Hayden Fuller
Fields and Waves HW4

1) Differential Surface Area and Volume Elements

We will be using differential surface area and volume elements in calculations throughout this course, so gaining a better understanding of where they come from can be very helpful. In this homework problem, you'll derive each of the differential surface area and volume elements in cylindrical and spherical coordinates.

The differential surface area elements can be derived by selecting a surface of constant coordinate (z=z0 in Cartesian coordinates for example) and then varying the other two coordinates to trace out a small 2D shape (x and y in Cartesian coordinates). The surface normal vector will be perpendicular to the surface swept by varying these two coordinates ($\hat{z}$ direction in this case). See the differential surface area element for the surface of constant z below:

The surface area element for the surface of constant z is then given by $d\mathbf{S}z=(dx)(dy)\hat{\mathbf{z}}$.

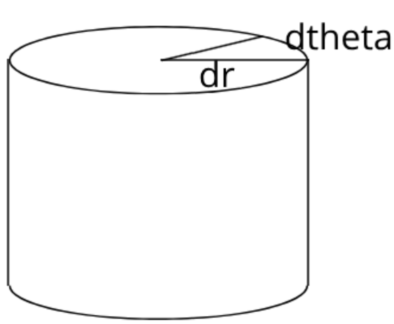
Performing the same procedure for a surface of constant x gives $d\mathbf{S}x=(dy)(dz)\hat{\mathbf{x}}$ and for a surface of constant y: $d\mathbf{S}y=(dx)(dz)\hat{\mathbf{y}}$.

The differential volume element is derived by varying each of the coordinates, then calculating the volume that those variations sweep out in space. In Cartesian coordinates, variations in x, y, and z sweep out a cube with sides of length dx, dy, and dz, giving the differential volume element dV = (dx)(dy)(dz).

- Select a coordinate that will be fixed (i.e. z if z = z0)
- Draw the surface that results from fixing that coordinate and varying the other two coordinates.
- Determine the area traced out by varying those two coordinates.
- Write the expression for the differential surface area element dS.
- After you have determined dS for the three surfaces of constant coordinate, determine the differential volume element using the same approach as above to determine the length of each side of the differential volume element shape and multiplying them together.

a) Cylindrical coordinates:
    i) Surface of constant z

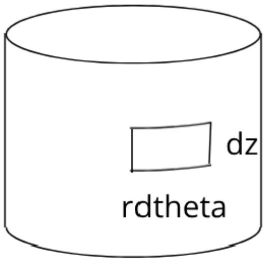
z is fixed
wedge, radius r, circumference phi (*r)



phi * r^2
dS_z=r * dr * dphi

ii) Surface of constant r

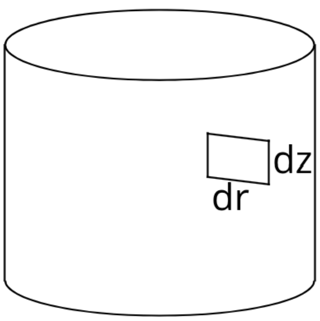
r is fixed
Rectangle, height z, width phi (*r)



phi * r * z
dS_r=r * dphi * dz

iii) Surface of constant φ

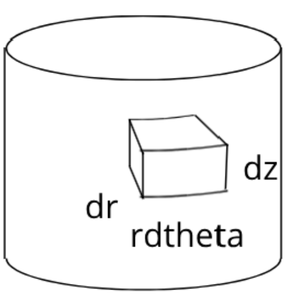
phi is fixed
Rectangle, height z, width r



r * z
dS_phi=dr * dz

iv) Differential volume dV

None are fixed
Square wedge prism, radius r, length z, circumference phi (*r)

