ECSE 2500
Lec 16
Mar. 23th

Topic: ① PDF / CDF of a function of continuous RV
② Markov / Chebyshev inequality
③ Introduce PMF/PDF of two RVs

□ PDF / CDF of a function of RVs

Talk about $Y = g(X) = X^2$. Now we are given CDF of X as $F_X(x)$, for all x. We want to find CDF / PDF of Y?

Answer: $F_Y(y) = P(Y \le y)$

$$= P(X^2 \le y)$$

$$= P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= P(X \le \sqrt{y}) - P(X \le -\sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y}), \quad y \ge 0$$

Once we have the CDF of Y, we can derive PDF of y via derivative

$$f_Y(y) = \frac{\partial F_Y(y)}{\partial y}$$

$$= \frac{\partial F_X(\sqrt{y})}{\partial \sqrt{y}} \cdot \frac{\partial \sqrt{y}}{y} - \frac{\partial F_X(-\sqrt{y})}{-\sqrt{y}} \cdot \frac{-\partial \sqrt{y}}{\partial y}$$

$$= f_X(\sqrt{y}) \frac{1}{2\sqrt{y}} + f_X(-\sqrt{y}) \frac{1}{2\sqrt{y}}$$

(*) $$= \frac{1}{2\sqrt{y}} \left(f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right)$$

Example: Assume X is a Gaussian RV with mean $\mu = 0$, variance $\sigma^2 = 1$. Let $Y = X^2$. What is PDF $f_Y(y)$?

Answer: PDF of X

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Plugging the Gaussian PDF into $(*)$

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{y}{2}} + \frac{1}{\sqrt{2\pi}}e^{-\frac{y}{2}}\right)$$

$$= \frac{1}{\sqrt{y}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}}, \quad y \geq 0$$

Remark: This is called a chi-square (χ^2) random variable with one degress of freedom.

- If $Y = g(X) = |X|$, then what is PDF/CDF of Y?

By definition of CDF,

$$F_Y(y) = P(Y \leq y)$$

$$= P(|X| \leq y)$$

$$= P(-y \leq X \leq y)$$

$$= P(X \leq y) - P(X \leq -y)$$

$$= F_X(y) - F_X(-y), \quad y \geq 0$$

With CDF, we can derive PDF of y as

$$f_Y(y) = \frac{dF_Y(y)}{dy} = f_X(y) + f_X(-y)$$

- We know CDF and PDF of X as $F_X(x)$ and $f_X(x)$. If $Y = g(X) = \begin{cases} X, & \text{if } X \geq 0 \\ 0, & \text{o.w.} \end{cases}$ what is the CDF / PDF of Y? $= (X)^+$

$$F_Y(y) = P(Y \leq y)$$

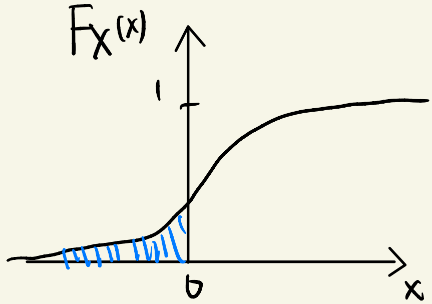
$$= \begin{cases} P(X \leq y) & \text{if } y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

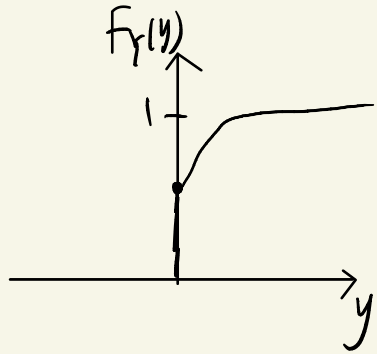
$$= \begin{cases} F_X(y), & \text{if } y \geq 0 \\ 0, & \text{o.w.} \end{cases}$$

$$f_Y(y) = \begin{cases} f_X(y), & \text{if } y \geq 0 \\ 0, & \text{o.w.} \end{cases}$$

