ECSE 2500
Lec 18
April 6

Plan: Joint CDF/PMF/PDF of two RVs

→ Marginal CDF/PMF/PDF of one RV

→ Independence of two RVs

□ From joint PMF of two RVs to Marginal PMF of each RV

Given joint PMF $P_{X,Y}(x,y)$, $\forall x \in S_X, y \in S_Y$, we want to

all values RV X can take

marginal PMF $$P_X(x) = \sum_{y_i \in S_Y} P_{X,Y}(x, y_i)$$

likewise, $$P_Y(y) = \sum_{x_i \in S_X} P_{X,Y}(x_i, y)$$

This definition means that we can look at the PMF of X and Y on their own, by ignoring the other variable.

$\boxed{\text{Example}}$ Flip coin n times. Define

One random experiment generates two RVs

$X(s) = \#$ of heads over n coin flipping

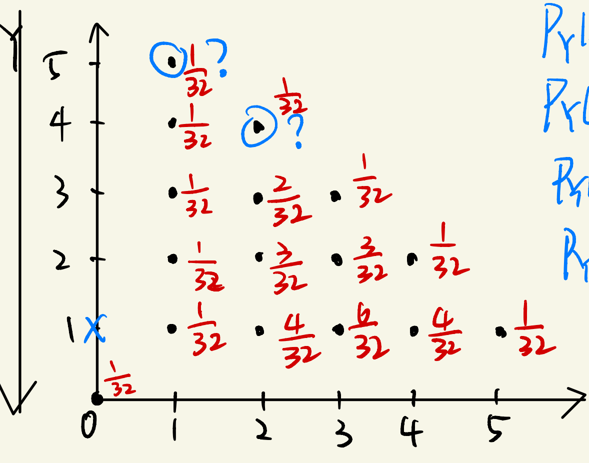
$Y(s) =$ position of the first head

$\boxed{\text{Copy from last lecture}}$

Joint PMF

$P_{XY}(x,y)$

Sum up



$P_Y(5) = \frac{1}{32}$

$P_Y(4) = \frac{2}{32}$

$P_Y(3) = \frac{4}{32}$

$P_Y(2) = \frac{8}{32}$

$P_Y(1) = \frac{16}{32}$

$P_Y(0) = \frac{1}{32}$

sum up

$P_X(0) = \frac{1}{32}$ $P_X(1) = \frac{5}{32}$ $P_X(2) = \frac{10}{32}$, $\cdots$ $P_X(5) = \frac{1}{32}$

Note: $P_X(x)$ is a Binomial distribution, and marginal PMF $P_Y(y)$ is a Geometric distribution.

This confirms that we can obtain the information of each RV from joint PMF.

In the same way, this is a notion of marginal CDF

$$F_X(x) = P(X \le x)$$
$$= F_{X,Y}(x, \infty) = P(X \le x, Y < \infty)$$

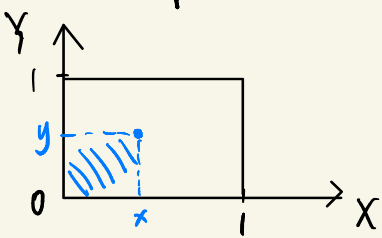
$$F_Y(y) = F_{X,Y}(\infty, y) = P(X < \infty, Y \le y)$$

Note: Just knowing the marginal PMF/CDF is not enough to obtain the joint PMF/CDF.

☐ Joint CDF / PDF of two continuous RVs

For discrete RVs, joint PMF $F_{X,Y}(x,y)$ is a "stairwise" function, but for continuous RVs, $F_{X,Y}(x,y)$ is a smooth increasing function.

Ex. Two Uniform random variables on unit square



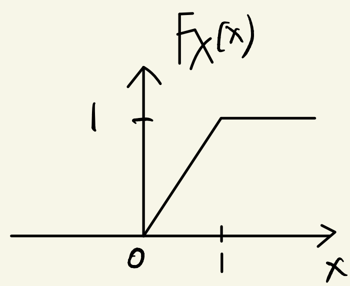
$$F_{X,Y}(x,y) = \begin{cases} 0, & \text{if } x \text{ or } y < 0 \\ xy, & \text{if } x, y \in [0,1] \\ x, & \text{if } x \in [0,1], y > 1 \\ y, & \text{if } x > 1, y \in [0,1] \\ 1, & \text{if } x, y > 1 \end{cases}$$

For continuous RVs, we can also define the notion of a marginal CDF, e.g.,

$$F_X(x) = P(X \le x) = P(X \le x, Y < \infty)$$
$$= F_{X,Y}(x, \infty)$$

We can apply it to the above joint CDF.

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } x \in [0,1] \\ 1, & \text{if } x > 1 \end{cases}$$



This is the CDF of one uniform RV!