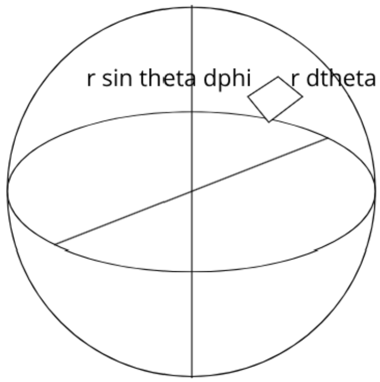


phi * r^2 * z
dV=r dr dphi dz

b) Spherical coordinates:

i) Surface of constant r

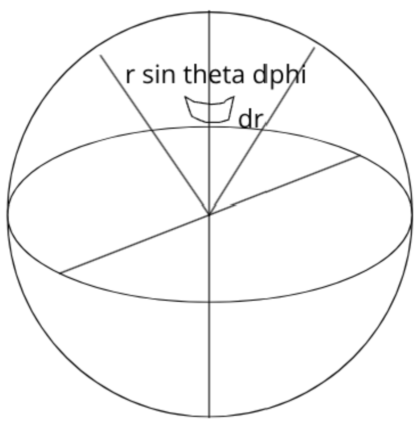
r is fixed
Square, height theta (*r), width phi (*r sin(theta))



theta * phi * r^2 * sin(theta)
dS_r= r^2 sin(theta) dtheta dphi

ii) Surface of constant θ

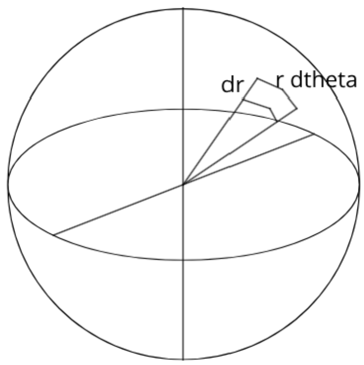
theta is fixed
Wedge, radius r, circumference phi (*r sin(theta))



r^2 * sin(theta) * phi
dS_theta=r sin(theta) dr dphi

iii) Surface of constant φ

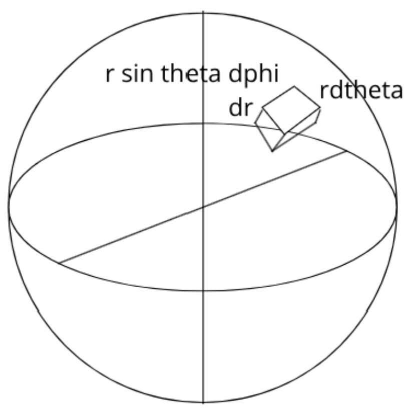
phi is fixed
Wedge, radius r, circumference theta (*r)



r^2 * theta
dS_phi=r dr dtheta

iv) Differential volume dV

None are fixed
Square wedge pyramid, radius r, circumference theta (*r), circumference phi (*r sin(theta))



r^r * sin(theta) * theta * phi
dV=r^2 sin(theta) dr dtheta dphi

2) Relationship between Divergence and Flux
During lecture, we said that divergence is a “local measure of flux”. In this problem, you will relate the idea of total flux through a closed surface to divergence. For this problem, we’ll be using the surfaces of a cube as our closed surface. The cube has side lengths Δx, Δy, and Δz.
    a) Find the total net flux through the surfaces (all 6) of the cube for a vector field F = <Fx, Fy, Fz>. The left-hand side of your equation will be “total net flux” and the right-hand side will be a mathematical expression. For example, the net flux through the surfaces of constant x at x = 0 and x = x0 is:

Net Flux$x$=
$\{[F(x0,y,z)\hat{x}\cdot(\Delta y)(\Delta z)\hat{x}]+[F(0,y,z)\hat{x}\cdot(\Delta y)(\Delta z)(-\hat{x})]\}$
$=[Fx(x0,y,z)-Fx(0,y,z)]\Delta y\Delta z$

Total net flux =
[F(x0,y,z)-F(0,y,z)] dy dz \hat x
+ [F(x,y0,z)-F(x,0,z)] dx dz \hat y
+ [F(x,y,z0)-F(x,y,0)] dx dy \hat z
=
[Fx(x0,y,z)-Fx(0,y,z)] dy dz
+ [Fy(x,y0,z)-Fy(x,0,z)] dx dz
+ [Fz(x,y,z0)-Fz(x,y,0)] dx dy

    b) Divide both sides of your equation from a) by ΔV = ΔxΔyΔz. Before dividing by ΔV, the left side of the equation should be the “total net flux” and the right side should be your expression from a) for the total net flux through the cube’s surfaces.

