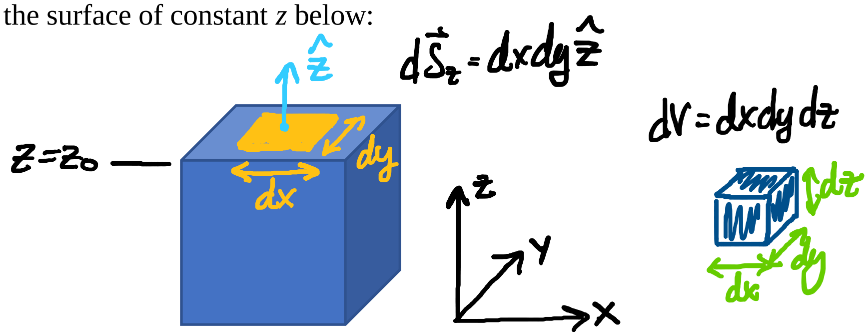
Homework 4

1. Differential Surface Area and Volume Elements

We will be using differential surface area and volume elements in calculations throughout this course, so gaining a better understanding of where they come from can be very helpful. In this homework problem, you'll derive each of the differential surface area and volume elements in cylindrical and spherical coordinates.

The differential surface area elements can be derived by selecting a surface of constant coordinate (z=z_0 in Cartesian coordinates for example) and then varying the other two coordinates to trace out a small 2D shape (x and y in Cartesian coordinates). The surface normal vector will be perpendicular to the surface swept by varying these two coordinates ($\hat{z}$ direction in this case). See the differential surface area element for the surface of constant z below:



The surface area element for the surface of constant z is then given by $d\boldsymbol{S}_z = (dx)(dy)\hat{\boldsymbol{z}}$. Performing the same procedure for a surface of constant x gives $d\boldsymbol{S}_x = (dy)(dz)\hat{\boldsymbol{x}}$ and for a surface of constant y: $d\boldsymbol{S}_y = (dx)(dz)\hat{\boldsymbol{y}}$.

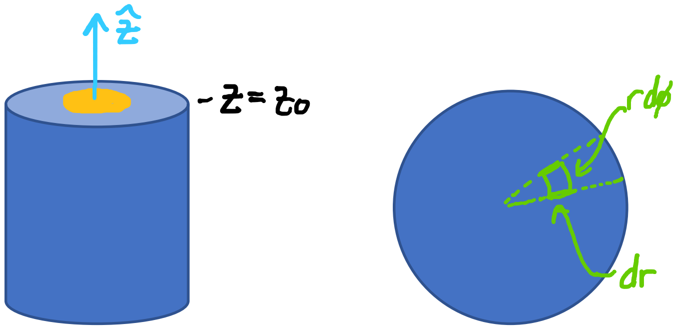
The differential volume element is derived by varying each of the coordinates, then calculating the volume that those variations sweep out in space. In Cartesian coordinates, variations in x, y, and z sweep out a cube with sides of length dx, dy, and dz, giving the differential volume element dV = (dx)(dy)(dz).

For this problem you will do the following in cylindrical and spherical coordinates:

1. Select a coordinate that will be fixed (i.e. z if $z = z_0$)
2. Draw the surface that results from fixing that coordinate and varying the other two coordinates.
3. Determine the area traced out by varying those two coordinates.
4. Write the expression for the differential surface area element d**S**.
5. After you have determined d**S** for the three surfaces of constant coordinate, determine the differential volume element using the same approach as above to determine the length of each side of the differential volume element shape and multiplying them together.

a) Cylindrical coordinates:

i. Surface of constant *z* (Example)

