Fields and Waves I

Lecture 2

Sine Waves on Transmission Lines

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Materials from other sources are referenced where they are used.
Those listed as Ulaby are figures from Ulaby's textbook.

Overview

- Course Administration
- Transmission Line History
- Transmission Line Model
- Traveling Waves
- Phasor Notation
- Wave Reflection
- Upcoming Studio Sessions
- Wrap-Up

Course Administration

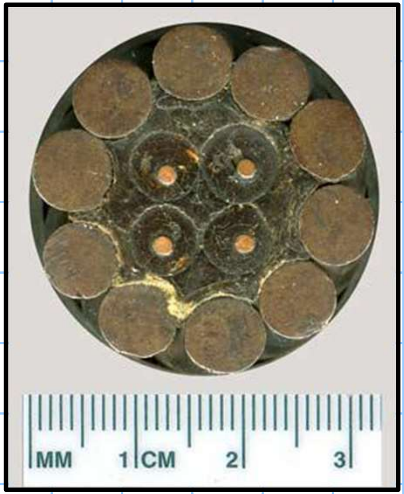
- Optional midterm retest exam: **7pm-9pm, Monday 11/6, Amos Eaton 214**
- Retest slots: **1pm-4pm Thursday, 1pm-4pm Friday, alternating weeks**
- 6pm Wednesday lab slot
- PLEASE DO the following on Gradescope if you haven't already done so:
  - Academic Integrity Policy
  - Digital Tools Acknowledgement

Transmission Line History

- Circuit theory was developed in the 18th and 19th centuries.
- One of the first practical applications was the telegraph (Francis Ronalds, 1816)
- Following this discovery, more and more long conductors were built for telegraphy

Transmission Line History

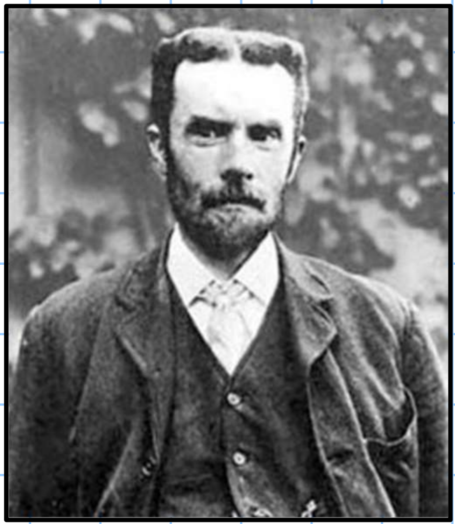
- As telegraphy expanded, there was demand for longer and longer cables
- First working undersea telegraph line was successfully laid across the English Channel in 1851. Other European undersea cables were then built, and the work of building a transatlantic cable began in the mid 1850s.
- These cables suffered from problems:
  - Voltage drop due to resistance
  - Signal delay
  - Signal distortion



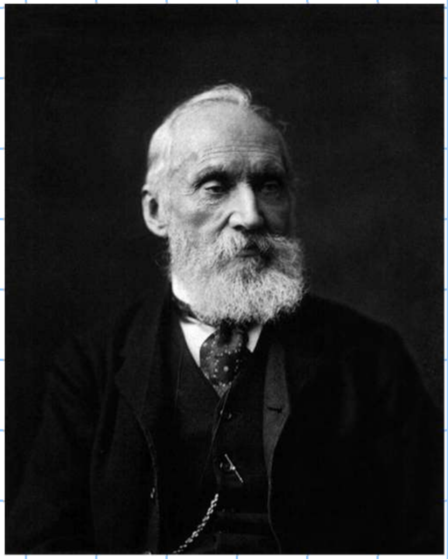
English Channel cable

Transmission Line History

- Several scientists and engineers worked to develop a formal electrical model of long conductors
- The equations that made up this model became known as the Telegrapher's Equations
- These equations allowed for intercontinental telegraph lines to develop, and they are used today for many new applications such as CPU microstrip lines



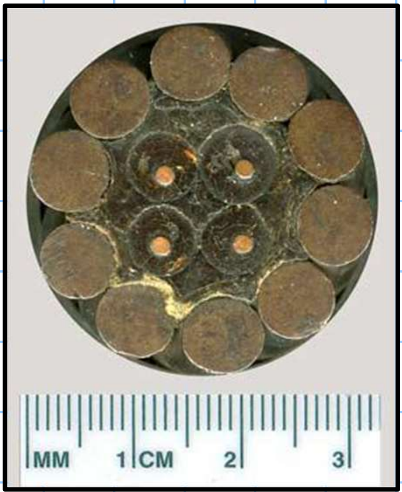
Oliver Heaviside



Lord Kelvin

Transmission Line Model

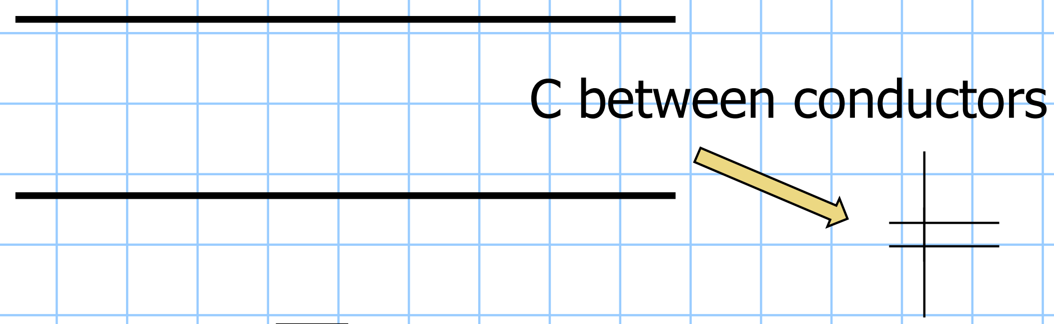
- The purpose of a transmission line is to **transmit signal and/or power from one place to another.**
- Examples include:
  - Power Lines (60Hz)
  - Coaxial Cables
  - Twisted Pairs
  - Interconnects inside a computer
- Note that these all have **two conductors.**



