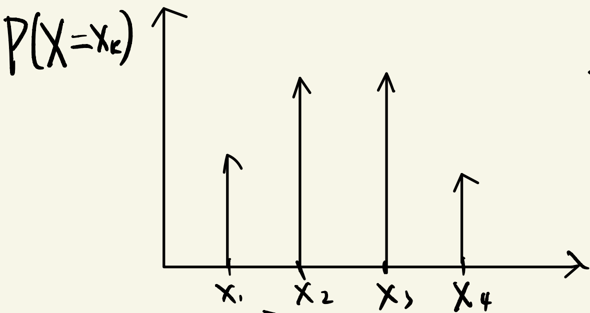
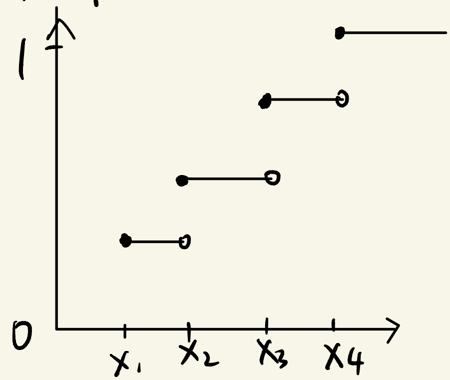
ECSE 4500
Lec 12
March 2nd

# Topic: Probability Density Function (PDF) for continuous RV.

For discrete RV X, $F_X(x) = P(X \le x)$



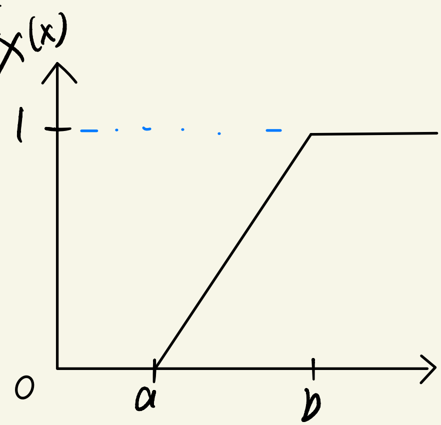

PMF




CDF

For continuous RV X,

PDF




CDF

Q: What is the notion similar to PMF?

It is PDF, which is the most useful and important way of discussing a continuous random variable.

▷ Recall the definition of CDF (both discrete/continuous)

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} P_X(y)\, dy$$

In the discrete RV case, the function $P_X(y)$ is a impulse function, so $F_X(x)$ is a step function.

□ The PDF is defined as

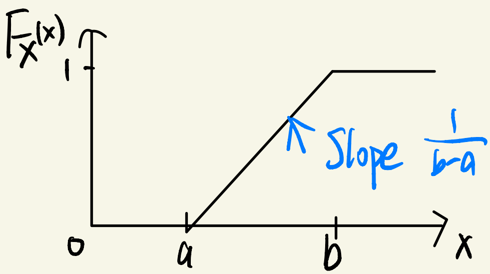
$$\boxed{f_X(x) = \frac{d}{dx} F_X(x)}$$

small case

big case

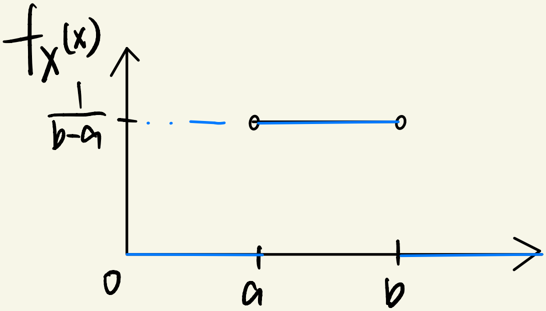
which makes sense since we got from the PMF to CDF via integration (summation).

$\boxed{\text{Example}}$ — Uniform (Continuous) Random Variable




$$= \begin{cases} \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 0, & \text{if } x \leq a \\ 1, & \text{if } x \geq b \end{cases}$$

Therefore, the PDF of this RV is




$$= \begin{cases} \frac{1}{b-a}, & \text{if } a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

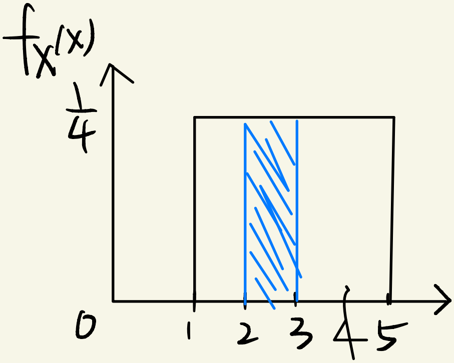
At points $x=a$ and $x=b$, the derivatives are undefined, but it doesn't really matter since $P(X=a) = P(x=b) = 0$. Often we casually "Connect the dots."

