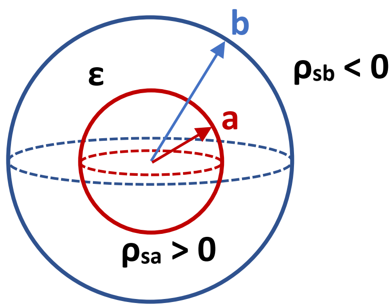
Homework 5

1. Spherical Capacitor

Consider the system below, which consists of a conducting spherical shell of radius *a* inside of a conducting spherical shell of radius *b*. The inner spherical shell has a surface charge density of $\rho_{sa} > 0$ on its surface, while the outer spherical has a surface charge density of $\rho_{sb} < 0$, such that the outer shell is at V = 0. The space between the conducting spherical shells is filled with a dielectric of permittivity ε.



a) Find the electric field in the region a < r < b.

Using Gauss's Law with a spherical Gaussian surface of radius r, where $a < r < b$:

$$\oint \vec{D}\cdot d\vec{S} = Q_{enc}$$

E field should point radially: $\vec{E} = E\hat{r}$

i) $\oint (\varepsilon E\hat{r})\cdot(r^2\sin\theta\, d\theta\, d\phi\, \hat{r}) = \varepsilon E r^2 \int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\phi$

$= 4\pi\varepsilon E r^2$

ii) $Q_{enc} = \int_0^{2\pi}\int_0^{\pi} \rho_{sa}\cdot a^2 \sin\theta\, d\theta\, d\phi = 4\pi a^2 \rho_{sa}$

iii) $4\pi\varepsilon E r^2 = 4\pi a^2 \rho_{sa} \rightarrow \boxed{\vec{E} = \frac{\rho_{sa}}{\varepsilon}\left(\frac{a}{r}\right)^2 \hat{r}}$

b) Find the potential V_{ab} between the two conducting spherical shells.

$$V_{ab} = V(a) - \cancel{V(b)}^{0} = -\int_a^b \vec{E}\cdot d\vec{l} = -\frac{\rho_{sa}}{\varepsilon}a^2\int_a^b 1/r^2\, dr = \boxed{\frac{\rho_{sa}}{\varepsilon}a^2\left(1/b - 1/a\right) = V_{ab}}$$

c) Calculate the capacitance of the system $C = Q/V_{ab}$.

$$C = \frac{Q}{V_{ab}} = \frac{4\pi a^2 \rho_{sa}}{\frac{\rho_{sa}}{\varepsilon}a^2\left(1/b - 1/a\right)} = \frac{4\pi\varepsilon}{\left(1/b - 1/a\right)} = \boxed{\frac{4\pi\varepsilon ab}{(a-b)} = C}$$

d) Calculate the total energy W_e stored in the electric field in the region a < r < b.

$$W_e = \frac{1}{2}\iiint \vec{D}\cdot\vec{E}\,dV \qquad \vec{E}(V_{ab}) = \frac{\rho_{sa}}{\varepsilon}\left(\frac{a}{r}\right)^2\hat{r} =$$

$$= \left(\frac{\varepsilon V_{ab}}{a^2(1/b - 1/a)}\right)\frac{1}{\varepsilon}\left(\frac{a}{r}\right)^2\hat{r} = \frac{V_{ab}}{(1/b - 1/a)}\frac{1}{r^2}\hat{r}$$

$$W_e = \frac{1}{2}\varepsilon\int_0^{2\pi}\int_0^{\pi}\int_a^b \frac{V_{ab}^2}{(1/b - 1/a)^2}\frac{1}{r^4}\cdot(r^2\sin\theta\,dr\,d\theta\,d\phi\cdot\hat{r}) = \frac{1}{2}\varepsilon V_{ab}^2\frac{4\pi}{(1/b-1/a)^2}\int_a^b r^{-2}dr$$

$$= \frac{-2\pi\varepsilon\,(1/b - 1/a)}{(1/b - 1/a)^2}V_{ab}^2$$

$$= \boxed{\frac{2\pi\varepsilon ab}{(b-a)}V_{ab}^2 = W_e}$$

e) From your expression in part d, find C.

$$W_e = \frac{1}{2}CV_{ab}^2 = \frac{1}{2}\left(\frac{4\pi\varepsilon ab}{(b-a)}\right)V_{ab}^2 \text{, So } \boxed{C = \frac{4\pi\varepsilon ab}{(b-a)}}$$

f) Do the two conducting spherical shells feel a net electrical force when a voltage is applied between them such that V(a) > V(b)? If so, in which direction does the force point?