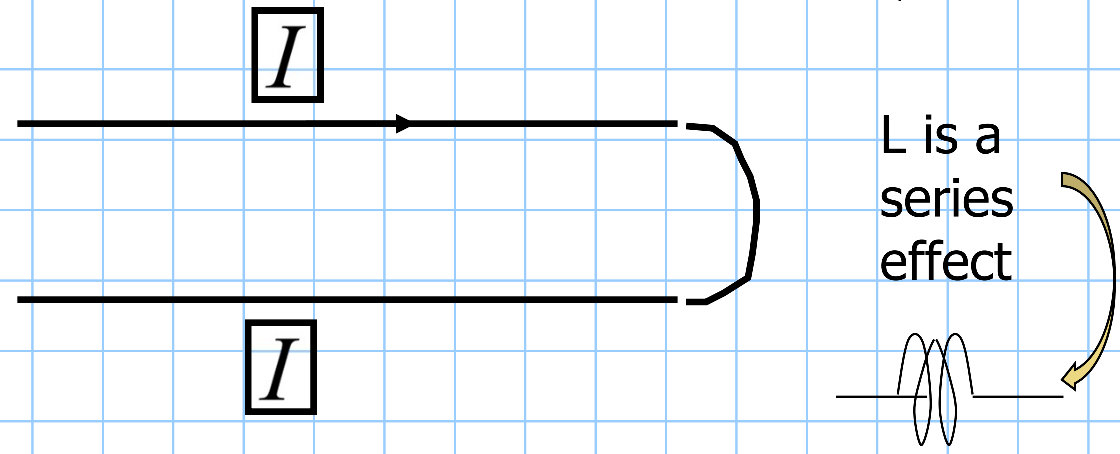
- In transmission line theory, we represent a transmission line as having certain properties **per unit length**.

Transmission Line Model

Cables have both ***L*** and ***C***:

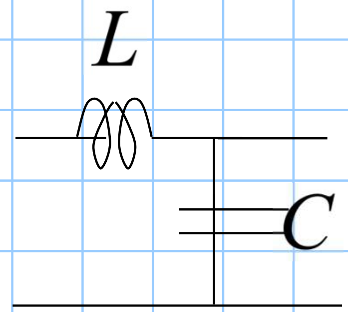


2 wire example:

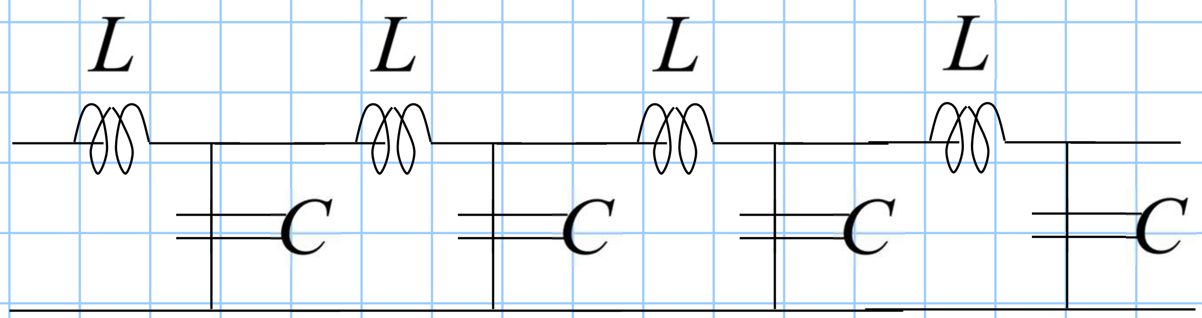


Transmission Line Model

Model of a short section:



Model the full length as:

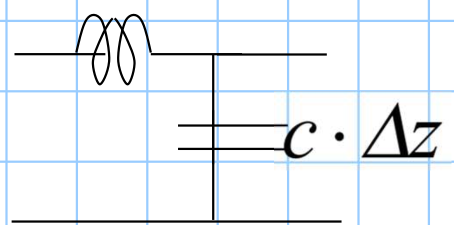


....etc.

Transmission Line Model

Since ***L*** and ***C*** are actually distributed through the length of the cable rather than discrete lumps of inductance/capacitance:

$l \cdot \Delta z$, represents a length of cable

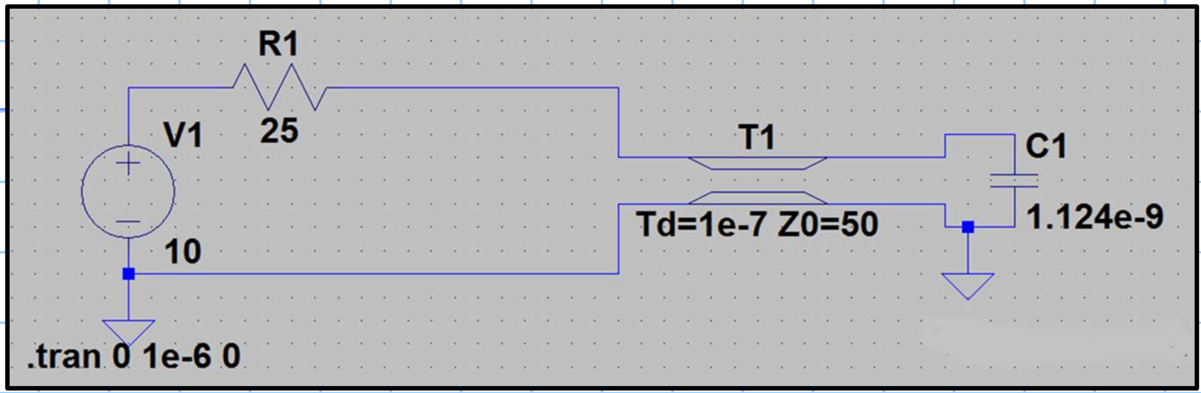


l = inductance/length

c = capacitance/length

This model works as long as **$\Delta z << \lambda$**

Transmission Line Model



- If transmission lines have no losses, we can treat them as a "black box" with just two quantities: the **time delay**, and the **characteristic impedance.**
- The time delay is how long a signal takes to propagate through the line. (Keep in mind that for higher frequency AC signals, a small time delay can mean a big phase change).
- Characteristic impedance is the voltage-to-current ratio for a signal on the line, *or,* what the impedance of the line "looks like" to a signal entering the line.

