# Fields and Waves I

## Lecture 13

Electric Boundary Conditions
Capacitance
Laplace and Poisson's Equations

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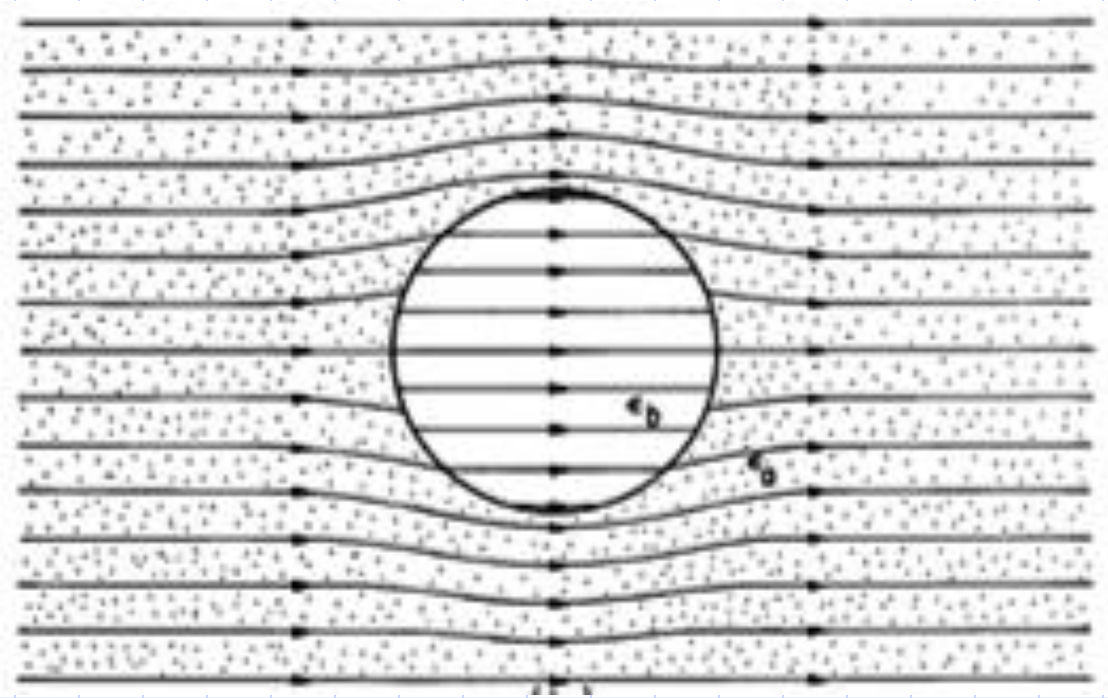
Materials from other sources are referenced where they are used.
Those listed as Ulaby are figures from Ulaby's textbook.

# Exam 1

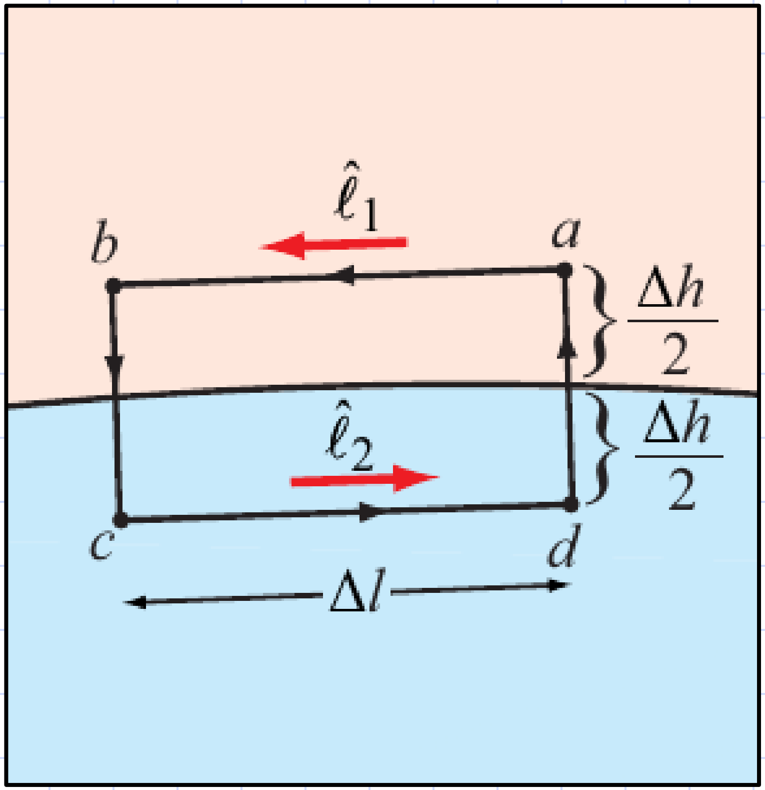
- Rework exams will be offered this week on the following skills:
  - Skill 1c (phasors)
  - Skill 1f (input impedance)
  - Skill 1b (wave properties)
  - Skill 2a (Smith chart)

# Boundary Conditions



Look at this picture again. How do electric fields behave at the boundary between two different dielectrics?

# Boundary Conditions



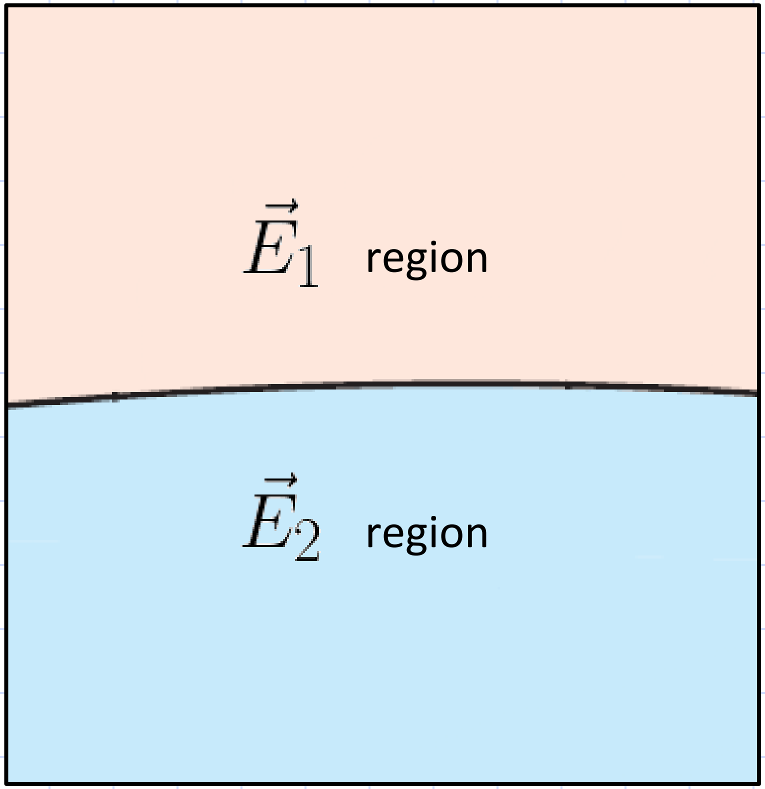

Ulaby

- We know that

$$\oint \vec{E} \cdot \vec{dl} = 0$$

- This will hold for any loop we choose, so we can choose $\Delta h \rightarrow 0$ so that the contribution of segments **bc** and **da** goes to zero.
- Note that we have chosen $\ell_1$ and $\ell_2$ tangent to the surface.

