Fields and Waves I

Lecture 25

Review

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Announcements

- End-of-Semester Survey
- Final Office Hours
- Free-Form Exam 3 Reworks

Review

Reflection Coefficient

$$\Gamma_{\perp} = \frac{E_{m1}^{-}}{E_{m1}^{+}} = \frac{\eta_2 \cos\theta_1 - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

$$1 + \Gamma_{\perp} = \tau_{\perp}$$

$$\Gamma_{\parallel} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1}$$

$$1 + \Gamma_{||} = \tau_{||} \frac{cos\theta_2}{cos\theta_1}$$

Widget showing angular dependence of reflection coefficient

Reflection Angular Dependence

Critical Angle

$$k_1 \sin \theta_c = k_2 \sin \theta_2$$

$$\sin \theta_2 = \sin 90° = 1$$

$$k_1 \sin \theta_c = k_2$$

$$\sin \theta_c = \frac{k_2}{k_1}$$

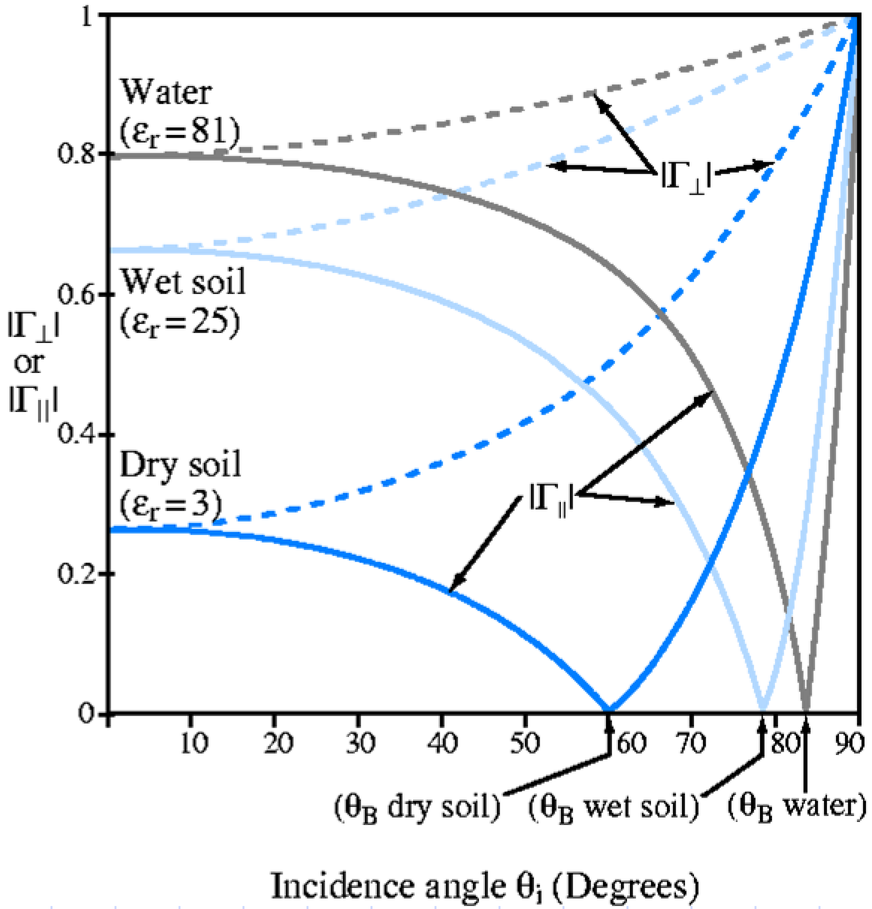
$$k = \omega \sqrt{\mu_o \varepsilon_o} \sqrt{\varepsilon_r}$$

$$\sin \theta_c = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

(Note that when $\varepsilon_2 > \varepsilon_1$, this will not have a real-valued answer.)

Reflection Angular Dependence

Note that the reflection varies with angle. Perpendicular reflects more than Parallel. There is also an angle for which there is no reflection for parallel polarization.



Reflection Angular Dependence

Brewster's Angle

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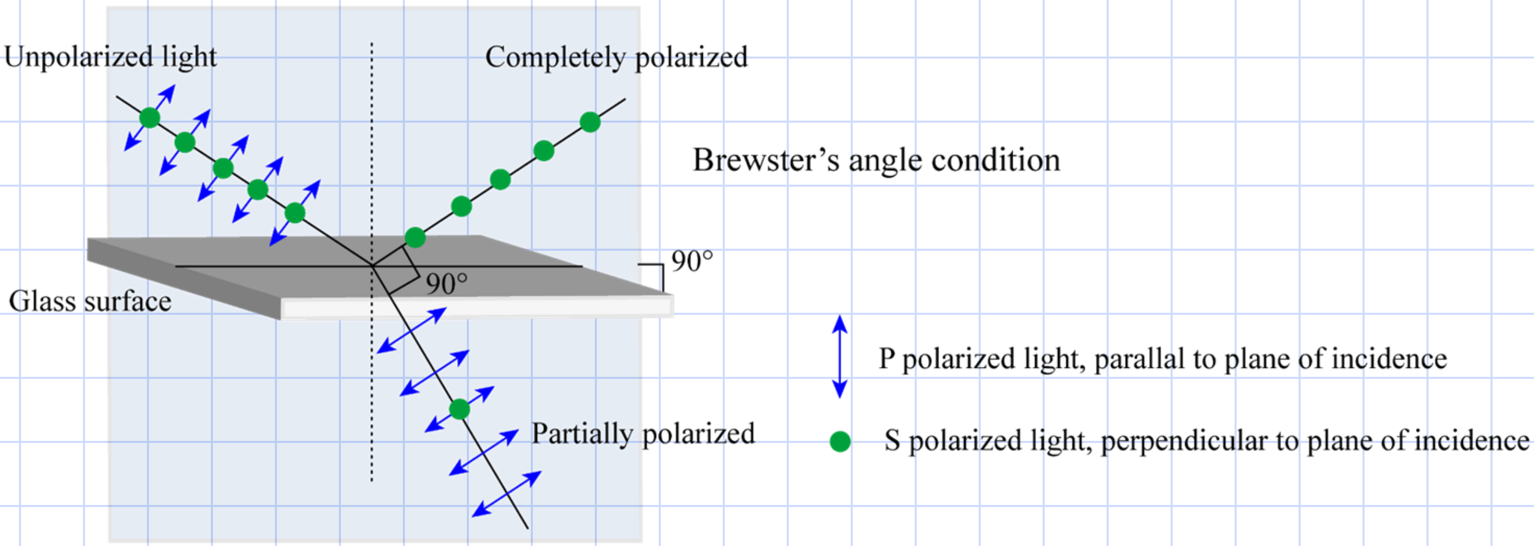


- At the Brewster's Angle or polarizing angle, the reflection coefficient is 0 for one polarization of light.
- This means that only polarized light is reflected.

Reflection Angular Dependence

Brewster's Angle

$$\Gamma_{\parallel} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1}$$

Set this equal to zero:

$$\boxed{\eta_2 \cos\theta_\mathrm{i} = \eta_1 \cos\theta_\mathrm{t}.}$$

… to derive this. Looks like Snell's law but with cosines instead of sines.

By once again combining the two equations, we get:

$$\boxed{\sin\theta_{\mathrm{B}\parallel} = \sqrt{\frac{1-(\epsilon_1\mu_2/\epsilon_2\mu_1)}{1-(\epsilon_1/\epsilon_2)^2}}\ .}$$

Note that there will be no Brewster angle if the permittivities are the same.

Reflection Angular Dependence

Brewster's Angle

Wikimedia Commons

When the permeabilities of the two materials are the same, this becomes

$$\theta_{B\|} = \sin^{-1}\sqrt{\frac{1}{1+(\epsilon_1/\epsilon_2)}}$$

$$= \tan^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}} \qquad (\text{for } \mu_1 = \mu_2).$$

Reflection Angular Dependence

Brewster's Angle

- The Brewster's angle is why glare from the surface of water can be easily blocked with sunglasses.

Wikipedia

Brewster Angle

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

$$\boxed{\eta_2 \cos\theta_i = \eta_1 \cos\theta_t .}$$

$$\Gamma_\perp = \frac{E_{m1}^-}{E_{m1}^+} = \frac{\eta_2 \cos\theta_1 - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

Combining these two equations and setting the incident angle to be the Brewster angle, we get:

$$\boxed{\sin\theta_{B\perp} = \sqrt{\frac{1-(\mu_1\epsilon_2/\mu_2\epsilon_1)}{1-(\mu_1/\mu_2)^2}} .}$$

(Ulaby pg. 375)

Brewster Angle

$$\boxed{\sin\theta_{\mathrm{B}\perp} = \sqrt{\frac{1-(\mu_1\epsilon_2/\mu_2\epsilon_1)}{1-(\mu_1/\mu_2)^2}}\ .}$$

Note that this is undefined when the permeabilities of the two materials are the same. But when they are different, some Brewster angle will exist.

Final Exam Review

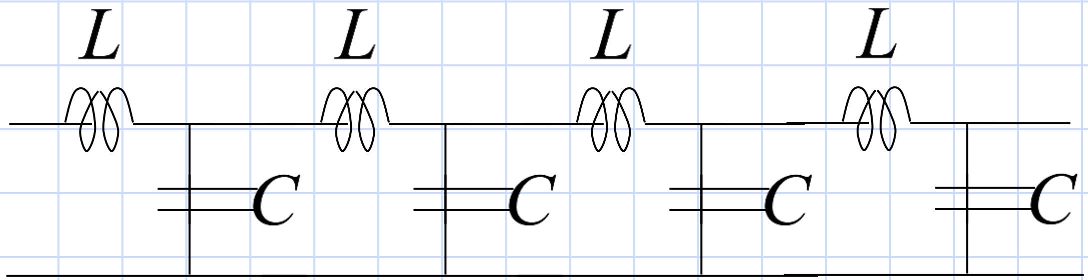
T-Line Equations

- What is the equivalent circuit of a transmission line (both the lossless and the lossy version)?

Final Exam Review

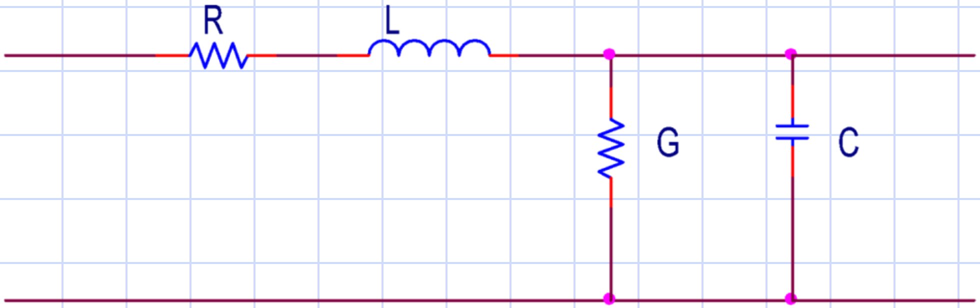
T-Line Equations

- Lossless



....etc.

