Fields and Waves I

Lecture 24

EM Waves at Oblique Incidence

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Announcements

- Final exam: Friday Dec 15, **10:30am-2:30pm**
- HW 8 due on Sunday
- Friday reworks will be announced ASAP (today or tomorrow) on Discord

These slides were prepared through the work of the following people:

- Kenneth A. Connor – ECSE Department, Rensselaer Polytechnic Institute, Troy, NY (Emeritus)
- J. Darryl Michael – GE Global Research Center, Niskayuna, NY
- Thomas P. Crowley – National Institute of Standards and Technology, Boulder, CO
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- Lale Ergene – ITU Informatics Institute, Istanbul, Turkey
- Jeffrey Braunstein – ECE Department, University at Albany
- James Lu - ECSE Department, Rensselaer Polytechnic Institute, Troy, NY
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Materials from other sources are referenced where they are used.
Those listed as Ulaby are figures from Ulaby’s textbook.

Agenda

- Review
- Oblique Incidence
- Snell's Law
- Reflection Angular Dependence
- Critical Angle
- Brewster Angle
- Wrap-Up

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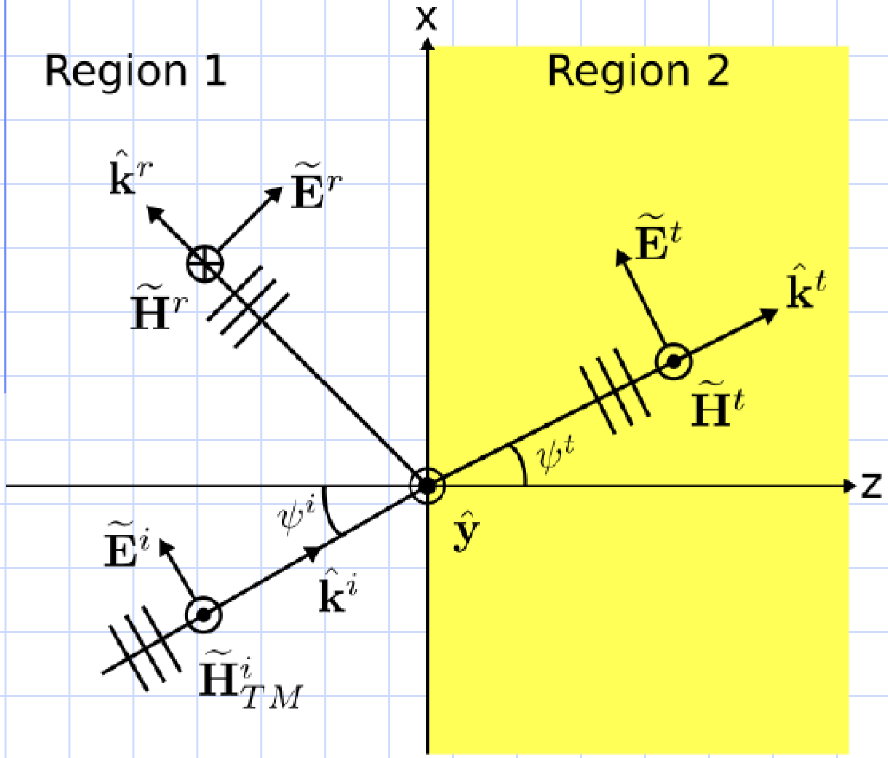
Review

- Last class we discussed the reflection of EM waves at boundaries at normal incidence.
- What does “normal incidence” mean, and how does it simplify calculation of reflection?

Oblique Incidence



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Now we consider EM wave transmission and reflection at oblique incidence ($\theta_i \neq 0$).

Reflection and transmission is no longer a one-dimensional problem (e.g. E_i and E_t in the +z direction and E_r in the -z direction)

Oblique Incidence

$$k = \omega\sqrt{\mu_0 \varepsilon}$$

$$\tilde{E}_1(z) = \hat{a}_x E_0^i \left(e^{-\gamma_1 z} + \Gamma e^{\gamma_1 z}\right)$$

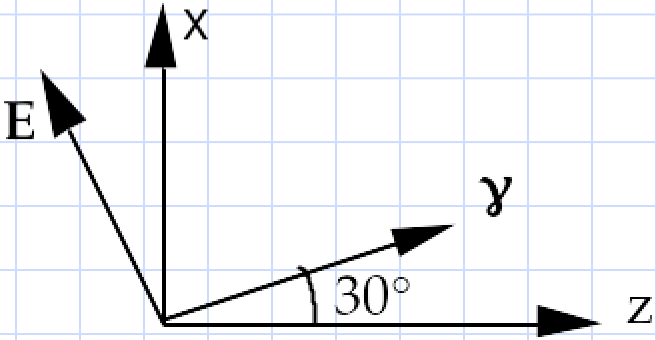
$$\gamma_1 = \alpha_1 + j\beta_1$$

- Note that k, ɣ are used interchangeably in wave equations (with **β** able to replace ɣ)
- For transmission line equations and normal incidence EM waves, we did not represent k / ɣ as having a direction, but for oblique incidence scenarios it becomes useful to represent it as having one, because it is a vector that may include x, y, and z components.

Oblique Incidence

Example 1

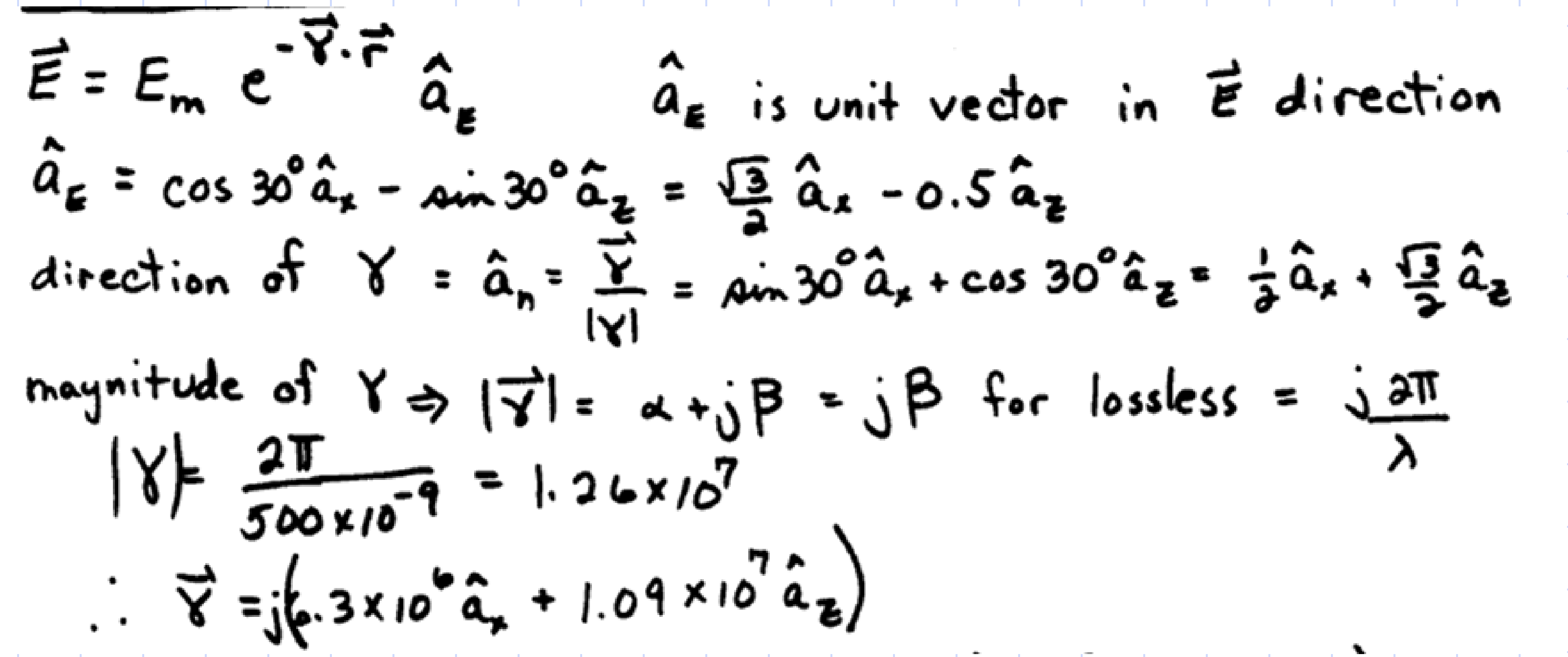
The direction of **E** and γ of a electromagnetic wave with $\lambda = 500\text{nm}$ are shown below. The wave is traveling through air. The electric field has a magnitude of 30 V/m. What are the **E** and **H** phasors?