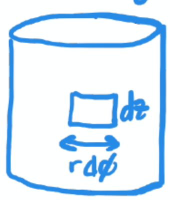


$$\text{Area} = (dr)(r d\phi)$$

$$d\vec{S}_z = r\,dr\,d\phi\,\hat{z}$$

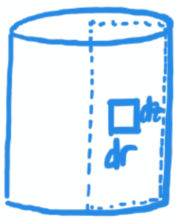
ii. Surface of constant *r*

surface of cylinder with radius r



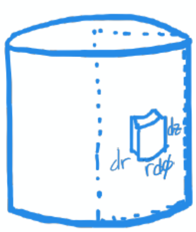
$$d\vec{S}_r = r\,d\phi\,dz\,\hat{r}$$

iii. Surface of constant φ



$$d\vec{S}_\phi = dr\,dz\,\hat{\phi}$$

iv. Differential volume dV



$$dV = r\,dr\,d\phi\,dz$$

b) Spherical coordinates:

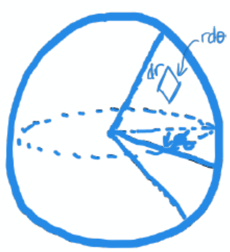
i. Surface of constant r

$$d\vec{S}_r = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$

ii. Surface of constant θ

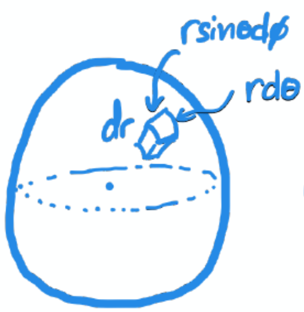
$$d\vec{S}_\theta = r \sin\theta \, dr \, d\phi \, \hat{\theta}$$

iii. Surface of constant φ



$$d\vec{S}_\phi = r \, dr \, d\theta \, \hat{\phi}$$

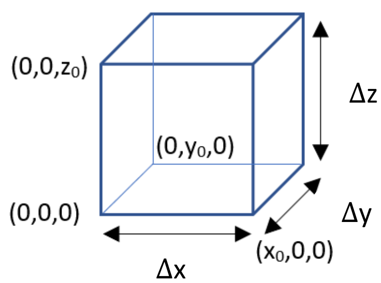
iv. Differential volume dV



$$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

2. Relationship between Divergence and Flux

During lecture, we said that divergence is a "local measure of flux". In this problem, you will relate the idea of total flux through a closed surface to divergence. For this problem, we'll be using the surfaces of a cube as our closed surface. The cube has side lengths Δx, Δy, and Δz.



a) Find the total net flux through the surfaces (all 6) of the cube for a vector field **F** = <F_x, F_y, F_z>. The left-hand side of your equation will be "total net flux" and the right-hand side will be a mathematical expression. For example, the net flux through the surfaces of constant x at $x = 0$ and $x = x_0$ is:

$$\text{Net Flux}_x = \{[\vec{F}(x_0, y, z)\hat{x} \cdot (\Delta y)(\Delta z)\hat{x}] + [\vec{F}(0, y, z)\hat{x} \cdot (\Delta y)(\Delta z)(-\hat{x})]\}$$
$$= [F_x(x_0, y, z) - F_x(0, y, z)]\Delta y \Delta z$$

Total net flux

$$= [\vec{F}(x_0,y,z) - \vec{F}(0,y,z)](\Delta y)(\Delta z)\cdot\hat{x} + [\vec{F}(x,y_0,z) - \vec{F}(x,0,z)](\Delta x)(\Delta z)\cdot\hat{y} + [\vec{F}(x,y,z_0) - \vec{F}(x,y,0)](\Delta x)(\Delta y)\cdot\hat{z}$$

$\Longrightarrow$

Total net flux

$$= [F_x(x_0,y,z) - F_x(0,y,z)]\Delta y\Delta z + [F_y(x,y_0,z) - F_y(x,0,z)]\Delta x\Delta z + [F_z(x,y,z_0) - F_z(x,y,0)]\Delta x\Delta y$$

b) Divide both sides of your equation from a) by $\Delta V = \Delta x\Delta y\Delta z$. Before dividing by ΔV, the left side of the equation should be the "total net flux" and the right side should be your expression from a) for the total net flux through the cube's surfaces.