# Boundary Conditions

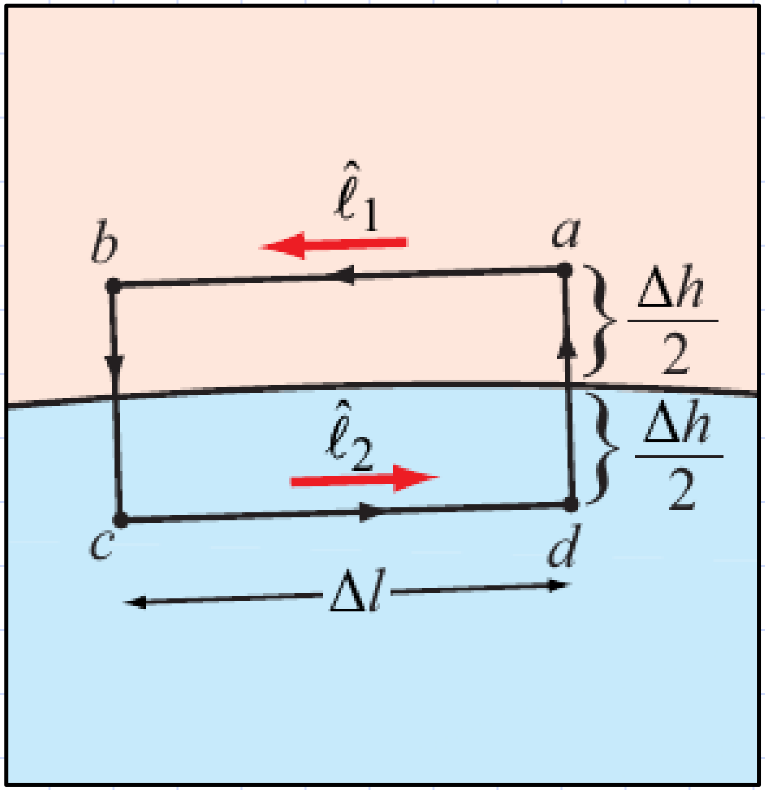



- Therefore we can write

$$\vec{E}_1 = \vec{E}_{1t} + \vec{E}_{1n}$$

$$\vec{E}_2 = \vec{E}_{2t} + \vec{E}_{2n}$$

# Boundary Conditions




Ulaby

- We chose $\ell_1$ and $\ell_2$ such that

$$\vec{E}_1 \cdot \hat{l}_1 = \vec{E}_{1t}$$

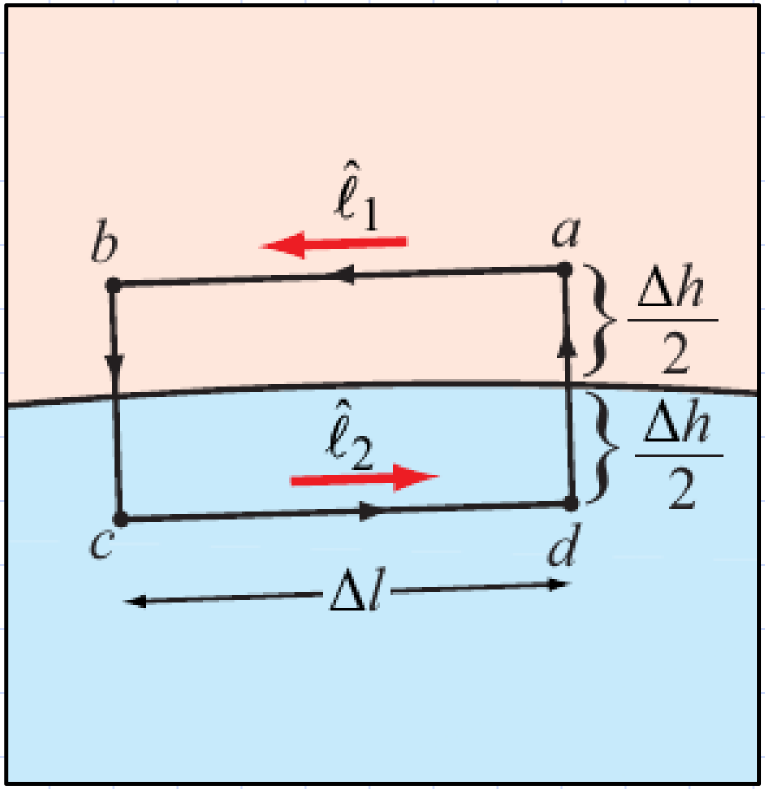
$$\vec{E}_2 \cdot \hat{l}_2 = \vec{E}_{2t}$$

- Now we simplify:

$$(\vec{E}_1 - \vec{E}_2) \cdot \hat{l}_1 = 0$$

$$\boxed{\vec{E}_{1t} = \vec{E}_{2t}}$$

# Boundary Conditions



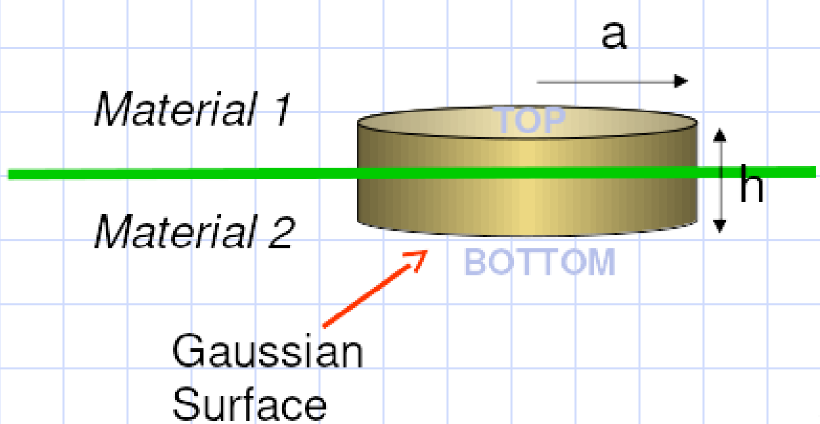

Ulaby

$$\vec{E}_{1t} = \vec{E}_{2t}$$

- So component of the E-field that is tangent to a media boundary is continuous across it.
- What about normal to the boundary?

