Fields and Waves I

Lecture 11

Electric Force, Potential and Voltage

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Materials from other sources are referenced where they are used.
Those listed as Ulaby are figures from Ulaby's textbook.

Exam 1

- Rework exams will be offered this week on the following skills:
  Skill 1c (phasors)
  Skill 1f (input impedance)
- **New slots available!**

Electrostatics

$$\nabla \cdot \vec{D} = \rho$$

We had stated that the points that contain charge have a non zero divergence, whereas the points with no charge have a divergence of zero. *(This is the differential form of Gauss's Law.)*

Electrostatics

$$\oint \vec{D} \cdot d\vec{S} = \int \rho dV$$

This could take the form of point charges or line, surface, or volume charge density

The integral form of Gauss's Law states that the flux through the circles is determined by the amount of charge inside. This can be rewritten as:

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q_{enc}}{\epsilon}$$

for empty space, $\epsilon = \epsilon_0$ (8.85e-12 farads per meter)

Electrostatics

Coulomb's Law

$\vec{E}$,of Q_1 is $= \dfrac{Q_1}{4\pi\varepsilon_0 R^2} \cdot \hat{a}_R$

Unit vector pointing away from Q_1

Then,

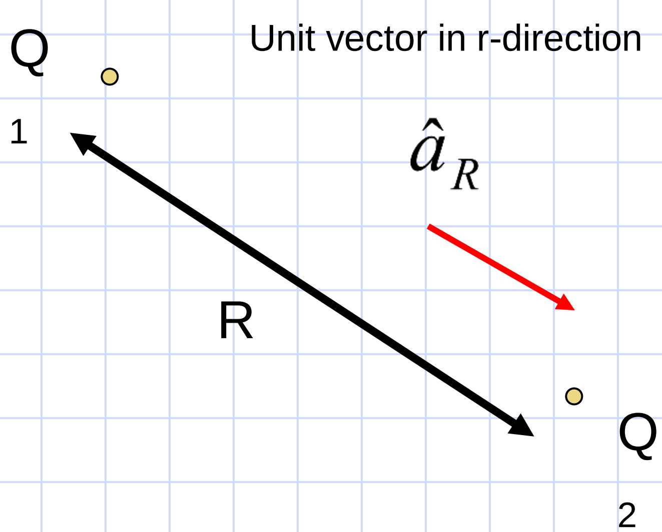
$$\vec{F}_{12} = Q_2 \cdot \vec{E}$$

- we work with E-Field because Maxwell's equations written in those terms

Electrostatics

Coulomb's Law

$\vec{F}$ (force), between point charges



$$\vec{F}_{12} = \frac{Q_1 \cdot Q_2}{4\pi\varepsilon_0 R^2} \cdot \hat{a}_R$$

Force on Charge 2 by Charge 1

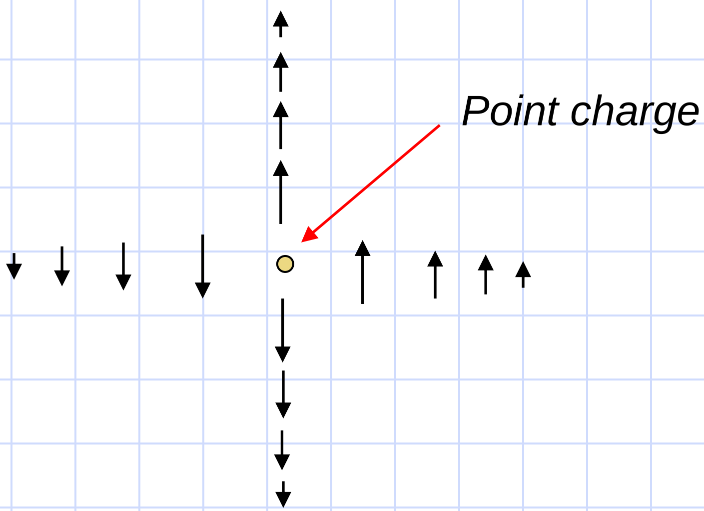
Electrostatics

$\vec{E}$, is a VECTOR Field *How do we represent it?*

- *Field points in the direction that a +q test charge would move*

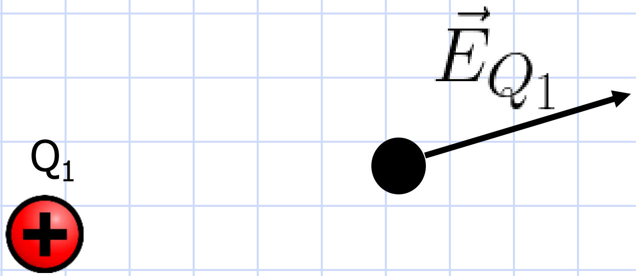
Represent using Arrows : Direction and Length

Proportional to Magnitude or strength of E-Field



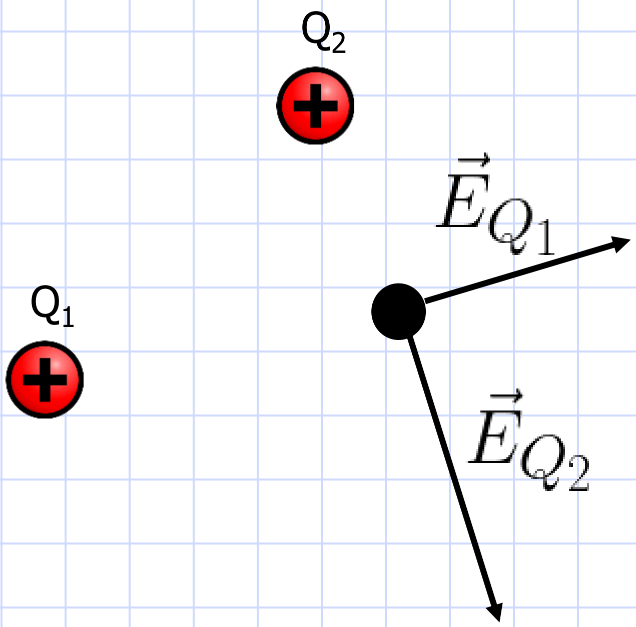
Electrostatics

To compute E-fields from multiple charges, apply superposition of fields.



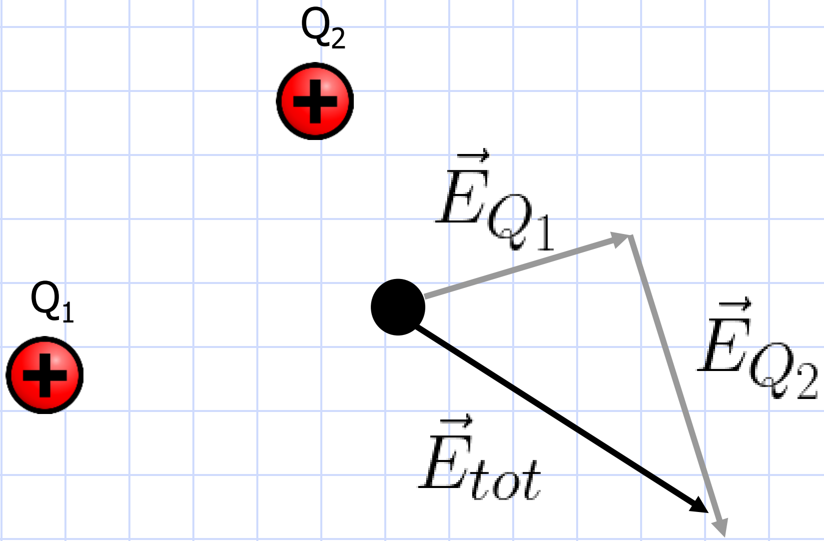
Electrostatics

To compute E-fields from multiple charges, apply superposition of fields.



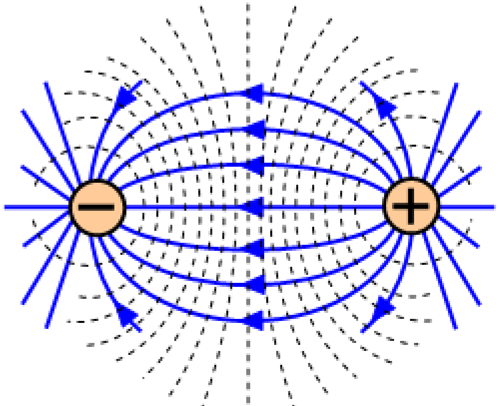
Electrostatics

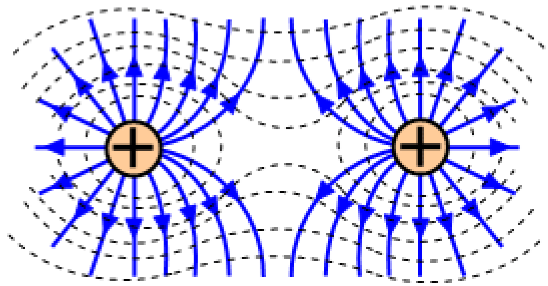
To compute E-fields from multiple charges, apply superposition of fields.