$$\frac{\text{Total net flux}}{\Delta V} = [F_x(x_0,y,z) - F_x(0,y,z)]/\Delta x + [F_y(x,y_0,z) - F_y(x,0,z)]/\Delta y + [F_z(x,y,z_0) - F(x,y,0)]/\Delta z$$

c) Each of your terms in the right-hand side of the expression for the total net flux should contain an expression of the form $F_x(x_0, y, z) - F_x(0, y, z)$ for each of the coordinates. For each of the coordinates, replace this expression with the equivalent of ΔF_x. This is the change in flux in the x direction over a distance Δx.

$$\frac{\text{Total net flux}}{\Delta V} = \frac{\Delta F_x}{\Delta x} + \frac{\Delta F_y}{\Delta y} + \frac{\Delta F_z}{\Delta z}$$

d) Now, take the limit of Δx, Δy, and Δz as they become infinitesimally small (i.e. $\Delta x \rightarrow \partial x$). Also do this for all the ΔF terms and ΔV terms. You should obtain the mathematical expression for the divergence of the vector field **F** on the right-hand side of your equation.

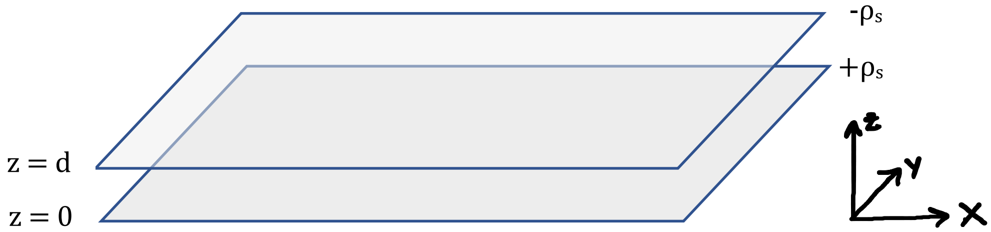
$$\frac{\text{Total net flux}}{dV} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \vec{\nabla} \cdot \vec{F}$$

e) Looking back at your equation, state in your own words what the divergence of a vector field *means* in terms of flux.

Divergence is the local change in flux of a vector field in an infinitesimal volume dV. Alternatively it is the net flux through the surfaces of a closed surface that is infinitesimally small, with volume dV.

3. Electrostatics Between Two Infinite Sheets of Charge

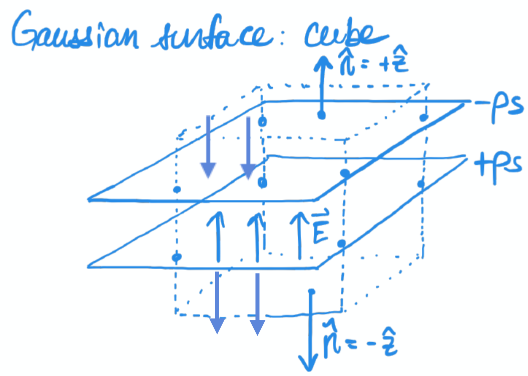
Consider the two infinite sheets of charge below. The top sheet of charge, located at z = d, has a charge density of $-\rho_s$ [C/m^2] and the bottom sheet located at z = 0, has a charge density of $+\rho_s$ [C/m^2].



a) Using symmetry arguments, determine which components of the electric field **E** = <E_x, E_y, E_z> are non-zero.

E has a non-zero component only in the **z** direction due to the **x** and **y** components of the electric field canceling (infinitely many point charges in an infinite plane).

b) Choose a Gaussian surface to use with Gauss's Law to determine the electric field generated by each of the sheets of charge. Draw a diagram that you would use to determine the electric field outside the sheets of charge. Show both sheets of charge, the location of the Gaussian surface, electric field lines, and all relevant surface normal vectors.



c) Use Gauss's Law to determine the electric field outside the sheets of charge.