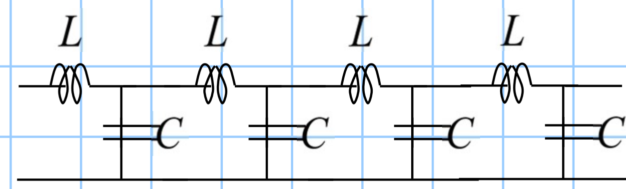
Transmission Line Model

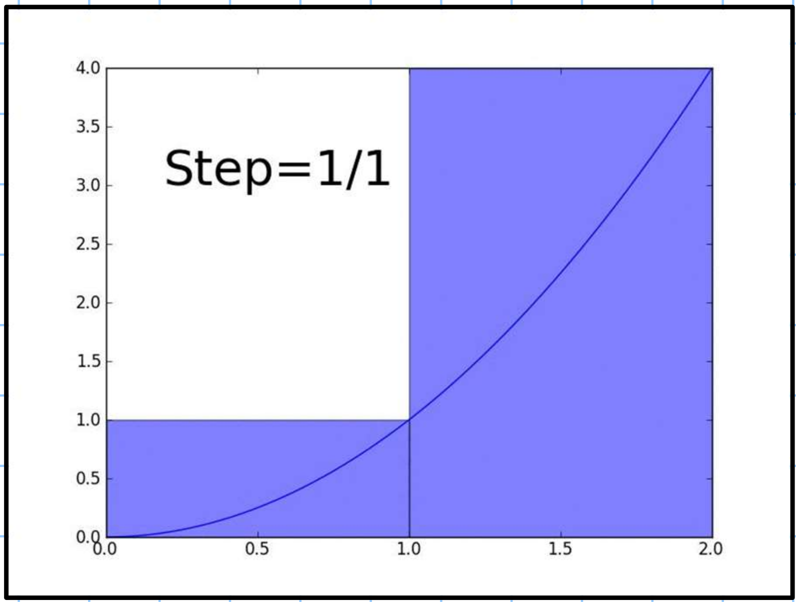
- This simple "black box" analysis is good enough for some purposes, such as calculating how long a signal will take to reach its destination.
- But doing **complete analysis** of transmission lines means being able to calculate the voltage and current anywhere on the line at any time. We do this using the Telegrapher's Equations.

Transmission Line Model

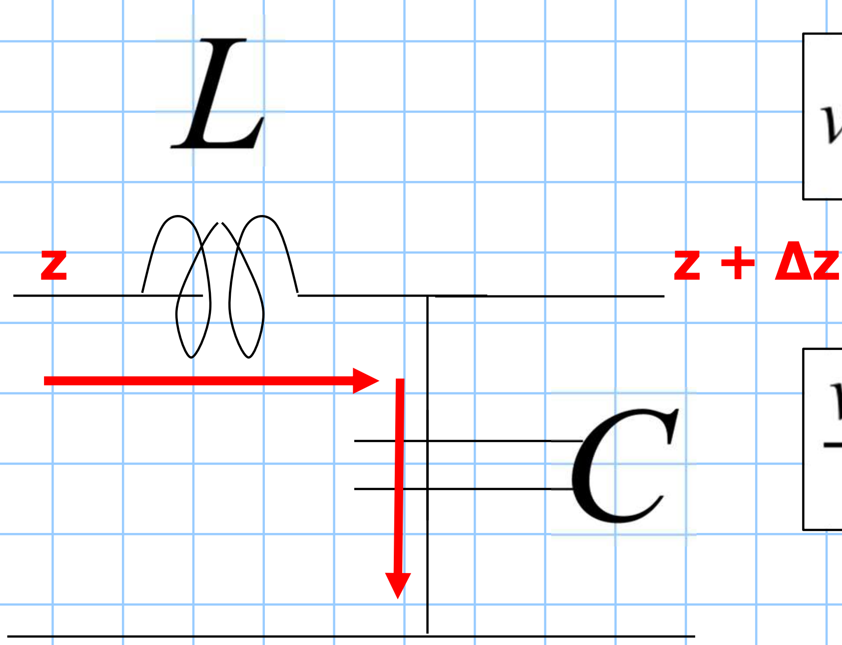
- Heaviside et. al. derived the telegrapher's equations by thinking of the lumped-element transmission line and letting the length that each element represented approach zero.

- This is a bit like the rectangle area approximation method of integrals in calculus.





Telegrapher's Equations



Applying Kirchhoff's voltage law:

$$\boxed{v(z) - l\,\Delta z\,\frac{\partial i}{\partial t} - v(z+\Delta z) = 0}$$

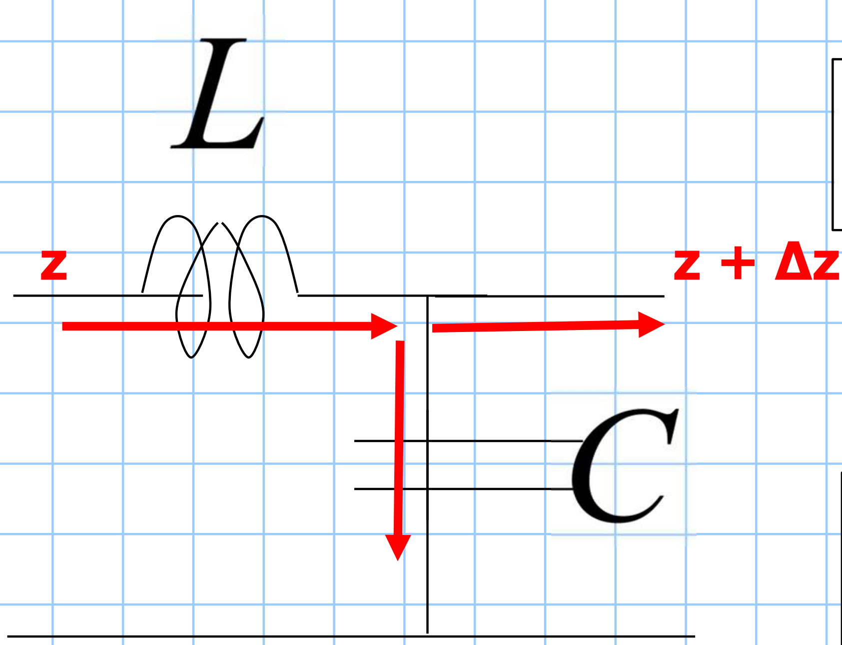
Rearranging:

$$\boxed{\frac{v(z+\Delta z) - v(z)}{\Delta z} = -l\,\frac{\partial i}{\partial t}}$$

