Fields and Waves I

Lecture 16

Magnetic Flux
Magnetic Materials

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Materials from other sources are referenced where they are used.
Those listed as Ulaby are figures from Ulaby's textbook.

Review

Magnetostatic Version of Maxwell's Equations

Integral Form

$$\oint \vec{H} \cdot d\vec{l} = \int \vec{J} \cdot d\vec{s} = I_{enc}$$

$$\oint \vec{B} \cdot d\vec{S} = 0$$

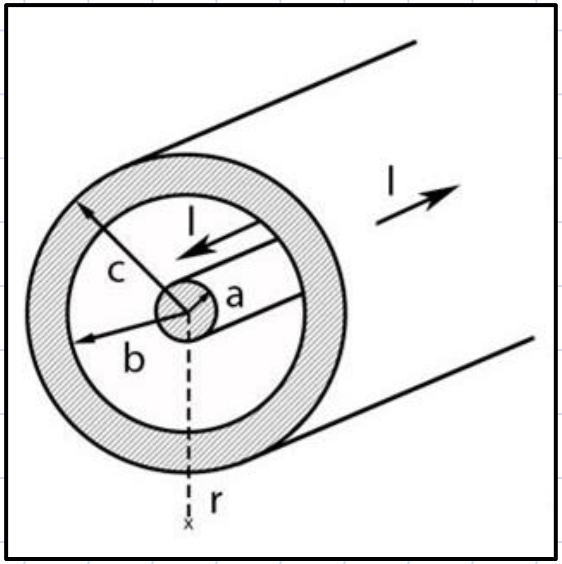
Differential Form

$$\nabla \times \vec{H} = \vec{J}$$

$$\nabla \cdot \vec{B} = 0$$

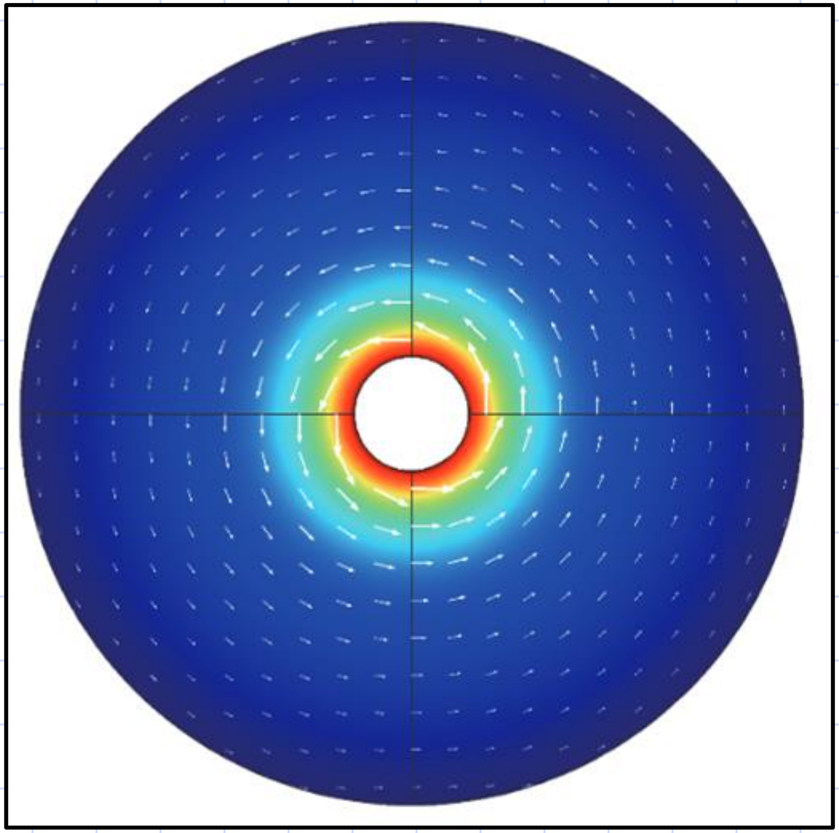
- Electrostatics are based on two Maxwell's equations, but simplified.
- Electrostatics are based on the other two Maxwell's equations, but simplified.
- In these simplified Maxwell's equations, electric and magnetic fields are separate. In the full Maxwell's equations, they are coupled.

Ampere's Law



What does the magnetic field look like for this conductive coaxial cable?

Ampere's Law



comsol.com

<u>The field between the conductors looks something like this.</u> (you can check this using the right hand rule)

A quick point: conductors do not shield magnetic fields in the same way they shield electric fields. Why is this?

Ampere's Law

$$\oint \vec{H} \cdot \vec{dl} = \int \vec{j} \cdot \vec{ds} = I_{net}$$

$$\vec{B} = \mu_o \vec{H}$$

for $r < a$:

$$\vec{J} = \frac{I}{\pi a^2} \hat{z}$$

$$H(2\pi r) = \left(\frac{I}{\pi a^2}\right) \pi r^2$$

$$\vec{H} = \frac{I}{2\pi a^2} r \hat{\phi}$$

$$\vec{B} = \frac{\mu_o I}{2\pi a^2} r \hat{\phi}$$

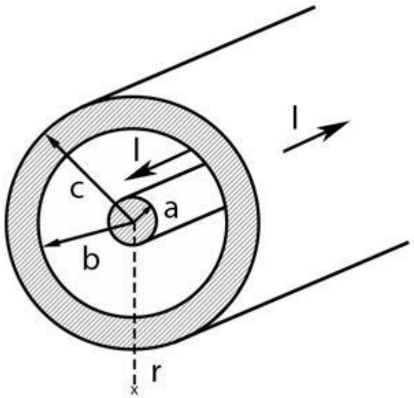
for $a < r < b$:

$$H(2\pi r) = I$$

$$\vec{B} = \frac{\mu_o I}{2\pi r} \hat{\phi}$$

Ampere’s Law

Magnetic Field of a Coaxial Cable



$$B = \frac{\mu_o I}{2\pi a^2} r \qquad (r < a)$$

$$B = \frac{\mu_o I}{2\pi r} \qquad (a < r < b)$$

$$B = \frac{\mu_o I}{2\pi r}\left(\frac{c^2 - r^2}{c^2 - b^2}\right) \qquad (b < r < c)$$

$$B = 0 \qquad (r > c)$$

I : Electric current μ_o: Permeability of free space

B : Magnetic field

ScienceFacts.net

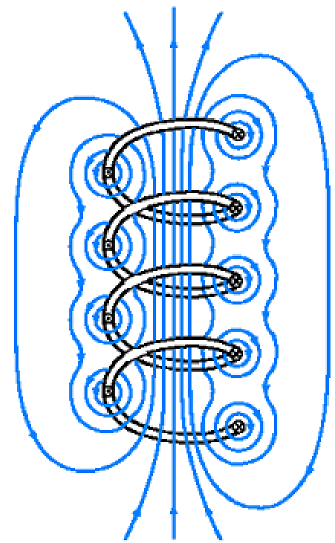
Ampere's law

$$\oint H \bullet dl = \int j \bullet ds = I_{net}$$

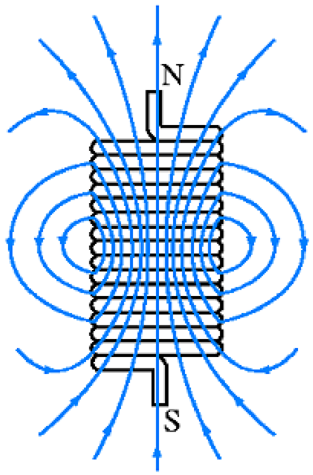
$$B = \mu_0 \cdot H$$

sciencefacts.net

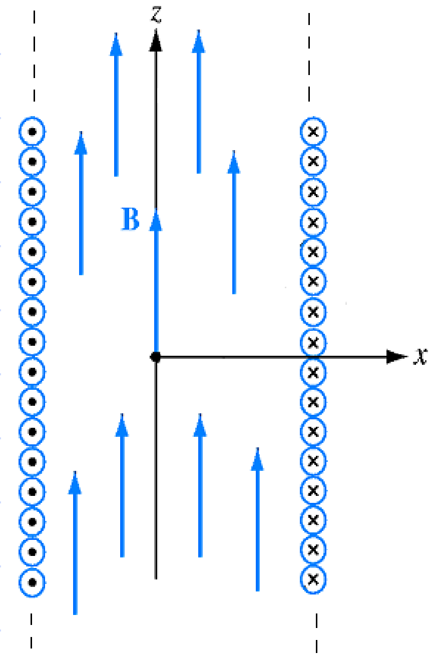
Ampere's Law



(a) Loosely wound solenoid



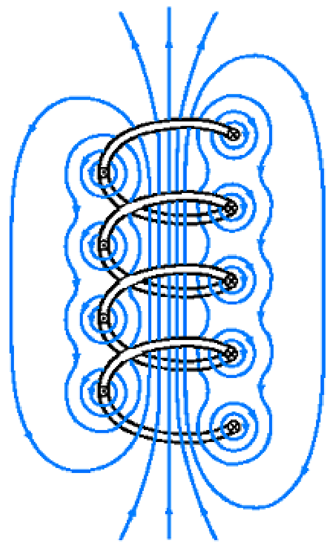
(b) Tightly wound solenoid

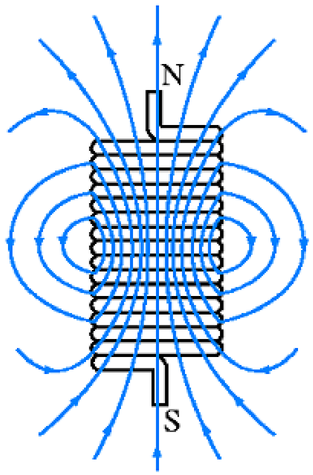


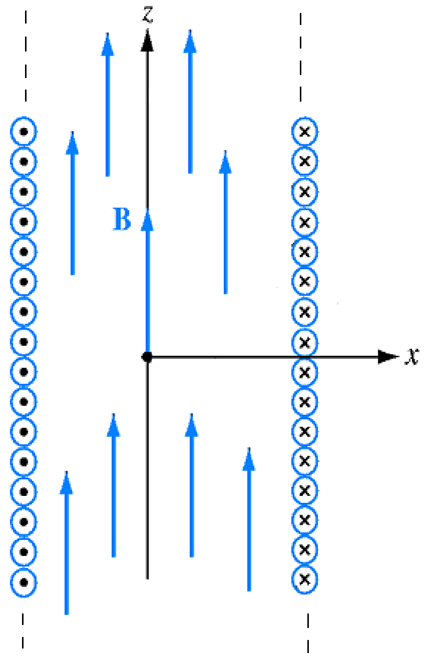
(c) Infinite tightly wound solenoid

Ulaby

Ampere’s Law







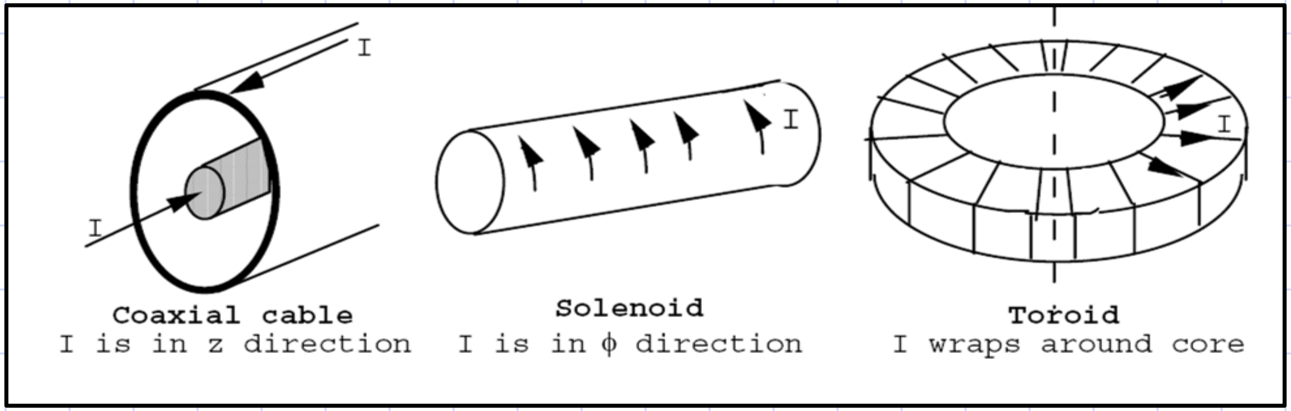
A simple geometric argument: An ideal solenoid has rotational symmetry in ϕ so the B-field can’t change in ϕ or point in r or ϕ. And because the ideal solenoid is very long, the B-field can’t change in z. So the B-field must change in r and point in z.