# Boundary Conditions

NORMAL COMPONENT




$$\oint D \bullet ds = Q_{enclosed}$$

Take h << a **(a thin disc)**

$$Q_{enclosed} = \rho_s \cdot A$$

$$\oint D \bullet ds = \int_{TOP} D \bullet ds + \int_{BOTTOM} D \bullet ds$$

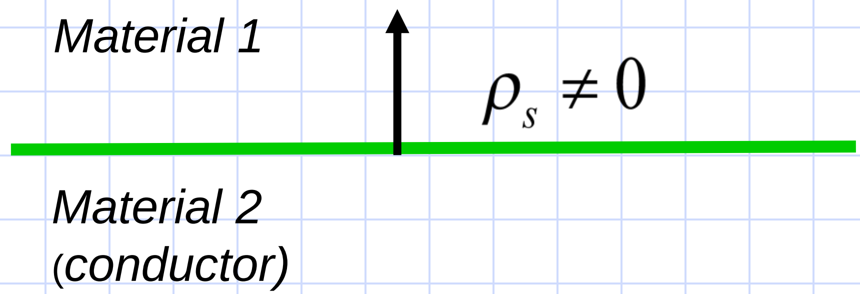
$$= (D_{1n} - D_{2n}) \cdot A$$

$$\therefore D_{1n} - D_{2n} = \rho_s$$

# Boundary Conditions

Case 1: REGION 2 is a CONDUCTOR, $D_2 = E_2 = 0$

$$\therefore D_{1n} = \rho_s$$

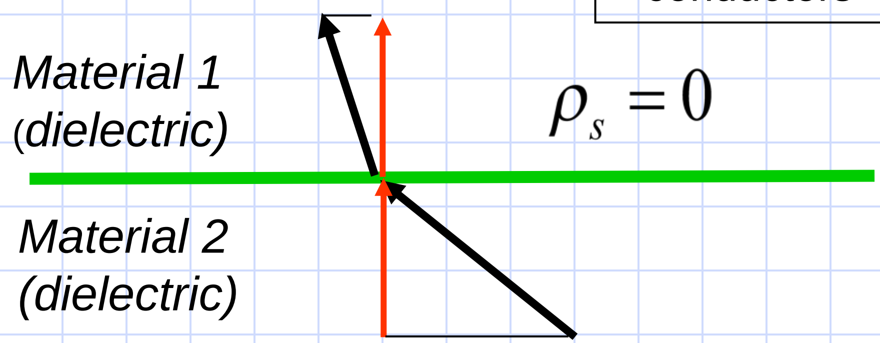



Case 2: REGIONS 1 & 2 are DIELECTRICS with $\rho_s = 0$

Can only really get $\rho_s$ with conductors

$$\therefore D_{1n} = D_{2n}$$

$$\therefore \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

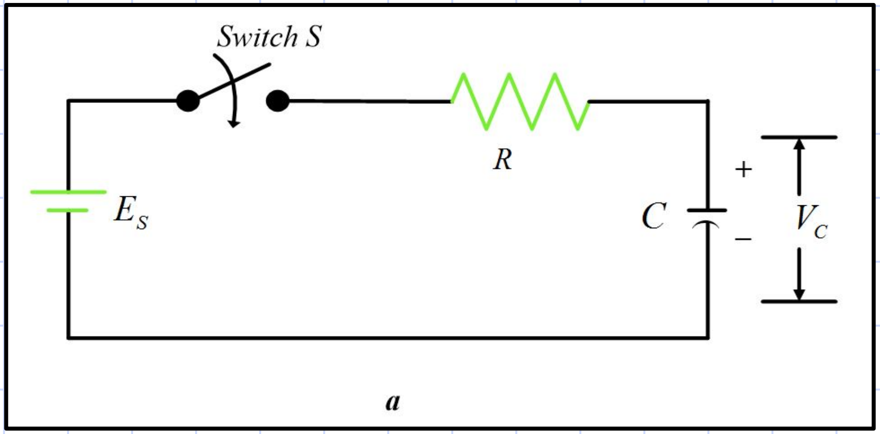


# Boundary Conditions

Do Lecture 13 Exercise 1 in groups of up to 4.

# Capacitance

- In earlier EE classes you did analysis on circuits containing capacitors, which have a property called capacitance. <u>What is capacitance, exactly?</u>



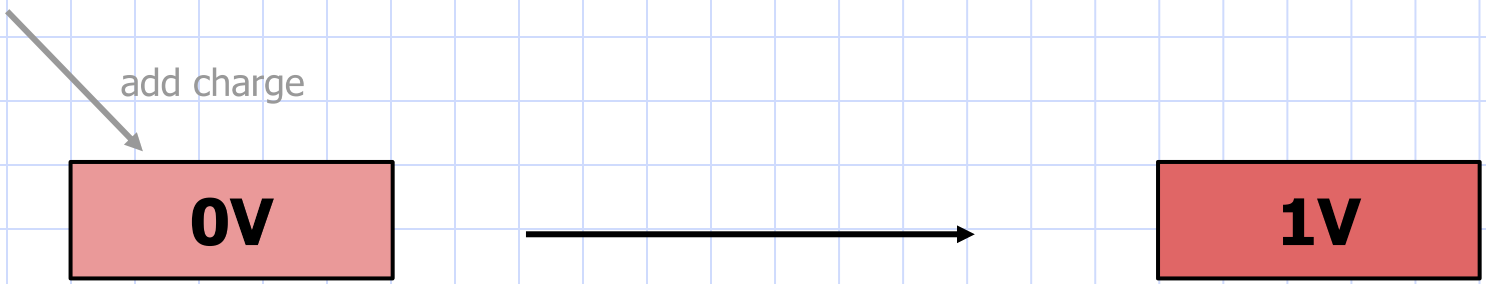

# Capacitance

- Fundamentally, capacitance describes the relationship between change in charge and change in voltage in an electrical system.
- There are two types of capacitance: self-capacitance and mutual capacitance.