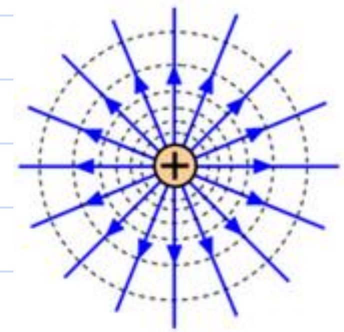


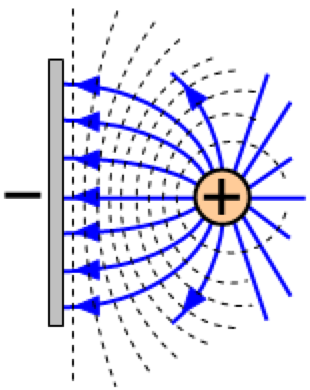
Electrostatics

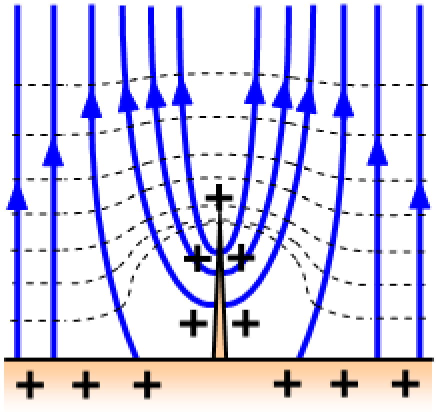
Some examples of E-Fields



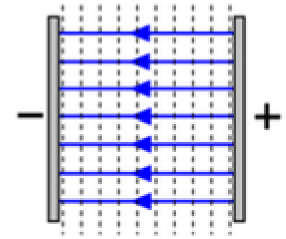






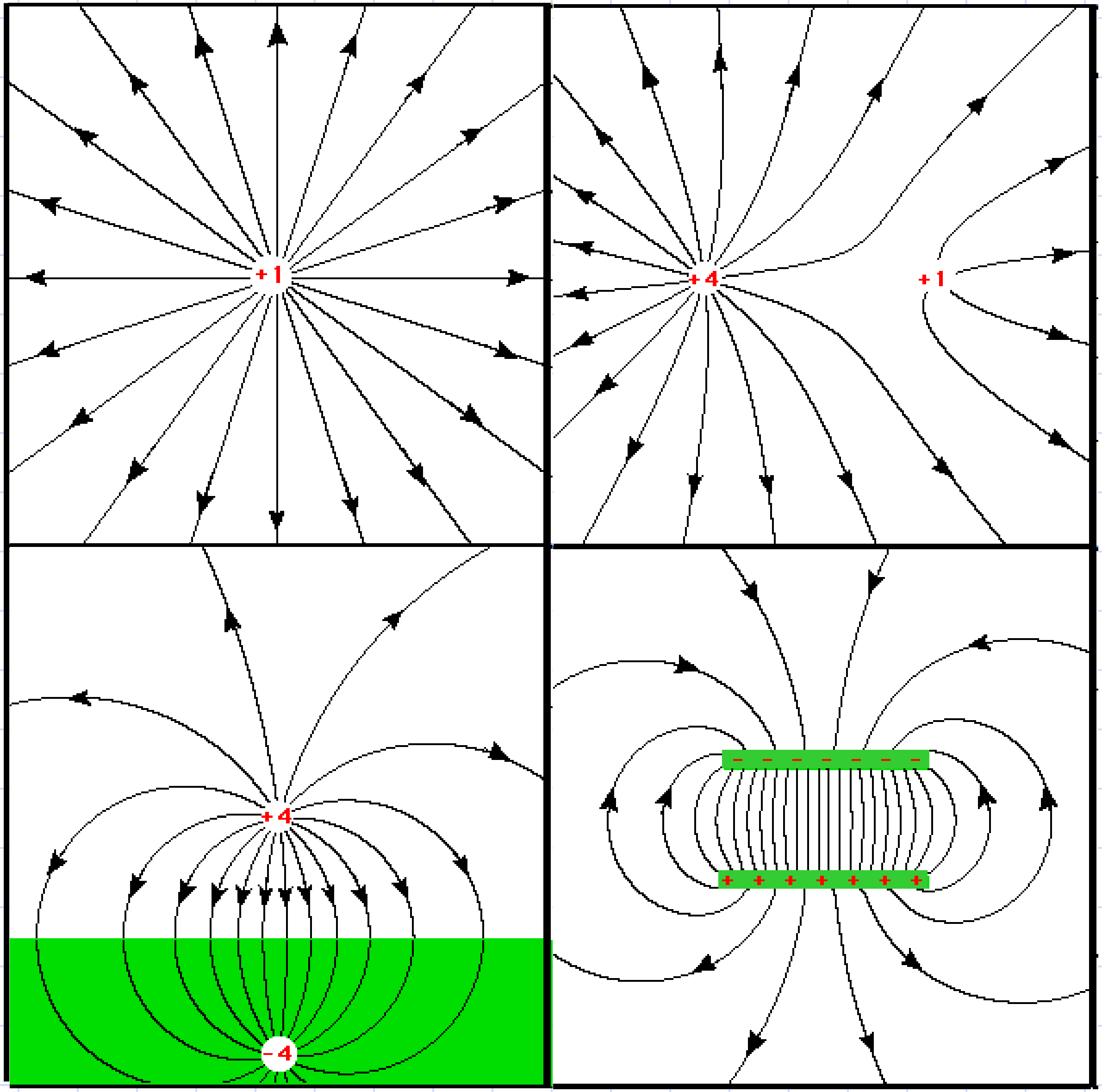


(resourcefulphysics.org)



http://shinliang.blogspot.com/2009/04/21-coulombs-law.html

Electrostatics

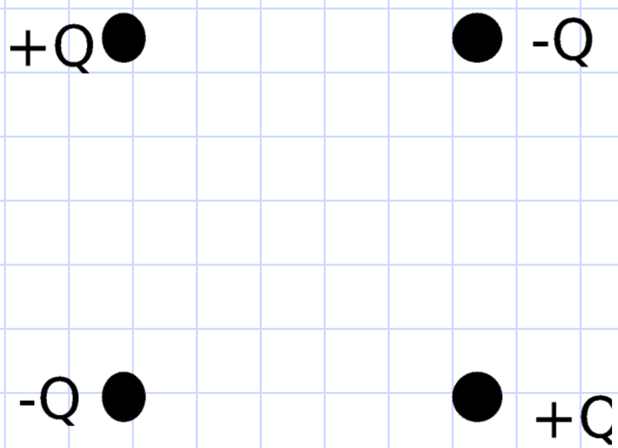


Note, in the upper right figure, that four times as many field lines leave the +4 positive charge as leave the +1 charge. All of the field lines end at infinity, as they do with a single positive charge.

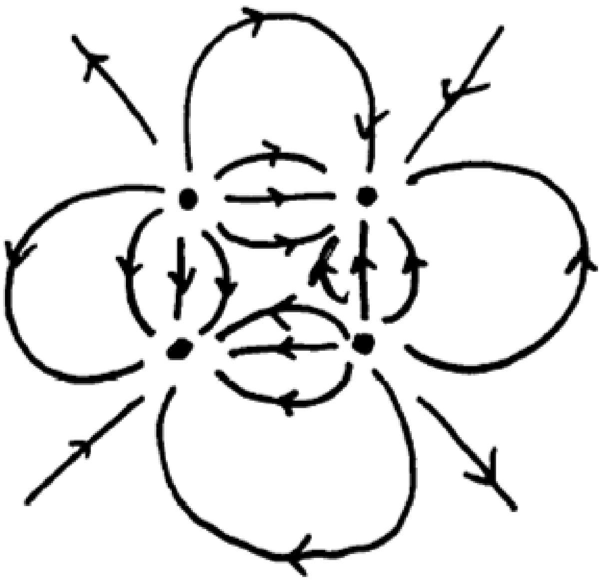
http://people.seas.harvard.edu/~jones/cscie129/nu_lectures/lecture6/field_vis/e_vis.html

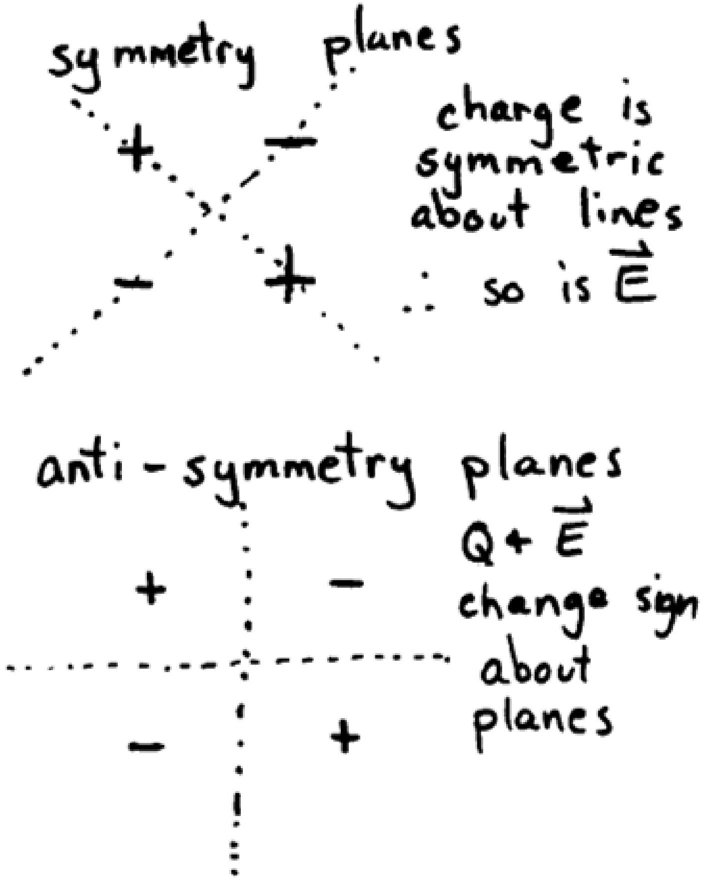
Electrostatics

Sketch the electric field lines for the electric quadrupole shown. Sketch the planes for which you expect the field to be symmetric. *After completing your sketch,* verify your result with the applet at https://davidawehr.com/projects/electric_field.html