Ulaby

Ampere's Law

Do Lecture 16, Exercise 1 in groups of up to 4.

Ampere's Law



coax $\vec{B} = B_\phi(r)\,\hat{a}_\phi$
solenoid $\vec{B} = B_z(r)\,\hat{a}_z$ } ignoring end effects
torus $\vec{B} = B_\phi(r,z)\,\hat{a}_\phi$ ← assume tightly wound

Ampere's Law

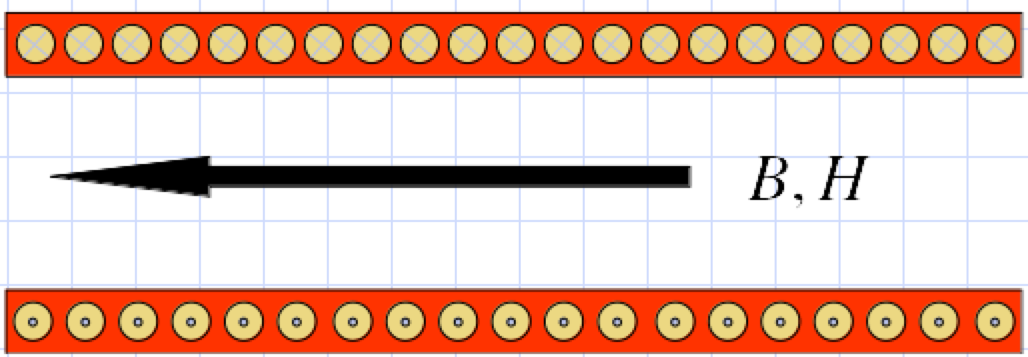
$$\oint H \bullet dl = \int j \bullet ds = I_{net}$$

or

$$\oint B \bullet dl = \mu_0 \cdot \int j \bullet ds = \mu_0 \cdot I_{net}$$

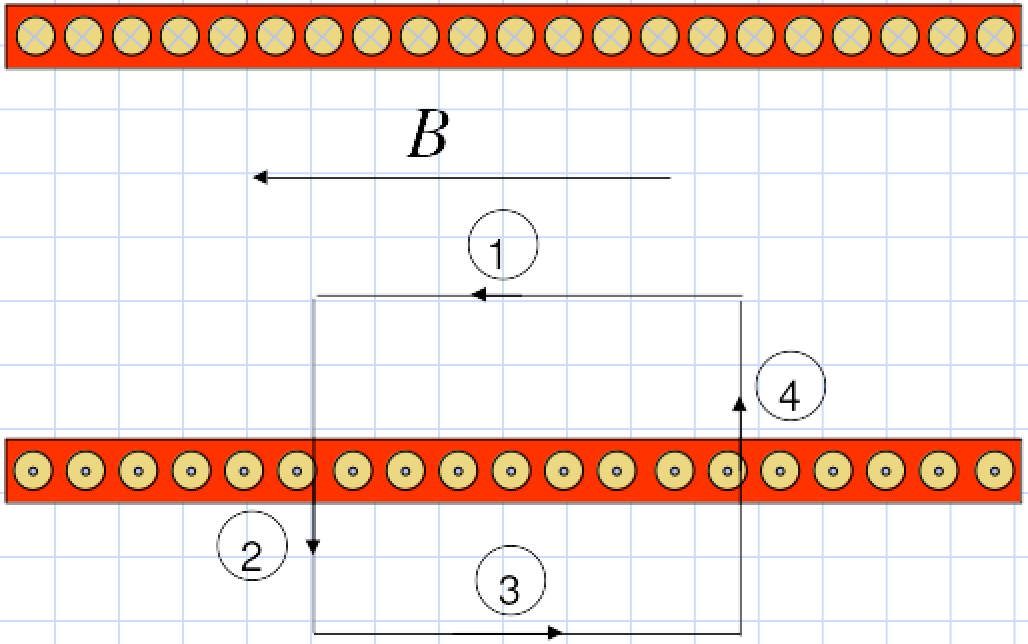
Approach similar to using Gauss' Law, use symmetry to get B-field out of integral

Example: Consider an infinite wire solenoid



sectional view

Ampere's Law



Find B

Solenoid has current I through **n** turns/length

STEP 1: Choose path for integral - $\oint B \bullet dl$

- Chosen paths are 1,2, 3 and 4 - they form a closed loop

Ampere's Law

STEP 2: Evaluate $\oint B \bullet dl$

- Segments 2 and 4 have $B \perp dl$

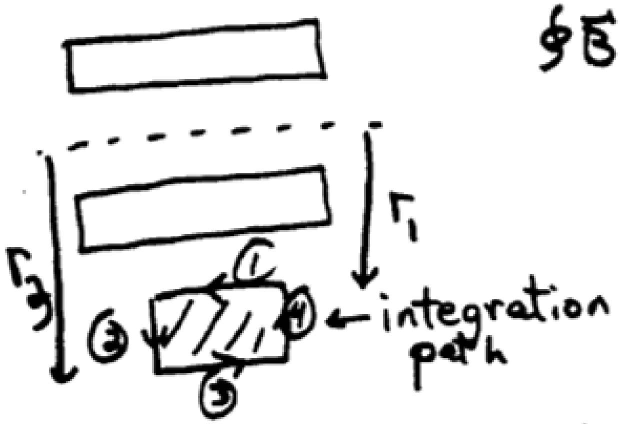
$$\therefore \oint B \bullet dl \Rightarrow 0$$

- Segment 3 has $B \rightarrow 0$ (will show later)

- Segment 1 has $\oint \vec{B} \bullet \vec{dl} = \int_0^l B_z \cdot dz = B_z \cdot l$

arbitrary length

Ampere's Law



$$\oint \vec{B}\cdot d\vec{l} = \int_{①} \vec{B}\cdot d\vec{l} + \int_{③} \vec{B}\cdot d\vec{l}$$

② + ④ → 0 since $\vec{B} \perp d\vec{l}$

$$= \int_0^l B_z(r_1)\,dz + \int_l^0 B_z(r_3)\,dz$$

$$= \{B_z(r_1) - B_z(r_3)\}\, l$$

$\mu_0 I_{net} = 0 \Rightarrow \therefore \oint \vec{B}\cdot d\vec{l} = 0$ and $B_z(r_1) = B_z(r_3)$

$\therefore$ B_z is constant outside solenoid since it must $\to 0$ as $r \to \infty$, it must be $\emptyset$ everywhere for $r > b$.

Ampere's Law

STEP 3: find I_{net}

- current passing through loop :

$$I_{net} = n \cdot l \cdot I$$

STEP 4: solve for B

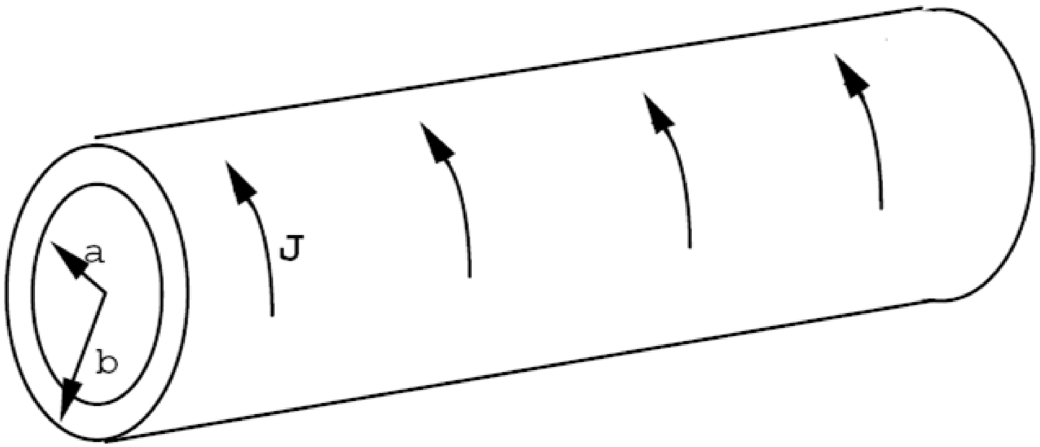
$$\oint B \bullet dl = \mu_0 \cdot \int j \bullet ds = \mu_0 \cdot I_{net}$$

$$\Rightarrow B_z \cdot l = \mu_0 \cdot n \cdot l \cdot I \quad \Rightarrow \quad B = \mu_0 \cdot n \cdot I \cdot \hat{a}_z$$

Ampere's Law

A long solenoid has a current density of $\mathbf{J} = J_0\, \mathbf{a}_\varphi$ for $a < r < b$ and is 0 everywhere else. Ignore end effects.

a. Find the magnetic flux density, $\mathbf{B}$ for $r < a$. Be sure to sketch the line integral paths you use. Assume $\mathbf{B} = 0$ for $r > b$.



b. Check your answer to part a. by evaluating $\nabla \bullet \mathbf{B}$ and $\nabla \times \mathbf{B}$.

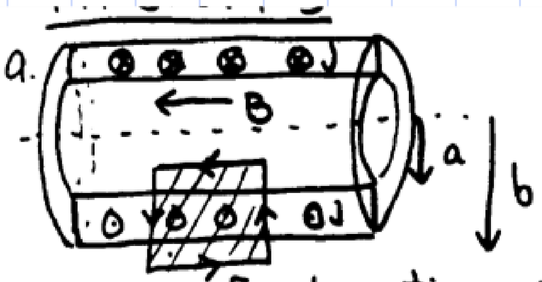
c. Find $\mathbf{B}$ for $a < r < b$. Sketch the line integral path you use.

d. Check your answer to part c. by evaluating $\nabla \bullet \mathbf{B}$ and $\nabla \times \mathbf{B}$.

e. Plot B_z vs r.

f. Show that $\mathbf{B} = 0$ for $r > b$.