- Lossy

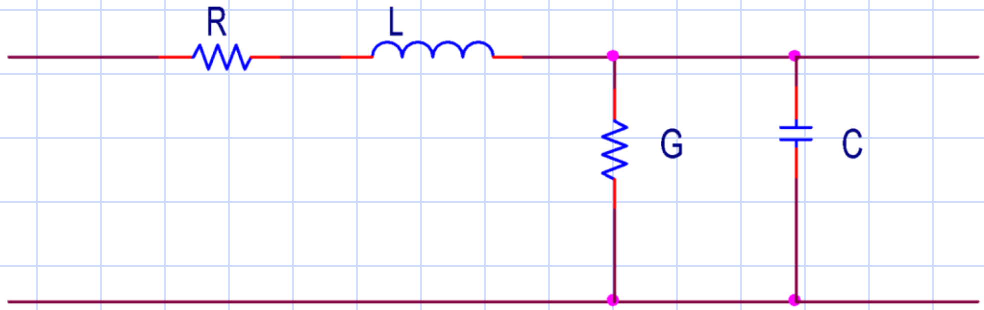


....etc.

Final Exam Review

T-Line Equations

- What parameters must be set to what values in order to turn this lossy t-line into a lossless t-line?



....etc.

Final Exam Review

T-Line Equations

- When can we approximate a lumped circuit model as being an accurate representation of a real transmission line?

Final Exam Review

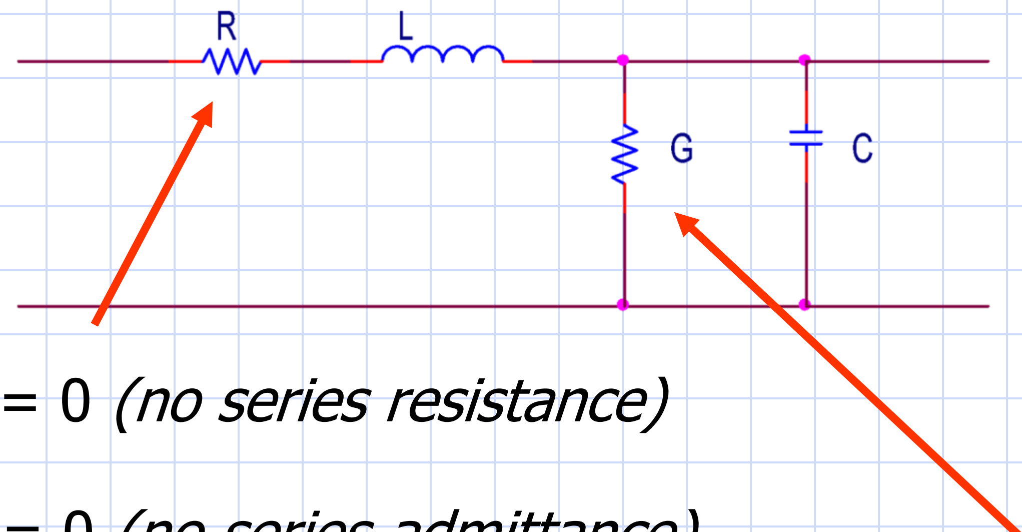
T-Line Equations

- When can we approximate a lumped circuit model as being an accurate representation of a real transmission line?
- *The model is accurate when the length that each lumped segment represents is much less than the wavelength of the signal on the line.*

Final Exam Review

T-Line Equations

- What parameters must be set to what values in order to turn this lossy t-line into a lossless t-line?



....etc.

R = 0 *(no series resistance)*

G = 0 *(no series admittance)*
(note that we are setting this admittance of this to zero, NOT resistance, which would short our t-line!)

Final Exam Review

Transmission Lines

- What are the telegrapher's equations, or transmission line equations?

Final Exam Review

T-Line Equations

$$v(z) = V^{+} e^{-\gamma z} + V^{-} e^{+\gamma z} \qquad v(z) = V^{+} e^{-\gamma z} + V^{-} e^{+\gamma z}$$

$$i(z) = I^{+} e^{-\gamma z} + I^{-} e^{+\gamma z} \qquad i(z) = \frac{V^{+}}{Z_0} e^{-\gamma z} - \frac{V^{-}}{Z_0} e^{+\gamma z}$$

$$v(z) = V^{+}\left(e^{-\gamma z} + \Gamma_L e^{+\gamma z}\right)$$

$$i(z) = \frac{V^{+}}{Z_0}\left(e^{-\gamma z} - \Gamma_L e^{+\gamma z}\right)$$

$$\boxed{\gamma = \alpha + j\beta}$$

These equations represent the lossy case. The lossless case is a special sub-case of the lossy case. How could we derive the lossless equations from these lossy equations?

Final Exam Review

T-Line Equations

$$v(z) = V^{+} e^{-j\beta z} + \Gamma_L V^{+} e^{+j\beta z}$$

$$i(z) = \frac{V^{+}}{Z_0} e^{-j\beta z} - \frac{\Gamma_L V^{+}}{Z_0} e^{+j\beta z}$$

Lossless equations emerge when we set $\alpha=0$.

Final Exam Review

Reflection Coefficient for T-Lines

- In plain English, what does the reflection coefficient represent?

Final Exam Review

Reflection Coefficient for T-Lines

- In plain English, what does the reflection coefficient represent?

When a wave hits an interface that causes it to reflect, the reflection coefficient tells us the amplitude ratio between the incident forward-traveling wave and the reflected backward-traveling wave.

Final Exam Review

Reflection Coefficient for T-Lines

- What does it mean when the reflection coefficient has a negative value?

- What does it mean when the reflection coefficient has a complex value?

Final Exam Review

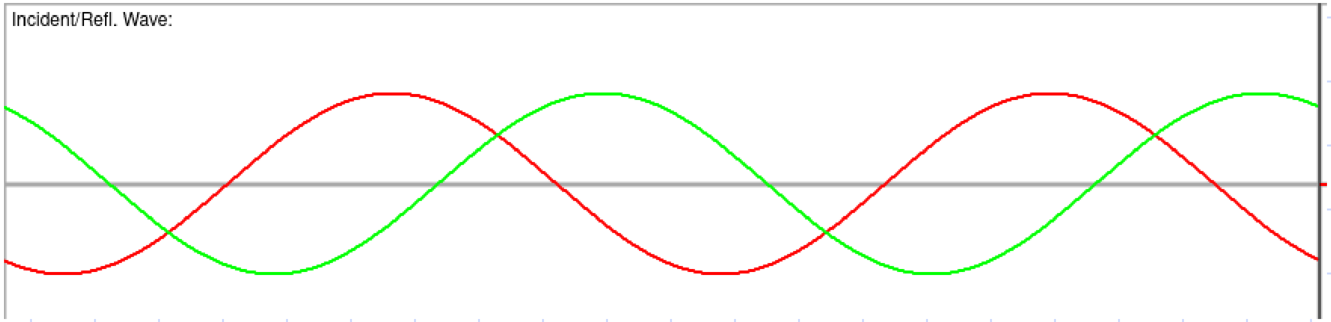
Reflection Coefficient for T-Lines

$$\Gamma = |\Gamma| e^{j\theta_r} = \Gamma_r + j\Gamma_i$$

$$\Gamma_r = |\Gamma| \cos\theta_r$$

$$\Gamma_i = |\Gamma| \sin\theta_r$$

- A negative Γ would mean that at the point of reflection, the reflected wave is 180 degrees out of phase with the incident wave.



Final Exam Review

Reflection Coefficient for T-Lines

$$\Gamma = |\Gamma| e^{j\theta_r} = \Gamma_r + j\Gamma_i$$

$$\Gamma_r = |\Gamma| \cos\theta_r$$

$$\Gamma_i = |\Gamma| \sin\theta_r$$

- A complex Γ would mean that at the point of reflection, the reflected wave phase-shifted from incident wave by some amount that is not 0 or 180 degrees.

Final Exam Review

Example

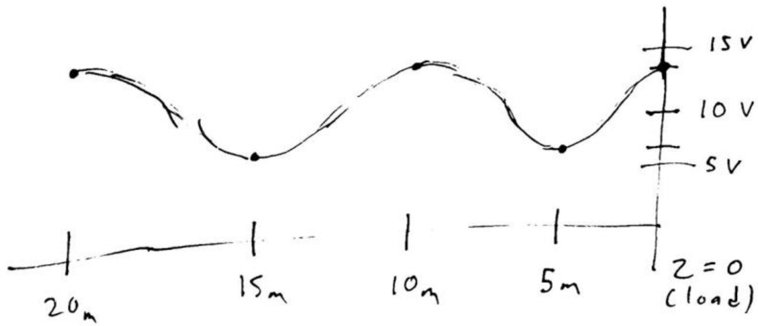
- Suppose that a transmission line has Z_0 = 50 and the amplitude of an incident voltage pulse is 10V with wavelength 20m. Choose a load that will cause this transmission line to have a real reflection coefficient that is not zero or infinity.
- Then calculate the reflection coefficient and draw the standing wave pattern. Calculate the standing wave ratio.

Final Exam Review

Example

If $Z_0 = 50$, Let $Z_L = 100$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{100-50}{100+50} = \frac{50}{150} = \frac{1}{3}$$



$V_0 = 10\,V$ $\quad V_0 + \Gamma V_0 = 13.33\,V$

$V_0 - \Gamma V_0 = 6.67\,V$

$$\frac{1+|\Gamma|}{1-|\Gamma|} = \frac{\frac{4}{3}}{\frac{2}{3}} = 2$$

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Pulses on T-Lines

- Now let's do transient analysis on this same line. Why information do we need that we don't have yet?

Final Exam Review

Pulses on T-Lines

- Now let's do transient analysis on this same line. Why information do we need that we don't have yet?

We need to know the source impedance as well as the load impedance, and we need to know the time delay of the line.

Final Exam Review

Pulses on T-Lines

- Suppose we put a very short 10V pulse into this line. What will the bounce diagram look like if the source impedance is match to the line, and what will it look like if the source impedance is the same as the load? Suppose that the time delay is 20ms.
- How would the bounce diagram analysis be different if the short 10V pulse were replaced with switching on a 10V DC source?

