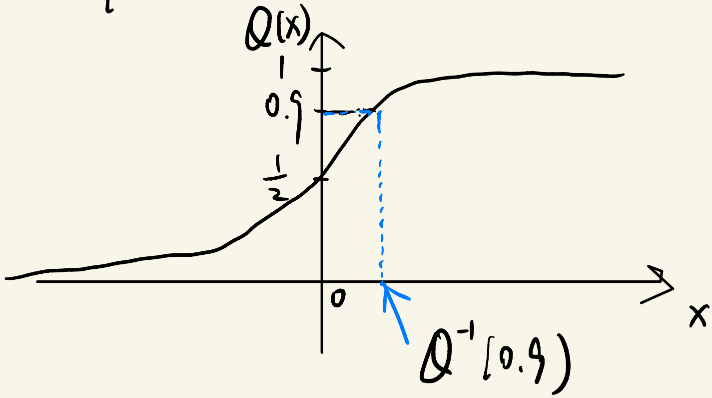
ECSE 2500
Lec 14
March 16

Topic: Inverse CDF of Gaussian RV
Expected values of Continuous RV

☐ Inverse CDF of Gaussian RV.

Why/Where it is useful?

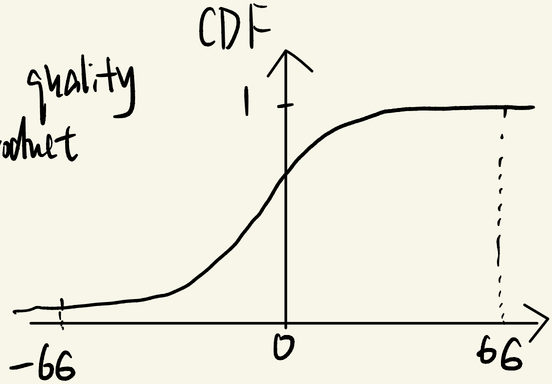
Definition / Concepts of inverse CDF



Look Table 4.3 for inverse CDF values

This inverse CDF is related to the "6 σ" fact in management / manufacturing / supply chain.

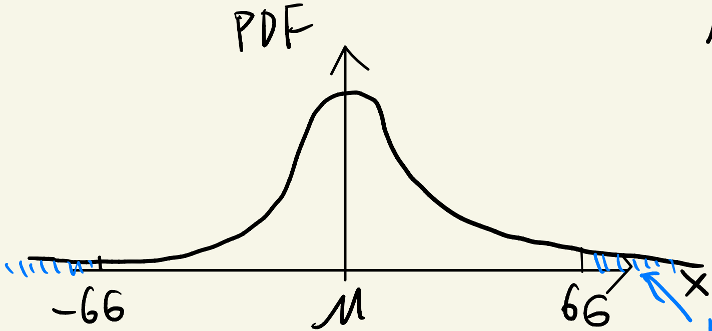
CDF of quality of a product



You want

$P(\text{Defective product}) \le 0.0000034$

(3 defects among 1 million products)



metric of product
(e.g., weight)

$$P(\text{Good product}) = P(\mu - 6\sigma < X < \mu + 6\sigma)$$

$\boxed{\text{Example:}}$ If X is a Gaussian RV with mean 5 and standard deviation $\sigma = 3$.

Find a threshold constant a such that

$$\underbrace{P(X < a)}_{\text{CDF } \Phi(a)} = 0.9.$$

Since $\Phi(a) = P(X < a) = 0.9$,

we know

$$Q(a) = 1 - \Phi(a) = 0.1$$

Check table 4.2

x	$Q(x)$
1.2	0.115
1.3	0.097

We approximate

$\tilde{a} \simeq 1.28$

But note that $Q(x)$ in table is for Gaussian RV with mean 0 and variance 1, so we need to translate the value a to Gaussian RV with mean 5 and variance 3^2.

$$\frac{a-5}{3} = \tilde{a} = 1.28 \Rightarrow a \simeq 8.84.$$

Example

Suppose Gaussian RV with mean 5 and variance 3^2. We want constant b

$$P(7 < X < b) = 0.2.$$

Answer:

$$P(7 < X < b)$$
$$= F_X(b) - F_X(7)$$
$$= \phi\left(\frac{b-5}{3}\right) - \phi\left(\frac{7-5}{3}\right)$$
$$= 1 - Q\left(\frac{b-5}{3}\right) - \left(1 - Q\left(\frac{7-5}{3}\right)\right)$$
$$= \underbrace{Q\left(\frac{2}{3}\right)}_{\text{Check Table}} - Q\left(\frac{b-5}{3}\right)$$
$$= 0.25 - Q\left(\frac{b-5}{3}\right) = 0.2$$
$$\Rightarrow Q\left(\frac{b-5}{3}\right) = 0.05$$

Check table

$$\Rightarrow \frac{b-5}{3} \approx 1.64 \Rightarrow b = 9.92$$

□ Gaussian or other Continous RV also has conditional CDF / PDF, similar to discrete RVs.