Ampere's Law



integration path

region is $d\vec{s}$ integral ($\vec{J}=0$ outside $a \to b$)

$\vec{B} = B_z(r)\hat{a}_z$

$\oint \vec{B}\cdot\vec{dl} = \mu_o I_{net} = \mu_o \int \vec{J}\cdot\vec{ds}$

$\oint \vec{B}\cdot\vec{dl} = \Sigma$ 4 segments

$\vec{B}\cdot\vec{dl} = 0$ on radial, $\vec{B}=0$ on ext.

$\therefore \oint \vec{B}\cdot\vec{dl} = \int_{internal} \vec{B}\cdot\vec{dl} = B_z l$

$$I_{net} = \int \vec{J}\cdot\vec{ds} = \int_0^l \int_a^b J_o\,\hat{a}_\phi \cdot dr\,dz\,\hat{a}_\phi = J_o(b-a)l$$

$$\oint \vec{B}\cdot\vec{dl} = \mu_o I_{net} \Rightarrow B_z l = \mu_o J_o (b-a) l \Rightarrow \boxed{\vec{B} = \mu_o J_o (b-a)\hat{a}_z}$$

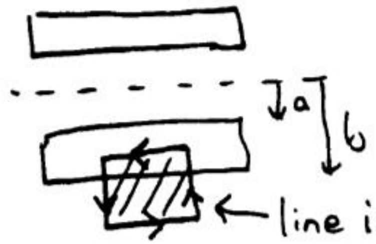
b. $\nabla\cdot\vec{B} = \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z} = 0$ since $B_r = B_\phi = \frac{\partial B_z}{\partial z} = 0$

$\nabla\times\vec{B} = 0$ because all 6 terms 0

$B_r = B_\phi = \frac{\partial B_z}{\partial \phi} = \frac{\partial B_z}{\partial r} = 0$

$\vec{J} = 0$ for $r < a$ + $\nabla\times\vec{B} = \mu_o \vec{J}$ ✓

Ampere’s Law

c. 

$\oint \vec{B}\cdot d\vec{l}$ is same as before $= B_z l$ on inner leg

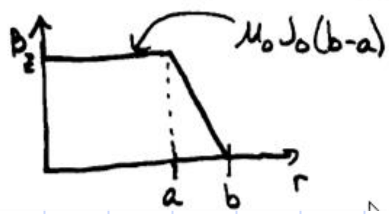
//// region is $\vec{J}\cdot d\vec{s}$ integral surface ($\vec{J}=0$ for $r>b$)

$$I_{net} = \int \vec{J}\cdot d\vec{s} = \int_0^l \int_r^b J_0 \, dr\, dz = J_0 (b-r) l$$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{net} \Rightarrow B_z l = J_0 (b-r) l \Rightarrow \boxed{\vec{B} = \mu_0 J_0 (b-r)\hat{a}_z}$$

d. $\nabla\cdot\vec{B} = 0$ since $B_r = B_\varphi = \frac{\partial B_z}{\partial z} = 0$

$$\nabla\times\vec{B} = -\frac{\partial B_z}{\partial r}\hat{a}_\varphi + \underset{\text{that}\to 0}{5 \text{ terms}} = -\frac{\partial}{\partial r}\left(\mu_0 J_0 (b-r)\right)\hat{a}_\varphi = \mu_0 J_0 \hat{a}_\varphi = \mu_0 \vec{J}$$

e. 

Ampere's Law

- Just like with Gauss' Law, a great deal of symmetry is necessary to use Ampere's Law to find B or H.
- Simplify everything before attempting a solution.
- There is an analog to using the electric potential, although for B, it is a bit more complex since it involves a vector potential instead of a scalar potential. It is still easier since the vector potential is in the direction of the current.

Magnetic Vector Potential

- We found that in electrostatic fields, it was useful to derive the concept of voltage from electric fields in order to solve problems. Can we do the same for magnetic fields?
- Yes! But it takes a different form from the electric field.
- For any vector A:

$$\nabla \cdot (\nabla \times \vec{A}) = 0$$

Magnetic Vector Potential

$$\nabla \cdot (\nabla \times \vec{A}) = 0$$

- We will define the vector potential in the following way:

$$\vec{B} = \nabla \times \vec{A}$$

- This satisfies our original equation because

$$\nabla \cdot \vec{B} = 0$$

Magnetic Vector Potential

What can we do with magnetic vector potential?

Remember that in electrostatics, we can only find analytical solutions using Gauss's law for simple geometries.

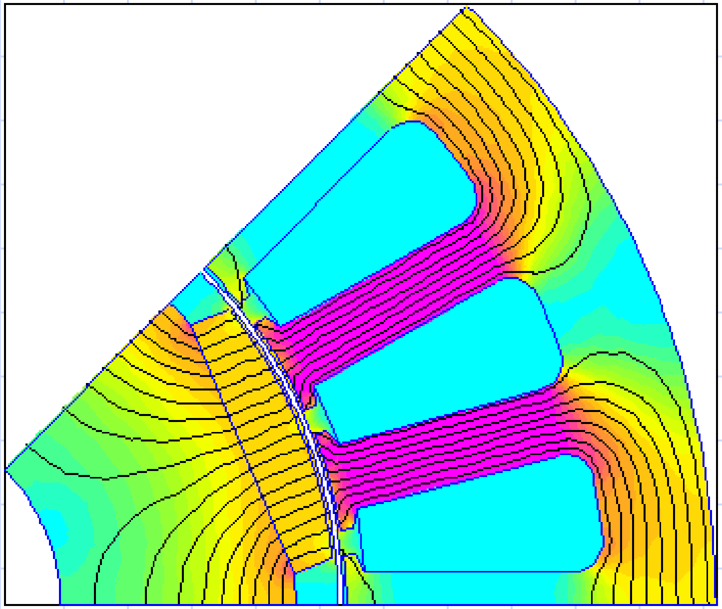
For more complicated geometries, we need a computer, Laplace / Poisson's Equations and finite element / finite difference methods.

Magnetic Vector Potential

The same is true for magnetostatics. For simple geometries, we can use Ampere's Law. To solve more complicated problems (i.e. computational magnetic problems), we need a concept of magnetic potential.

We can derive a vector Poisson's Equation:

$$\nabla^2 \vec{A} = -\mu \vec{J}$$

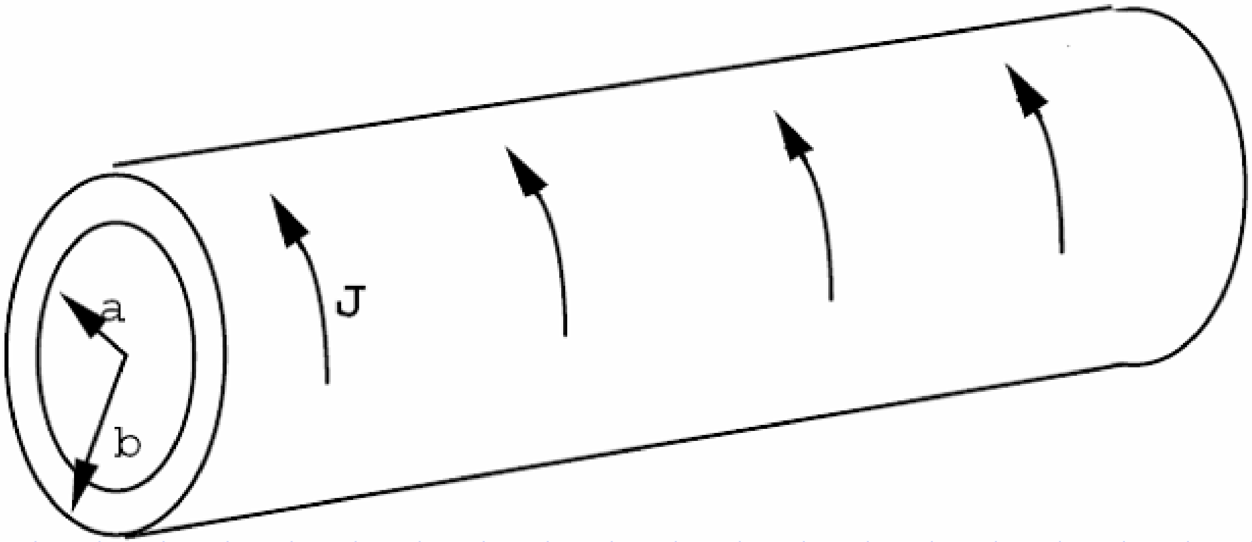


femm.info

Magnetic Vector Potential

The current density, $\mathbf{J} = J_0\,\mathbf{a}_\varphi$ for $a < r < b$ and is 0 everywhere else.

$$\mathbf{B} = \begin{cases} \mu_0 J_0 (b - a)\,\mathbf{a}_z & \text{for } r \le a \\ \mu_0 J_0 (b - r)\,\mathbf{a}_z & \text{for } a \le r \le b \\ 0 & \text{for } b \le r. \end{cases}$$



Magnetic Vector Potential

Check that $\mathbf{B} = \nabla \times \mathbf{A}$ if the magnetic vector potential, $\mathbf{A}$ is given by:

$$\mathbf{A} = \begin{cases} \mu_0 J_0 (b - a)\, r / 2\, \mathbf{a}_\varphi & \text{for } r \le a \\ \mu_0 J_0 (r\, b/2 - r^2/3 - a^3/6r)\, \mathbf{a}_\varphi & \text{for } a \le r \le b \\ \mu_0 J_0 (b^3 - a^3)/ 6r\, \mathbf{a}_\varphi & \text{for } b \le r \end{cases}$$

Magnetic Vector Potential

b. $\nabla \times \vec{A} = \hat{a}_z \frac{1}{r}\left[\frac{d}{dr}(rA_\phi)\right] + 5\ \cancel{\text{terms}} \to 0$

$$r<a \quad = \hat{a}_z \frac{1}{r}\left[\frac{d}{dr}\left(r\,\frac{\mu_0 J_0 (b-a)\,r}{2}\right)\right] = \hat{a}_z \frac{\mu_0 J_0 (b-a)}{2r}\frac{d}{dr}(r^2) =$$

$$= \boxed{\mu_0 J_0 (b-a)\,\hat{a}_z} = \vec{B}\ \checkmark$$

$a<r<b$:

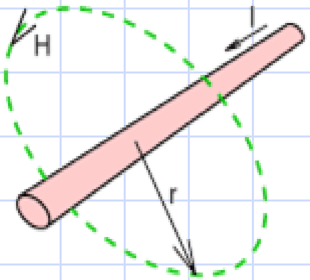
$$\nabla \times \vec{A} = \hat{a}_z \frac{1}{r}\left[\frac{d}{dr}\left\{\mu_0 J_0\left(\frac{r^2 b}{2} - \frac{r^3}{3} - \frac{\cancel{r}a^3}{6\cancel{r}}\right)\right\}\right]$$

$$= \frac{\mu_0 J_0}{r}\hat{a}_z\left[\frac{\cancel{2}rb}{\cancel{2}} - \frac{\cancel{3}r^2}{\cancel{3}}\right] = \boxed{\mu_0 J_0 (b-r)\,\hat{a}_z} = \vec{B}\ \checkmark$$

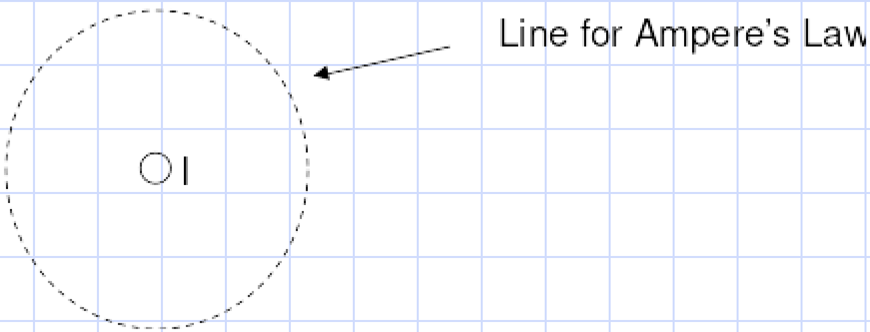
$r>b$

$$\nabla \times \vec{A} = \hat{a}_z \frac{1}{r}\left[\frac{d}{dr}\left\{\cancel{r}\,\frac{\mu_0 J_0 (b^3-a^3)}{6\cancel{r}}\right\}\right] = \boxed{0} = \vec{B}$$

Vector Potential and Magnetic Field



Determine the field for a long straight wire carrying current I.