In a differential area of size $ds = a^2\sin\theta\,d\theta\,d\phi$ on the surface of the inner sphere, the differential charge is $dQ = \rho_{sa}\,ds$. As a result, the differential force felt at ds due to $\vec{E}$ is

$$d\vec{F} = dQ\,\vec{E} = \rho_{sa}a^2\left(\frac{\rho_{sa}}{\varepsilon}\left(\frac{a^2}{a^2}\right)\hat{r}\right)\cdot a^2\sin\theta\,d\theta\,d\phi$$

$$= \frac{\rho_{sa}^2 a^4}{\varepsilon}\sin\theta\,d\theta\,d\phi\,\hat{r}$$

To find the total force on the inner sphere, integrate $d\vec{F}$ across the surface: $\vec{F} = \int_0^{2\pi}\int_0^{\pi} d\vec{F} = \boxed{\frac{4\pi\rho_{sa}^2 a^4}{\varepsilon}\hat{r} = \vec{F}}$

So the conductors feel a force in the radial direction ($+\hat{r}$ for inner conductor and $-\hat{r}$ for outer conductor).

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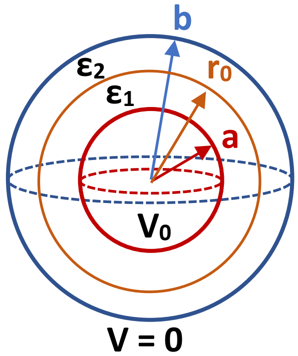
2. Spherical Capacitor with Two Dielectrics

Consider a spherical capacitor similar to the one in Question #1, except that the space between the two conducting spherical shells is now filled with two different dielectrics, as shown below.



The inner spherical shell (still of radius a) now has a potential V_0, while the outer spherical shell (still of radius b) is grounded. From $r = a$ to $r = r_0$, the space is filled with a dielectric of permittivity $\varepsilon = \varepsilon_1$, while from $r = r_0$ to $r = b$, the space is filled with a dielectric of permittivity $\varepsilon = \varepsilon_2$. The charge density in the region $a < r < b$ is zero.

a) Using Laplace's Equation, find the general solution for V(r) in the region $a < r < r_0$ (denoted V_1) and in the region $r_0 < r < b$ (denoted V_2). The "general solution" is the functional form of the solution without specific values for the constants that arise after integration. You will solve for these constants in later steps using boundary conditions.

$$\vec{\nabla}_r^2 V_1 = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV_1}{dr}\right) = 0 \quad \text{for } a<r<r_0$$
$$\vec{\nabla}_r^2 V_2 = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV_2}{dr}\right) = 0 \quad \text{for } r_0<r<b$$

$$V_1(r) = -\frac{C_1'}{r} + C_1 \ ; \ V_2(r) = -\frac{C_2'}{r} + C_2$$

b) Write down all of the boundary conditions you will need to find the unique solution for V_1 and V_2. Hint: since you should have 4 unknown constants to solve for in your answer from part a, you will need 4 boundary conditions: one at $r = a$, one at $r = b$, and two at $r = r_0$.

Boundary Conditions

1) $V_1(r=a) = V_0$
2) $V_2(r=b) = 0$
3) $D_{1n}\big|_{r=r_0} = D_{2n}\big|_{r=r_0}$

$$\varepsilon_1 \frac{\partial V_1}{\partial r}\bigg|_{r=r_0} = \varepsilon_2 \frac{\partial V_2}{\partial r}\bigg|_{r=r_0}$$

4) $V_1(r=r_0) = V_2(r=r_0)$

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c) Using the boundary conditions from part b and your general solutions from part a, find the unique solution for $V_1(r)$ for $a < r < r_0$ and $V_2(r)$ for $r_0 < r < b$.

