# Fields and Waves I

## Lecture 12

Electric Materials and Interface Conditions

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Materials from other sources are referenced where they are used.
Those listed as Ulaby are figures from Ulaby's textbook.

# Gauss's Law

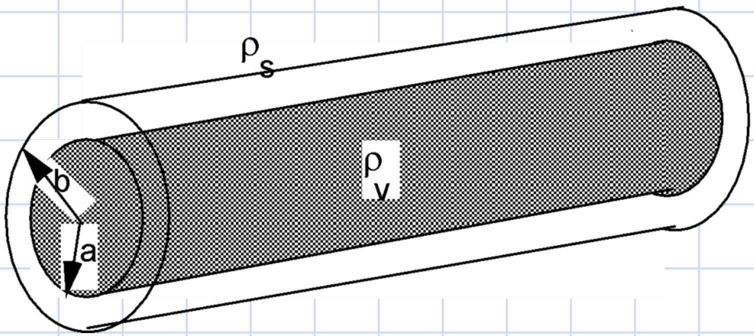
- Recognize the coordinate system.
- Using symmetry, determine which components of the field exist.
- Identify a Gaussian surface for which the sides are either parallel to or perpendicular to the field components. This surface is arbitrary in size.
- Determine the total charge within that surface. The charges can be distributed on lines, surfaces or in volumes.

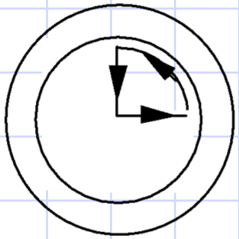
# Gauss's Law

**Full Gauss' Law Solution**

A charge distribution with *cylindrical* symmetry is shown. The inner cylinder has a uniform charge density $\rho_v \, {}^{C}\!/_{m^3}$.

The outer shell has a surface charge density $\rho_s \, {}^{C}\!/_{m^2}$ such that the total charge on the outer shell is the negative of the total charge in the inner cylinder. Ignore end effects.



integration contour for part c.

# Gauss's Law

a. Find the electric field for all *r*.

a. Check your answer by evaluating the divergence and curl of the electric field.

a. What is the closed line integral of the electric field around the contour shown?

# Gauss's Law

From symmetry $\vec{E} = E_r(r)\,\hat{a}_r$

$$\oint \vec{E}\cdot\vec{ds} = \int_{side} \vec{E}\cdot\vec{ds} + \cancelto{0}{\int_{ends} \vec{E}\cdot\vec{ds}}$$

$\vec{ds}$ in $\hat{a}_z$

Gaussian surface

$$\oint \vec{E}\cdot\vec{ds} = \int_0^{l}\int_0^{2\pi} E_r\, r\, d\phi\, dz = E_r\, r \int_0^{l}\int_0^{2\pi} d\phi\, dz$$

$$= 2\pi r l\, E_r$$

$$Q_{enc} = \iiint \rho\, dv = \int_0^{l}\int_0^{2\pi}\int_0^{r} \rho_v\, r\, dr\, d\phi\, dz = \rho_v\, \pi r^2 l$$

$$\oint \vec{E}\cdot\vec{ds} = Q_{enc}/\varepsilon_0 \implies 2\pi r l\, E_r = \frac{\rho_v \pi r^2 l}{\varepsilon_0} \Rightarrow \boxed{\vec{E} = \frac{\rho_v r}{2\varepsilon_0}\,\hat{a}_r}$$

# Gauss's Law

$a<r<b$ $\oint \vec{E}\cdot d\vec{s}$ integral is same $= 2\pi r \ell E_r$

Gaussian integration surface

$Q_{enc} \Rightarrow$ for $0<r<a$ ~~it is~~ $\rho_v = \rho_v$

$a<r$ $\rho = 0$

$\therefore$ r integral in $Q_{enc}$ is $0 \to a$

$Q_{enc} = \rho_v \pi a^2 \ell$ $\therefore$ $2\cancel{\pi} r \cancel{\ell} E_r = \dfrac{\rho_v \cancel{\pi} a^2 \cancel{\ell}}{\varepsilon_0}$

$$\boxed{\vec{E} = \frac{\rho_v a^2}{2 r \varepsilon_0}\hat{a}_r \quad a<r<b}$$

$b<r$ $Q_{enc} = 0$ since $Q_{outer} = -Q_{inner}$

$\therefore 2\pi r \ell E_r = 0$ & $\vec{E} = 0$

Summary

$$\boxed{\vec{E} = \begin{cases} \dfrac{\rho_v r}{2\varepsilon_0}\hat{a}_r & r<a \\ \dfrac{\rho_v a^2}{\varepsilon_0 2 r}\hat{a}_r & a<r<b \\ 0 & b<r \end{cases}}$$

# Gauss's Law

Divergence and curl expressions for cylindrical coordinates:

Divergence $\vec{\nabla}\bullet\vec{F} = \frac{1}{r}\frac{\partial(r\cdot F_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$

Curl $\vec{\nabla}\times\vec{F} = \left(\frac{1}{r}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right)\hat{r} + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)\hat{\phi} + \frac{1}{r}\left(\frac{\partial(r\cdot F_\phi)}{\partial r} - \frac{\partial F_r}{\partial \phi}\right)\hat{z}$

c. $\nabla\cdot\vec{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) +$ $E_\phi$ and $E_z$ terms $\rightarrow 0$ since $E_\phi = E_z = 0$

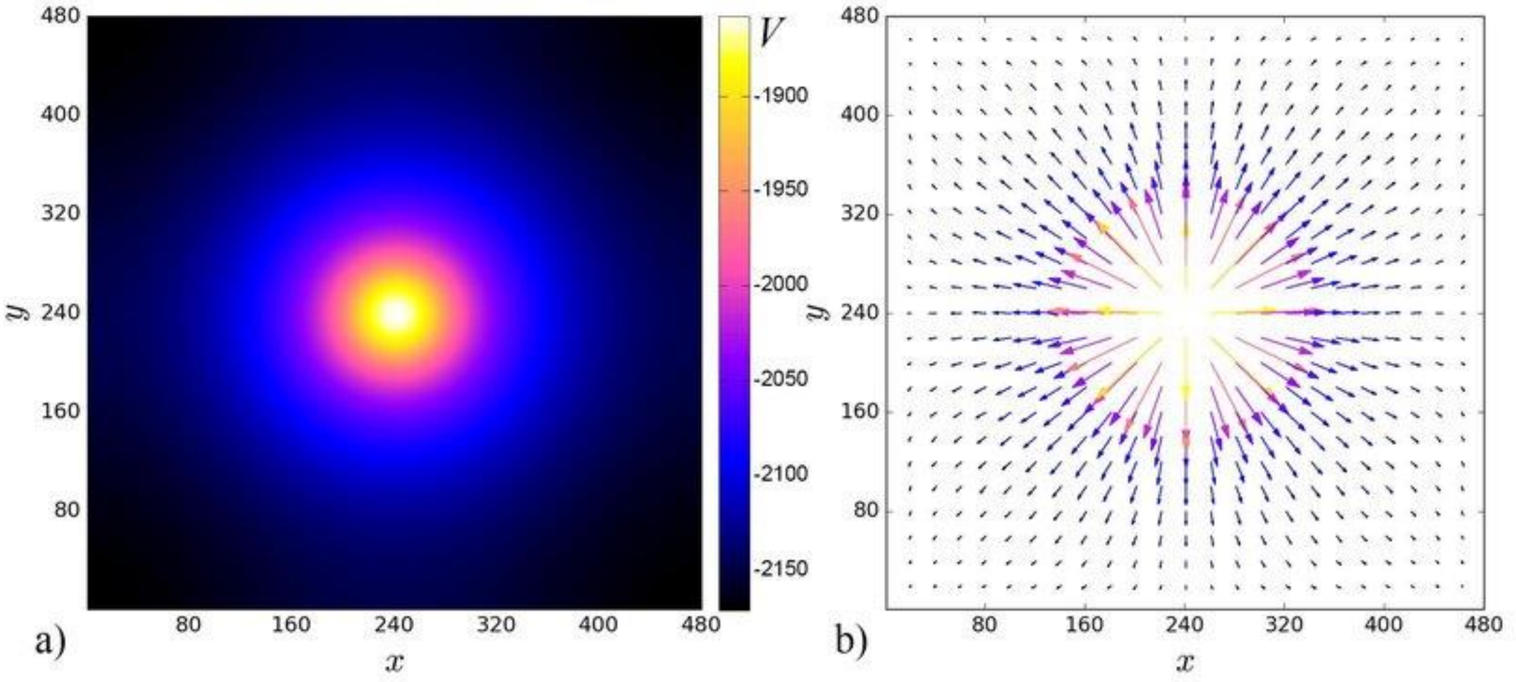
$r<a$ $\quad \nabla\cdot\vec{E} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\rho_v r^2}{2\varepsilon_0}\right) = \frac{1}{r}\frac{2\rho_v r}{2\varepsilon_0} = \rho_v/\varepsilon_0$ ✓

$a<r<b$ $\quad \nabla\cdot\vec{E} = \frac{1}{r}\frac{\partial}{\partial r}\left(\cancel{r}\frac{\rho_v a^2}{2\cancel{r}}\right) = 0$ ✓ THERE is no charge here

$r>b$ $\quad \nabla\cdot\vec{E} = 0$

$\nabla\times\vec{E} = \frac{\partial E_r}{\partial z}\hat{a}_\varphi - \frac{1}{r}\frac{\partial E_r}{\partial \varphi}\hat{a}_z + \cancel{E_\varphi + E_z \text{ terms}} \rightarrow 0$ $= 0$ since $E_r$ not function of $\varphi$ or $z$

# Electric Potential




If the electric field looks like this (right), how do we derive the electric potential (left)? (Lauricella)

# Electric Potential

- From vector calculus,

$$\nabla \times \nabla f = 0 \text{ for any scalar field f.}$$

- Introducing the electric scalar potential:

Since $\nabla \times \vec{E} = 0$, we can find a vector field such that

$$\vec{E} = -\nabla V \qquad \nabla \times \nabla V = \nabla \times \vec{E} = 0$$

$$V(P_2) - V(P_1) = -\int_{P_1}^{P_2} \vec{E} \cdot d\vec{l}$$

# Electric Potential

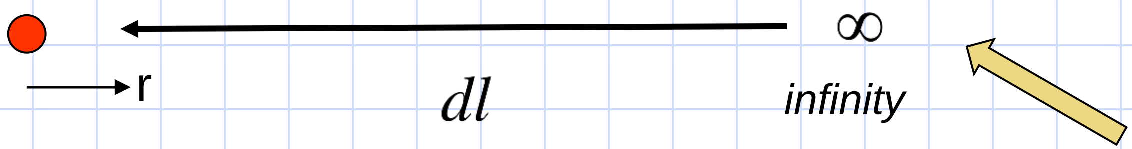
Example: Use case of <u>point charge</u> at origin and obtain potential everywhere from E-field

*Spherical Geometry*

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2} \cdot \hat{a}_r$$

Point charge
at (0,0,0)

*Integration Path*




Reference:
V=0 at infinity

# Electric Potential

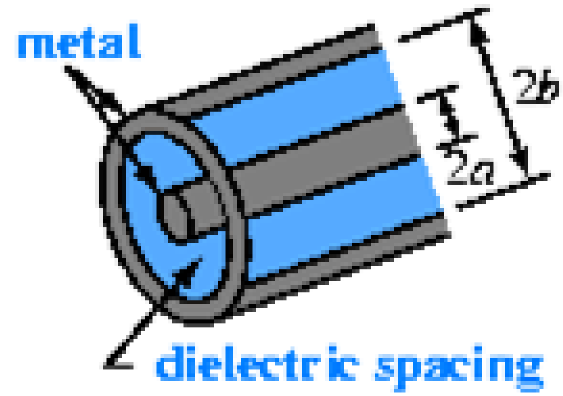
- Charge-Voltage Method

$$Q = C\,V$$

$$Q \rightarrow \vec{E} \rightarrow V$$

Maxwell's first equation

Maxwell's second equation



(a) Coaxial line

# Electric Potential

The integral for computing the potential of the point charge is:

$$V(r) - \underbrace{V(r=\infty)}_{0} = -\int_{r=\infty}^{r} \vec{E} \bullet d\vec{l}$$

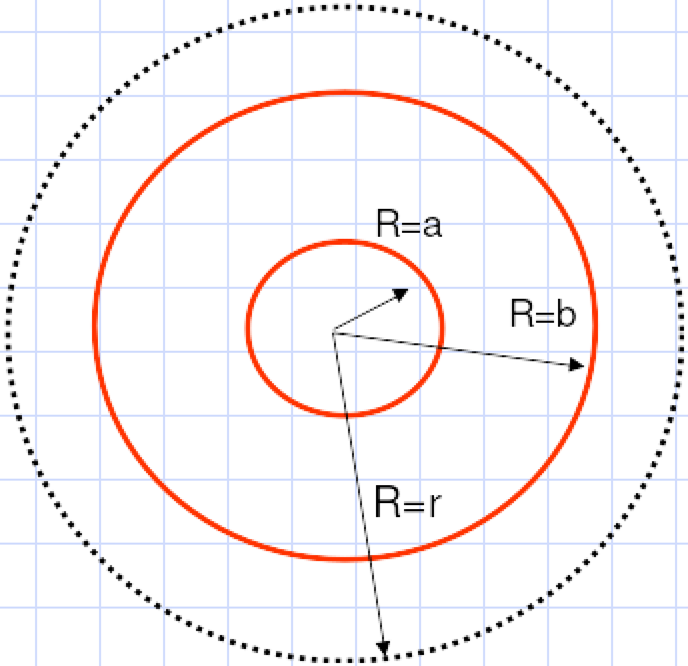
$$\therefore V(r) = -\int_{r=\infty}^{r} E \cdot dr$$

$$= -\int_{r=\infty}^{r} \frac{q}{4\pi\varepsilon_0 r^2} \cdot dr \quad \Rightarrow \quad V(r) = \frac{q}{4\pi\varepsilon_0 r}$$

# Electric Potential

- What is the potential expression for this E-field from a shielded conductor with a grounded

$$\vec{E} = \begin{cases} \dfrac{\rho_v r}{2\varepsilon_0}\hat{a}_r & r<a \\ \dfrac{\rho_v a^2}{\varepsilon_0 2r}\hat{a}_r & a<r<b \\ 0 & r>b \end{cases}$$