y axis is out of the page

Oblique Incidence

Example 1



$\vec{E} = E_m e^{-\vec{\gamma}\cdot\vec{r}}\,\hat{a}_E$ $\hat{a}_E$ is unit vector in $\vec{E}$ direction

$\hat{a}_E = \cos 30^\circ \hat{a}_x - \sin 30^\circ \hat{a}_z = \frac{\sqrt{3}}{2}\hat{a}_x - 0.5\,\hat{a}_z$

direction of γ : $\hat{a}_n = \frac{\vec{\gamma}}{|\gamma|} = \sin 30^\circ \hat{a}_x + \cos 30^\circ \hat{a}_z = \frac{1}{2}\hat{a}_x + \frac{\sqrt{3}}{2}\hat{a}_z$

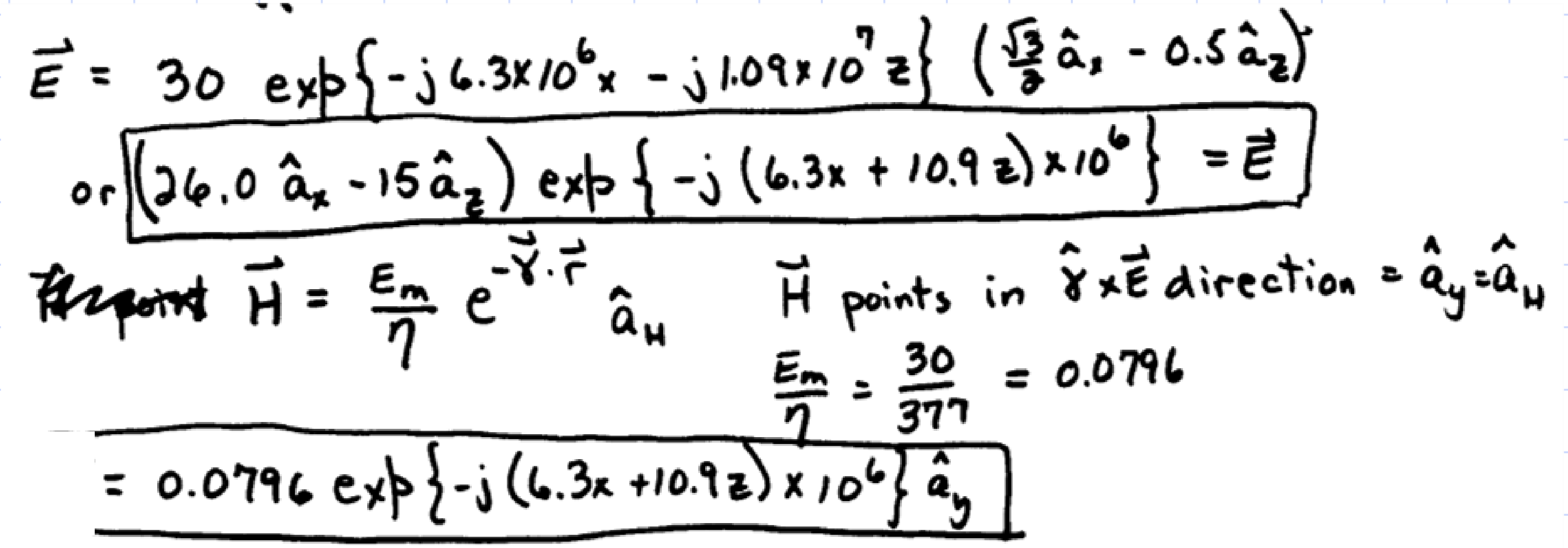
magnitude of $\gamma \Rightarrow |\vec{\gamma}| = \alpha + j\beta = j\beta$ for lossless $= j\frac{2\pi}{\lambda}$

$|\gamma| = \frac{2\pi}{500\times10^{-9}} = 1.26\times10^{7}$

$\therefore\ \vec{\gamma} = j(6.3\times10^{6}\,\hat{a}_x + 1.09\times10^{7}\,\hat{a}_z)$

Oblique Incidence

Example 1



$\vec{E} = 30 \exp\{-j\, 6.3\times 10^6 x - j\, 1.09\times 10^7 z\} \left(\frac{\sqrt{3}}{2}\hat{a}_x - 0.5\hat{a}_z\right)$

or $\boxed{(26.0\,\hat{a}_x - 15\,\hat{a}_z) \exp\{-j\,(6.3x + 10.9z)\times 10^6\} = \vec{E}}$

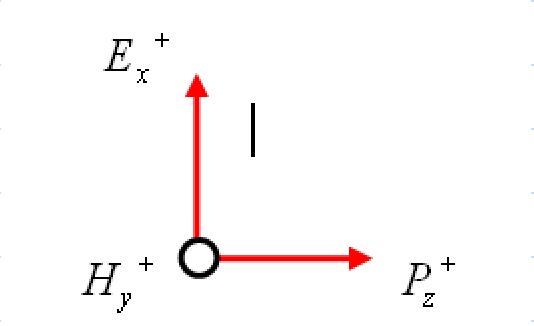
~~[illegible]~~ $\vec{H} = \frac{E_m}{\eta} e^{-\vec{\gamma}\cdot\vec{r}}\,\hat{a}_H$ $\quad$ $\vec{H}$ points in $\hat{\gamma}\times\vec{E}$ direction $= \hat{a}_y = \hat{a}_H$

$\frac{E_m}{\eta} = \frac{30}{377} = 0.0796$

$\boxed{= 0.0796 \exp\{-j\,(6.3x + 10.9z)\times 10^6\}\,\hat{a}_y}$

Oblique Incidence

In phasor form we have had $E_x(z) = E_m^+ e^{-j\beta z}$



$$H_y(z) = \frac{E_m^+ e^{-j\beta z}}{\eta}$$

We can generalize this with $e^{-jk \cdot r}$

$$r = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z \qquad k = k_x\hat{a}_x + k_y\hat{a}_y + k_z\hat{a}_z$$

$$k \cdot r = \beta\hat{a}_z \cdot \left(x\hat{a}_x + y\hat{a}_y + z\hat{a}_z\right) = \beta z$$

k = direction wave is traveling. r = direction in which we measure. (usually the same as k)

Oblique Incidence

Let's consider the case in which EM wave's propagation direction has both an x and a z component. This is a bit easier than a wave with x, y and z propagation components and is sufficient to show the mathematics of oblique incidence.

