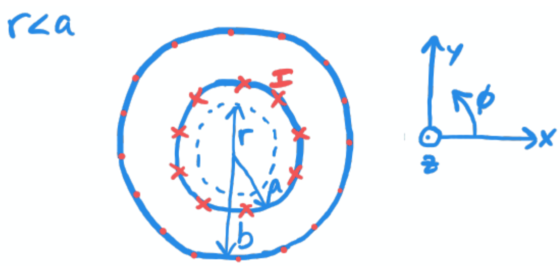
Homework 6 Solutions

1. Ampere's Law: Magnetic Field of a Toroid

Shown below is a toroid of inner radius *a* and outer radius *b*. The toroid lies parallel to the x-y plane and has a magnetic permeability of μ. The wire carries a current *I* and is wrapped *N* times around the toroid.



a) Draw a diagram indicating the geometry and location of the Amperian loop you will use to solve for **B** for r < a.



b) Using Ampere's Law, determine **B** for r < a. Justify any mathematical simplifications you made to the form of the **B** field using geometrical arguments (as we did in class).

Since the toroid is symmetrical in ϕ, B will not change in ϕ. Using the right hand rule, we can determine that B points along the $-\hat{\phi}$ direction, so $\vec{B} = B_\phi(r,z)$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \rightarrow B_\phi(r,z)\cdot 2\pi r = \overset{I_{enc}}{0}$$

So $\boxed{\vec{B} = 0 \text{ for } r<a}$

c) Repeat each of the steps in parts a and b to determine **B** for a < r < b.

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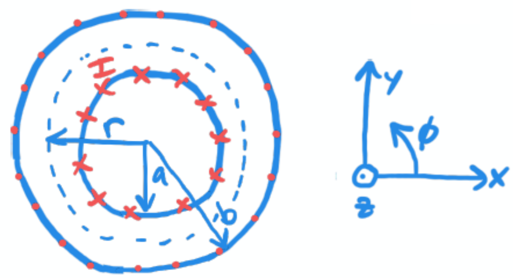
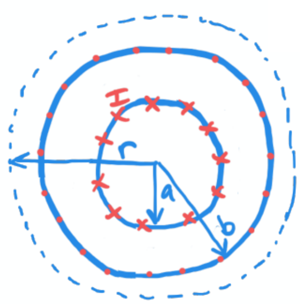
$a < r < b$



$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

$$B_\phi(r,z)\cdot 2\pi r = \mu_0 N I \rightarrow \boxed{\vec{B} = -\frac{\mu_0 N I}{2\pi r}\hat{\phi},\ a \le r \le b}$$

d) Repeat each of the steps in parts a and b to determine **B** for r > b.



$$\oint \vec{B}\cdot d\vec{l} = I_{enc}$$

$$B_\phi(r,z)\, 2\pi r = 0$$

So $\boxed{\vec{B} = 0 \text{ for } r > b}$

e) Verify that your solutions in parts b, c, and d satisfy $\vec{\nabla}\cdot\vec{B} = 0$.

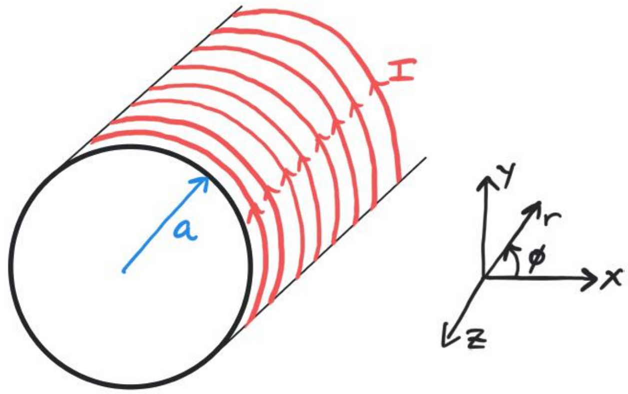
$\vec{\nabla}\cdot\vec{B} = \frac{1}{r}\frac{dB_\phi}{d\phi} \rightarrow$ Since none of the expressions for $\vec{B}$ contain a ϕ dependence, $\frac{dB_\phi}{d\phi} = 0$ in all regions and $\vec{\nabla}\cdot\vec{B} = 0$.