# Electric Potential

$$\vec{E} = \begin{cases} \dfrac{\rho_v r}{2\varepsilon_0}\hat{a}_r & r<a \\ \dfrac{\rho_v a^2}{\varepsilon_0 2r}\hat{a}_r & a<r<b \\ 0 & r>b \end{cases}$$

$V(b) = 0$

$\therefore$ FOR $\boxed{r>b \quad V=0}$ since $\vec{E}=0$

$$a<r<b \qquad V(r) - \underbrace{V(b)}_{0} = -\int_b^r E_r\,dr = -\int_b^r \frac{\rho_v a^2}{\varepsilon_0 2r}\,dr = -\frac{\rho_v a^2}{2\varepsilon_0}\ln\frac{r}{b}$$

$$\boxed{V(r) = \frac{\rho_v a^2}{2\varepsilon_0}\ln\frac{b}{r} \qquad a<r<b}$$

# Electric Potential

& $a<r<b$ $\quad \vec{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{a}_r = -\frac{\partial}{\partial r}\left(\frac{\rho_v a^2}{2\varepsilon_0}\ln\frac{b}{r}\right)\hat{a}_r$

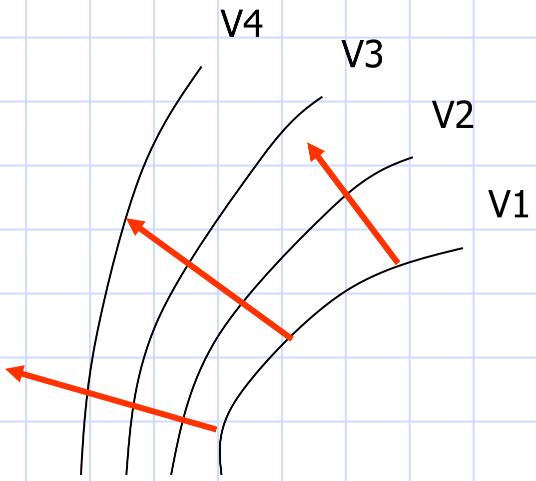
$$= -\frac{\rho_v a^2}{2\varepsilon_0}\,\frac{-b/r^2}{b/r}\hat{a}_r = \frac{\rho_v a^2}{2\varepsilon_0 r}\hat{a}_r = \text{original } \vec{E} \checkmark$$

$\cdot\ V(0) - V(a) = -\int_a^0 \vec{E}\cdot d\vec{l} \Rightarrow V(0) = V(a) - \int_a^0 \frac{\rho_v r}{2\varepsilon_0}\,dr$

$$V(0) = \underbrace{\frac{\rho_v a^2}{2\varepsilon_0}\ln\frac{b}{a}}_{\substack{\text{set } r=a \text{ in} \\ a<r<b \text{ solution}}} - \frac{\rho_v}{2\varepsilon_0}\frac{r^2}{2}\Big]_a^0 = \boxed{\frac{\rho_v a^2}{2\varepsilon_0}\ln\frac{b}{a} + \frac{\rho_v a^2}{4\varepsilon_0}}$$

# Electric Potential

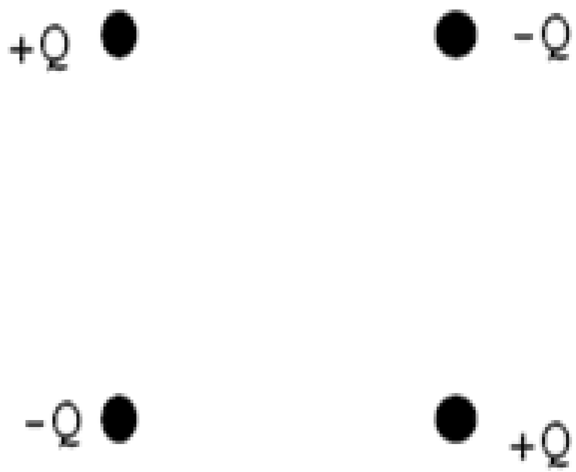
$$\vec{E} = -\nabla V$$

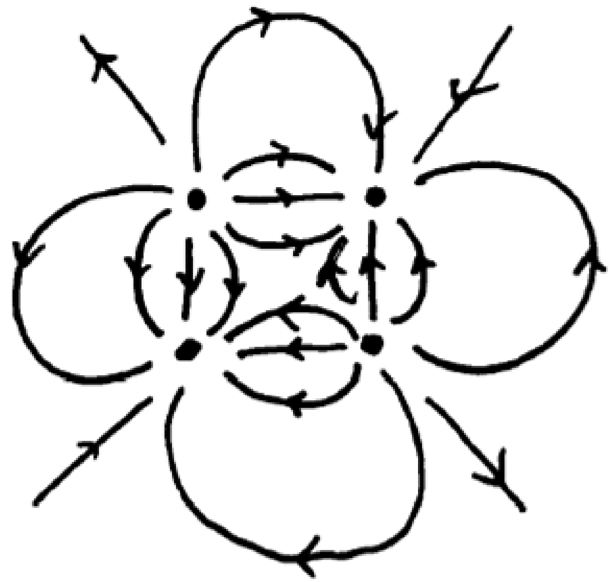



- Gradient points in the direction of largest change
- Therefore, E-field lines are perpendicular (normal) to constant V surfaces

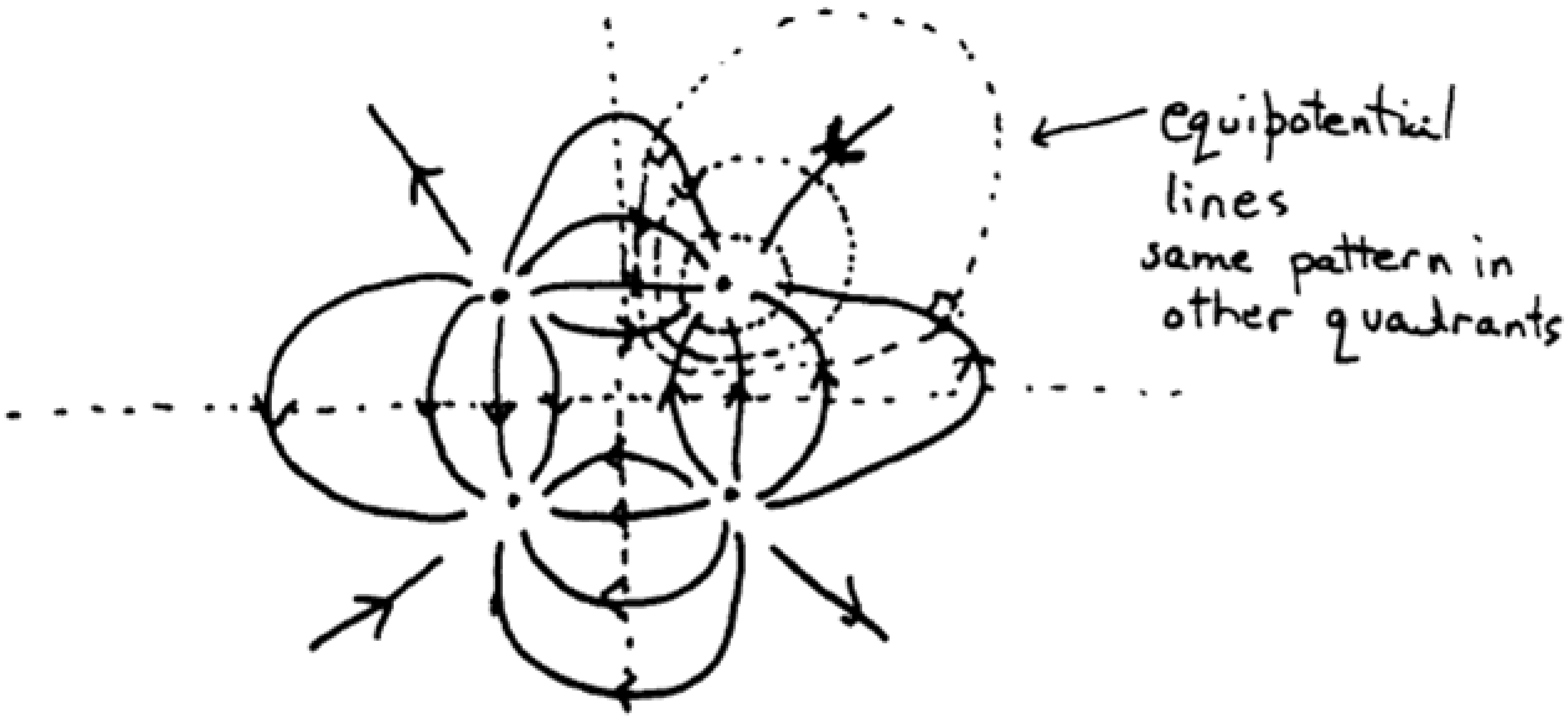
# Electric Potential

- Plot a set of equipotentials for this quadrupole.



# Electric Potential



# Electric Potential

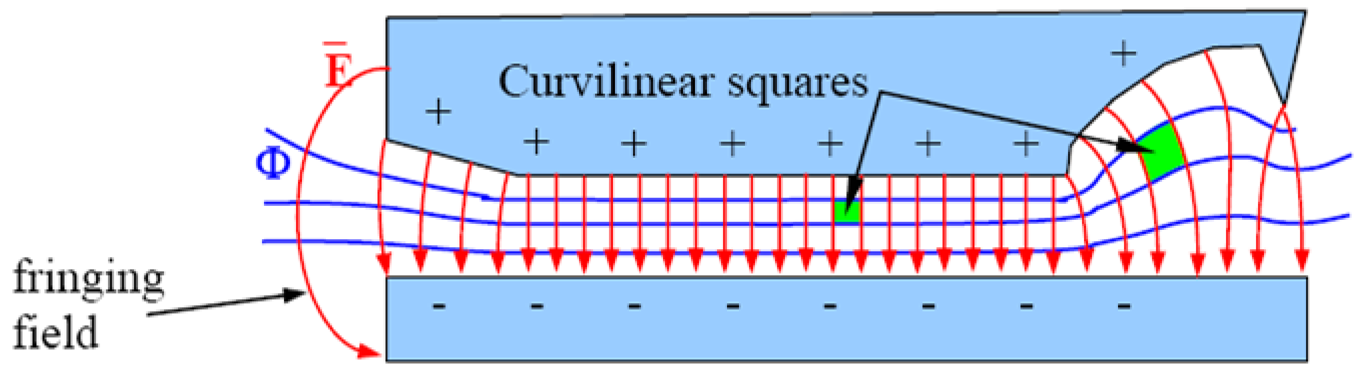



Figure R11-3. Graphical field mapping of $\bar{E}$ and F between charged conductors

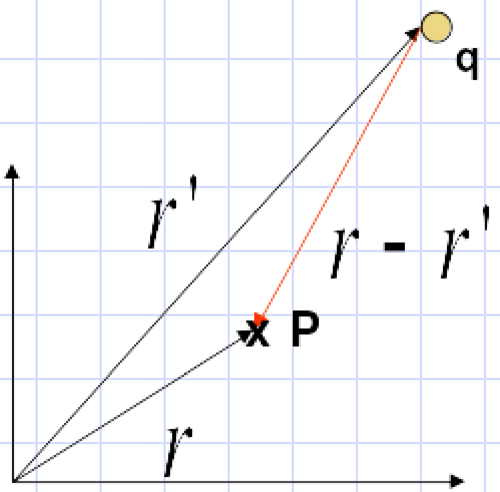
http://ocw.mit.edu/NR/rdonlyres/Electrical-Engineering-and-Computer-Science/6-013Electromagnetics-and-ApplicationsFall2002/922D1A06-9AC9-4076-B1F5-066EE896043C/0/Rec11Notes.pdf

# Electric Potential

## Potential of a single charge

For the case of a point charge:

$$V = \frac{q}{4\pi\epsilon_0 r} = \frac{q}{4\pi\epsilon_0 (r - r')} = V(\vec{r})$$




$\vec{r}$ , is field point where we are measuring/calculating V

$\vec{r}$ , is location of charge

# Electric Potential

For a charge distribution:

$$V(r) = \int\int\int \frac{\rho(r') \cdot dv'}{4\pi\epsilon_0 |r - r'|}$$

Volume charge distribution

$$V(r) = \int \frac{\rho(r') \cdot dl'}{4\pi\epsilon_0 |r - r'|}$$

Line charge distribution

# Review

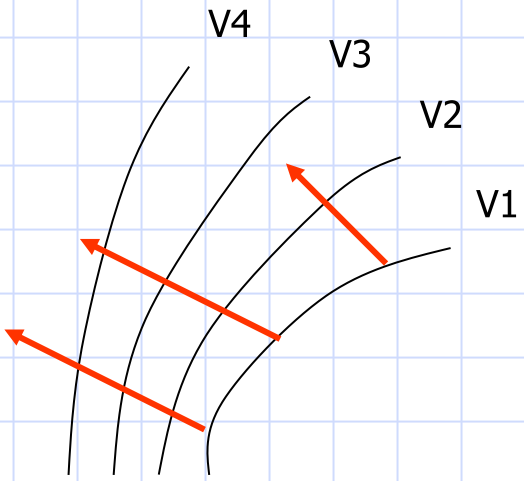
## Using Gauss' Law to find E

- Recognize the coordinate system.
- Using symmetry, determine which components of the field exist.
- Identify a Gaussian surface for which the sides are either parallel to or perpendicular to the field components. This surface is arbitrary in size.
- Determine the total charge within that surface. The charges can be distributed on lines, surfaces or in volumes.

# Review

Electric Potential

$$\vec{E} = -\nabla V$$




- Gradient points in the direction of largest change
- Therefore, E-field lines are perpendicular (normal) to constant V surfaces

# Electric Materials

Previously we said:

$$\oint \vec{D} \cdot ds = \int \rho \cdot dv$$

$$\vec{D} = \epsilon \vec{E}$$

- What about situations in which $\boldsymbol{\varepsilon} \neq \boldsymbol{\varepsilon}_0$ ?
- Materials other than free space can have a wide range of $\boldsymbol{\varepsilon}$.