$$k = k_x \hat{a}_x + 0\hat{a}_y + k_z \hat{a}_z = k_x \hat{a}_x + k_z \hat{a}_z$$

$$E(r) = E_m^+ e^{-jk \cdot r} = \hat{a}_E E_m^+ e^{-jk_x x - jk_z z}$$

$$\vec{H}(\vec{r}) = \hat{a}_H \frac{E_m^+ e^{-jk_x x - jk_z z}}{\eta}$$

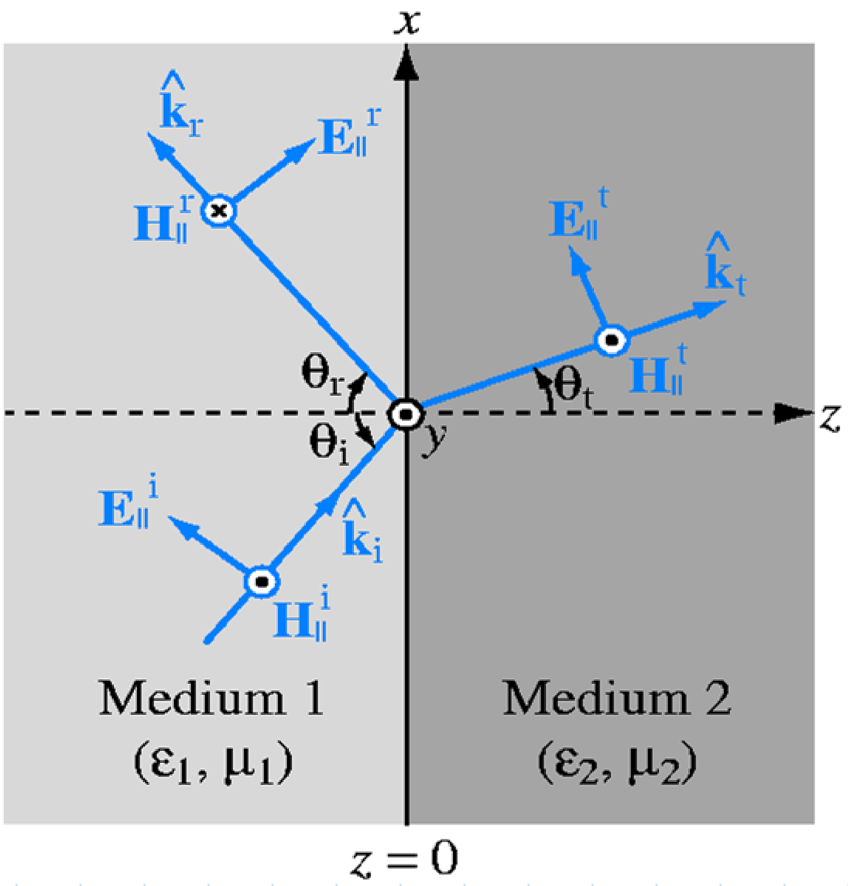
There are two possible orientations for the E and H fields in this scenario.

Oblique Incidence

Parallel Polarization

- For the first choice, we can assume that the electric field is directed in the plane of incidence (in this case the x-z plane.)

- This is called parallel polarization since E is parallel to this plane of interaction

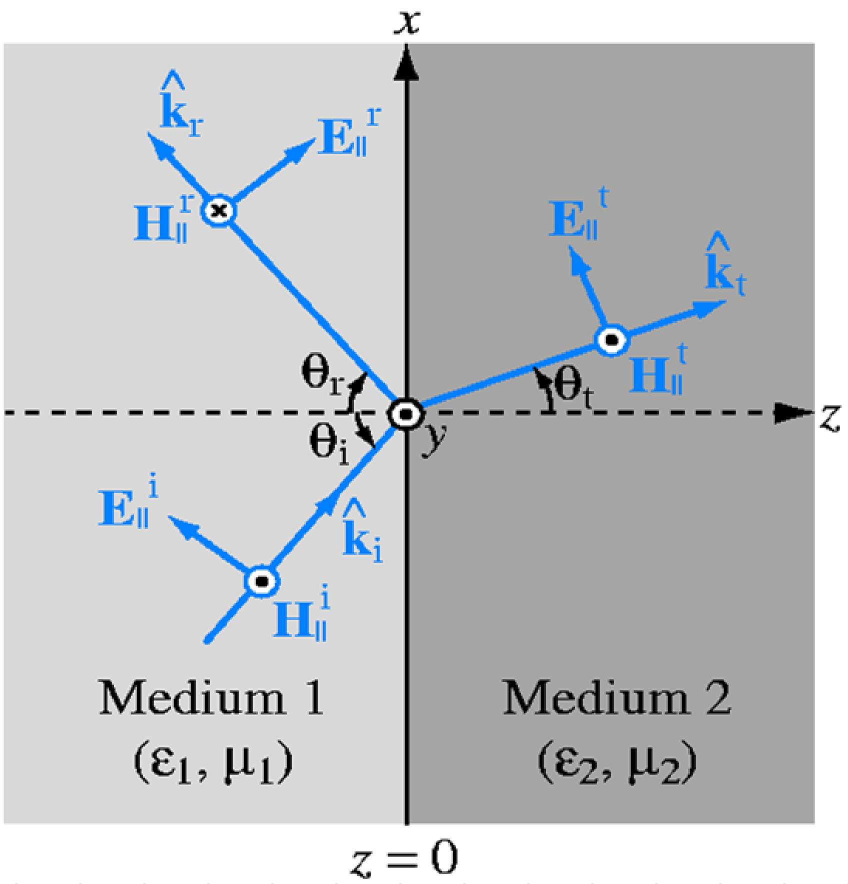


Ulaby

Oblique Incidence

Parallel Polarization

- Note that H is only tangent to the boundary while E has both normal and tangential components.



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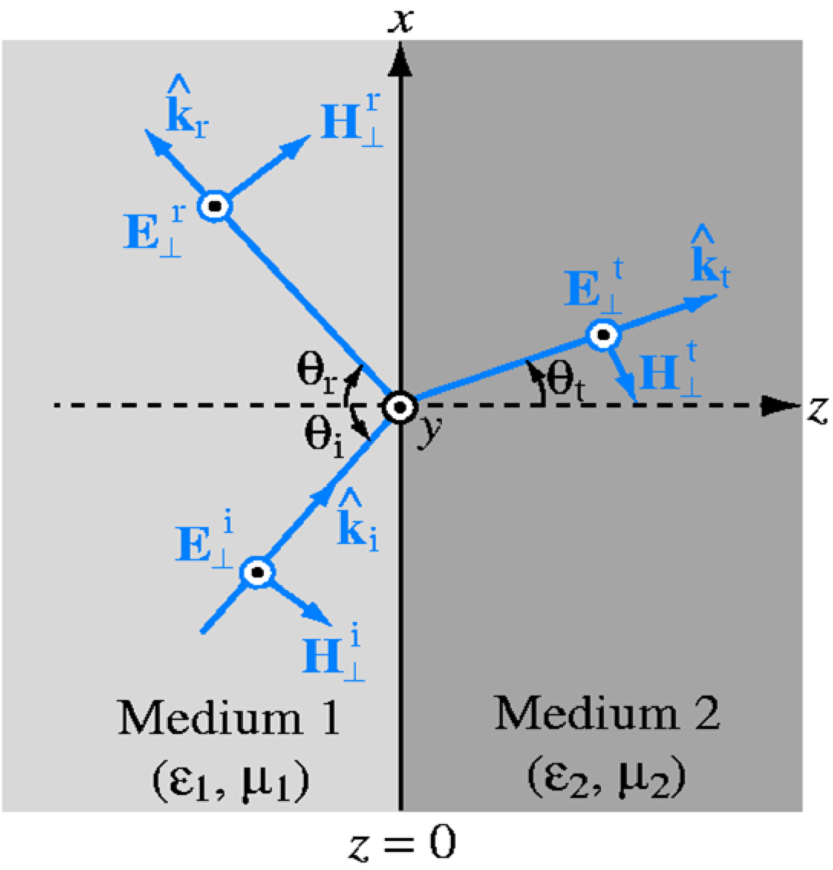
Oblique Incidence

Perpendicular Polarization

- For the second choice, we can assume that the electric field in directed out of the plane of incidence.
- This is called perpendicular polarization since E is perpendicular to this plane of the interaction.
- Note that E is only tangential while H has both components.



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Oblique Incidence

With the two possible polarizations, we have two sets of boundary conditions. Thus, they will behave differently.

Note also that the combination of the two polarizations gives us all possible vector components for E and H.

Now we must apply the boundary conditions to determine how the incident, reflected and transmitted waves relate to one another.

VERY IMPORTANT POINT: Our first task is to match the phase and then we will match the amplitudes. The matching of the phase will allow us to derive one of the most fundamental laws of optics.