Letting Δz → 0:

$$\boxed{\frac{\partial v}{\partial z} = -l\,\frac{\partial i}{\partial t}}$$

Telegrapher's Equations



Applying Kirchhoff's current law:

$$\boxed{i(z) - c\,\Delta z \frac{\partial v}{\partial t} - i(z+\Delta z) = 0}$$

Similarly letting Δz → 0:

$$\boxed{\frac{\partial i}{\partial z} = -c\frac{\partial v}{\partial t}}$$

Telegrapher's Equations

Either x or z could be used to refer to the spatial dimension

$$\frac{\partial}{\partial x} V(x,t) = -L \frac{\partial}{\partial t} I(x,t) \boxed{- R\, I(x,t)}$$

$$\frac{\partial}{\partial x} I(x,t) = -C \frac{\partial}{\partial t} V(x,t) \boxed{- G\, V(x,t)}$$

These equations are clearly coupled… so how do we actually find V(x,t) and I(x,t)?

We did not derive these terms but they appear in the lossy version of the equations. Ignore for now.

Telegrapher's Equations

To find the solution, combine the two equations:

$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial}{\partial z}(-l \cdot \frac{\partial I}{\partial t}) = -l \cdot \frac{\partial}{\partial t}(\frac{\partial I}{\partial z}) = lc\frac{\partial^2 V}{\partial t^2}$$

Obtain the following PDE:

$$\frac{\partial^2 V}{\partial z^2} = lc\frac{\partial^2 V}{\partial t^2}$$

<u>This is a wave equation!</u>

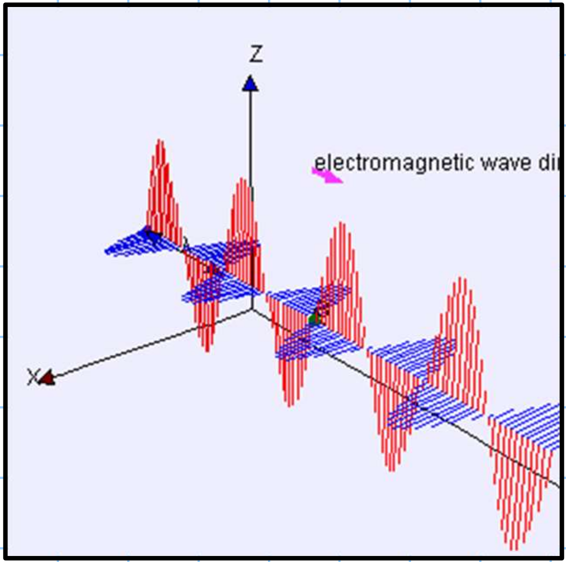
Why Study Electromagnetism (E&M)?

The "Waves" Part of Fields & Waves

Electromagnetic fields propagate as waves. They are governed by the electromagnetic wave equations, which are derived from Maxwell's Equations and can be written as:

$$\frac{\partial^2 \vec{E}}{\partial t^2} = c^2 \nabla^2 \vec{E} \qquad \frac{\partial^2 \vec{B}}{\partial t^2} = c^2 \nabla^2 \vec{B}$$

where **c** is the speed of light / EM waves. (Note that c depends on the medium the wave is propagating through!)



Source: https://en.wikipedia.org/wiki/Electromagnetic_radiation

Telegrapher's Equations

$$\frac{\partial^2 V}{\partial z^2} = lc\frac{\partial^2 V}{\partial t^2} \rightarrow \frac{\partial^2 V}{\partial t^2} = \frac{1}{lc}\frac{\partial^2 V}{\partial z^2} \rightarrow \frac{\partial^2 V}{\partial t^2} = u^2\frac{\partial^2 V}{\partial z^2}$$

In this standard form of the wave equation, u represents velocity. So for our transmission line voltage waves,

$$u = \frac{1}{\sqrt{lc}}$$

Traveling Waves

Sinusoids represent a solution to these equations:

$$V(t,z) = A\cos(\omega t \mp (\omega/u)z) = A\cos(\omega t \mp \boldsymbol{\beta} z)$$

$$\omega = 2\pi f \qquad \beta = \frac{\omega}{u}$$

phase constant

$$\text{(change in phase per unit length)} = \frac{\text{angular velocity (change in phase per unit time)}}{\text{wave velocity (unit length per unit time)}}$$

Traveling Waves

Sinusoids represent a solution to these equations:

$$V(t,z) = A\cos(\omega t \mp (\omega/u)z) = A\cos(\omega t \mp \boldsymbol{\beta} z)$$

$$\omega = 2\pi f \qquad \beta = \frac{\omega}{u}$$

Also note that the phase constant (and therefore velocity) can be either positive or negative. This represents the fact that the wave can be traveling either forward or backward.

Traveling Waves

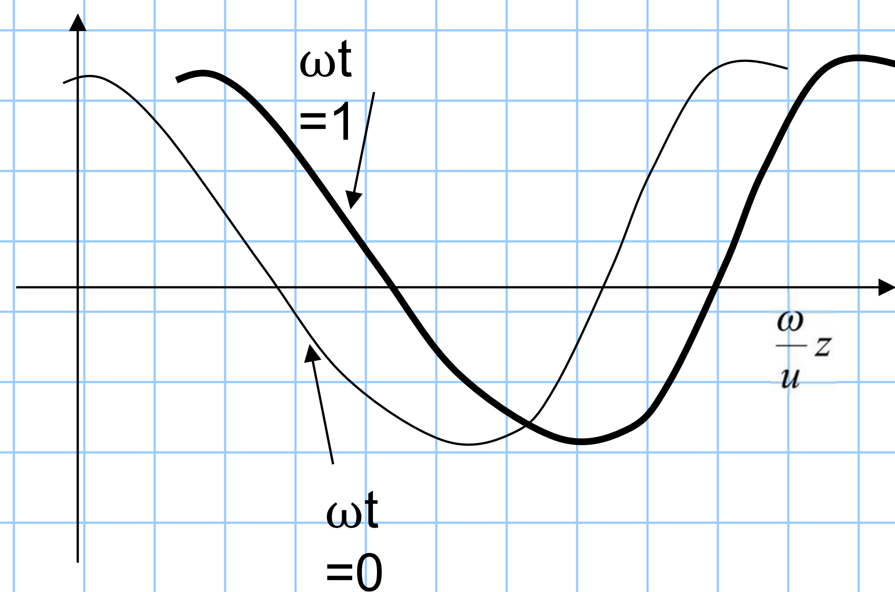
Functions that move with velocity ***u***

