MATH-2400 DIFFEQ F 2022 Crib Sheet Exam 3 Friday, December 9, 2022 Hayden Fuller Notes:

SECOND ORDER: $y'' + p(t)y' + q(t)y = g(t)\,;\ y(t) = C_1 y_1(t) + C_2 y_2(t)$

wronskion: $w(t) = det[y_1, y_2, //, y_1', y_2'] = y_1 y_2' - y_1' y_2 \neq 0$ (linearly independent, not multiples of each other)

$L[y_1] = y_1'' + p(t)y_1' + q(t)y = g(t)$

CONSTANT COEFFICIENT: $ay'' + by' + cy = 0\,;\ y(t) = e^{rt}\,;\ ar^2 + br + c = 0$

CASE 1 $r_1, r_2 \in R,\ r_1 \neq r_2\,;\ b^2 - 4ac > 0\,;\ y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

CASE 2 $r_1, r_2 \in R,\ r_1 = r_2\,;\ b^2 - 4ac = 0\,;\ y(t) = C_1 e^{rt} + C_2 t e^{rt}$

REDUCTION OF ORDER:

$y_2 = y_1 h\,,\ y_2' = y_1' h + y_1 h'\,,\ y_2'' = y_1'' h + 2y_2' h' + y_1 h''$

$y_1'' h + 2y_2' h' + y_1 h'' + p(t)(y_1' h + y_1 h') + q(t)(y_1 h) = (y'' + py' + qy)h + (2y' + py)h' + yh''$ (should) $= (2y' + py)h' + yh''$,

$u = h'\,;\ yu' + (2y' + py)u = 0$

get u to one side, integrate to find u (has C); integrate to get h (has D) and plug in to $y_2 = y_1 h$; choose C and D to be easy.

CASE 3 $r_1, r_2 \notin R\,;\ b^2 - 4ac < 0\,;\ \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = r = \lambda \pm i\omega\,;$

$y_1^c = e^{\lambda + i\omega} = e^{\lambda t}(\cos(\omega t) + i \sin(\omega t))$

$y_2^c = e^{\lambda - i\omega} = e^{\lambda t}(\cos(\omega t) - i \sin(\omega t))$

$y_1(t) = e^{\lambda t}\cos(\omega t) \qquad y_2(t) = e^{\lambda t}\sin(\omega t)$

CAUCHY-EULER

$ax^2 y'' + bxy' + cy = 0\,;\ y = x^r\,;\ ar(r-1) + br + c = 0$ roots $r_1\,, r_2$

CASE 1: $y = C_1 x^{r_1} + C_2 x^{r_2}$

CASE 2: $y = C_1 x^r + C_2 x^r \ln(x)$

CASE 3: $y = C_1 x^\lambda \cos(\omega \ln(x)) + C_2 x^\lambda \sin(\omega \ln(x))$

polar: $y = Re^{\lambda t}\cos(\omega t - \phi)\,;\ y(t) = (R\cos(\phi))e^{\lambda t}\cos(\omega t) + (\sin(\phi))e^{\lambda t}\sin(\omega t)$

$R\cos\phi = C_1\,,\ R\sin\phi = C_2$

METHOD OF UNDETERMINED COEFFICIENTS (must be constant coefficient)

$ae^{bt} \to Ae^{bt}$

$a\cos(ct) + b\sin(ct) \to A\cos(ct) + B\sin(ct)$

$at^n \to A_{n+1}t^n + \ \dots\ + A_1$

$g(t) = P_n(t)e^{at}(\alpha\cos(bt) + \beta\sin(bt))\ ;\ y_p(t) = Q_n(t)e^{at}\cos(bt) + R_n(t)e^{at}\sin(bt)$

addition of $g_1\,,\ g_2 \dots$ results in addition of solutions

you just guessed y_p, derive y_p' and yh_p'', multiply t if resonance

plug those in to $L[y_p] = g$ and solve for A's and B's; sets of (A's and B's) for each term

plug those into guess for y_p and $y(t) = y_h + y_p$, plug in ICs to solve for $C_1,\ C_2$

VARIATION OF PARAMETERS (must be in standard form)

must know homo $y_1,\ y_2\ ;\ y_p(t) = u_1 y_1 + u_2 y_2\ ;\ u_1' = \frac{-y_2 g}{W}\,; u_2' = \frac{y_1 g}{W}$

$u = \int u' dt + A_n$, plug in to get y_p, choose $A_1,\ A_2$ to make it easy.