→ PDF For a continuous RV, we can obtain PDF by differentiating CDF.

$$F_X(x) = \int_{-\infty}^{x} f_X(z)\,dz$$

$$f_X(x) = \frac{d}{dx} F_X(x)$$

For the uniform RVs case,

$$f_X(x) = \begin{cases} 0, & \text{otherwise} \\ 1, & x \in [0,1] \end{cases}$$

For the general case,

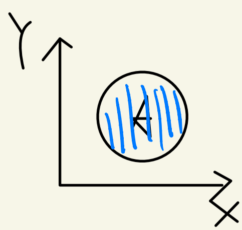
$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)\,du\,dv$$

$$f_{X,Y}(x,y) = \frac{d^2}{dx\,dy} F_{X,Y}(x,y)$$

The joint PDF also has the property like PDF for single RV, e.g.,

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

Probability of $(X,Y) \in A$ is



$$P(A) = \iint_A f_{X,Y}(x,y)\,dx\,dy$$

In particular, for a "Product-type" event,

$$P(a \le X \le b,\ c \le Y \le d) = \int_a^b \int_c^d f_{X,Y}(x,y)\,dx\,dy$$

Example: Jointly uniform RVs on $[0,1]\times[0,1]$
(see the example in the beginning of class)

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{dx\,dy} = \begin{cases} 0, & \text{otherwise} \\ \underset{\uparrow}{1}, & \text{if } x,y\in[0,1] \end{cases}$$

area of (x,Y) that covers

If we want to calculate joint PDF of two RVs uniformly distributed over $[-2,5]\times[1,3]$

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{14}, & \text{if } x\in[-2,5],\ y\in[1,3] \\ 0, & \text{otherwise} \end{cases}$$

This is because integration of joint PDF is 1.

☐ From joint PDF to marginal PDF

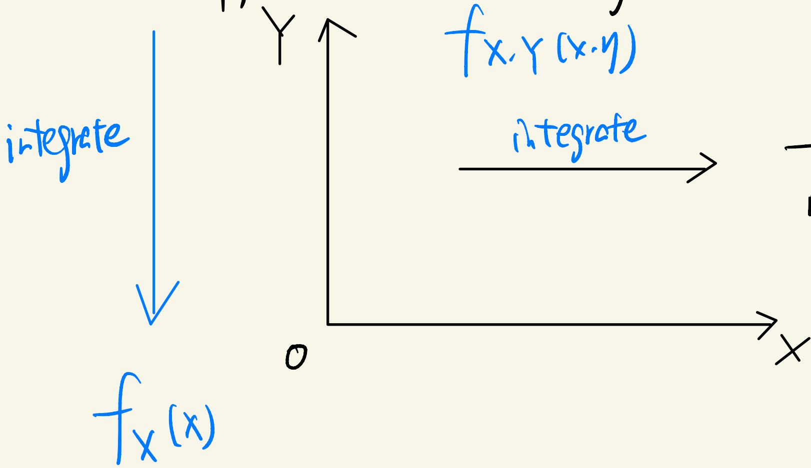
Marginal of X

$$f_X(x) = \frac{d}{dx}F_X(x) = \frac{d}{dx}\underline{F_{X,Y}(x,\infty)}$$

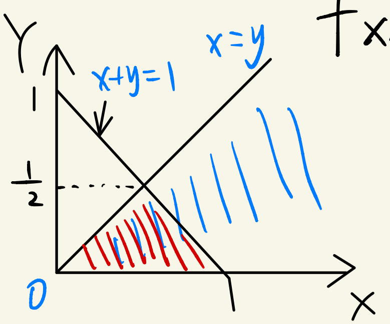
$$= \frac{d}{dx}\underline{\int_{-\infty}^{x}\int_{-\infty}^{+\infty} f_{X,Y}(u,v)\,du\,dv}$$

$$= \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy \qquad \text{integrate over } y$$

This is, to get the marginal in X, we integrate along Y (columns); to get the marginal in Y, we integrate along X (rows)



Example: We have a joint PDF



$$f_{X,Y}(x,y)=\begin{cases} c\,e^{-(x+y)}, & 0\le y\le x<\infty \\ 0, & \text{otherwise}\end{cases}$$

1) What is value of c for joint PDF $f_{X,Y}(x,y)$ to be a valid PDF?

i) $f_{X,Y}(x,y)\ge 0$

ii) $\iint f_{X,Y}(x,y)\,dx\,dy=1$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} c\, e^{-(x+y)}\, dx\, dy$$

integration by parts $= c \cdot \frac{1}{2} = 1 \iff c = 2$

2) What is the marginal PDF $f_X(x)$?

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy$$

$$= \int_{-\infty}^{+\infty} 2e^{-(x+y)}\, dy$$

$$= 2e^{-x} \cdot \left(-e^{-y}\right)\Big|_{y=0}^{y=x}$$

$$= \begin{cases} 2e^{-x}\left(1-e^{-x}\right), & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

3) What is $P(X+Y \leq 1)$?

$$P(X+Y\leq 1) = \int_0^{\frac{1}{2}}\left(\int_y^{1-y} 2e^{-x}\, dx\right) e^{-y}\, dy$$

$$= \int_0^{\frac{1}{2}} 2e^{-y}\left(-e^{-x}\right)\Big|_y^{1-y}\, dy$$

$$= \int_0^{\frac{1}{2}} 2e^{-y}\left(e^{-y} - e^{-(1-y)}\right) dy = 1 - 2e^{-1}$$

Important example of joint PDF of two continuous RVs

One RV $\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Joint Gaussian RVs

$$f_{X,Y}(x,y) = \frac{1}{2\pi |\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}\left(\begin{bmatrix} x \\ y \end{bmatrix} - \underline{\mu}\right)^T \Sigma^{-1} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \underline{\mu}\right)}$$

mean vector $\underline{\mu} = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix} \in \mathbb{R}^2$; $\Sigma = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix} \in \mathbb{R}^{2\times 2}$

covariance matrix

critical parameter $\rho \in [-1,1]$
captures correlation between X and Y.