ECSE 2500
Lec 4
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Topic: Applications of Axioms of probability and conditional probability

□ Sanity check

The events $\{X=1\}, \{X=2\}, \cdots \{X=n\}, \cdots$ are all disjoint, and therefore from Axiom 3, we have

$$P(S) = \sum_{n=1}^{\infty} P(X=n)$$

$$= \sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right)$$

$$= \frac{1}{6} \times \sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^{n-1}$$

$$= \frac{1}{6} \times \frac{1}{1-\frac{5}{6}} = \frac{1}{6} \times 6$$

$$= 1$$ ← Satisfy Axiom 2

Here we used the important formula for the sum of a Geometric Series

$$\boxed{\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \text{ given } 0 < a < 1.}$$

These kinds of counting problems can be approached with ideas from $\boxed{\text{Combinatorics.}}$

Key things to know:

$\boxed{\text{1. k-tuples}}$. Problems where there are n_1 choices for the first variable, n_k choices for the k th variable. Then there are in total

$$n_1 \times n_2 \times \cdots \times n_k = \prod_{i=1}^{k} n_i$$

distinct ordered k-tuples.

$$\begin{array}{ccccccc} n_1 & & n_2 & & n_3 & & n_k \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (\quad & , & & , & & , \; \cdots \; , & \quad) \\ \text{1st} & & \text{2nd} & & \text{3rd} & & k\text{th} \end{array}$$

$\boxed{\text{Example}}$ License plate: 3 letters followed by 4 digits. How many possible license plates?

DMV New York State

$\boxed{\text{A B C 1 2 2 4}}$

26 possibles 26 26 10 10 10 10 possibilities

$26^3 \times 10^4$ in total license plates

Why this is related to probability?

Q: What is the probability a randomly selected license plate is of the form

$$\boxed{A\_\ \_\ \_\ 7\_\ \_}$$?

We can tackle this problem by computing

$$\text{Prob} = \frac{\text{number of plates having } AXXX7XX}{\text{total number of license plates}}$$

$$= \frac{26^2 \times 10^3}{26^3 \times 10^4}$$

$$= \frac{1}{260}$$

2. Ordered samples with replacement.

Choose k objects from a set of n with replacement, i.e., pick objects, observe its properties, put it back (so it can be selected again next time). Then applying (1) there are n^k ordered k-tuples.

$$(Ob\,1,\ Ob\,2,\ \ldots,\ Ob\,k)$$

$\uparrow$ n choices $\uparrow$ n $\uparrow$ n choices

Example We choose a password by randomly selecting 4 8-bit ASCII characters. How many passwords are there?

$$(\text{Password}\,1,\ Ps\,2,\ Ps\,3,\ Ps\,4)$$

$\uparrow$ 2^8 choices $\uparrow$ 2^8 $\uparrow$ 2^8 $\uparrow$ $2^8 = 256$

8 bits

$(256)^4$ total number of passwords.

3. Ordered samples without replacements

Choose k objects from a set of n objects.
But object cannot be chosen again after
it is picked. So # of possible objects
decreases by 1 each time we select an object.

By Model (1), there are

ordered samples without replacements

$$n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)$$

$$= \frac{n!}{(n-k)!} = \frac{n \cdot (n-1) \cdot \cdots \cancel{\cdot} \cancel{1}}{\cancel{(n-k)} \cdot \cancel{(n-k-1)} \cdot \cancel{1}}$$

Example

We choose a password by randomly selecting
4 8-bit ASCII characters such that
no characters can be used twice.
How many possible passwords?

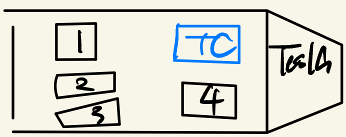
$$(256) \cdot (255) \cdot (254) \cdot (253)$$

4. Permutations of objects

How many ways are there to arrange n distinct objects? Like the previous problem with $k=n$ (i.e., keep drawing from n objects without replacements until no more options).

There are $n!$ permutations of n distinct objects.

$\boxed{\text{Example}}$ A car has 4 seats. How many ways are there for 4 people to sit in this car?



$4 \times 3 \times 2 \times 1$ arrangements

5. Sampling without replacement or ordering

This is for problems where we don't care about the order of the objects and is typically posed as

" How many ways are there to choose k out of n objects ?

Without ordering, "1 2 3" and "1 3 2" are same

We can use the notation

$\boxed{C_k^n}$ to denote " n choose k "

$$C_k^n = \frac{n!}{(n-k)!\,k!}$$ much smaller than $\frac{n!}{(n-k)!}$

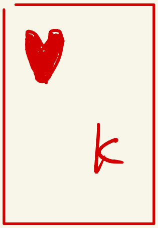
Ordered samples

Example

How many 3-person groups can be formed from a pool of 10 candidates ?

$$C_3^{10} = \frac{10!}{7!\,3!} = \frac{10 \times 9 \times 8}{3 \times 2} = 120$$

Example Choose 5 cards at random from the deck of 52 cards. What is the probability that exactly 2 are Hearts?



13 cards per color

There are C_5^{52} ways to chose 5 cards out of 52 cards.

There are C_2^{13} ways of selecting 2 cards from all 13 heart cards.

There are C_3^{39} ways of selecting 3 non-heart cards.

Therefore, the probability

$$P(\text{exactly 2 cards out of 5 cards are heart}) = \frac{C_2^{13}\, C_3^{39}}{C_5^{52}}$$

□ Assign probabilities to events in a continous sample space (Next)

So far we could enumerate and assign a probability to each discrete outcome.

When we have a continuous sample space, we cannot assign a probability to an exact outcome; so we must assign a probability to a set of outcomes.

Example: Randomly pick a continuous number between 0 and 1.

What is the probability $P(\frac{1}{2})$?

It must be 0 ! It seems counterintuitive at first glance. But think about it. Suppose each real number was equally likely to be selected. Since there are infinitely many numbers in $[0,1]$, the sum of probabilities will be infinite! This violates Axiom 2 .