Gauss's Law

$$\oint \vec{D}\cdot d\vec{s} = Q_{enc}$$

$$= \varepsilon_0 \iint (E_z \cdot \hat{z})\cdot(ds\,\hat{z}) + \varepsilon_0 \iint (E_z\hat{z})\cdot(-ds\,\hat{z})$$

$$= \varepsilon_0 E_z \Delta x \Delta y - \varepsilon_0 E_z \Delta x \Delta y = 0$$

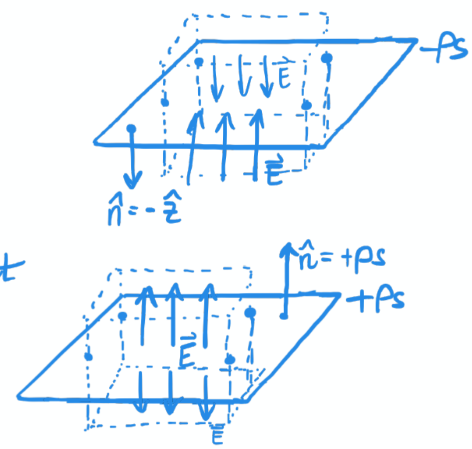
Alternatively: the total enclosed charge is $\Delta x \Delta y \rho_s - \Delta x \Delta y \rho_s = 0$

So $\vec{E} = 0$ for $z<0,\ z>d$

d) For each sheet of charge, draw the sheet of charge, the location of the Gaussian surface you'll use to calculate the electric field from that sheet, the direction of the electric field lines from each sheet, and the relevant surface normal vectors.

Top Sheet

Bottom Sheet



e) Use Gauss's Law to determine the electric field between the sheets of charge (using superposition).

For top sheet:

$$\oint \vec{D}\cdot d\vec{s} = Q_{enc}$$

↳ $Q_{enc} = -\rho_s A$ bottom surface   top surface

↳ $\oint \vec{D}\cdot d\vec{s} = \iint (\varepsilon_0 E_z \hat{z})(-\hat{z}\,ds) + \iint (-\varepsilon_0 E_z \hat{z})\cdot(\hat{z}\,ds)$

$= -2\varepsilon_0 A E_z$

↳ $-2\varepsilon_0 A E_z = -\rho_s A$: $\vec{E}_z^{top} = \frac{\rho_s}{2\varepsilon_0}$

For bottom sheet:

↳ $Q_{enc} = +\rho_s A$

↳ $\oint \vec{D}\cdot d\vec{s} = +2\varepsilon A E_z$

↳ $\vec{E}_z^{bot} = \frac{\rho_s}{2\varepsilon_0}\hat{z}$

$$\vec{E} = \vec{E}_z^{top} + \vec{E}_z^{bot} = \boxed{\frac{\rho_s}{\varepsilon_0}\hat{z}} \text{ for } 0<z<d$$

f) Determine the potential outside and between the sheets of charge.

Potential: $V = -\int \vec{E} \cdot d\vec{l}$

• Outside the sheets of charge: $z<0$, $\vec{E}=0$

So, if $V(-\infty)=0$

$V(0) - \cancel{V(-\infty)}^{0} = -\int_{-\infty}^{0} \vec{E} \cdot d\vec{l}^{\,0} = \underline{0 \quad \text{for } z<0}$

• Between the sheets of charge

$\vec{E}_z = \frac{\rho_s}{\varepsilon_0} \hat{z}$

$V(z) - \cancel{V(0)}^{0} = -\int_0^z \left(\frac{\rho_s}{\varepsilon_0} \hat{z}\right)(dz\,\hat{z}) = -\left[\frac{\rho_s z}{\varepsilon_0}\right]$

$\boxed{V(z) = -\frac{\rho_s z}{\varepsilon_0} \text{ for } 0<z<d}$

• Outside the sheets of charge: $z>d$, $\vec{E}=0$

$V(z) - V(d) = -\int_d^z \vec{E} \cdot d\vec{l}^{\,0}$, so $V(z) = V(d)$ for $z>d$

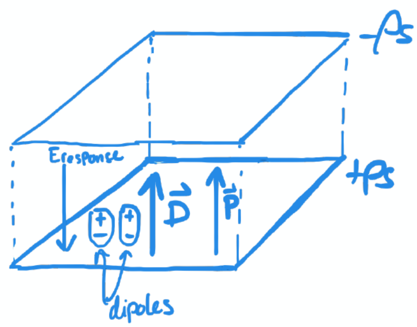
$\boxed{V(z) = -\frac{\rho_s d}{\varepsilon_0} \text{ for } z \geq d}$