# Capacitance

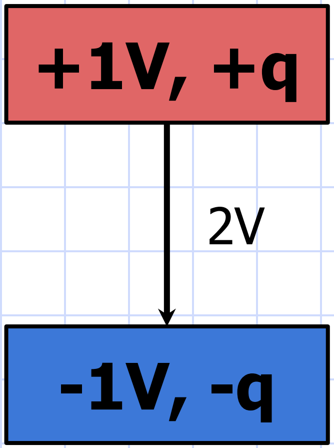
- Self-capacitance describes how much charge is required to increase the electric potential of some conductor by 1V
- In this case there is *no parallel plate*. We just need a reference point from which to define the voltage of the conductor.

$$C = \frac{q}{V} = \frac{dq/dt}{dV/dt} = \frac{i(t)}{dV/dt}$$



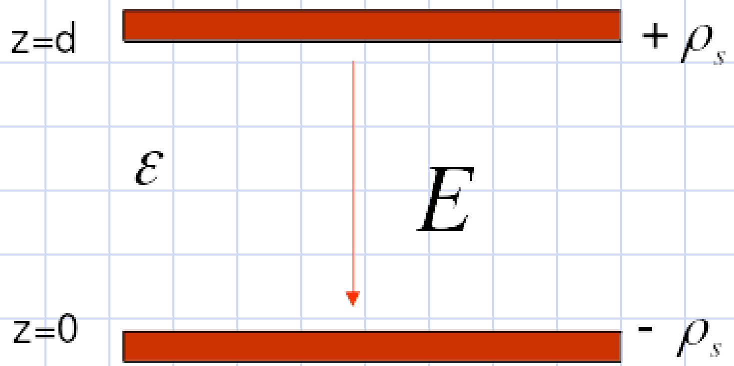
# Capacitance

- Mutual capacitance describes the charge-voltage relationship between two conductors. (This applies to the parallel plate capacitors that you are familiar with.)
- In this case, the charge in the capacitance equation is an equal and opposite charge on *both* plates, and the voltage is *between* them.




$$C = \frac{q}{V} = \frac{dq/dt}{dV/dt} = \frac{i(t)}{dV/dt}$$

# Capacitance




Use Gauss' Law,

$$\vec{E} = -\frac{\rho_s}{\varepsilon} \cdot \hat{a}_z$$

$$V_{top} - V_{bottom} = -\int_0^d \vec{E} \bullet d\vec{l} = \int_0^d \frac{\rho_s}{\varepsilon} \cdot dz$$

$$= \frac{\rho_s \cdot d}{\varepsilon}$$

Note: $V \propto \rho_s$

$V \propto Q$ ← very general result

$$\therefore C = \frac{Q}{V} = \frac{\rho_s \cdot A}{\rho_s \frac{d}{\varepsilon}} = \varepsilon \cdot \frac{A}{d}$$

C of Parallel Plate capacitor

# Capacitance

$$C = \varepsilon \cdot \frac{A}{d}$$

Parallel Plate Capacitance

To get large C

- increase A
- increase ε
- decrease d

This is how electrolytics increase C

Define: $$C = \frac{Q}{V}$$

charge on 1 conductor

ΔV between conductors

Note that: $$\frac{d}{dt}(Q = CV) \Rightarrow I = C\frac{dV}{dt}$$

# Capacitance

- The energy stored in capacitors is stored in the E-field

Define stored energy: $W_e = \frac{1}{2} \cdot CV^2$

Substitute values of C and V for parallel plate capacitor

$$W_e = \frac{1}{2} \cdot CV^2 = \frac{1}{2} \cdot \left( \varepsilon \frac{A}{d} \right) \cdot \left( \left| \vec{E} \right| \cdot d \right)^2 = \underbrace{\frac{1}{2} \cdot \varepsilon \left| \vec{E} \right|^2}_{\text{Energy Density}} \cdot \underbrace{Ad}_{\text{Volume}}$$

# Capacitance

In general we can write the total stored energy as:

$$W_e = \frac{1}{2}\int \epsilon|\vec{E}|^2 dv$$

or

$$W_e = \frac{1}{2}\int (\vec{D}\cdot\vec{E}) dv$$

Volume integral

# Capacitance

Use the Energy Formulation to compute C for the Parallel Plate Capacitor

We know that, $\vec{E} = -\frac{V_0}{d} \cdot \hat{a}_z$ (E in terms of V is needed)

Compute TOTAL ENERGY:

$$W_e = \frac{1}{2} \cdot \int \varepsilon \cdot \left( -\frac{V_0}{d} \right)^2 \cdot dv = \frac{1}{2} \cdot \varepsilon \cdot \left( \frac{V_0}{d} \right)^2 \cdot Ad$$

$$= \frac{1}{2} \cdot \varepsilon \frac{A}{d} \cdot V_0^2$$

$$C = \varepsilon \frac{A}{d}$$

# Capacitance

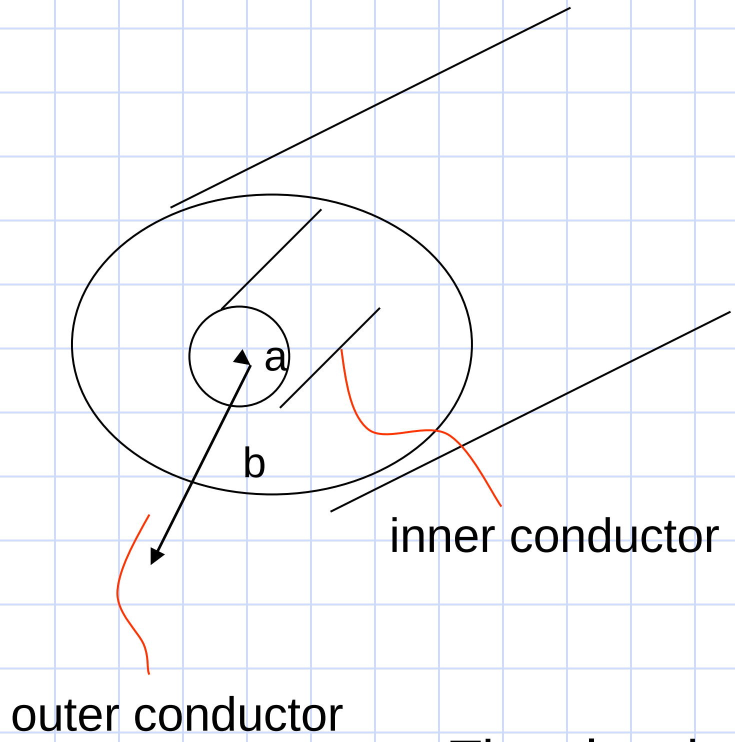
Consider a coaxial cable with inner radius, *a*, and outer radius, *b*, length *l*.