$$V_1(r) = -\frac{C_1'}{r} + C_1 \quad \text{and} \quad V_2(r) = -\frac{C_2'}{r} + C_2$$

$$\left.\begin{aligned} &\text{BC\#1}: \quad V_0 = -\frac{C_1'}{a} + C_1 \\ &\text{BC\#2}: \quad 0 = -\frac{C_2'}{b} + C_2 \end{aligned}\right\} \begin{aligned} &\#2-\#1: \\ &-V_0 = -\frac{C_2'}{b} + \frac{C_1'}{a} + C_2 - C_1 \end{aligned}$$

$$\text{BC\#3}: \quad \varepsilon_1\left(\frac{C_1'}{r_0^2}\right) = \varepsilon_2\left(\frac{C_2'}{r_0^2}\right)$$

$$\varepsilon_1 C_1' = \varepsilon_2 C_2'$$

$$\text{BC\#4}: \quad -\frac{C_1'}{r_0} + C_1 = -\frac{C_2'}{r_0} + C_2$$

$$C_1' - r_0 C_1 = C_2' - r_0 C_2$$

$$r_0(C_2 - C_1) = C_2' - C_1'$$

· Start w/ BC#4 because it has all unknowns

$$r_0\underbrace{(C_2 - C_1)}_{\text{Via BC\#2-BC\#1}} = \underbrace{C_2' - C_1'}_{\text{Via BC\#3}}$$

$$r_0\left(\frac{C_2'}{b} - \frac{C_1'}{a} - V_0\right) \qquad C_2'\left(1 - \frac{\varepsilon_2}{\varepsilon_1}\right)$$

Via BC#3

$$r_0\left(\frac{C_2'}{b} - \frac{\varepsilon_2}{\varepsilon_1}\frac{C_2'}{a} - V_0\right) = C_2'\left(1 - \frac{\varepsilon_2}{\varepsilon_1}\right)$$

$$C_2'\left(r_0/b - \frac{\varepsilon_2}{\varepsilon_1}\frac{r_0}{a} + \frac{\varepsilon_2}{\varepsilon_1} - 1\right) = r_0 V_0$$

$$\boxed{C_2' = \frac{r_0 V_0}{\left(r_0/b - \frac{\varepsilon_2}{\varepsilon_1}\frac{r_0}{a} + \frac{\varepsilon_2}{\varepsilon_1} - 1\right)}} \xrightarrow[\text{BC\#3}]{C_1' = \frac{\varepsilon_2}{\varepsilon_1}C_2'} \boxed{C_1' = \frac{\frac{\varepsilon_2}{\varepsilon_1} r_0 V_0}{\left(r_0/b - \frac{\varepsilon_2}{\varepsilon_1}\frac{r_0}{a} + \frac{\varepsilon_2}{\varepsilon_1} - 1\right)}}$$

From BC#2: $\underline{C_2} = \dfrac{C_2'}{b} = \boxed{\dfrac{r_0 V_0}{\left(r_0/b - \frac{\varepsilon_2}{\varepsilon_1}\frac{r_0}{a} + \frac{\varepsilon_2}{\varepsilon_1} - 1\right)}\dfrac{1}{b}}$

From BC#1: $\underline{C_1} = \dfrac{C_1'}{a} + V_0 = \boxed{\dfrac{\frac{\varepsilon_2}{\varepsilon_1} r_0 V_0}{\left(r_0/b - \frac{\varepsilon_2}{\varepsilon_1}\frac{r_0}{a} + \frac{\varepsilon_2}{\varepsilon_1} - 1\right)}\dfrac{1}{a} + V_0}$

Plug in constants to find the unique solutions

$$\boxed{\begin{aligned} V_1(r) &= \frac{\frac{\varepsilon_2}{\varepsilon_1} r_0 V_0}{\left(r_0/b - \frac{\varepsilon_2}{\varepsilon_1}\frac{r_0}{a} + \frac{\varepsilon_2}{\varepsilon_1} - 1\right)}\left\{1/a - 1/r\right\} + V_0 \\ V_2(r) &= \frac{r_0 V_0}{\left(r_0/b - \frac{\varepsilon_2}{\varepsilon_1}\frac{r_0}{a} + \frac{\varepsilon_2}{\varepsilon_1} - 1\right)}\left\{1/b - 1/r\right\} \end{aligned}}$$