Conditional CDF for continuous RVs

$$F_X(x \mid X \in [a,b]) = P(X \le x \mid X \in [a,b])$$

$$= \frac{P(X \le x, X \in [a,b])}{P(X \in [a,b])}$$

$$= \begin{cases} 0 & , \text{if } x < a \\ 1 & , \text{if } x > b \\ \dfrac{P(X \in [a,x])}{P(X \in [a,b])} & ; \text{if } x \in [a,b] \end{cases}$$

$$= \begin{cases} 0 & , \text{if } x < a \\ 1 & , \text{if } x > b \\ \dfrac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)} & , \text{if } x \in [a,b] \end{cases}$$

Likewise, Conditional PDF for continuous RV

$$f_X(x \mid x \in [a,b]) = \frac{d}{dx} F_X(x \mid x \in [a,b])$$

$$= \begin{cases} 0, & \text{if } x > b \text{ or } x < a \\ \dfrac{f_X(x)}{F_X(b) - F_X(a)}, & \text{if } x \in [a, b] \end{cases}$$

☐ Expected values (Mean, Variance) of Continuous RVs

Recall that for discrete RVs,

$$E[X] = \sum_{\underbrace{x_k \in S_X}_{\text{a set of all possible values}}} \underbrace{x_k}_{\text{value of } X} \underbrace{P_X(x_k)}_{\text{PMF}}$$

For continuous RVs, we replace the summation with integration, e.g.,

$$E[X] = \int_{-\infty}^{\infty} \underbrace{y}_{\text{value of } X} \underbrace{f_X(y)}_{\text{PDF of a value } X \text{ can take}} \, dy$$

Warning: We may hand-calculate integration.

Note that the above definition of expected value is not contradictory to the definition for discrete RVs.

We can reduce the $E[X]$ defined for Continuous RVs to $E[X]$ defined for discrete RVs.

Definition $E[X]$ for continuous RVs

$$E[X] = \int_{-\infty}^{\infty} y \underbrace{f_X(y)}\, dy$$

If X is discrete RV

$$\overset{\searrow}{=} \int_{-\infty}^{\infty} y \underbrace{\sum_{x_k \in S_X} P_X(x_k)\, \delta(y - x_k)\, dy}$$

$$= \sum_{x_k \in S_X} P_X(x_k) \int_{-\infty}^{\infty} y\, \delta(y - x_k)\, dy$$

$$= \sum_{x_k \in S_X} x_k P_X(x_k) \leftarrow$$

Definition of $E[X]$ for discrete RVs

$\boxed{\text{Trick of integration}}$ – Integration by parts

$$\int u\, dv = uv - \int v\, du$$

or more generally

$$\int f(x) \underbrace{g'(x)}_{\text{derivative of } g(x)} dx = f(x) g(x) - \int g(x) f'(x)\, dx$$

$\boxed{\text{Example:}}$ If X is a uniform continuous RV on $[a, b]$, what is $E[X]$?

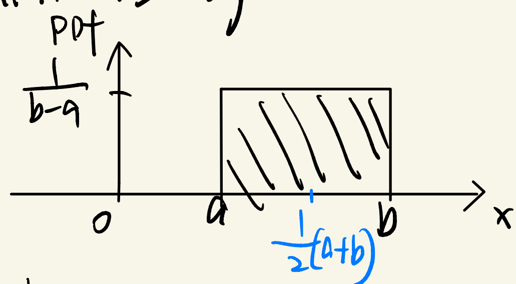
$$E[X] = \int_a^b y f_X(y)\, dy$$

$$= \int_a^b y \frac{1}{b-a}\, dy$$

$$= \frac{1}{b-a} \frac{1}{2} y^2 \Big|_a^b = \frac{1}{b-a}\left(\frac{1}{2}b^2 - \frac{1}{2}a^2\right)$$

$$= \frac{1}{2}(a+b)$$

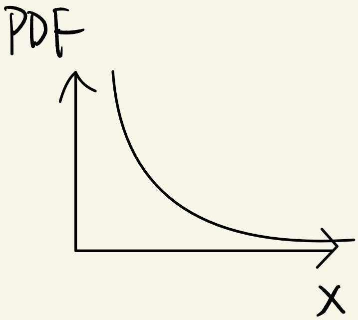
This also make senses since the uniform distribution is symmetric



In general, the mean of any RVs with symmetric PDF/PMF is the mid-point of PDF/PMF!

For other RVs with more involved PDF, we need to use trick of integration by parts.

$\boxed{\text{Example}}$ X is Exponential RV with



PDF $f_X(x) = \lambda e^{-\lambda x}, \quad x > 0$

What is $E[X]$?

$$E[X] = \int_0^\infty \underbrace{y \lambda e^{-\lambda y}}_{u \cdot v'} dy$$

$\boxed{\begin{aligned} u &= y \\ v &= -e^{-\lambda y} \end{aligned}}$

$$= \underbrace{-ye^{-\lambda y}\Big|_0^\infty} - \int_0^\infty -e^{-\lambda y} dy$$

$$= 0 - 0 - \frac{1}{\lambda} e^{-\lambda y}\Big|_0^\infty$$

$$= \frac{1}{\lambda}$$ ← verify the rate of exponential RV.

λ : Expected # of events per time unit

$\frac{1}{\lambda}$: Expected time unit between two events