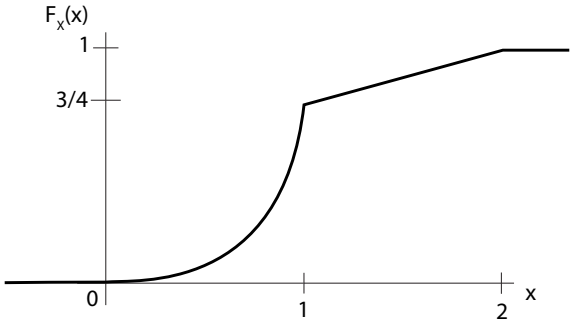
Rensselaer Polytechnic Institute
Department of Electrical, Computer, and Systems Engineering
ECSE 2500: Engineering Probability, Spring 2023
Homework #4 Solutions

1. (a) The CDF is illustrated below. The first area is part of a cubic function, and the second area is a straight line segment.



Grading criteria: 5 points in total
-0 point: correct
-1 point: Missing or wrong at the point x=0
-1 point: Missing or wrong at the point x=1
-1 point: Missing or wrong at the point x=2
-5 point: completely incorrect or blank

(b) Since $\frac{3}{2}$ is in the second (linear) piece, we can easily compute that $P\left(X < \frac{3}{2}\right) = F_X\left(\frac{3}{2}\right) = \frac{7}{8} = 0.875$.

Grading criteria: 5 points in total
-0 point: correct
-3 point: plug into the wrong formula/interval
-5 point: completely incorrect or blank

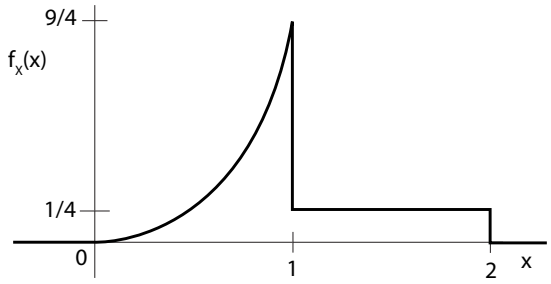
(c) To compute $P\left(X \in \left[\frac{1}{2}, \frac{9}{8}\right]\right)$, note that $\frac{1}{2}$ and $\frac{9}{8}$ are in different pieces of the CDF. The answer is $F_X\left(\frac{9}{8}\right) - F_X\left(\frac{1}{2}\right) = \frac{25}{32} - \frac{3}{32} = \frac{11}{16} = 0.6875$.

Grading criteria: 5 points in total
-0 point: correct
-2 point: wrong or missing when calculating CDF of x=1/2
-2 point: wrong or missing when calculating CDF of x=9/8
-5 point: completely incorrect or blank

(d) To compute the PDF we take the derivative (in this case, piece by piece):

$$f_X(x) = \begin{cases} \frac{9}{4}x^2 & x \in [0,\ 1] \\ \frac{1}{4} & x \in [1,\ 2] \\ 0 & \text{otherwise} \end{cases}$$

The PDF is sketched below; remember to show that it's 0 outside of the key region! In this way we can see that 1/4 of the probability is in the region $X \in [1,\ 2]$ and 3/4 of the probability is in the region $X \in [0,\ 2]$.

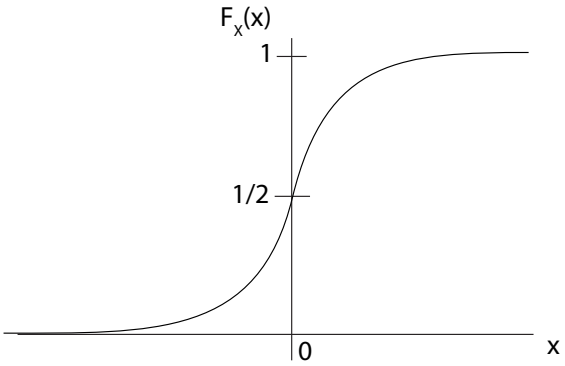


Grading criteria: 5 points in total
-0 point: correct
-3 point: the formula of PDF is wrong or missing
-3 point: the sketch is wrong or missing
-1 point: missing or wrong at the point x=0
-1 point: missing or wrong at the point x=1
-1 point: missing or wrong at the point x=2
-10 point: completely incorrect or blank

2. (a) We can compute that $F_X(-\infty) = \frac{1}{2}\left(\frac{-1}{1}\right) + \frac{1}{2} = 0$, and that $F_X(\infty) = \frac{1}{2}\left(\frac{\infty}{\infty}\right) + \frac{1}{2} = 1$.