2. Magnetic Vector Potential for a Solenoid

The solenoid below has a radius *a* and is oriented parallel to the z-axis. A current *I* flows in the φ direction and there are *n* wire windings per unit length of the solenoid. The magnetic vector potential is defined via the following two expressions:

$$\oint \vec{A} \cdot d\vec{l} = \oiint \vec{B} \cdot d\vec{S} \quad (1)$$

$$\vec{\nabla} \times \vec{A} = \vec{B} \quad (2)$$



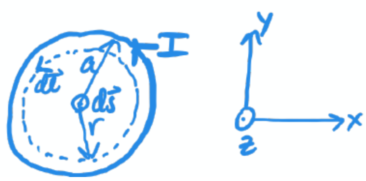
a) Which components of **A** will be non-zero?

Since $\vec{A}$ always points in the direction of the current, we will have $\vec{A} = A_\phi \hat{\phi}$.

b) Draw a diagram of the loop and surface you will use to determine **A** for the region r < a, using equation (1) above.

$\oint \vec{A} \cdot d\vec{l} = \oiint \vec{B} \cdot d\vec{s}$

Since $\vec{B}$ points along z, the surface $d\vec{s}$ should be perpendicular to $\hat{z}$ and its contour should have a radius $r < a$



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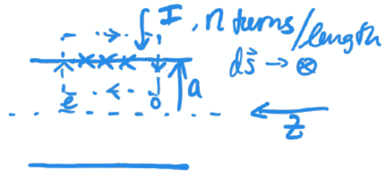
c) Calculate **A** for r < a.

for $r<a$: $\oint \vec{A}\cdot d\vec{l} = \iint \vec{B}\cdot d\vec{s}$

Calculation of $\vec{B}$:



$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$

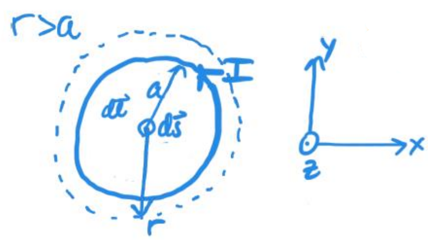
$B_z l = \mu_0 I n l$

$\vec{B_z} = \mu_0 I n \hat{z}$

$\oint A\cdot d\vec{l} = \iint \vec{B}\cdot d\vec{s}$

$A_\phi \cdot 2\pi r = (\mu_0 I n)(\pi r^2) \rightarrow \boxed{\vec{A} = \frac{\mu_0 I n r}{2}\hat{\phi}}$

d) Draw a diagram of the loop and surface you will use to determine **A** for the region r > a.



e) Calculate **A** for r > a.

$r>a$

$\oint \vec{A}\cdot d\vec{l} = \iint \vec{B}\cdot d\vec{s}$

$A_\phi \cdot 2\pi r = (\mu_0 I n)(\pi a^2)$

$\boxed{\vec{A} = \frac{\mu_0 I n a^2}{2r}\hat{\phi}, \; r>a}$

f) Verify that your solution satisfies the definition of the magnetic vector potential via equation (2) above in each of the regions.