Electrostatics





Electrostatics

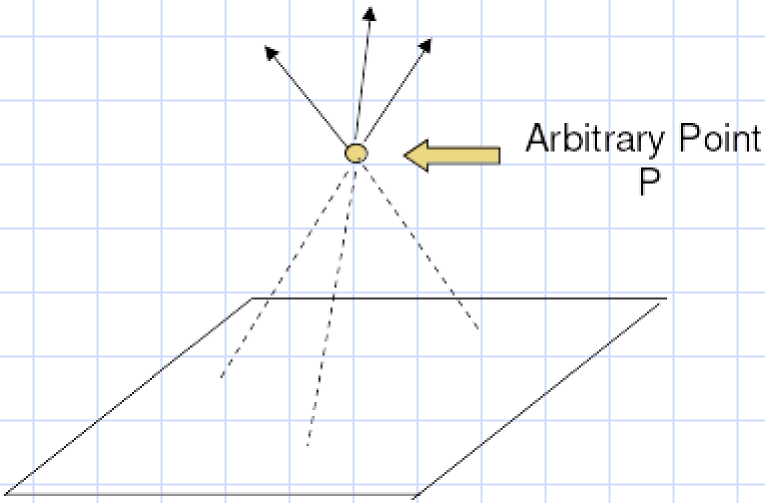
Previously we said:

$$\oint \vec{D} \cdot ds = \int \rho \cdot dv$$

$$\vec{D} = \epsilon \vec{E}$$

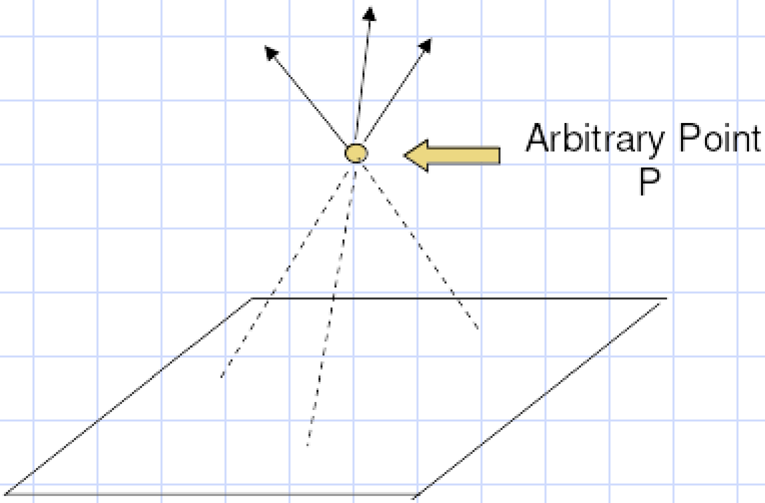
Can we understand this in terms of electric field lines?

Gauss's Law



What would the electric field be at an arbitrary point suspended over an infinite surface of uniform charge per area?

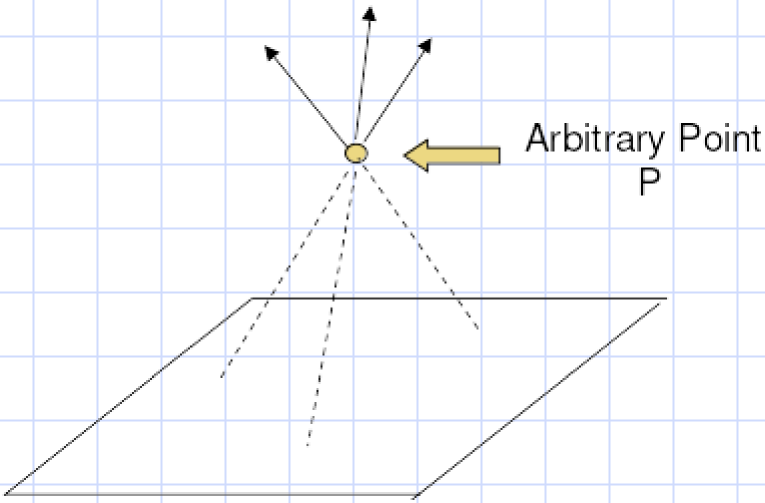
Gauss's Law



What would the electric field be at an arbitrary point suspended over an infinite surface of uniform charge per area?

First question: what direction will the field point in?

Gauss's Law



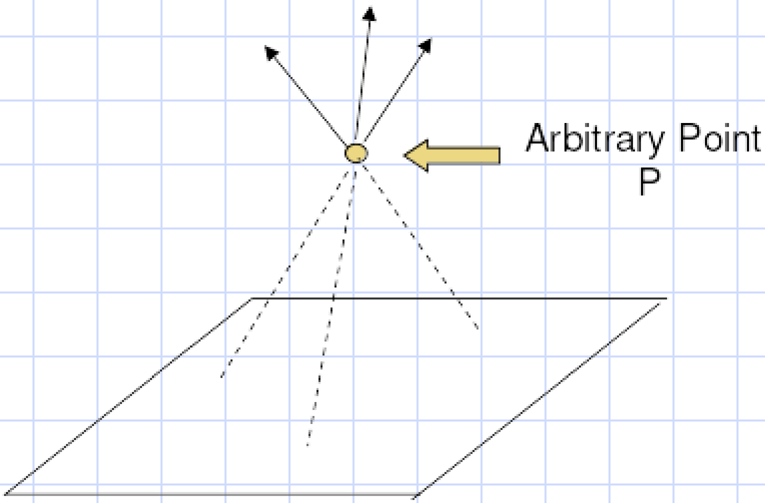
What would the electric field be at an arbitrary point suspended over an infinite surface of uniform charge per area?

<u>First question</u>: what direction will the field point in?

The +z direction since all non-z components will cancel.

$$\vec{E} = E_z(z) \cdot \hat{a}_z$$

Gauss's Law

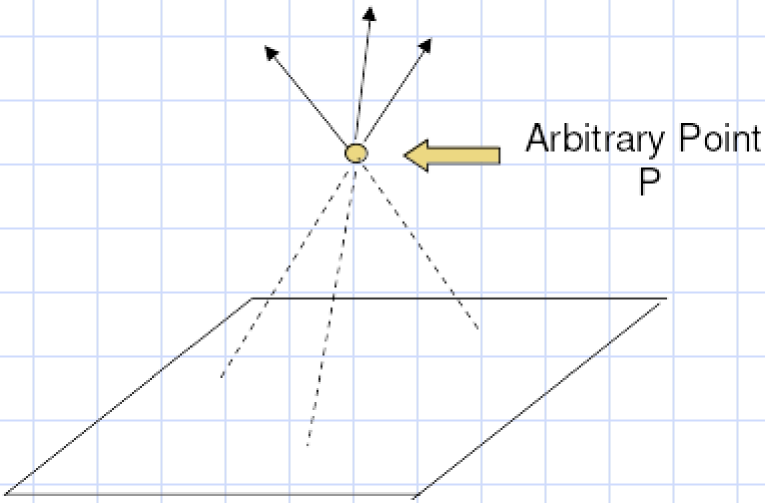


$$\vec{E} = E_z(z) \cdot \hat{a}_z$$

What would the electric field be at an arbitrary point suspended over an infinite surface of uniform charge per area?

<u>Second question</u>: should the field be uniform everywhere above the surface at the same z?

Gauss's Law



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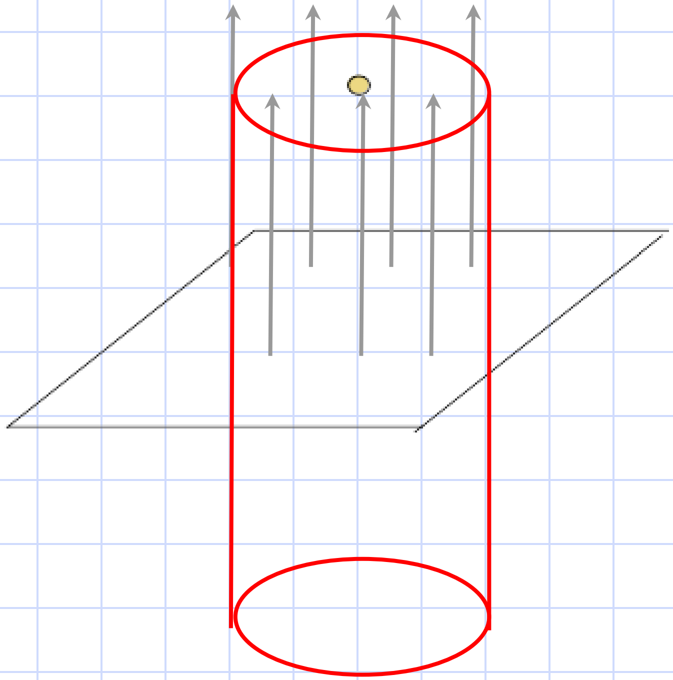
What would the electric field be at an arbitrary point suspended over an infinite surface of uniform charge per area?

<u>Second question</u>: should the field be uniform everywhere above the surface at the same z?

Yes!