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d) Calculate the electric field and displacement field for the region a < r < b. Are the relevant boundary conditions for E and D in part b satisfied by this solution?

$$\left.\begin{aligned}\vec{E}_1 &= -\vec{\nabla}_r V_1 = -\frac{\frac{\varepsilon_2}{\varepsilon_1} r_0 V_0}{\left(\frac{r_0}{b} - \frac{\varepsilon_2}{\varepsilon_1}\frac{r_0}{a} + \frac{\varepsilon_2}{\varepsilon_1} - 1\right)}\frac{1}{r^2}\hat{r} \\ \vec{D}_1 &= \varepsilon_1 \vec{E}_1 = -\frac{\varepsilon_2 r_0 V_0}{\left(\frac{r_0}{b} - \frac{\varepsilon_2}{\varepsilon_1}\frac{r_0}{a} + \frac{\varepsilon_2}{\varepsilon_1} - 1\right)}\frac{1}{r^2}\hat{r}\end{aligned}\right\} \text{for } a<r<r_0$$

$$\left.\begin{aligned}\vec{E}_2 &= -\vec{\nabla}_r V_2 = -\frac{r_0 V_0}{\left(\frac{r_0}{b} - \frac{\varepsilon_2}{\varepsilon_1}\frac{r_0}{a} + \frac{\varepsilon_2}{\varepsilon_1} - 1\right)}\frac{1}{r^2}\hat{r} \\ \vec{D}_2 &= \varepsilon_2 \vec{E}_2 = -\frac{\varepsilon_2 r_0 V_0}{\left(\frac{r_0}{b} - \frac{\varepsilon_2}{\varepsilon_1}\frac{r_0}{a} + \frac{\varepsilon_2}{\varepsilon_1} - 1\right)}\frac{1}{r^2}\hat{r}\end{aligned}\right\} \text{for } r_0<r<b$$

Checking BC#3: $D_{1n} = D_{2n}$? yes, since $\vec{D}_1 = \vec{D}_2$ everywhere

e) Calculate the total charge on each of the conducting spherical shells in terms of V_0.

Total charge on inner sphere:

$$D_{1n}(r=a) = \rho_{sa} > 0$$

$$\rho_{sa} = -\frac{\varepsilon_2 r_0 V_0}{\left(\frac{r_0}{b} - \frac{\varepsilon_2}{\varepsilon_1}\frac{r_0}{a} + \frac{\varepsilon_2}{\varepsilon_1} - 1\right)}\frac{1}{a^2}$$

$$\boxed{Q_a = 4\pi a^2 \cdot \rho_{sa} = -\frac{4\pi \varepsilon_2 r_0 V_0}{\left(\frac{r_0}{b} - \frac{\varepsilon_2}{\varepsilon_1}\frac{r_0}{a} + \frac{\varepsilon_2}{\varepsilon_1} - 1\right)}}$$

$$D_{2n}(r=b) = \rho_{sb} < 0$$

$$\rho_{sb} = \frac{\varepsilon_2 r_0 V_0}{\left(\frac{r_0}{b} - \frac{\varepsilon_2}{\varepsilon_1}\frac{r_0}{a} + \frac{\varepsilon_2}{\varepsilon_1} - 1\right)}\frac{1}{b^2}$$

$$\boxed{Q_b = 4\pi b^2 \cdot \rho_{sb} = \frac{4\pi \varepsilon_2 r_0 V_0}{\left(\frac{r_0}{b} - \frac{\varepsilon_2}{\varepsilon_1}\frac{r_0}{a} + \frac{\varepsilon_2}{\varepsilon_1} - 1\right)}}$$

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f) Calculate the capacitance between the two conducting spherical shells.

$$\boxed{C = \frac{Q_a}{V_0} = -\frac{4\pi\varepsilon_2 r_0}{\left(r_0/b - \frac{\varepsilon_2}{\varepsilon_1}\frac{r_0}{a} + \frac{\varepsilon_2}{\varepsilon_1} - 1\right)}}$$