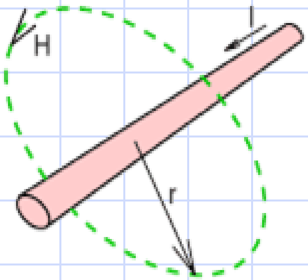
$$\oint \vec{H} \cdot \vec{dl} = I_{enclosed}$$

$$\frac{B_\phi 2\pi r}{\mu_0} = I$$

$$B_\phi = \frac{\mu_0 I}{2\pi r}$$

http://www.ee.surrey.ac.uk/Workshop/advice/coils/terms.html

Vector Potential and Magnetic Field



A reminder:

$$B \equiv \nabla \times A$$

$$\text{Curl } \vec{\nabla}\times\vec{F} = \left(\frac{1}{r}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right)\hat{r} + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)\hat{\phi} + \frac{1}{r}\left(\frac{\partial (r \cdot F_\phi)}{\partial r} - \frac{\partial F_r}{\partial \phi}\right)\hat{z}$$

Vector Potential and Magnetic Field

The magnetic vector potential can be determined from first principles or from the magnetic field. We will do the latter.

$$B_\phi = \frac{\mu_o I}{2\pi r} = -\frac{\partial A_z}{\partial r} \Rightarrow A_z = -\frac{\mu_o I}{2\pi}\ln(r) + const$$

From the curl expression

Specifying the zero reference will determine this constant

Note that the vector potential is always in the direction of the current

Magnetic Flux

Previously we used: $\oint H \bullet dl = \int j \bullet ds = I_{net}$

Now we will look at the effect of

$$\nabla \bullet B = 0 \Rightarrow \underbrace{\oint B \bullet ds}_{\text{This surface integral encloses the volume}} = 0$$

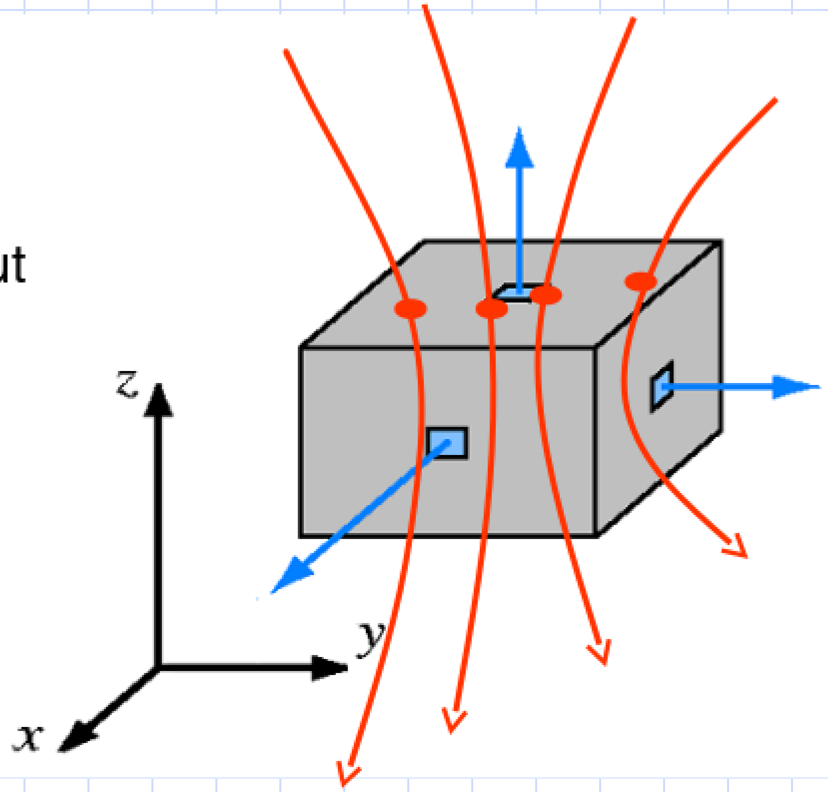
Recall, $\nabla \bullet B = 0 \Rightarrow \int \nabla \bullet B \cdot dv = \oint B \bullet ds = 0$

Magnetic Flux

$$\therefore \oint B \bullet ds = \int_{left} B \bullet ds + \int_{right} B \bullet ds + \int_{up} B \bullet ds + \ldots = 0$$

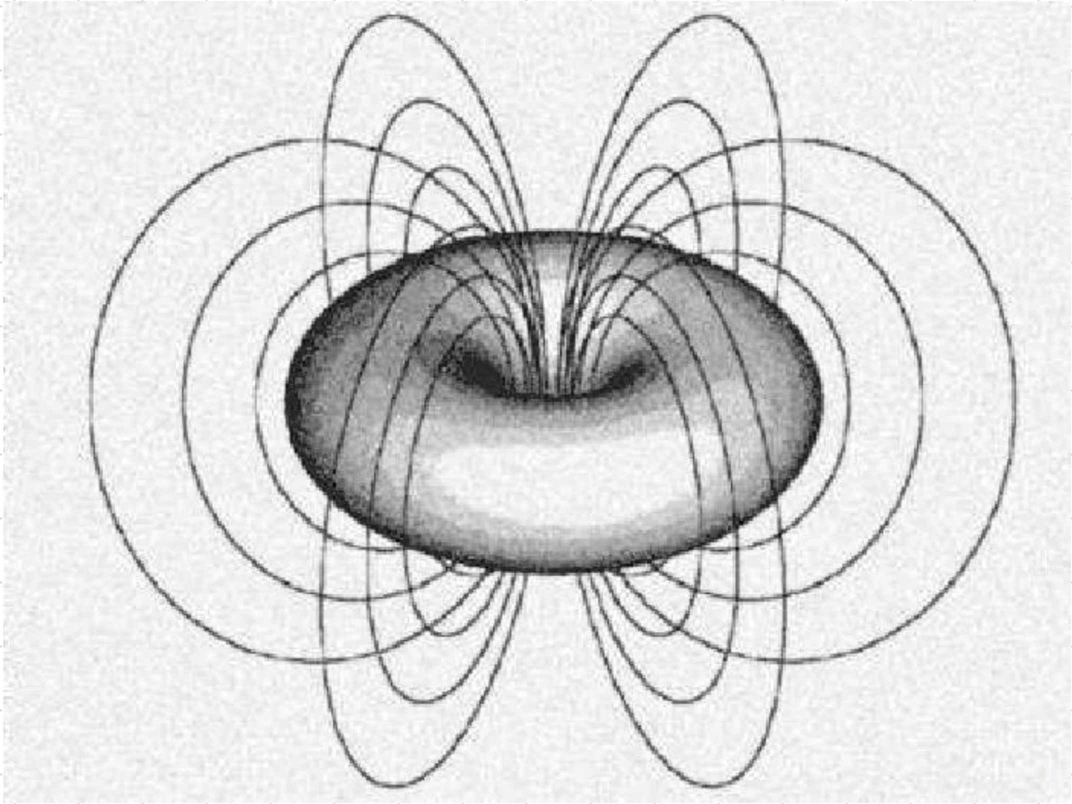
The flux is conservative:

flux coming in = flux going out



Magnetic Flux

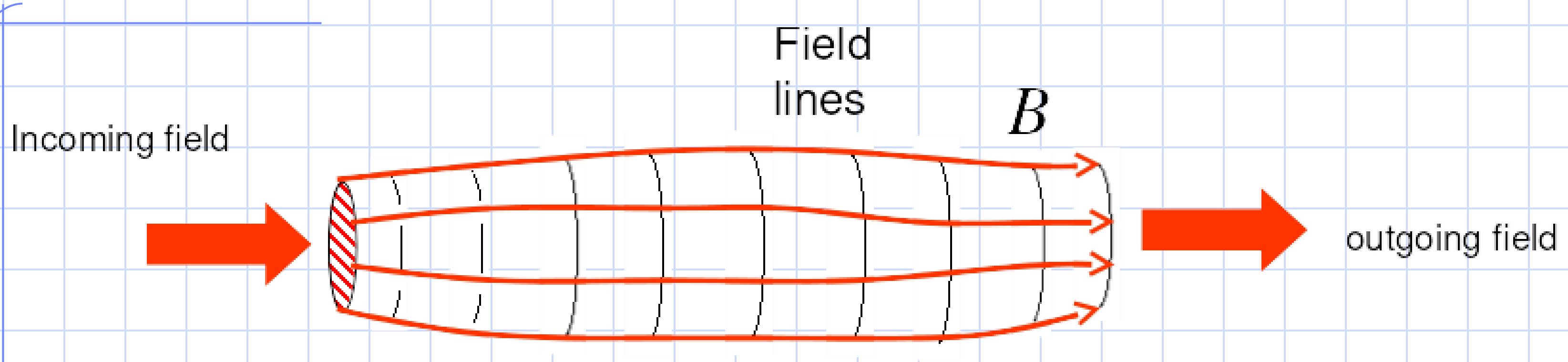
- B-field and H-field lines close on themselves (no beginning or end)
- This is in contrast to D-field and E-field lines which start and end on charges



Spivey et. al. (2000)