> Grading criteria: 5 points in total
> -0 point: correct
> -2 point: proof of $F_x(-\infty) = 0$ is wrong or missing
> -2 point:proof of $F_x(\infty) = 1$ is wrong or missing
> -5 point: completely incorrect or blank

(b) The CDF is illustrated below. If you're a machine learning person, you may recognize this as a (scaled and shifted) tanh or hyberbolic tangent function.



> Grading criteria: 5 points in total
> -0 point: correct
> -1 point: $F_x(\infty)$ is wrong or missing
> -1 point:$F_x(-\infty)$ is wrong or missing
> -1 point:$F_x(x = 0)$ is wrong or missing
> -5 point: completely incorrect or blank

(c) We can compute that

$$\begin{aligned} P(-2 < X < 1) &= F_X(1) - F_X(-2) \\ &= \frac{1}{2}\left(\frac{e^2-1}{e^2+1}\right) + \frac{1}{2} - \frac{1}{2}\left(\frac{e^{-4}-1}{e^{-4}+1}\right) - \frac{1}{2} \\ &= \frac{1}{2}\left(\frac{e^2-1}{e^2+1}\right) - \frac{1}{2}\left(\frac{e^{-4}-1}{e^{-4}+1}\right) \\ &= 0.8628 \end{aligned}$$

> Grading criteria: 5 points in total
> -0 point: correct
> -2 point: $F_x(x = 1)$ is wrong or missing
> -2 point: $F_x(x = 2)$ is wrong or missing
> -5 point: completely incorrect or blank

(d) Trick question! We know that for a continuous random variable, the probability of getting any exact value is 0. In terms of the CDF, this is like looking at the limit $\lim_{h\to 0} F_X(2+h) - F_X(2) = 0$.

> Grading criteria: 5 points in total
> -0 point: correct

-3 point: No formula or detailed explanation
-5 point: completely incorrect or blank

(e) To get the PDF from the CDF, we need to differentiate:

$$\begin{aligned} f_X(x) &= \frac{d}{dx} F_X(x) \\ &= \frac{d}{dx} \frac{1}{2} \left(\frac{e^{2x}-1}{e^{2x}+1} \right) + \frac{1}{2} \\ &= \frac{1}{2} \left(\frac{(e^{2x}+1)\cdot 2e^{2x} - (e^{2x}-1)\cdot 2e^{2x}}{(e^{2x}+1)^2} \right) \\ &= \frac{2e^{2x}}{(e^{2x}+1)^2} \end{aligned}$$

Grading criteria: 5 points in total
-0 point: correct
-2 point: the formula is wrong or missing
-2 point: the final answer is wrong or missing
-5 point: completely incorrect or blank

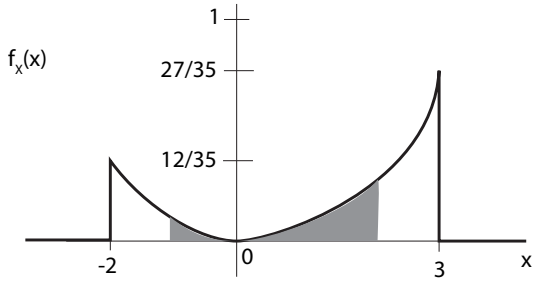
3. (a) We need the PDF to integrate to 1, i.e.,

$$\begin{aligned} \int_{x=-\infty}^{\infty} f_X(x)\, dx &= \int_{x=-2}^{3} cx^2\, dx \\ &= \frac{1}{3} cx^3 \Big]_{x=-2}^{x=3} \\ &= c\left(9 + \frac{8}{3}\right) \\ &= \frac{35}{3} c \\ &= 1 \end{aligned}$$

Thus we must have $c = \frac{3}{35}$.