# Electric Materials

Conductors and Dielectrics

<u>Superconductors</u>

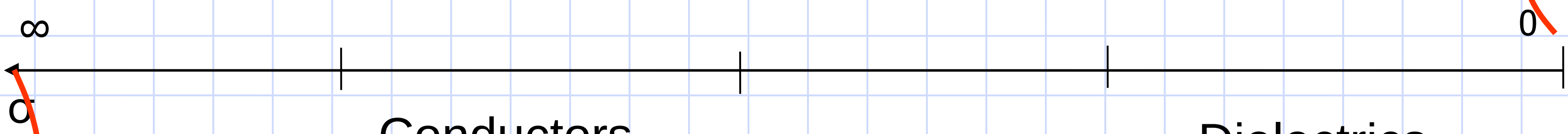
$\sigma \sim \infty$

<u>Semiconductors</u>

mid σ's

$\sigma$ (for Germanium) ~ 2.2 S/m

Perfect dielectric




<u>Conductors</u>

<u>Dielectrics</u>

High conductivities;
$\sigma$ (for Copper) ~ $5.8\times10^{7}$ S/m

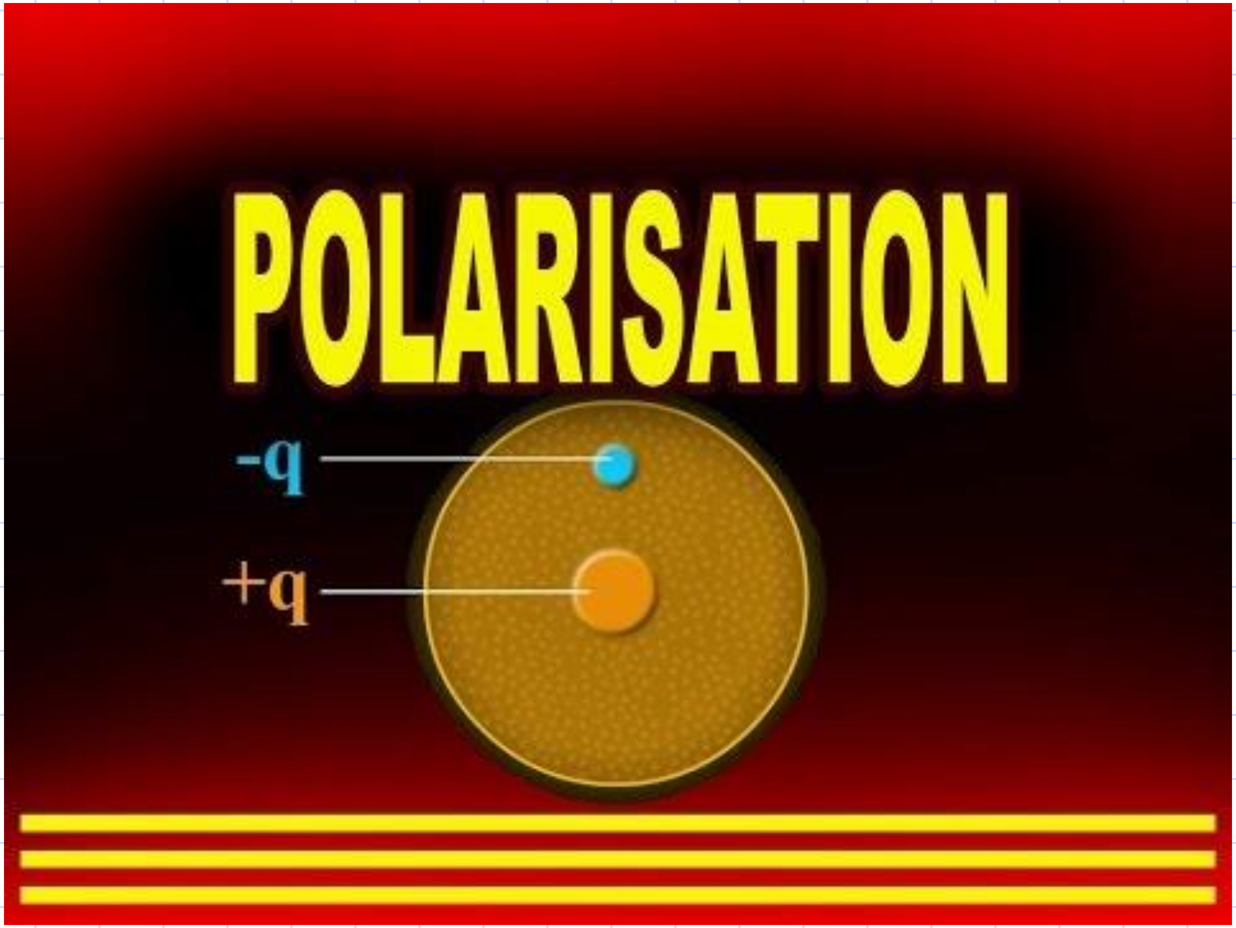
Low conductivities;
$\sigma$ (for Rubber) ~ $1\times10^{-15}$ S/m

Perfect Conductor

Permittivities, $\varepsilon = 1\text{-}100\varepsilon_0$

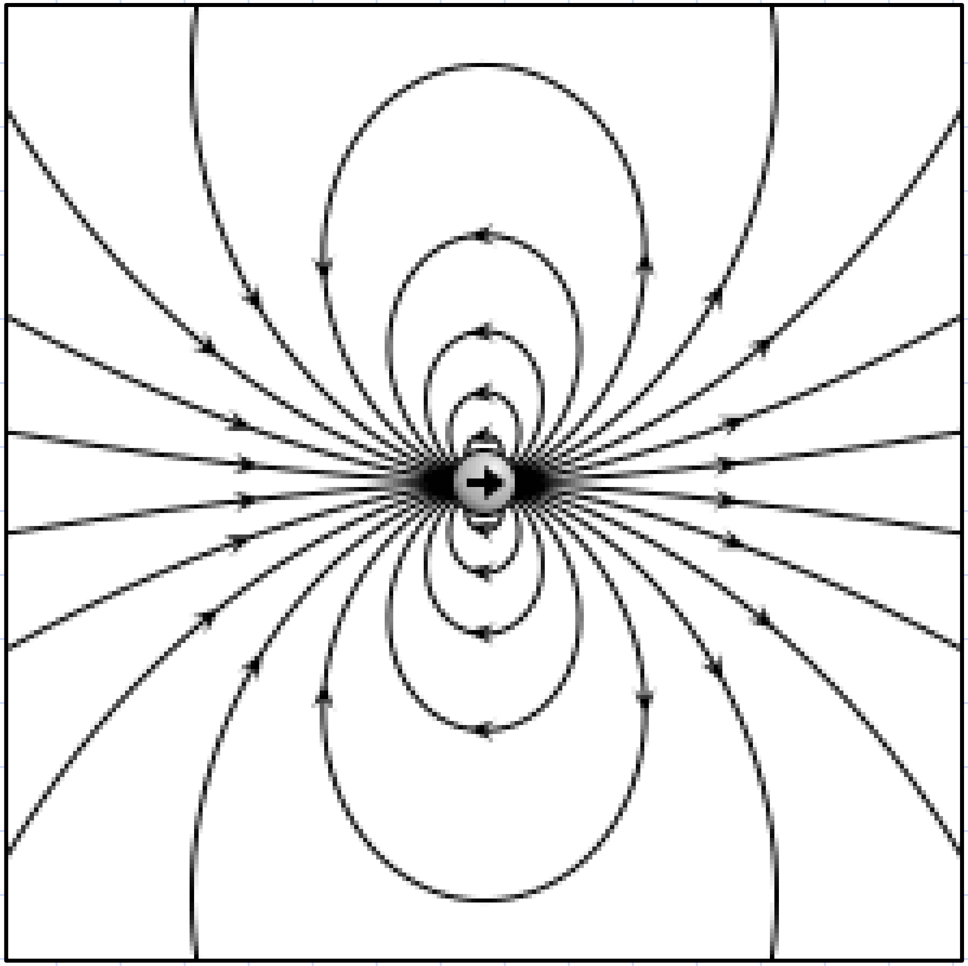
*Note: $\varepsilon_0$ is the permittivity of free space / vacuum = $8.854 \times 10^{-12}$ F/m*

# Electric Materials




Physics4Students

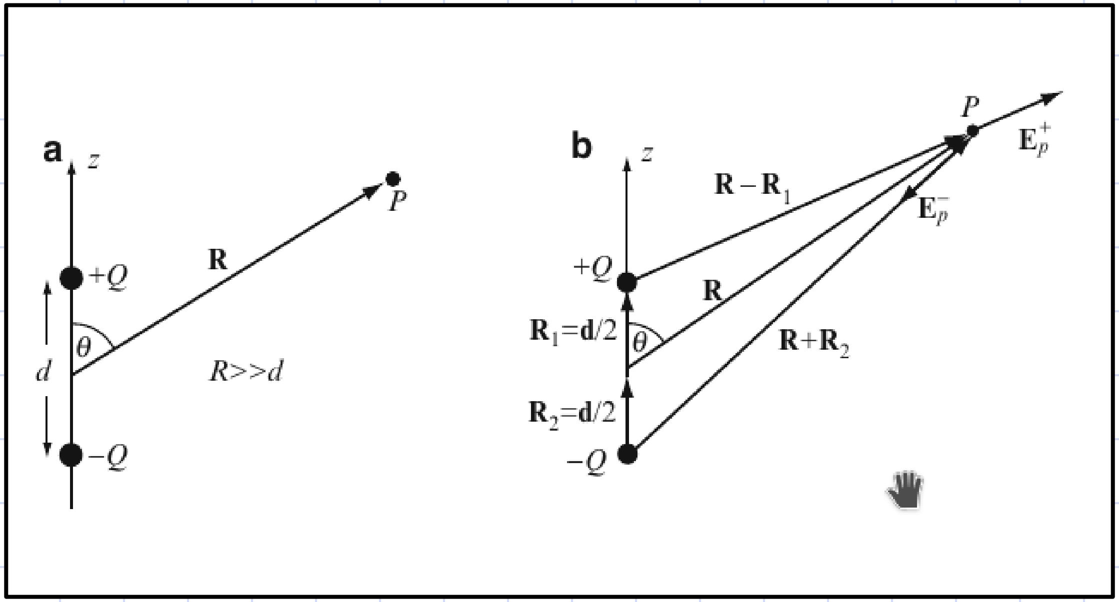
# Electrostatics



Wikipedia

When two opposite charges are very close together relative to some much larger distance d from which we measure their field, we call them an electric dipole.

# Electrostatics




Superposition allows us to write an expression for the electric field caused by two opposite-value point charges, separated by distance d, at a distance R away.

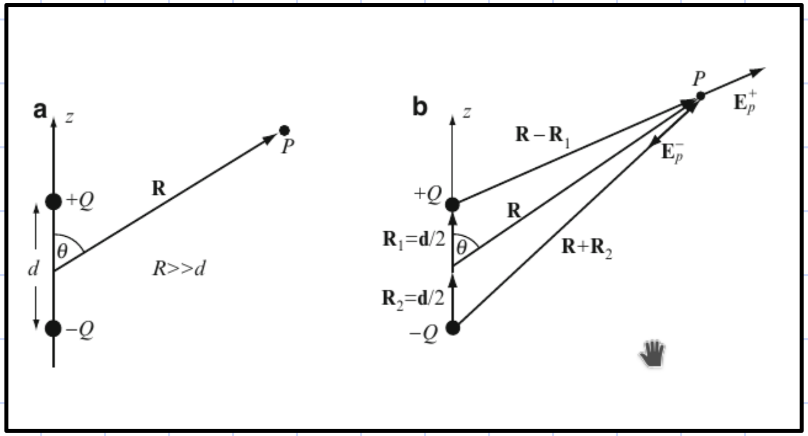
$$\mathbf{E}_d = \mathbf{E}_p^+ + \mathbf{E}_p^- = \frac{Q}{4\pi\varepsilon_0}\left(\frac{(\mathbf{R}-\mathbf{d}/2)}{|\mathbf{R}-\mathbf{d}/2|^3} - \frac{(\mathbf{R}+\mathbf{d}/2)}{|\mathbf{R}+\mathbf{d}/2|^3}\right) \quad \left[\frac{\mathrm{N}}{\mathrm{C}}\right]$$

**Note: here, we will use bold letters to represent vectors.**

**Which coordinate system is this?**

Ida

# Electrostatics




… after some algebra, infinite series math and limits, we can write

$$\mathbf{E}_d \approx \frac{1}{4\pi\varepsilon_0 R^3}\left[3\frac{Rp\cos\theta}{R^2}\hat{\mathbf{R}}R - p\left(\hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta\right)\right] = \frac{p}{4\pi\varepsilon_0 R^3}\left(\hat{\mathbf{R}}2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\right) \quad [\mathrm{N/C}]$$

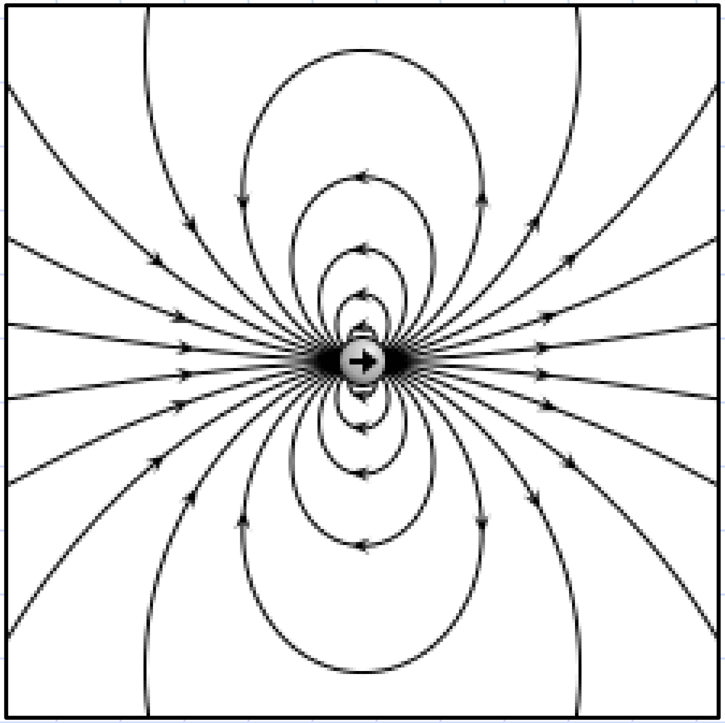
We let $\mathbf{p} = Q\mathbf{d}$ and call **p** the dipole moment.