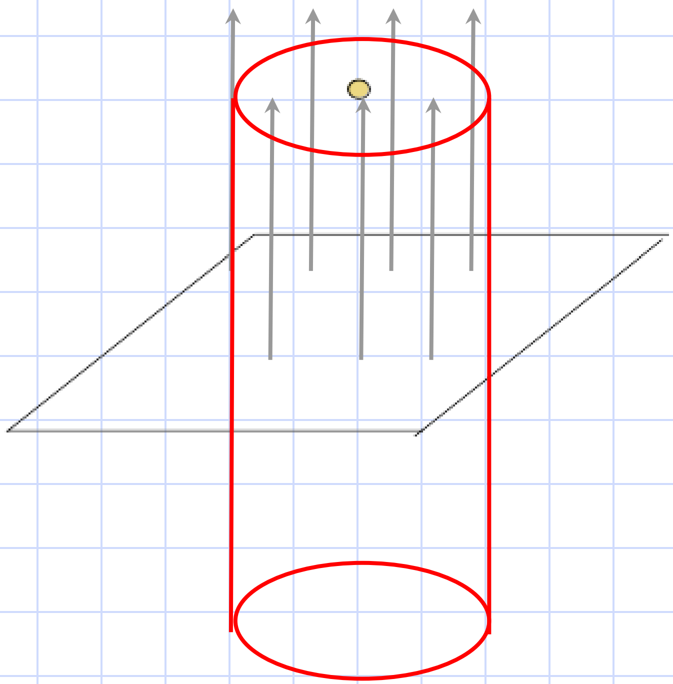
Gauss's Law



What would the electric field be at an arbitrary point suspended over an infinite surface of uniform charge per area?

<u>Third question</u>: If we make a Gaussian surface and measure the flux through it, what sides have net flux?

Gauss's Law

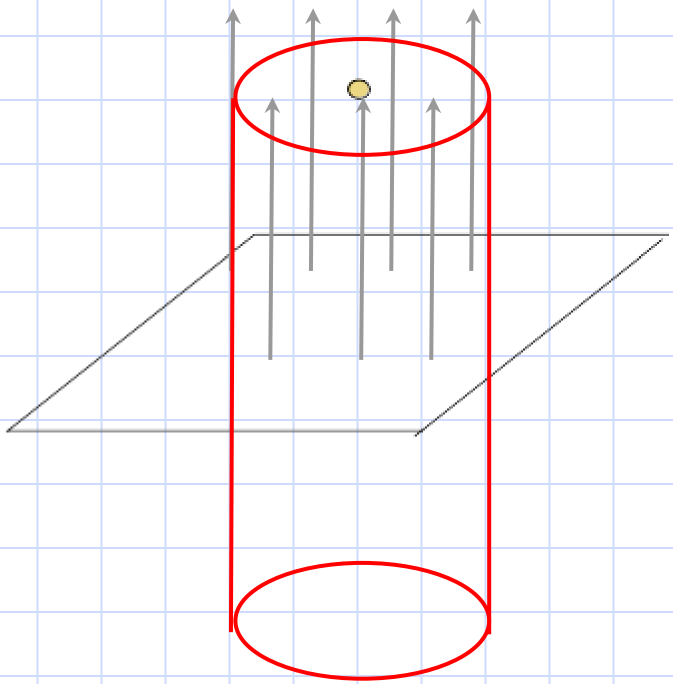


What would the electric field be at an arbitrary point suspended over an infinite surface of uniform charge per area?

Third question: If we make a Gaussian surface and measure the flux through it, what sides have net flux?

The ends!

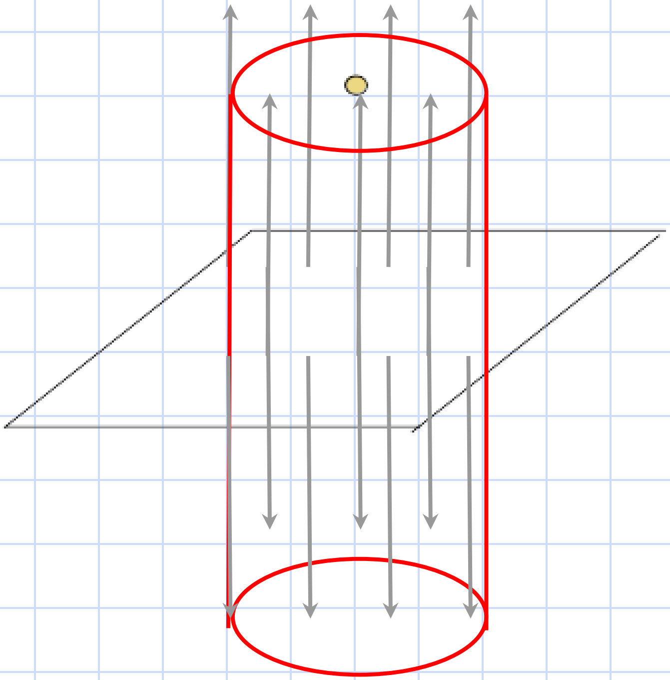
Gauss's Law



What would the electric field be at an arbitrary point suspended over an infinite surface of uniform charge per area?

Fourth question: Are there any lines we forgot to draw?

Gauss's Law

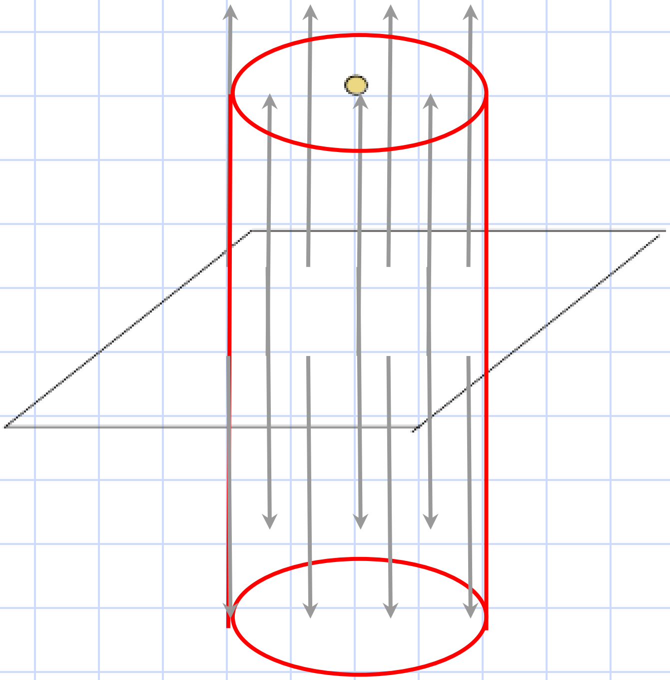


What would the electric field be at an arbitrary point suspended over an infinite surface of uniform charge per area?

Fourth question: Are there any lines we forgot to draw?

Yes, the equal and opposite ones on the underside!

Gauss's Law



What would the electric field be at an arbitrary point suspended over an infinite surface of uniform charge per area?

<u>Fifth question</u>: What is the electric field as calculated by Gauss's law for this surface?

Gauss's Law

Let r = cylinder radius

$$\oint \vec{D} \cdot \vec{ds} = Q_{encl}$$

$$2 \vec{E} \varepsilon_0 \pi r^2 = \rho \pi r^2$$

$$\vec{E} = \frac{\rho}{2\varepsilon_0} \hat{z}$$

Gauss's Law

Let r = cylinder radius

$$\oint \vec{D} \cdot \vec{ds} = Q_{encl}$$

$$2 \vec{E} \varepsilon_0 \pi r^2 = \rho \pi r^2$$

$$\vec{E} = \frac{\rho}{2\varepsilon_0} \hat{z}$$

<u>Sixth question</u>:
Will this be the same for any Gaussian surface we choose?

Gauss's Law

Let r = cylinder radius

$$\oint \vec{D} \cdot \vec{ds} = Q_{encl}$$

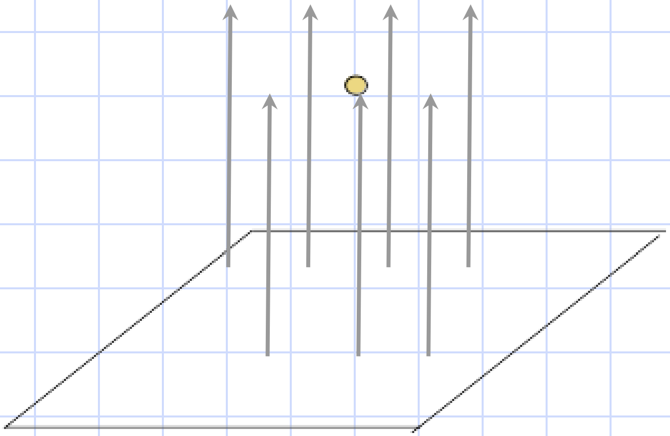
$$2 \vec{E} \varepsilon_0 \pi r^2 = \rho \pi r^2$$

$$\vec{E} = \frac{\rho}{2\varepsilon_0} \hat{z}$$

<u>Sixth question</u>:
Will this be the same for any Gaussian surface we choose?

Yes!

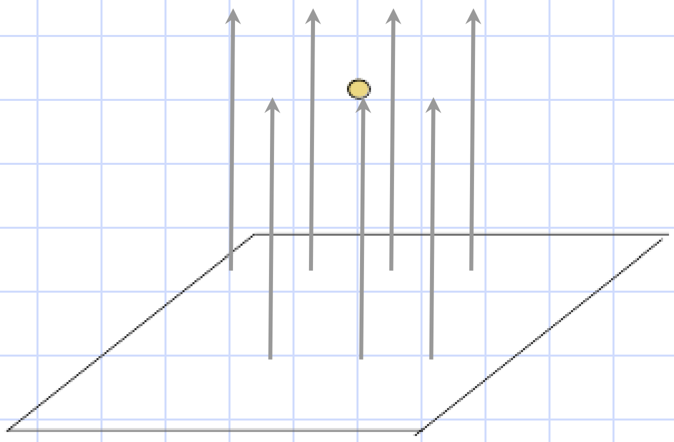
Gauss's Law



$$\vec{E} = \frac{\rho}{2\epsilon_0}\hat{z}$$

This is an interesting answer because we know that field strength generally falls of with distance, and this expression has no dependence on distance at all. Does this make sense?

