Fields and Waves I

Lecture 9

Fields & Math

Electric Fields

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These slides were prepared through the work of the following people:

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- J. Darryl Michael – GE Global Research Center, Niskayuna, NY
- Thomas P. Crowley – National Institute of Standards and Technology, Boulder, CO
- Sheppard J. Salon – ECSE Department, Rensselaer Polytechnic Institute, Troy, NY (Emeritus)
- Lale Ergene – ITU Informatics Institute, Istanbul, Turkey
- Jeffrey Braunstein – ECE Department, University at Albany
- James Lu - ECSE Department, Rensselaer Polytechnic Institute, Troy, NY
- James Dylan Rees - ECSE Department, Rensselaer Polytechnic Institute, Troy, NY

Materials from other sources are referenced where they are used.
Those listed as Ulaby are figures from Ulaby's textbook.

Exam 1

- Solutions released this afternoon
- Average: 42.25/50 (84.5%)
- Median: 44/50 (88%)
- Stdev: 5.58/50 (11.1%)

- Rework exams will be offered next week on the following skills:
  Skill 1c (phasors)
  Skill 1f (input impedance)

Field Math

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation $\mathbf{A} =$	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of $\mathbf{A}$ $\quad \lvert\mathbf{A}\rvert =$	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\quad \overrightarrow{OP_1} =$	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P(R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product $\quad \mathbf{A}\cdot\mathbf{B} =$	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product $\quad \mathbf{A}\times\mathbf{B} =$	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length $\quad d\mathbf{l} =$	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume $\quad dV =$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

Field Math

Maxwell's Equations

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

Source: spectrumscientifics.wordpress.com

Gauss' Law: Electric charge gives rise to an electric field.

Field Math

Divergence Operator

- Points of electric charge are "sources" or "sinks" of electric field lines.
- In mathematical terms, we say that the *divergence* of a point is proportional to how much charge it has.

Notation: $div A = \nabla \bullet A$ *NOT a DOT product but has similar features*

Result is a SCALAR, composed of derivatives

$$\nabla \bullet \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$ in Cartesian coordinates

Field Math

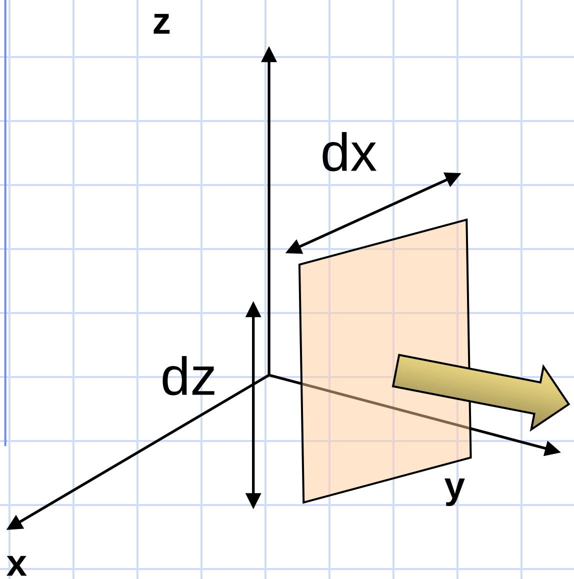
Maxwell's Equations

What does that field actually look like?