**d** points from the negative charge to the positive.

# Electrostatics

$$\frac{p}{4\pi\varepsilon_0 R^3}\left(\hat{\mathbf{R}}2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\right) \quad [\mathrm{N/C}]$$

$$\mathbf{p} = Q\mathbf{d}$$



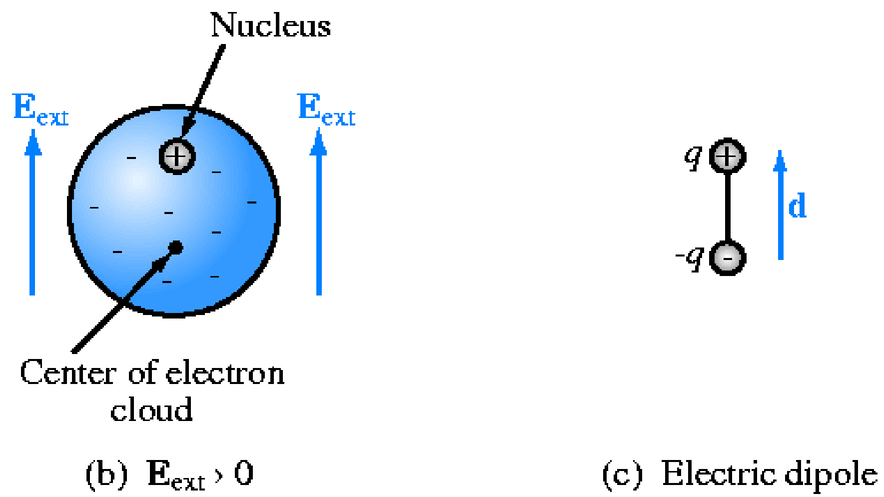
**A few things to note:**

- This is an inverse cube expression instead of the standard inverse square expression of Coulomb's Law
- Field intensity is maximum at points aligned with the dipole and minimum perpendicular to the dipole
- Field intensity is proportional the distance between the charges
- Anything that causes the charges to move apart will proportionally increase their dipole field

Wikipedia

# Electric Materials

$$\vec{D} = \epsilon \vec{E}$$




Define: $p = q \cdot d$ = dipole moment

$E_{response} \propto -\sum p_i = P$ ⟵ Polarization

$\boxed{E_{response}}$ partially cancels applied Field

$$\therefore E_{total} = E_{external} + E_{response} \neq 0$$

# Electric Materials

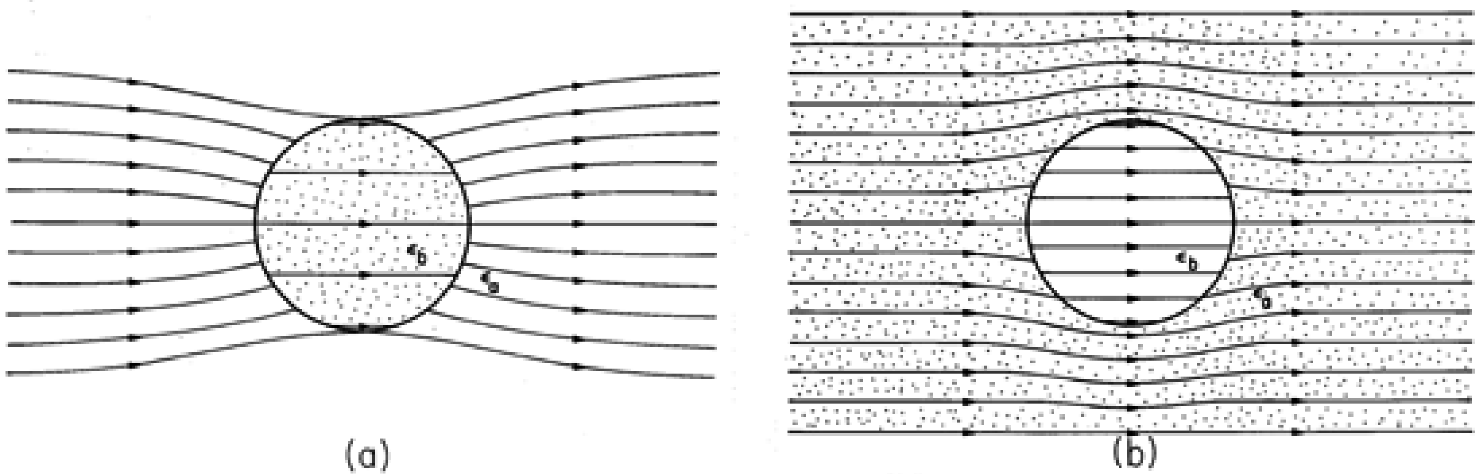
- Different materials have different permittivities
- Permittivities will also depend on field frequency, temperature, humidity, and other factors

| Material | Relative Permittivity |
|---|---|
| Paper | 1.4 |
| Concrete | 4.5 |
| Silicon | 11.68 |
| Water | 50-90 |
| Titanium dioxide | 86-173 |
| Calcium copper titanate | >250,000 |

Wikipedia

# Electric Materials

StackExchange




a.) $\varepsilon_b > \varepsilon_a$
b.) $\varepsilon_b < \varepsilon_a$

Note the effect on relative E-field density.

# Electric Materials

- Why aren't any metals listed in the table of dielectric constants?
- Because metals (generally) aren't dielectrics; they're conductors
- In dielectrics, E-fields cause electrons and nuclei to "flex" and form dipoles, but the electrons generally remain bound to the nuclei.
- In conductors, that isn't the case - a large number of electrons are free to move around under the influence of a field.

# Electric Materials

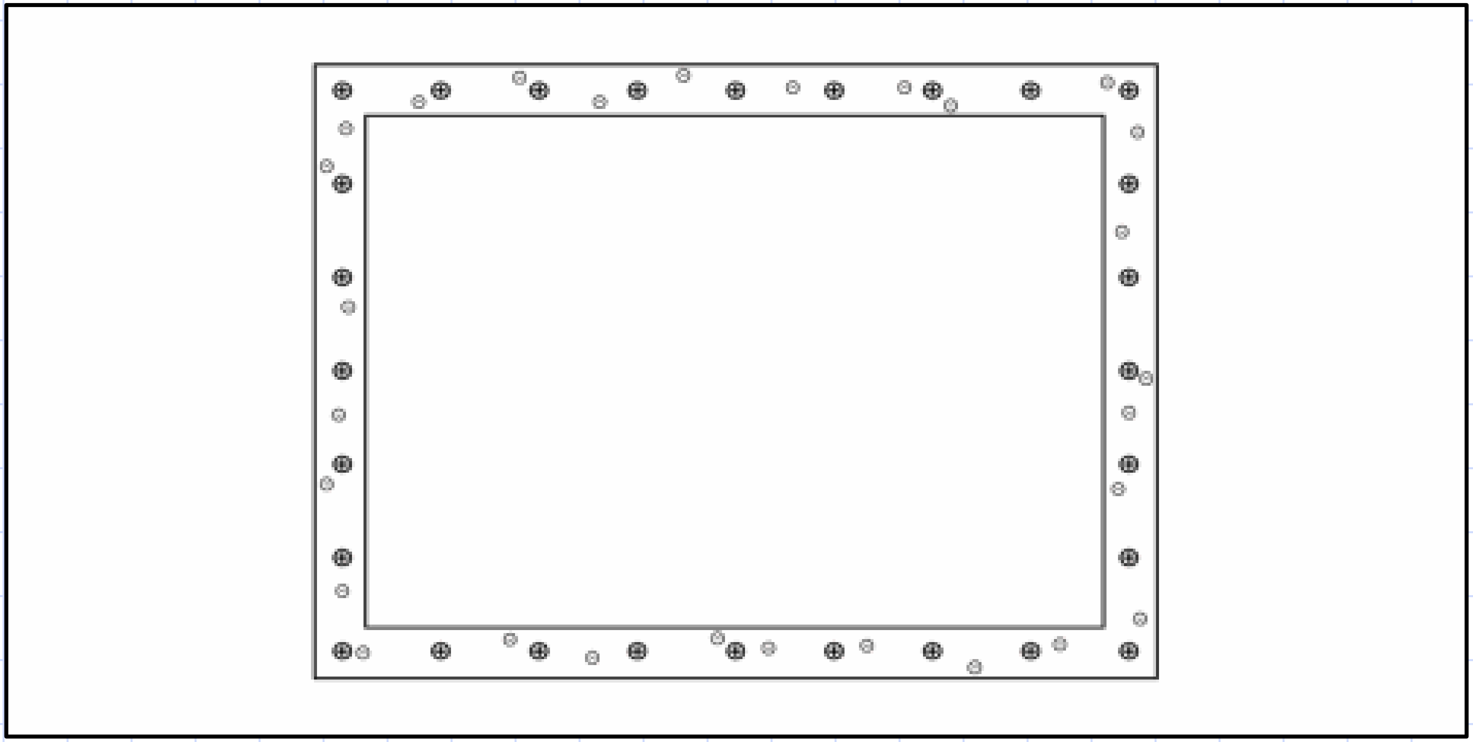
- We have an expression for the energy stored in an electric field:

$$W = \frac{1}{2}\epsilon|\vec{E}|^2$$

- Inside a material, charge carriers will attempt to *reach the lowest possible energy state*. Moving in such a way as to minimize this field will minimize energy (but whether or not the carriers actually move depends on how well bound to their atoms they are.)

# Electric Materials

- Charge carriers in a conductor move in such a way as to minimize the E-field that they enclosed
- This is how a Faraday cage works



Wikipedia

# Electric Materials

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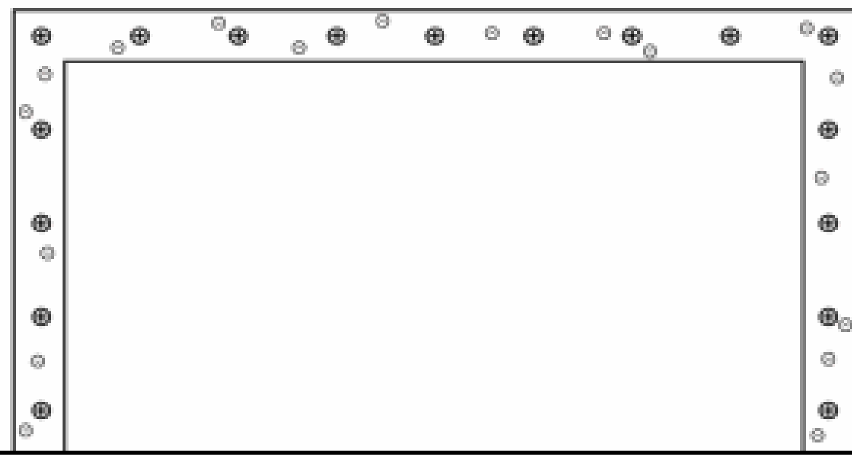
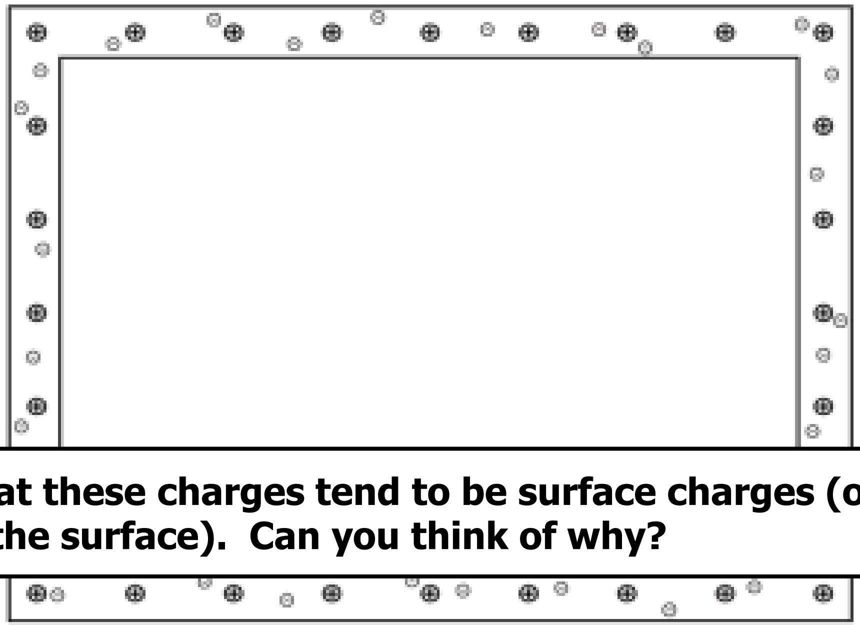


**Side note: a Faraday cage is not a perfect shield in all cases. Do you think a Faraday cage is more likely to pass EM waves at low frequencies or high frequencies? Why?**

Wikipedia

# Electric Materials

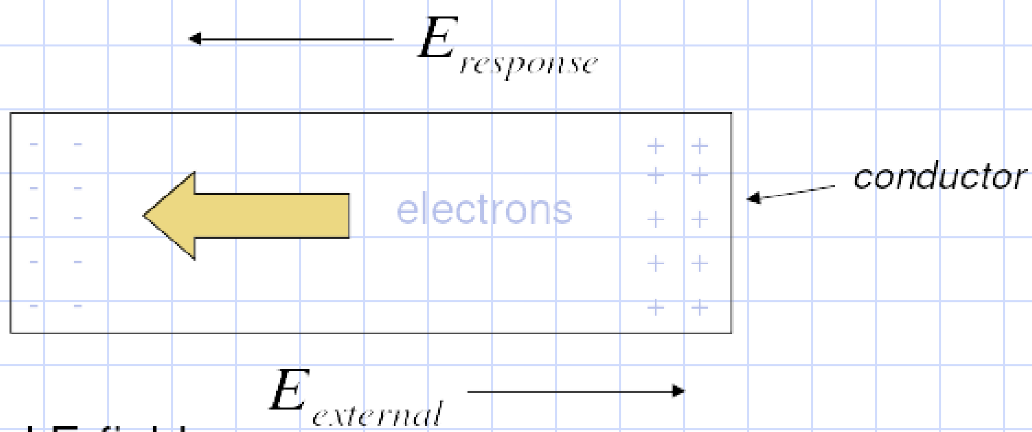
- Charge carriers in a conductor move in such a way as to minimize the E-field that they enclosed
- This is how a Faraday cage works



