ECSE 2500
Feb. 6th
Lec 8

Plan: Discrete Random Variable

□ Random variable — Discrete

In many engineering applications, we are interested in some numerical attributes of the outcome, e.g.,

# of heads in 5 coin flips;
# of packets transmission in a given time interval;
# of failure/outage in a residential area

A random variable X is a function that assigns each outcome in the sample space $s \in S$ to a real number $X(s)$

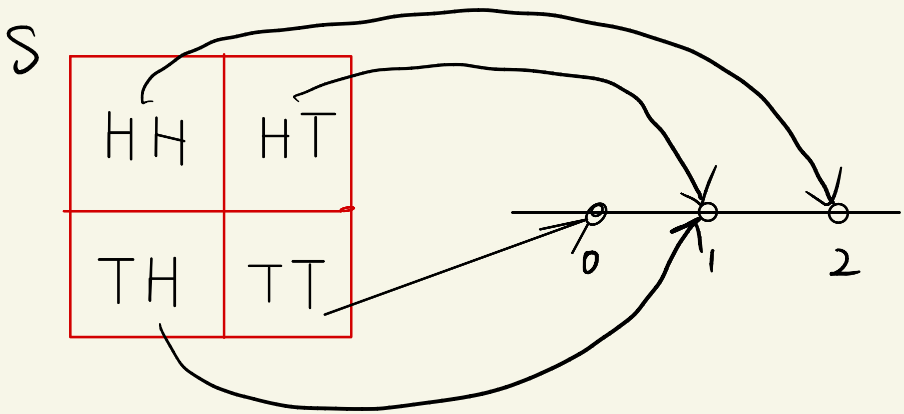
$$\boxed{X : S \to \mathbb{R}}$$

Ex: $S = \{HH, HT, TH, TT\}$

Introduce a random variable X

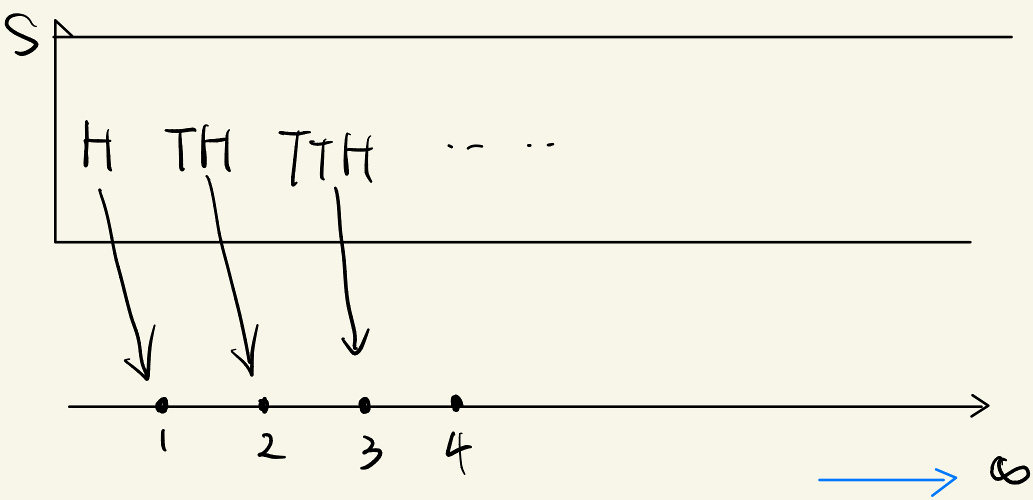
s	$X(s)$
HH	2
HT	1
TH	1
TT	0

Def: A $\boxed{\text{discrete}}$ random variable X is a function from S to a discrete set of values on the real space $\mathbb{R}$ / real line.



- We can have an infinite number of possible values.

Example: Flip a coin until you get a head.
Record the number of times we flipped before that.



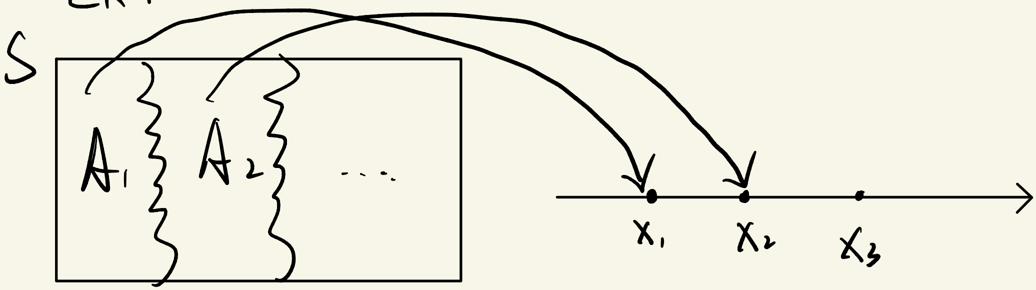
Remember that we defined **probability** on **events**; i.e., subsets of sample space S. But we can see how the variable X inherits this concept.