https://davidawehr.com/projects/electric_field.html

Field Math

Surface Integrals

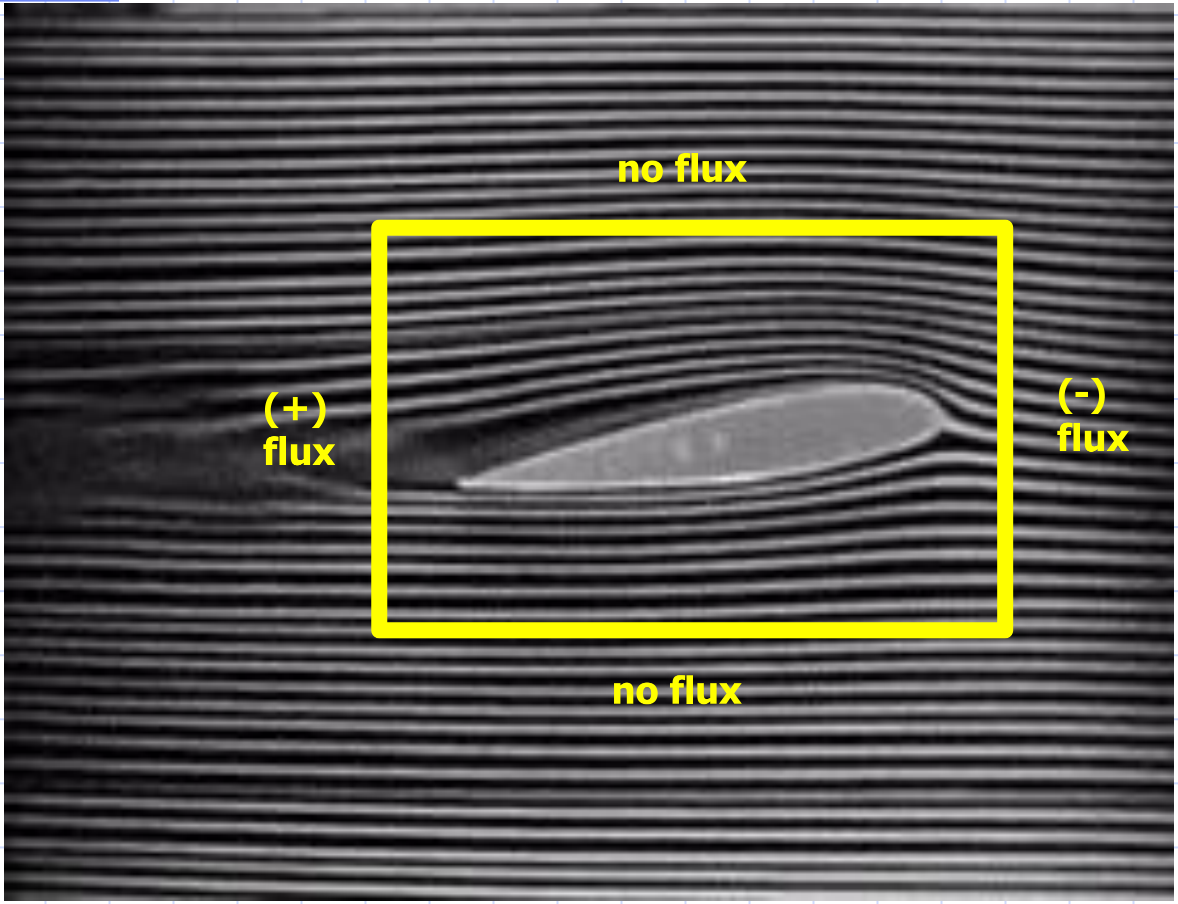


- An important property in field calculations is **flux:** how *much* field is flowing through a surface.
- Whenever we have a vector field, we can calculate a flux by taking the following integral:

$$\boxed{\text{Flux} = \iint_S \vec{\mathbf{F}} \cdot \hat{\mathbf{n}} \underbrace{dS}_{\text{surface area element}}}$$

- Note that the total flux will depend on both the area of the surface through which the field is flowing and the magnitude of the field.