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$$\vec{\nabla}\times\vec{A} = \vec{B} \rightarrow \vec{\nabla}\times\vec{A} = \frac{1}{r}\frac{\partial(rA_{\phi})}{\partial r}\hat{z}$$

· for $r<a$, $\vec{A} = \frac{\mu_0 I n r}{2}\hat{\phi}$ $\vec{B}$ for $r<a$

So: $\vec{\nabla}\times\vec{A} = \frac{\mu_0 I n}{2r}\frac{\partial(r\cdot r)}{\partial r}\hat{z} = \boxed{\mu_0 I n \hat{z}}$

· for $r>a$, $\vec{A} = \frac{\mu_0 I n a^2}{2r}\hat{\phi}$ $\vec{B}$ for $r>a$

$$\vec{\nabla}\times\vec{A} = -\frac{\mu_0 I n a^2}{2r}\frac{\partial}{\partial r}(r/r) = \boxed{0}$$

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3. Magnetic Materials

a) What are the three main classifications of materials in terms of their magnetization properties? By which physical property are materials categorized into these groups? How do they differ in terms of the internal magnetization field that is induced by an externally applied magnetic field **H**?

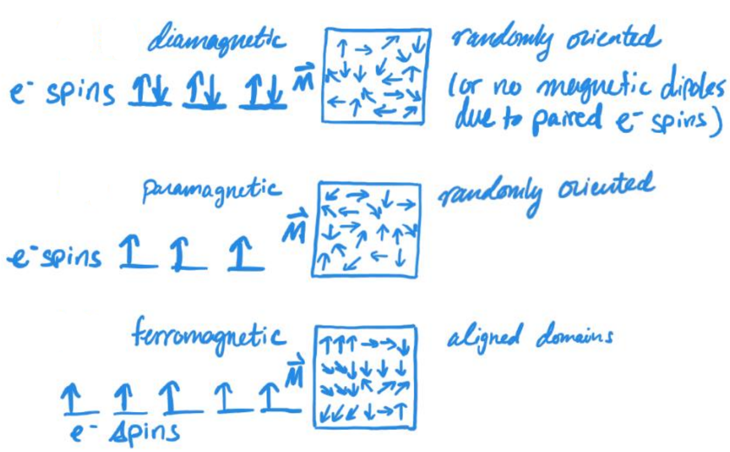
The three main classifications are diamagnetic, paramagnetic, and ferromagnetic. They are categorized based on their magnetic susceptibility χ_m:

- diamagnetic : $\chi_m < 0$, $|\mu_r| \sim 1$
- paramagnetic : $\chi_m > 0$, $|\mu_r| \sim 1$
- ferromagnetic : $\chi_m \gg 1$, $|\mu_r| \gg 1$

In terms of the internal magnetic field that's produced in response to an externally-applied magnetic field :

- diamagnetic : internal field opposes applied field
- paramagnetic : internal field aligns with applied field
- ferromagnetic : hysteresis : the internal field can oppose or align with the applied field, based on its magnetization history

b) Sketch the typical orientation of the magnetic dipoles in each type of material in the absence of an externally applied magnetic field (i.e. **H** = 0).



c) Sketch the typical orientation of the magnetic dipoles in each type of material in the presence of an externally applied magnetic field (i.e. **H** ≠ 0).