Oblique Incidence

Matching the Phase

$$k = k_x \hat{a}_x + k_y \hat{a}_y + k_z \hat{a}_z$$

$$r = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$$

The incident electric field:

$$E_1^+ = E_{m1}^+ e^{-jk_1^+ \cdot \vec{r}} = E_{m1}^+ e^{-jk_1 \sin\theta_i x - jk_1 \cos\theta_i z}$$

The reflected electric field:

$$E_1^- = E_{m1}^- e^{-jk_1^- \cdot \vec{r}} = E_{m1}^- e^{-jk_1 \sin\theta_r x + jk_1 \cos\theta_r z}$$

To match the phase of the terms at z = 0:

$$-jk_1 \sin\theta_i x = -jk_1 \sin\theta_r x$$

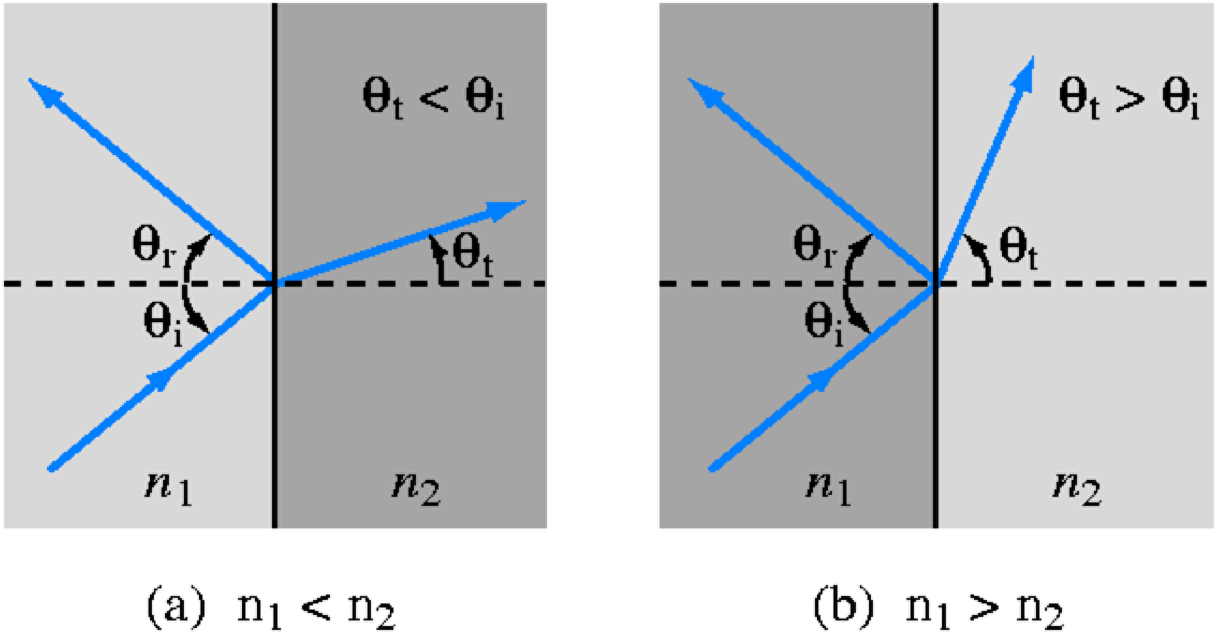
$$\Rightarrow \theta_i = \theta_r \equiv \theta_1$$

Why must this phase be matched?

Oblique Incidence

Matching the Phase

Thus, we have that the angle of incidence equals the angle of reflection, a result that all of us have seen before. Now, we need to see what happens to the transmitted angle.



(a) $n_1 < n_2$ (b) $n_1 > n_2$

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Oblique Incidence

Matching the Phase

Consider now all three waves – incident, reflected and transmitted:

$$E_1^+ = E_{m1}^+ e^{-jk_1^+\vec{r}} = E_{m1}^+ e^{-jk_1\sin\theta_1 x - jk_1\cos\theta_1 z}$$

$$E_1^- = E_{m1}^- e^{-jk_1^-\vec{r}} = E_{m1}^- e^{-jk_1\sin\theta_1 x + jk_1\cos\theta_1 z}$$

$$E_2^+ = E_{m2}^+ e^{-jk_2^+\vec{r}} = E_{m2}^+ e^{-jk_2\sin\theta_t x - jk_2\cos\theta_t z}$$

Matching the phases at z = 0:

$$-k_1\sin\theta_1 x = -k_2\sin\theta_t x$$

$$\Rightarrow \quad k_i\sin\theta_i = k_1\sin\theta_1 = k_2\sin\theta_t = k_2\sin\theta_2$$

Oblique Incidence

Matching the Phase

This is Snell’s Law:

$$k_1 \sin\theta_1 = k_2 \sin\theta_2$$

To put it in its more normal form:

$$k = \omega\sqrt{\mu_0\varepsilon} = \omega\sqrt{\mu_0\varepsilon_0}\boxed{\sqrt{\varepsilon_r}} = \frac{\omega}{c}\sqrt{\varepsilon_r} = \boxed{\frac{\omega}{c}n}$$

$$\Rightarrow \quad n_1 \sin\theta_1 = n_2 \sin\theta_2$$

(n = refractive index)

Oblique Incidence

Matching the Phase

The following also holds:

(n = refractive index)

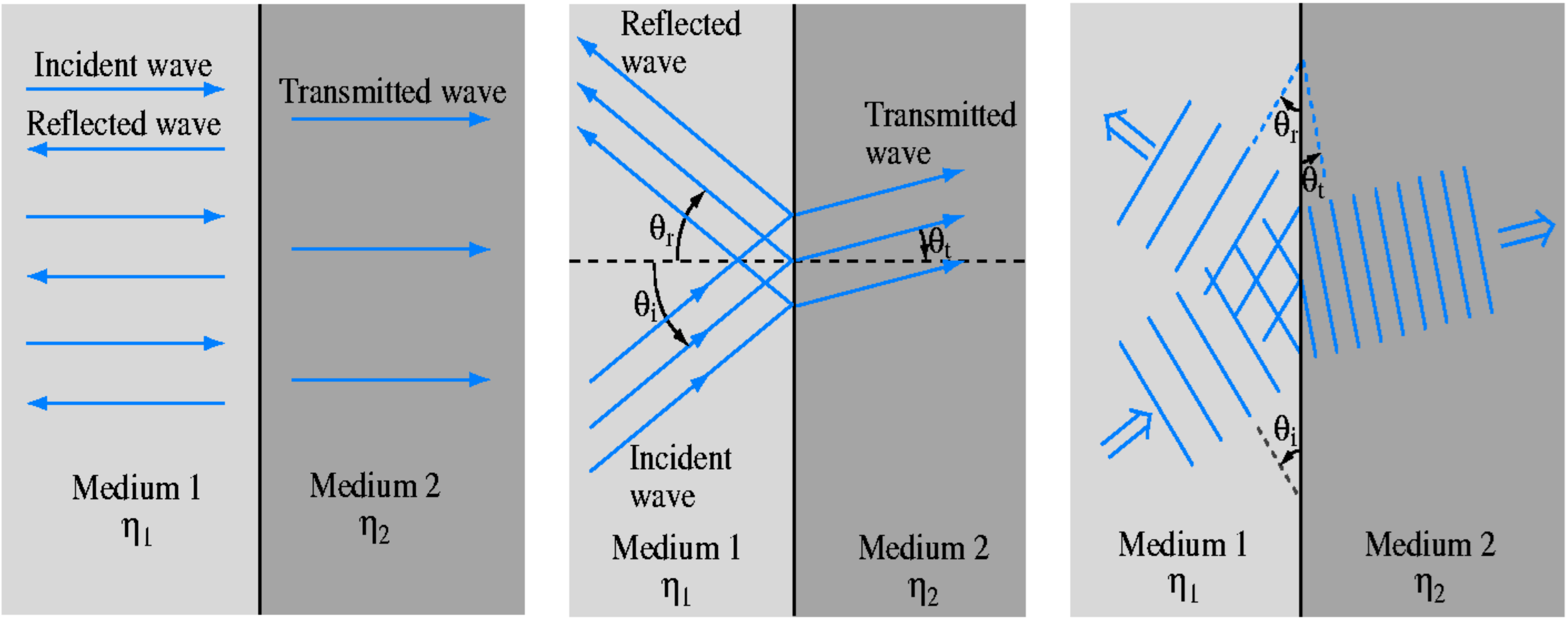
$$\eta = \frac{\eta_0}{n}$$

Therefore:

$$\frac{1}{\eta_1} sin\theta_1 = \frac{1}{\eta_2} sin\theta_2$$

Snell's Law

There are many useful wave representations:



(a) Normal incidence

(b) Ray representation of oblique incidence

(c) Wavefront representation of oblique incidence

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Oblique Incidence

Matching the Phase

Do Lecture 24, Exercise 1 in groups of up to 4.

