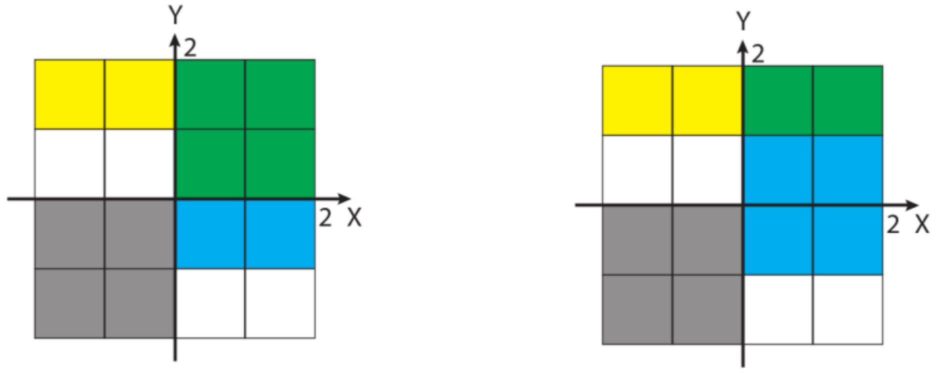
Rensselaer Polytechnic Institute
Department of Electrical, Computer, and Systems Engineering
ECSE 2500: Engineering Probability, Fall 2022
Homework #2 Solutions

1. The events and their intersections are sketched below.



The left subfigure shows events A (blue + green squares) and B (yellow and green squares). We can visually read off that

$$P(A) = \tfrac{1}{2} \qquad P(A \cap B) = \tfrac{1}{3}$$

$$P(B) = \tfrac{1}{2}$$

Since $P(A \cap B) \neq P(A)P(B)$, events A and B are not independent.

Grading criteria: 5 points in total
-1 point: 1 probability or conclusion is incorrect or missing
-2 point: 2 probabilities or conclusion are incorrect or missing
-3 point: 3 probabilities or conclusion are incorrect or missing
-4 point: 3 probabilities and conclusion are incorrect or missing
-5 point: all outcomes are incorrect or missing

The right subfigure shows events A (blue + green squares) and C (yellow and green squares). Again, we can visually read off that

$$P(A) = \tfrac{1}{2} \qquad P(A \cap C) = \tfrac{1}{6}$$

$$P(C) = \tfrac{1}{3}$$

Since $P(A \cap C) = P(A)P(C)$, events A and C are independent. Another way of thinking of this is that event C makes up the same proportion of the sample space as it does of event A (and vice versa).

Grading criteria: 5 points in total
-1 point: 1 probability or conclusion is incorrect or missing
-2 point: 2 probabilities or conclusion are incorrect or missing
-3 point: 3 probabilities or conclusion are incorrect or missing

-4 point: 3 probabilities and conclusion are incorrect or missing
-5 point: all outcomes are incorrect or missing

Events B and C are definitely not independent since event C is a subset of B. That is, if C happened, we know for sure that B happened.

Grading criteria: 5 points in total
-1 point: 1 probability or conclusion is incorrect or missing
-2 point: 2 probabilities or conclusion are incorrect or missing
-3 point: 3 probabilities or conclusion are incorrect or missing
-4 point: 3 probabilities and conclusion are incorrect or missing
-5 point: all outcomes are incorrect or missing

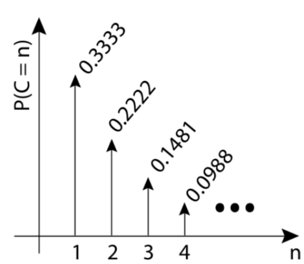
2. (a) This is like a sequence of coin flips, where we keep flipping the coin until the first win, so it's a geometric random variable.

Grading criteria: 5 points in total
-2 point: some outcomes are incorrect or missing
-5 point: all outcomes are incorrect or missing

(b) The event $C = n$ means that we fail $n-1$ times in a row and succeed on the nth try, i.e.,

$$p_C(n) = \left(\frac{2}{3}\right)^{n-1} \cdot \frac{1}{3}, \quad n = 1, 2, \ldots, \infty$$

Note that the possible values for n start at 1 (i.e., we must defend at least one case).
The first few values are $p_C(1) = 0.3333$, $p_C(2) = 0.2222$, $p_C(3) = 0.1481$, and $p_C(4) = 0.0988$. The PMF is sketched below.



Grading criteria: 10 points in total
-2 point: 1-2 steps are incorrect or missing
-4 point: 3-4 steps are incorrect or missing
-6 point: 5-6 steps are incorrect or missing
-10 point: all steps are incorrect or missing

(c) Remember that the formula for a finite geometric sum is:

$$\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a}$$

Applying this to our problem, we have:

$$
\begin{aligned}
\sum_{k=1}^{10} P(C=k) &= \sum_{k=1}^{8} \left(\frac{2}{3}\right)^{k-1} \cdot \frac{1}{3} \\
&= \frac{1}{3} \sum_{k=1}^{8} \left(\frac{2}{3}\right)^{k-1} \\
&= \frac{1}{3} \sum_{k=0}^{7} \left(\frac{2}{3}\right)^{k} \\
&= \frac{1}{3} \frac{1-\left(\frac{2}{3}\right)^8}{1-\frac{2}{3}} \\
&= 1 - 0.0390 \\
&= 0.9610
\end{aligned}
$$

Grading criteria: 15 points in total
-5 point: 1-3 steps are incorrect or missing
-10 point: 4-6 steps are incorrect or missing
-15 point: all steps are incorrect or missing

3. (a) This is a binomial distribution (number of successes in a given number of trials).