Total net flux / dV =
[Fx(x0,y,z)-Fx(0,y,z)] /dx
+ [Fy(x,y0,z)-Fy(x,0,z)] /dy
+ [Fz(x,y,z0)-Fz(x,y,0)] /dz

    c) Each of your terms in the right-hand side of the expression for the total net flux should contain an expression of the form $Fx(x0,y,z)-Fx(0,y,z)$ for each of the coordinates. For each of the coordinates, replace this expression with the equivalent of ΔFx. This is the change in flux in the x direction over a distance Δx.

Total net flux / dV = dFx/dx + dFy/dy + dFz/dz

d) Now, take the limit of Δx, Δy, and Δz as they become infinitesimally small (i.e. $\Delta x \rightarrow \partial x$). Also do this for all the ΔF terms and ΔV terms. You should obtain the mathematical expression for the divergence of the vector field F on the right-hand side of your equation.

Total net flux / dV = dFx/dx + dFy/dy + dFz/dz = \nabla F

e) Looking back at your equation, state in your own words what the divergence of a vector field means in terms of flux.

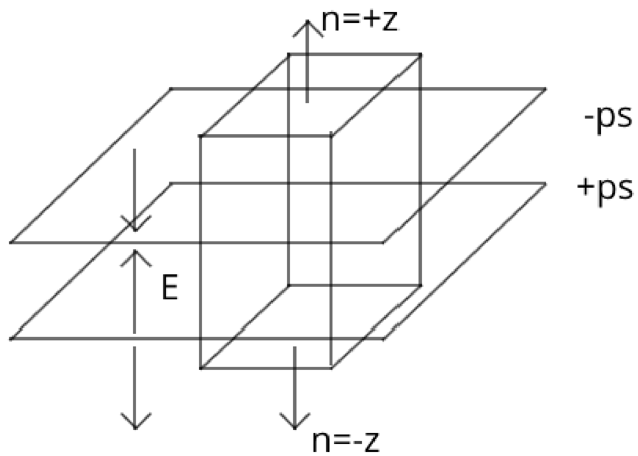
I like to think of it as a “three-dimensional derivative”, as opposed to giving the change in a scalar value over an infinitesimal length, it gives the change in a three dimensional vector over an infinitesimal volume.

3) Electrostatics Between Two Infinite Sheets of Charge
   Consider the two infinite sheets of charge below. The top sheet of charge, located at z = d, has a charge density of -ρs [C/m2] and the bottom sheet located at z = 0, has a charge density of +ρs [C/m2].
   a) Using symmetry arguments, determine which components of the electric field E = <Ex, Ey, Ez> are non-zero.

Ex and Ez are zero because the plane extends infinitely in those directions

   b) Choose a Gaussian surface to use with Gauss's Law to determine the electric field generated by each of the sheets of charge. Draw a diagram that you would use to determine the electric field outside the sheets of charge. Show both sheets of charge, the location of the Gaussian surface, electric field lines, and all relevant surface normal vectors.