Snell’s Law

Reflection and Transmission Coefficients

$$E_1^- = E_{m1}^- e^{-jk_1^- \cdot \vec{r}} = E_{m1}^- e^{jk_1 \sin\theta_r x + jk_1 \cos\theta_r z}$$

Recall that we determined in Lecture 21 that tangential components of electric field should be continuous across the boundary. We can write this as:

$$\tilde{E}_y^i + \tilde{E}_y^r = \tilde{E}_y^t$$

(at z=0, the boundary)

Written out in expanded form:

$$E_{m0}^i e^{-jk_1 x sin\theta_1} + E_{m0}^r e^{-jk_1 x sin\theta_1} = E_{m0}^t e^{-jk_2 x sin\theta_2}$$

Snell's Law

Reflection and Transmission Coefficients

Meanwhile, for the magnetic field:

$$\tilde{H}_y^i + \tilde{H}_y^r = \tilde{H}_y^t$$

Cosine terms appear because H is in the plane and may not be truly tangent to the interface; only the component shown below is tangent.

Written out in expanded form:

$$-\frac{E_{m0}^i}{\eta_1} cos\theta_1 e^{-jk_1 x sin\theta_1} + \frac{E_{m0}^r}{\eta_1} cos\theta_1 e^{-jk_1 x sin\theta_1}$$

$$= -\frac{E_{m0}^t}{\eta_2} cos\theta_2 e^{-jk_2 x sin\theta_2}$$

Snell’s Law

Reflection and Transmission Coefficients

Ulaby

$$E^{\mathrm{i}}_{\perp 0} + E^{\mathrm{r}}_{\perp 0} = E^{\mathrm{t}}_{\perp 0},$$

$$\frac{\cos\theta_{\mathrm{i}}}{\eta_1}\left(-E^{\mathrm{i}}_{\perp 0} + E^{\mathrm{r}}_{\perp 0}\right) = -\frac{\cos\theta_{\mathrm{t}}}{\eta_2}\, E^{\mathrm{t}}_{\perp 0}.$$

By algebraically combining the E and H tangential boundary conditions we can get the following:

$$\Gamma_{\perp} = \frac{E_{m1}^{-}}{E_{m1}^{+}} = \frac{\eta_2 \cos\theta_1 - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

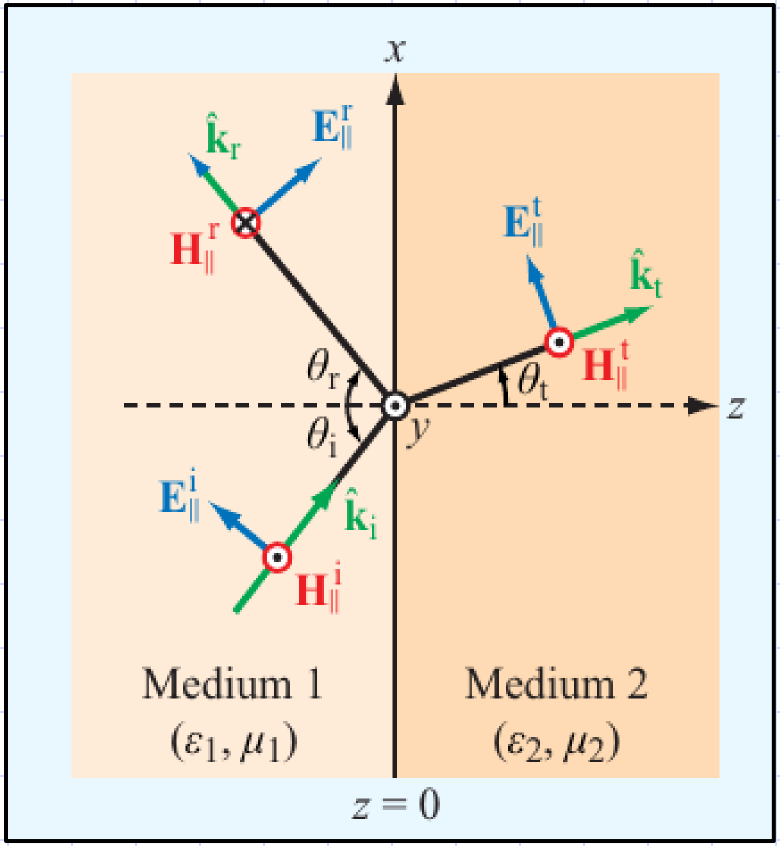
$$\tau_{\perp} = \frac{E_{m2}^{+}}{E_{m1}^{+}} = \frac{2\eta_2 \cos\theta_1}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

$$1 + \Gamma_{\perp} = \tau_{\perp}$$

Snell’s Law

Reflection and Transmission Coefficients

- Now, consider the second (parallel) case.
- We will skip the derivation (which follows very similar steps) and show the results.



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Snell’s Law

Reflection and Transmission Coefficients

Parallel Polarization:

$$\Gamma_{\|} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1}$$

$$\tau_{\|} = \frac{2\eta_2 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1}$$

$$1 + \Gamma_{||} = \tau_{||}\frac{cos\theta_2}{cos\theta_1}$$

Snell's Law

Example 2

A plane wave described by $E_i = 100cos(\omega t - \pi x - 1.73\pi z)\hat{a}_y$ is incident upon a dielectric material with $\varepsilon_r = 4$.

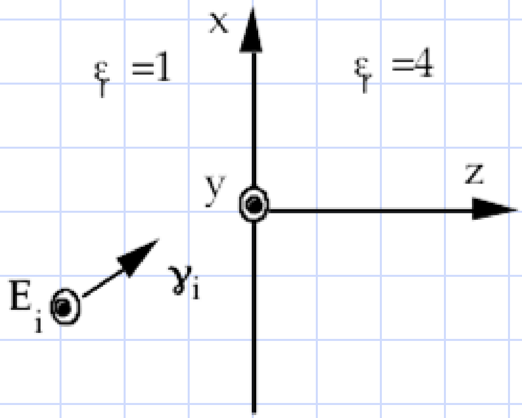
a.) Write E_i in phasor form.

b.) What are γ_i and θ_i ?

c.) What are γ_t and θ_t ?