The outer conductor was grounded, and inner conductor had voltage $V = V_0$.

The relation between surface charge density on the inner conductor and $V_0$ is $\rho_{sa} = \varepsilon V_0/(a \ln(b/a))$.

- Find the capacitance using $C = Q/V$.
- Find the stored energy in the system by integrating the energy density over the system volume.
- Find the capacitance using the stored energy from part b.

# Capacitance




In a previous class, for coaxial cable:

$$V_{ab} = \frac{\rho_{sa} \cdot a}{\varepsilon} \cdot \ln \frac{b}{a}$$

The electric field is given by E = V0 / (r ln(b /a )) ar.

# Capacitance

Surface Charge Density

$$C = \frac{Q}{V} = \frac{2\pi a l \rho_{sa}}{V} = \frac{2\pi a l}{V_0} \frac{\varepsilon V_0}{a \ln \frac{b}{a}} = \boxed{\frac{2\pi \varepsilon l}{\ln b/a}}$$

Note the length in this expression. The practical result is for capacitance per unit length, for which we drop the length.

# Capacitance

$$W_e = \int \frac{1}{2}\varepsilon E^2 dv = \int_0^{\ell}\int_0^{2\pi}\int_a^b \frac{1}{2}\varepsilon \frac{V_0^2}{r^2(\ln\frac{b}{a})^2}\, r\,dr\,d\phi\,dz$$

$$= \frac{1}{2}\varepsilon \frac{V_0^2}{(\ln b/a)^2}\int_0^{\ell}\int_0^{2\pi}\int_a^b \frac{dr}{r}\,d\phi\,dz = \frac{1}{2}\varepsilon\frac{V_0^2}{(\ln b/a)^2}\cdot \ln\frac{b}{a}\,2\pi\ell$$

$$= \boxed{\frac{\pi\varepsilon\ell V_0^2}{\ln b/a}}$$

$$\frac{1}{2}CV^2 = W_e \Rightarrow \quad C = \frac{2W_e}{V^2} = \frac{2}{V_0^2}\frac{\pi\varepsilon\ell V_0^2}{\ln b/a} = \boxed{\frac{2\pi\varepsilon\ell}{\ln b/a}}$$

Same Result

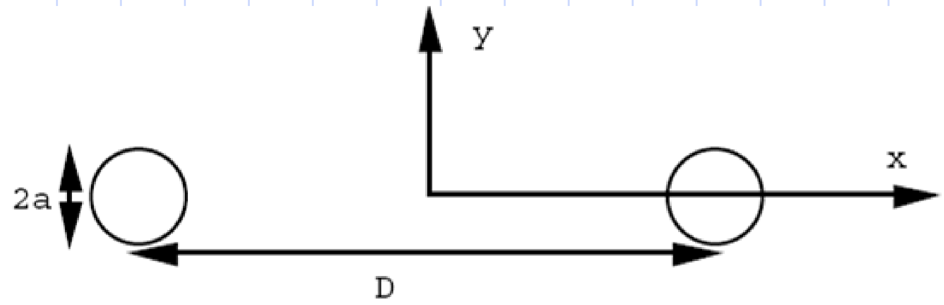
# Capacitance

The objective of this problem is to determine the capacitance between 2 conducting wires.

We will assume there are equal magnitude, opposite sign, uniform surface charge densities on the two wires.

What is the voltage difference between the wires?

What is the capacitance ?



# Capacitance

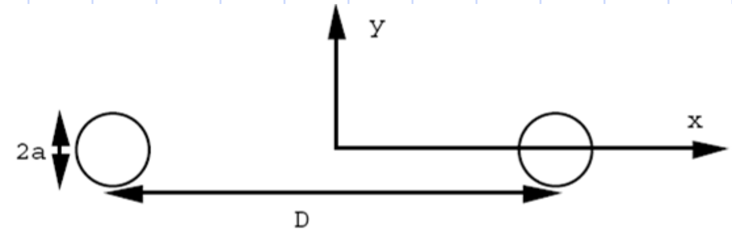
- Electric Field for a Single Wire Located at $x=x_o$ and $y=y_o$

$$E_R(R) = \frac{\rho_s a}{\varepsilon R}$$

$$R = \sqrt{\left(x - x_o\right)^2 + \left(y - y_o\right)^2}$$

# Capacitance

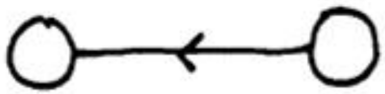
- This yields an equation for the electric field on the x-axis between the two wires with same charge (the wires are parallel to the z-axis):




$$\vec{E} = \frac{\rho_s a}{\epsilon}\left(\frac{1}{x+\frac{D}{2}} + \frac{1}{\frac{D}{2}-x}\right)\hat{a}_x$$

$$|x| < \left(\frac{D}{2} - a\right)$$ and y=0

# Capacitance



$x = \frac{D}{2} - a \rightarrow -\frac{D}{2} + a$

$$V_{LR} = -\int_{D/2-a}^{-D/2+a} \frac{\rho_s a}{\varepsilon_0} \left\{ \frac{1}{x+\frac{D}{2}} + \frac{1}{\frac{D}{2}-x} \right\} dx = -\frac{\rho_s a}{\varepsilon_0} \left\{ \ln\left(x+\frac{D}{2}\right) - \ln\left(\frac{D}{2}-x\right) \right]_{D/2-a}^{-D/2+a}$$

$$= -\frac{\rho_s a}{\varepsilon} \left\{ \ln a - \ln(D-a) - \left(\ln(D-a) - \ln a\right) \right\} = \boxed{\frac{2\rho_s a}{\varepsilon} \ln \frac{D-a}{a}}$$

$$C = \frac{Q}{V} = \frac{\cancel{\rho_s}\, 2\pi \cancel{a}\, l}{\frac{2\cancel{\rho_s}\, \cancel{a}}{\varepsilon} \ln\left(\frac{D-a}{a}\right)} = \boxed{\frac{\pi \varepsilon l}{\ln\left(\frac{D}{a}-1\right)}}$$