Grading criteria: 10 points in total
-0 point: correct
-3 point: the formula of the integral of PDF equals 1 is wrong or missing
-3 point: the final answer is wrong or missing
-10 point: completely incorrect or blank

(b) The PDF is illustrated below. It looks like a quadratic function that turns on at -2 and off at 3.



Grading criteria: 5 points in total
-0 point: correct
-2 point: the point -2 is wrong or missing
-2 point: the point 3 is wrong or missing
-5 point: completely incorrect or blank

(c) $P(X \in [-1,2])$ corresponds to the shaded region in the figure above, which we can compute as

$$\begin{aligned}
&\int_{x=-1}^{2} \frac{3}{35} x^2 \, dx \\
&= \frac{1}{35} x^3 \Big]_{x=-1}^{x=2} \\
&= \frac{8}{35} + \frac{1}{35} \\
&= \frac{9}{35}
\end{aligned}$$

Grading criteria: 5 points in total
-0 point: correct
-2 point: the formula of PDF is wrong or missing
-2 point: the answer is wrong or missing
-5 point: completely incorrect or blank

(d) To get the CDF from the PDF, we need to integrate. Below $x = -2$ the CDF is 0 and above $x = 3$ the CDF is 1, so we only need to figure out the "middle piece":

$$\begin{aligned}
F_X(x) &= \int_{y=-\infty}^{x} f_X(y) \, dy \\
&= \int_{y=-2}^{x} \frac{3}{35} y^2 \, dy \\
&= \frac{1}{35} y^3 \Big]_{y=-2}^{y=x} \\
&= \frac{1}{35} x^3 + \frac{8}{35}
\end{aligned}$$

So the CDF overall is:

$$F_X(x) = \begin{cases} 0 & x < -2 \\ \frac{1}{35} x^3 + \frac{8}{35} & x \in [-2,3] \\ 1 & x > 3 \end{cases}$$

Grading criteria: 5 points in total
-0 point: correct
-2 point: CDF in [-2, 3] is wrong or missing
-2 point: the final answer is wrong or missing
-5 point: completely incorrect or blank

4. The exponential PDF with $\lambda = \frac{1}{3}$ is

$$f_X(x) = \frac{1}{3} e^{-\frac{1}{3}x} u(x)$$

(a)

$$\begin{aligned} P(X > 2) &= \int_{x=2}^{\infty} f_X(x)\, dx \\ &= \int_{x=2}^{\infty} \frac{1}{3} e^{-\frac{1}{3}x}\, dx \\ &= -e^{-\frac{1}{3}x} \Big]_{x=2}^{x=\infty} \\ &= e^{-\frac{2}{3}} \\ &= 0.5134 \end{aligned}$$

Grading criteria: 5 points in total
-0 point: correct
-2 point: the PDF of exponential RV is wrong or missing
-2 point: the formula of P(x>2) is wrong or missing
-2 point: the final answer is wrong or missing
-8 point: completely incorrect or blank

(b) Recall that if the arrival times X between events are exponentially distributed, the number of events N in a given interval is a Poisson random variable. Here, $\lambda = \frac{1}{3}$ and $t = 5$, so the probability is

$$\begin{aligned} P(N = 4) &= e^{-\lambda t} \frac{(\lambda t)^4}{4!} \\ &= e^{-\frac{5}{3}} \frac{\frac{5}{3}^4}{4!} \\ &= 0.0607 \end{aligned}$$

Grading criteria: 5 points in total
-0 point: correct
-3 point: the PDF of Poisson RV is wrong or missing
-2 point: the final answer is wrong or missing
-8 point: completely incorrect or blank

(c) Now we want to compute λ so that X is equally likely to be on either side of 10, which is the same thing as solving

$$\int_{x=10}^{\infty} \lambda e^{-\lambda x} = \frac{1}{2}$$

Using the same logic as in part (a), this reduces to

$$e^{-10\lambda} = \frac{1}{2}$$

or $\lambda = \frac{\log 2}{10} = 0.0693$.

Grading criteria: 5 points in total
-0 point: correct
-3 point: the formula is wrong or missing
-3 point: the final answer is wrong or missing
-9 point: completely incorrect or blank