Rensselaer Polytechnic Institute
Department of Electrical, Computer, and Systems Engineering
ECSE 2500: Engineering Probability, Spring 2023
Homework #8 Solutions

1. (a) First we need to compute the marginal in Y. In the range $y \in [0,1]$,

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx \\ &= \frac{6}{19}\int_0^2 x^2 + y^3\,dx \\ &= \frac{6}{19}\left(\frac{1}{3}x^3 + xy^3\Big]_{x=0}^{x=2}\right) \\ &= \frac{6}{19}\left(\frac{8}{3} + 2y^3\right) \end{aligned}$$

So

$$f_Y(y) = \begin{cases} \frac{16}{19} + \frac{12}{19}y^3 & y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Then the conditional is the joint divided by the marginal:

$$f_{X \mid Y}(x \mid y) = \frac{\frac{6}{19}(x^2+y^3)}{\frac{16}{19} + \frac{12}{19}y^3} = \frac{6x^2+6y^3}{16+12y^3}$$

For $0 \le x \le 2, 0 \le y \le 1$, 0 otherwise. Note that here, we think of y like a fixed number (the observed outcome of Y), and X as a random variable that depends on this number.

Grading criteria: 10 points in total
-0 point: correct
-3 point: the formula is wrong or missing
-3 point: wrong or missing in the process
-3 point: the final answer is wrong or missing
-10 point: completely incorrect or blank

(b) Now we compute the expected value $E(X \mid Y = y)$, which will be a function of y.

$$\begin{aligned} E(X \mid Y = y) &= \int_0^2 x\frac{6x^2+6y^3}{16+12y^3}\,dx \\ &= \frac{1}{16+12y^3}\left(\frac{6}{4}x^4 + 3x^2y^3\Big]_{x=0}^{x=2}\right) \\ &= \frac{24+12y^3}{16+12y^3} \end{aligned}$$

This is only valid for $0 \le y \le 1$, and is 0 otherwise.

Grading criteria: 10 points in total
-0 point: correct
-3 point: the formula is wrong or missing
-3 point: wrong or missing in the process
-3 point: the final answer is wrong or missing
-10 point: completely incorrect or blank

(c) Now we compute $E(X)$ using the law of iterated expectations. We need the marginal PDF of Y computed in part a.

$$\begin{aligned} E(X) &= E(E(X \mid Y)) \\ &= \int_0^1 \frac{24+12y^3}{16+12y^3} \cdot \left(\frac{16}{19} + \frac{12}{19}y^3\right) dy \\ &= \int_0^1 \frac{12}{19}(2+y^3)\, dy \\ &= \frac{12}{19}\left(2y + \frac{1}{4}y^4\Big]_{y=0}^{y=1}\right) \qquad = \frac{12}{19} \cdot \frac{9}{4} \\ &= \frac{27}{19} \end{aligned}$$

Grading criteria: 10 points in total
-0 point: correct
-3 point: the formula is wrong or missing
-3 point: wrong or missing in the process
-3 point: the final answer is wrong or missing
-10 point: completely incorrect or blank

2. (a) $E(Z) = E(X) + E(Y) = 1 + \frac{1}{3} = \frac{4}{3}$.

Grading criteria: 5 points in total
-0 point: correct
-2 point: the formula is wrong or missing
-2 point: wrong or missing in the process
-2 point: the final answer is wrong or missing
-5 point: completely incorrect or blank

(b) Since X and Y are independent, $\text{Var}(Z) = \text{Var}(X) + \text{Var}(Y)$. We can take these variances from our table of common random variables to get $\text{Var}(Z) = \frac{1}{3} + \frac{1}{9} = \frac{4}{9}$.

Grading criteria: 5 points in total
-0 point: correct
-2 point: the formula is wrong or missing
-2 point: wrong or missing in the process
-2 point: the final answer is wrong or missing
-5 point: completely incorrect or blank

(c) To get the full PDF of Z, we need to convolve the PDFs of X and Y... just like in Signals! There will be three cases. When $z < 0$, the PDF is 0. We can solve the other cases using the exponential CDF without doing a lot of integration. When $z \in [0, 2]$, we have partial overlap between the PDFs:

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f_Y(x) f_X(z-x)\ dx \\
&= \int_0^z \frac{1}{2} \cdot 3e^{-3x}\ dx \\
&= \frac{1}{2}(1-e^{-3z})
\end{aligned}$$

When $z > 2$, we have full overlap between the PDFs:

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f_Y(x) f_X(z-x)\ dx \\
&= \int_{z-2}^z \frac{1}{2} \cdot 3e^{-3x}\ dx \\
&= \frac{1}{2} e^{-3z}(e^6 - 1)
\end{aligned}$$

So the overall PDF is:

$$f_Z(z) = \begin{cases} 0 & z < 0 \\ \frac{1}{2}(1-e^{-3z}) & z \in [0,2] \\ \frac{1}{2}e^{-3z}(e^6-1) & z > 3 \end{cases}$$