Grading criteria: 5 points in total
-2 point: some outcomes are incorrect or missing
-5 point: all outcomes are incorrect or missing

(b) We compute the probability of getting exactly 4 wins in 7 cases as

$$
\begin{aligned}
P(W=4) &= \binom{7}{4} \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^4 \\
&= 35 \cdot \frac{8}{2187} \\
&= \frac{280}{2187} \\
&= 0.1280
\end{aligned}
$$

Grading criteria: 10 points in total
-2 point: 1-2 steps are incorrect or missing
-4 point: 3-4 steps are incorrect or missing
-6 point: 5-6 steps are incorrect or missing
-10 point: all steps are incorrect or missing

(c) To compute $P(W \geq 5)$ we need to add up several outcomes:

$$\begin{aligned}
P(W \geq 5) &= P(W=5) + P(W=6) + P(W=7) \\
&= \binom{7}{5}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^5 + \binom{7}{6}\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^6 + \binom{7}{7}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^7 \\
&= 21 \cdot \frac{4}{2187} + 7 \cdot \frac{2}{2187} + 1 \cdot \frac{1}{2187} \\
&= \frac{99}{2187} \\
&= 0.0453
\end{aligned}$$

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> -6 point: 5-6 steps are incorrect or missing
> -10 point: all steps are incorrect or missing

(d) We will need the probability that the second case was won in trial 4 **and** exactly 4 cases were won. This means that we have to get exactly one win in the first three cases, one wins in the fourth case, and two win in the final three cases. The probability of this is:

$$\left(\binom{3}{1}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)\right) \cdot \left(\frac{1}{3}\right) \cdot \left(\binom{3}{2}\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^2\right) = \frac{8}{243}$$

From (b) we know

$$\begin{aligned}
P(W=4) &= \binom{7}{4}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^4 \\
&= 35 \cdot \frac{8}{2187} \\
&= \frac{280}{2187} \\
&= 0.1280
\end{aligned}$$

The conditional probability is thus

$$\begin{aligned}
P(\text{2nd win on case 4} \mid W=4) &= \frac{P(\text{2nd win on case 4 and } W=4)}{P(W=4)} \\
&= \frac{\frac{8}{243}}{\frac{280}{2187}} \\
&= \frac{9}{35}.
\end{aligned}$$

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> -6 point: 5-6 steps are incorrect or missing
> -10 point: all steps are incorrect or missing

4. Bernoulli's theorem says that

$$P(\left(\left|\frac{k}{n}-p\right|>\epsilon\right)<\frac{p(1-p)}{n\epsilon^2}$$

or equivalently

$$P(\left(\left|\frac{k}{n}-p\right|<\epsilon\right)>1-\frac{p(1-p)}{n\epsilon^2}$$

Applying this to our problem, $p=\frac{1}{3}$ and $n=6000$. We expect to get $6000\cdot\frac{1}{3}=2000$ wins, so the condition $k\in[1800,2200]$ is equivalent to $\left|\frac{k}{n}-p\right|<\frac{200}{6000}$, making $\epsilon=\frac{1}{30}$. So the lower bound from the right hand side is $1-\frac{2}{3}\cdot\frac{1}{3}\cdot\frac{3}{20}=\frac{29}{30}=0.97$. So this is pretty likely! (Later in the semester we'll see how we can get an even more accurate value for this probability).

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> -10 point: all steps are incorrect or missing

5. (a) In a 2-hour practice, we expect an average of 4 jokes, so the random variable has PMF

$$P(X=k)=\frac{4^k}{k!}e^{-4}, k=0,1,2,\ldots$$

We want to compute

$$P(X=0)=\frac{4^0}{0!}e^{-4}=e^{-4}=0.0183$$

> Grading criteria: 5 points in total
> -2 point: some outcomes are incorrect or missing
> -5 point: all outcomes are incorrect or missing

(b) In a 90-minute game, we expect an average of 3 jokes, so the random variable has PMF

$$P(X=k)=\frac{3^k}{k!}e^{-3}, k=0,1,2,\ldots$$

We want to compute

$$\begin{aligned}P(X\geq 3)&=1-P(X=0)-P(X=1)-P(X=2)\\&=1-\frac{(3)^0}{0!}e^{-3}-\frac{(3)^1}{1!}e^{-3}-\frac{(3)^2}{2!}e^{-3}\\&=1-0.0498-0.1494-0.2240\\&=0.5768\end{aligned}$$

Grading criteria: 5 points in total
-2 point: some steps are incorrect or missing
-5 point: all steps are incorrect or missing