**Also note that these charges tend to be surface charges (or minimal depth from the surface). Can you think of why?**

Wikipedia

# Electric Materials




Apply external E-field,

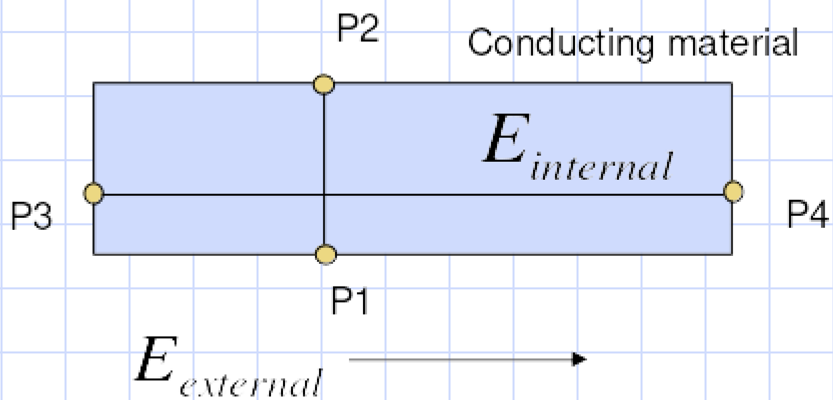
- Force on electrons causes free electrons to move
- Charge displacement causes response E-field (opposite to applied external E-field)

$$E_{total} = E_{external} + E_{response}$$

The electrons keep moving until, $E_{total} = 0$

# Electric Materials

The inside of a conductor is also an equipotential region




$$V(P_2) = V(P_1) = V(P_3) = V(P_4)$$

$$V(P_2) - V(P_1) = V(P_3) - V(P_4) = 0$$

$$= -\int_{P_1}^{P_2} \vec{E}_{internal} \bullet d\vec{l} \Rightarrow \vec{E}_{internal} = 0$$

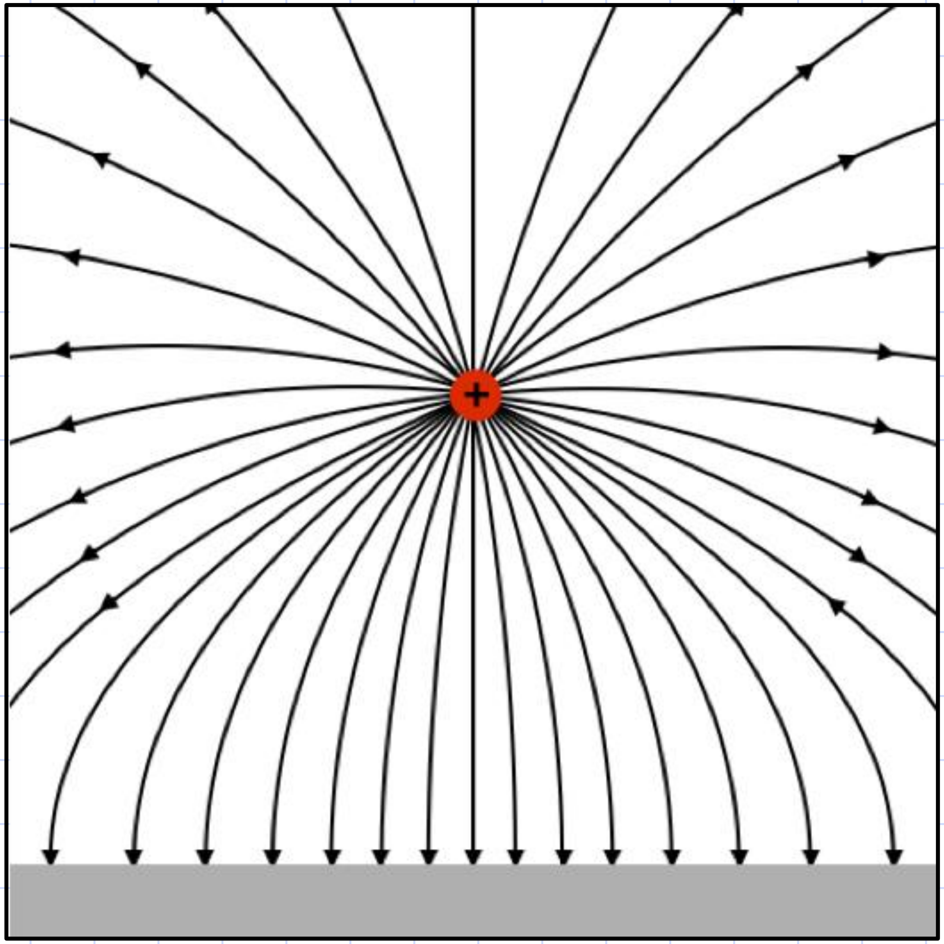
# Electric Materials

$$\vec{D} = \epsilon \vec{E}$$

$$\epsilon = \frac{|\vec{D}|}{|\vec{E}|}$$

- Since the E-field is zero in an ideal conductor, **ε** is effectively infinite!
- Hence, permittivity isn't a meaningful measurement for conductors

# Electric Materials



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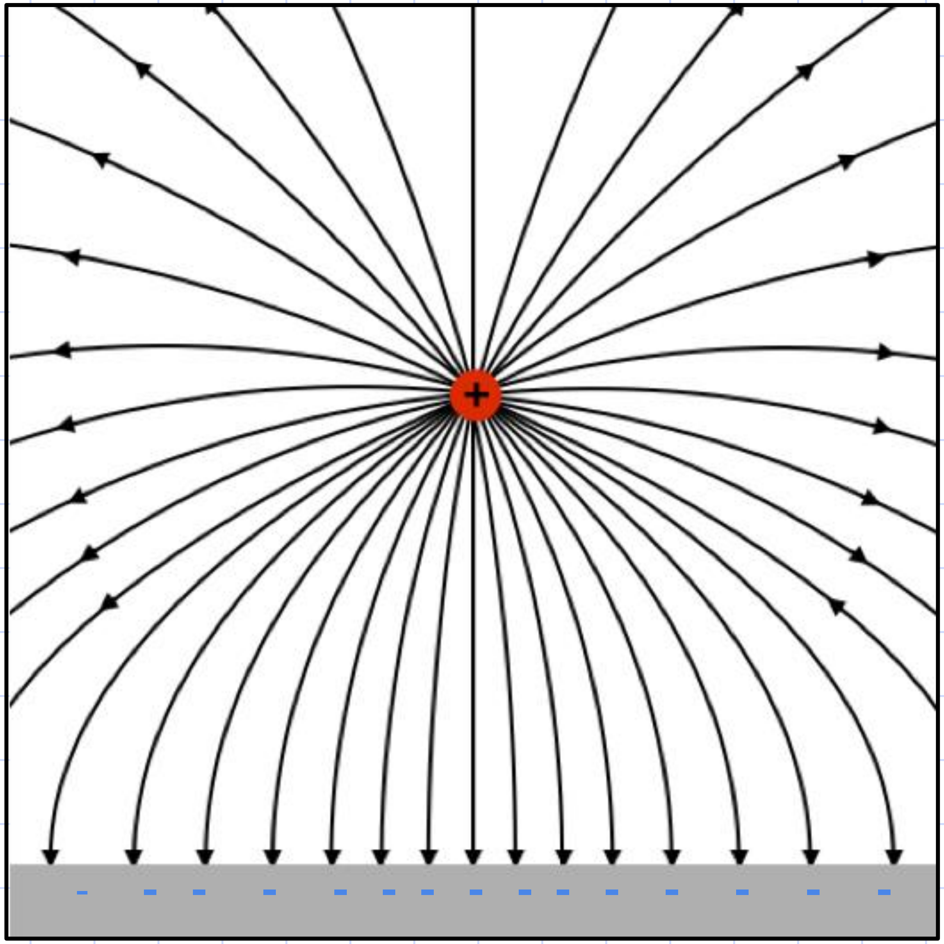
- Previously we looked at a charge suspended over an infinite plane.
- What if that plane is a conductor? How will the plane behave in this scenario?

# Electric Materials

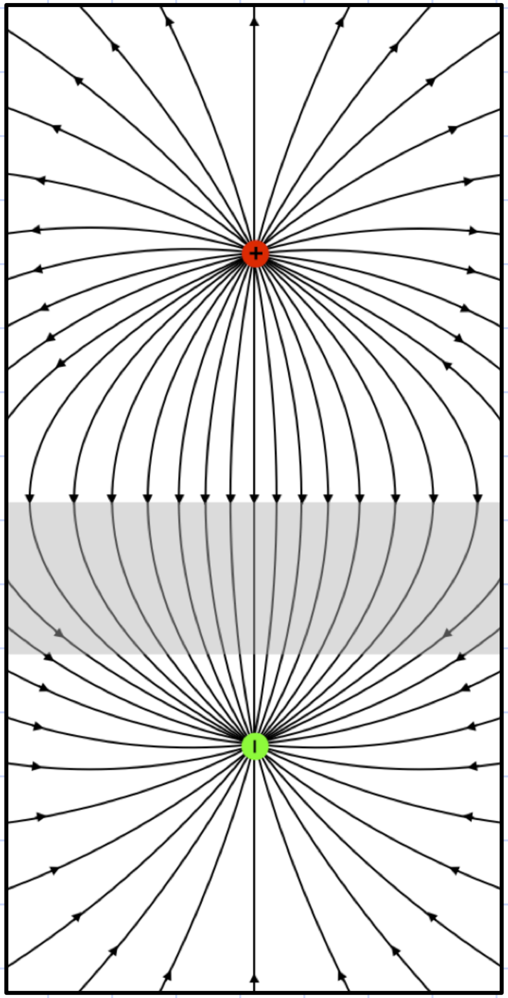



Physics LibreTexts

- As described before, the plane will accumulate charge near the surface that negates the E-field of the (+) charge within the material.
- How do we draw field lines for a situation like this? We need to take all charges and charge distributions into account 🤔

# Electric Materials

## Method of Images

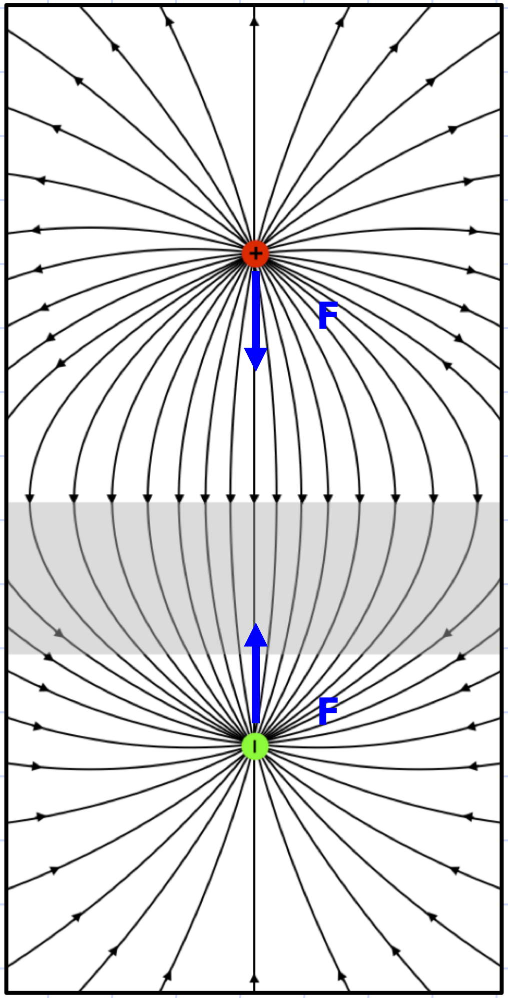


Physics LibreTexts

- Turns out that the conductor's surface charge influences the field in the exact same way as if we removed the conductive plan and added a a "mirror image" of the (+) charge at the same distance from the surface and with opposite charge.

# Electric Materials

## Method of Images




[Physics LibreTexts]

- All fields and forces will be the same as if this were the case above the plane, and this works for distributions of charge as well.
- When solving problems this way, we call it the “method of images”.

# Electric Materials

Do Lecture 12, Exercise 1 in groups of up to 4.

# Electric Materials

Examples of free charges:

- electrons in a conductor
- $\rho_s$ on conductor
- electron beam
- electrons or holes in a doped region of semi-conductor

Gauss' Law uses just free charge

$$\oint D \bullet ds = \int \rho \cdot dv = Q_{enclosed}$$ *Most general form*

# Electric Materials

Displacement / flux density in dielectrics

Define: $D \equiv \varepsilon_0 E_{TOTAL} + P$ → subtracts out bound charge

Displacement Flux Density ( C/m² )

Electric Field (V/m)

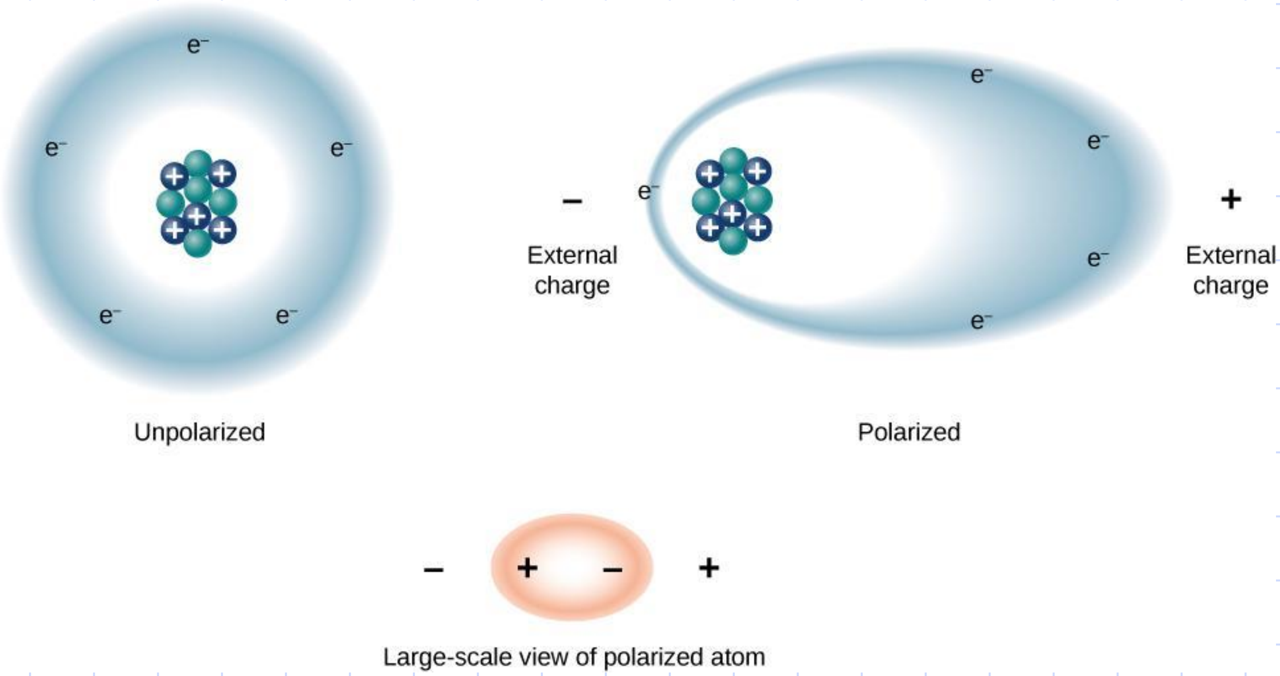
$E_{TOTAL}$ is due to bound/dielectric charge and free charge

$P$ is due to bound/dielectric charge only and opposite sign

$D$ is due to free charge only

# Electric Materials

- Keep in mind that polarization occurs through the pushing of bound electrons (in reality, an electron cloud) via the force of an electric field.