Surface integrals and flux

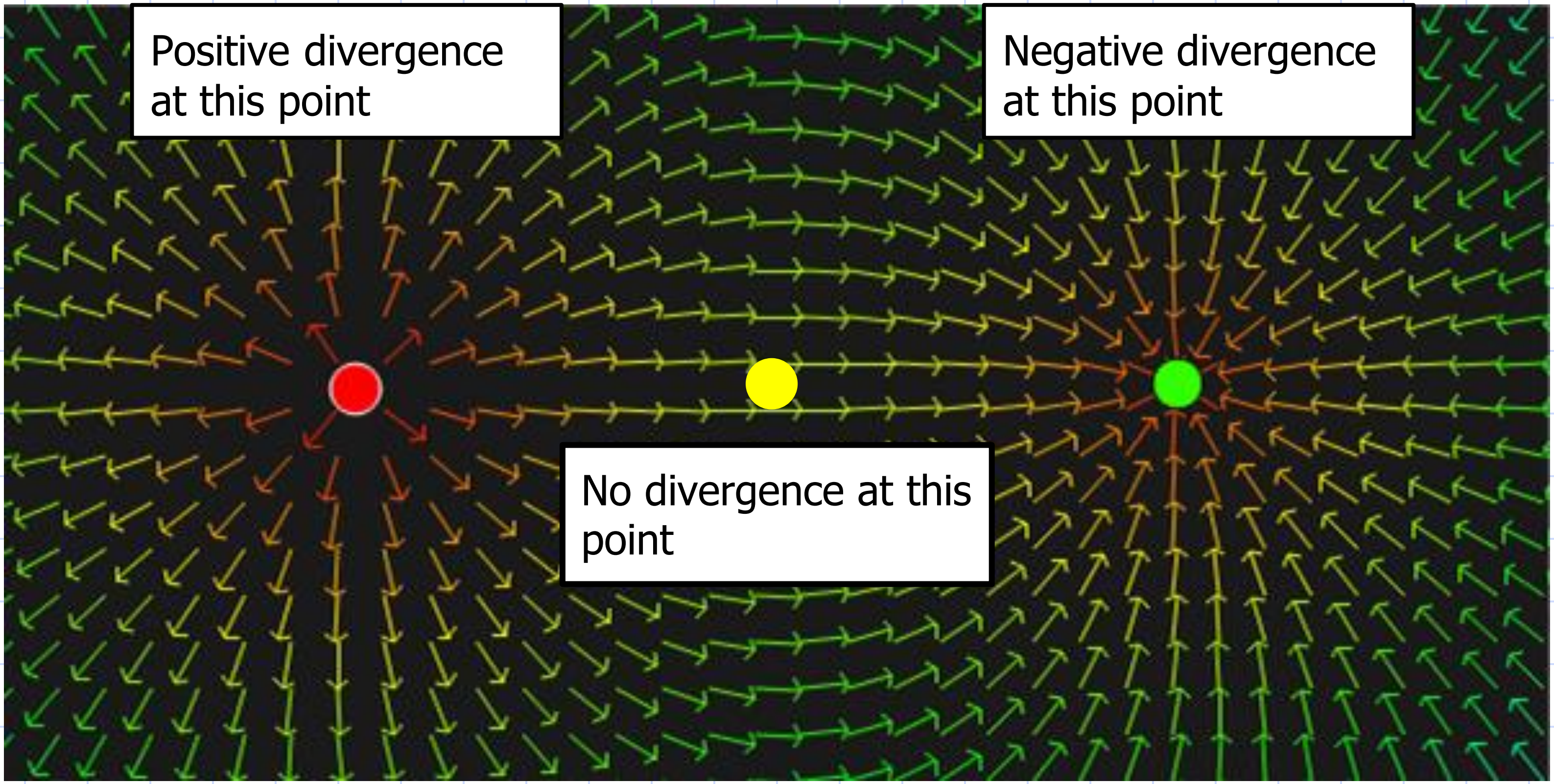


Gfycat

What is the total flux in the image above?