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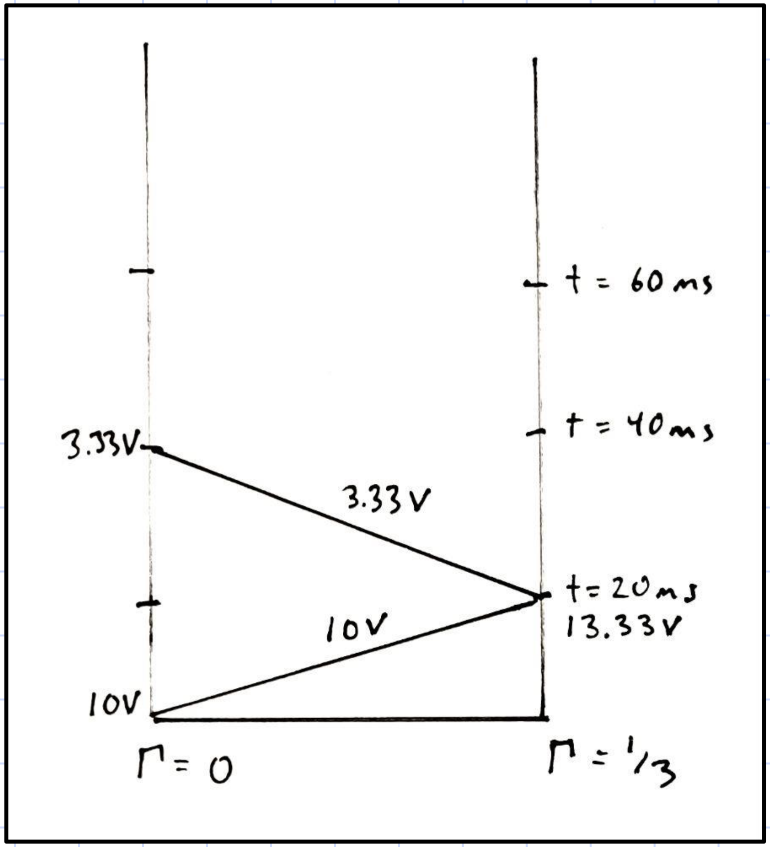
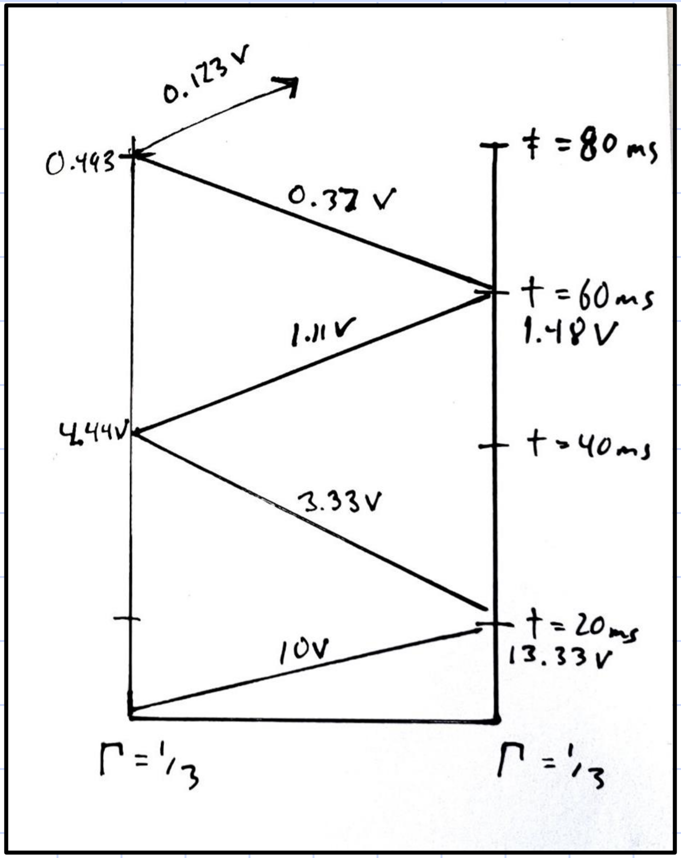
Pulses on T-Lines



Matched Source



Source Equals Load

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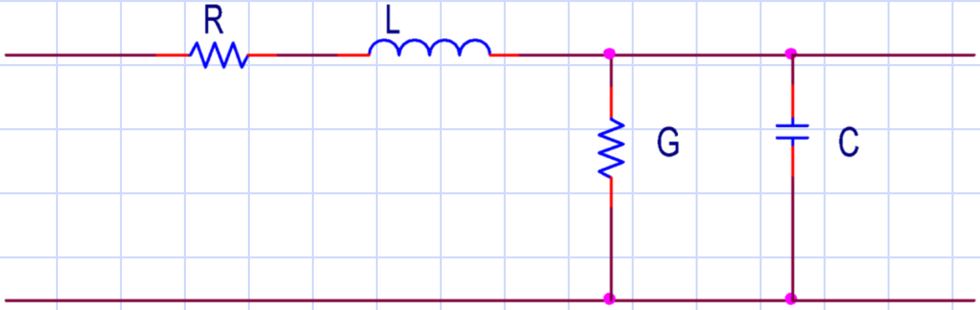
Lossy T-Lines

For a lossless system, Z_o represents $= \dfrac{\hat{V}}{\hat{I}}$

$$Z_o = \sqrt{\frac{l}{c}}$$



Replace ***jωl*** with ***r + jωl*** Replace ***jωc*** with ***g + jωc***

Characteristic Impedance

$$Z_0 = \sqrt{\frac{r + j\omega l}{g + j\omega c}}$$

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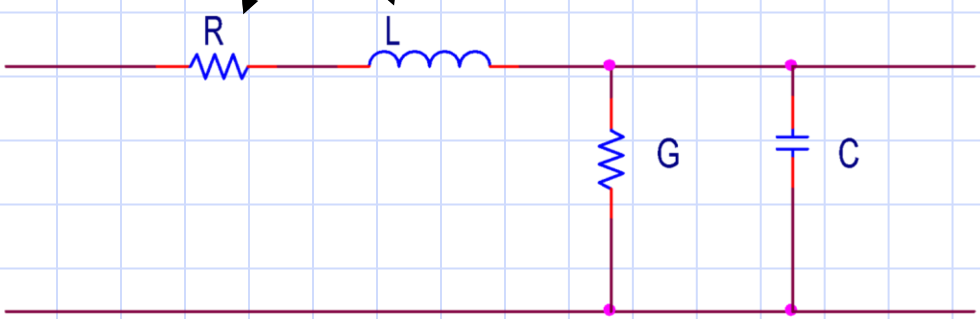
Low-Loss T-Lines

When the impedance of this R is nonzero but much less than the impedance of this L, we consider the T-line to be low-loss



$$Z_0 = \sqrt{\frac{r + j\omega l}{g + j\omega c}}$$

Likewise, when the admittance of G is nonzero but much less than the admittance of C, we consider the t-line to be low-loss.

Final Exam Review

Low-Loss T-Lines

- Using the Binomial Theorem $\sqrt{1+x} \approx 1 + \frac{x}{2}$ for *x<<1*.

$$Z_o = \sqrt{\frac{r + j\omega l}{g + j\omega c}} \approx \sqrt{\frac{r + j\omega l}{j\omega c}} \approx \sqrt{\frac{j\omega l}{j\omega c}}\sqrt{1 + \frac{r}{j\omega l}} \approx \sqrt{\frac{l}{c}}\left(1 - j\frac{r}{2\omega l}\right)$$

$$\gamma = \sqrt{(r + j\omega l)(g + j\omega c)} \approx \sqrt{(r + j\omega l)(j\omega c)} \approx \sqrt{(j\omega l)(j\omega c)}\sqrt{1 + \frac{r}{j\omega l}}$$

$$\gamma = \alpha + j\beta \approx j\omega\sqrt{lc}\left(1 - j\frac{r}{2\omega l}\right)$$

Final Exam Review

Low-Loss T-Lines

- Using the Binomial Theorem $\sqrt{1+x} \approx 1 + \frac{x}{2}$ for *x<<1*.

$$Z_o = \sqrt{\frac{r + j\omega l}{g + j\omega c}} \approx \sqrt{\frac{r + j\omega l}{j\omega c}} \approx \sqrt{\frac{j\omega l}{j\omega c}}\sqrt{1 + \frac{r}{j\omega l}} \approx \sqrt{\frac{l}{c}}\left(1 - j\frac{r}{2\omega l}\right)$$

The above expression is useful if you need to calculate the small phase of Z_0. However, if you only care about magnitude, you can say $Z_o \approx \sqrt{\frac{l}{c}}$

Final Exam Review

Low-Loss T-Lines

- The propagation and attenuation constants become

$$j\beta \approx j\omega\sqrt{lc} \qquad \alpha \approx \omega\sqrt{lc}\left(\frac{r}{2\omega l}\right) = \frac{r}{2\sqrt{\frac{l}{c}}} = \frac{r}{2Z_0}$$

Final Exam Review

Distortionless T-Lines

For practical lines, the conductance per unit length *g* is negligible. Thus, we will add loss between the conductors so that

$$\frac{r}{l} = \frac{g}{c}$$

This is called the **Heaviside condition** and it can be achieved with periodic lumped shunt resistors.

Final Exam Review

Example

- Suppose you have a lossy transmission line with:
  - r′ = 0.01 Ω/m
  - c′ = 1nF / m
  - l = 500nH / m
  - Signal frequency = 1 MHz

Figure out α and β for this line, then come up with a value of parallel shunt resistor that would cause the line to behave in a distortionless manner. Since placing these resistors is essentially creating a lumped line t-line model, they must be placed such that the lumped line approximation is reasonable.

Final Exam Review

Example

Is this like low-loss?

$$\frac{|r'|}{|j\omega l'|} = \frac{0.01}{2\pi \cdot 10^6 \cdot 500 \cdot 10^{-9}} = 0.003 \checkmark$$

low loss

$$Z_0 = \sqrt{\frac{l}{c}} = \sqrt{\frac{500 \times 10^{-9}}{10 \times 10^{-9}}} = 22.36\ \Omega$$

$$\alpha = \frac{r}{2Z_0} = \frac{0.01}{2(22.36)} = 2.24 \times 10^{-4}\ Np/m$$

$$\beta = \omega\sqrt{lc} = (10^6)\sqrt{lc} = 0.140\ rad/m$$

Final Exam Review

Example

We need Δx between shunt resistors to be much less than the wavelength (let's say 1%)

$$\lambda = \frac{2\pi}{\beta} = 44.72 \text{ m}$$

so shunt resistors should appear every 0.44 m.

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Example

We need

$$\frac{r'}{\ell'} = \frac{g'}{c'} \Rightarrow \frac{0.01}{500 \times 10^{-9}} = \frac{g'}{10^{-9}}$$

$$g' = 2 \times 10^{-5}\ \Omega^{-1} m^{-1}$$

$$g' \cdot 0.44\,m = 8.8 \times 10^{-6}\ \Omega^{-1}$$

$$\frac{1}{g' \cdot 0.44\,m} = 114\ k\Omega$$

Final Exam Review

T-Line Input Impedance

- What does the input impedance of a load look like through a half wavelength of transmission line?

- How about a for quarter wavelength of transmission line?

- How about other lengths of transmission line?

Final Exam Review

T-Line Input Impedance

- What does the input impedance of a load look like through a half wavelength of transmission line?

  It looks like the load impedance, as though the line isn't there.

- How about a for quarter wavelength of transmission line?

  It looks like Z_0^2 / Z_L (an "impedance transformer").

- How about other lengths of transmission line?

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan \beta L}{Z_0 + jZ_L \tan \beta L}$$

Final Exam Review

Smith Charts

- Explain, in plain English, what a Smith Chart is and how it can be used.