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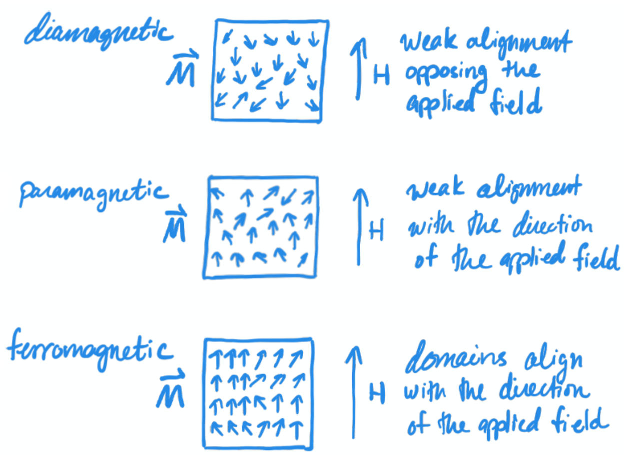
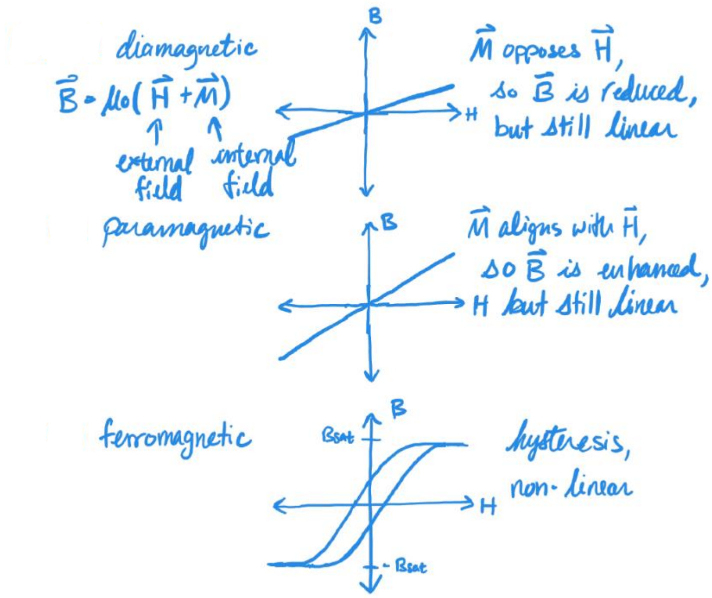


d) Assuming **M** points in the same direction as **H** (true for paramagnetic materials and can be true for ferromagnetic materials), explain why the **B** field is larger inside these materials where $\mu > \mu_0$ than in free space where $\mu = \mu_0$. Your reasoning should be based on your sketches in part c.

The $\vec{B}$ field is larger due to the magnetic moments in the material aligning with the applied field, which adds to the B field associated with the H field in free space.

e) Sketch a B-H magnetization diagram for each type of material.



f) Ferromagnetic materials exhibit what is called "magnetic hysteresis". What does this mean in terms of the B-H magnetization curve for this type of material? What does this physically mean in terms of your sketches in part c?

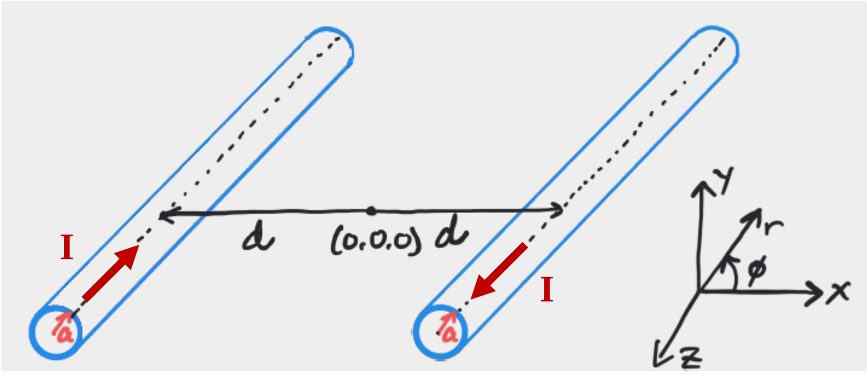
- Magnetic hysteresis is an effect whereby the B field in the material for a given applied H field depends on what the B field was previously (i.e. $\vec{B} \neq \mu\vec{H}$). Additionally, the B-H magnetization curve splits into two curves: one for H increasing and one for H decreasing.

- This means that the magnetic dipoles that become aligned after applying an H field remain aligned to some degree after $\vec{H}$ is reduced or removed

g) Given a non-magnetized piece of a ferromagnetic material, how would you turn it into a permanent magnet?

Apply a strong H field to align the magnetic dipole moments, then some residual B field will remain, creating a permanent magnet.

4. Inductance of a Parallel Wire Transmission Line and Faraday's Law



a) Calculate the total **B** field between the wires, in the x-z plane.