Example: $cos(\omega t \pm \frac{\omega}{u} z)$

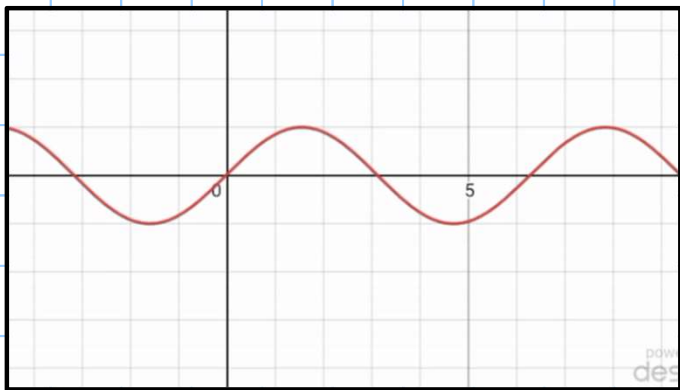
At t=0, $cos(-\frac{\omega}{u} z)$

Wave moving to the right

At ωt =1 $cos(1-\frac{\omega}{u} z)$



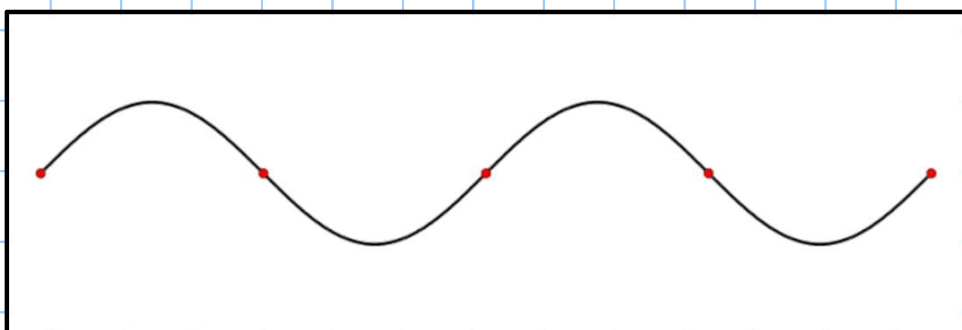
Traveling Waves



Traveling wave (depends on t and z in the argument of one sinusoid)

$$A \cos(\omega t \mp \boldsymbol{\beta} z)$$

in other words you can relate t and z with u.

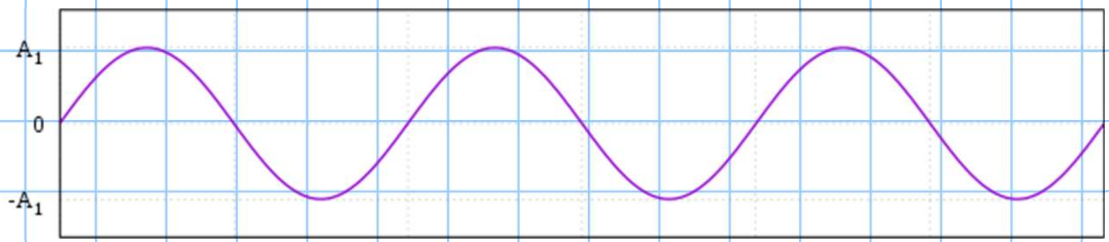


Standing wave (depends on t and z but as two different terms)

$$A \sin(\omega t)\cos(\boldsymbol{\beta} z)$$

No relationship between t and z.

Traveling Waves



Static sine wave varies in space but not time.

A cos ($\boldsymbol{\beta}$z)

What would A sin(ωt) look like?

Traveling Waves

$$\cos\left(\omega t - \frac{\omega}{u} z\right) = \cos\left(2\pi f t - \frac{2\pi f}{u} z\right)$$

- Consider one other property. What is the distance required to change the phase of this expression by 2π ?

Traveling Waves

$$\cos\left(\omega t - \frac{\omega}{u} z\right) = \cos\left(2\pi f t - \frac{2\pi f}{u} z\right)$$

- Consider one other property. What is the distance required to change the phase of this expression by 2π ?
- In other words, this is an expression for wavelength.

$$\beta z = \frac{\omega}{u} z = \frac{2\pi f}{u} z = 2\pi$$

$$\beta\lambda = \frac{\omega}{u}\lambda = \frac{2\pi f}{u}\lambda = 2\pi \qquad \lambda = \frac{2\pi}{\beta} = \frac{u}{f}$$

Traveling Waves

Some Properties Summarized

Solutions look like $A \cos(\omega t \mp \boldsymbol{\beta} z)$

$$\beta = \frac{\omega}{u} = \omega\sqrt{lc} = \frac{2\pi}{\lambda}$$

$$\omega = 2\pi f = \frac{2\pi}{T}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{u}{f} \qquad u = \frac{1}{\sqrt{lc}}$$

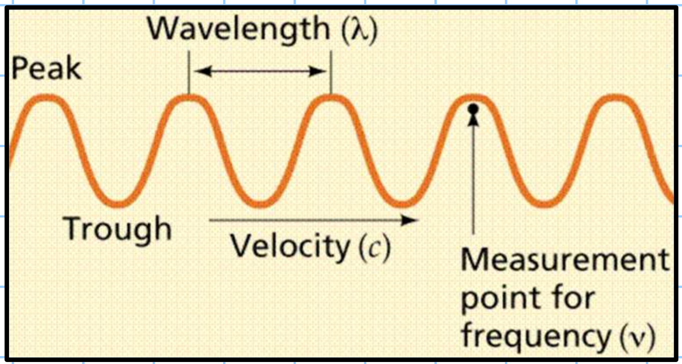


Figure from http://www.emc.maricopa.edu/

Traveling Waves

Do Lecture 4 Exercise 1 on Gradescope in groups of up to 4.