Final Exam Review

Smith Charts

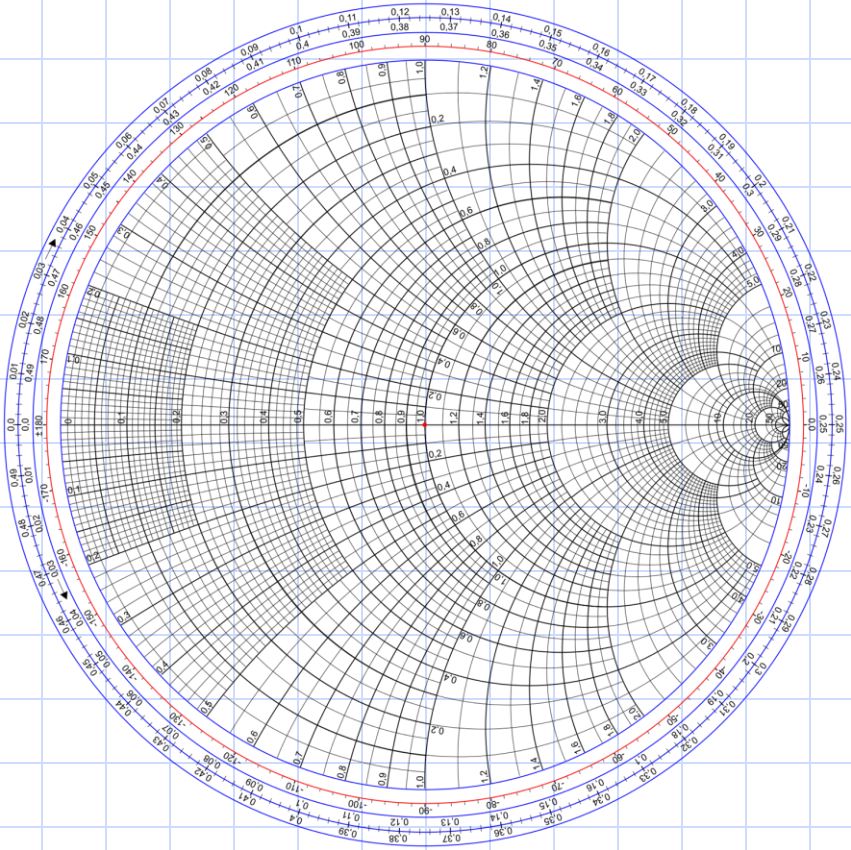
- Explain, in plain English, what a Smith Chart is and how it can be used.

A Smith Chart allows you to graphically convert between reflection coefficient and impedance and account for how traveling through a length of transmission line will "transform" how either of these two quantities look at the input.

Final Exam Review

Smith Charts

- On this special chart, we normalize impedances as fractions of a t-line's characteristic impedance.
- Then we plot them on the special non-Cartesian grid made of of circles and arcs. We can then read off our reflection coefficient in Cartesian coordinates.



Final Example Review

Example

- Suppose a transmission line has $Z_0 = 50\Omega$. Pick some arbitrary load impedance, express it as a normalized impedance, plot it on the Smith chart, and read off the reflection coefficient.
- Then read off what this impedance would look like a quarter wavelength from the load.

Final Exam Review

Example

- Let Z_L = 100Ω.
- Normalized impedance: z_L = 100Ω/50Ω = 2.
- A quarter wavelength away, this will look like z'_L = 0.5, or Z_L = 25Ω.

Final Exam Review

Example

- Now, match your chosen load with an open-circuit stub.

Final Exam Review

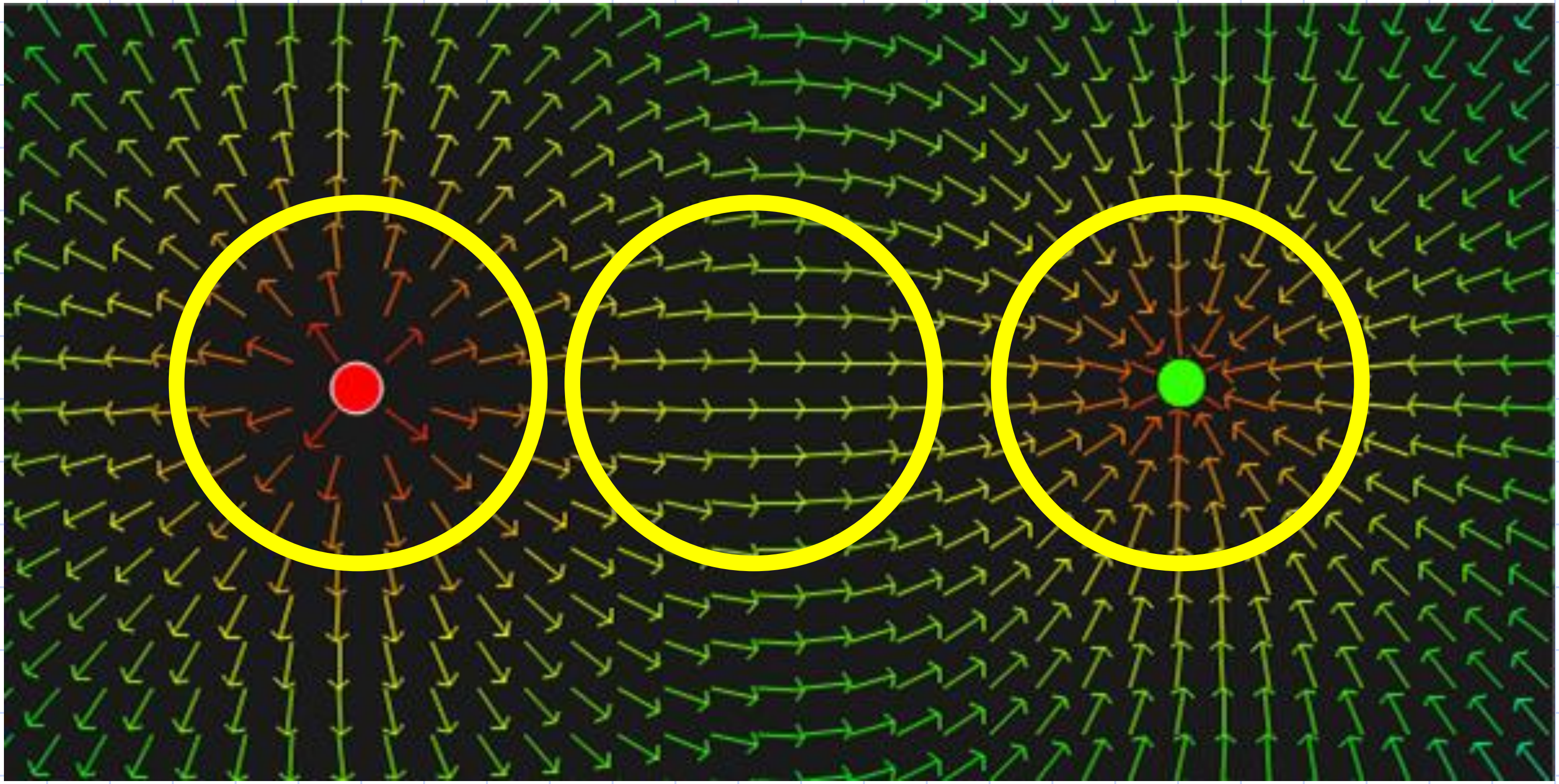
Example

- First, calculate the normalized load admittance $y_L = 1/z_L = 0.5$
- Next, rotate y_L onto the matching circle. $y'_L = 1 + 0.7j$
- How long must an open circuit stub be to have input impedance $-0.7j$? *About 0.401 wavelengths.*

Final Exam Review

Field Math

- Which these circles represent positive vs. negative vs. no flux?



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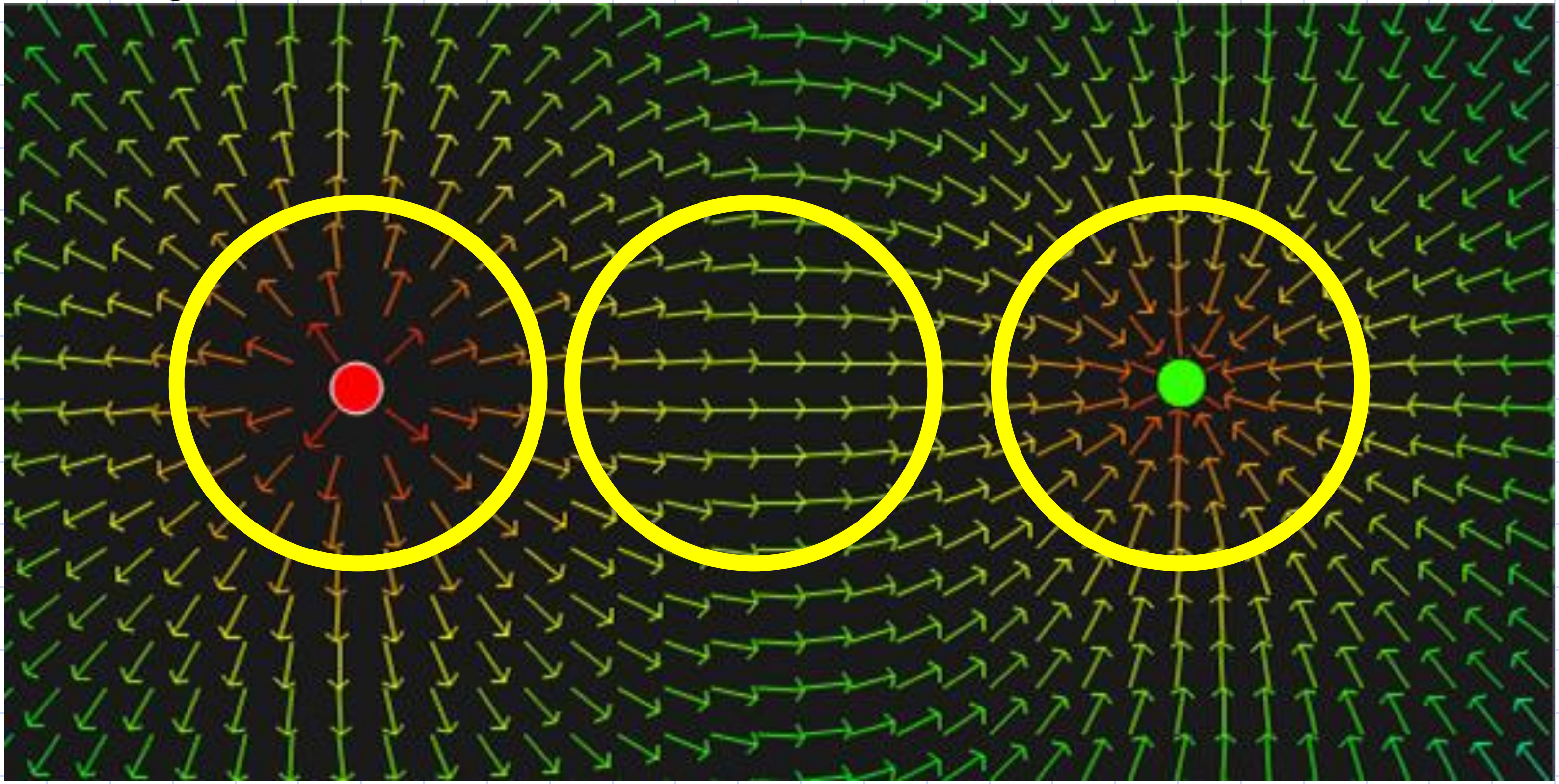
Field Math

- Explain where in this picture you see positive, negative, and zero divergence.