Field Math

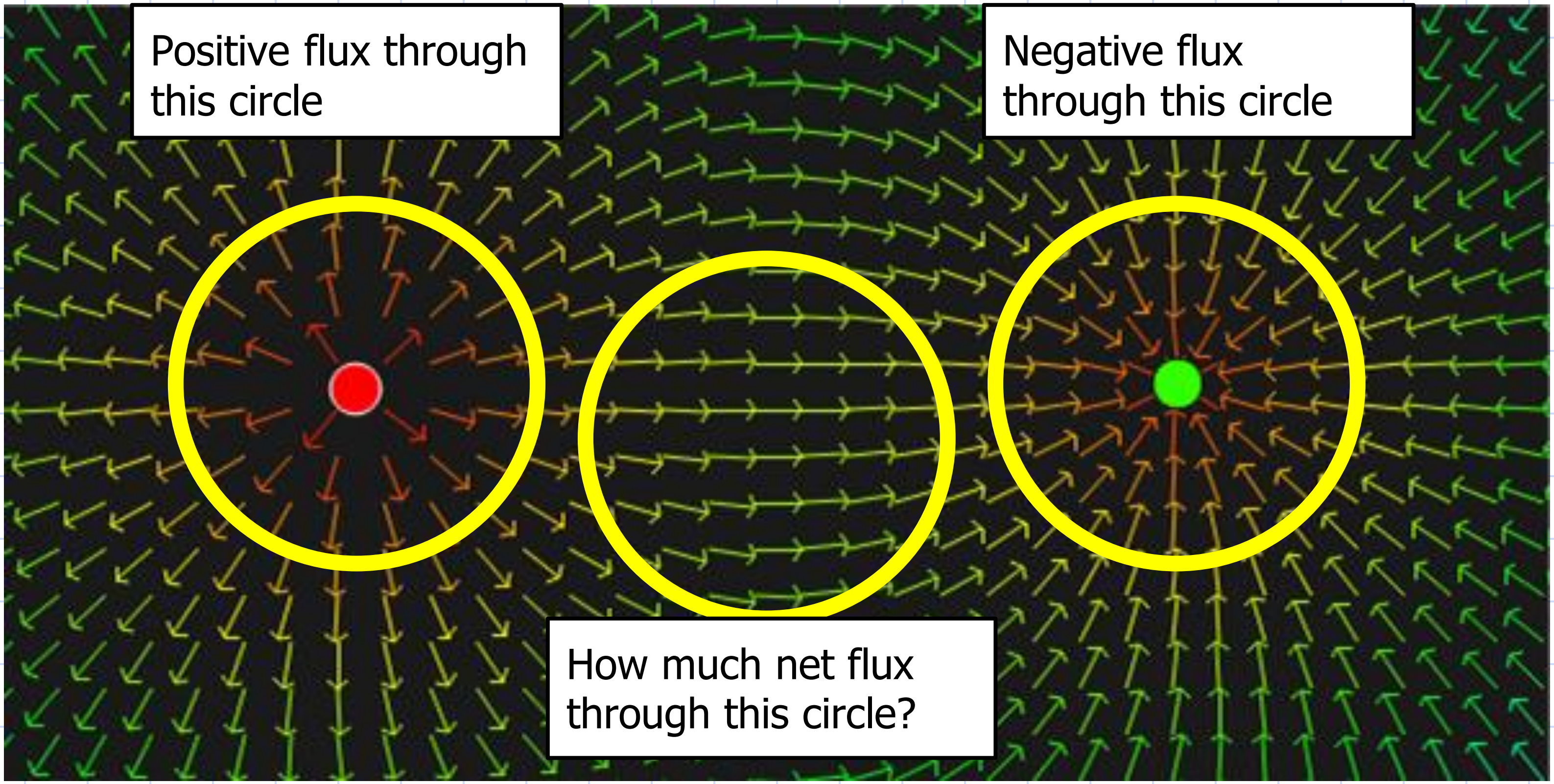
Maxwell's Equations



Field Math

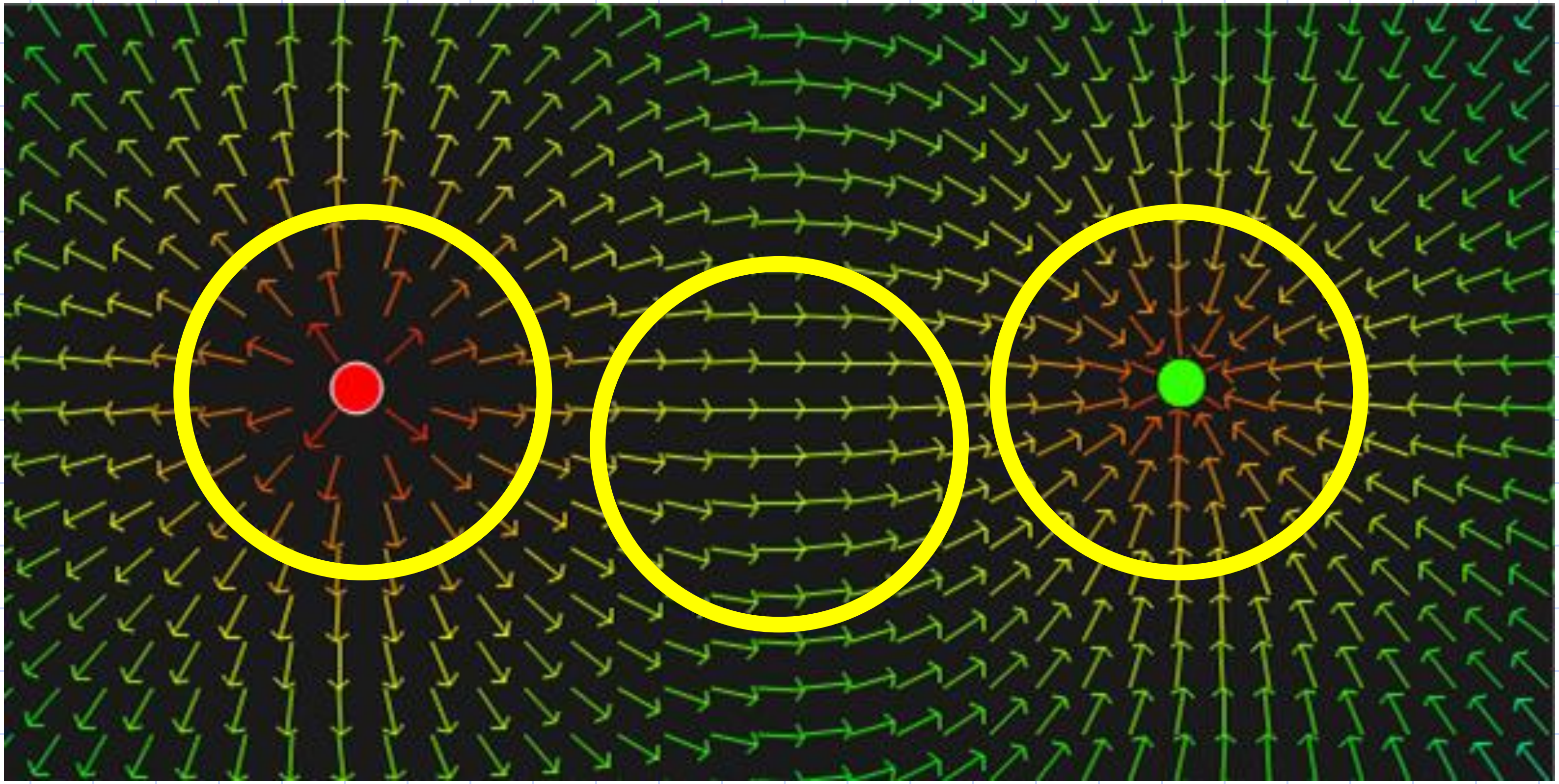
Maxwell's Equations

Flux = magnitude of flow through a surface



Field Math

Maxwell's Equations



You can see that the flux through any of these three circles has a relationship to the amount of charge inside. (To consider the effect of all the charge inside each circle, we must do the integral of the divergence operator inside the circle.)

