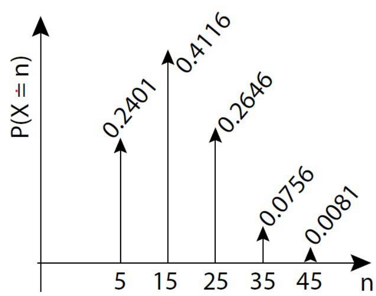
Rensselaer Polytechnic Institute
Department of Electrical, Computer, and Systems Engineering
ECSE 2500: Engineering Probability, Fall 2022
Homework #3 Solutions

1. (a) The table below gives the possible values of X and the corresponding probabilities. The PMF is also sketched below.

# members	# photographers X	probability
0	5	$\binom{4}{0}(0.3)^0(0.7)^4 = 0.2401$
1	15	$\binom{4}{1}(0.3)^1(0.7)^3 = 0.4116$
2	25	$\binom{4}{2}(0.3)^2(0.7)^2 = 0.2646$
3	35	$\binom{4}{3}(0.3)^3(0.7)^1 = 0.0756$
4	45	$\binom{4}{4}(0.3)^4(0.7)^0 = 0.0081$



(b) We can compute the expected value directly as

$$E(X) = (5)(0.2401) + (15)(0.4116) + (25)(0.2646) + (35)(0.0756) + (45)(0.0081) = 17$$

(c) We can compute the variance directly as

$$\begin{aligned}\text{Var}(X) &= (5-17)^2(0.2401) + (15-17)^2(0.4116) + (25-17)^2(0.2646) \\ &\quad + (35-17)^2(0.0756) + (45-17)^2(0.0081) \\ &= 84\end{aligned}$$

(d) X is linearly related to Y, a binomial random variable with $n = 4$ and $p = 0.3$, by the formula $X = 10Y + 5$.

(e) Since $E(X) = E(10Y + 5) = 10E(Y) + 5$, and $E(Y) = np = 4(0.3) = 1.2$, we get $E(X) = 17$, the same as in part (b). Similarly, $\text{Var}(X) = \text{Var}(10Y + 5) = 10^2\text{Var}(Y)$ and $\text{Var}(Y) = np(1-p) = 4(0.3)(0.7) = 0.84$, also giving $\text{Var}(X) = 84$.

2. (a) The problem is telling us that the conditional PMF of Y given {Marc} is uniform, and the conditional PMF of Y given {Stephen} is binomial.

   Therefore from the properties of these random variables, we know that $E(Y \mid \text{Marc}) = 20$ (i.e., the middle value of the distribution), and $E(Y \mid \text{Stephen}) = np = 30$. Then we can just apply the theorem on total probability to compute that

$$\begin{aligned} E(Y) &= E(Y \mid \text{Marc}) \cdot P(\text{Marc}) + E(Y \mid \text{Stephen}) \cdot P(\text{Stephen}) \\ &= (20)(0.4) + (30)(0.6) \\ &= 26 \end{aligned}$$

   (b) (10 points) Computing $E(Y^2)$ directly is a bit of a pain. However, we know that $\text{Var}(Y) = E(Y^2) - (E(Y))^2$, or $E(Y^2) = \text{Var}(Y) + (E(Y))^2$. From the table in the book, we know that $\text{Var}(Y \mid \text{Marc}) = \frac{21^2-1}{12} = \frac{110}{3}$ (i.e., $L = 21$ in the table), and $\text{Var}(Y \mid \text{Stephen}) = np(1-p) = \frac{15}{2}$. Since we computed the conditional expected values in part (a), we can compute that

$$\begin{aligned} E(Y^2 \mid \text{Marc}) &= \frac{110}{3} + (20)^2 = \frac{1310}{3} \\ E(Y^2 \mid \text{Stephen}) &= \frac{15}{2} + (30)^2 = \frac{1815}{2} \end{aligned}$$

   and thus

$$\begin{aligned} E(Y^2) &= E(Y^2 \mid \text{Marc}) \cdot P(\text{Marc}) + E(Y^2 \mid \text{Stephen}) \cdot P(\text{Stephen}) \\ &= \left(\frac{1310}{3}\right)(0.4) + \left(\frac{1815}{2}\right)(0.6) \\ &= \frac{4315}{6} \\ &= 719.17 \end{aligned}$$

3. (a) We need the first few values of the PMF: $p_X(1) = 0.3$, $p_X(2) = 0.21$, $p_X(3) = 0.147$, $p_X(4) = 0.1029$, and $p_X(5) = 0.0720$. The sum of these (i.e., $P(X \le 5)$) is 0.8319. The conditional PDF is obtained by renormalizing by this value over the 4 outcomes so the probabilities sum to 1:

$$\begin{aligned} p_X(1 \mid X \le 5) &= (0.3)/(0.8319) &&= 0.3606 \\ p_X(2 \mid X \le 5) &= (0.21)/0.8319) &&= 0.2524 \\ p_X(3 \mid X \le 5) &= (0.147)/(0.8319) &&= 0.1767 \\ p_X(4 \mid X \le 5) &= (0.1029)/(0.8319) &&= 0.1237 \\ p_X(5 \mid X \le 5) &= (0.0720)/(0.8319) &&= 0.0866 \end{aligned}$$

   (b) The conditional expected value is computed in the usual way from the conditional PMF:

$$E(X) = (1)(0.3606) + (2)(0.2524) + (3)(0.1767) + (4)(0.1237) + (5)(0.0866) = 2.32$$

   (c) The conditional PMF $p_X(x \mid X > 5)$ will have infinitely many values, but here we can leverage the **memoryless property** of the geometric random variable. That is,

$$p_X(x \mid X > 5) = p_X(x-5) = (0.3)(0.7)^{x-6} \quad x = 6, 7, \ldots, \infty$$

Put a different way, the probability of 6 sightings given that more than 5 have occurred is the same as the probability that there was 1 sighting in the first place. We can show this is true by computing $P(X > 5) = 1 - P(X \leq 5) = 0.1681$ from part (a). Then

$$P(X = 6 \mid X > 5) = \frac{P(X = 6)}{P(X > 5)} = \frac{(0.3)(0.7)^5}{0.1681} = \frac{0.0504}{0.1681} = 0.3 = P(X = 1)$$