If we rearrange C: $C_1 = \frac{4\pi\varepsilon_1 a r_0}{(r_0 - a)}$; $C_2 = \frac{4\pi\varepsilon_2 r_0 b}{(b - r_0)}$

$$\frac{1}{C_{eff}} = -\frac{\left(r_0/b - \frac{\varepsilon_2}{\varepsilon_1}\frac{r_0}{a} + \frac{\varepsilon_2}{\varepsilon_1} - 1\right)}{4\pi\varepsilon_2 r_0} = -\frac{(\varepsilon_1 r_0 a - \varepsilon_2 r_0 b + \varepsilon_2 ab - \varepsilon_1 ab)}{4\pi\varepsilon_1\varepsilon_2 r_0 ab}$$

$$= -\frac{\varepsilon_1(r_0 a - ab) + \varepsilon_2(ab - r_0 b)}{4\pi\varepsilon_1\varepsilon_2 r_0 ab} = \frac{\varepsilon_1 a(b - r_0)}{4\pi\varepsilon_1\varepsilon_2 r_0 ab} + \frac{\varepsilon_2 b(r_0 - a)}{4\pi\varepsilon_1\varepsilon_2 r_0 ab}$$

$$= \underbrace{\frac{b - r_0}{4\pi\varepsilon_2 b r_0}}_{1/C \text{ for } a<r<r_0} + \underbrace{\frac{r_0 - a}{4\pi\varepsilon_1 a r_0}}_{1/C \text{ for } r_0<r<b} = \frac{1}{C_1} + \frac{1}{C_2}$$

← same form for C as in Problem 1

So, the total capacitance of the sphere is the capacitances of the regions $a<r<r_0$ and $r_0<r<b$ added in series.

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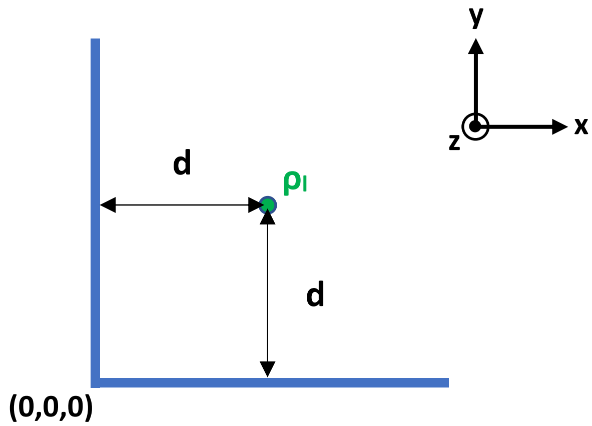
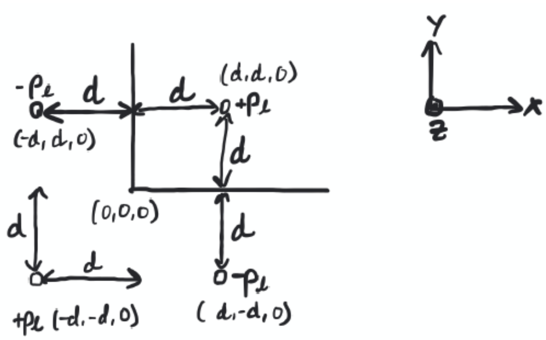
3. Method of Images

An infinite line of charge (line charge density ρ_l) is oriented along the z-axis. It is located a distance *d* above a grounded, conducting half-plane oriented in the x-z plane and a distance *d* away from a grounded, conducting half-plane oriented in the y-z plane, as shown below. Consider the half-planes to be infinite.



a) Draw the locations, magnitudes, and polarities of the image lines of charge with respect to each of the two grounded conductors.



b) Find the electric field at an arbitrary point P = (x, y, 0).