Example: Event $A := \underbrace{\{X \geq 3\}}_{\text{a set of real values}}$ in flip coins until a head

$$P(A) = P(\{s \mid X(s) \geq 3\})$$
$$= P(\underbrace{\{TTH, TTTH, TTTTH, \cdots\}}_{\text{a set of outcomes}})$$

We can see that the random variable induces a partition of events on S.

Ex:

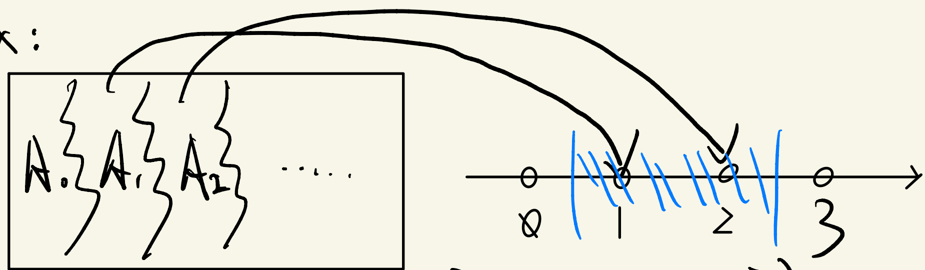


$$A_i = \{ s \in S \mid X(s) = x_i \}.$$

Then for any subset B of the real value, we can define probability $P(X \in B)$ as

$$\boxed{P(X \in B) = P(\{ s \in S \mid X(s) \in B \})}$$

Ex:



$$P(X \in [\tfrac{1}{2}, \tfrac{5}{2}])$$
$$= P(A_1) + P(A_2)$$

Therefore, from now on, we will exclusively talk about random variables since they assign a numerical value to each outcome of experiment and will allow us to talk about averages, variance in an unified way.

☐ The key concept of discrete random variable

— Probability Mass Function (PMF)

The PMF of a discrete variable X is defined as

$$P_X(x) = P(X=x) = P(\{s \mid X(s)=x\}),$$

Big case Small case: value/output
Random variable of random variable

for any $x \in R$

Since we focus on discrete random variable,

$P_X(x)$ is non-zero only at a set of values

$S_X := \{x_1, x_2, x_3, \cdots\}$

We can think of S_X as the sample space of an experiment and we assign probability to events.

Some properties of PMF for discrete random variables:

1) $P_X(x) \geq 0$ for all x

2) $\sum_{x \in S_X} P_X(x) = 1$

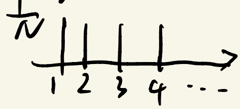
3) $P(X \in B) = \sum_{x \in B} P_X(x)$, for $B \subseteq S_X$

All three properties can be derived from three Axioms.

☐ Seven important random variables with their corresponding PMFs

① Uniform random variable on $\{1, 2, 3, \cdots N\}$

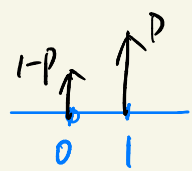
$$P_X(x) = \frac{1}{N}, \qquad x \in \{1, 2, \cdots N\}$$



② Bernoulli random variable: maps successes and failures to 1's and 0's,

$$P_X(0) = 1 - p, \quad P_X(1) = p$$

failure   success



③ Geometric random variable: Measure # of Bernoulli trials before a success.

$$P_X(1) = p$$

$$P_X(2) = (1-p)p \qquad P_X(x) = (1-p)^{x-1}p,$$

$$P_X(3) = (1-p)^2 p \qquad \text{for all } x$$

$$\cdots$$

④ Binomial random variable: Measure # of success in n Bernoulli trials.

$$P_X(k) = P(k \text{ successes in } n \text{ trials})$$

$$= C_k^n \, p^k (1-p)^{n-k}, \quad k = 0, 1, 2 \cdots n$$

Preview: If we don't know the PMF, we can estimate it by doing a large number of random experiments and count

$$\frac{\#\text{ of time } X = x_k}{\underbrace{\#\text{ of experiments}}_{n}} \xrightarrow{n \to \infty} P_X(x_k)$$

⑤ Poisson Random Variable:

ex: # of cars passing in a crossroad

# of phone calls in a given cell

# of packets arriving in an interval

PMF of Poisson RV

$$P_X(k) = \frac{\alpha^k}{k!} e^{-\alpha}, \quad k = 0, 1, 2, \cdots$$

where α is the average number of events in a green interval.

Ex: X is the number of packets arriving in a given 10-minute window;

α is the average number of packets expected arriving in 10-min window.

Estimate from historic data

What is P(more than 4 packets arrived in 10 min) given that average $\alpha = 2$?

$$P(X \geq 4) = 1 - P(X \leq 3) = 1 - \sum_{k=0}^{3} \frac{2^k}{k!} e^{-2}.$$