MATH-2400 DIFFEQ F 2022 Crib Sheet Exam 2 Friday, November 4, 2022 Hayden Fuller Notes:

SECOND ORDER: $y'' + p(t)y' + q(t)y = g(t)$ ; $y(t) = C_1 y_1(t) + C_2 y_2(t)$

wronskion: $w(t) = det[y_1, y_2, //, y_1', y_2'] = y_1 y_2' - y_1' y_2 \neq 0$ (linearly independent, not multiples of each other)

$L[y_1] = y_1'' + p(t)y_1' + q(t)y = g(t)$

CONSTANT COEFFICIENT: $ay'' + by' + cy = 0$ ; $y(t) = e^{rt}$ ; $ar^2 + br + c = 0$

CASE 1 $r_1, r_2 \in R$, $r_1 \neq r_2$ ; $b^2 - 4ac > 0$ ; $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

CASE 2 $r_1, r_2 \in R$, $r_1 = r_2$ ; $b^2 - 4ac = 0$ ; $y(t) = C_1 e^{rt} + C_2 t e^{rt}$

REDUCTION OF ORDER:

$y_2 = y_1 h$ , $y_2' = y_1' h + y_1 h'$ , $y_2'' = y_1'' h + 2y_2' h' + y_1 h''$

$y_1'' h + 2y_2' h' + y_1 h'' + p(t)(y_1' h + y_1 h') + q(t)(y_1 h) = (y'' + py' + qy)h + (2y' + py)h' + yh''$ (should) $= (2y' + py)h' + yh''$ ,

$u = h'$ ; $yu' + (2y' + py)u = 0$

get $u$ to one side, integrate to find $u$ (has C); integrate to get $h$ (has D) and plug in to $y_2 = y_1 h$ ; choose $C$ and $D$ to be easy.

CASE 3 $r_1, r_2 \notin R$ ; $b^2 - 4ac < 0$ ; $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = r = \lambda \pm i\omega$ ;

$y_1^c = e^{\lambda + i\omega} = e^{\lambda t}(\cos(\omega t) + i \sin(\omega t))$

$y_2^c = e^{\lambda - i\omega} = e^{\lambda t}(\cos(\omega t) - i \sin(\omega t))$

$y_1(t) = e^{\lambda t} \cos(\omega t)$ $\qquad y_2(t) = e^{\lambda t} \sin(\omega t)$

CAUCHY-EULER

$ax^2 y'' + bxy' + cy = 0$ ; $y = x^r$ ; $ar(r-1) + br + c = 0$ roots $r_1$ , $r_2$

CASE 1: $y = C_1 x^{r_1} + C_2 x^{r_2}$

CASE 2: $y = C_1 x^r + C_2 x^r \ln(x)$

CASE 3: $y = C_1 x^\lambda \cos(\omega \ln(x)) + C_2 x^\lambda \sin(\omega \ln(x))$

polar: $y = Re^{\lambda t} \cos(\omega t - \phi)$ ; $y(t) = (R\cos(\phi))e^{\lambda t} \cos(\omega t) + (\sin(\phi))e^{\lambda t} \sin(\omega t)$

$R \cos \phi = C_1$ , $R \sin \phi = C_2$

METHOD OF UNDETERMINED COEFFICIENTS (must be constant coefficient)

$ae^{bt} \to Ae^{bt}$

$a\cos(ct) + b\sin(ct) \to A\cos(ct) + B\sin(ct)$

$at^n \to A_{n+1} t^n + \ ... \ + A_1$

$g(t) = P_n(t) e^{at} (\alpha \cos(bt) + \beta \sin(bt))$ ; $y_p(t) = Q_n(t) e^{at} \cos(bt) + R_n(t) e^{at} \sin(bt)$

addition of $g_1$ , $g_2$ ... results in addition of solutions

you just guessed $y_p$ , derive $y_p'$ and $yh_p''$ , multiply t if resonance

plug those in to $L[y_p] = g$ and solve for A's and B's; sets of (A's and B's) for each term

plug those into guess for $y_p$ and $y(t) = y_h + y_p$ , plug in ICs to solve for $C_1$ , $C_2$

VARIATION OF PARAMETERS (must be in standard form)

must know homo $y_1$ , $y_2$ ; $y_p(t) = u_1 y_1 + u_2 y_2$ ; $u_1' = \frac{-y_2 g}{W}$ ; $u_2' = \frac{y_1 g}{W}$

$u = \int u' dt + A_n$ , plug in to get $y_p$ , choose $A_1$ , $A_2$ to make it easy.

LINEAR OSCILATOR

$mu'' + cu' + ku = F_0 \cos(\omega t)$

$\omega_0 = \sqrt{\frac{k}{m}}$ ; $kx = mg$ ; $c = \frac{F}{v}$ ; $r = \pm\sqrt{\frac{-k}{m}}$

$u(t) = e^{\lambda t}(C_1 \cos(\cos(\omega t) + \sin(\omega t))$

undetermined coefficients oscilator: $D = c^2 \omega^2 + (k - m\omega^2)^2$

$u_p(t) = A\cos(\omega t) + B\sin(\omega t)$ ; $u_p' = -A\omega \sin(\omega t) + B\omega \cos(\omega t)$ ; $u_p'' = -\omega^2 A \cos(\omega t) - \omega^2 B \sin(\omega t)$

max amplitude? $R' = 0$ , $D = 0$ , $R = \sqrt{A^2 + B^2}$ , $\omega$ sorta close to $\omega_0$

FREE UNDAMPED

$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) = R\cos(\omega_0 t - \phi)$

$C_1 = u(0) = R\cos(\phi)$ ; $C_2 = \frac{u'(0)}{\omega_0} = R\sin(\phi)$

$= \frac{2\pi}{\omega_0}$ ; frequency $= \frac{\omega_0}{2\pi} = \frac{1}{period}$ ; amplitude $= R = \sqrt{C_1^2 + C_2^2}$ ; $\phi = \arctan(\frac{C_2}{C_1})$

FREE DAMPED

$u = e^{rt}$ ; $r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$ ; sqrt: ($< 0$ under, not strong, overshoots, disipating wave) ($= 0$ cryticially, perfect, decays as fast as possible without overshooting) ($> 0$ over, too strong, slower, slow decay

FORCED UNDAMPED

$\omega_0 \neq \omega$ small wave. $\omega_0 \approx \omega$ bigger wave. $\omega_0 = \omega$ resonance, grows linearly

FORCED DAMPED

$u_p(t) = A\cos(\omega t) + B\sin(\omega t)$

LAPLACE TRANSFORM

$F(s) = \int_0^\infty f(t)e^{-st}dt$

linear, $L(af(t) + bg(t)) = aL(f(t)) + bL(g(t))$

$ay'' + by' + cy = f(t)\,;\ a(s^2Y - sy_0 - y_0') + b(sY - y_0) + c(Y) = F\,;\ Y = \frac{F + asy_0 + ay_0' + by_0}{as^2 + bs + c}$