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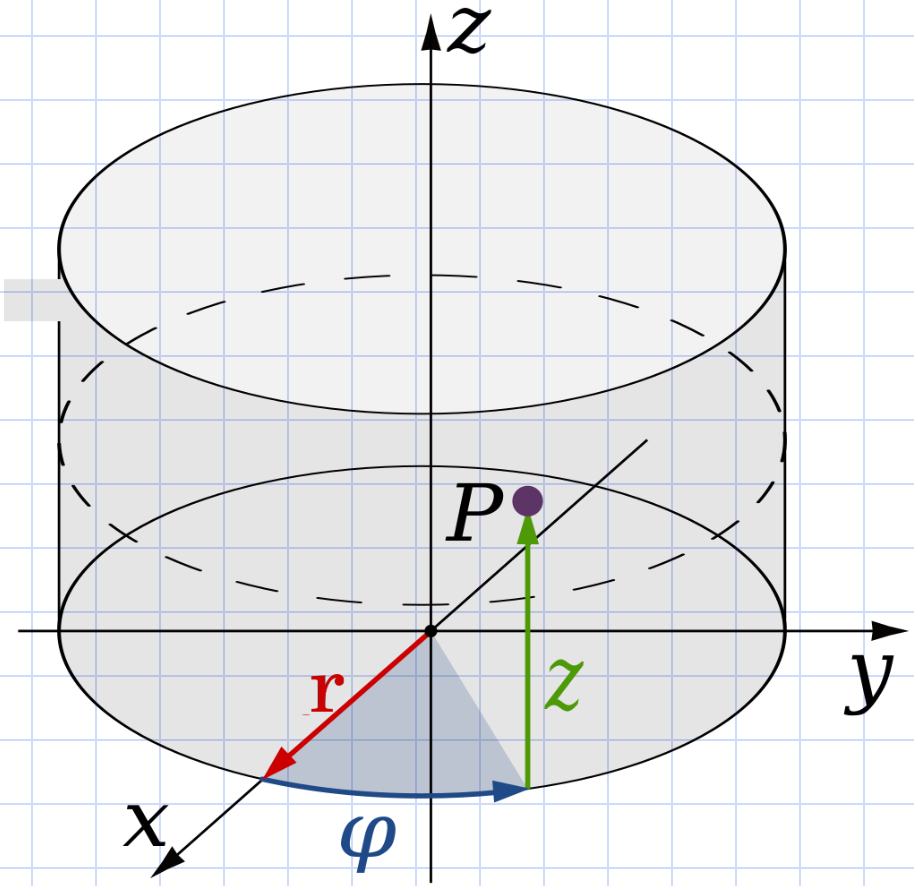
Field Math

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation $\mathbf{A} =$	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A $\lvert\mathbf{A}\rvert =$	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1} =$	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P(R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}} \cdot \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}} \times \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product $\mathbf{A} \cdot \mathbf{B} =$	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product $\mathbf{A} \times \mathbf{B} =$	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length $d\mathbf{l} =$	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume $dV =$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

Final Exam Review

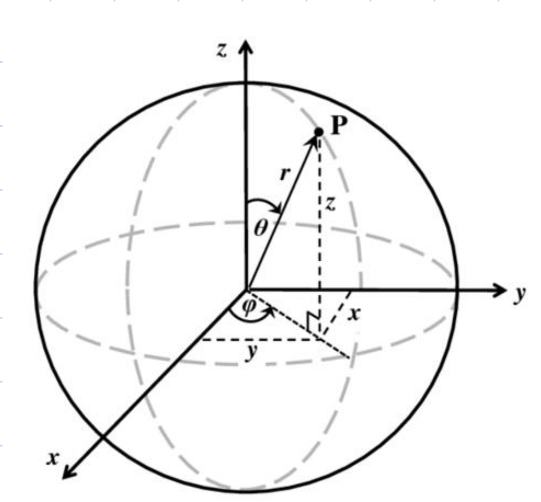
Field Math

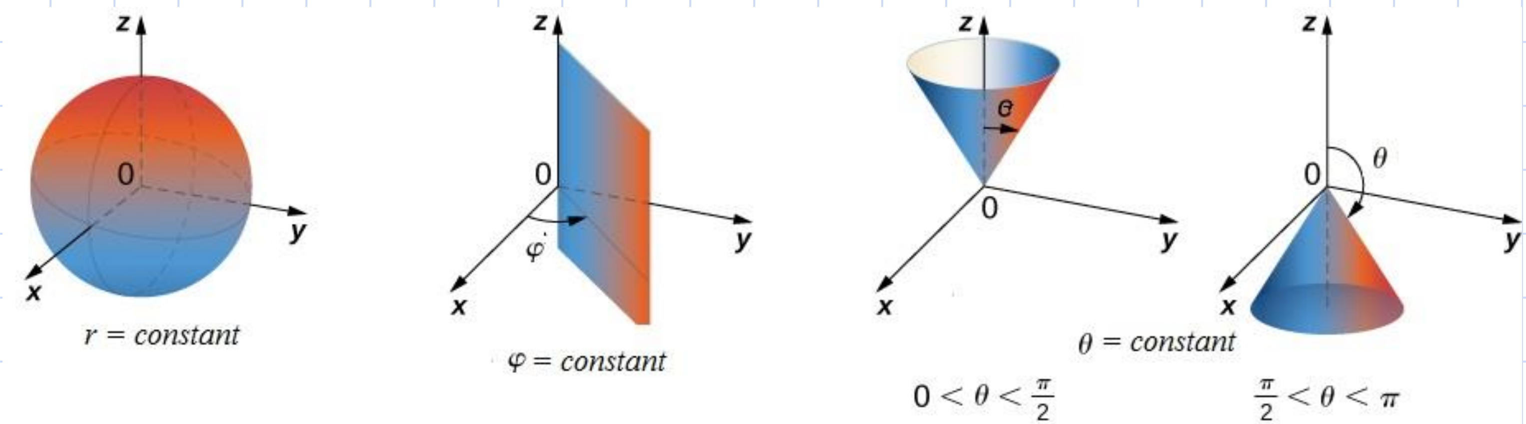


- How do we describe the three possible types of surface differential elements in cylindrical coordinates?

Final Exam Review

Field Math





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Electrostatics

- Describe in plain English what "electrostatics" means.

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Electrostatics

- Describe in plain English what "electrostatics" means.

A subset of Maxwell's equations in which all charge is at rest. This means no magnetic fields and no currents

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Example

3. Charged Plane (12 points)

Consider an infinitely thin sheet of positive charge floating in space. It has an area of 10m by 10m and its charge density is one nanocoulomb per square centimeter.

a.) (4 pts) What will the electric field (E) be for a point above the plane's surface and very close to it? *(Hint: You are being asked for an effective approximation).*

b.) (4 pts) What will the electric field (E) be for a point very far (much greater than 10m) above the plane's surface? *(This is also an approximation.)*

c.) (4 pts) Suppose that there is a conductive plane hovering below the charged plane as shown below. Both planes have the same dimensions in the x and y directions, and the conductive plane does not have any net charge. Assume that the distance from point P to either plane is very small compared to the dimensions of the plane. What is the electric field at point P? *(Hint: how does a conductive plane respond to a charge above it?)*

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Example

③ Charged Plane

a.) Very close to the plane's surface, it looks like an infinite charged plane.

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{z} \qquad \text{(pointing up)}$$

$$\sigma = \frac{1\,\text{nC}}{(0.01\text{m})^2} = 1 \times 10^{-5}\ \text{C/m}^2$$

$$\vec{E} = 5.65 \times 10^{5}\ \text{V/m}\ \hat{z}$$

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Example

b.) At these distances, the plane can be approximated as a point with total charge Q.

$$Q = (1 \times 10^{-5}\, C/m^2)\cdot(10m)^2 = 0.001\,\mathrm{C}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{0.001\,\mathrm{C}}{r^2}\hat{r}$$

$$\vec{E} = \left(\frac{8.987\times 10^6}{z^2}\right)\hat{z}\ \mathrm{V/m}$$

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Example

The field due to the charged plane is

$$\vec{E} = \frac{-\sigma}{2\varepsilon_0}\hat{z}$$ (points down because we are below the plane)

The effect of the conductive plane can be determined using the method of images. The charged plane causes a negative charge to appear on the conducting plane's surface.

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Example

This charge looks like a plane of negative charge with density σ below P.

Thus,

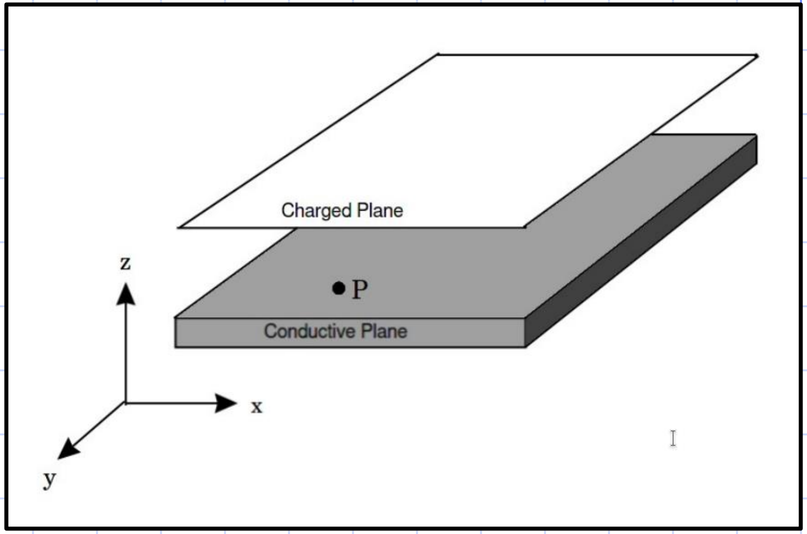
charged plane

conductive plane

$$\vec{E}_{tot} = \frac{-\sigma}{2\varepsilon_0}\hat{z} + \frac{(-\sigma)}{2\varepsilon_0}\hat{z}$$

$$\vec{E}_{tot} = -1.129\times10^{6}\,\hat{z} \text{ V/m}$$

(pointing down)



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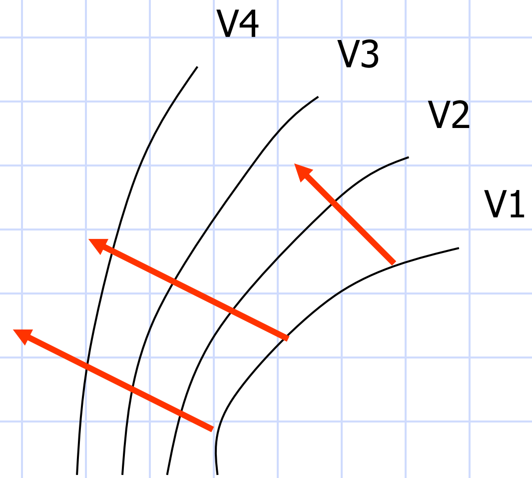
Example

$$\vec{E} = -\nabla V$$



- Gradient points in the direction of largest change
- Therefore, E-field lines are perpendicular (normal) to constant V surfaces

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Example

7. Capacitance (18 pts)

Consider a spherical capacitor with a layered structure. Its innermost layer is a grounded spherical conductor of radius 1mm. This is covered by a 1mm thick layer of dielectric with permittivity $3\varepsilon_0$, then a 1mm thick layer of permittivity $6\varepsilon_0$, a 1mm layer of permittivity $9\varepsilon_0$, and finally at the outermost layer, 1mm thick shell of conductor.

a.) (8 pts) Calculate the capacitance of this sphere.