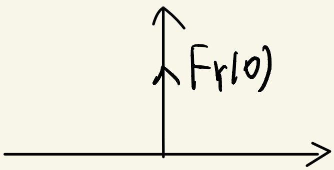
$$f_Y(y) = F_X(0)\delta(y) + f_X(y)u(y)$$

Visualization

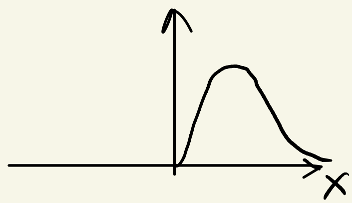




$f_Y(y) =$



$+$



Example: X is a Gaussian RV with mean $\mu = 2$, and variance $\sigma^2 = 4$. If $Y = (X)^+$, compute PDF of Y $f_Y(y)$?

Using the conclusion in the above derivation

$$f_Y(y) = F_X(0)\,\delta(y) + f_X(y)\,u(y)$$
$$= \phi\left(\frac{0-2}{2}\right)\delta(y) + \frac{1}{\sqrt{2\pi}\,2} e^{-\frac{(y-2)^2}{4}} u(y)$$

$\boxed{\text{Summary}}$ In general, if $Y = g(X)$ and the derivative $g'(x)$ exists, the general form of PDF of Y can be represented as follows:

Let y be a given value of Y and let $\{x_1, x_2, \dots, x_k\}$ denote the set of values of x that $g(x) = y$, then we can write

$$\boxed{f_Y(y) = \sum_{i=1}^{k} f_X(x_i) \cdot \frac{1}{|g'(x_i)|}}$$

Apply to $g(X) = X^2$? Verify it offline.

□ Some inequalities/bound about CDF/PDF.

Say we only know RV X is non-negative with mean μ. Can we say something about $P(X)$?

Markov inequality

$$P(X \geq a) \leq \frac{E[X]}{a}$$

non-negative

Proof:

By definition of mean of X,

$$\mu = E[X] = \int_0^\infty x f_X(x)\,dx$$

$$= \int_0^a x f_X(x)\,dx + \int_a^\infty x f_X(x)\,dx$$

$$\geq \int_a^\infty x f_X(x)\,dx$$

$$\geq a \int_a^\infty f_X(x)\,dx = a\,P(X \geq a)$$

$$\Leftrightarrow \quad P(X \geq a) \leq \frac{E[X]}{a}$$

Markov inequality gives us an upper bound on the probability of RV X from its mean E[X].

Example: (Application of Markov inequality)

Printer gets crashed if it gets more than 1000 requests/sec. We measure the average workload of the printer as 50 requests/sec. What is the estimate of P(Printer gets crashed)?

Answer: $P(\text{Crash}) = P(\underset{\substack{\uparrow \\ \#\text{ of requests per sec}}}{X} \geq 1000)$

Using Markov inequality

$$\leq \frac{E[X]}{1000} = \frac{1}{20}$$

→ We can get a tighter bound on the probability if we also know the variance of the RV.

Let $Y = (X - \underset{\substack{\uparrow \\ \text{mean of } X}}{\mu})^2$. Then $E[Y] = \underset{\substack{\uparrow \\ \text{Variance of } X}}{\sigma}^2$.

We can apply the Markov inequality to Y.

This will give

$$P(Y \geq a^2) \leq \frac{E[Y]}{a^2}$$

$(X-\mu)^2$

$$= \frac{\sigma^2}{a^2}$$

$$\Leftrightarrow \quad P(X-\mu \geq a \text{ or } X-\mu \leq -a) \leq \frac{\sigma^2}{a^2}$$

$$\Leftrightarrow \quad P(|X-\mu| \geq a) \leq \frac{\sigma^2}{a^2}$$

$\boxed{\text{Chebyshev inequality}}$

Example: Average height is 3.5 Ft, and the variance is $\sigma = \frac{1}{2}$ Ft.

What is the $P(\text{kid} \geq 9\text{Ft})$?

$$P(|X-3.5| \geq 5.5) \leq \frac{\frac{1}{4}}{(5.5)^2}$$

$$= 0.008$$ small!