Cube



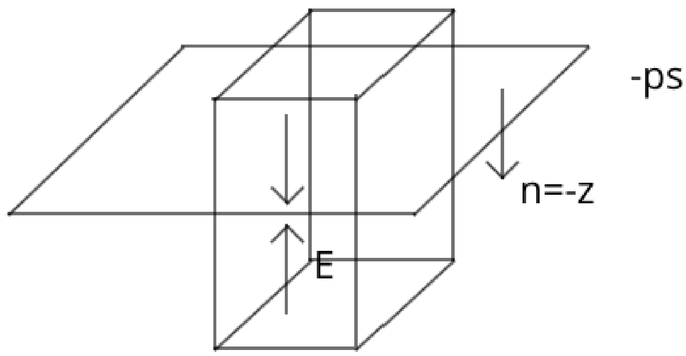
The cube is some size greater than d, extending both above d and below 0, and is oriented square to the infinite plane and coordinate system.
The electric field between the sheets is vertical and pointed upwards, from positive sheet to negative sheet. The field above the sheets is vertical and pointed down towards the negative sheet. The field below the sheets is vertical and pointed down away fro mthe positive sheet.
The normal vectors point out and away from the cube. The top normal is pointed upwards and the bottom is pointed downwards

   c) Use Gauss's Law to determine the electric field outside the sheets of charge.

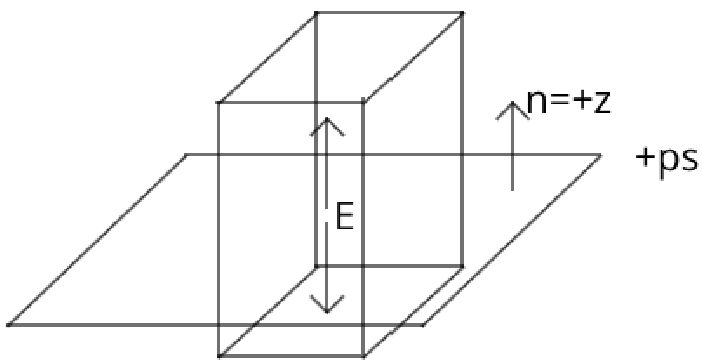
\oint D dS = Q_enclosed = dxdy(+ps) + dxdy(-ps) = 0

   d) For each sheet of charge, draw the sheet of charge, the location of the Gaussian surface you'll use to calculate the electric field from that sheet, the direction of the electric field lines from each sheet, and the relevant surface normal vectors.

Top:



Bottom:



Top: cube around a section of the sheet, normal vector facing downwards, electric field vertical and pointing towards the sheet.
Bottom: cube around a section of the sheet, normal vector facing upwards, electric field vertical and pointing away from the sheet.

e) Use Gauss's Law to determine the electric field between the sheets of charge (using superposition).

Top: \oint D dS = Q_enclosed = dxdy(-ps)
Bottom: \oint D dS = Q_enclosed = dxdy(+ps)
Superposition: ps/e_0 z

f) Determine the potential outside and between the sheets of charge.