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Example

Now we calculate voltage from the outer conductor to the inner:

$$V = -\int \vec{E} \cdot \vec{dl}$$

$$V = -\int_{0.004}^{0.003} \frac{1}{4\pi(9\epsilon_0)} \frac{Q}{r^2}\, dr - \int_{0.003}^{0.002} \frac{1}{4\pi(6\epsilon_0)} \frac{Q}{r^2}\, dr$$

$$- \int_{0.002}^{0.001} \frac{1}{4\pi(3\epsilon_0)} \frac{Q}{r^2}\, dr$$

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Permittivity

- In plain English, what is permittivity?

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Permittivity

- In plain English, what is permittivity?

Permittivity describes how strong the E-field will be in a material in response to an external D-field.

$$\vec{D} = \epsilon \vec{E}$$

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- In plain English, what is magnetostatics?

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Magnetostatics

- In plain English, what is magnetostatics?

Magnetostatics is a subset of Maxwell's Equations in which charges move only as constant currents. This means that we have static electric fields and static magnetic fields.

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Displacement Current

- What is displacement current, and how do you calculate it?
- What is skin depth, and how do you calculate it?

Maxwell's Equations

Full Version

Added term in curl H equation for time varying electric field that gives a magnetic field.

$$\oint \vec{H} \cdot d\vec{l} = \int \vec{J} \cdot d\vec{S} + \int \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\oint \vec{B} \cdot d\vec{S} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int \vec{B} \cdot d\vec{S}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\oint \vec{D} \cdot dS = \oint \rho dv = Q_{encl}$$

$$\nabla \cdot \vec{D} = \rho$$

First introduced by Maxwell in 1873

Maxwell's Equations

The total displacement current between the capacitor plates

$$I_d = -\epsilon \int \frac{\partial}{\partial t}\left(-\frac{V(t)}{d}\right)\hat{z} \cdot d\vec{S} = \frac{\epsilon \pi a^2}{d}\frac{\partial V(t)}{\partial t}$$

Using phasor notation for the voltage and current

$$V(t) = \mathrm{Re}\left(V_o e^{j\omega t}\right) \qquad I_D = j\omega \frac{\varepsilon \pi a^2}{d} V_o$$

Lossy EM Waves

Skin Depth, δ_s

shows how well an electromagnetic wave can penetrate into a conducting medium

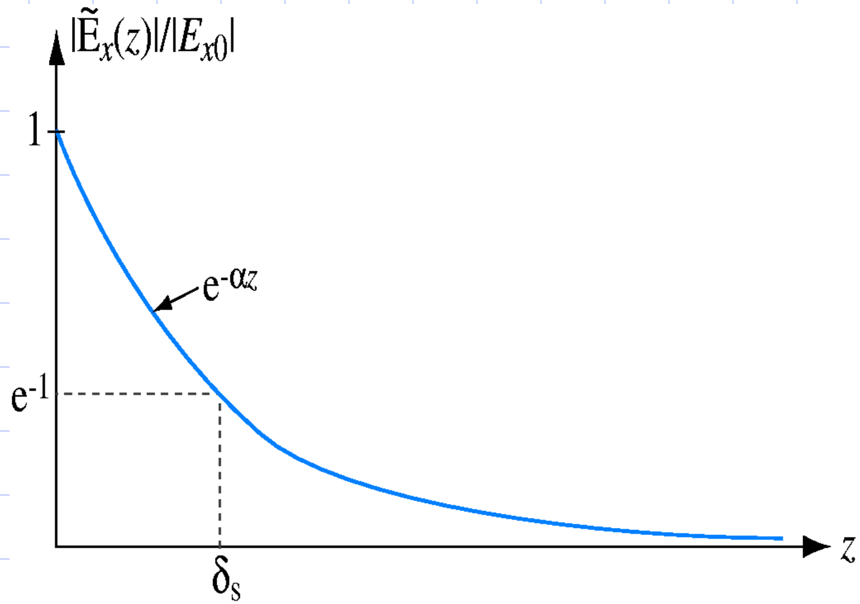
Skin Depth $$\delta_s = \frac{1}{\alpha} \quad [\text{m}]$$

Perfect dielectric:

$\sigma=0 \qquad \alpha=0 \qquad \delta_s=\infty$

Perfect Conductor:

$\sigma=\infty \qquad \alpha=\infty \qquad \delta_s=0$



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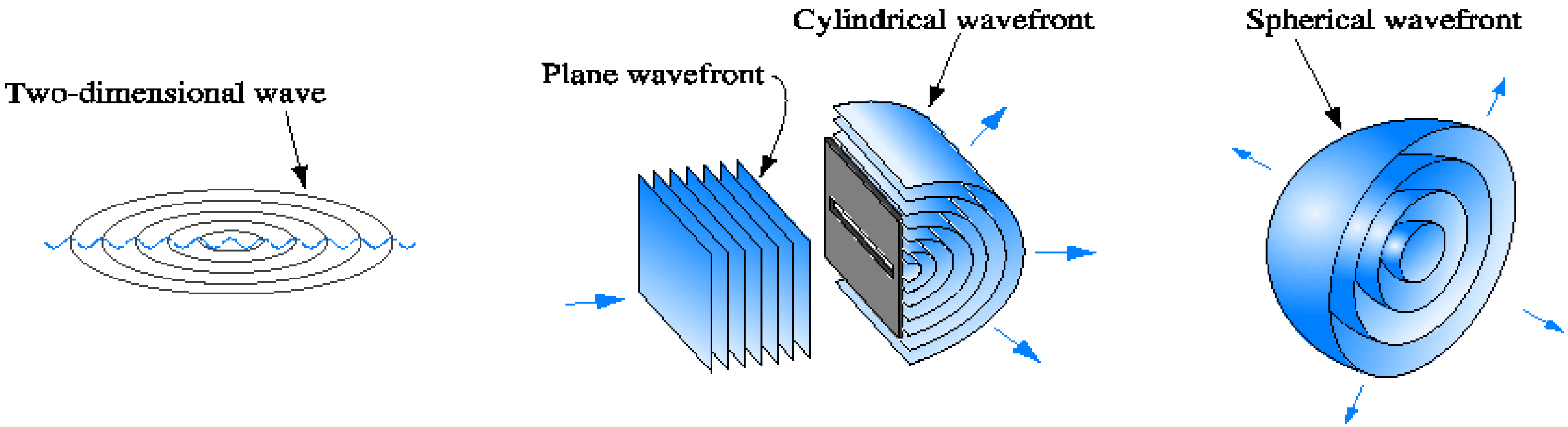
Plane Waves

- What is a plane wave?
- What is intrinsic impedance?
- What happens when a plane wave passes through two regions of different intrinsic impedance, and why?

Electromagnetic Waves

Some Typical Waves



(a) Circular waves (b) Plane and cylindrical waves (c) Spherical wave

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Figure 1-10

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Plane Waves

- What is a Poynting vector, and how do you calculate it?
- How do you calculate the average power in a Poynting vector?

EM Wave Power Transmission

- Poynting Vector S , is defined

$$S = E \times H$$ [W/unit area]

S is along the propagation direction of the wave

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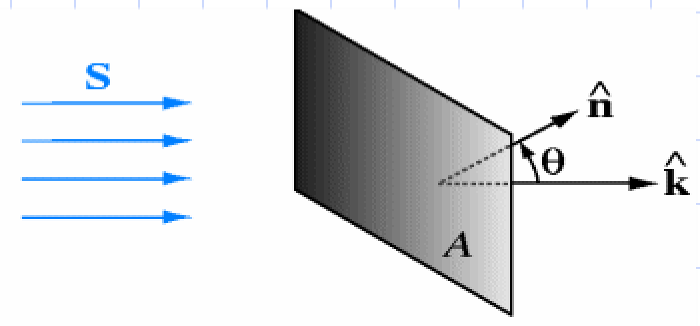
Total power



$$P = \int_A S \cdot \hat{a}_n \, dA$$ [W]

OR $P = |S| A \cos\theta$ [W]

Average power density of the wave [W/m²]

EM Power Transmission

Plane wave in a Lossless Medium

$$\tilde{E}(z) = \tilde{E}_x^+(z)\hat{a}_x + \tilde{E}_y^+(z)\hat{a}_y$$

$$\tilde{E}(z) = (E_{x0}\hat{a}_x + E_{y0}\hat{a}_y)e^{-jkz}$$

$$\tilde{H}(z) = \frac{1}{\eta}\hat{a}_Z \times \tilde{E} = \frac{1}{\eta}(-E_{y0}\hat{a}_x + E_{x0}\hat{a}_y)e^{-jkz}$$

$$S_{av} = \hat{a}_z \frac{1}{2\eta}\left(\left|E_{x0}\right|^2 + \left|E_{y0}\right|^2\right)$$

$$\boxed{S_{av} = \hat{a}_z \frac{\left|\tilde{E}\right|^2}{2\eta}} \qquad [\mathrm{W/m^2}]$$

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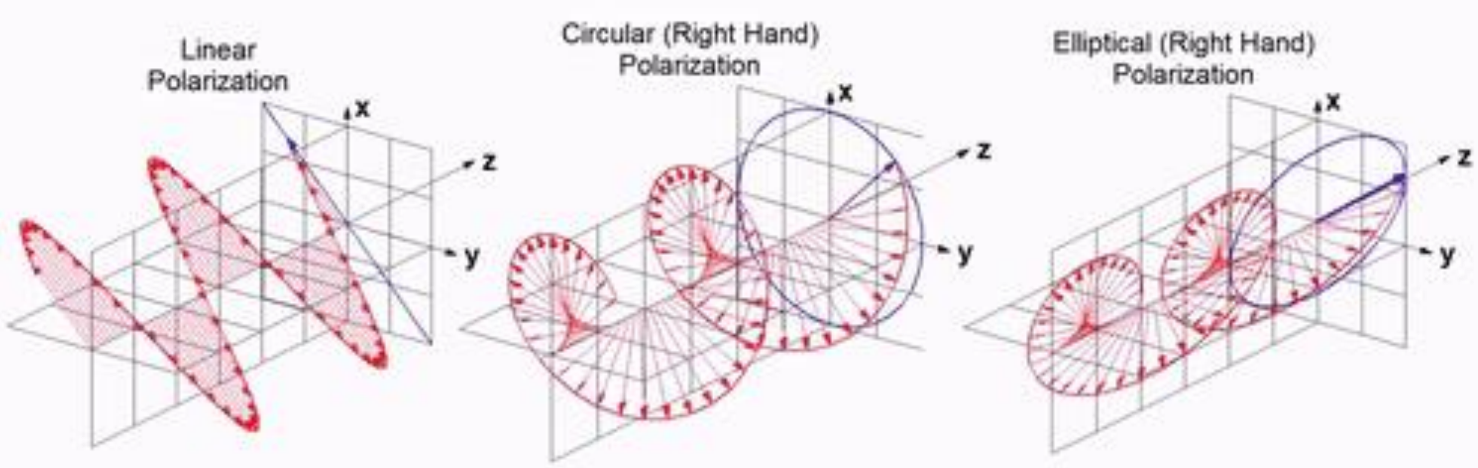
Polarization

- What is the difference between linear, circular, and elliptical polarization?
- What is the difference between left-hand circular and right-hand circular polarization?