4. Dielectrics and the Effect of Polarization

For this problem, you will be investigating the effect of adding a dielectric between the sheets of charge from Problem 3.

a) Replace the air ($\varepsilon = \varepsilon_0$) between the sheets of charge with a dielectric, whose permittivity is given by $\varepsilon = \varepsilon_0\varepsilon_r$, where $\varepsilon_r > 1$. Draw a diagram that includes:

   i. both sheets of charge and their charge densities
   ii. the direction of the displacement field ($\mathbf{D_0} = \varepsilon_0\mathbf{E}$) between the sheets of charge as determined in Problem 3
   iii. the orientation of the dipoles in the dielectric
   iv. the orientation of the electric field generated by the dipoles $\mathbf{E}_{\mathbf{response}}$
   v. the direction of the polarization field $\mathbf{P}$ (remember that $\mathbf{P}$ points in the direction of the electric dipole moment $\mathbf{d}$)



b) Write the expression for the displacement field in the dielectric $\mathbf{D}$ as a function of the electric field generated by the sheets of charge, $\mathbf{E}$, and the polarization field, $\mathbf{P}$.

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$

c) Replace $\mathbf{D}$ with $\varepsilon_0\varepsilon_r\mathbf{E}$ in your expression from b) and solve for the polarization field, $\mathbf{P}$. (Hint: it should be expressed in terms of ε_0, ε_r, and $\mathbf{E}$). What is the polarization field when $\varepsilon_r = 1$? What does this case physically correspond to?

$$\varepsilon_r\varepsilon_0\vec{E} = \varepsilon_0\vec{E} + \vec{P}$$

$$\vec{P} = \varepsilon_0\vec{E}(\varepsilon_r - 1)$$

$\vec{P} = 0$ when $\varepsilon_r = 1$. This is the case of free space, where there are no molecules or atoms to polarize (vacuum).

d) Consider Gauss's Law for both the case of air between the sheets of charge and a dielectric between the sheets of charge. In both cases, the total charge is the same, but the displacement **D** field is different ($D_0 = \varepsilon_0\mathbf{E}$ for air, and $\mathbf{D} = \varepsilon_0\varepsilon_r\mathbf{E}$ with $\varepsilon_r > 1$ for a dielectric).

$$\oint \mathbf{D_0} \cdot d\mathbf{S} = \oint \mathbf{D} \cdot d\mathbf{S} = Q_{enc}$$

What must be true about the magnitude of the electric field in a dielectric (call this $\mathbf{E}_{\mathbf{dielectric}}$) relative to the magnitude of the electric field in air (call this $\mathbf{E_0}$) in order for Gauss's Law to hold? What causes this change in the magnitude of the electric field in a dielectric?

If the total enclosed charge is the same in both cases, then we have

$$\oint \varepsilon_0 \vec{E}_{air} \cdot d\vec{s} = \oint \varepsilon_0 \varepsilon_r \vec{E}_{dielectric} \cdot d\vec{s}$$

If $\varepsilon_r > 1$, then $\vec{E}_{dielectric} < \vec{E}_{air}$

The polarization of the molecules in the dielectric sets up an electric field $\vec{E}_{response}$ that opposes the direction of the applied electric field $\vec{E}$, reducing $\vec{E}$ compared to the case of air / free space.

e) Calculate the potential V_d between the sheets of charge in the dielectric (you may make use of your result from Problem 3). Is it larger or smaller than in the case of an air gap between the sheets of charge?

From problem 3:

$$V(z) = -\frac{\rho_s z}{\varepsilon}, \text{ so } |V_d| = \frac{\rho_s d}{\varepsilon_0 \varepsilon_r}$$

Since $\varepsilon_r > 1$, this potential difference is smaller than in the case of air.