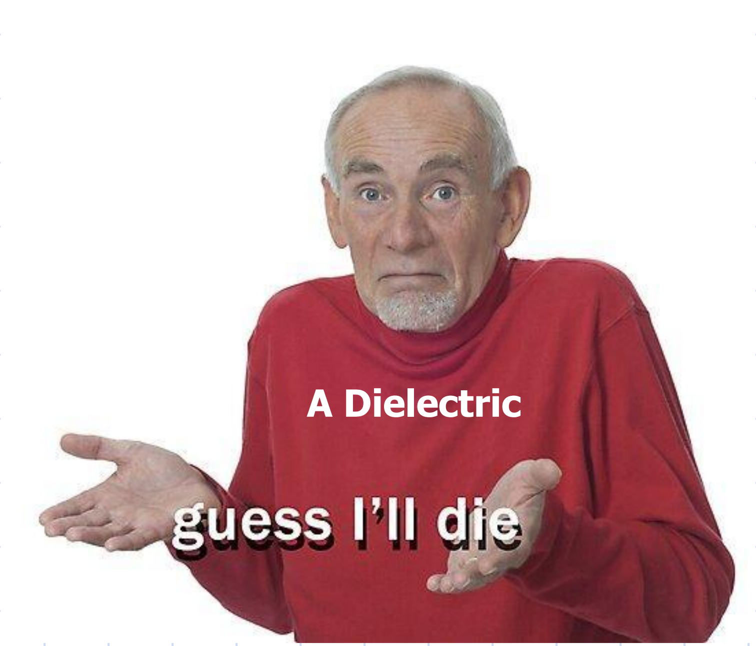

LibreTexts

# Electric Materials

- If your electric field intensity is high enough, electrons will eventually become unbound from their atoms, forging a conductive path through the material (which can become permanent if it changes the material composition occur).

# Electric Materials

- In most cases, this is very bad result.



# Electric Materials

Dielectric Breakdown

# Electric Materials

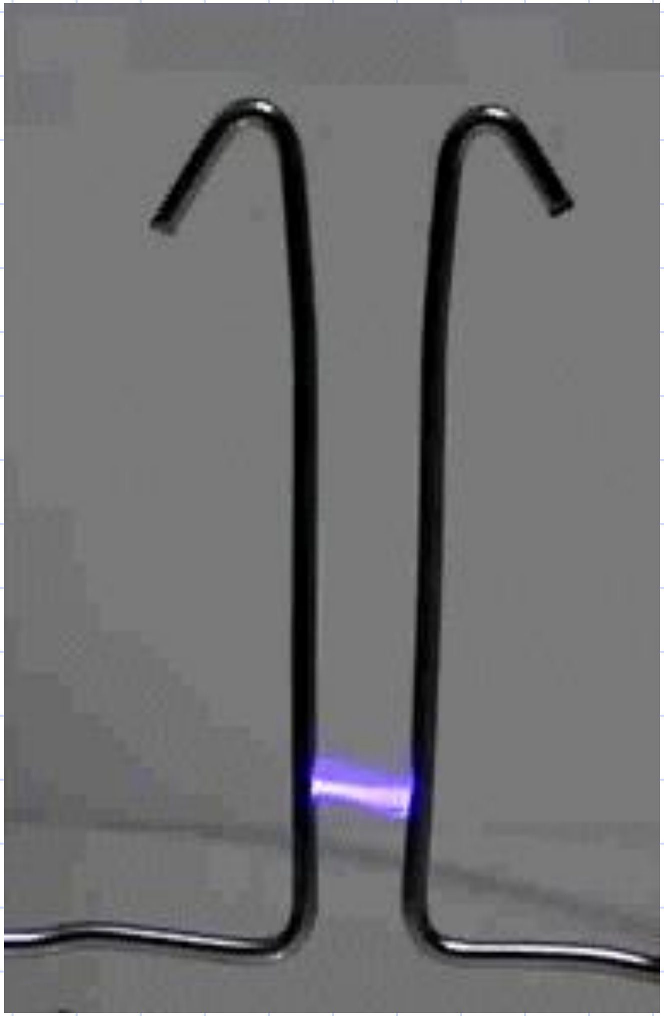
Dielectric Breakdown

- A dielectric will break down if

$$\vec{E}_{applied} > \vec{E}_{d-strength}$$

- As electrons begin to flow, they collide with and dislodge more electrons, leading to more current in an avalanche process.

- For two cables separated by an air gap, this breakdown field value is 30kV/cm.

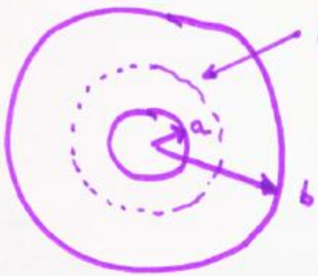


"Jacob's Ladder" (Gifer)

# Electric Materials

- A coaxial cable has an inner conductor with radius *a* , an outer conductor with radius *b* , and an insulating material with a relative permittivity of $\varepsilon_r = 2.6$.
  - Assume the outer conductor is grounded.
  - Assume that the inner conductor has a surface charge density of $\rho_{sa}$
- Find **D** and **E** between the conductors.
- Find the voltage difference, $V_{ab}$, between the conductors in terms of $\rho_{sa}$.
- In reality, we can control $V_{ab}$, not $\rho_{sa}$. Rewrite the expressions for **D** and **E** in terms of $V_{ab}$.

# Electric Materials




Gaussian Surface

$$\vec{D} = D_r(r)\,\hat{a}_r$$

$$\oint \bar{D}\cdot d\bar{S} = Q_{encl}$$

$$D_r(r)\,2\pi r l \qquad \rho_{sa}\,2\pi a l$$

$$\Rightarrow \hat{a}_r D_r(r) = \rho_{sa}\frac{a}{r}\,\hat{a}_r = \vec{D}$$

$$\vec{E} = \frac{\bar{D}}{\varepsilon} = \frac{\rho_{sa}}{\varepsilon}\frac{a}{r}\,\hat{a}_r \qquad \varepsilon = 2.6\,\varepsilon_0$$

$$V(a) - \underbrace{V(b)}_{0} = -\int_b^a E_r\,dr = -\frac{\rho_{sa}}{\varepsilon}\,a\int_b^a \frac{dr}{r}$$

$$= \frac{\rho_{sa}\,a}{\varepsilon}\ln\frac{b}{a} = V_{ab}$$

# Electric Materials

$$\Rightarrow \quad \rho_{sa} = \frac{\varepsilon V_{ab}}{a \ln b/a}$$

$$\text{or} \quad \vec{D} = \hat{a}_r \frac{\varepsilon V_{ab}}{\cancel{a} \ln b/a} \frac{\cancel{a}}{r} = \hat{a}_r \frac{\varepsilon V_{ab}}{\ln b/a} \frac{1}{r}$$

$$\vec{E} = \hat{a}_r \frac{V_{ab}}{\ln b/a} \frac{1}{r}$$

# Electric Materials

- Assume that *a* = 1 cm, *b* = 2 cm, and the dielectric material is polystyrene with $\varepsilon_r = 2.6$ and a dielectric strength of $2 \times 10^7$ V/m.

- At what value of $V_{ab}$ will the cable fail and where (what radii) will the failure occur?

. Cable fails if $|\vec{E}| > 2\times10^{7}$ V/m at any location

$$|\vec{E}| = \frac{V_{ab}}{\ln b/a}\,\frac{1}{r}$$ ⇐ This has its largest value at $r=a$

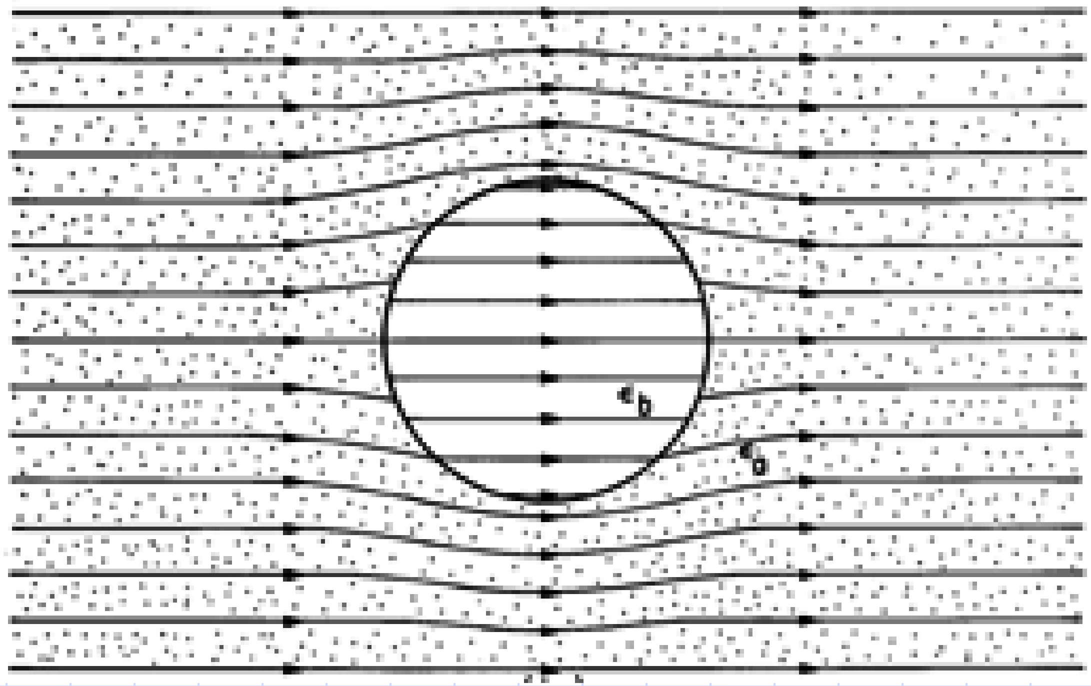
cable fails if $|\vec{E}| = \dfrac{V_{ab}}{\ln b/a}\,\dfrac{1}{a} = 2\times10^{7}$ V/m

This occurs when $V_{ab} = a \ln\frac{b}{a}\,(2\times10^{7}\ \text{V/m}) =$

$= (0.01)\ln 2\,(2\times10^{7}) = \boxed{139\ \text{kV}}$

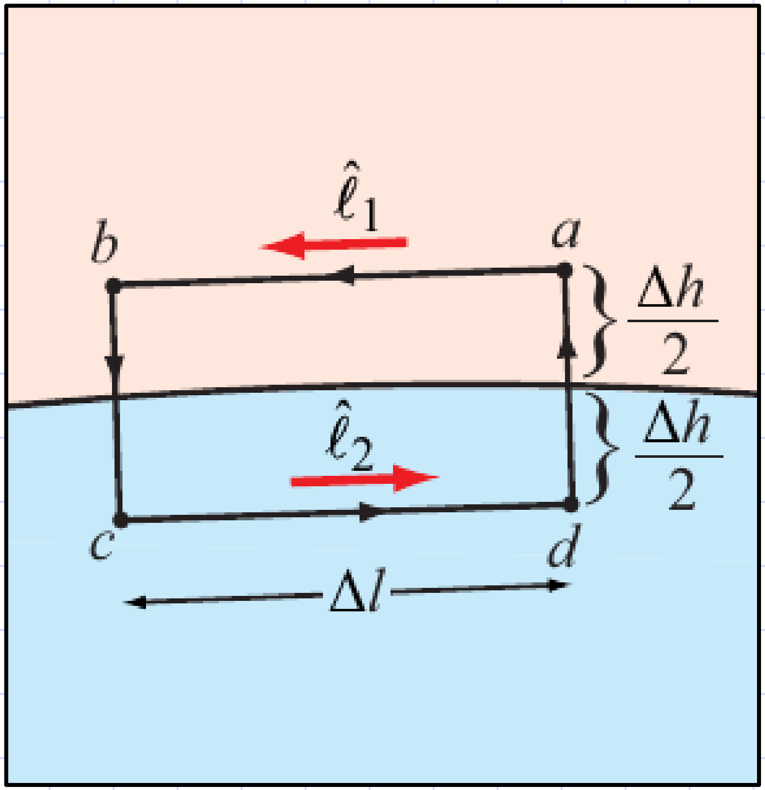
$\boxed{\text{Failure occurs at } r=a}$

# Boundary Conditions



Look at this picture again. How do electric fields behave at the boundary between two different dielectrics?