· $\vec{E}$ field from infinite line of charge at (0,0,0), as seen at an arbitrary point P=(x,y,0)

$$\oint \vec{D}\cdot d\vec{s} = Q_{enc}$$

$$\hookrightarrow \varepsilon E 2\pi r l = l\rho_l$$

$$\vec{E} = \frac{\rho_l}{2\pi\varepsilon r}\hat{r}$$

· If the line of charge is located at $(+d,+d,0)$

then $\underset{\substack{\uparrow \\ OP}}{\vec{r}_P} = \langle x, y, 0\rangle$, $\underset{\substack{\uparrow \\ O \to (d,d,0)}}{\vec{r}_d} = \langle d, d, 0\rangle$ and $\vec{r} = \vec{r}_P - \vec{r}_d$

$= \langle x-d, y-d, 0\rangle$

So $|\vec{r}| = |\langle x-d, y-d, 0\rangle| = \sqrt{(x-d)^2+(y-d^2)}$

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\langle x-d, y-d, 0\rangle}{\sqrt{(x-d)^2+(y-d)^2}}$$

Then $\vec{E}$ at $P=(x,y,0)$ for a line charge at $(d,d,0)$ is then

$$\vec{E} = \frac{\rho_\ell/2\pi}{\varepsilon\sqrt{(x-d)^2+(y-d)^2}}\,\frac{\langle x-d, y-d, 0\rangle}{\sqrt{(x-d)^2+(y-d)^2}} = \frac{\rho_\ell}{2\pi\varepsilon}\,\frac{\langle x-d, y-d, 0\rangle}{[(x-d)^2+(y-d)^2]}$$

We can then add the contributions of the line charges at $(d,-d,0)$ $(-d,+d,0)$, and $(-d,-d,0)$:

$$\boxed{\vec{E}_{tot} = +\frac{\rho_\ell}{2\pi\varepsilon}\frac{\langle x-d, y-d, 0\rangle}{(x-d)^2+(y-d)^2} - \frac{\rho_\ell}{2\pi\varepsilon}\frac{\langle x-d, y+d, 0\rangle}{(x-d)^2+(y+d)^2} - \frac{\rho_\ell}{2\pi\varepsilon}\frac{\langle x+d, y-d, 0\rangle}{(x+d)^2+(y-d)^2} + \frac{\rho_\ell}{2\pi\varepsilon}\frac{\langle x+d, y+d, 0\rangle}{(x+d)^2+(y+d)^2}}$$

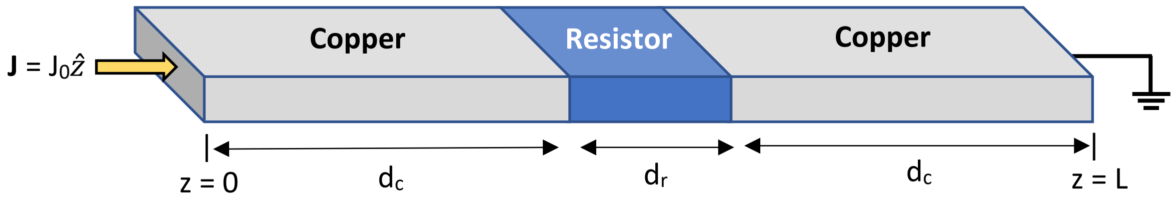
Check : $\vec{E}_{tot}$ should point along $-\hat{y}$ for $(x,0,0)$

$\vec{E}_{tot}$ should point along $-\hat{x}$ for $(0,y,0)$

4. Current and Resistance

The diagram below shows a rectangular copper wire segment (length d_c and conductivity $\sigma = \sigma_c$) connected in series with a rectangular resistor (length d_r and resistivity $\rho = \rho_r = 1/\sigma_r$) and another segment of rectangular copper wire (length d_c and $\sigma = \sigma_c$). The cross-sectional area of the wires and resistor is A.

A uniform current density $J = J_0\hat{z}$ is injected into the wire at z = 0, while the other end of the wire/resistor/wire system at z = L is grounded.



a) Find the electric field in each region (the two copper wires and the resistor).