Magnetic Flux

Flux Tubes



Along sides: $B \perp ds$

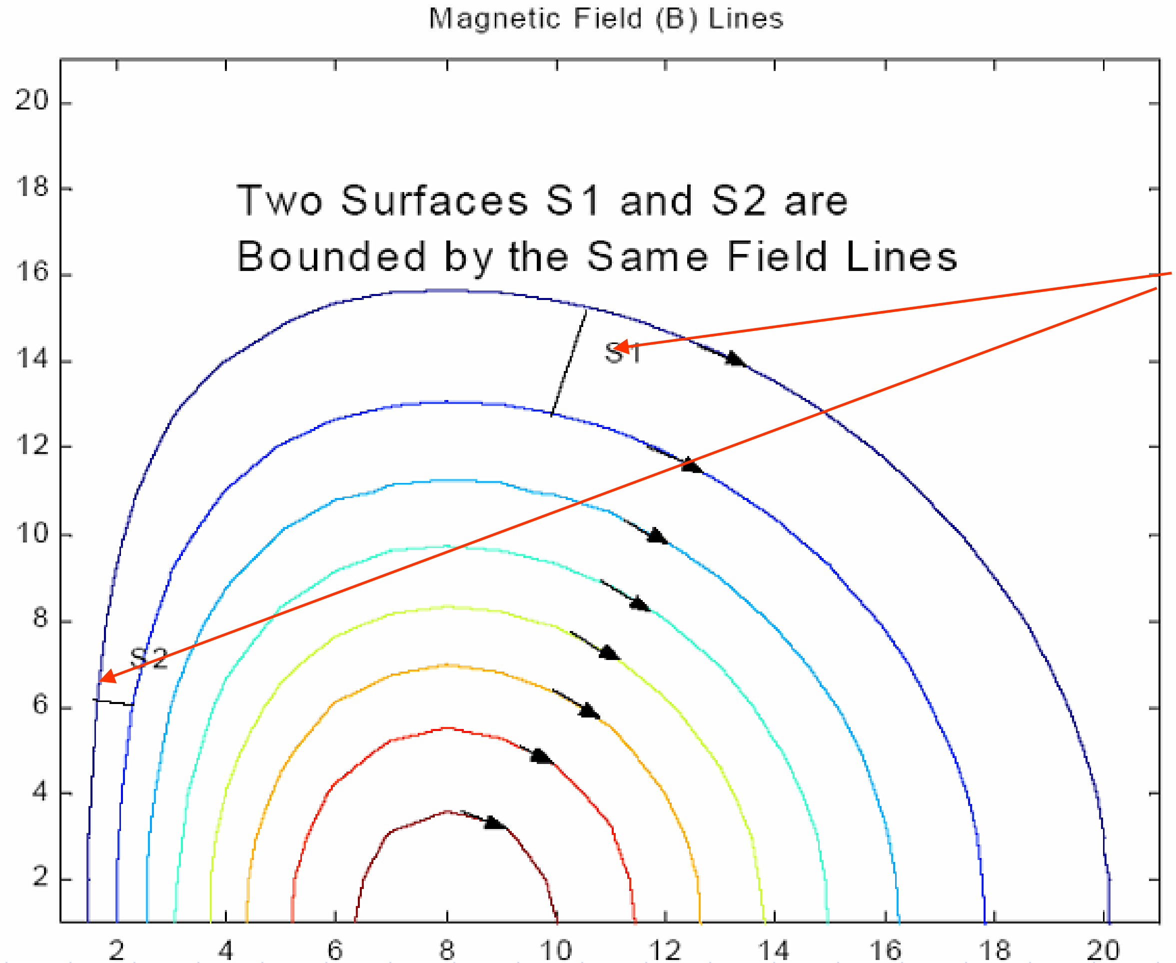
$$\therefore B \bullet ds = 0$$

$$\therefore \oint B \bullet ds = \int_{left} B \bullet ds + \int_{right} B \bullet ds$$

Define Flux, $\Psi \equiv \int B \bullet ds$, enters from left and leaves to the right

Magnetic Flux

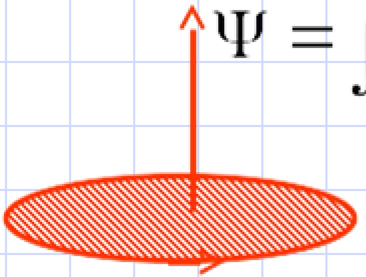
Flux Tubes



The same flux passes through both surfaces since they are in the same flux tube

Magnetic Flux

$$\Psi = \int B \bullet ds$$



$$\Psi = \oint A \bullet dl$$

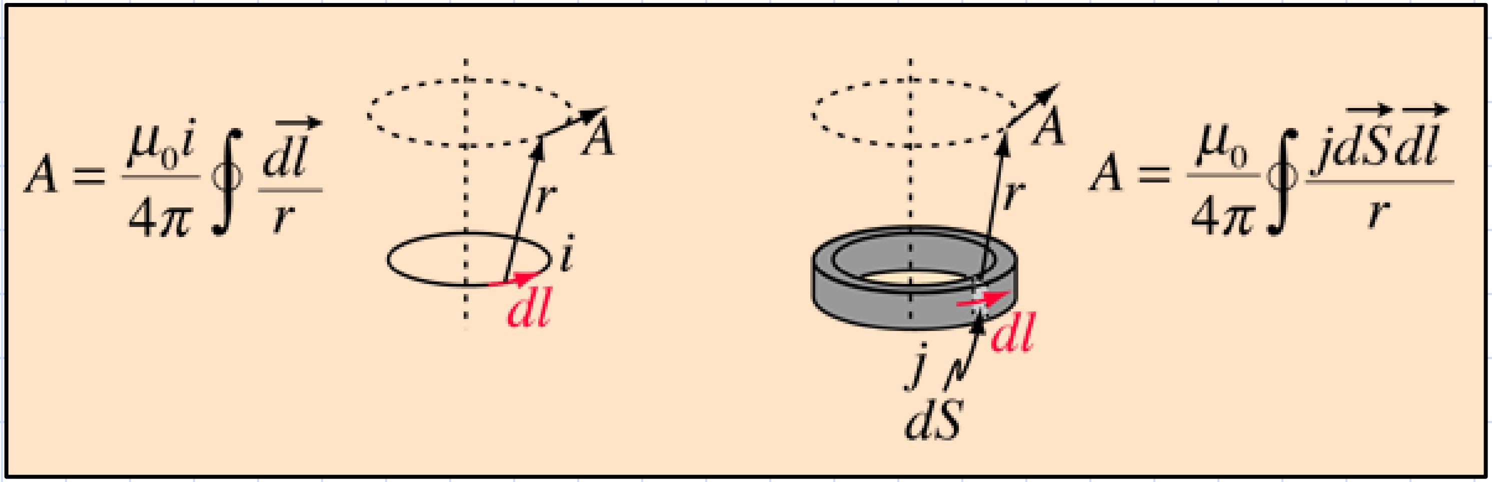
A is related to Flux

$$\Psi = \int B \bullet ds = \int \nabla \times A \bullet ds = \oint A \bullet dl$$

After some math....

Alternative way to find FLUX

Magnetic Flux

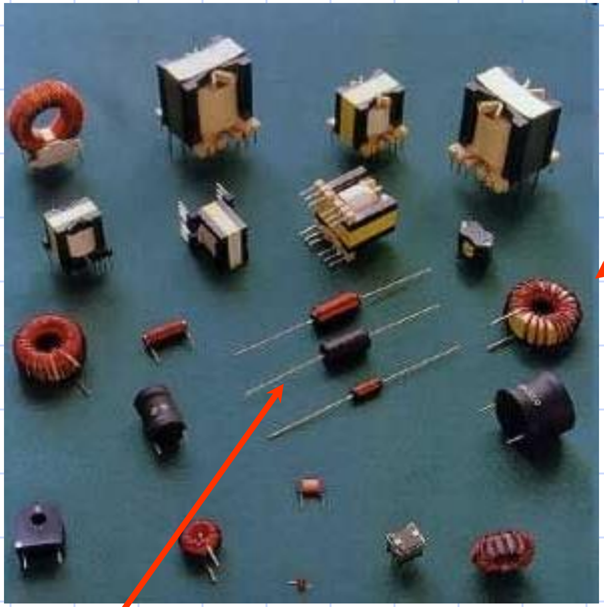


http://hyperphysics.phy-astr.gsu.edu

Magnetic Flux

Torus

The geometries of magnetic fields influence the design of inductors.



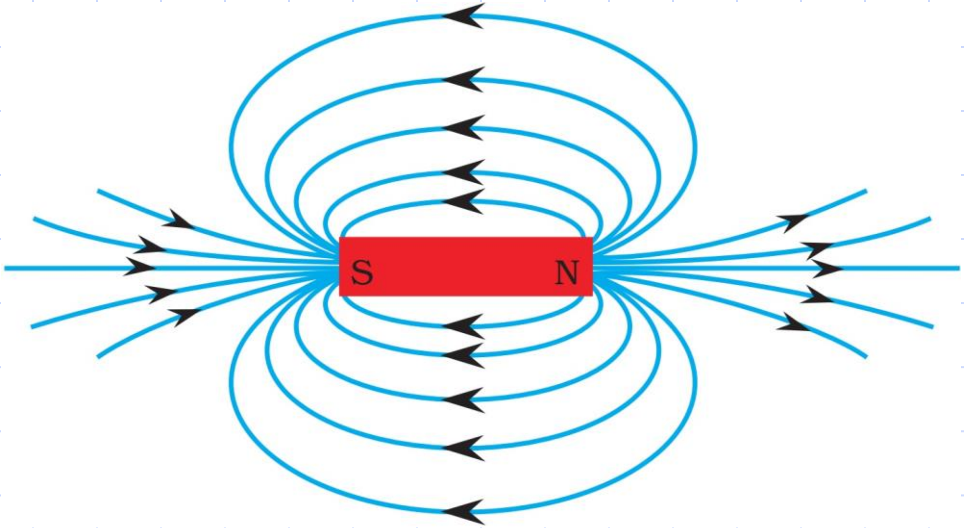
Solenoid

http://www.directindustry.fr/prod/lcr-electronics/assemblage-de-cables-electriques-pour-applications-telecom-donnees-35095-214564.html#prod_214564

http://www.magasia.com.tw/inductor.html

Magnetic Materials

- In a permanent magnet, the magnetic moments of the atoms align and add together to create a measurable total field. External fields can help realign the atoms and their total field.
- Ferromagnetic materials do this especially well, but virtually any material can become magnetic in a large field.