Gauss's Law



$$\vec{E} = \frac{\rho}{2\epsilon_0}\hat{z}$$

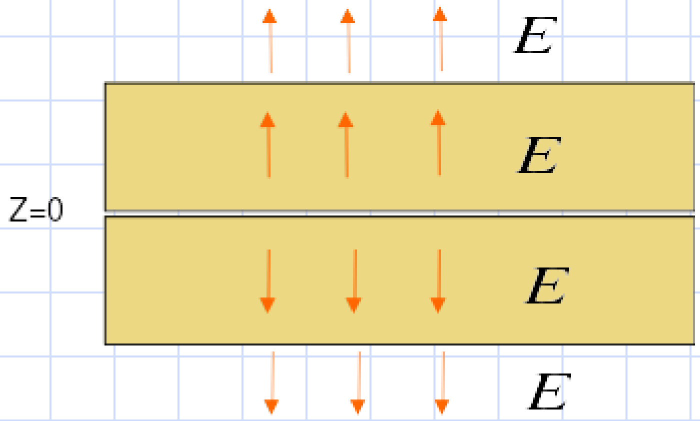
This is an interesting answer because we know that field strength generally falls of with distance, and this expression has no dependence on distance at all. Does this make sense?

Yes… if the plan is truly infinite, it should look exactly the same no matter how high above it you are!

Electrostatics

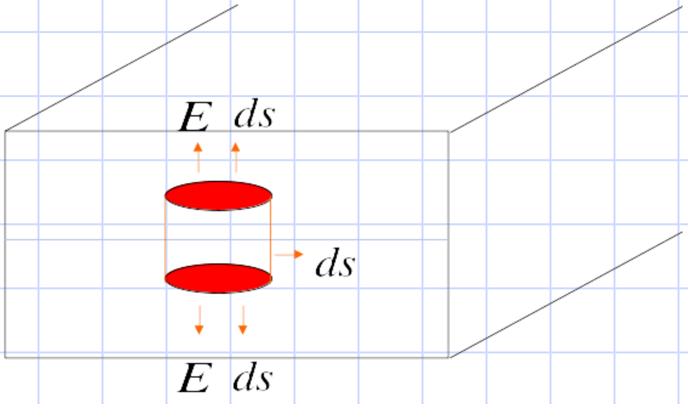
Do Lecture 11 Exercise 1 in groups of up to 4.

Gauss's Law



Let's say that we now consider the thickness of the infinite plane. How do we find an expression for the field *inside* the plane?

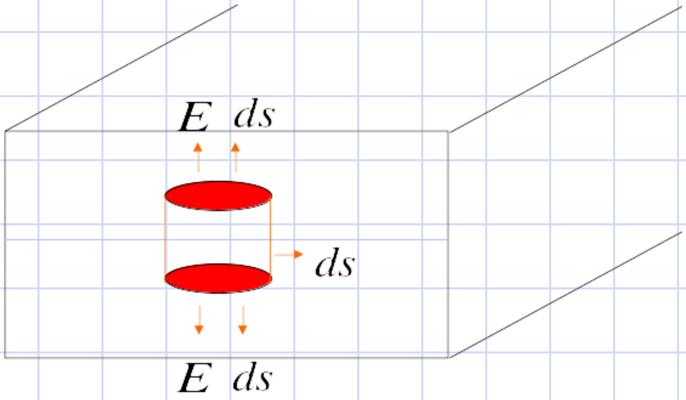
Gauss's Law



Charge density ρ is per unit volume here.

Start the Gaussian surface in the center of the plane and increase its length toward the surface.

Gauss's Law



Charge density ρ is per unit volume here.

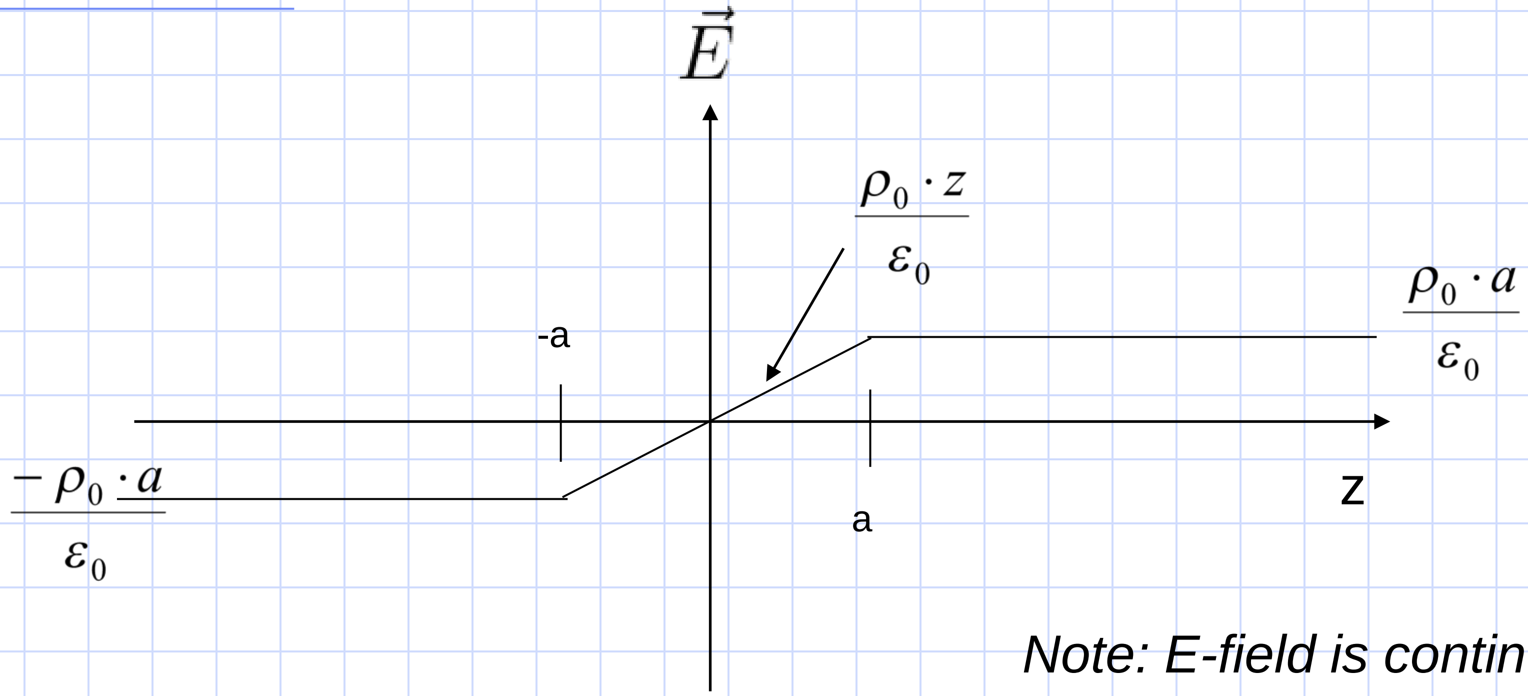
Start the Gaussian surface in the center of the plane and increase its length toward the surface.

$$\oint \vec{D}\cdot\vec{ds} = Q_{encl}$$

$$2\vec{E}\varepsilon_0 \pi r^2 = \rho \pi r^2 h$$

$$\vec{E} = \frac{\rho h}{2\varepsilon_0}$$

Gauss's Law



Note: E-field is continuous

Plot of E-field as a function of z for planar example

Gauss's Law

- Recognize the coordinate system.
- Using symmetry, determine which components of the field exist.
- Identify a Gaussian surface for which the sides are either parallel to or perpendicular to the field components. This surface is arbitrary in size.
- Determine the total charge within that surface. The charges can be distributed on lines, surfaces or in volumes.

Gauss's Law

- Evaluate the electric flux passing through the Gaussian surface.
  - If the field is parallel to the surface $\oint D \cdot dS = 0$
  - If the field is perpendicular to the surface,

$$\oint D \cdot dS = D_i \oint dS_i = D_i S_i$$

  where the subscript refers to the direction of the surface.
  - Note that a high level of symmetry is necessary to make these simplifications.