# Boundary Conditions



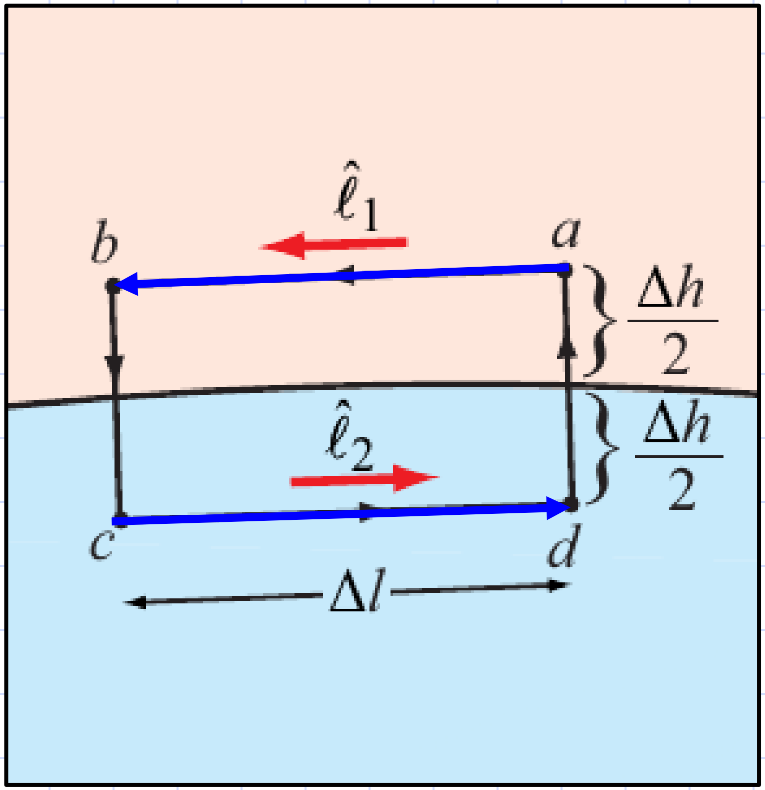

Ulaby

- We know that

$$\oint \vec{E} \cdot \vec{dl} = 0$$

- This will hold for any loop we choose, so we can choose $\Delta h \rightarrow 0$ so that the contribution of segments **bc** and **da** goes to zero.
- Note that we have chosen $\ell_1$ and $\ell_2$ tangent to the surface.

# Boundary Conditions

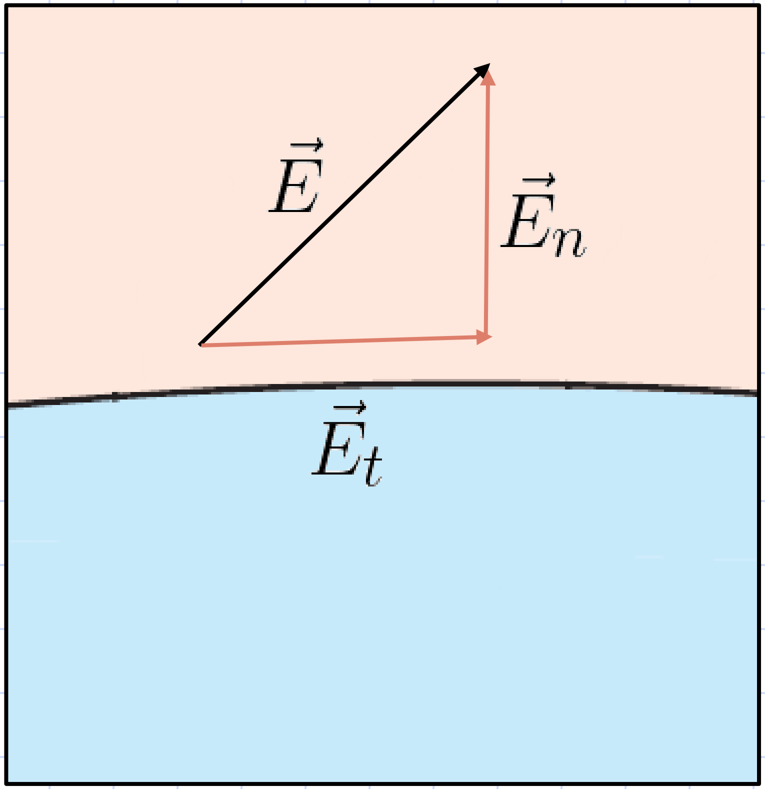



Ulaby

- We can then write

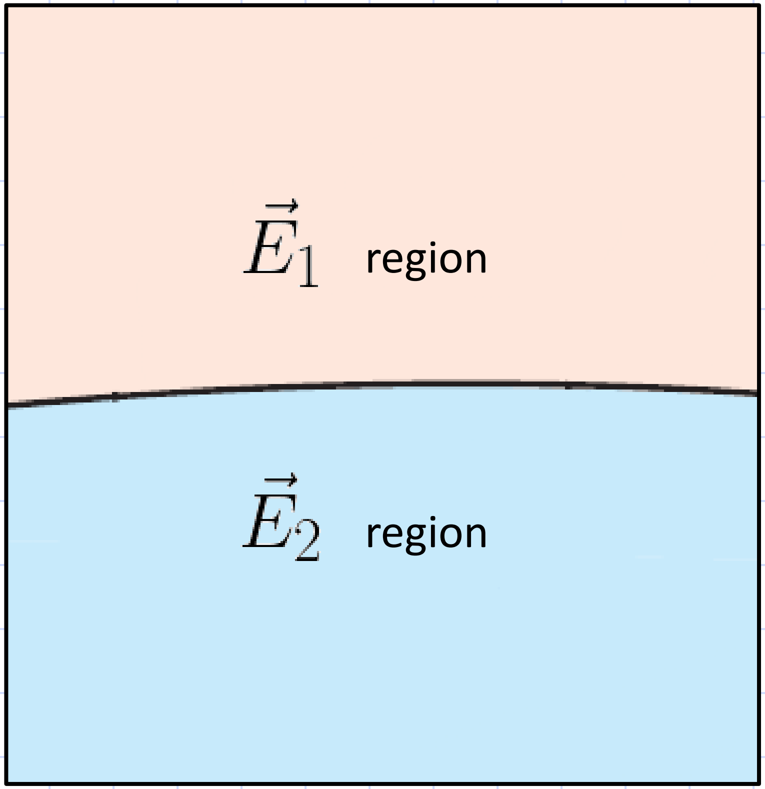
$$\int_a^b \vec{E}_1 \cdot \hat{l}_1 + \int_c^d \vec{E}_2 \cdot \hat{l}_2 = 0$$

# Boundary Conditions




- Note that we can easily decompose any E-field vector into its tangent and normal components relative to the material boundary

# Boundary Conditions

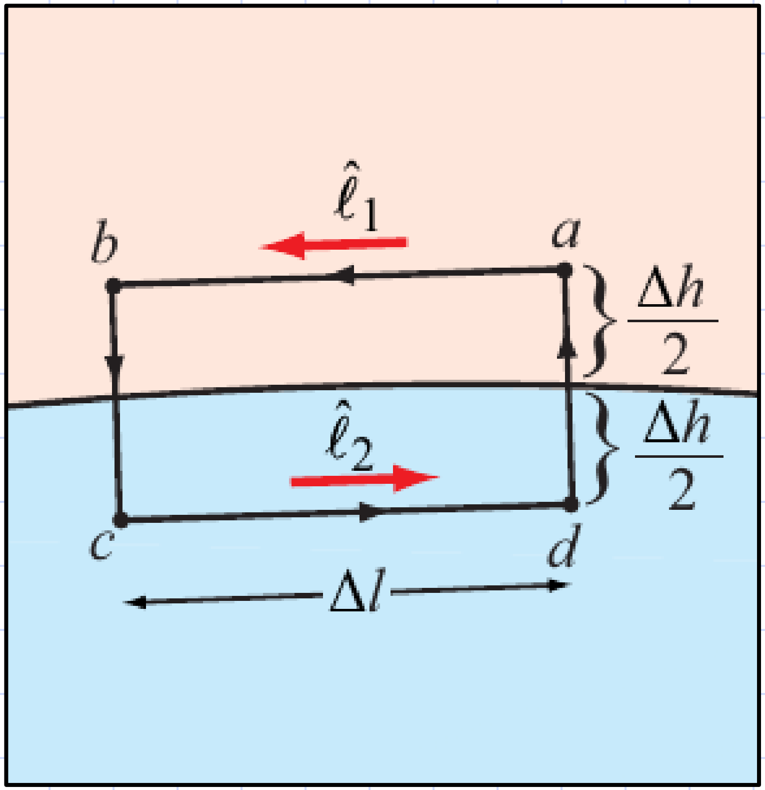



- Therefore we can write

$$\vec{E}_1 = \vec{E}_{1t} + \vec{E}_{1n}$$

$$\vec{E}_2 = \vec{E}_{2t} + \vec{E}_{2n}$$

# Boundary Conditions




Ulaby

- This equation:

$$\int_a^b \vec{E}_1 \cdot \hat{l}_1 + \int_c^d \vec{E}_2 \cdot \hat{l}_2 = 0$$

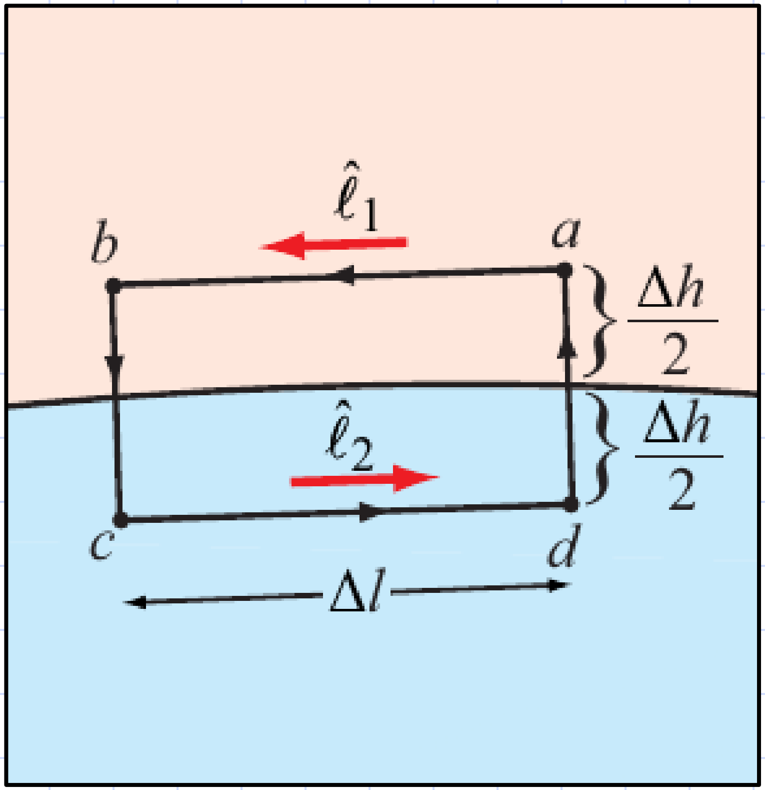
- then becomes:

$$(\vec{E}_1 - \vec{E}_2) \cdot \hat{l}_1 = 0$$

because

$$\hat{l}_1 = -\hat{l}_2$$

# Boundary Conditions




Ulaby

- We chose $\ell_1$ and $\ell_2$ such that

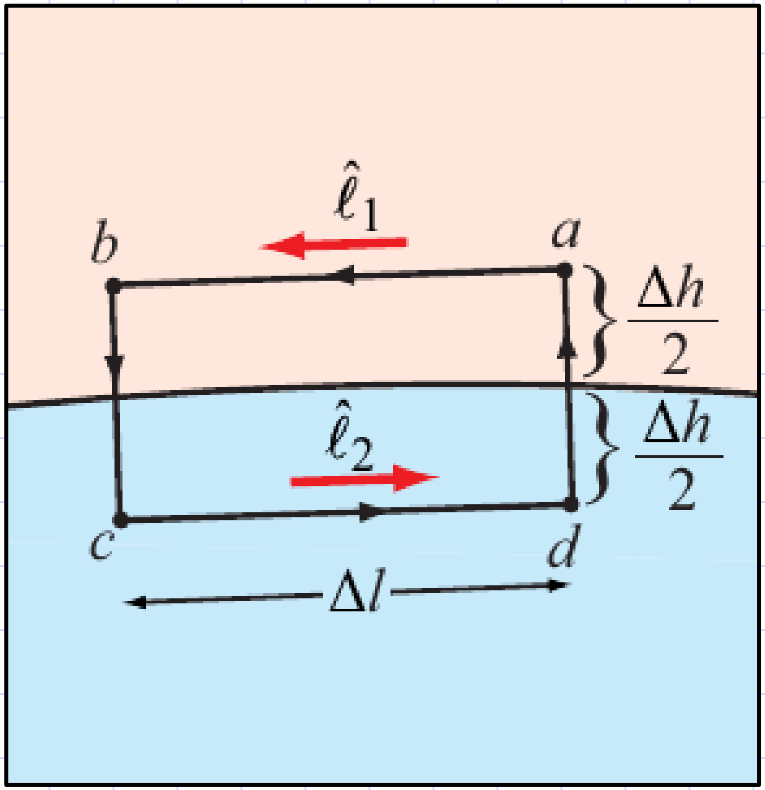
$$\vec{E}_1 \cdot \hat{l}_1 = \vec{E}_{1t}$$
$$\vec{E}_2 \cdot \hat{l}_2 = \vec{E}_{2t}$$

- Now we simplify:

$$(\vec{E}_1 - \vec{E}_2) \cdot \hat{l}_1 = 0$$

$$\boxed{\vec{E}_{1t} = \vec{E}_{2t}}$$

# Boundary Conditions



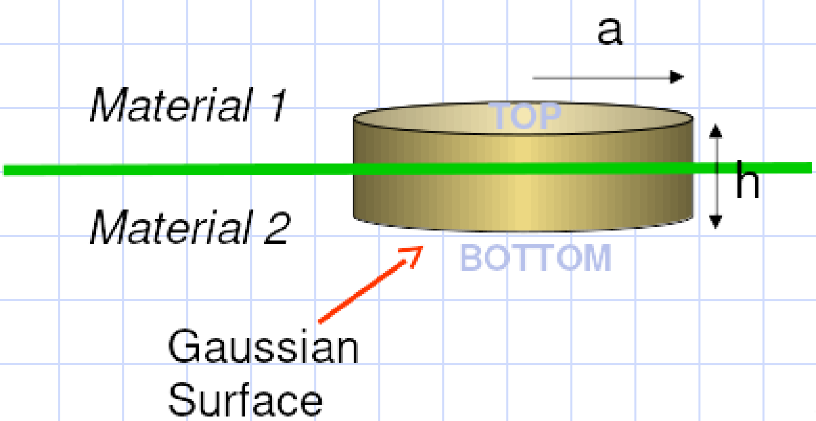

Ulaby

$$\vec{E}_{1t} = \vec{E}_{2t}$$

- So component of the E-field that is tangent to a media boundary is continuous across it.
- What about normal to the boundary?