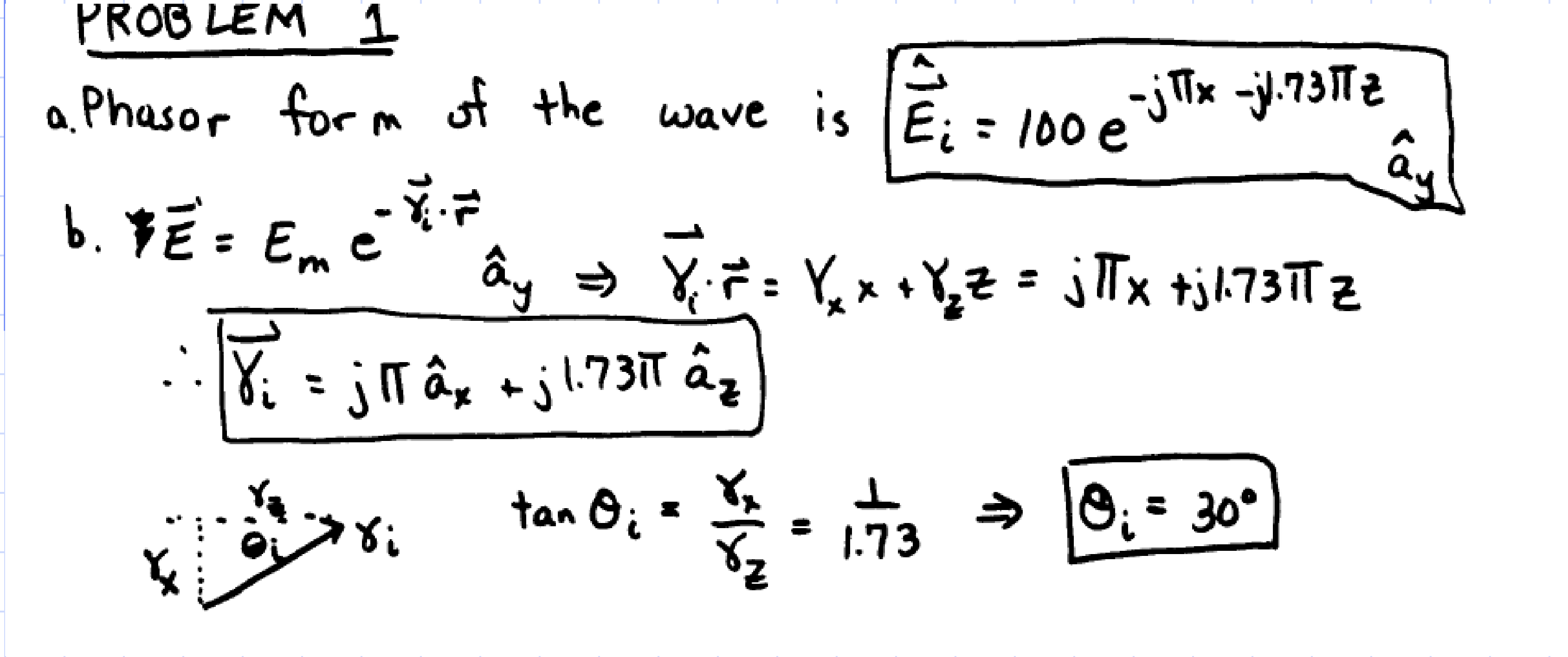
d.) What are the reflection and transmission coefficients?

e.) Write the total electric field phasors in both regions.



Snell's Law

Example 2



PROBLEM 1

a. Phasor form of the wave is $\boxed{\hat{\vec{E}}_i = 100 e^{-j\pi x - j1.73\pi z} \hat{a}_y}$

b. $\vec{E} = E_m e^{-\vec{\gamma}_i \cdot \vec{r}} \hat{a}_y \Rightarrow \vec{\gamma}_i \cdot \vec{r} = \gamma_x x + \gamma_z z = j\pi x + j1.73\pi z$

$\therefore \boxed{\vec{\gamma}_i = j\pi \hat{a}_x + j1.73\pi \hat{a}_z}$

$\tan\theta_i = \frac{\gamma_x}{\gamma_z} = \frac{1}{1.73} \Rightarrow \boxed{\theta_i = 30°}$

Snell's Law

Example 2

C. $\gamma_1 \sin\theta_i = \gamma_2 \sin\theta_t \Rightarrow \sin\theta_t = \frac{\gamma_1}{\gamma_2} \sin 30° = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\, \frac{1}{2} = \sqrt{\frac{1}{4}}\, \frac{1}{2} = \frac{1}{4}$

$\theta_t = \sin^{-1} \frac{1}{4} = \boxed{14.5° = \theta_t}$

$|\vec{\gamma}_t| = j\beta_2 = j\omega\sqrt{\mu_2 \varepsilon_2} = j\omega\sqrt{\mu_0 4\varepsilon_0} = j\underbrace{\omega\sqrt{\mu_0\varepsilon_0}}_{\beta_1} \times 2 = j\beta_1 2 = 2|\vec{\gamma}_1|$

$= 2|\vec{\gamma}_1| = 2\sqrt{\pi^2 + (1.73\pi)^2} = 4\pi$

$\vec{\gamma}_t = |\vec{\gamma}_2| \sin\theta_t\, \hat{a}_x + |\vec{\gamma}_2| \cos\theta_t\, \hat{a}_z = \boxed{j(3.14\hat{a}_x + 12.16\hat{a}_z)}$

Snell’s Law

Example 2

d. $\perp$ polarization

$$\Gamma_\perp = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$

$$\Gamma_\perp = \frac{\sqrt{3}/4 - .968}{\sqrt{3}/4 + .968} = \boxed{-.382}$$

$$\eta_2\cos\theta_i = \sqrt{\frac{\mu_0}{4\epsilon_0}}\cos 30^\circ$$

$$= \eta_0\,\frac{\sqrt{3}}{4}$$

$$\eta_1\cos\theta_t = \eta_0\cos 14.5^\circ$$

$$= .968\,\eta_0$$

$$\boxed{\tau_\perp = 1+\Gamma_\perp = 0.618}$$

Snell's Law

Example 2

e. $$\vec{E}_1 = E_m e^{-\vec{\gamma}_i \cdot \vec{r}}\,\hat{a}_y + \Gamma E_m e^{-\vec{\gamma}_r \cdot \vec{r}}\,\hat{a}_y$$

$$\vec{\gamma}_r = j\,(\pi \hat{a}_x - 1.73\pi \hat{a}_z)$$

(change sign of $\hat{a}_z$ component
of $\vec{\gamma}_i$)

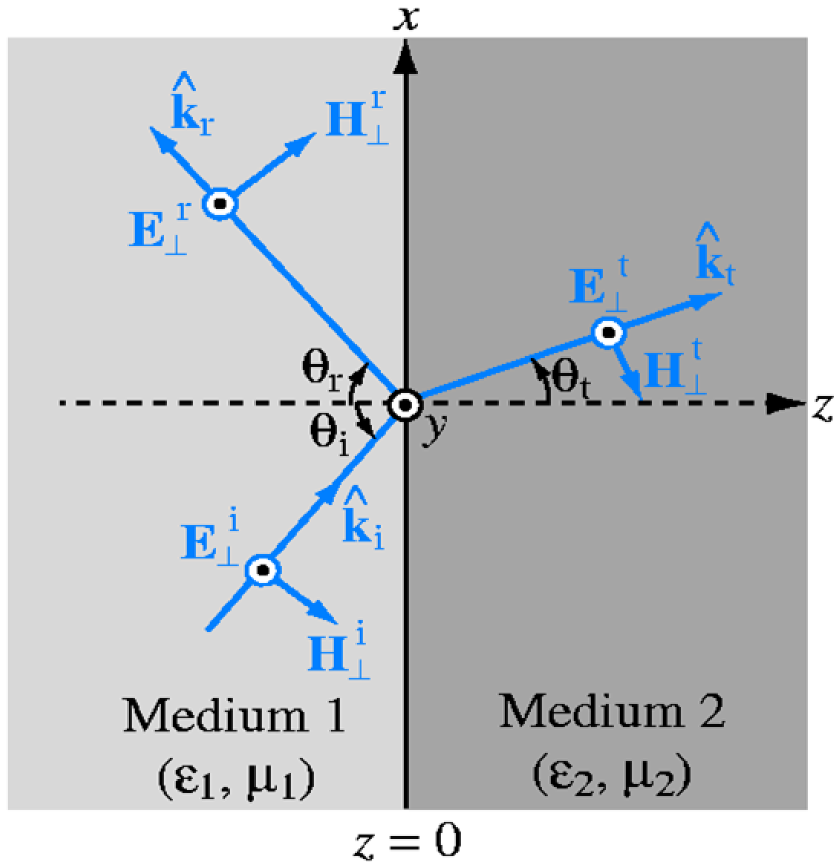
$$\vec{E}_1 = \left\{100\, e^{-j\pi x} e^{-j1.73\pi z} - 38.2\, e^{-j\pi x} e^{+j1.73\pi z}\right\}\hat{a}_y$$

$$= \boxed{100\, e^{-j\pi x}\left\{e^{-j1.73\pi z} - 0.382\, e^{+j1.73\pi z}\right\}\hat{a}_y}$$

$$\vec{E}_2 = T_\perp E_m e^{-\vec{\gamma}_t \cdot \vec{r}} = \boxed{61.8\, e^{-j(\pi x + 12.16 z)}\,\hat{a}_y = \vec{E}_2}$$

Review

- We want to calculate the field magnitude(s) and direction of the EM wave in Medium 2.
- What information do we need, and what is the procedure to do this?