□ The usage of PDF? $X$ is continuous RV

$$P(a \le X \le b) = \int_a^b f_X(x)\,dx$$

If we know PDF of X

That is, the probability of $X$ lying in an interval $[a, b]$ is the area under the interval,




$$P(2 \le X \le 3)$$

$$= \int_2^3 f_X(x)\,dx$$

$$= \int_2^3 \frac{1}{4}\,dx = \frac{1}{4}$$

This can be also viewed from the fundamental theorem of caculus:

$$\int_a^b f_X(x)\,dx = \int_{-\infty}^b f_X(x)\,dx - \int_{-\infty}^a f_X(x)\,dx$$

$$= F_X(b) - F_X(a)$$

$$= P(a \le X \le b)$$

(Being a little sloppy about two endpoints)

Put another way,

$$P(x < X \le x+h) = F_X(x+h) - F_X(x)$$

a small constant

$$= \frac{F_X(x+h) - F_X(x)}{h} \cdot h \overset{h \to 0}{=} F_X'(x) \cdot h$$

$$= f_X(x) \cdot h$$

Before we give more examples, one more thing I want all of you to be careful is that

**Notice:** PDF $f_X(x)$ is not the "Probability of X getting the value x!"

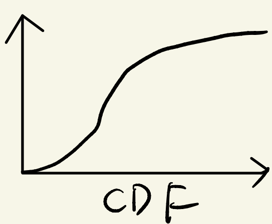
**Why?** The probability of continuous RV getting any particular value x is $0$!

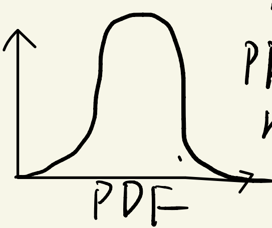
**What it is?** The idea of PDF describes the "density" of probability near the value x.

If the PDF is high near x, we expect to get more outcome around this area!

## ☐ Properties of PDF

1) $f_X(x) \geq 0$ for all $x$



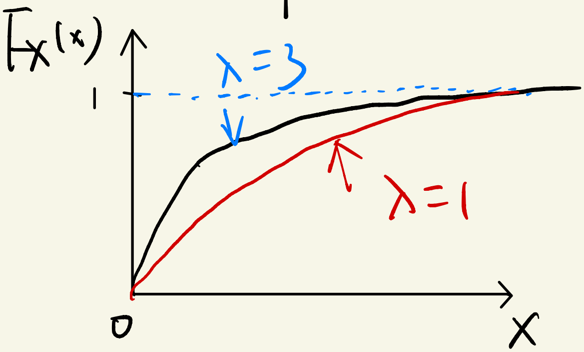




CDF is non-decreasing; PDF is nonnegative.

2) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$, $= F_X(\infty) = 1$

☐ Some new examples of continuous RVs

1) The exponential random variable has CDF



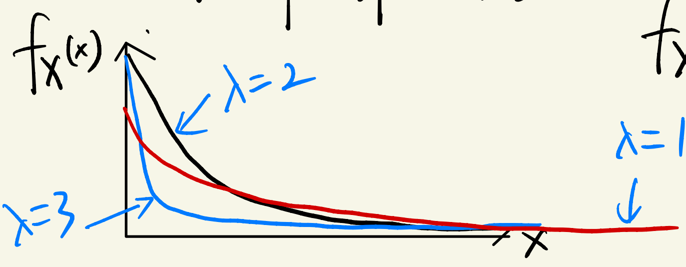

$$F_X(x) = \begin{cases} 1-e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

As parameter $\lambda$ decreases, the CDF becomes close to linear.

Q: What real-world examples the RV can model?

- How long does a device last?

The PDF of exponential RV




$$f_X(x) = \begin{cases} 0, & x < 0 \\ \lambda e^{-\lambda x}, & x \geq 0 \end{cases}$$

positive constant

– Time between two school buses arrive

Additional discussion about Exponential RV.

We often care about events of the form

$$P(X > x)$$

e.g., Probability devices last longer than x years.

$$P(X > x) = 1 - P(X \leq x)$$

$$= 1 - F_X(x)$$

Plug in the CDF of exponential RV

$$= e^{-\lambda x}, \quad x \geq 0$$

A connection between the Poisson Discrete Random variable and the exponential random variable.

Let $N$ (integer) denote the number of events happen in $t$ seconds (e.g., packet arrivals), where the Poisson parameter $\alpha = \lambda t$ ($\lambda$ is the average arrival rate in Packets/second).

Q: What is the probability that the next arrival occurs $t$ or more seconds later?

Equivalently, $P(N=0) = \frac{\alpha^0}{0!} e^{-\alpha} = e^{-\alpha} = e^{-\lambda t}$

PMF of Poisson RV

$= P(X > t)$

where $X$ denotes the time until the next packet arrival.

That is, $N=0$ and $X>t$ are equivalent events. One looks at it as # of packets and the other as the time to packet arrival.

☐ Other examples of continuous RVs.

1) Gamma RV

$$f_X(x) = \frac{\lambda(\lambda x)^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, \quad 0 < x < \infty$$

with

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

2) Beta RV

$$f_X(x) = \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)} x^{a-b}(1-x)^{b-1}, \quad 0<x<1$$