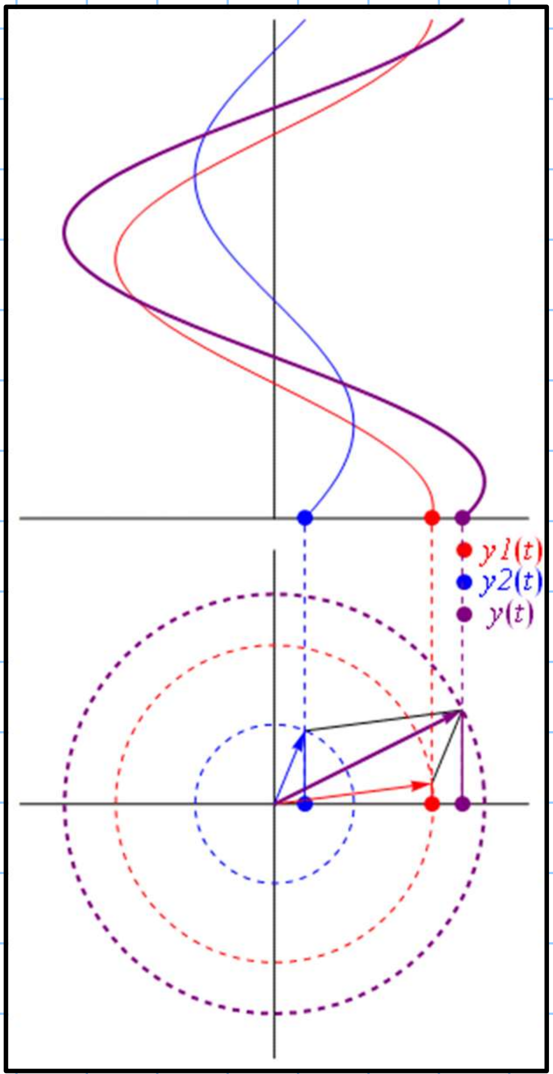
Phasor Notation

- What are phasors?
- In short, they are the **periodically time-variant version of vectors.**
- Working in phasor domain temporarily collapses the periodic time-variant component, allowing a linear expression to be written.
- This is useful for solving current or voltage expressions in linear circuits (see Ulaby pg. 37-38)
- Especially useful in power engineering, where electric grids can contain *many* different time-varying voltages/currents

$$\boxed{A\cos\left(\omega t+\beta x+\phi_0\right)\Leftrightarrow \tilde{A}e^{j\left(\beta x+\phi_0\right)}}$$

Phasor Notation

- At right is a visualization of phasor addition.
- Working in phasor domain allows you to add these two sinusoidal functions together easily without dealing with sine and cosine addition formulas, etc.
- Many useful properties can be derived for converting between phasor and time domain (see Ulaby pg. 41)



https://en.wikipedia.org/wiki/File:Sumafasores.gif

Phasor Notation

- To convert to space-time form from the phasor form, multiply by $e^{j\omega t}$ and take the real part.

$$\boxed{f(z,t) = \mathrm{Re}[\tilde{A}e^{\mp j\beta z}e^{j\omega t}] = A\cos(\omega t \mp \beta z)}$$

- If A is complex,

$$\tilde{A} = |\tilde{A}|e^{j\theta_A}$$

$$\boxed{f(z,t) = \mathrm{Re}[|\tilde{A}|e^{j\theta_A}e^{\mp j\beta z}e^{j\omega t}] = |\tilde{A}|\cos(\omega t \mp \beta z + \theta_A)}$$

Example

A phasor voltage is given by:

$$\tilde{V} = j5$$

How do we find V(z,t)?

Example

How do we find V(z,t)?

$$\tilde{V} = j5 = 5e^{j\frac{\pi}{2}}$$

$$V(z,t) = Re\left\{5e^{j\omega t}e^{j\frac{\pi}{2}}\right\}$$

$$V(z,t) = 5cos(\omega t + \frac{\pi}{2}) = -5sin(\omega t)$$

Phasor Notation

$$\frac{\partial i}{\partial z} = -c\frac{\partial v}{\partial t}$$

$$\frac{\partial v}{\partial z} = -l\frac{\partial i}{\partial t}$$

We have the following phasor property:

$$\frac{d}{dt}(z(t)) \Leftrightarrow j\omega\widetilde{Z}$$

Therefore:

$$-\frac{d\widetilde{V}(z)}{dz} = j\omega l\,\widetilde{I}(z)$$

$$-\frac{d\widetilde{I}(z)}{dz} = j\omega c\,\widetilde{V}(z)$$

Phasor Notation

Combining the two equations, we get two wave equations:

$$\boxed{\frac{\partial^2 \widetilde{V}(z)}{\partial z^2} - (j\omega l)(j\omega c)\widetilde{V}(z) = 0}$$

$$\boxed{\frac{\partial^2 \widetilde{I}(z)}{\partial z^2} - (j\omega l)(j\omega c)\widetilde{I}(z) = 0}$$

We make a substitution:

$$\boxed{u^2 = (j\omega l)(j\omega c)} \qquad \boxed{u = \sqrt{(j\omega l)(j\omega c)}}$$

Phasor Notation

General solutions of these wave equations:

$$\boxed{\tilde{V}(z) = V_o^+ e^{-uz} + V_o^- e^{uz}}$$

$$\boxed{\tilde{I}(z) = I_o^+ e^{-uz} + I_o^- e^{uz}}$$

Combining these with our earlier equations...

$$\boxed{-\frac{d\tilde{V}(z)}{dz} = j\omega l\,\tilde{I}(z)} \qquad \boxed{-\frac{d\tilde{I}(z)}{dz} = j\omega c\,\tilde{V}(z)}$$

... we can now derive:

$$\boxed{\tilde{I}(z) = \frac{u}{j\omega l}\left(V_o^+ e^{-uz} - V_o^- e^{uz}\right)}$$

note the minus sign that appears for the backwards traveling current wave!