Review

Reflection Coefficient

$$\Gamma_{\perp} = \frac{E_{m1}^{-}}{E_{m1}^{+}} = \frac{\eta_2 \cos\theta_1 - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

$$1 + \Gamma_{\perp} = \tau_{\perp}$$

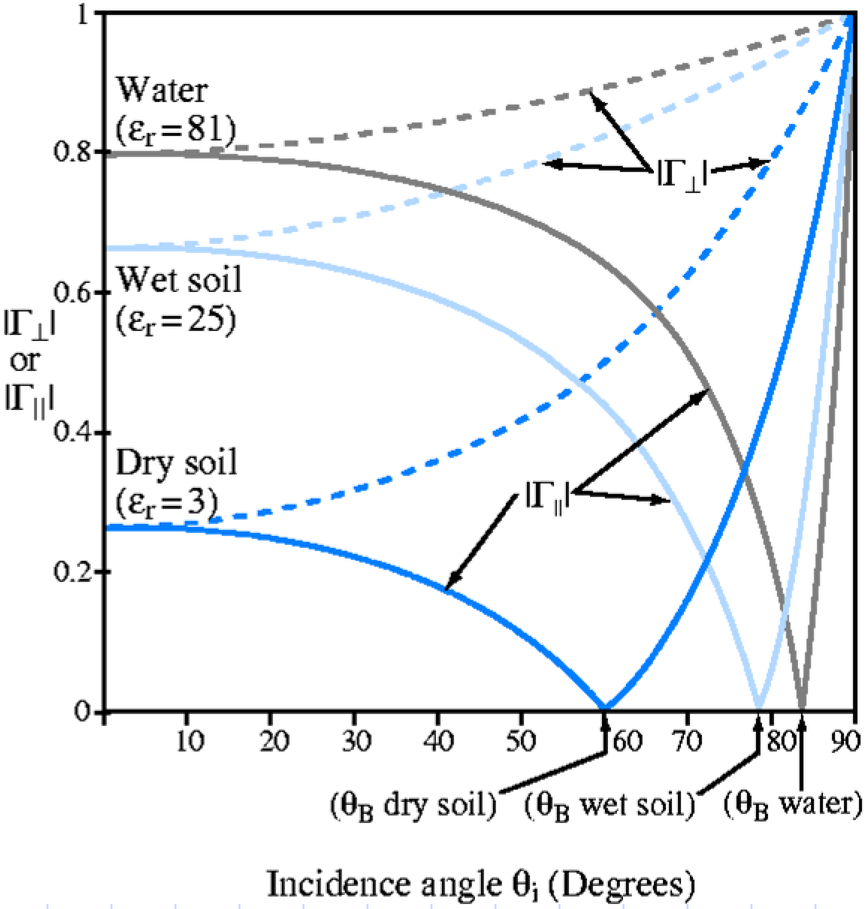
$$\Gamma_{\|} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1}$$

$$1 + \Gamma_{||} = \tau_{||} \frac{cos\theta_2}{cos\theta_1}$$

Widget showing angular dependence of reflection coefficient

Reflection Angular Dependence

Note that the reflection varies with angle. Perpendicular reflects more than Parallel. There is also an angle for which there is no reflection for parallel polarization.

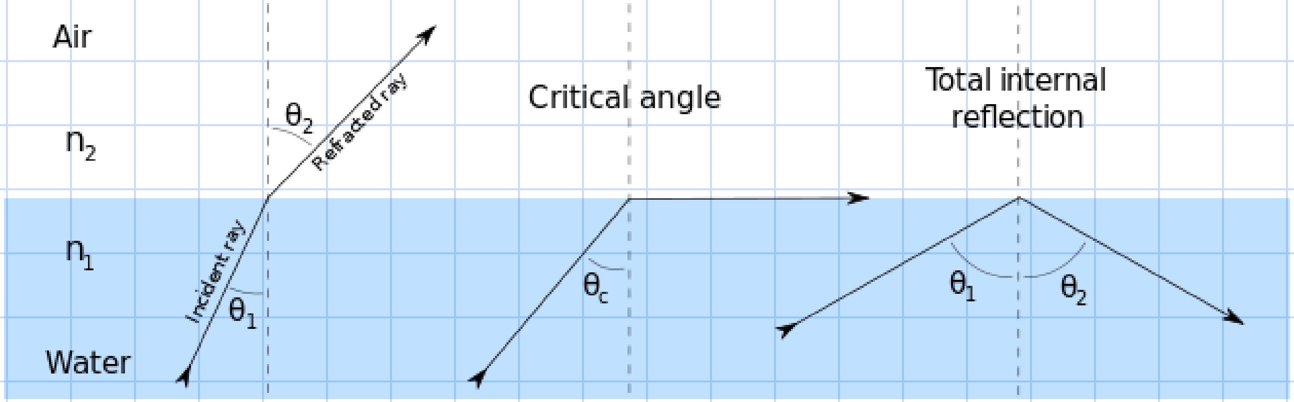


Reflection Angular Dependence

Critical Angle

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

- As angle of incidence increases, angle of transmission will also increase.
- If incident EM wave is inside the area of higher n, each degree of change in θ_1 results in more than 1 degree of change to θ_2.



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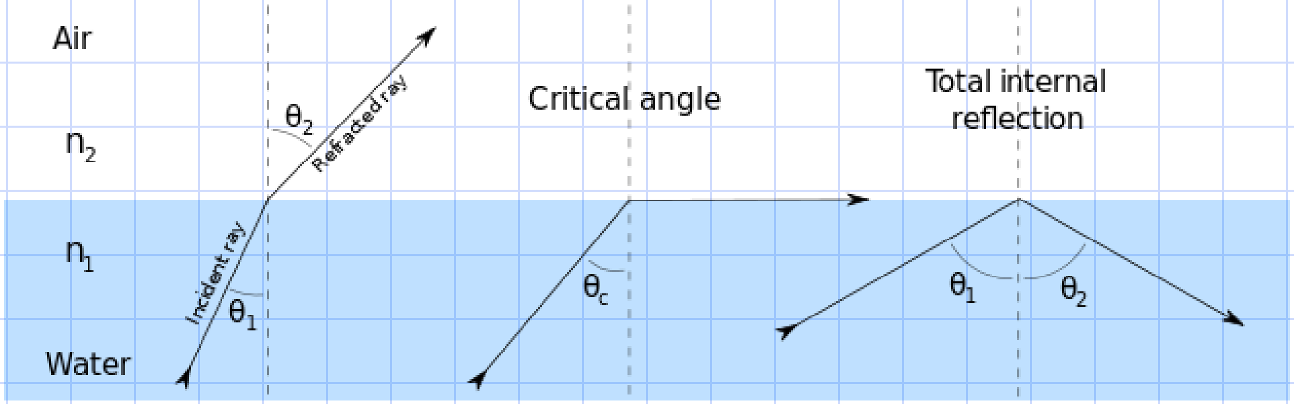
Reflection Angular Dependence

Critical Angle

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

- When transmission angle reaches 90 degrees, there will be no transmission.
- Beyond this angle, there will be total reflection.



