ECSE 2500

Lec 20

April 13

Topic: Covariance of two RVs

Conditional PDF of random variable

- [ ] Given a joint PDF/PMF/CDF of two RVs, what to determine if they are correlated?

→ Compute Covariance $Cov(X, Y) = 0$?

Example: $f_{X,Y}(x, y) = x + y$, if $x, y \in [0, 1]$.

What is $Cov(X, Y)$? Correlated?

Answer: Using the definition of Cov.

$$Cov(X, Y) = E[XY] - E[X]E[Y]$$

① $E[XY]$

$$= \int_0^1 \int_0^1 xy\, f_{XY}(x,y)\, dx\, dy$$

$$= \int_0^1 \int_0^1 xy\,(x+y)\, dx\, dy$$

$$= \int_0^1 \int_0^1 x^2 y\, dx\, dy + \int_0^1 \int_0^1 x y^2\, dx\, dy$$

$$= \int_0^1 \frac{1}{3} x^3 y \Big|_{x=0}^{x=1} dy + \int_0^1 \frac{1}{3} x y^3 \Big|_{y=0}^{y=1} dx$$

$$= \frac{1}{3}$$

② $E[X]$

$$= \int_0^1 x\, f_X(x)\, dx$$

$$= \int_0^1 x \underbrace{\int_0^1 f_{X,Y}(x,y)\, dy}_{f_X(x)}\, dx$$

$$= \int_0^1 \int_0^1 x(x+y)\, dx\, dy = \int_0^1 \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 y\right)\Big|_0^1 dy = \frac{7}{12}$$

③ $E[Y]$

$$= \int_0^1 y f_Y(y)\, dy$$

$$= \int_0^1 y \underbrace{\int_0^1 f_{X,Y}(x,y)\, dx}_{f_Y(y)}\, dy$$

$$= \int_0^1 y \int_0^1 (x+y)\, dx\, dy = \frac{7}{12}$$

Then $\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$

$$= \frac{1}{3} - \left(\frac{7}{12}\right)^2 \neq 0$$

They are correlated!

Note: By checking the Cov value above, you may find $\mathrm{Cov}(X,Y) < 0$.

This is in contrast with $\mathrm{Var}(X) \geq 0$.

☐ Correlation coefficients (e.g., ρ in Gaussian)

We often normalize the covariance of two RVs so that we pay less attention on their magnitude and immediately tell how related two RVs are.

Definition of correlation coefficients are

$$\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)}\ \sqrt{Var(Y)}} = \frac{Cov(X,Y)}{\sigma_X \ \sigma_Y}$$

The nice thing about this definition is that $\rho_{X,Y} \in [-1, 1]$

To see this point, we can consider

$$E\left[\left(\frac{X-\mu_X}{\sigma_X} - \frac{Y-\mu_Y}{\sigma_Y}\right)^2\right] \geq 0$$

$$= E\left[\left(\frac{X-\mu_X}{\sigma_X}\right)^2 + \left(\frac{Y-\mu_Y}{\sigma_Y}\right)^2 - \frac{(X-\mu_X)(Y-\mu_Y)}{\sigma_X \sigma_Y}\right]$$

$$= E\left[\left(\frac{X-\mu_X}{\sigma_X}\right)^2\right] + E\left[\left(\frac{Y-\mu_Y}{\sigma_Y}\right)^2\right] - 2\rho_{XY}$$

$$= \frac{E[(X-\mu_X)^2]}{\sigma_X^2} + \frac{E[(Y-\mu_Y)^2]}{\sigma_Y^2} - 2\rho_{XY}$$

$$= 2 - 2\rho_{XY} \geq 0 \Rightarrow \rho_{XY} \leq 1$$

$$\Rightarrow \text{Show } E\left[\left(\frac{X-\mu_X}{\sigma_X} + \frac{Y-\mu_Y}{\sigma_Y}\right)^2\right] \geq 0$$

$$\Rightarrow \rho_{XY} \geq -1$$

Q: How to intepret ρ_{XY}?