Phasor Notation

Our two phasor expressions are now related....

$$\boxed{\widetilde{V}(z) = V_o^+ e^{-uz} + V_o^- e^{uz}}$$

$$\boxed{\widetilde{I}(z) = \frac{u}{j\omega l}\left(V_o^+ e^{-uz} - V_o^- e^{uz}\right)}$$

... such that the voltage/current ratio is a constant. (In other words, we have an expression for **impedance!**)

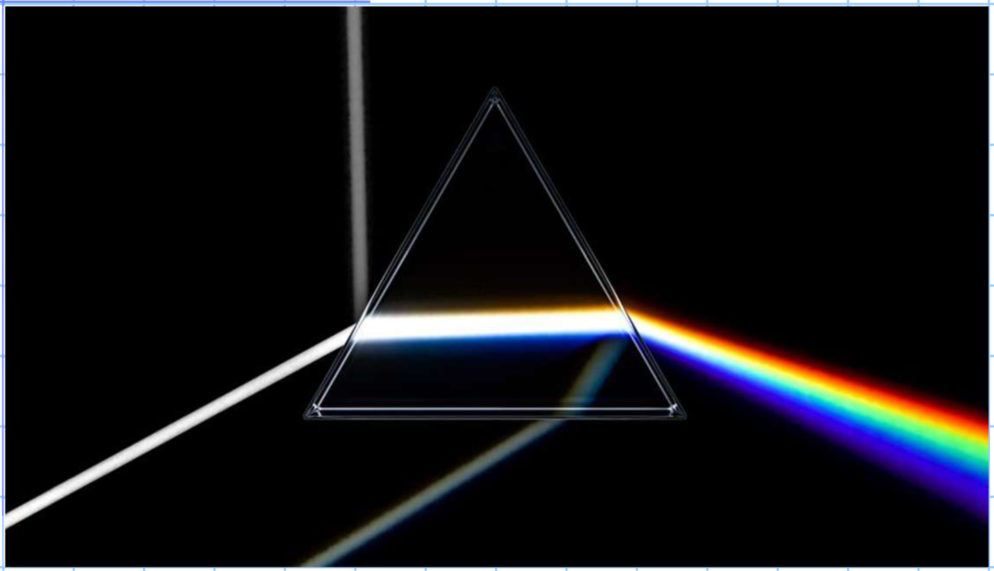
$$\frac{V_0^+}{I_0^+} = \frac{-V_0^-}{I_0^-} = \frac{j\omega l}{u} = \frac{j\omega l}{\sqrt{(j\omega l)(j\omega c)}} = \sqrt{\frac{l}{c}}$$

Phasor Notation

- As we discussed earlier, one high-level properties of transmission lines is their **characteristic impedance.**
- Characteristic impedance can be described in a few ways:
  - The impedance that a signal "sees" at the input of transmission line when the load is far away
  - The ratio of voltage and current for a signal propagating on the line (think Ohm's Law)

$$Z_0 = \sqrt{\frac{L'}{C'}}$$

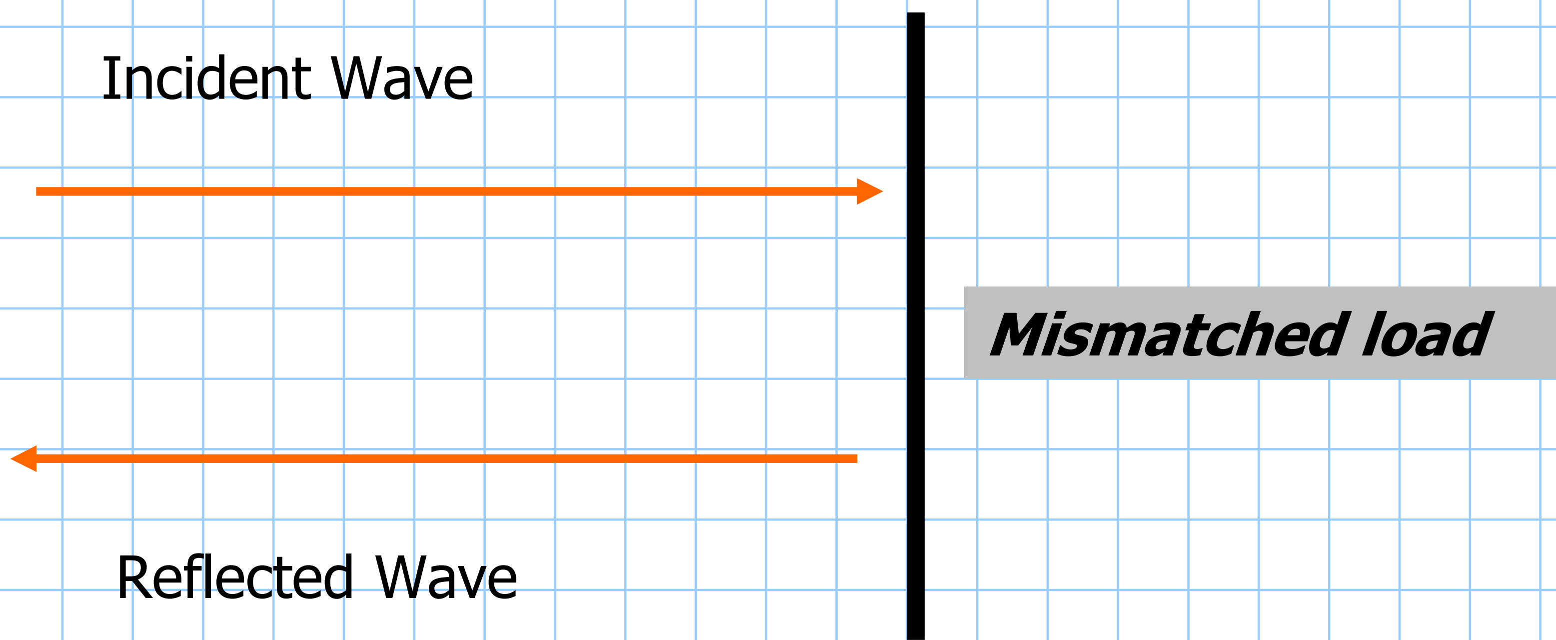
Wave Reflection



- When light enters a prism, some light will be transmitted in while some will be reflected. This is because the impedance of the prism to light is different to that of air.