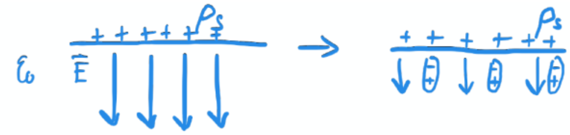
f) Calculate $C = \frac{|Q|}{|\Delta V|}$ for an area A of the sheets and compare it with the answer you would get in free space. You should see that adding a dielectric between the sheets of charge enables the storage of the same amount of charge for a smaller potential difference.

Return to your diagram from part a) and look at the charge distribution in the dielectric near the sheets of charge. How is it possible that the same charge density on the sheets of charge leads to a smaller potential difference between them when a dielectric is the medium? (Expressed in another way, how does inserting a dielectric medium enable the storage of more charge for a given potential difference than air?)

$|Q_{total}|$ for a single sheet: $\rho_s A$

$$|V_d| = \frac{\rho_s d}{\varepsilon_0 \varepsilon_r} \rightarrow C = \frac{\rho_s A \varepsilon_0 \varepsilon_r}{\rho_s d} = \boxed{\frac{\varepsilon_0 \varepsilon_r A}{d}}$$

When the dielectric becomes polarized, the response field reduces the magnitude of the total E field in the dielectric, which in turn, reduces the potential difference between the sheets. The local bound charge density near the sheets however, makes up for the lower E field by redistributing itself to maintain the charge density in the sheets:



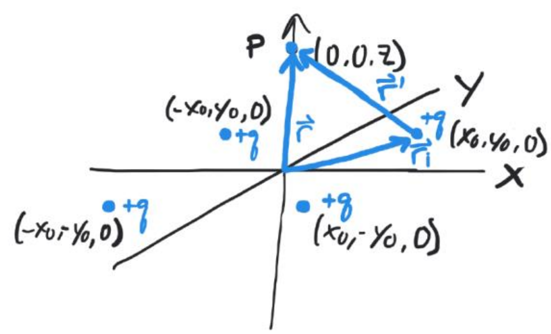
In the case of $\rho_s > 0$, the (-) side of the dipoles ensure that the $\vec{D}$ field (flux) is the same across the boundary:

$$D_0 = \varepsilon_0 \vec{E}_0$$
$$D_d = \varepsilon_0 \varepsilon_r \vec{E}_d$$
$$D_0 A = D_d A = \rho_s A$$
$$\varepsilon_0 \vec{E}_0 = \varepsilon_0 \varepsilon_r \vec{E}_d = \rho_s A$$
$$\varepsilon_0 \vec{E}_0 = \varepsilon_0 \vec{E}_d + \vec{P} = \rho_s A$$
$$\varepsilon_0 \vec{E}_0 = \varepsilon_0 \vec{E}_d - \rho_b \vec{d} = \rho_s A$$

5. Coulomb's Law

A square with vertices $(-x_0, -y_0, 0)$, $(+x_0, -y_0, 0)$, $(-x_0, +y_0, 0)$ and $(+x_0, +y_0, 0)$ has a point charge +q at each vertex. You will calculate the electric field at a location (0, 0, z) and the force on a charge placed at that point. The center of the square is located at (0, 0, 0).

a) Draw a diagram containing each of the 4 charges, the vector **r** pointing from the origin to point P = (0, 0, z), the vector $\mathbf{r}_i$ pointing from the origin to one of the charges at vertex $v_i = (x_i, y_i, 0)$, and the vector pointing from v_i to P. What is the expression for the vector pointing from v_i to P in terms of **r** and $\mathbf{r}_i$?



$$\vec{r}\,' = \vec{r} - \vec{r}_i$$

b) In which direction does the electric field at P = (0, 0, z), due to the four charges point? Consider all values of *z*. Give your reasoning.