$V=-\int E dl$

Outside: $E=0$, so $V=0$

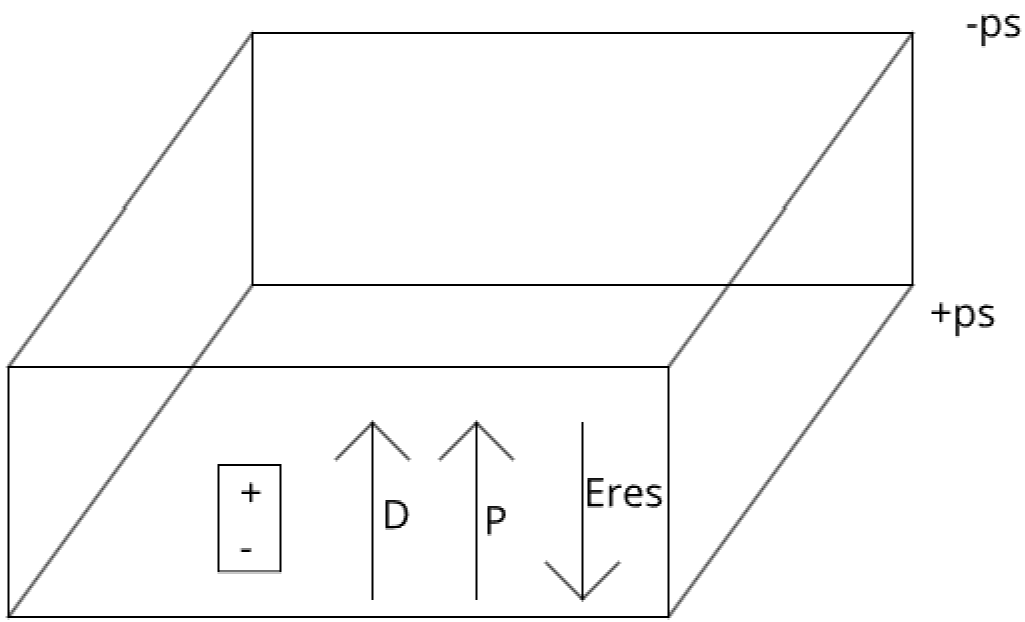
Inside: $E=ps/e_0 z$

$V=-\int_0^z ps/e_0 z dz z = -ps/e_0 z$

$V=-ps z / e_0$

4) Dielectrics and the Effect of Polarization
For this problem, you will be investigating the effect of adding a dielectric between the sheets of charge from Problem 3.

a) Replace the air ($\varepsilon = \varepsilon 0$) between the sheets of charge with a dielectric, whose permittivity is given by $\varepsilon = \varepsilon 0 \varepsilon r$, where $\varepsilon r > 1$. Draw a diagram that includes:
   i) both sheets of charge and their charge densities
   ii) the direction of the displacement field ($D0 = \varepsilon 0 E$) between the sheets of charge as determined in Problem 3
   iii) the orientation of the dipoles in the dielectric
   iv) the orientation of the electric field generated by the dipoles Eresponse
   v) the direction of the polarization field P (remember that P points in the direction of the electric dipole moment d)



b) Write the expression for the displacement field in the dielectric D as a function of the electric field generated by the sheets of charge, E, and the polarization field, P.

D=e0 E + P

c) Replace D with $\varepsilon 0 \varepsilon r E$ in your expression from b) and solve for the polarization field, P. (Hint: it should be expressed in terms of $\varepsilon 0$, εr, and E). What is the polarization field when $\varepsilon r = 1$? What does this case physically correspond to?

E0 er E = e0 E + P
P=e0 E (er-1)
er = 1 => P=0, free space

d) Consider Gauss's Law for both the case of air between the sheets of charge and a dielectric between the sheets of charge. In both cases, the total charge is the same, but the displacement D field is different ($D0 = \varepsilon 0E$ for air, and $D = \varepsilon 0\varepsilon rE$ with $\varepsilon r > 1$ for a dielectric).

$\oint \mathbf{D0}\cdot d\mathbf{S}=\oint \mathbf{D}\cdot d\mathbf{S}=Qenc$

What must be true about the magnitude of the electric field in a dielectric (call this Edielectric) relative to the magnitude of the electric field in air (call this E0) in order for Gauss's Law to hold? What causes this change in the magnitude of the electric field in a dielectric?

\oint e0 E0 ds = \oint e0 er Edielectric ds

er>1 => Edielectric < E0

The polarization of the dielectric creates a field that opposes the applied field

e) Calculate the potential Vd between the sheets of charge in the dielectric (you may make use of your result from Problem 3). Is it larger or smaller than in the case of an air gap between the sheets of charge?

V= -ps z / e_0, so replacing e_0 when er>1, V decreases

smaller

f) Calculate $C = |Q||\Delta V|$ for an area A of the sheets and compare it with the answer you would get in free space. You should see that adding a dielectric between the sheets of charge enables the storage of the same amount of charge for a smaller potential difference.

Return to your diagram from part a) and look at the charge distribution in the dielectric near the sheets of charge. How is it possible that the same charge density on the sheets of charge leads to a smaller potential difference between them when a dielectric is the medium? (Expressed in another way, how does inserting a dielectric medium enable the storage of more charge for a given potential difference than air?)