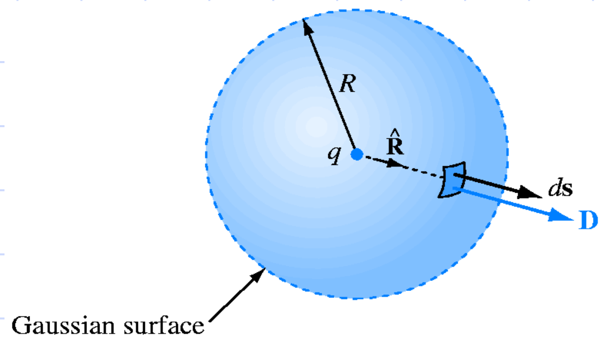
Gauss's Law

- Now equate the two sides of Gauss' Law to find E:

$$D_i S_i = \oint D \cdot dS = \int \rho \, dv = Q_{encl}$$

Gauss’s Law

Examples of typical Gaussian surfaces



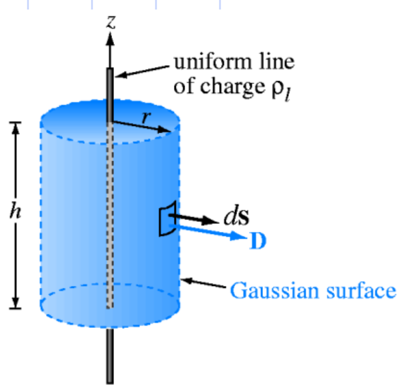
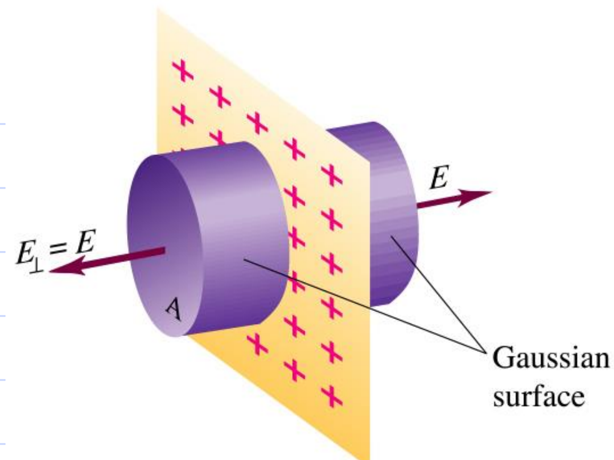


Figure 4-10



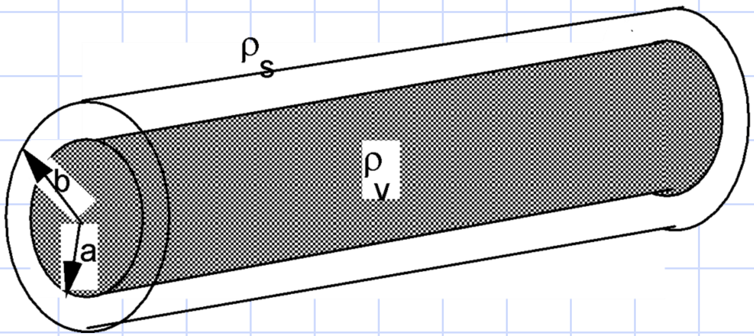
Ulaby

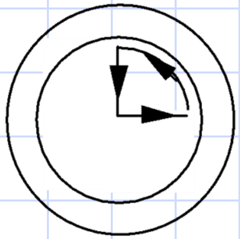
Gauss's Law

Full Gauss' Law Solution

A charge distribution with *cylindrical* symmetry is shown. The inner cylinder has a uniform charge density $\rho_v \, C/m^3$.

The outer shell has a surface charge density $\rho_s \, C/m^2$ such that the total charge on the outer shell is the negative of the total charge in the inner cylinder. Ignore end effects.





integration contour for part c.

Gauss's Law

a. Find the electric field for all *r*.

a. Check your answer by evaluating the divergence and curl of the electric field.

a. What is the closed line integral of the electric field around the contour shown?

Gauss's Law

From symmetry $\vec{E} = E_r(r)\,\hat{a}_r$

Gaussian surface

$$\oint \vec{E}\cdot\vec{ds} = \int_{side} \vec{E}\cdot\vec{ds} + \int_{ends} \cancelto{0}{\vec{E}\cdot\vec{ds}}$$

ds in $\hat{a}_z$

$$\oint \vec{E}\cdot\vec{ds} = \int_0^{l}\int_0^{2\pi} E_r\, r\, d\phi\, dz = E_r\, r \int_0^{l}\int_0^{2\pi} d\phi\, dz$$

$$= 2\pi r l\, E_r$$

$$Q_{enc} = \iiint \rho\, dv = \int_0^{l}\int_0^{2\pi}\int_0^{r} \rho_v\, r\, dr\, d\phi\, dz = \rho_v\, \pi r^2 l$$

$$\oint \vec{E}\cdot\vec{ds} = Q_{enc}/\varepsilon_0 \Longrightarrow 2\pi r l E_r = \frac{\rho_v \pi r^2 l}{\varepsilon_0} \Rightarrow \boxed{\vec{E} = \frac{\rho_v\, r}{2\varepsilon_0}\,\hat{a}_r}$$

Gauss's Law

$a<r<b$ $\oint \vec{E}\cdot d\vec{s}$ integral is same $= 2\pi r l E_r$

Gaussian integration surface

$Q_{enc} \Rightarrow$ for $0<r<a$ ~~it is~~ $\rho_v = \rho_v$

$a<r$ $\rho = 0$

$\therefore$ r integral in Q_{enc} is $0 \to a$

$Q_{enc} = \rho_v \pi a^2 l$

$\therefore$ $2\cancel{\pi} r \cancel{l} E_r = \frac{\rho_v \cancel{\pi} a^2 \cancel{l}}{\varepsilon_0}$

$$\boxed{\vec{E} = \frac{\rho_v a^2}{2 r \varepsilon_0}\hat{a}_r \quad a<r<b}$$

$b<r$ $Q_{enc}=0$ since $Q_{outer} = -Q_{inner}$

$\therefore$ $2\pi r l E_r = 0$ & $\vec{E} = 0$

Summary

$$\boxed{\vec{E} = \begin{cases} \frac{\rho_v r}{2\varepsilon_0}\hat{a}_r & r<a \\ \frac{\rho_v a^2}{\varepsilon_0 2 r}\hat{a}_r & a<r<b \\ 0 & b<r \end{cases}}$$

Gauss's Law

Divergence and curl expressions for cylindrical coordinates:

$$\text{Divergence}\quad \vec{\nabla}\bullet\vec{F} = \frac{1}{r}\frac{\partial(r\cdot F_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

$$\text{Curl}\quad \vec{\nabla}\times\vec{F} = \left(\frac{1}{r}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right)\hat{r} + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)\hat{\phi} + \frac{1}{r}\left(\frac{\partial(r\cdot F_\phi)}{\partial r} - \frac{\partial F_r}{\partial \phi}\right)\hat{z}$$

c. $\nabla\cdot\vec{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \underbrace{E_\phi \text{ and } E_z \text{ terms}}_{\to\, 0}$ since $E_\phi = E_z = 0$

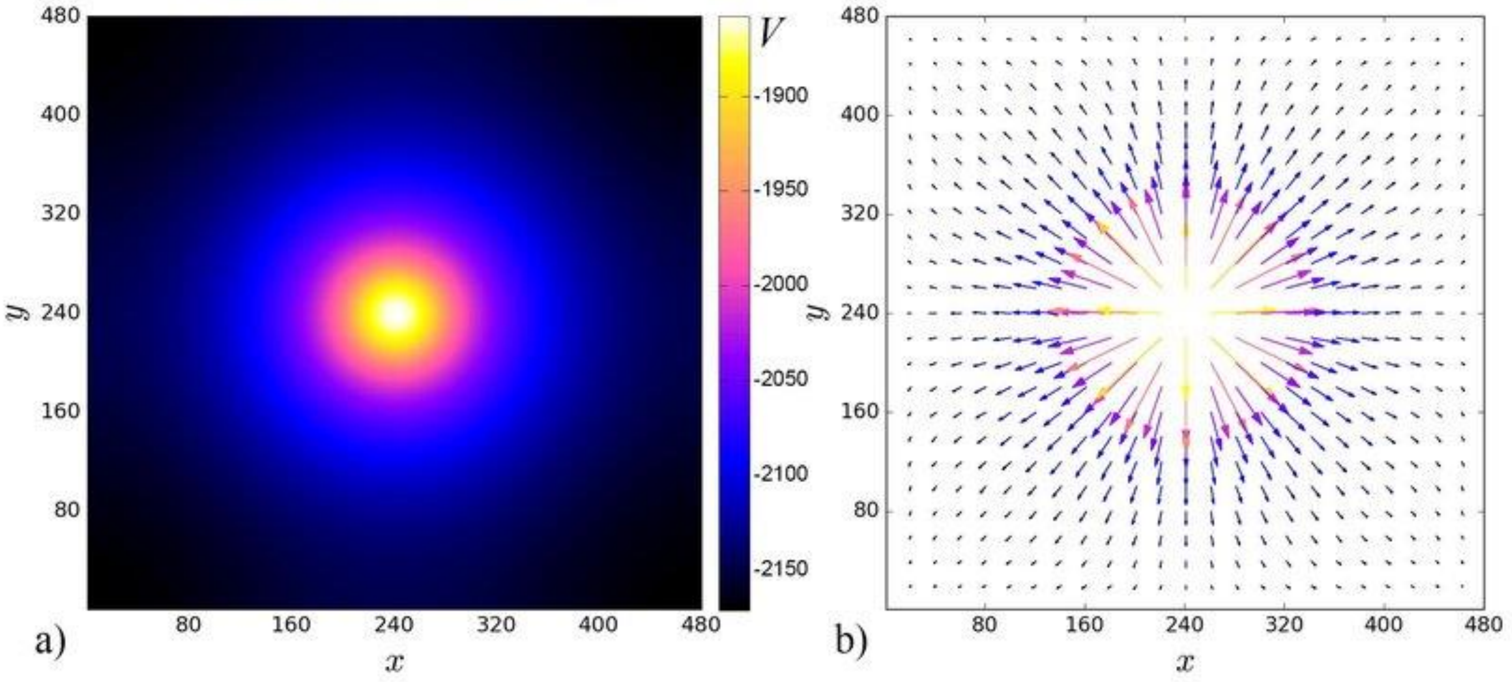
$r<a \quad \nabla\cdot\vec{E} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\rho_v r^2}{2\varepsilon_0}\right) = \frac{1}{r}\,\frac{2\rho_v r}{2\varepsilon_0} = \rho_v/\varepsilon_0$ ✓