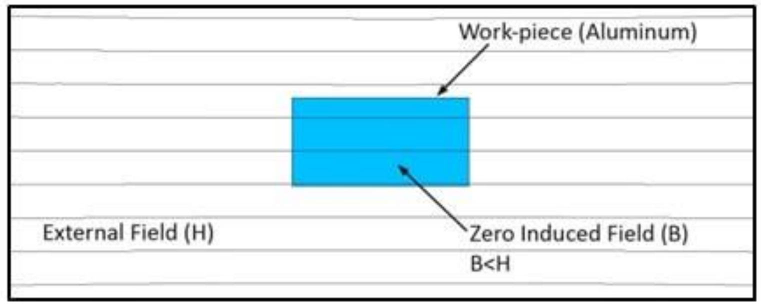


Magnetic Materials

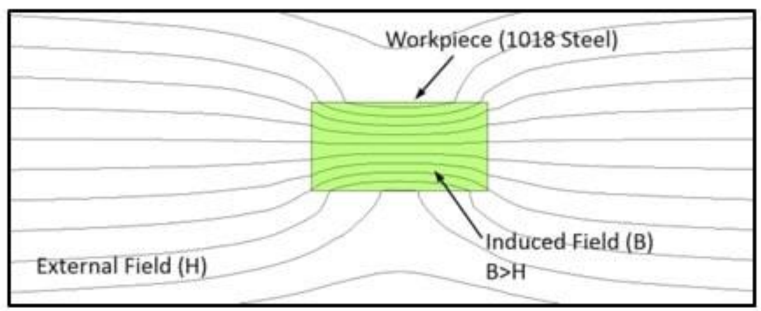
$$\vec{B} = \mu \vec{H}$$

Magnetic field lines are bent by materials with higher permeability

Applied Field H
Permeability $\mu = 1$
Induced Field $B = 0$
$B < H$



Applied Field H
Permeability $\mu > 1$
Induced Field $B = \mu \times H$
$B > H$



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Magnetic Materials

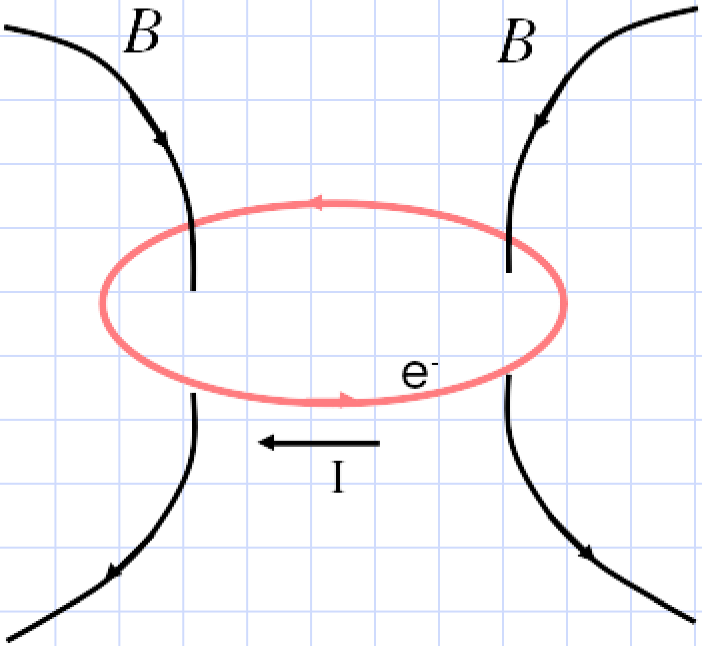
Magnetic properties tend to either very strong or very weak

Vast majority of materials

- ferromagnets (Fe) and permanent magnets
- exhibit strong non-linear effects
- demonstrate "memory" or hysteresis effects

Use simple, linear model

Magnetic Materials



In Quantum mechanics, atoms have "spin"

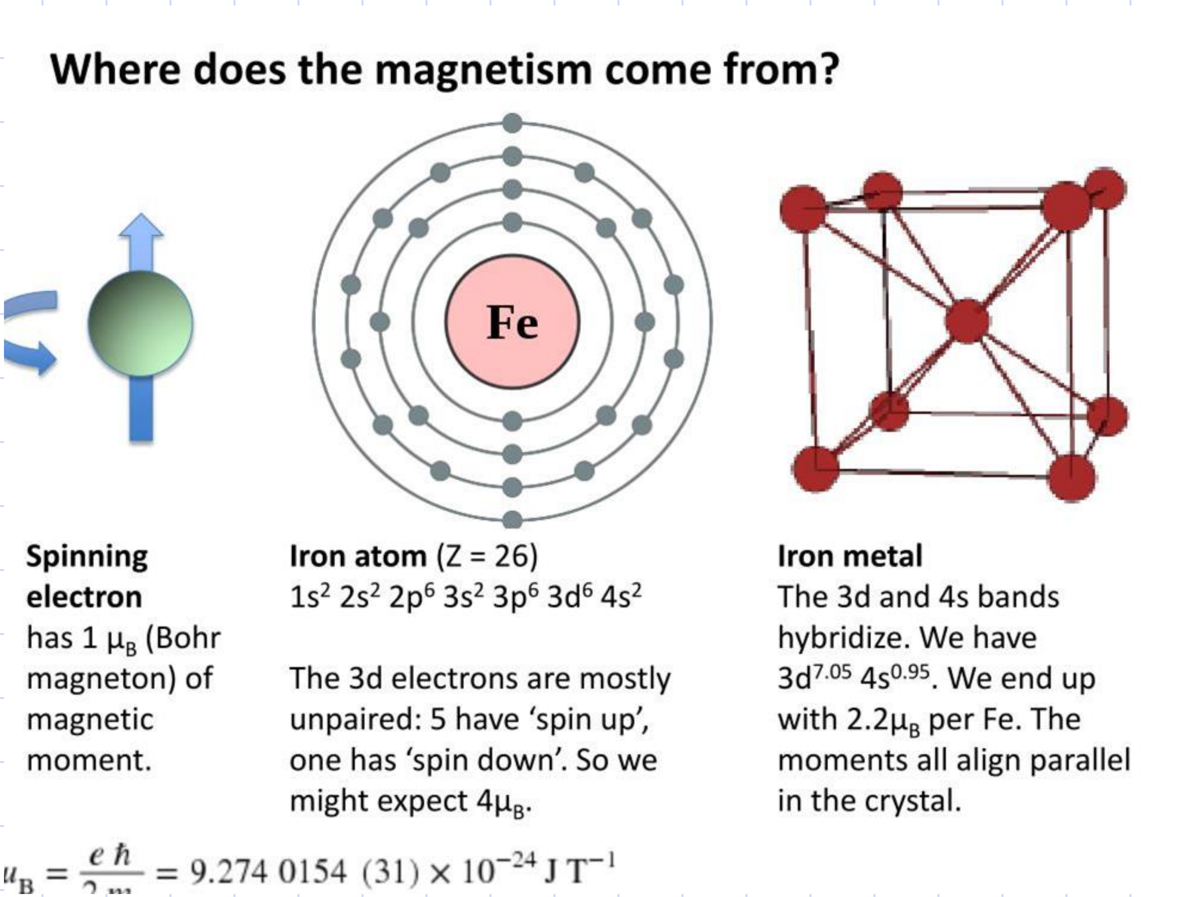
In the classical picture:

- electron orbits the nucleus
- acts like a current loop

Usually in most materials, the loops has random orientation - so the net effect is small

In ferromagnets, neighboring atoms have spins that are aligned - strong effects

Magnetic Materials



Ferromagnetic materials include

- iron
- cobalt
- neodymium
- ferric oxide
- nickel
- many others

Ross 2014

Magnetic Materials

In general, one can write:

$$M = \sum_i m_i$$

magnetization

Sum over all atoms

$$B = \mu_0 \cdot (H + M)$$ where,

B -flux due to all sources

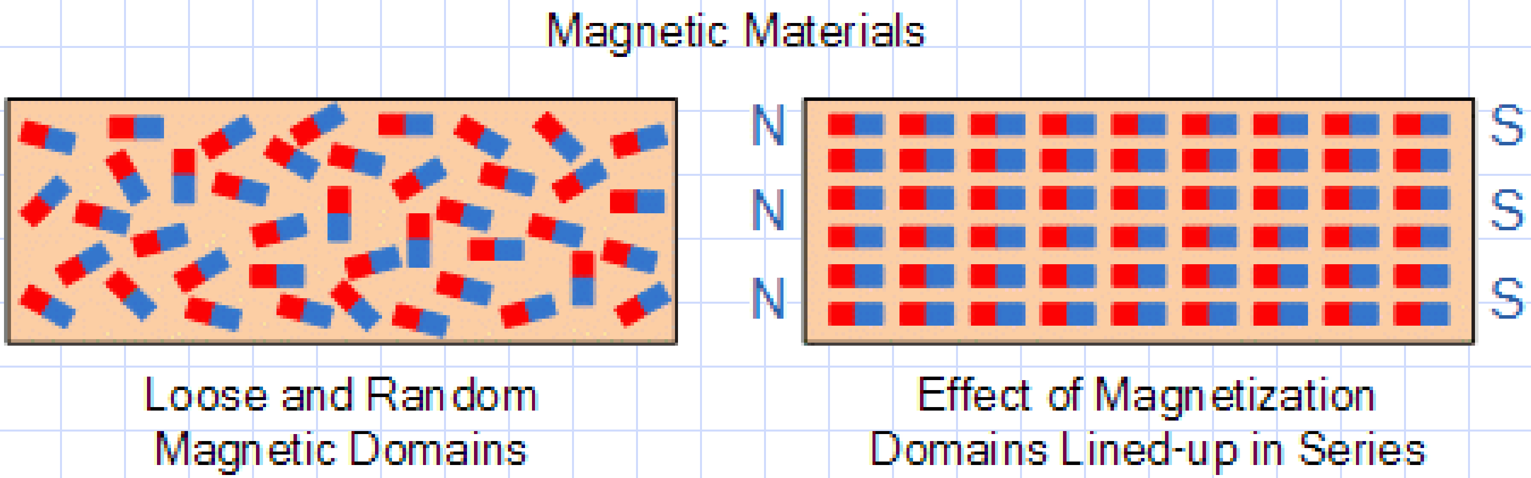
M -flux due to atomic sources

H -flux due to free current

(e.g. conduction current, e-beam)

Magnetic Materials

- Alignment of individual atomic/molecular magnetic moments leads to the evolution of a macroscopic magnetic field



