ECSE 4500

Lec 9

April 10

Topic: Joint PDF/PMF/CDF of two RVs

Independence / Correlation of two RVs

□ Joint Gaussian Random Variables

Joint Gaussian RVs

$$\rightarrow f_{X,Y}(x,y) = \frac{1}{2\pi|\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}\left(\begin{bmatrix}x\\y\end{bmatrix}-\underline{\mu}\right)^T \Sigma^{-1}\left(\begin{bmatrix}x\\y\end{bmatrix}-\underline{\mu}\right)}$$

mean vector $\underline{\mu} = \begin{bmatrix}\mu_x\\ \mu_y\end{bmatrix} \in R^2$; $\Sigma = \begin{bmatrix}\sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2\end{bmatrix} \in R^{2\times 2}$

Covariance matrix

critical parameter $\rho \in [-1,1]$

captures correlation between X and Y.

Expand the joint PDF as

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]}$$

If the correlation constant $\rho = 0$,

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y} e^{-\frac{1}{2}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]}$$

Factorize

$$= \boxed{\frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}} \cdot \boxed{\frac{1}{\sqrt{2\pi}\,\sigma_Y} e^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}}}$$

PDF of single Gaussian RV (μ_X, σ_X^2)

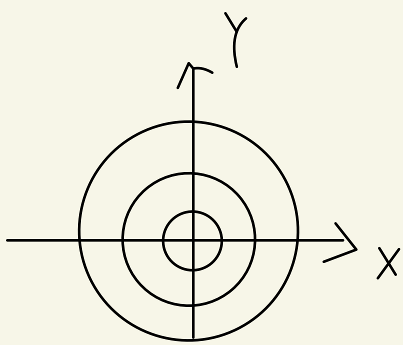
PDF of single Gaussian (μ_Y, σ_Y^2)

Further simplified case

1) $\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = 1$, $\rho = 0$

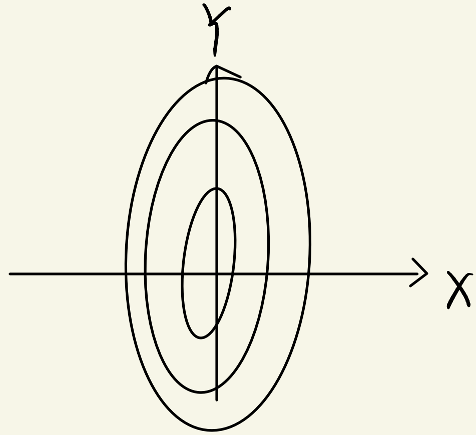
$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$$

Round circle



2) $\mu_X = \mu_Y = 0, \quad \sigma_X = 1, \quad \sigma_Y = 2, \quad \rho = 0$

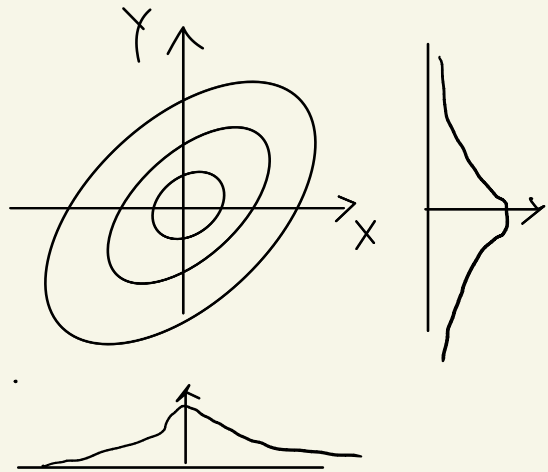
Ellipses



3) $\mu_X = \mu_Y = 0, \quad \sigma_X = \sigma_Y = 1, \quad \rho = \frac{1}{2}$

A rotated ellipse

We can obtain information of individual RVs X and Y by marginalizing joint PDF.



□ Independence of two RVs

Before we define the concept of independence for RVs, we first revisit the concept of independence of events.

Recall that we say two events A and B are independent if

$$P(A \cap B) = P(A) \cdot P(B)$$

Now, we say two random variables X and Y are independent if for any events A and B,

$$P(X \in A \text{ and } Y \in B) = P(X \in A) \cdot P(Y \in B)$$

As a consequence, we have

$$F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y), \quad \forall x, y$$

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y), \quad \forall x, y$$

Note: If two RVs are independent, their CDF and PDF can be factorized by the product of CDF/PDF of X and Y.