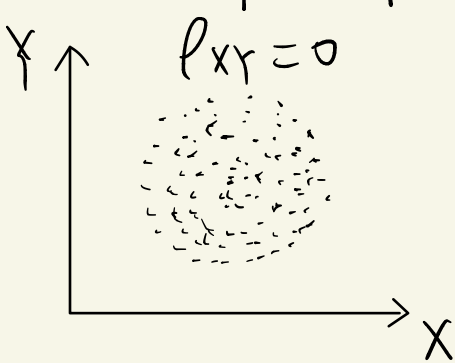
$\rho_{XY} = 0 \Rightarrow Cov(X, Y) = 0$, X, Y uncorrelated

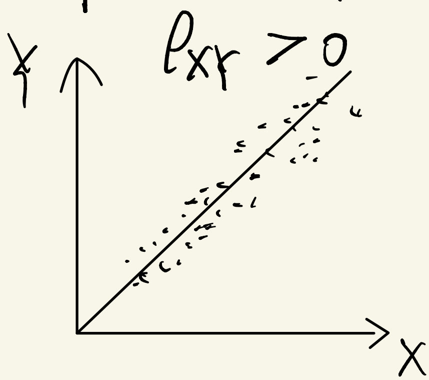
$\rho_{XY} = \pm 1 \Rightarrow Y$ is completely predictable given X

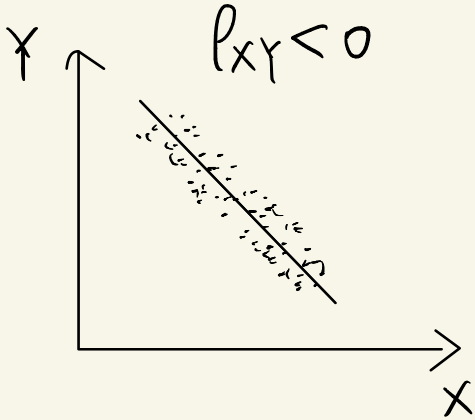
$\rho_{XY} > 0 \Rightarrow$ If X is above its mean, Y is likely also above its mean.

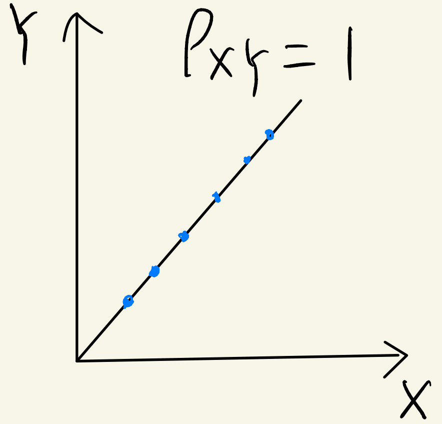
$\rho_{XY} < 0 \Rightarrow$ If X is above its mean, Y is likely below its mean.

One way to visualize this is to look at a scatterplot of values of X and Y:



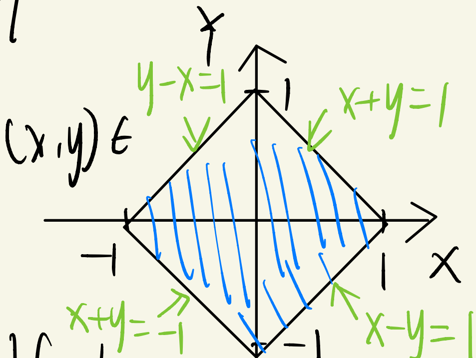






Example: Given joint PDF

$$\boxed{f_{X,Y}(x,y) = \frac{1}{2}}, \ \forall \ (x,y) \in$$



Q: What $Cor(X,Y)$? If they are correlated?