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Magnetic Materials

Maxwell's equation: $\oint H \bullet dl = I_{net} = \int j \bullet ds$

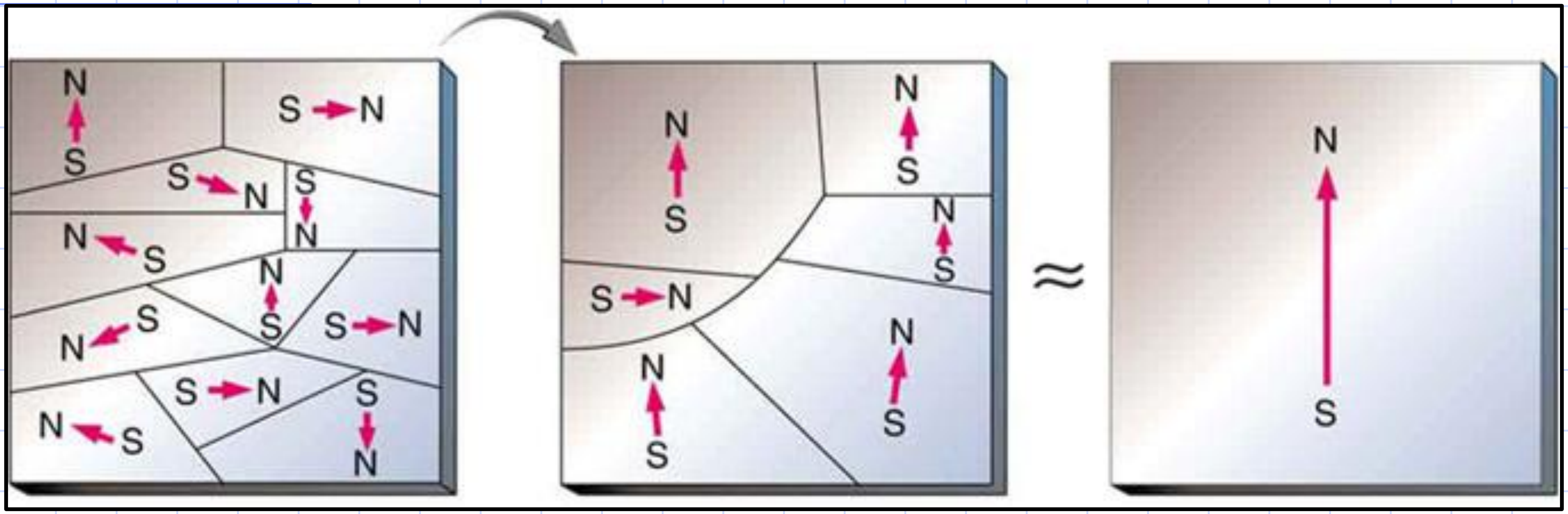
these are free-currents

- we can determine <u>H</u> without determining <u>M</u>

In most general form: $B = \mu_0 \cdot (H + M)$

We need M to determine <u>B</u>

Magnetic Materials



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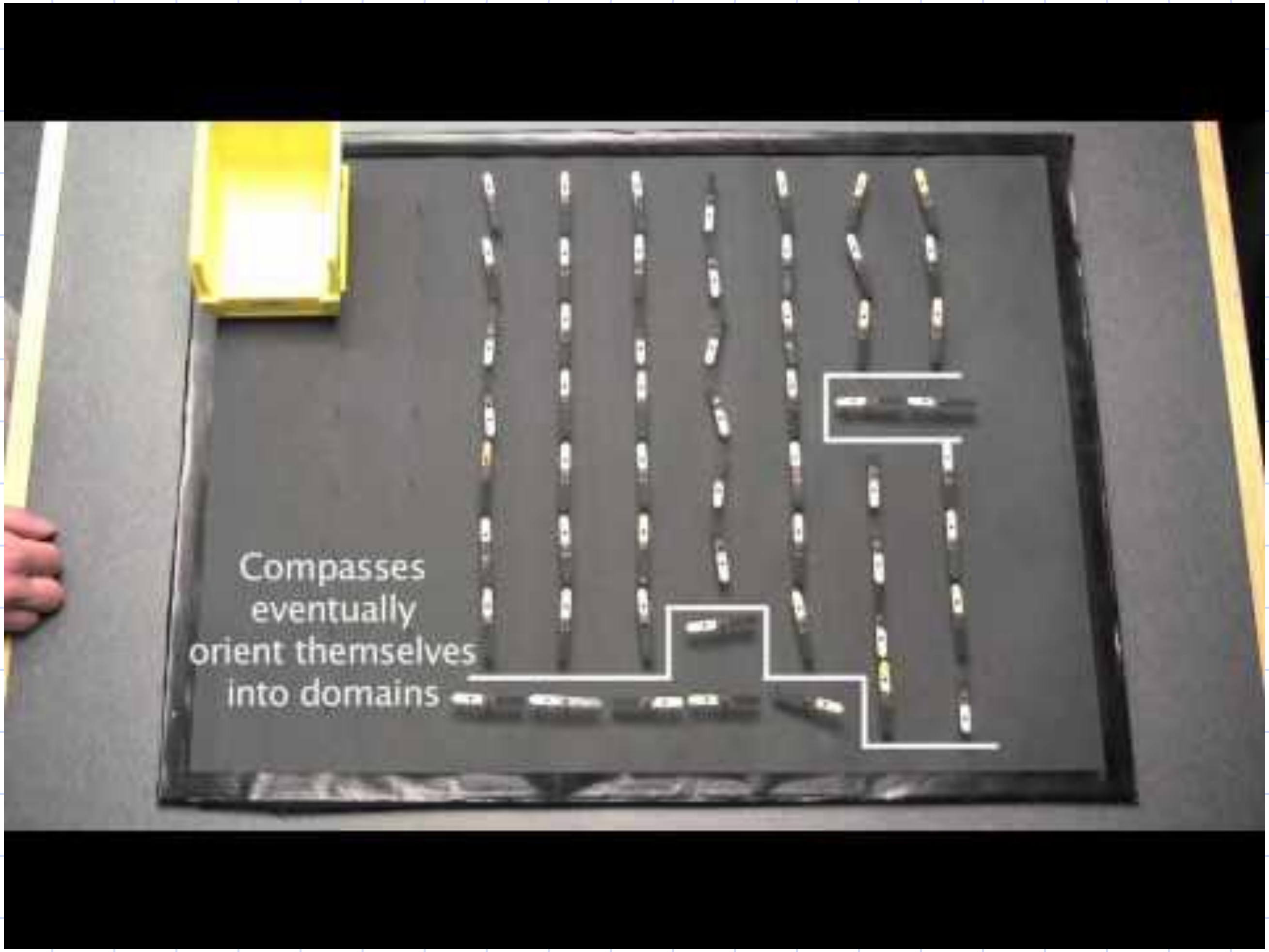
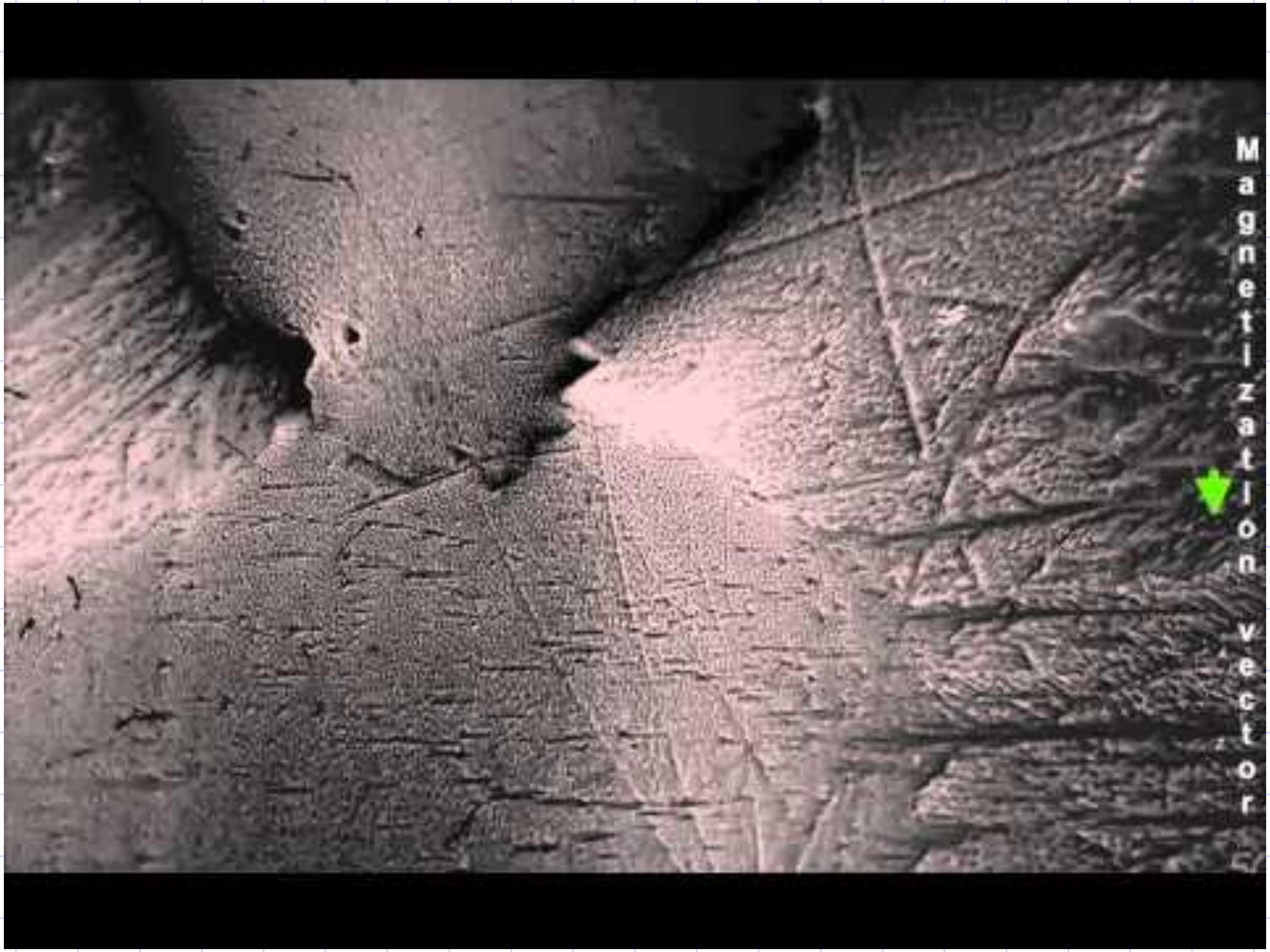
- Ferromagnetic materials tend to form into "magnetic domains" - subregions in which all atoms are aligned and have a net field
- As applied field gets stronger, these domains can merge and realign to produce a stronger magnetization field in the direction of the applied field

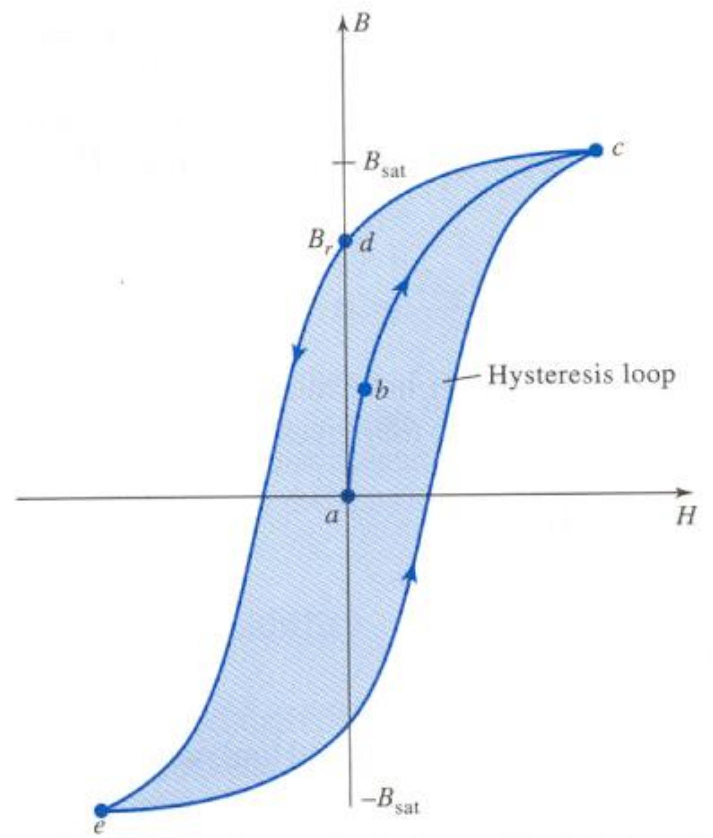
Magnetic Materials



Magnetic Materials



Magnetization Curves



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Typical B-H Curve for materials

Typical values

$1 < B_{sat} < 2$

Magnetization Curves

- "Hard" magnetic materials hold magnetization very well; they require more applied field to align and require more applied field in the opposite direction to de-align
- "Hard" magnetic materials also continue to remain aligned and produce a magnetic field once the external field is removed
- "Soft" magnetic materials are easier to magnetize but also do not hold magnetization as well once the external field is removed
- The energy required to realign magnetic materials manifests as an energy loss