Injected current density $\vec{J} = J_0\hat{z}$ must be the same in each segment, so: $\vec{J} = \sigma\vec{E}$

$$\boxed{\begin{aligned} \vec{E}_{cu,1} &= \frac{J_0}{\sigma_c}\hat{z} \\ \vec{E}_r &= \frac{J_0}{\sigma_r}\hat{z} \\ \vec{E}_{cu,2} &= \frac{J_0}{\sigma_c}\hat{z} \end{aligned}}$$

b) What is the ratio of the electric field in the resistor to the electric field in the copper wire segments?

$$\frac{|\vec{E}_r|}{|\vec{E}_{cu}|} = \frac{\cancel{J_0}}{\sigma_r} \cdot \frac{\sigma_c}{\cancel{J_0}} = \boxed{\frac{\sigma_c}{\sigma_r}}$$

c) If $\sigma_c > \sigma_r$, what can you conclude about the magnitude of the electric field in a resistor as compared to the electric field in a copper wire? Explain why this physically makes sense in terms of the current density and conductivity. Does this align with what you know about the potential drop across a resistor in a circuit, as compared with the potential drop across a copper wire?

If $\sigma_c > \sigma_r$, then $|\vec{E}_r| > |\vec{E}_{cu}|$. In terms of the conductivity and current density, if J_0 must be constant everywhere and the current density sees a higher resistance, more force is required to push the current density through it. Since $\vec{F} = q\vec{E}$, the proper physical response is a larger E field. In terms of circuits, we see a larger voltage drop across circuit elements with lower conductivities.

d) Using the electric field, find the potential dropped across each of the segments (the two wires and the resistor).

$$V(L-d_c) - \cancel{V(L)}^{\,0} = -\int_{L}^{L-d_c} \vec{E}\cdot d\vec{l} = -\int_{L}^{L-d_c} \left(\frac{J_0}{\sigma_c}\hat{z}\right)\cdot\left(-dz\,\hat{z}\right) = \frac{J_0}{\sigma_c}(d_c) = \frac{J_0}{\sigma_c}d_c$$

$$V(L-d_c-d_r) - V(L-d_c) = -\int^{L-d_c-d_r} \left(\frac{J_0}{\sigma_r}\hat{z}\right)\cdot\left(-dz\,\hat{z}\right) = \frac{J_0}{\sigma_r}d_r$$

$$V(0) - V(L-d_c-d_r) = -\int_{L-d_c-d_r}^{0} \left(\frac{J_0}{\sigma_c}\hat{z}\right)\cdot\left(-dz\,\hat{z}\right) = \frac{J_0}{\sigma_c}d_r$$

$$\boxed{V(0) - V_L = J_0\left(\frac{2}{\sigma_c}d_c + \frac{1}{\sigma_r}d_r\right)}$$

e) Using the geometry of the wires and resistor, calculate R for each.

$$R = \frac{l}{\sigma A} \rightarrow \begin{cases} R_c = \dfrac{d_c}{\sigma_c A} \\ R_r = \dfrac{d_r}{\sigma_r A} \end{cases}$$

f) Using the voltage divider equation and the resistances you calculated in part e, calculate the voltage drop across the resistor. Does it match what you found in part d?

$$V_R = V_L \frac{R_r}{2R_c + R_r} = J_0\left(\frac{2}{\sigma_c}d_c + \frac{1}{\sigma_r}d_r\right)\frac{\frac{d_r}{\sigma_r A}}{\frac{2}{\sigma_c A} + \frac{d_r}{\sigma_r A}}$$

$$= \boxed{\frac{J_0 d_r}{\sigma_r}}$$

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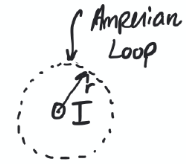
5. Ampere's Law

a) Using Ampere's law, find the H field and B field at an arbitrary point P = (x, y, z) of an infinitely long wire which is oriented along the z-axis and located in the x-y plane at a point (x_0, y_0). The wire is carrying a current **I** = I$\hat{z}$ and is surrounded by free space.



$$\oint \vec{H} \cdot d\vec{l} = I_{enc}$$

$$2\pi r H = I \rightarrow \vec{H} = \frac{I}{2\pi r}\hat{\phi}$$

$$\boxed{\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}}$$

$$\left.\begin{aligned} OP &= \langle x, y, z \rangle = \vec{r}_P \\ 0 &\rightarrow \langle x_i, y_i, 0 \rangle = \vec{r}_1 \end{aligned}\right\} \vec{r} = \langle x - x_i, y - y_i, z \rangle$$

$$|\vec{r}| = \sqrt{(x-x_i)^2 + (y-y_i)^2 + z^2}$$

what is $\hat{\phi}$? it points in the direction of increase of $\phi \rightarrow$ counter clock wise