Field Math

Divergence Operator

Divergence Theorem:

$$\oint \vec{A} \cdot \vec{ds} = \int (\nabla \cdot \vec{A}) \cdot dV$$

Volume integral on right is volume enclosed by surface on the left

In plain English:

"The amount and magnitude of field sources or sinks inside a volume will determine the flux through the volume's surface."

Field Math

Divergence Operator

"Global" quantities

$$\int A \bullet ds$$ Measures Flux through any surface

$$\oint A \bullet ds$$ Measures Flux through closed surfaces

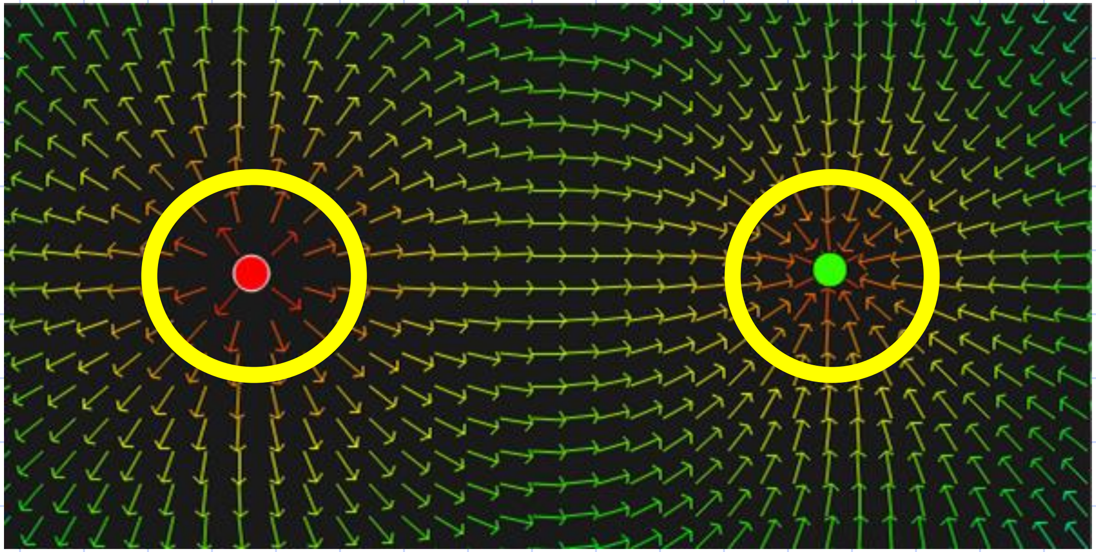
is related to

$$\nabla \bullet A$$, is a "local" measure of flux property

Divergence = degree to which a given point is a source or a sink

Field Math

Maxwell's Equations



$$\oint \vec{A} \cdot \vec{ds} = \int (\nabla \cdot \vec{A}) \cdot dV$$

One last important point:

The source of flux could be a *single point* with a nonzero divergence, or the divergence could come from an *area*. This parallels the fact that electric charges can be treated as points, but it is often more accurate to speak of an *area of charge distribution*.

Field Math

Divergence Operator

Calculate $\nabla \bullet \mathbf{A}$ for each of the vectors below.

a. $\mathbf{A} = x^2 y\, \mathbf{a}_x + c^2 x\, \mathbf{a}_z$

b. $\mathbf{A} = c/r^2\, \mathbf{a}_r + e^{-j\beta r} \sin\theta / r\, \mathbf{a}_\varphi$

a. $\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \cancelto{0}{\frac{\partial A_y}{\partial y}} + \frac{\partial A_z}{\partial z} = \frac{\partial}{\partial x}(x^2 y) + \cancelto{0}{\frac{\partial}{\partial z}(c^2 x)} = \boxed{2xy}$

b. $\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\cancelto{0}{\frac{\partial}{\partial \theta}(A_\theta \sin\theta)} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$

$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\cancel{r^2}\frac{c}{\cancel{r^2}}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\frac{e^{-j\beta r}\sin\theta}{r}\right) = \boxed{0}$

Field Math

Divergence Operator

Divergence operator simulation:

http://em8e.eecs.umich.edu/jws/ch3/mod3_3/mod3_3_webstart.jnlp

Field Math

What's the difference between these two whirlpools?