Magnetization Curves

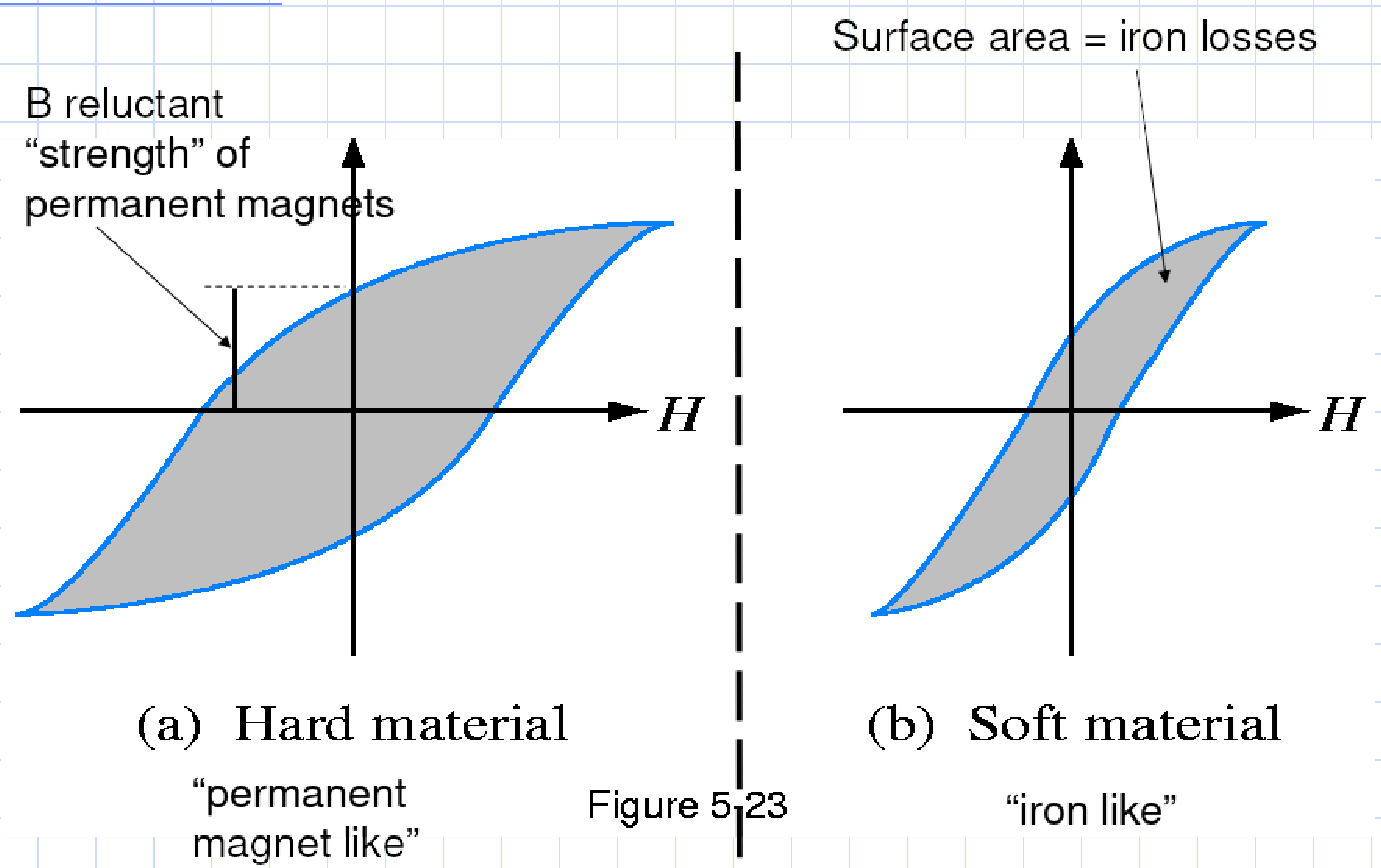
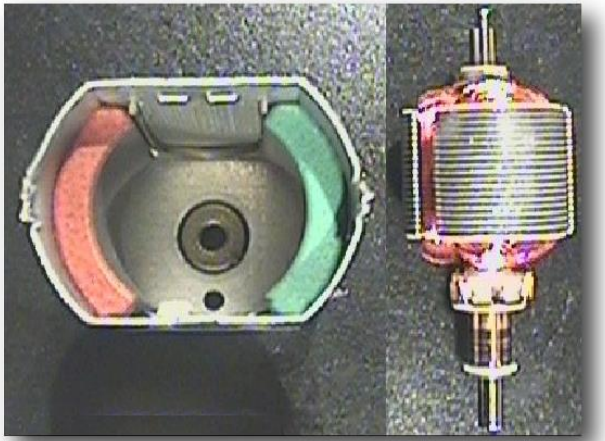


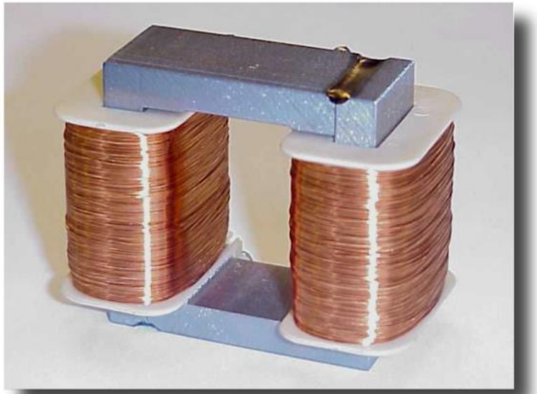
Figure 5.23

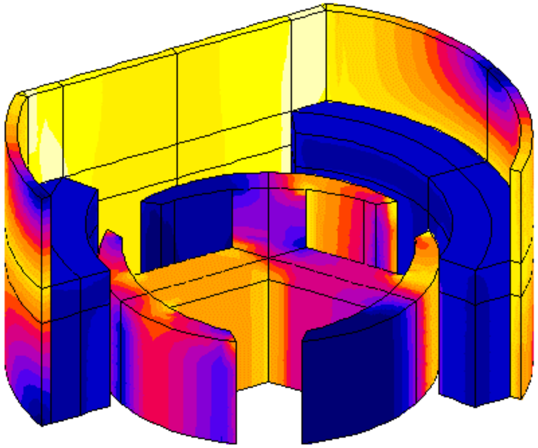
Magnetization Curves

Applications

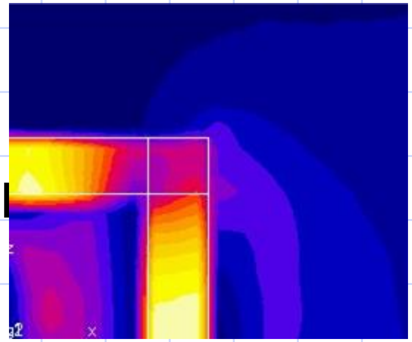


Small transformer





Small DC motor



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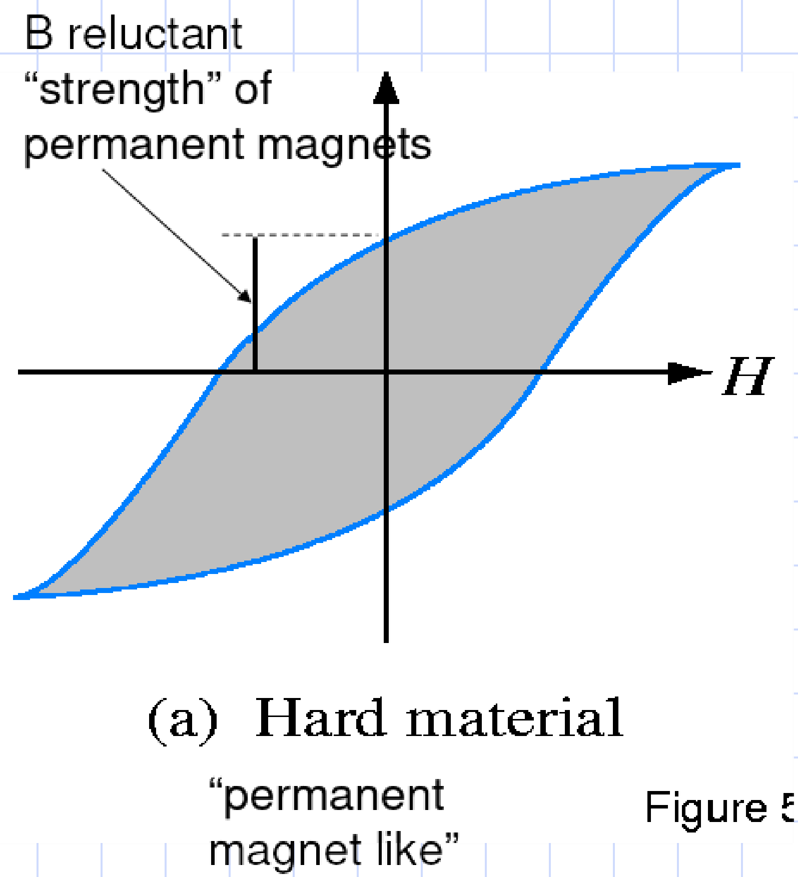
Magnetization Curves

Hysteresis Loss



- H-field has units of amperes per meter, B-field has units of teslas

$$T = \frac{V \cdot s}{m^2} = \frac{N}{A \cdot m} = \frac{J}{A \cdot m^2} = \frac{H \cdot A}{m^2} = \frac{Wb}{m^2} = \frac{kg}{C \cdot s} = \frac{N \cdot s}{C \cdot m} = \frac{kg}{A \cdot s^2}$$

- Product of H-field and B-field has units of energy density (J/m^3)
- Area inside the hysteresis curve also has units of J/m^3 and represents hysteresis losses

Magnetization Curves

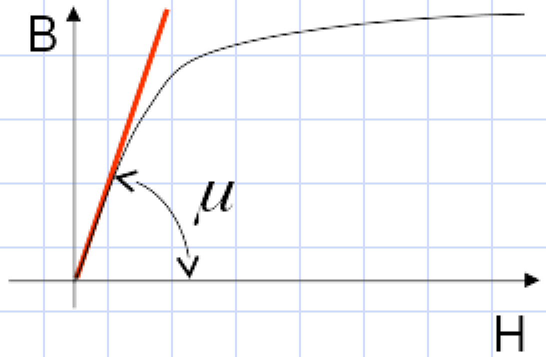
Applications

Hysteresis losses manifest as noise and heat as magnetized atoms realign

Magnetization Curves

Applications

Magnetization Curves



Linear Approximation

If, $M \propto H \quad \Rightarrow \boxed{B = \mu \cdot H = \mu_r \cdot \mu_0 \cdot H}$

$$\mu_0 = 4\pi \times 10^{-7}\ H/m$$

Values, $\mu_r = 1$ most materials (plastic, aluminum, copper)

$\mu_r \approx 10^3$ to 10^5 for iron

Recall :

In electrostatics, most materials have moderate effect $1 < \varepsilon < 10$