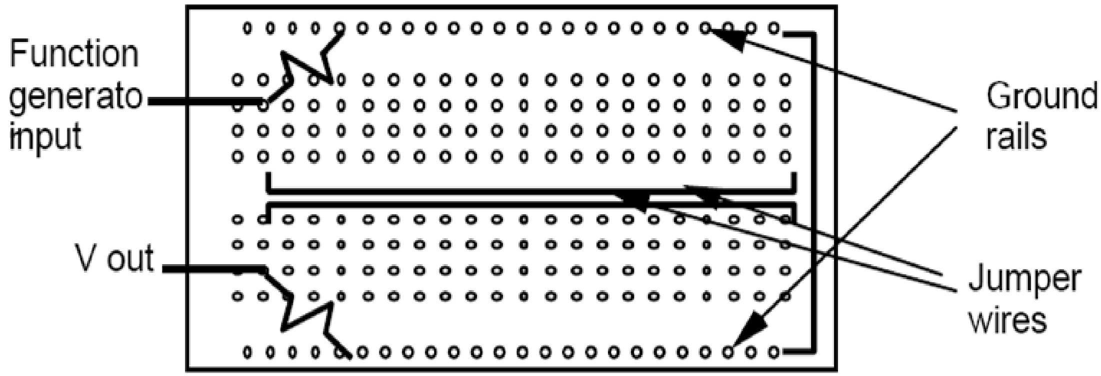
$|Q|$ on 1 conductor

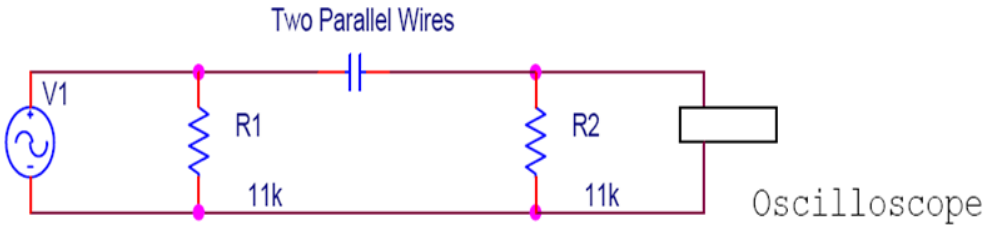
V between conductors

$$\boxed{C/l = \frac{\pi \varepsilon}{\ln\left(\frac{D}{a}-1\right)}}$$

# Capacitance

- Any 2 conductors have capacitance (such as wires on a breadboard)
- Estimate the capacitance of the setup below



# Capacitance

Dimensions for experiment (yours may differ)


$a$ = wire radius

$b$ = wire + insulation radius = $a + \Delta$

$\ell = 0.12$

guess $\varepsilon_r = 2$

$D = 2b = 2a + 2\Delta \qquad \frac{D}{a} - 1 = 2 + 2\frac{\Delta}{a} - 1 = 1 + 2\frac{\Delta}{a}$

approx. $\frac{\Delta}{a} = 0.5 \Rightarrow \frac{D}{a} - 1 = 2$

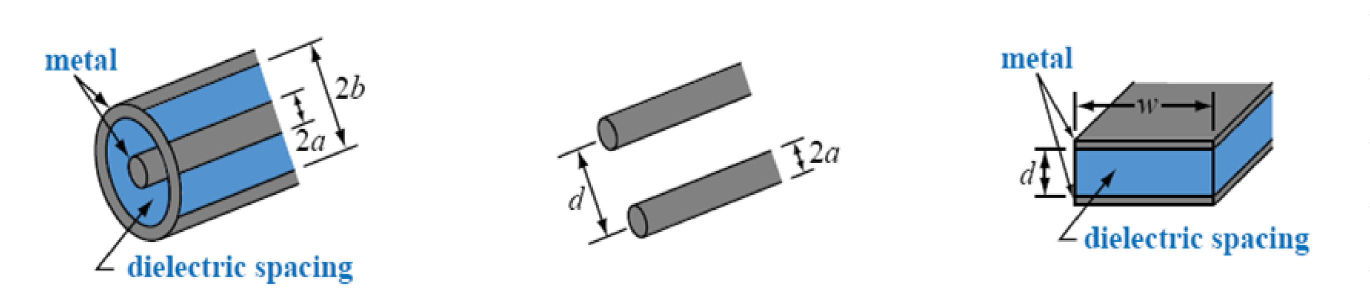
$$C = \frac{\pi \varepsilon \ell}{\ln\left(\frac{D}{a} - 1\right)} = \frac{\pi\, 2\varepsilon_0\, (0.12)}{\ln 2} = \boxed{9.6\ \text{pF}}$$

FORMULA ABOVE IS FOR $D/a \gg 1$

MORE ACCURATE VERSION $C = \dfrac{\pi \varepsilon \ell}{\ln\left\{\frac{D}{2a} + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right\}} = \boxed{6.9\ \text{pF}}$

# Capacitance

From Lecture 5




- Capacitance per unit length: *c F/m*

$$\frac{2\pi\varepsilon}{\ln\frac{b}{a}}$$

$$\frac{\pi\varepsilon}{\ln\left[\frac{d}{2a}+\sqrt{\left(\frac{d}{2a}\right)^2-1}\right]}$$

$$\approx\frac{\pi\varepsilon}{\ln\left[\frac{d}{a}\right]} \quad \text{for } d >> 2a$$

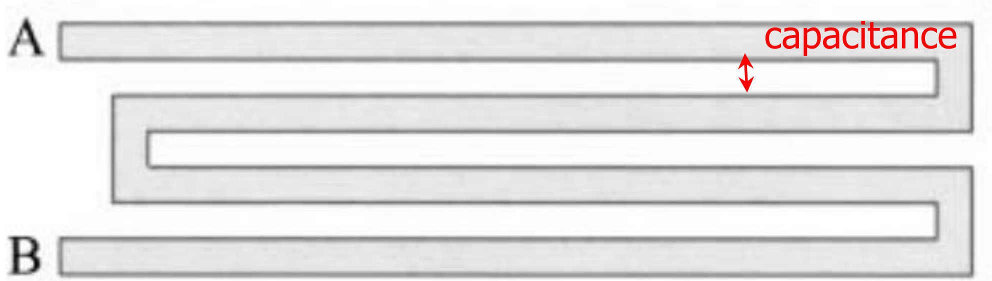
$$\frac{\varepsilon w}{d}$$

# Capacitance

Do Lecture 13, Exercise 2 in groups of up to 4.