# Boundary Conditions

NORMAL COMPONENT




$$\oint D \bullet ds = Q_{enclosed}$$

Take h << a **(a thin disc)**

$$Q_{enclosed} = \rho_s \cdot A$$

$$\oint D \bullet ds = \int_{TOP} D \bullet ds + \int_{BOTTOM} D \bullet ds$$

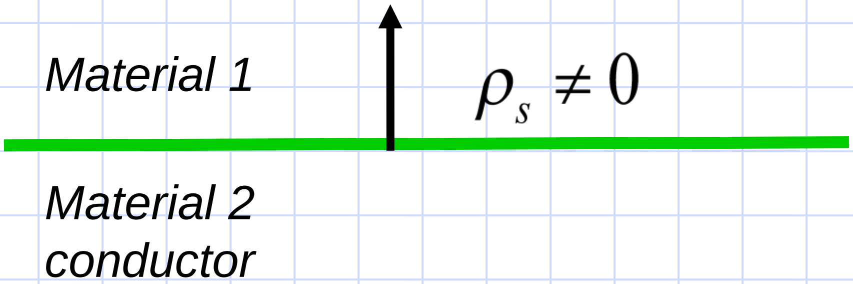
$$= (D_{1n} - D_{2n}) \cdot A$$

$$\therefore D_{1n} - D_{2n} = \rho_s$$

# Boundary Conditions

Case 1: REGION 2 is a CONDUCTOR, $D_2 = E_2 = 0$

$$\therefore D_{1n} = \rho_s$$

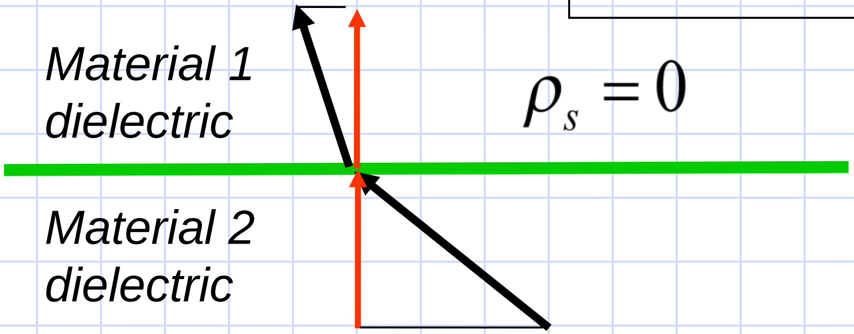



Case 2: REGIONS 1 & 2 are DIELECTRICS with $\rho_s$ = 0

Can only really get $\rho_s$ with conductors

$$\therefore D_{1n} = D_{2n}$$

$$\therefore \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

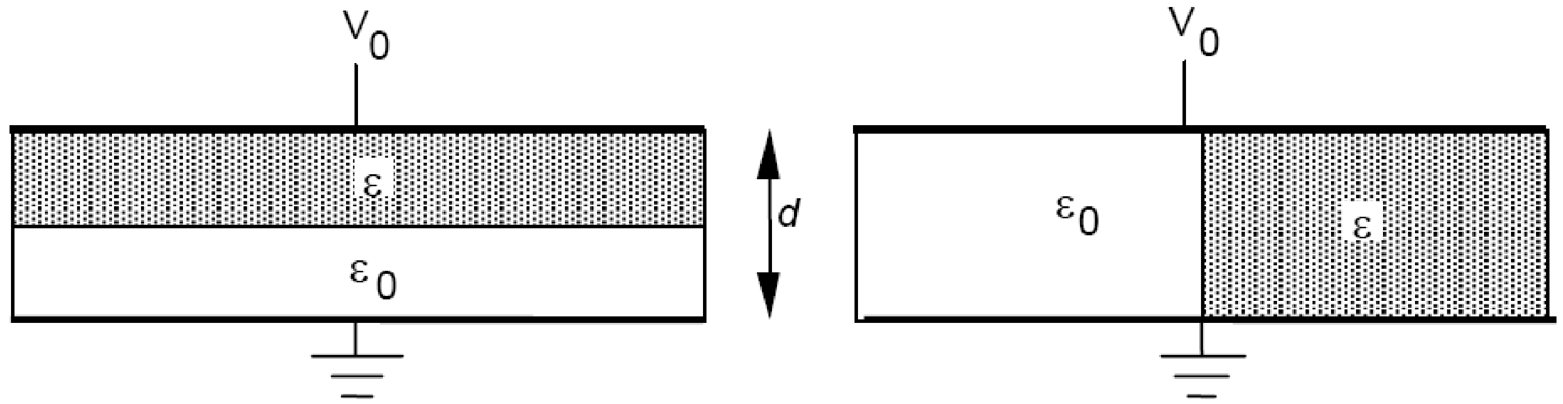


# Boundary Conditions

Consider the two parallel plate geometries below. Assume that the plate dimensions are large compared to the separation *d* and ignore fringe effects. For the two figures, the electric field in the air region, (specified by $\varepsilon_o$) is given by:

$\mathbf{E} = -(V_0/d) * (2\varepsilon_r/(1+\varepsilon_r))\ \mathbf{a}_z$ figure on left

$\mathbf{E} = -(V_0/d)\ \mathbf{a}_z$ figure on right



# Boundary Conditions

- For both cases, find **E** in the dielectric region. Find **D** in both regions. Within a given region, **D** and **E** do not vary with position.
- Find the charge density on the plates at all locations.

# Boundary Conditions

a. Boundary conditions $E_{1t} = E_{2t}$ ; $D_{1n} = D_{2n} \Rightarrow E_{2n} = \frac{\varepsilon_1}{\varepsilon_2} E_{1n}$

Left : E is normal $\therefore \vec{E}_{diel} = \frac{\varepsilon_o}{\varepsilon_r \varepsilon_o} E_{air} = \boxed{-\frac{V_o}{d} \frac{2}{1+\varepsilon_r} \hat{a}_z}$

Right : E is tangential $\therefore \vec{E}_{diel} = \vec{E}_{air} = \boxed{-\frac{V_o}{d} a_z}$

$\vec{D} = \epsilon E$

Left: $\boxed{\vec{D}_{diel} = -\frac{V_o}{d} \frac{2\varepsilon_r \varepsilon_o}{1+\varepsilon_r} \hat{a}_z}$

same

$\vec{D}_{air} = \varepsilon_o \vec{E}_{air}$

$\boxed{\vec{D}_{air} = -\frac{V_o}{d} \frac{2\varepsilon_r \varepsilon_o}{1+\varepsilon_r} \hat{a}_z}$

Right: $\boxed{\vec{D}_{diel} = -\frac{\varepsilon_r \varepsilon_o V_o}{d} \hat{a}_z}$

$\boxed{\vec{D}_{air} = -\frac{\varepsilon_o V_o}{d} \hat{a}_z}$

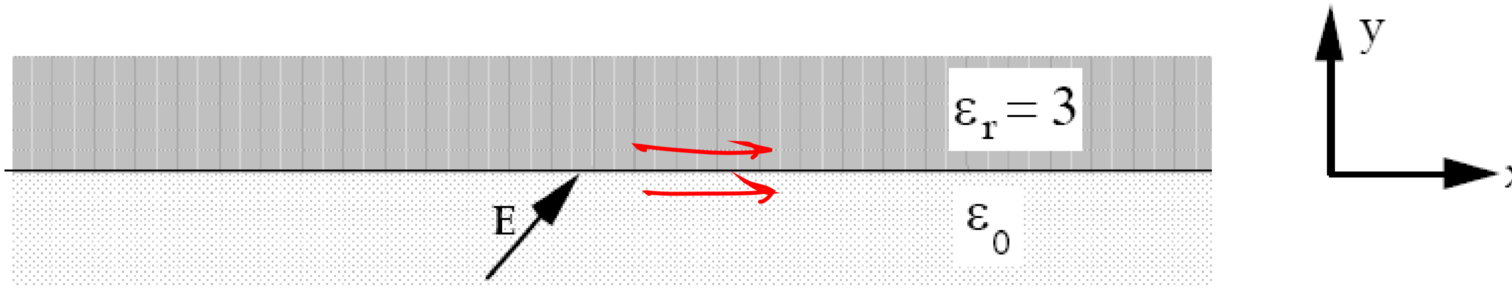
b. Boundary conditions at conductor-dielectric $D_n = \rho_s$

Left : $\boxed{\rho_s = \pm \frac{2\varepsilon_r \varepsilon_o}{1+\varepsilon_r} \frac{V_o}{d}}$ + on top
− on bottom

Right : $\boxed{\rho_{s,diel} = \pm \frac{\varepsilon_r \varepsilon_o V_o}{d}}$ $\boxed{\rho_{s,air} = \pm \frac{\varepsilon_o V_o}{d}}$ + on top
− on bottom

# Boundary Conditions

The **E** field on the air side of a dielectric-dielectric boundary is **E** = 100 $\mathbf{a}_x$ + 100 $\mathbf{a}_y$. What is **E** on the dielectric side?



# Boundary Conditions

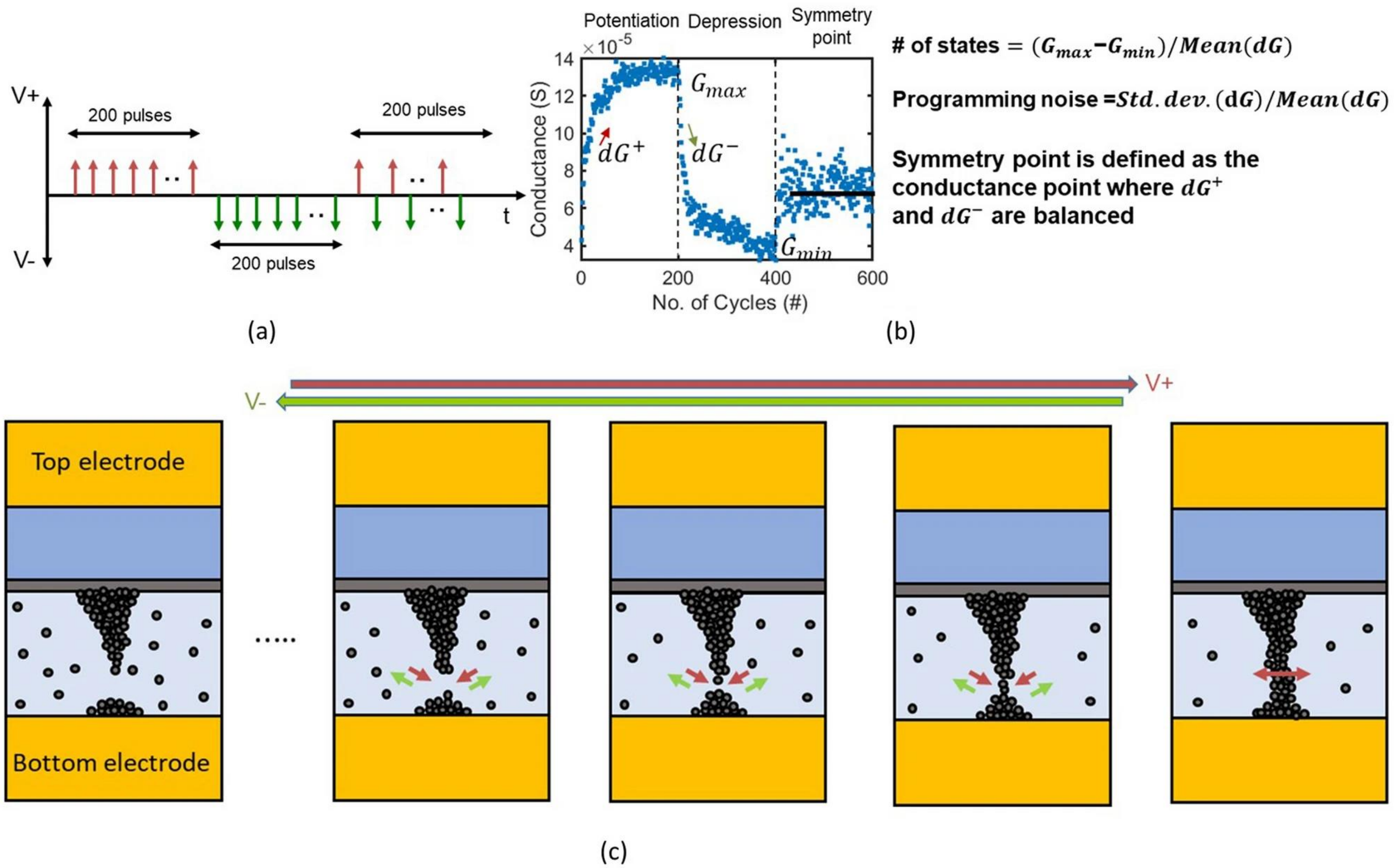
$E_{1t} = E_{2t} \Rightarrow E_{1x} = E_{2x} \Rightarrow \therefore E_{2x} = 100$

Air = Region 1
Diel = Region 2

$D_{1n} = D_{2n} \Rightarrow \varepsilon_o E_{1y} = 3\varepsilon_o E_{2y} \Rightarrow E_{2y} = \frac{E_{1y}}{3} = \frac{100}{3} = 33\frac{1}{3}$

$$\boxed{\vec{E}_2 = 100\,\hat{a}_x + 33\tfrac{1}{3}\,\hat{a}_y}$$

# Dielectric Breakdown as a Device Principle




Abedin, Minhaz, et al. "Material to system-level benchmarking of CMOS-integrated RRAM with ultra-fast switching for low power on-chip learning." *Scientific Reports* 13.1 (2023): 14963.