Example. $f_{X,Y}(x,y) = \begin{cases} 1, & \text{if } (x,y) \in [0,1] \\ 0, & \text{else} \end{cases}$

$$= \left(\begin{cases} 1, & \text{if } x \in [0,1] \\ 0, & \text{else} \end{cases} \right) \cdot \left(\begin{cases} 1, & \text{if } y \in [0,1] \\ 0, & \text{else} \end{cases} \right)$$

$$= f_X(x) \cdot f_Y(y)$$

Example of two RVs not independent.

Joint Gaussian with $(\rho \neq 0)$ $\quad f_{X,Y}(2,5) \neq f_X(2) \cdot f_Y(5)$

In summary, for two Gaussian RVs, they are independent if and only if constant in Covariance matrix $\rho = 0$. Otherwise, knowing X also tells some information about Y.

Once we know the independence which is a property of PDF/CDF of two RVs, next we will talk about the correlation of two RVs, which is a property regarding the joint expectation and expectation of individual RVs.

□ Expectation of two RVs

For one RV, we have

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x)\, dx$$

For two RVs, we extend this to

$$E[g(X,Y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y) \underbrace{\boxed{f_{X,Y}(x,y)}}_{\text{joint PDF}}\, dx\, dy$$

The nice thing about expectation is that we can simplify via linearity

$$E[aX+bY] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (ax+by) f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} ax\, f_{X,Y}(x,y)\,dx\,dy \; +$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} by\, f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{+\infty} ax \underbrace{\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy}_{f_X(x)}\,dx \; +$$

$$\int_{-\infty}^{+\infty} by \underbrace{\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx}_{f_Y(y)}\,dy$$

$$= \int_{-\infty}^{+\infty} ax\, f_X(x)\,dx + \int_{-\infty}^{+\infty} by\, f_Y(y)\,dy$$

$$= E[aX] + E[bY]$$

So we can decompose any sum on scalar multiple two RVs into this way.

Recall that for one RV, we have defined moments and central moments. They can be defined similarly for two RVs, given by

one RV, kth moment: $E[X^k] = \int_{-\infty}^{+\infty} x^k f_X(x)\,dx$

two RVs, (j,k)th moment: $E[X^j Y^k] = \iint x^j y^k f_{X,Y}(x,y)\,dx\,dy$

one RV, kth central moment:

$$E[(X-\mu)^k] = \int_{-\infty}^{+\infty} (x-\mu)^k f_X(x)\,dx$$

two RVs, (j,k)th central moments:

$$E[(X-\mu_X)^j (Y-\mu_Y)^k] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x-\mu_X)^j (y-\mu_Y)^k f_{X,Y}(x,y)\,dx\,dy$$

The most important relationship is $j=k=1$,

$\boxed{\text{Correlation}}$ of X and Y: $E[XY]$

$\boxed{\text{Covariance}}$ of X and Y: $\mathrm{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$

Note: $\mathrm{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$

$$= E[XY - X\mu_Y - Y\mu_X + \mu_X\mu_Y]$$

$$= E[XY] - \mu_Y \underbrace{E[X]}_{\mu_X} - \mu_X \underbrace{E[Y]}_{\mu_Y} + \mu_X\mu_Y$$

$$= E[XY] - \mu_X\mu_Y$$

Note: If X and Y are independent, then

$$E[XY] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f_X(x) f_Y(y)\, dx\, dy$$

$$= \left(\int_{-\infty}^{+\infty} x f_X(x)\, dx\right)\left(\int_{-\infty}^{+\infty} y f_Y(y)\, dy\right)$$

$$= E[X]\cdot E[Y] = \mu_X\mu_Y$$

$$\Rightarrow \mathrm{Cov}(X,Y) = E[XY] - \mu_X\mu_Y = 0$$

If X and Y are independent, Covariance $= 0$.

Definition of uncorrelated

$\Rightarrow$ If covariance $= 0$, we say X and Y are uncorrelated!

Student Q: If reverse is true?

A: No, if X and Y are uncorrelated, they are not necessarly independent!

Special case: $X = Y$,

$$\begin{aligned} Cov(X, X) &= E[X \cdot X] - \mu_X \mu_X \\ &= E[X^2] - (E[X])^2 \\ &= Var(X) \end{aligned}$$

Therefore, it makes sense that we call Cov as Covariance, and covariance of X with itself is variance.