ECSE 2500
Jan. 26
Lec5

Topic: Probability in continuous space and Conditional Probability

☐ Probability in Continuous Sample Space

Because $P(\text{Exact one outcome in Continuous Space}) = 0$,

a natural idea is to assign based on the interval, e.g., X continuous number $[0, 1]$,

Define $P(a \le x \le b) = \underbrace{b - a}_{\ge 0}$, for $0 \le b \le a \le 1$.

Axiom 1

This is called the uniform distribution.

It says that the probability of getting a value in an interval equals the width/length of that intervals. Therefore,

$P(\underbrace{[0, 1]}_{S}) = 1 - 0 = 1$ ✓ Validates Axiom 2

$P([0, \frac{1}{2}]) = \frac{1}{2}$

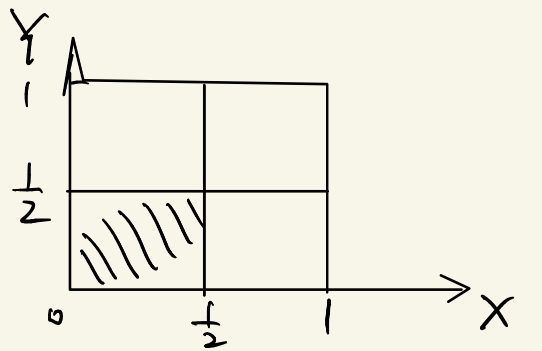
More example on uniform distribution, which can be evaluated geometrically.

$\boxed{\text{Example}}$ Pick two numbers X and Y. Each of them is between 0 and 1.

What is

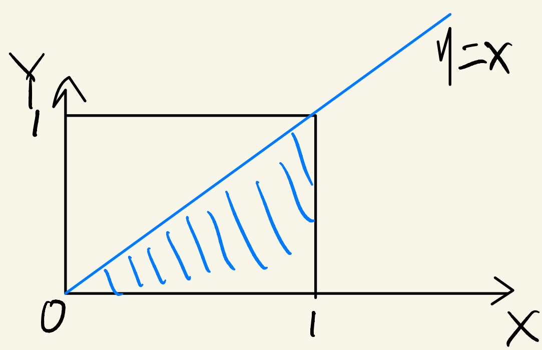
$$P\left(X \le \tfrac{1}{2},\ Y \le \tfrac{1}{2}\right)?$$

$$= \frac{1}{4}$$

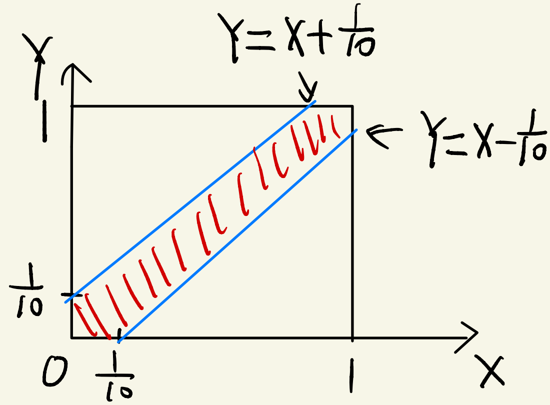


$$P(X \ge Y)$$

$$= \frac{1}{2}$$



$$P\left(\underbrace{|X-Y| < \tfrac{1}{10}}\right)\ ?$$



$$= \underbrace{1 - 2 \times \left(\tfrac{9}{10}\right)^2 \times \tfrac{1}{2}}_{\frac{81}{100}}$$

$$= \frac{19}{100}$$

- An interesting example on discrete sample space

The birthday problem

P(at least 2 students of a class of 60 students have the same birthday) ?

= 1 − P(no 2 students share the same birthday)

$$= 1 - \frac{\text{all combination of 60 different birthdays}}{\text{All possible combination of 60 birthdays}}$$

$$= 1 - \frac{365 \cdot 364 \cdot 363 \cdots \cdot (365 - 60 + 1)}{365^{60}}$$

$= 1 - 0.0008 = 0.9992$ ← large probability

□ Introduce Conditional Probability

Motivation: We often have encountered problems of the following form, "What is the probability of X, given that I observed that Y happens?"

Any kind of diagnosis problems where we see some symptoms and want to figure out the causes/disease will have the conditional probability

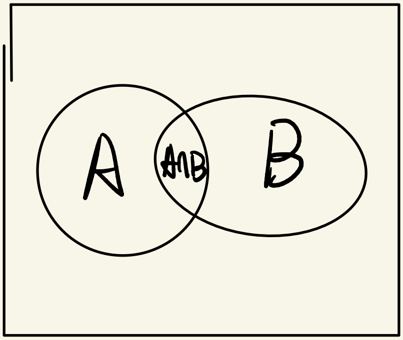
$$P(\text{Covid-19} \mid \text{Test kit is positive})$$

Mathematical definition

We have two events A and B, and we define the notation

$$P(A \mid B)$$

to measure the probability of event A given that the event B happens.



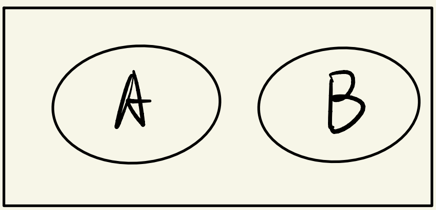
We ~~define~~ that

$$P(A|B) := \frac{P(A \cap B)}{P(B)}$$

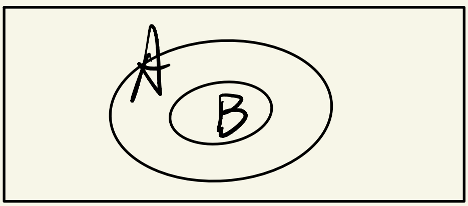
(as long as $P(B) > 0$)

This makes sense; It relates the sample space of event B to A and normalize so that $P(B) = 1$.

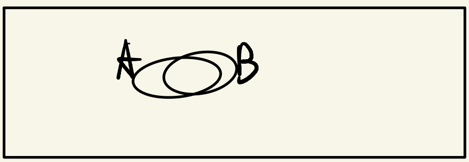
How likely is $A \cap B$ relative to B?



$$P(A|B) = 0$$



$$P(A|B) = 1$$



$P(A), P(B)$ both small, but $P(A|B)$ is large

Rare disease

Rare symptom

Or, we may get very little information about Event A given Event B.

$$\underset{\text{Weather in LA}}{P(\overset{\text{Weather in Albany}}{A} \mid B)} = P(A)$$

In this case, we call Event A and B are independent.

☐ Consider again Rolling Two 6-sided Dice and Recording their Sum.

Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$P(\underbrace{\text{Second Die is Even}}_{A} \mid \underbrace{\text{Dice sum is 4}}_{B})$$

$$= \frac{P(A \cap B)}{P(B)}$$

$$= \frac{\frac{1}{36}}{\frac{1}{12}} = \frac{1}{3}$$

$$P(B) = \frac{1}{12}$$

$$P(A \cap B) = P(\{2,2\}) = \frac{1}{36}$$

$$P(\underbrace{\text{Sum is} \geq 8}_{A} \mid \underbrace{\text{One Die is} \geq 4}_{B})$$

$$= \frac{P(A \cap B)}{P(B)}$$

$$P(B) = \frac{\quad}{36}$$

1	1
1	2
1	3
1	4
1	5
1	6
2	1
2	2

⋮