$\hat{\phi} = +\hat{y}$ when $\phi = 0$

$\hat{\phi} = -\hat{x}$ when $\phi = \pi/2$

$$\hat{\phi} = \hat{y}\cos\phi - \hat{x}\sin\phi$$

$$\phi = \tan^{-1}(y/x)$$

Using the trigonometric identities: $\sin(\arctan(y/x)) = \frac{y/x}{\sqrt{1+(y/x)^2}} = \frac{y}{\sqrt{x^2+y^2}}$

$\cos(\arctan(y/x)) = \frac{1}{\sqrt{1+(y/x)^2}} = \frac{x}{\sqrt{x^2+y^2}}$

$$\hat{\phi} = \left\langle -\frac{y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}, 0 \right\rangle$$

$$\vec{H} = \frac{I}{2\pi\sqrt{(x-x_i)^2+(y-y_i)^2+z^2}}\left\langle -\frac{y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}, 0 \right\rangle$$

$$\vec{B} = \frac{\mu_0 I}{2\pi\sqrt{(x-x_i)^2+(y-y_i)^2+z^2}}\left\langle -\frac{y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}, 0 \right\rangle$$

b) Find the H field and B field at an arbitrary point P = (x, y, z) of two infinitely long wires oriented along the z-axis. One is located in the x-y plane at ($-x_0$, $-y_0$) and is carrying current **I** = -I$\hat{z}$ and the other is located at (x_0, y_0) and is carrying current **I** = I$\hat{z}$.

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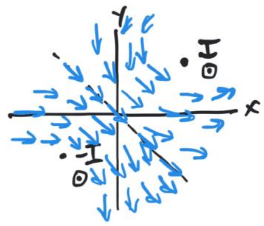
Add a wire at $(-x_i,-y_i)$ with $I_2 = -I$

$$\vec{H} = \frac{I}{2\pi}\left\{\frac{1}{\sqrt{(x-x_i)^2+(y-y_i)^2+z^2}} - \frac{1}{\sqrt{(x+x_i)^2+(y+y_i)^2+z^2}}\right\}\left\langle -\frac{y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}, 0\right\rangle$$

$$\vec{B} = \frac{\mu_0 I}{2\pi}\left\{\frac{1}{\sqrt{(x-x_i)^2+(y-y_i)^2+z^2}} - \frac{1}{\sqrt{(x+x_i)^2+(y+y_i)^2+z^2}}\right\}\left\langle \frac{y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}, 0\right\rangle$$

c) Sketch the total magnetic field around each of the wires. Is the magnetic field zero at any point between the wires? If so, where?



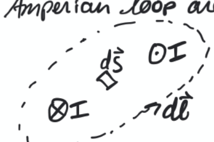
$\vec{H}_{total}$ is not zero anywhere between the wires.

d) According to Maxwell's Equations (magnetostatics version), what physical quantity do you obtain when you calculate the curl of the H field? If you were to integrate the curl of your H field from part c over the surface bounded by an Amperian loop containing both wires (likely a circle): $\oiint (\nabla \times \vec{H}) \cdot d\vec{S}$, what value would you get? Hint: look at Ampere's Law and Stokes' Theorem – you don't have to actually do the calculation.

$\vec{\nabla}\times\vec{H} = \vec{J}$, the current density (differential form of Ampere's Law)

Using Ampere's Law: $\oint \vec{H}\cdot d\vec{l} = \oint(\vec{J}\cdot d\vec{s}) = \oint(\vec{\nabla}\times\vec{H})\cdot d\vec{s} = I_{enc}$

So, if we draw an Amperian loop around the two wires,



$I_{enc} = 0$

If $I_{enc} = 0$, then $\oint(\vec{J}\cdot d\vec{s})\cdot d\vec{s} = I_{enc} = 0$. This just tells us that if we integrate the current density passing through the surface bounded by our Amperian loop, we get 0 (i.e. total enclosed current is 0).

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