Vd=ps d / e0er

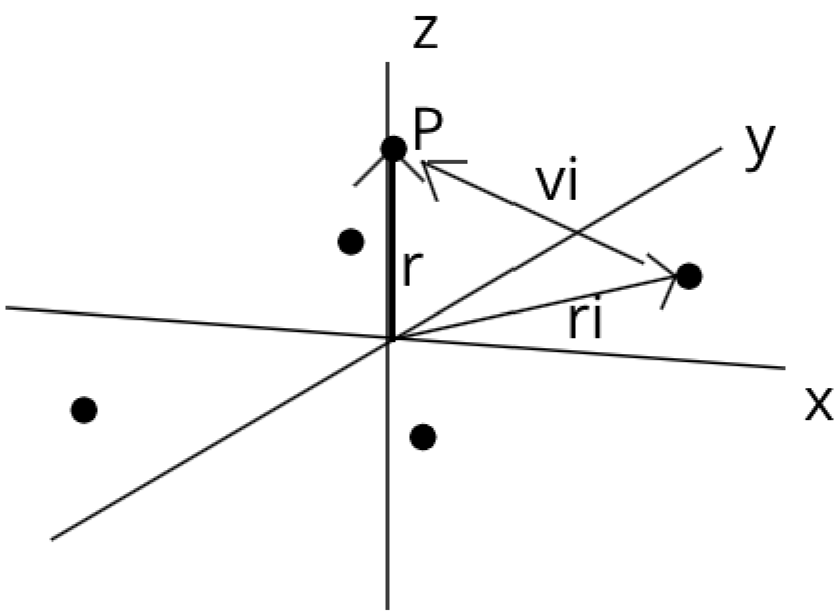
C=psAe0er / psd = e0erA/d

The polarization of the dielectric creates an opposing field, decreasing the potential difference between the plates, while retaining the same charge, allowing for more charge at the same potential difference.

5) Coulomb's Law

A square with vertices (-x0, -y0, 0), (+x0, -y0, 0), (-x0, +y0, 0) and (+x0, +y0, 0) has a point charge +q at each vertex. You will calculate the electric field at a location (0, 0, z) and the force on a charge placed at that point. The center of the square is located at (0, 0, 0).

a) Draw a diagram containing each of the 4 charges, the vector r pointing from the origin to point P = (0, 0, z), the vector ri pointing from the origin to one of the charges at vertex vi = (xi, yi, 0), and the vector pointing from vi to P. What is the expression for the vector pointing from vi to P in terms of r and ri?



vi=r-ri

b) In which direction does the electric field at P = (0, 0, z), due to the four charges point? Consider all values of z. Give your reasoning.

For z>0, E points up, away from the positive charges, and vertical due to symmetry.

c) Write the expression for the magnitude of the electric field at point P = (0, 0, z), generated by a single point charge located at one of the vertices vi = (xi, yi, 0). Express your answer in terms of xi, yi and z0.

E=\hat r q/(4pie0 r^2)

E=q/(4 pi e_0 (xi^2 + yi^2 + zi^2))

d) Write the expression for the electric field at point P = (0, 0, z), due to all charges at the vertices of the square.

E=4 * \hat z * q/((4 pi e_0 zi^2)^3/2)

E=\hat z * q/((pi e_0 zi^2)^3/2)

e) If you were to place a point charge +q at P, what is the force F felt by the charge?

F=qE

F=\hat z * q^2/((pi e_0 zi^2)^3/2)

f) If you wanted to maximize the force on the charge by changing its position z, where would you place it so that it feels the maximum force? If you were to change the positions of the charges at the corners of the square ±x0 and ±y0, where would you place them to maximize the force on the charge?

The Z position to maximize force would be z=1/sqrt(2) * sqrt(x0^2+y0^2), maximizing the vertical component from each charge.

The x0 and y0 positions to maximize force would be 0 and 0 so there’s nothing wasted on horizontal components