Grading criteria: 10 points for everyone

3. (a) First we need the mean and variance of a cow's weight:

$$\begin{aligned}
E(X) &= \int_0^2 x \cdot \frac{3}{2}x - \frac{3}{4}x^2\ dx \\
&= \frac{1}{2}x^3 - \frac{3}{16}x^4 \Big]_{x=2}^{x=0} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(X) &= E(X^2) - (E(X))^2 \\
&= \int_0^2 x^2 \cdot \frac{3}{2}x - \frac{3}{4}x^2\ dx - 1 \\
&= \frac{3}{8}x^4 - \frac{3}{20}x^5 \Big]_{x=2}^{x=0} - 1 \\
&= \frac{1}{5}
\end{aligned}$$

Now if we let $S_{100} = \sum_{i=1}^{100} X_i$ and $M_{100} = \frac{1}{100}\sum_{i=1}^{100} X_i$, we want to estimate

$$
\begin{aligned}
P(S_{100} \in [85, 115]) &= P(|S_{100} - 100| \leq 15) \\
&= P(|M_{100} - 1| \leq \frac{15}{100}) \\
&\geq 1 - \frac{1}{100 \cdot \left(\frac{15}{100}\right)^2} \frac{1}{5} \\
&= 1 - \frac{20}{225} \\
&= \frac{205}{225} \\
&= 0.91
\end{aligned}
$$

Here we're using the key Chebyshev bound that

$$P(|M_n - \mu| < \epsilon) \geq 1 - \frac{1}{n\epsilon^2}\sigma^2$$

Grading criteria: 10 points in total
-0 point: correct
-3 point: the formula is wrong or missing
-3 point: wrong or missing in the process
-3 point: the final answer is wrong or missing
-10 point: completely incorrect or blank

(b) The Central Limit Theorem says that

$$P(|S_n - n\mu| \leq z\sigma\sqrt{n}) = 1 - 2Q(z)$$

Here, $\mu = 1$, $\sigma^2 = \frac{1}{5}$, $n = 100$ and $z\sigma\sqrt{n} = 15$. That is, $z = \frac{15}{10}\sqrt{5} = 3.35$. So

$$P(|S_{100} - 100| \leq 15) = 1 - 2Q(3.35) = 1 - 2(0.000404) = 0.999192$$

Grading criteria: 10 points in total
-0 point: correct
-3 point: the formula is wrong or missing
-3 point: wrong or missing in the process
-3 point: the final answer is wrong or missing
-10 point: completely incorrect or blank

(c) i. We can see the actual probability is much higher than we estimated with Chebyshev. The Central Limit Theorem is much more accurate when n is large since it does a good job of actually reflecting the PDF of S_n.

Grading criteria: 5 points in total
-0 point: correct
-2 point: the formula is wrong or missing
-2 point: wrong or missing in the process
-2 point: the final answer is wrong or missing
-5 point: completely incorrect or blank

4. (a) The cost to make the N shipments is:

$$\begin{aligned} E(Z \mid N) &= E(X_1 + X_2 + \cdots + X_N) \\ &= E(X_1) + E(X_2) + \cdots + E(X_N) \\ &= NE(X_i) \\ &= N \cdot \frac{1}{2}(0.3 + 1.2) \\ &= 0.75N \end{aligned}$$

The second step follows from the fact that the X_i's are all independent and identically distributed, and the third step follows from the information that this distribution is uniform on $[0.3, 1.2]$ dollars.

Grading criteria: 10 points in total
-0 point: correct
-3 point: the formula is wrong or missing
-3 point: wrong or missing in the process
-3 point: the final answer is wrong or missing
-10 point: completely incorrect or blank

(b) By the law of iterated expectations,

$$E(Z) = E(E(Z \mid N)) = E(0.75N) = 0.75E(N)$$

What is $E(N)$? It is the expected number of orders in a span of 30 days, where the interval between arrivals is exponentially distributed with mean 5 days (i.e., $\lambda = \frac{1}{5}$). Thus the expected number of arrivals per day is $\frac{1}{5}$ and we expect $30 \cdot \frac{1}{5} = 6$ arrivals in 30 days. This means that N is a Poisson random variable with $\alpha = 6$, which means that $E(N) = \alpha = 6$. Then

$$E(Z) = \$0.75 \cdot 6 = \$4.50$$

Grading criteria: 8 points in total
-0 point: correct
-3 point: the formula is wrong or missing
-3 point: wrong or missing in the process
-2 point: the final answer is wrong or missing
-8 point: completely incorrect or blank

(c) Since it costs RTR \$4.50 on average for the raw monthly shipping and handling, they should charge \$9.50 per month to make a \$5 profit.

Grading criteria: 7 points in total
-0 point: correct
-3 point: the formula is wrong or missing
-2 point: wrong or missing in the process
-2 point: the final answer is wrong or missing
-7 point: completely incorrect or blank