+ B-field from a single wire at the origin

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

$$B_\phi(r)2\pi r = \mu_0 I$$

$$B_\phi(r) = \frac{\mu_0 I}{2\pi r}\hat{\phi} \quad \text{if } I = |I|\hat{z}$$

+ If we move the wire to a location $\vec{r}_0 = \langle x_0, y_0\rangle$ and observe the $\vec{B}$ field at a location $\vec{r} = \langle x, y\rangle$, the vector pointing from (x_0, y_0) to (x, y) is $\vec{r}' = \vec{r} - \vec{r}_0 = \langle x - x_0, y - y_0\rangle$

+ Using this,

$$B_\phi(\vec{r}') = \frac{\mu_0 I}{2\pi|\vec{r}'|}\hat{\phi}$$

+ for the wire at $x = -d$: $\vec{r} = \langle x + d, 0\rangle$

$$B_\phi(\vec{r}') = -\frac{\mu_0 I}{2\pi|x+d|}\hat{y} \sim \hat{\phi}(y=0) = -\hat{y}$$

+ for the wire at $x = +d$;

$$B_\phi(\vec{r}') = -\frac{\mu_0 I}{2\pi|x-d|}\hat{y} \sim \hat{\phi}(y=0) = -\hat{y}$$

$$\boxed{\vec{B}_{total} = -\frac{\mu_0 I}{2\pi}\left\{\frac{1}{|x+d|} + \frac{1}{|x-d|}\right\}\hat{y}}$$

for $-d + a < r < d - a$

b) Calculate the magnetic flux Φ in a plane spanning the space between the wires (-d + a < x < d - a) in the x-z plane.

$$\Phi = \iint \vec{B} \cdot \underbrace{d\vec{S}}_{-\hat{y}\,dx\,dz} = \frac{\mu_0 I}{2\pi} \int_0^{\ell} \left\{ \int_{-d+a}^{d-a} \left[\frac{1}{|x+d|} + \frac{1}{|x-d|} \right] dx \right\} dz$$

$$= \frac{\mu_0 I \ell}{2\pi} \left\{ \int_{-d+a}^{d-a} \frac{dx}{x+d} + \int_{-d+a}^{d-a} \frac{dx}{-(x-d)} \right\}$$

$$= \frac{\mu_0 I \ell}{2\pi} \left\{ \ln[x+d] \Big|_{-d+a}^{d-a} - \ln[x-d] \Big|_{-d+a}^{d-a} \right\}$$

$$= \frac{\mu_0 I \ell}{2\pi} \left\{ \ln\left(\frac{2d-a}{a}\right) - \ln\left(\frac{-a}{-2d+a}\right) \right\}$$

$$= \frac{\mu_0 I \ell}{2\pi} \left\{ \ln\left[\left(\frac{2d-a}{a}\right) \cdot \left(\frac{2d-a}{a}\right) \right] \right\}$$

$$= \boxed{\frac{\mu_0 I \ell}{\pi} \ln\left[\frac{2d}{a} - 1\right] = \Phi}$$

c) Calculate the inductance of the two-wire system via $L = \frac{\Lambda}{I}$, where $\Lambda = N\Phi$ is the total flux linkage and N is the number of wire loops that the flux links.

$L = \frac{\Lambda}{I} = \frac{\Psi}{I}$ since there is only one loop we're considering for flux linkage

$$\boxed{L = \frac{\mu_0 \ell}{\pi} \ln\left[\frac{2d}{a} - 1\right]}$$

d) Calculate a numerical value for the inductance L and inductance per unit length *l* if the length of the wires we're considering is 0.1m, *a* = 0.5mm, and *d* = 0.5cm.