# Capacitance

- In Ulaby, capacitance is generally described between two isolated conductors separated by a dielectric. But it also occurs between different parts of a single conductor.
- In circuit design, this unwanted effect is called "parasitic capacitance."
- Circuit designers choose wire geometries to reduce this effect (and other effects)




(a) A serpentine pattern

bjpcjp.github.io

# Capacitance

From Lecture 5

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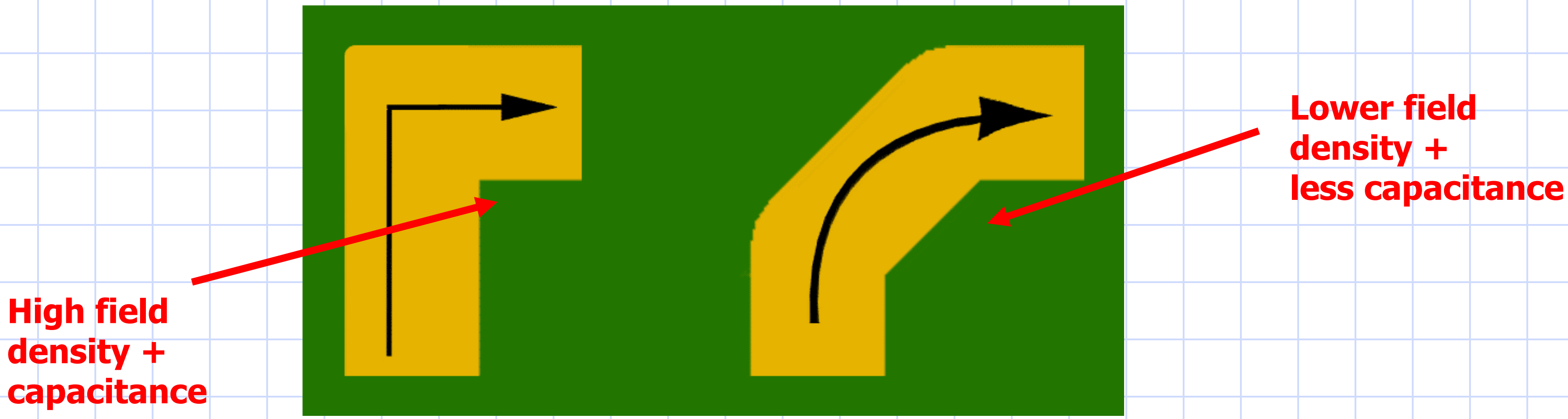
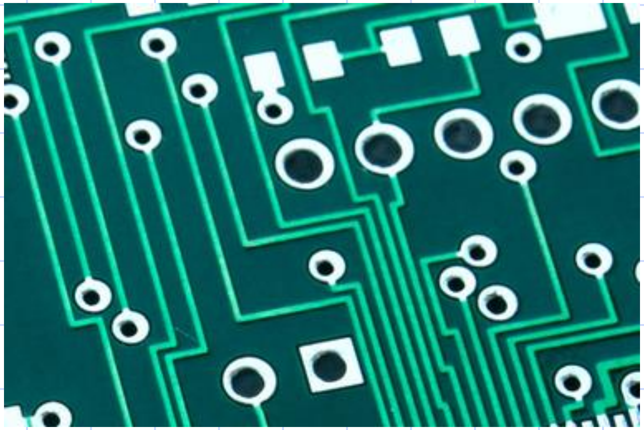


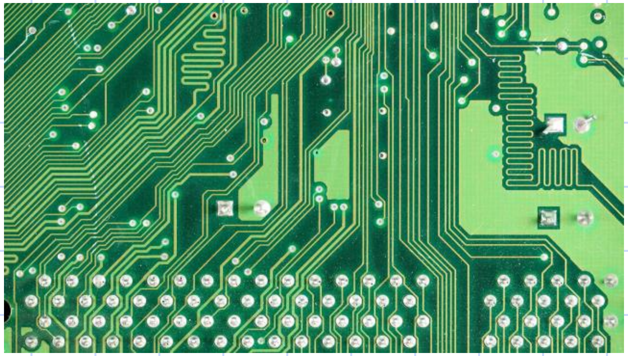

electronics.stackexchange.com

# Capacitance

From Lecture 5

- This has inspired changes in circuit design, such as the 45-degree angles in the PCB at right.
- This effect can be relevant for precise timing / digital circuit applications but is often overestimated





Hackaday

# Laplace and Poisson Equations

- Integral Form

$$\oint \vec{D} \cdot \vec{ds} = \int \rho \cdot dv$$

$$\oint \vec{E} \cdot \vec{dl} = 0$$

- Differential Form

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \times \vec{E} = 0$$

What if we try to rewrite Maxwell's Equations in terms of voltage?

$$\vec{E} = -\nabla V$$

# Laplace and Poisson Equations

- First, the curl equation

$$-\nabla V$$

$$\vec{E} = -\nabla V \Rightarrow \nabla \times \vec{E} = 0$$

since $\nabla \times \left(\nabla f\right) = 0$

- Next, the divergence equation

$$\nabla \cdot \vec{D} = \nabla \cdot (\epsilon \vec{E}) = -\epsilon(\nabla \cdot (\nabla V)) = \rho$$

$$\Rightarrow \nabla^2 V = -\frac{\rho}{\varepsilon} \quad \& \quad \nabla^2 V = 0$$

Laplacian of V = divergence of gradient of V

# Laplace and Poisson Equations

- Laplace's Equation: $$\nabla^2 V = 0$$

- Poisson's Equation $$\nabla^2 V = -\frac{\rho}{\varepsilon}$$

$$\nabla^2 = \begin{bmatrix} \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{bmatrix} \bullet \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

*(in cartesian coordinates)*

# Laplace and Poisson Equations

- Cylindrical Laplacian Operator:

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial V}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

- Spherical Laplacian Operator:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}(sin\theta\frac{\partial V}{\partial \theta}) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

# Laplace and Poisson Equations

- Laplace's Equation and Poisson's Equation are general mathematical expressions that allow us to solve scalar fields if we know the boundary conditions. They are used for solving for:
  - Voltage
  - Heat
  - Gravity
  - Aspects of fluid flow
  - Various abstract mathematical fields
  - etc.