LINEAR OSCILATOR

$mu'' + cu' + ku = F_0\cos(\omega t)$

$\omega_0 = \sqrt{\frac{k}{m}}\ ;\ kx = mg\ ;\ c = \frac{F}{v}\ ;\ r = \pm\sqrt{\frac{-k}{m}}$

$u(t) = e^{\lambda t}(C_1\cos(\cos(\omega t) + \sin(\omega t))$

undetermined coefficients oscilator: $D = c^2\omega^2 + (k - m\omega^2)^2$

$u_p(t) = A\cos(\omega t) + B\sin(\omega t)\,;\ u_p' = -A\omega\sin(\omega t) + B\omega\cos(\omega t)\,;\ u_p'' = -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t)$

max amplitude? $R' = 0\,,\ D = 0\,,\ R = \sqrt{A^2 + B^2}\,,\ \omega$ sorta close to ω_0

FREE UNDAMPED

$u(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) = R\cos(\omega_0 t - \phi)$

$C_1 = u(0) = R\cos(\phi)\,;\ C_2 = \frac{u'(0)}{\omega_0} = R\sin(\phi)$

$= \frac{2\pi}{\omega_0}$; frequency $= \frac{\omega_0}{2\pi} = \frac{1}{period}$; amplitude $= R = \sqrt{C_1^2 + C_2^2}\ ;\ \phi = \arctan(\frac{C_2}{C_1})$

FREE DAMPED

$u = e^{rt}\,;\ r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$; sqrt: ($< 0$ under, not strong, overshoots, disipating wave) ($= 0$ cryticially, perfect, decays as fast as possible without overshooting) (> 0 over, too strong, slower, slow decay

FORCED UNDAMPED

$\omega_0 \neq \omega$ small wave. $\omega_0 \approx \omega$ bigger wave. $\omega_0 = \omega$ resonance, grows linearly

FORCED DAMPED

$u_p(t) = A\cos(\omega t) + B\sin(\omega t)$
LAPLACE TRANSFORM
$F(s) = \int_0^\infty f(t)e^{-st}dt$
linear, $L(af(t) + bg(t)) = aL(f(t)) + bL(g(t))$
$ay'' + by' + cy = f(t)$; $a(s^2Y - sy_0 - y_0') + b(sY - y_0) + c(Y) = F$; $Y = \frac{F+asy_0+ay_0'+by_0}{as^2+bs+c}$