$$L = \frac{4\pi \times 10^{-7}\,\text{H/m} \cdot 0.1\,\text{m}}{\pi} \ln\left[\frac{2 \cdot 0.5\times10^{-2}\,\text{m}}{0.5\times10^{-3}\,\text{m}} - 1\right]$$

$$= 1.18\times10^{-7}\,\text{H} = \boxed{118\,\text{nH}}$$

$$L' = \frac{118\,\text{nH}}{0.1\,\text{m}} = \boxed{1.18\,\mu\text{H/m}}$$

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e) Faraday's law states: $V_{emf} = -\frac{d\Phi}{dt}$. If the current of the wires is now time-dependent, such that I = I(t), use your expression from part b to write V_{emf} in the form $V_{emf} = -A\frac{dI}{dt}$, where A is a constant. What is this constant A? Does this expression for V_{emf} look familiar from circuit theory?

$$V_{emf} = -\frac{d\Phi}{dt} = \underbrace{-\frac{\mu_0 l}{\pi} \ln\left[\frac{2d}{a} - 1\right]}_{\text{inductance } L} \frac{dI(t)}{dt}$$

$V_L = L\frac{dI}{dt}$ is the voltage across an inductor

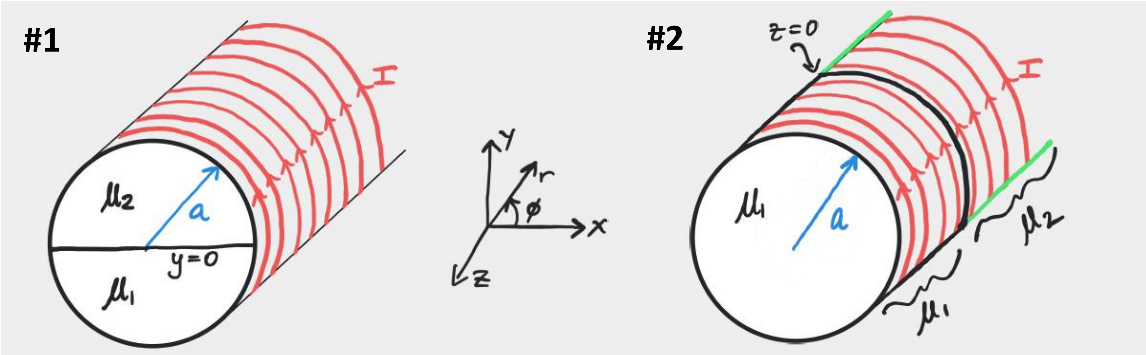
f) If $I(t) = I_0 \sin(2\pi f t)$, what is the EMF that would be generated across each of the wires? In which direction does the **E** field associated with the EMF point in each of the wires?

$$\boxed{V_{emf} = -2\mu_0 l I_0 f \ln\left[\frac{2d}{a} - 1\right] \sin(2\pi f t)}$$

The induced $\vec{E}$ field points opposite to the change in the current $\pm\hat{z}$ direction

5. Magnetic Boundary Conditions

Two solenoids are shown below: solenoid #1 consists of two materials with different magnetic permeabilities: μ_1 for y < 0 and r < a and μ_2 for y > 0 and r < a. Solenoid #2 also consists of two materials with different magnetic permeabilities: μ_1 for z < 0 and r < a and μ_2 for z > 0 and r < a.



a) Calculate **B** and **H** for solenoid #1 in both regions for r < a.

· For region 1, Ampere's Law gives

$$\oint \vec{H}\cdot d\vec{l} = I_{enc}$$

$$H_1 \cdot l = n \cdot l \cdot I$$

$$\boxed{\vec{H}_1 = nlI\hat{z}}$$

· Since $\hat{z}$ lies parallel to the boundary and $H_{2t} = H_{2t}$ for tangential components of H,

$$\boxed{\vec{H}_2 = nlI\hat{z}}$$

$$\boxed{\begin{aligned}\vec{B}_1 &= \mu_1\vec{H}_1 = \mu_1 nlI\hat{z}\\ \vec{B}_2 &= \mu_2\vec{H}_2 = \mu_2 nlI\hat{z}\end{aligned}}$$

· There is no B-field normal to $d\vec{s}$ in either field so the boundary condition $B_{2n} = B_{1n}$ is satisfied.