A: $$E[XY] = \int_{-1}^{+1} \int_{-(1-x)}^{(1-x)} xy \cdot \frac{1}{2} \, dy \, dx$$

$$+ \int_{-1}^{+\infty} \int_{-(x+1)}^{(x+1)} xy \cdot \frac{1}{2} \, dy \, dx$$

$$= \int_0^1 \frac{1}{2} x \cdot \frac{1}{2} y^2 \Big|_{-(1-x)}^{1-x} dx + \int_{-1}^{0} \frac{1}{2} x \cdot \frac{1}{2} y^2 \Big|_{-(x+1)}^{(x+1)} dx$$

$= 0 + 0 = 0$

$$E[X] = \int_{-1}^{+1} f_X(x)\, dx$$

$$f_X(x) = \int_{-1}^{+1} f_{X,Y}(x,y)\, dy$$

skip derivation

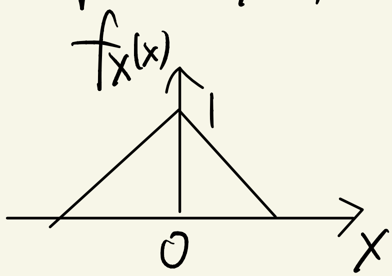
$$= 1 - |x|, \quad \text{if } -1 \le x \le 1$$

Likewise, we can find

$$f_Y(y) = 1 - |y|, \quad \text{if } -1 \le y \le 1$$

$$\Rightarrow E[X] = 0$$

since PDF symmetric around 0.



$$E[Y] = 0$$

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y] = 0$$

Note: X and Y are uncorrelated!

Q: Are they independent?

Necessary conditions of independent RVs

✓ $f_{X,Y}(x,y) = f_X(x) f_Y(y)$

$F_{X,Y}(x,y) = F_X(x) F_Y(y)$

Check $f_{X,Y}(\frac{3}{4}, \frac{1}{4}) = \frac{1}{2}$

$f_X(\frac{3}{4}) \cdot f_Y(\frac{1}{4}) = \frac{1}{4} \cdot \frac{3}{4}$

$f_{X,Y}(x,y) \neq f_X(x) f_Y(y)$

□ Conditional PDFs.

We have talked about conditional probability in terms of events in the sample space

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Also if B_i is a partition of sample space S

$$\left(\bigcup_{i=1}^{n} B_i = S, \; B_i \cap B_j = \phi \right)$$

then

$$P(A) = \sum_{i=1}^{N} P(A|B_i) P(B_i)$$

(Law of total probability)

Now we are interested in

$$P(Y \in A \mid X = x)$$

We first assume X is discrete, so $P(X = x_k) > 0$ for $S_X = \{x_1, x_2, \cdots x_k\}$.

If Y is also a discrete RV, we can look at their joint PMF and determine the conditional PMF, e.g.,

$$P(Y=y_j \mid X=x_k) = \frac{P(X=x_k \text{ and } Y=y_j)}{P(X=x_k)}$$

Using the marginal/joint PMF notions,

$$P_{Y|X}(y_j \mid x_k) = \frac{P_{X,Y}(x_k, y_j)}{P_X(x_k)} = \frac{\text{joint}}{\text{Marginal}}$$

Example: Recall the experiment of coin toss 3 times. Let

X = # of heads over 3 times

Y = position of first head

Outcome	HHH	HHT	HTH	THH	HTT	THT	TTH	TTT
X	3	2	2	2	1	1	1	0
Y	1	1	1	2	1	2	3	0

We can compute marginal PMFs

x	$P_X(x)$
0	$\frac{1}{8}$
1	$\frac{3}{8}$
2	$\frac{3}{8}$
3	$\frac{1}{8}$

y	$P_Y(y)$
0	$\frac{1}{8}$
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$

We can compute joint PMF

(x,y)	$P_{X,Y}(x,y)$
$(0,0)$	$\frac{1}{8}$
$(1,1)$	$\frac{1}{8}$
$(1,2)$	$\frac{1}{8}$
$(1,3)$	$\frac{1}{8}$
$(2,1)$	$\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$
$(2,2)$	$\frac{1}{8}$
$(3,1)$	$\frac{1}{8}$
else	0

We can use joint PMFs and marginal PMFs to compute conditional PMFs.

$P(Y|X)$

$X \backslash Y$	0	1	2	3
0	$P(Y=0\|X=0)$			
1	$P(Y=0\|X=1)$			
2				
3				

$P(X|Y)$

$Y \backslash X$	0	1	2	3
0				
1	$P(X=0\|Y=1)$			
2				
3				