Field Math

What's the difference between these two whirlpools?


Circulation in opposite directions.
Their velocity fields have opposing curl.

Field Math

Curl Operator

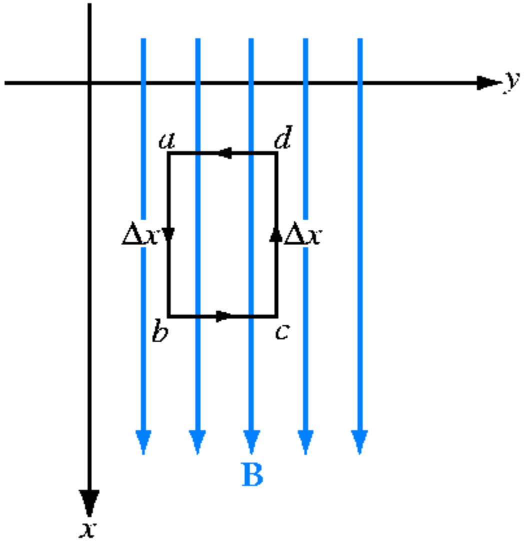
Curl can be calculated at a point using the following expression for Cartesian coordinates (Ulaby pg. 166).
Similar expressions exist for the other coordinate systems.

$$\nabla \times \mathbf{B} = \hat{\mathbf{x}}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}. \quad (3.105)$$

Field Math

Curl Operator

NOTATION: $\oint$ Implies a CLOSED LOOP Integral



(a) Uniform field

$$\oint \vec{B} \cdot \vec{dl}$$ measures circulation (related to Curl)

Example of a uniform field B in the x direction

$$\text{circulation} = \oint \vec{B} \cdot \vec{dl} \; =$$

$$\int_a^b B\hat{x} \bullet \hat{x}dx + \int_b^c B\hat{x} \bullet \hat{y}dy + \int_c^d B\hat{x} \bullet \hat{x}dx + \int_d^a B\hat{x} \bullet \hat{y}dy = 0$$