$a<r<b \quad \nabla\cdot\vec{E} = \frac{1}{r}\frac{\partial}{\partial r}\left(\cancel{r}\,\frac{\rho_v a^2}{2\cancel{r}}\right) = 0$ ✓ THERE is no charge here

$r>b \quad \nabla\cdot\vec{E} = 0$

$\nabla\times\vec{E} = \frac{\partial E_r}{\partial z}\hat{a}_\phi - \frac{1}{r}\frac{\partial E_r}{\partial \phi}\hat{a}_z + \cancel{E_\phi + E_z \text{ terms}}^{\,0} = 0$ since E_r not function of ϕ or z

Electric Potential



If the electric field looks like this (right), how do we derive the electric potential (left)? (Lauricella)

Electric Potential

- From vector calculus,

$$\nabla \times \nabla f = 0 \text{ for any scalar field f.}$$

- Introducing the electric scalar potential:

Since $\nabla \times \vec{E} = 0$, we can find a vector field such that

$$\vec{E} = -\nabla V \qquad \nabla \times \nabla V = \nabla \times \vec{E} = 0$$

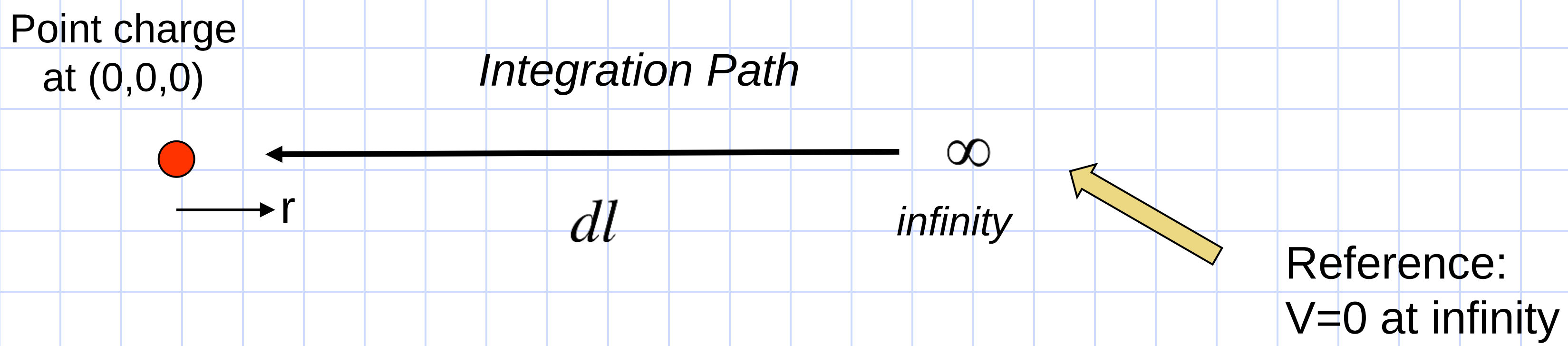
$$V(P_2) - V(P_1) = -\int_{P_1}^{P_2} \vec{E} \cdot d\vec{l}$$

Electric Potential

Example: Use case of point charge at origin and obtain potential everywhere from E-field

Spherical Geometry

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2} \cdot \hat{a}_r$$



Electric Potential

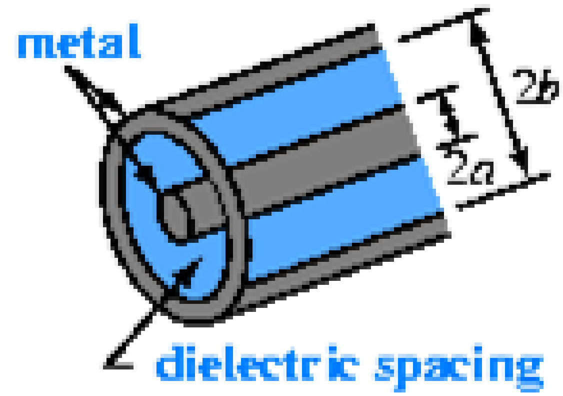
- Charge-Voltage Method

$$Q = C\,V$$

$$Q \to \vec{E} \to V$$

Maxwell's first equation

Maxwell's second equation



(a) Coaxial line

Electric Potential

The integral for computing the potential of the point charge is:

$$V(r) - V(r=\infty) = -\int_{r=\infty}^{r} \vec{E} \bullet d\vec{l}$$

0

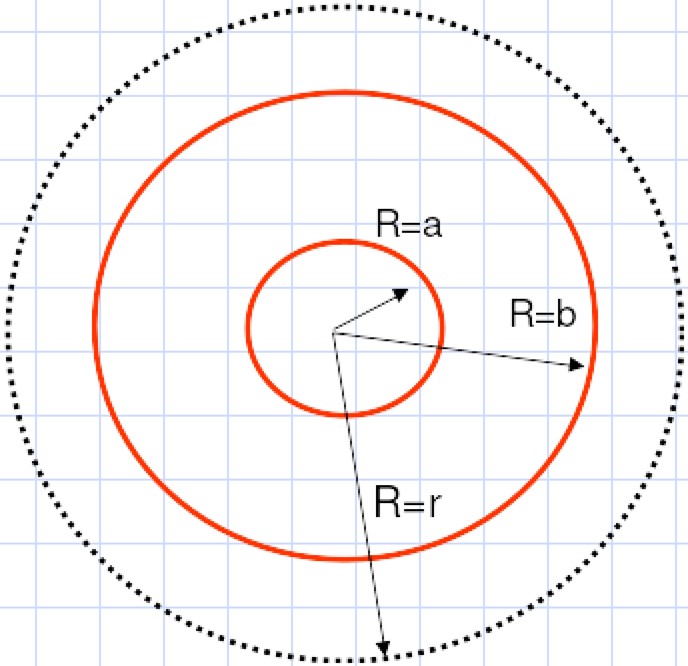
$$\therefore V(r) = -\int_{r=\infty}^{r} E \cdot dr$$

$$= -\int_{r=\infty}^{r} \frac{q}{4\pi\varepsilon_0 r^2} \cdot dr \quad \Rightarrow \quad V(r) = \frac{q}{4\pi\varepsilon_0 r}$$

Electric Potential

- What is the potential expression for this E-field from a shielded conductor with a grounded exterior?

$$\vec{E} = \begin{cases} \dfrac{\rho_v r}{2\varepsilon_0}\hat{a}_r & r<a \\ \dfrac{\rho_v a^2}{\varepsilon_0 2r}\hat{a}_r & a<r<b \\ 0 & r>b \end{cases}$$



Electric Potential

$$\vec{E} = \begin{cases} \dfrac{\rho_v r}{2\varepsilon_0}\,\hat{a}_r & r<a \\[2ex] \dfrac{\rho_v a^2}{\varepsilon_0 2r}\,\hat{a}_r & a<r<b \\[2ex] 0 & r>b \end{cases}$$

$V(b) = 0$

$\therefore$ FOR $\boxed{r>b \quad V=0}$ since $\vec{E}=0$

$$a<r<b \qquad V(r) - \underbrace{\cancel{V(b)}}_{0} = -\int_b^r E_r\,dr = -\int_b^r \frac{\rho_v a^2}{\varepsilon_0 2r}\,dr = -\frac{\rho_v a^2}{2\varepsilon_0}\ln\frac{r}{b}$$

$$\boxed{V(r) = \frac{\rho_v a^2}{2\varepsilon_0}\ln\frac{b}{r} \qquad a<r<b}$$

Electric Potential

& $a<r<b \quad \vec{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{a}_r = -\frac{\partial}{\partial r}\left(\frac{\rho_v a^2}{2\varepsilon_0}\ln\frac{b}{r}\right)\hat{a}_r$

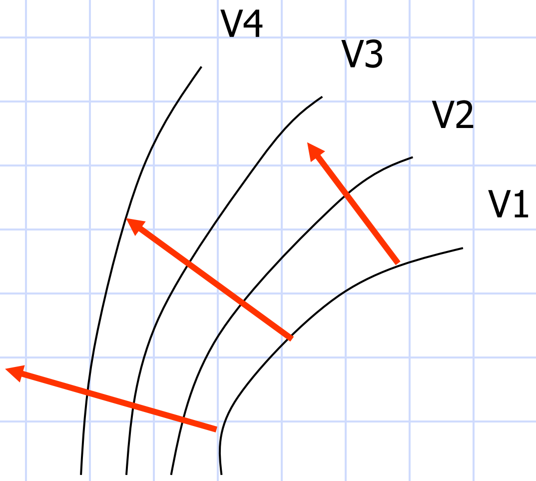
$= -\frac{\rho_v a^2}{2\varepsilon_0}\ \frac{-b/r^2}{b/r}\hat{a}_r = \frac{\rho_v a^2}{2\varepsilon_0 r}\hat{a}_r = \text{original } \vec{E}$ ✓

- $V(0) - V(a) = -\int_a^0 \vec{E}\cdot d\vec{l} \Rightarrow V(0) = V(a) - \int_a^0 \frac{\rho_v r}{2\varepsilon_0}\,dr$

$$V(0) = \underbrace{\frac{\rho_v a^2}{2\varepsilon_0}\ln\frac{b}{a}}_{\substack{\text{set } r=a \text{ in} \\ a<r<b \text{ solution}}} - \frac{\rho_v}{2\varepsilon_0}\frac{r^2}{2}\Big]_a^0 = \boxed{\frac{\rho_v a^2}{2\varepsilon_0}\ln\frac{b}{a} + \frac{\rho_v a^2}{4\varepsilon_0}}$$