b) For solenoid #2:

i. Calculate **H** and **B** in region #1 (where $\mu = \mu_1$) using Ampere's Law (or simply use the expression for **H** of a solenoid from a previous problem).

ii. Apply the normal boundary condition for **B** to find **B** in region #2 (where $\mu = \mu_2$), then calculate **H** in region 2. You should find that your expression for **H** in region #2 is not what you would get by solving for it using Ampere's Law in region #2.

iii. Find the current density at the surface of the solenoid in region #2 via the tangential boundary condition for **H** between region #2 and air. Is this different from the current density flowing at the surface of the solenoid in region #1?

$\vec{H}_1 = nlI\hat{z}$
$\vec{B}_1 = \mu_1\vec{H}_1 = \mu_1 nlI\hat{z}$

· Via the normal boundary condition for B,
$\vec{B}_2 = \vec{B}_1 = \mu_1 nlI\hat{z}$
$\boxed{\vec{H}_2 = \frac{\vec{B}_2}{\mu_2} = \frac{\mu_1}{\mu_2}nlI\hat{z}}$

· According to Ampere's Law,
$\vec{H}_2$ due to the current flowing through the N wire windings should be $\vec{H}_2 = nlI\hat{z}$

· At the material 2 / air boundary, we have $H_{2t} - H_{air,t} = J_s$
$\frac{\mu_1}{\mu_2}nlI - 0 = J_{S2}$, so $\boxed{J_{S_2} = \frac{\mu_1}{\mu_2}nlI}$

· At the material 1 / air boundary:

$H_{1t} - H_{air,t} = J_{S1}$
$\boxed{nlI = J_{S1}}$

· The two surface current densities differ, telling us there's an additional surface current density on the core in region #2.

c) In electrostatics, we had the boundary conditions:

$$\begin{cases} D_{2n} - D_{1n} = \rho_s \\ E_{2t} - E_{1t} = 0 \end{cases}$$

which stated in mathematical terms that for:

i) the normal component: the net electric flux density passing through the boundary is equal to the free surface charge density at that location (net flux is zero unless there's a source at the boundary: Gauss's Law says $\vec{\nabla} \cdot \vec{D} = \rho_s \rightarrow$ electric field lines originate from or end on free charges)

ii) the tangential component: the electric field must be continuous across the boundary because the curl of the electric field is zero $\vec{\nabla} \times \vec{E} = 0$ (the E field has no rotation)

Using the physical interpretations of Maxwell's equations for magnetostatics as your basis, explain why the difference in the normal B field across the boundary is always zero and the tangential magnetic field is equal to the surface current density at the boundary.

$$\begin{cases} \hat{n} \cdot (\vec{B}_2 - \vec{B}_1) = 0 \\ \hat{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{J}_s \end{cases}$$

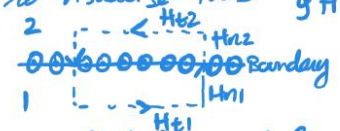
· The B field normal to the boundary must be continuous across the boundary due to the condition

$$\oiint \vec{B} \cdot d\vec{s} \iff \vec{\nabla} \cdot \vec{B} = 0$$

which states that the net magnetic flux through a closed surface is 0. This means that there can be no $\vec{B}$ field lines that originate or end at a single point → there are no magnetic monopoles (unlike for $\vec{E}$ fields)

· $\vec{\nabla} \times \vec{H} = \vec{J}$ · tells us that a current density gives rise to a magnetic field perpendicular to the direction of current flow.

The integral form of Ampere's law gives us a way to visualize this $\oint \vec{H} \cdot d\vec{l} = \oiint \vec{J} \cdot d\vec{s}$



If we have a sheet of current flowing at the surface, the resulting H field must be tangential to the boundary.

d) A surface current density can only exist at the boundary between certain types of materials. Which types of materials can have a surface current density at a boundary?

Surface currents can only exist at the boundaries of perfect conductors and superconductors.

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