Electromagnetic Waves

Some Typical Waves



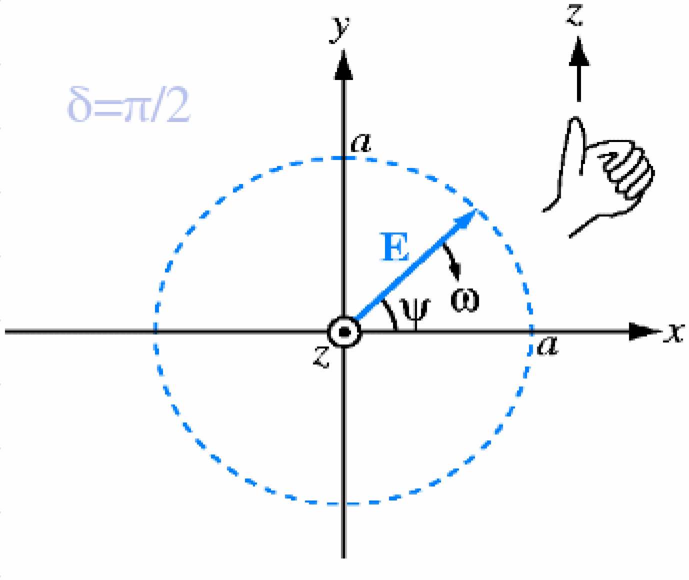
Source: [Gfycat]

Wave Polarization

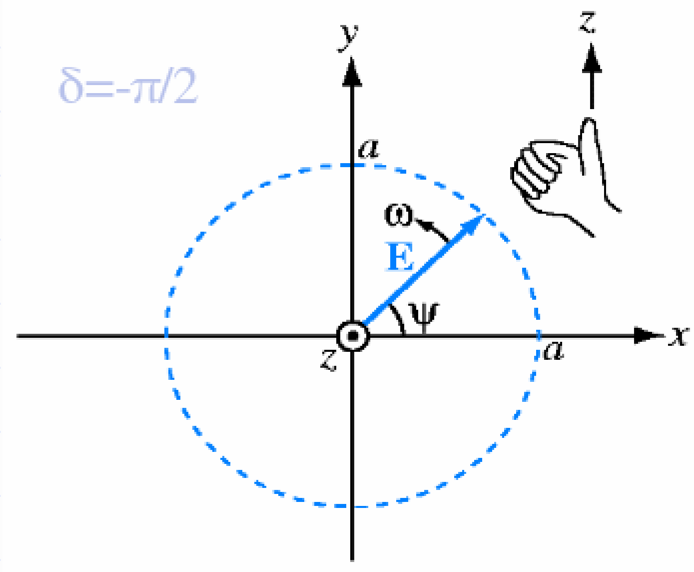
Circular Polarization

A wave is said to be circularly polarized if

- the magnitudes of $\tilde{E}_x(z)$ and $\tilde{E}_y(z)$ are equal and
- The phase difference is $\delta=\pm\pi/2$



LHC polarization



RHC polarization

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Polarization

$$\tan 2\gamma = (\tan 2\psi_0) \cos\delta \quad (-\pi/2 \le \gamma \le \pi/2),$$

$$\sin 2\chi = (\sin 2\psi_0) \sin\delta \quad (-\pi/4 \le \chi \le \pi/4),$$

where ψ_0 is an ***auxiliary angle*** defined by

$$\tan\psi_0 = \frac{a_y}{a_x} \qquad \left(0 \le \psi_0 \le \frac{\pi}{2}\right).$$

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▶ Positive values of χ, corresponding to $\sin\delta > 0$, are associated with left-handed rotation, and negative values of χ, corresponding to $\sin\delta < 0$, are associated with right-handed rotation. ◀

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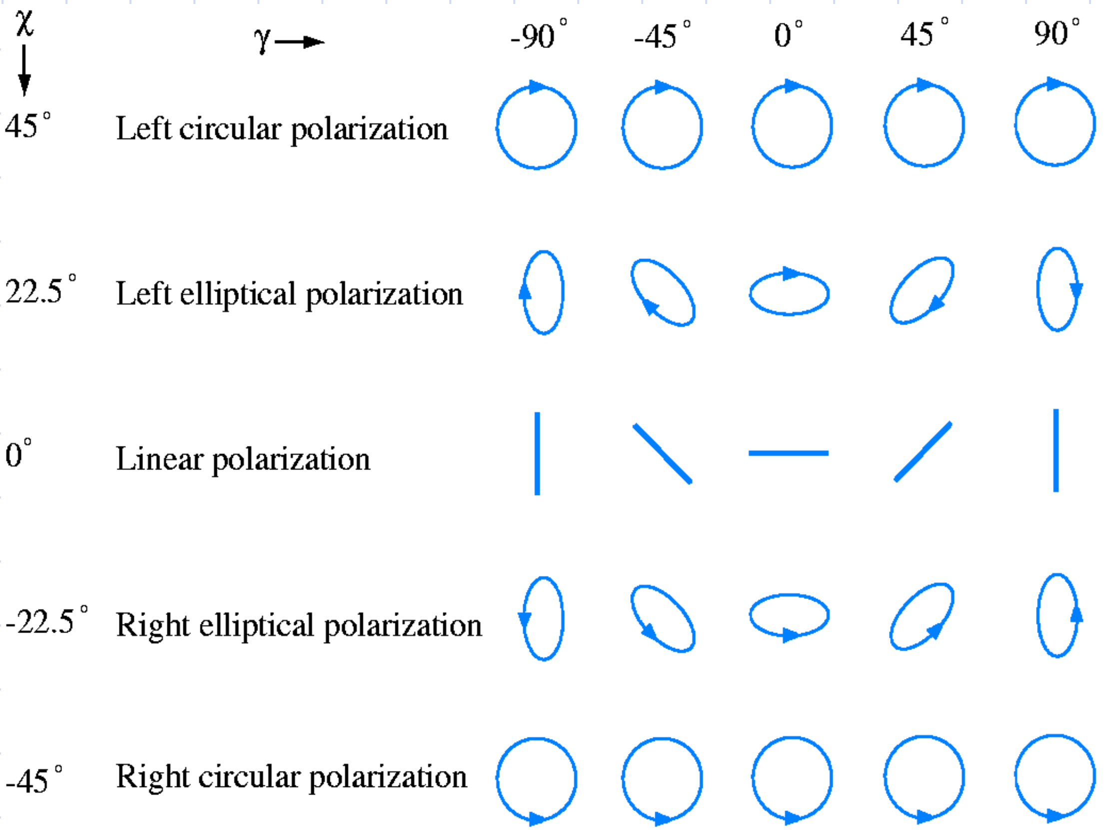
Polarization



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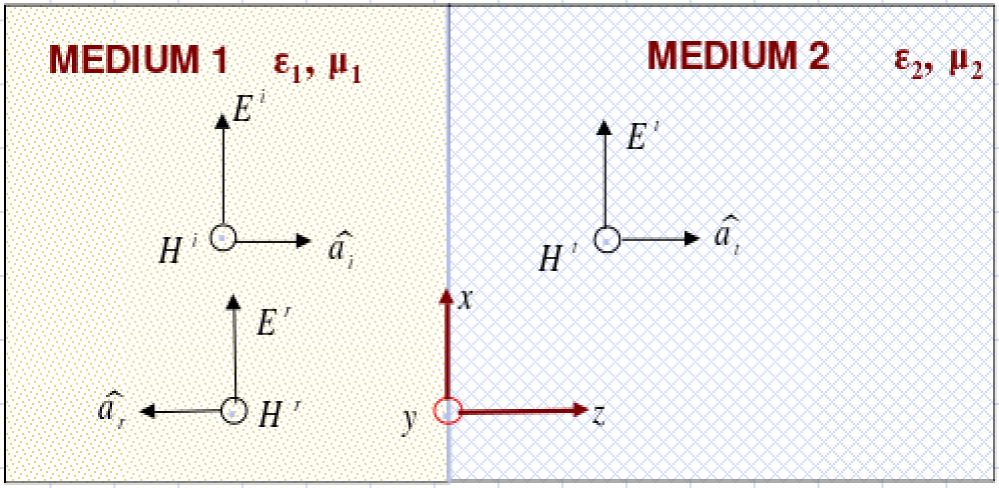
Oblique Incidence

- How do you calculate the reflection between two media of different intrinsic impedance?

Review

Lossless Media

- two lossless, homogenous, dielectric media



z=0

k (or wavenumber) is often used for EM waves but it is functionally the same as β (phase constant)

<u>Incident wave</u> $\tilde{E}^{i}(z) = \hat{a}_x E_0^i e^{-jk_1 z}$

$$k_1 = \omega\sqrt{\mu_1 \varepsilon_1}$$

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

EM Waves and Boundaries

Reflection and Transmission Coefficients

$$\Gamma = \frac{E_0^r}{E_0^i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$ Normally incident

$$\tau = \frac{E_0^t}{E_0^i} = \frac{2\eta_2}{\eta_2 + \eta_1}$$ Normally incident

Γ and τ are real for lossless dielectric media

$$\tau = 1 + \Gamma$$ Normally incident

For nonmagnetic media

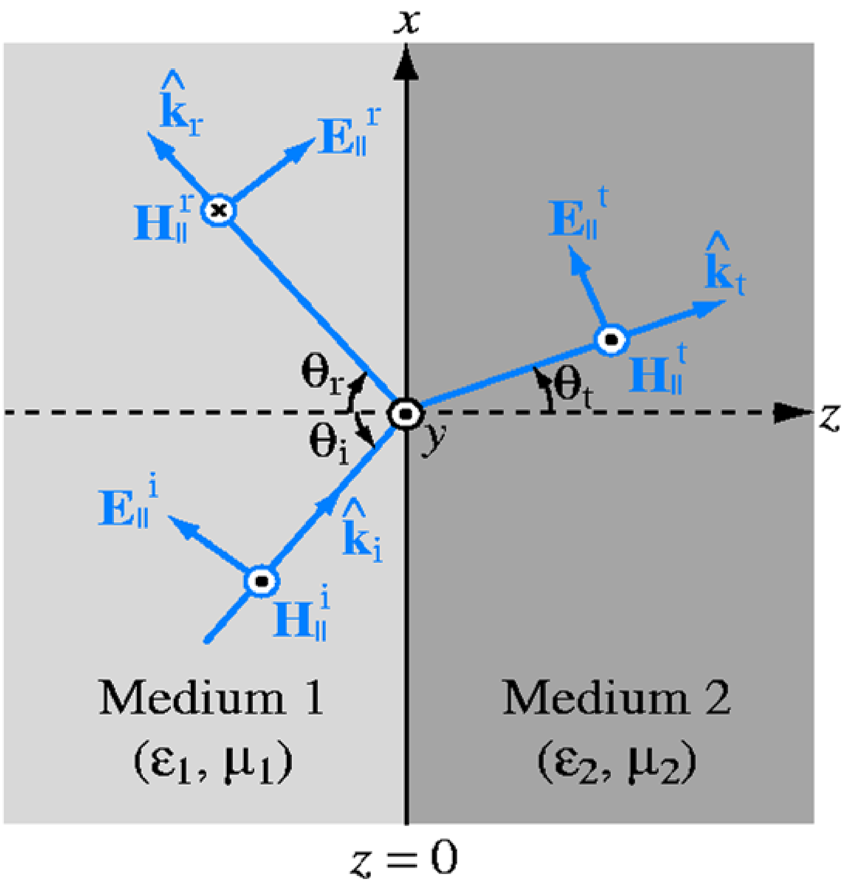
$$\Gamma = \frac{E_0^r}{E_0^i} = \frac{\sqrt{\varepsilon_{r1}} - \sqrt{\varepsilon_{r2}}}{\sqrt{\varepsilon_{r1}} + \sqrt{\varepsilon_{r2}}}$$

Oblique Incidence

Parallel Polarization

- For the first choice, we can assume that the electric field is directed in the plane of incidence (in this case the x-z plane.)