Electric Potential

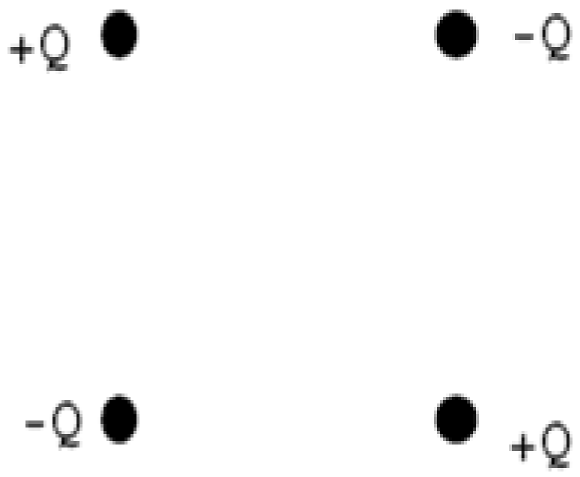
$$\vec{E} = -\nabla V$$

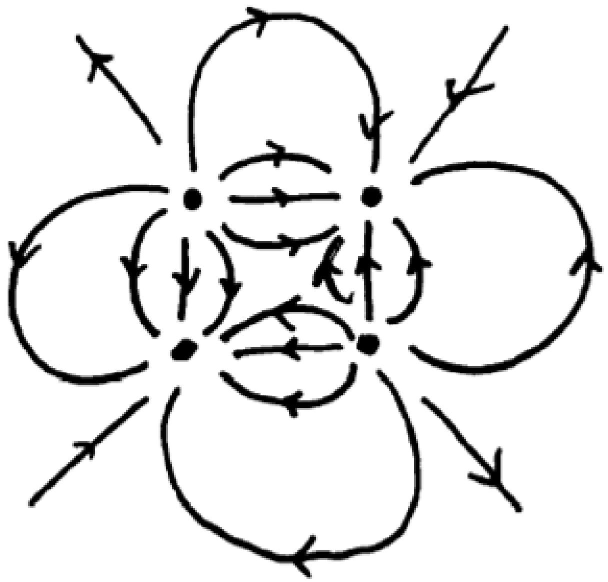


- Gradient points in the direction of largest change
- Therefore, E-field lines are perpendicular (normal) to constant V surfaces

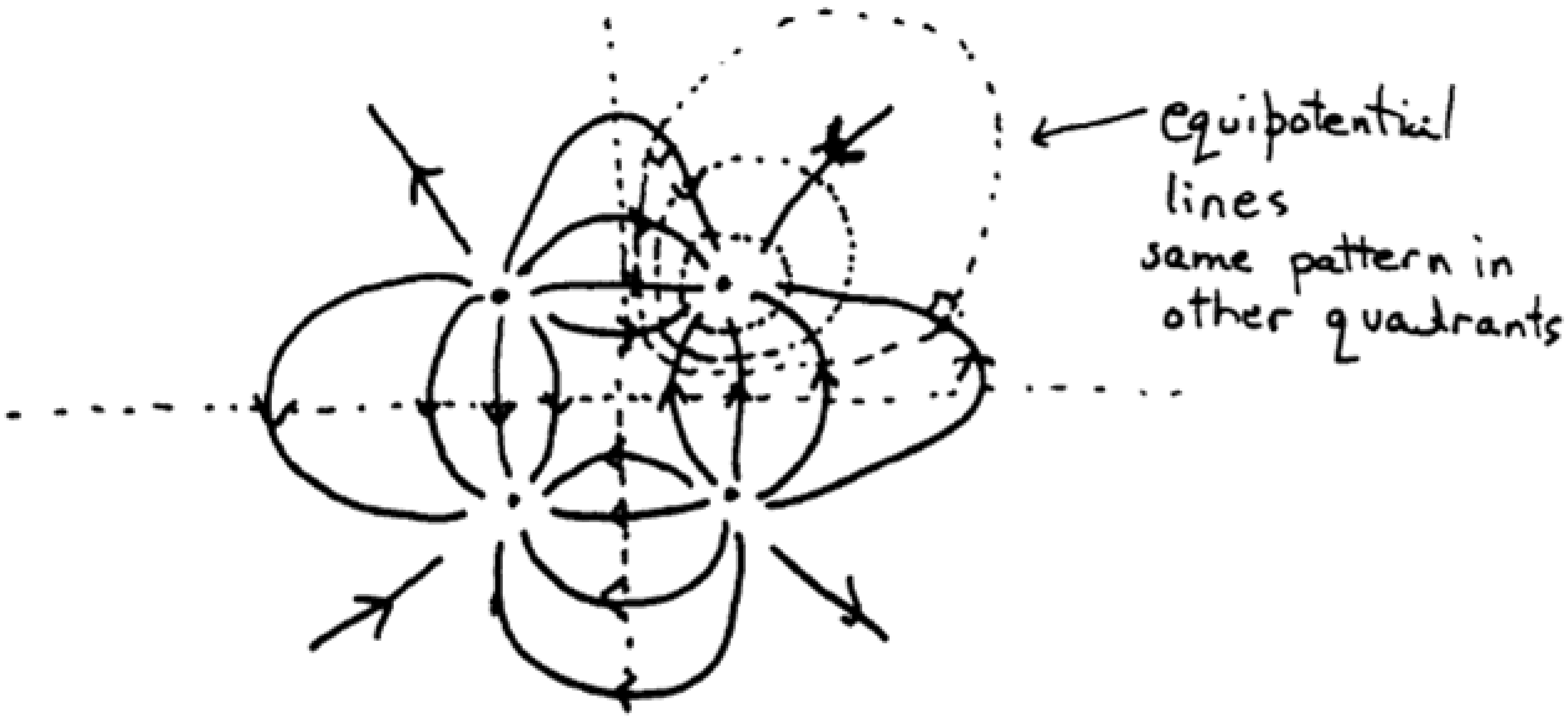
Electric Potential

- Plot a set of equipotentials for this quadrupole.





Electric Potential



Electric Potential

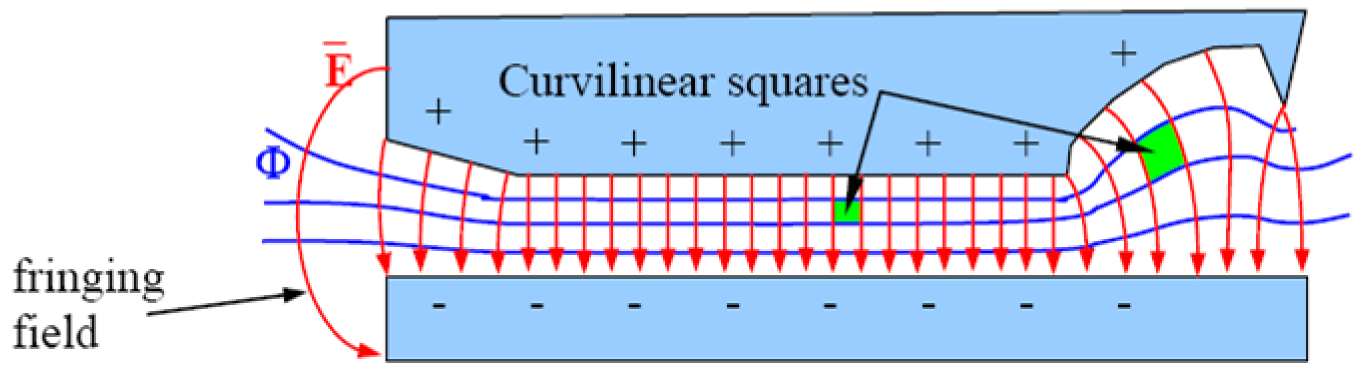
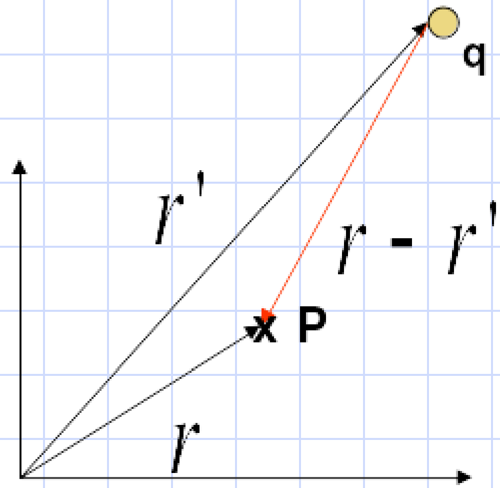


Figure R11-3. Graphical field mapping of $\bar{E}$ and F between charged conductors

http://ocw.mit.edu/NR/rdonlyres/Electrical-Engineering-and-Computer-Science/6-013Electromagnetics-and-ApplicationsFall2002/922D1A06-9AC9-4076-B1F5-066EE896043C/0/Rec11Notes.pdf

Electric Potential

Potential of a single charge



For the case of a point charge:

$$V = \frac{q}{4\pi\epsilon_0 r} = \frac{q}{4\pi\epsilon_0 (r - r')} = V(\vec{r})$$

$\vec{r}$, is field point where we are measuring/calculating V

$\vec{r}$, is location of charge

Electric Potential

For a charge distribution:

$$V(r) = \int\int\int \frac{\rho(r') \cdot dv'}{4\pi\epsilon_0 |r - r'|}$$

Volume charge distribution

$$V(r) = \int \frac{\rho(r') \cdot dl'}{4\pi\epsilon_0 |r - r'|}$$

Line charge distribution