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Reflection Angular Dependence

Critical Angle

$$k_1 \sin \theta_c = k_2 \sin \theta_2$$

$$\sin \theta_2 = \sin 90^\circ = 1$$

$$k_1 \sin \theta_c = k_2$$

$$\sin \theta_c = \frac{k_2}{k_1}$$

$$k = \omega \sqrt{\mu_0 \varepsilon_0} \sqrt{\varepsilon_r}$$

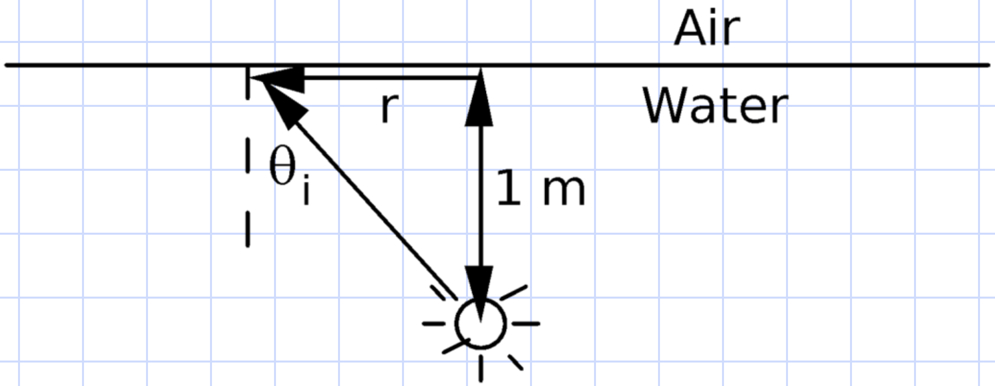
$$\sin \theta_c = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

(Note that when $\varepsilon_2 > \varepsilon_1$, this will not have a real-valued answer.)

Reflection Angular Dependence

Example 3

For visible light, the index of refraction for water is n = 1.33. If we put a light source 1 meter under water and observe it from above the surface of the water, what is the largest θ_i for which light will be transmitted?
How large will the circle of illumination be?



Reflection Angular Dependence

Example 3

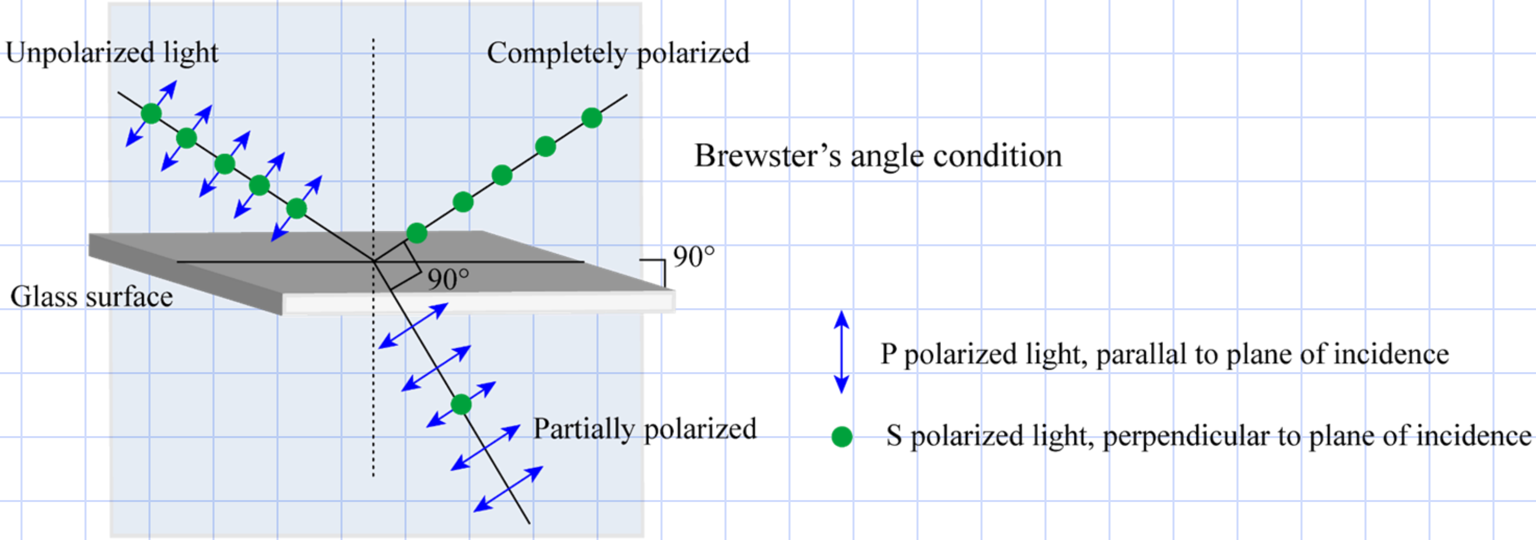
$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right) = \sin^{-1}\left(\frac{1}{1.33}\right) = \boxed{48.8^\circ}$$

$$\tan\theta_i = r = \boxed{1.14\ \text{m}}$$

Reflection Angular Dependence

Brewster's Angle

[Wikimedia Commons]



- At the Brewster's Angle or polarizing angle, the reflection coefficient is 0 for one polarization of light.
- This means that only polarized light is reflected.

Reflection Angular Dependence

Brewster's Angle

$$\Gamma_{\|} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1}$$

Set this equal to zero:

$$\boxed{\eta_2 \cos\theta_{\mathrm{i}} = \eta_1 \cos\theta_{\mathrm{t}}.}$$

… to derive this. Looks like Snell's law but with cosines instead of sines.

By once again combining the two equations, we get:

$$\boxed{\sin\theta_{\mathrm{B}\|} = \sqrt{\frac{1-(\epsilon_1\mu_2/\epsilon_2\mu_1)}{1-(\epsilon_1/\epsilon_2)^2}}\ .}$$

Note that there will be no Brewster angle if the permittivities are the same.

Reflection Angular Dependence

Brewster's Angle

Wikimedia Commons

When the permeabilities of the two materials are the same, this becomes

$$\begin{aligned}\theta_{B\parallel} &= \sin^{-1}\sqrt{\frac{1}{1+(\epsilon_1/\epsilon_2)}}\\ &= \tan^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}} \qquad (\text{for } \mu_1 = \mu_2).\end{aligned}$$

Reflection Angular Dependence

Brewster’s Angle

- The Brewster’s angle is why glare from the surface of water can be easily blocked with sunglasses.

Wikipedia

Brewster Angle

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

$$\boxed{\eta_2 \cos \theta_{\mathrm{i}} = \eta_1 \cos \theta_{\mathrm{t}}.}$$

$$\Gamma_{\perp} = \frac{E_{m1}^{-}}{E_{m1}^{+}} = \frac{\eta_2 \cos \theta_1 - \eta_1 \cos \theta_2}{\eta_2 \cos \theta_1 + \eta_1 \cos \theta_2}$$

Combining these two equations and setting the incident angle to be the Brewster angle, we get:

$$\boxed{\sin \theta_{\mathrm{B}\perp} = \sqrt{\frac{1 - (\mu_1 \epsilon_2 / \mu_2 \epsilon_1)}{1 - (\mu_1/\mu_2)^2}}\ .}$$

(Ulaby pg. 375)

Brewster Angle

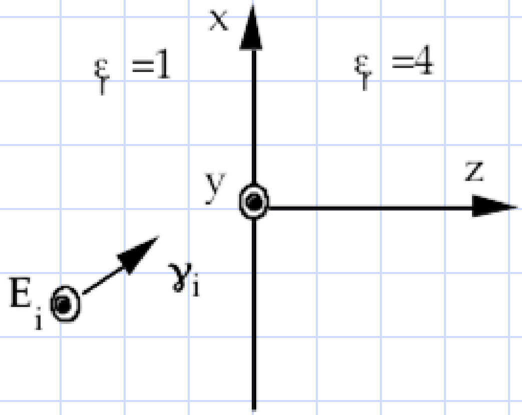
$$\boxed{\sin\theta_{\mathrm{B}\perp} = \sqrt{\frac{1-(\mu_1\epsilon_2/\mu_2\epsilon_1)}{1-(\mu_1/\mu_2)^2}}\ .}$$

Note that this is undefined when the permeabilities of the two materials are the same. But when they are different, some Brewster angle will exist.

Example 2

A plane wave described by $E_i = 100cos(\omega t - \pi x - 1.73\pi z)\hat{a}_y$ is incident upon a dielectric material with ε_r = 4.

a.) Write E_i in phasor form.
b.) What are γ_i and θ_i ?
c.) What are γ_t and θ_t ?
d.) What are the reflection and transmission coefficients?
e.) Write the total electric field phasors in both regions.



Brewster Angle

Do Lecture 24, Exercise 2 in groups of up to 4.