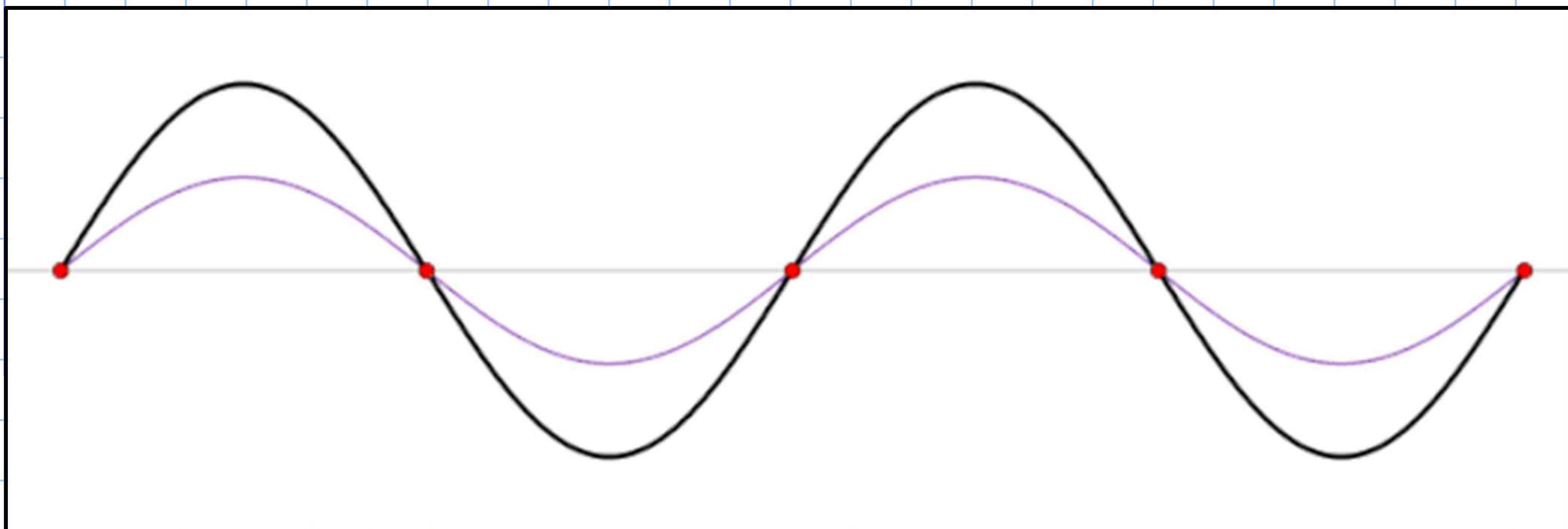
- In the same manner, a signal that travels down a transmission line will reflect at the load if the characteristic impedance of the line and the impedance of the load are not the same.

Wave Reflection



These two waves interfere with each other and produce a **standing wave.**

Wave Reflection



These two waves interfere with each other and produce a **standing wave.**

Wave Reflection

$$V = V^{+} e^{-j\cdot\beta\cdot z} + V^{-} e^{+j\cdot\beta\cdot z}$$

Forward Wave

$cos(\omega \cdot t - \beta \cdot z)$

Backward Wave

$cos(\omega \cdot t + \beta \cdot z)$

TIME DOMAIN

V_{max} occurs when Forward and Backward Waves are in Phase

⇨ *CONSTRUCTIVE INTERFERENCE*

V_{min} occurs when Forward and Backward Waves are out of Phase

⇨ *DESTRUCTIVE INTERFERENCE*

Wave Reflection

Define the Reflection Coefficient:

$$\boxed{V_0^+ \Gamma = V_0^-}$$

This can be done in terms of either current or voltage:

$$\boxed{\Gamma = \frac{V_0^-}{V_0^+} = \frac{-I_0^-}{I_0^+}}$$

Wave Reflection

We can define load impedance as follows:

$$\boxed{Z_L = \frac{\tilde{V}_L}{\tilde{I}_L}}$$

Meanwhile we previously said:

$$\boxed{\tilde{V}(z) = V_o^+ e^{-uz} + V_o^- e^{uz}}$$

$$\boxed{\tilde{I}(z) = I_o^+ e^{-uz} + I_o^- e^{uz}}$$

Wave Reflection

When z=0 (i.e. we are at the load), we get:

$$\widetilde{V}(0) = V_o^+ + V_o^-$$

$$\tilde{I}_L = \tilde{I}(z=0) = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}$$

Doing some algebra...

$$V_o^- = \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) V_o^+$$

Wave Reflection

When z=0 (i.e. we are at the load), we get:

$$\boxed{\tilde{V}(0) = V_o^+ + V_o^-}$$

$$\boxed{\tilde{I}(0) = I_o^+ - I_o^- = \frac{I_o^+}{Z_o} - \frac{I_o^-}{Z_o}}$$

Doing some algebra...

$$\boxed{V_o^- = \left(\boxed{\frac{Z_L - Z_0}{Z_L + Z_0}}\right) V_o^+} \rightarrow \Gamma$$

Wave Reflection

At short circuit, load impedance is zero.

$$\boxed{\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{0 - Z_0}{0 + Z_0} = -1}$$

At open circuit, load impedance is *very big.*

$$\boxed{\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{very\ big - Z_0}{very\ big + Z_0} = 1}$$

Wrap-Up

- **PLEASE DO the following on Gradescope if you haven't already done so:**
  - Academic Integrity Policy
  - Digital Tools Acknowledgement
  - Fields & Waves Intro Survey

- Come to class on Wednesday **with LTSpice installed** so we can do some demos

Wrap-Up

- Come to class on Monday **with LTSpice installed** so we can do some demos
- First Studio Session will be next Wednesday at the same Zoom link we use for lecture.
- HW 1 is now on LMS/Gradescope, due next Sept 10 at 11:59pm
- Reading assignments are on the shared drive