# Laplace and Poisson Equations

## Boundary Conditions

- In General

$$D_{n1} - D_{n2} = \rho_s \qquad E_{t1} = E_{t2}$$

- Dielectric-Dielectric

$$D_{n1} = D_{n2} \qquad E_{t1} = E_{t2}$$

- Conductor-Dielectric

$$D_{n1} = \rho_s \qquad E_{t1} = 0$$

# Laplace and Poisson Equations

## Boundary Conditions

- Dielectric-Dielectric

$$D_{n1} = D_{n2} \qquad E_{t1} = E_{t2}$$

- Writing in terms of voltage:

$$\varepsilon_1 \frac{\partial V_1}{\partial n} = \varepsilon_2 \frac{\partial V_2}{\partial n} \qquad V_1 = V_2$$

here, n represents direction of boundary normal

Boundary Conditions

- Dielectric-Dielectric

$$D_{n1} = D_{n2} \qquad E_{t1} = E_{t2}$$

- Writing in terms of voltage:

$$\varepsilon_1 \frac{\partial V_1}{\partial n} = \varepsilon_2 \frac{\partial V_2}{\partial n} \qquad V_1 = V_2$$

this limit ensures voltage continuity

# Laplace and Poisson Equations

## Boundary Conditions

- Conductor-Dielectric

$$D_{n1} = \rho_s \qquad E_{t1} = 0$$

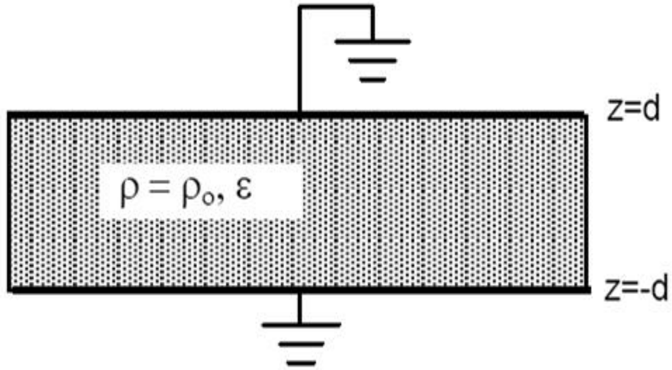
$$\varepsilon_1 \frac{\partial V_1}{\partial n} = \rho_s \qquad V_1 = const$$

# Laplace and Poisson Equations

- Coulomb’s Law is already a solution
- All other voltage expressions can be checked with one of these equations
- This is the most common way of finding electric fields

# Laplace and Poisson Equations

A charged region of a semiconductor is sandwiched between two grounded conductors as shown below.




Solve for V(z) directly using Poisson's Equation

Find **E** and **D**

Find the charge density on the conductors

# Laplace and Poisson Equations

a. $\nabla^2 V = -\rho/\varepsilon \Rightarrow$ ~~sk~~ $V = V(z)$, $\therefore \nabla^2 V = \frac{d^2 V}{dz^2}$

$\frac{dV}{dz} = -\frac{\rho z}{\varepsilon} + C_1 \Rightarrow \quad \therefore \; -\frac{\rho z^2}{2\varepsilon} + C_1 z + C_2$

$V(d) = 0 = -\frac{\rho d^2}{2\varepsilon} + C_1 d + C_2$

$V(-d) = 0 = -\frac{\rho d^2}{2\varepsilon} - C_1 d + C_2$

Add eq. $-\frac{\rho d^2}{\varepsilon} + 2C_2 = 0$

$\hookrightarrow C_2 = \frac{\rho d^2}{2\varepsilon}$

subtract eq. $2C_1 d = 0 \Rightarrow C_1 = 0$

$V = -\frac{\rho z^2}{2\varepsilon} + \frac{\rho d^2}{2\varepsilon} = \boxed{\frac{\rho}{2\varepsilon}(d^2 - z^2)}$

b. $\vec{E} = -\nabla V = -\frac{\partial V}{\partial z}\hat{a}_z = -\frac{\rho}{2\varepsilon}(-2z)\hat{a}_z = \boxed{\frac{\rho z}{\varepsilon}\hat{a}_z}$

$\vec{D} = \varepsilon\vec{E} = \boxed{\rho z\,\hat{a}_z}$

c. Boundary condition $D_n = \rho_s$

if $\rho_s > 0$ D points into surface at both $\pm d \Rightarrow \therefore \rho_s < 0$

$\boxed{\rho_s = -\rho d}$ on both

# Laplace and Poisson Equations

A coaxial cable has an inner conductor (at r = *a* ) held at voltage $V_0$ and an outer conductor (at r = *b* ) that is grounded. There is no charge other than the surface charge on the conductors.

Solve for V(r) directly using Laplace's Equation

Solve for **E** and **D**

What is the charge density on the two conductors?

What is the capacitance per unit length?

# Laplace and Poisson Equations

a. $\nabla^2 V = 0 \qquad V = V(r) \Rightarrow \therefore \nabla^2 V = \frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0$

$\frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0 \Rightarrow r\frac{dV}{dr} = c_1 \ ; \quad \frac{dV}{dr} = \frac{c_1}{r} \ ; \quad \underline{V = c_1 \ln r + c_2}$

$V(b) = 0 = c_1 \ln b + c_2 \Rightarrow c_2 = -c_1 \ln b \Rightarrow V = c_1 \ln\frac{r}{b}$

$V(a) = V_0 = c_1 \ln\frac{a}{b} \Rightarrow c_1 = \frac{V_0}{\ln a/b} \Rightarrow V = \frac{V_0}{\ln\frac{a}{b}}\ln\frac{r}{b} = \boxed{\frac{V_0}{\ln\frac{b}{a}}\ln\frac{b}{r}}$

# Laplace and Poisson Equations

b. $\vec{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{a}_r = \frac{-V_o}{\ln b/a}\frac{1}{b/r}\left(-\frac{b}{r^2}\right)\hat{a}_r = \boxed{\frac{V_o}{r\ln b/a}\hat{a}_r}$

$\vec{D} = \varepsilon\vec{E} = \boxed{\frac{\varepsilon V_o}{r\ln b/a}\hat{a}_r}$

c. Boundary conditions $D_n = \rho_s$

if $V_o > 0$ $\vec{E}$ points from $a \to b$

$\therefore \rho_{sa} > 0 \quad \rho_{sb} < 0$

$\boxed{\rho_{sa} = \frac{\varepsilon V_o}{a\ln b/a} \qquad \rho_{sb} = \frac{-\varepsilon V_o}{b\ln b/a}}$

d. $C = \frac{Q}{V} = \frac{\rho_{sa}\,2\pi a l}{V_o} = \frac{\varepsilon \cancel{V_o}}{\cancel{a}\ln b/a}\frac{2\pi \cancel{a} l}{\cancel{V_o}} = \frac{2\pi\varepsilon l}{\ln b/a}\ ; \boxed{\frac{C}{l} = \frac{2\pi\varepsilon}{\ln b/a}}$

# Laplace and Poisson Equations

- ...but as engineers, you all know that integrals can be very difficult to evaluate for all but very simple geometries.
- So how do we solve for V(r) when the geometry is more complex?
- We rely on numerical methods
  - Finite Difference
  - Finite Elements
  - Method of Moments
  - Etc.