Using Symmetry arguments, all $\hat{x}$ and $\hat{y}$ components of the $\vec{E}$ field cancel at (0,0,z)

$$\vec{E}(0,0,z) = \begin{cases} |E_z|\hat{z} & \text{for } z>0 \\ 0 & \text{for } z=0 \\ -|E_z|\hat{z} & \text{for } z<0 \end{cases}$$

c) Write the expression for the magnitude of the electric field at point P = (0, 0, z), generated by a single point charge located at one of the vertices $v_i = (x_i, y_i, 0)$. Express your answer in terms of x_i, y_i and z_0.

·Electric field at P due to a single charge +q at a point $(x_i, y_i, 0)$.

$$\vec{E}(r) = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$$

· We want our radial distance r to be measured from $(x_i, y_i, 0)$ to $P = (0,0,z)$

so $r = |\vec{r}'| = |\langle 0,0,z\rangle - \langle x_i, y_i, 0\rangle|$

$= |\langle -x_i, -y_i, z\rangle|$

$= \sqrt{x_i^2 + y_i^2 + z^2}$

$$|\vec{E}(x_i, y_i, z)| = \frac{q}{4\pi\varepsilon_0 (x_i^2 + y_i^2 + z^2)}$$

d) Write the expression for the electric field at point P = (0, 0, z), due to all charges at the vertices of the square.

$$\vec{E}_P(0,0,z) = |E_P(0,0,z)|\,\hat{r}$$

$$= \left(|E_{q_1}|\hat{r}_{q_1} + |E_{q_2}|\hat{r}_{q_2} + |E_{q_3}|\hat{r}_{q_3} + |E_{q_4}|\hat{r}_{q_4}\right)$$

$$\hat{r} = \frac{\vec{r}'}{|\vec{r}|} = \frac{\langle -x_i, -y_i, z\rangle}{\sqrt{x_i^2 + y_i^2 + z^2}} \rightarrow \hat{r}_{q_1} + \hat{r}_{q_2} + \hat{r}_{q_3} + \hat{r}_{q_4} = \frac{4z}{\sqrt{x_0^2 + y_0^2 + z^2}}\hat{z}$$

$$\boxed{\vec{E}_P(0,0,z) = \frac{qz}{\pi\varepsilon_0 (x_0^2 + y_0^2 + z^2)^{3/2}}\hat{z}}$$

e) If you were to place a point charge +q at P, what is the force **F** felt by the charge?

$$\vec{F} = q\vec{E}$$

$$\boxed{\vec{F} = \frac{q^2 z}{\pi\varepsilon_0 (x_0^2 + y_0^2 + z^2)^{3/2}}\hat{z}}$$

f) If you wanted to maximize the force on the charge by changing its position *z*, where would you place it so that it feels the maximum force? If you were to change the positions of the charges at the corners of the square $\pm x_0$ and $\pm y_0$, where would you place them to maximize the force on the charge?

In terms of z, the maximum force would be felt at:

$$\frac{d\vec{F}}{dz} = 0 = \frac{q^2}{\pi\varepsilon_0}\left\{\frac{(x_0^2+y_0^2+z^2)^{3/2} - \frac{3}{2}z(x_0^2+y_0^2+z^2)^{1/2}(2z)}{(x_0+y_0^2+z^2)^3}\right\}$$

$$= \frac{q^2}{\pi\varepsilon_0}\left\{\frac{1}{(x_0^2+y_0^2+z^2)^{3/2}} - \frac{3z^2}{(x_0^2+y_0^2+z^2)^{5/2}}\right\}$$

$$= \frac{q^2}{\pi\varepsilon_0}\frac{1}{(x_0^2+y_0^2+z^2)^{1/2}}\left\{1 - \frac{3z^2}{(x_0^2+y_0^2+z^2)}\right\} = 0$$

$$\rightarrow \quad 1 - \frac{3z^2}{(x_0^2+y_0^2+z^2)} = 0$$

$$x_0^2+y_0^2+z^2 = 3z^2 \rightarrow x_0^2+y_0^2 = 2z^2$$

$$\boxed{z_{Fmax} = \sqrt{\frac{x_0^2+y_0^2}{2}}}$$

In terms of x_0 and y_0, we can maximize the force by minimizing x_0 and $y_0 \rightarrow \boxed{x_0 = y_0 = 0}$