BOUNDARY VALUE PROBLEMS
given DE $y'' + Ay' - 3y = f(x)$ and boundary conditions such as $y(0) = 0$ or $y'(1) = 0$
homo sols $y = e^{rt}$; $y_h = C_1e^{r_1t} + C_2e^{r_2t}$; particular sol, method of undetermined coefficients
apply boundary conditions to solve for C_1 and C_2
EIGENVALUE PROBLEM
find general sol, $y = e^{rt}$; $y'' + \lambda y = 0$; $y(0) = 0$; $y(L) = 0$; apply BCs,what λs sols exist for,
$\lambda = p^2 = (\frac{n\pi}{L})^2$; check for trivials at zero and stuff
HEAT EQUATION
$u_t = Du_{xx}$; $0 \le x \le L$; $t \ge 0$; $u(0,t) = 0$; $u_x(L,t) = 0$; $u(x,0) = f(x)$
let $u(x,t) = F(x)G(t)$; $u_t = FG'$; $u_{xx} = F''G$; $FG' = DF''G$; $\frac{G'}{DG} = \frac{F''}{F} = -\lambda$; $F'' + \lambda F = 0 = G' + \lambda DG$
apply BCs to find BCx and BCt ; $u(0,t) = 0 = F(0)G(t)$; $F(0) = 0$; $u_x(L,t) = 0 = F'(L)G(t)$; $F(L) = 0$,
apply these BCs to functions of F and G; $F(x) = \sin(\frac{n\pi x}{L})$; $G(t) = Ae^{-(\frac{n\pi}{L})^2Dt}$; $u = GF$
$u(x,t) = \sum_{n=1}^\infty A_n e^{-(\frac{n\pi}{L})^2Dt}\sin(\frac{n\pi x}{L})$; apply IC $u(x,0) =?$; build ? of parts from sum
with A_n 's, build $u(x,t)$
FOURIER SINE SERIES $S(x) = \sum_{n=1}^\infty b_n\sin(\frac{n\pi x}{L})$; $b_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx$; average at jumps
FOURIER COS SERIES $C(x) = \frac{1}{2}a_0 + \sum_{n=1}^\infty a_n\cos(\frac{n\pi x}{L})$; $a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})dx$
WAVE EQUATION
$u_{tt} = c^2u_{xx}$; $0 \le x \le L$; $t \ge 0$; $u(x,0) = f(x)$; $u_t(x,0) = g(x)$; $u_{tt} = FG''$; $u_{xx} = F''G$; $\frac{G''}{c^2G} = \frac{F''}{F} = -\lambda$
$F'' + \lambda F = 0$; $G'' + \lambda c^2G = 0$; $\lambda = (\frac{n\pi}{L})^2$; $G'' + (\frac{n\pi c}{L})^2G = 0$; $G = e^{rt}$; $r^2 + (\frac{n\pi c}{L})^2 = 0$; $r = \pm i(\frac{n\pi c}{L})$
$G = A\cos(\frac{n\pi c}{L}t + B\sin(\frac{n\pi c}{L}t)$; $u(x,t) = \sum +n = 1^\infty(A\cos(\frac{n\pi c}{L}t + B\sin(\frac{n\pi c}{L}t))\sin(\frac{n\pi x}{L})$
apply ICs $u(x,0) = \sum_{n=1}^\infty A_n\sin(\frac{n\pi x}{L}) = f(x)$; $A_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx$
$u_t(x,t) = \sum_{n=1}^\infty(-\frac{n\pi c}{L}A_n\sin(\frac{n\pi c}{L}t) + \frac{n\pi c}{L}B_n\cos(\frac{n\pi c}{L}t))\sin(\frac{n\pi x}{L})$; $u_t(x,0) = \sum_{n=1}^\infty \frac{n\pi c}{L}B_n\sin(\frac{n\pi x}{L}) = g(x)$
$B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin(\frac{n\pi x}{L}dx$
LINEAR ALGEBRA
$p(\lambda) = \det(A - \lambda I) = \lambda^2 - (a+d)\lambda + \det(A)$; $(A - \lambda_nI)x_n = 0$; $x_n = [\alpha \;\; \beta]^T$
IVP: $x' = [1 \;\; -2/n3 \;\; -4]x$; $x(0) = [-2 \;\; 1]^T$; find eigenvals r_n and vectors z_n ; $x(t) = C_1z_1e^{r_1t} + C_2z_2e^{r_2t}$
apply IC to find C_n ; PHASE PLOT: REAL: $x(t) = u(t) + v(t)$; $x = u(t)$; $y = v(t)$; $r_1 < 0 < r_2$, saddle;
$r_1 < r_2 < 0$, sink, tangent r_2 at 0 ; $0 < r_1 < r_2$, source, tangent r_1 at 0
IMAGINARY: $r = \lambda \pm \mu i$; $a =$ real part of z_n ; $b =$imaginary part of z_n ; $x_1 = (a+ib)e^{(\lambda+i\mu)t}$
$u(t) = (a\cos(\mu t) - b\sin(\mu t))e^{\lambda t}$; $v(t) = (b\cos(\mu t) + a\sin(\mu t))e^{\lambda t}$
$\lambda > 0$, source; $\lambda < 0$, sink; $\lambda = 0$, center; a¿-b¿-a¿b¿a?; bottom left positive = counter clockwise.