- This is called parallel polarization since E is parallel to this plane.



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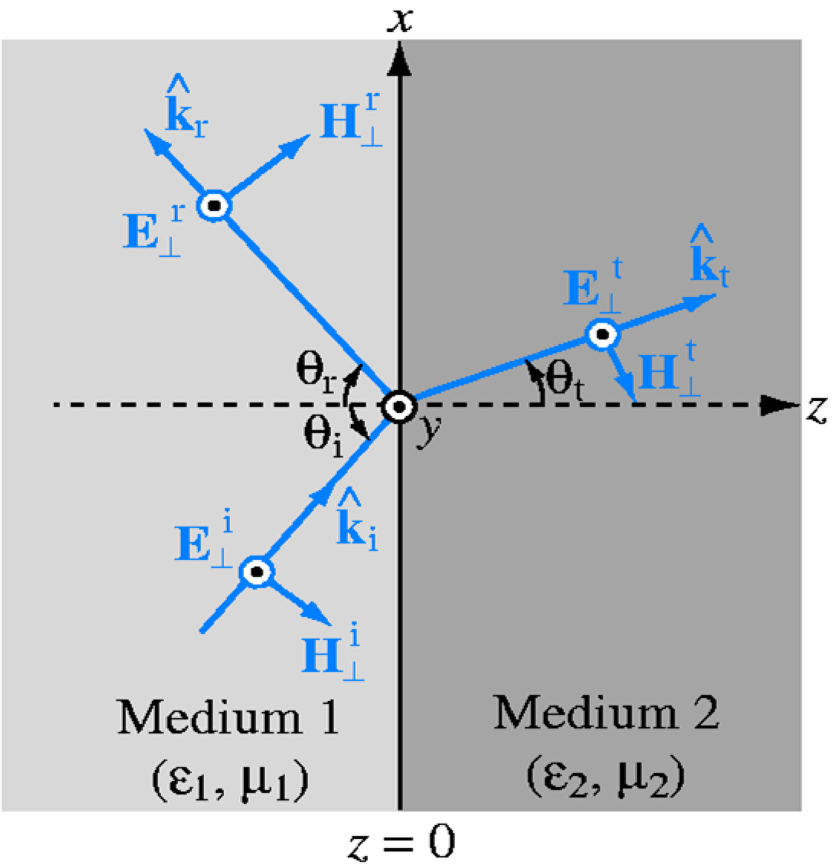
Oblique Incidence

Perpendicular Polarization

- For the second choice, we can assume that the electric field in directed out of the plane of incidence.
- This is called perpendicular polarization since E is perpendicular to this plane.
- Note that E is only tangential while H has both components.



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Snell's Law

Reflection and Transmission Coefficients

By algebraically combining the E and H tangential boundary conditions we can get the following:

$$\Gamma_{\perp} = \frac{E_{m1}^{-}}{E_{m1}^{+}} = \frac{\eta_2 \cos\theta_1 - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

$$\tau_{\perp} = \frac{E_{m2}^{+}}{E_{m1}^{+}} = \frac{2\eta_2 \cos\theta_1}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

$$1 + \Gamma_{\perp} = \tau_{\perp}$$

Snell's Law

Reflection and Transmission Coefficients

Parallel Polarization:

$$\Gamma_{\|} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1}$$

$$\tau_{\|} = \frac{2\eta_2 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1}$$

$$1 + \Gamma_{||} = \tau_{||} \frac{cos\theta_2}{cos\theta_1}$$

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Special Angles

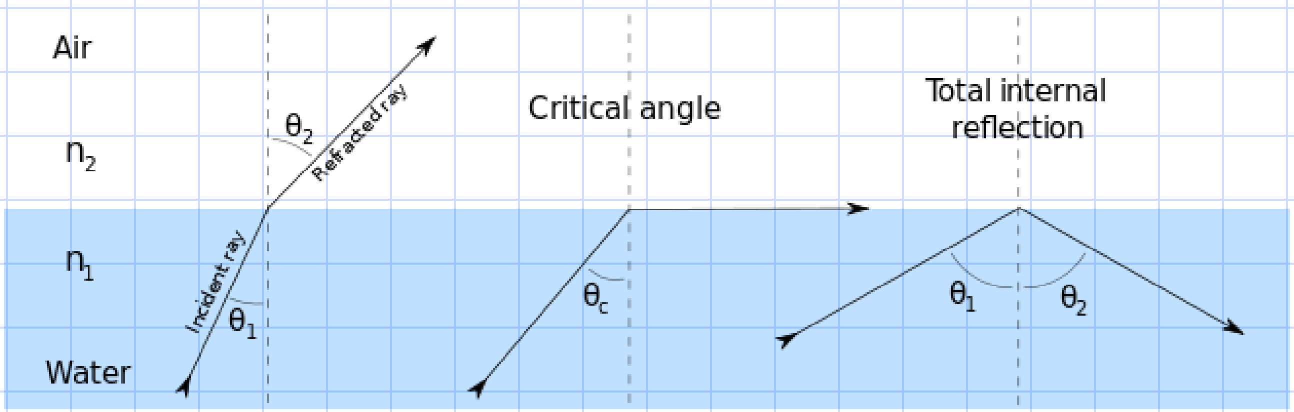
- What is a critical angle and how do you calculate it?
- What is a Brewster's angle and how do you calculate it?

Reflection Angular Dependence

Critical Angle

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

- When transmission angle reaches 90 degrees, there will be no transmission.
- Beyond this angle, there will be total reflection.



physics.stackexchange.com

Reflection Angular Dependence

Critical Angle

$$k_1 \sin \theta_c = k_2 \sin \theta_2$$

$$\sin \theta_2 = \sin 90^\circ = 1$$

$$k_1 \sin \theta_c = k_2$$

$$\sin \theta_c = \frac{k_2}{k_1}$$

$$k = \omega \sqrt{\mu_0 \varepsilon_0} \sqrt{\varepsilon_r}$$

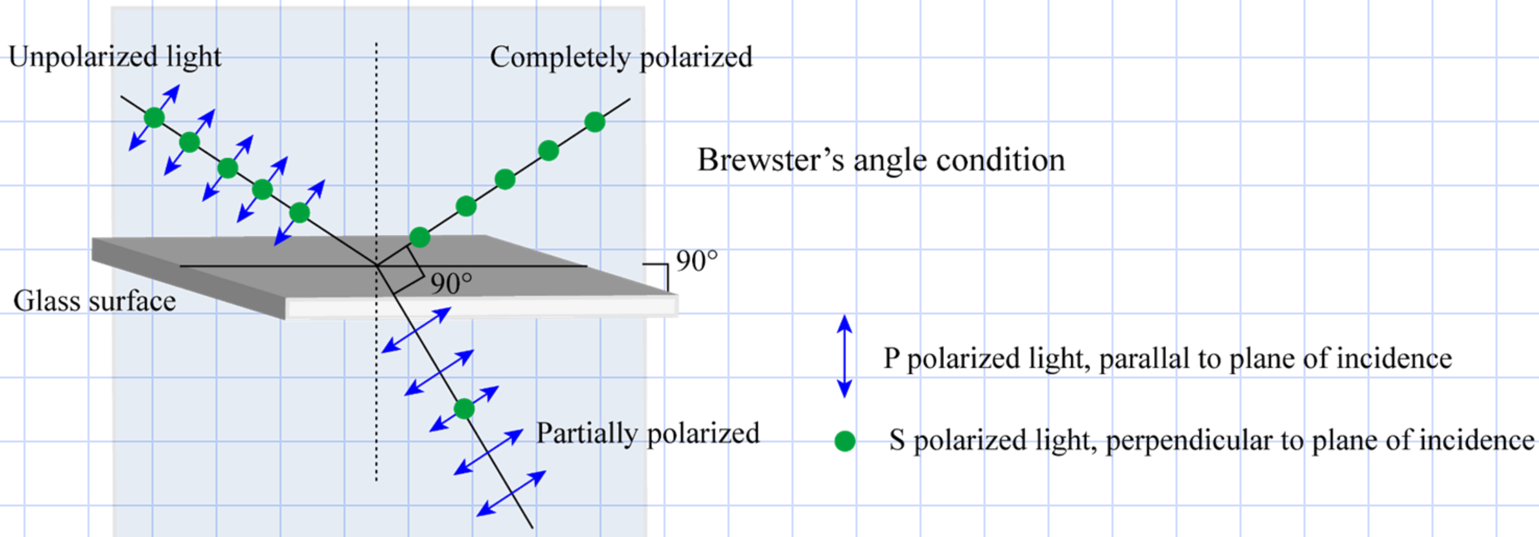
$$\sin \theta_c = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

(Note that when $\varepsilon_2 > \varepsilon_1$, this will not have a real-valued answer.)

Reflection Angular Dependence

Brewster's Angle

Wikimedia Commons



- At the Brewster's Angle or polarizing angle, the reflection coefficient is 0 for one polarization of light.
- This means that only polarized light is reflected.

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Brewster's Angle

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

$$\boxed{\eta_2 \cos\theta_\mathrm{i} = \eta_1 \cos\theta_\mathrm{t}.}$$

$$\Gamma_\perp = \frac{E_{m1}^{-}}{E_{m1}^{+}} = \frac{\eta_2 \cos\theta_1 - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

Combining these two equations and setting the incident angle to be the Brewster angle, we get:

$$\boxed{\sin\theta_{\mathrm{B}\perp} = \sqrt{\frac{1 - (\mu_1\epsilon_2/\mu_2\epsilon_1)}{1 - (\mu_1/\mu_2)^2}}\,.}$$

(Ulaby pg. 375)

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Brewster's Angle

$$\sin \theta_{B\perp} = \sqrt{\frac{1 - (\mu_1 \epsilon_2 / \mu_2 \epsilon_1)}{1 - (\mu_1/\mu_2)^2}}\ .$$

Note that this is undefined when the permeabilities of the two materials are the same.
But when they are different, some Brewster angle will exist.

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Brewster's Angle

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

$$\boxed{\eta_2 \cos\theta_i = \eta_1 \cos\theta_t .}$$

$$\Gamma_{\|} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1}$$

By once again combining the two equations, we get:

$$\boxed{\sin\theta_{B\|} = \sqrt{\frac{1-(\epsilon_1\mu_2/\epsilon_2\mu_1)}{1-(\epsilon_1/\epsilon_2)^2}} \; .}$$

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Brewster's Angle

When the permeabilities of the two materials are the same, this becomes

$$\theta_{B\|} = \sin^{-1}\sqrt{\frac{1}{1+(\epsilon_1/\epsilon_2)}}$$

$$= \tan^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}} \qquad (\text{for } \mu_1 = \mu_2).$$