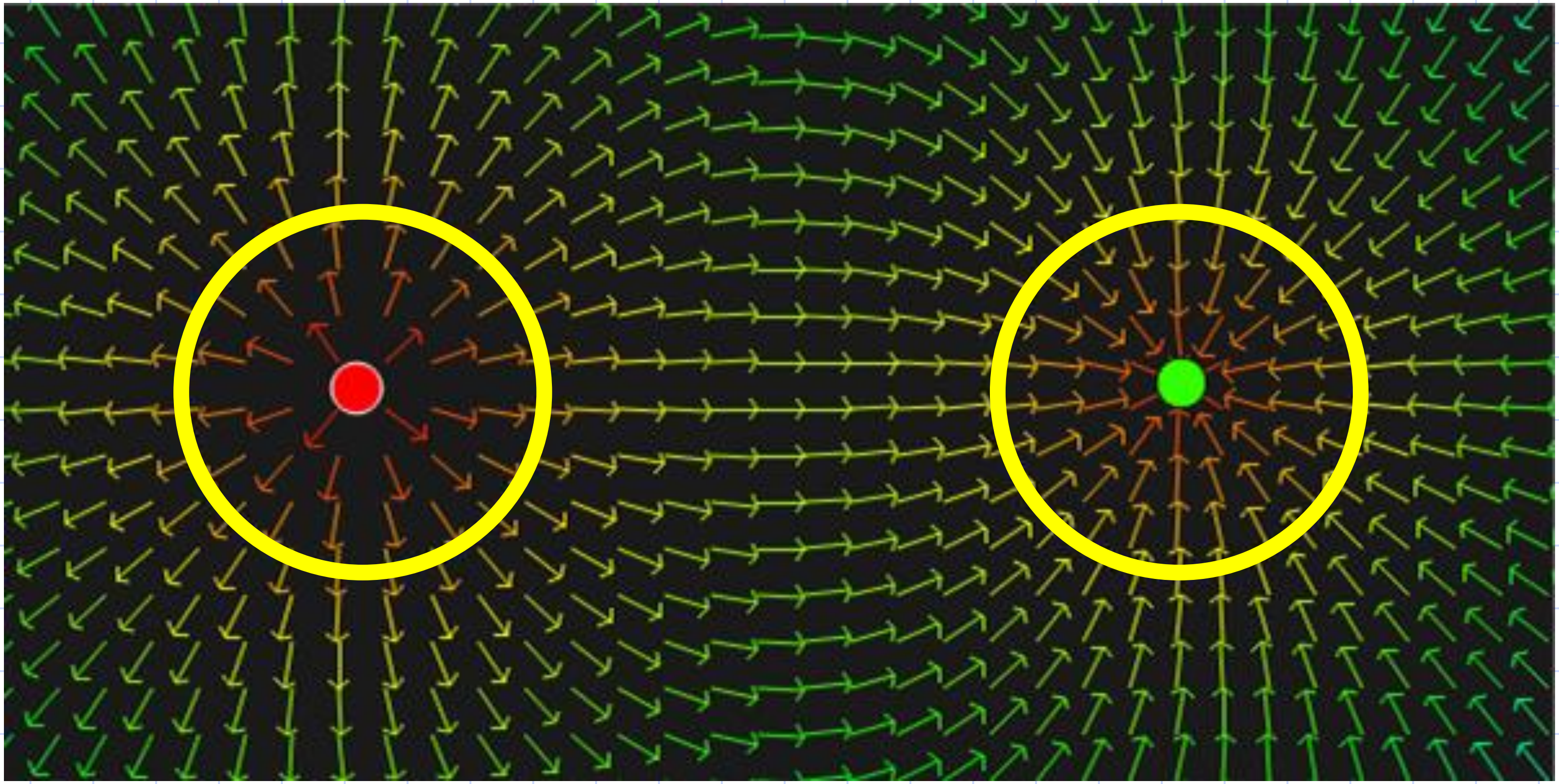
Field Math

The eye of the whirlpool is a place where the curl is nonzero. Whereas the point charges were *sources* or *sinks* of field lines, the whirlpool eye is a point that causes *circulation* of a vector field (in this case velocity) around it.

What is the curl inside the yellow circle?

Field Math

Maxwell's Equations



What is the curl of these two circles?

Field Math

Curl Operator

- The curl operator

NOTATION: $\nabla \times B$ Result of this operation is a VECTOR

This is NOT a CROSS-PRODUCT

- Stokes's theorem:

$$\oint \vec{B} \cdot \vec{dl} = \int\int \nabla \times \vec{B} \cdot ds$$

Surface integral on right is surface enclosed by line on the left

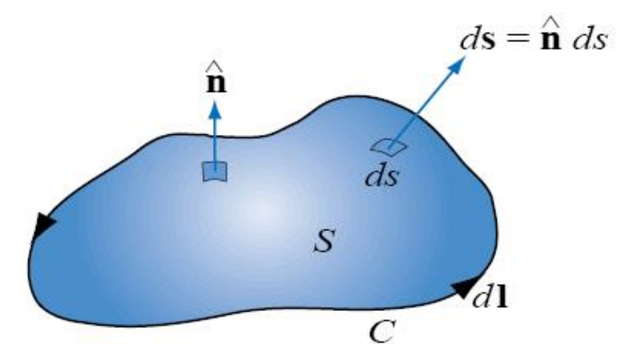


Figure 3-23

Field Math

Curl Operator

$$\oint \vec{B} \cdot \vec{dl} = \int\int \nabla \times \vec{B} \cdot ds$$

In plain English:

“The amount of **B** pointing in a loop around the outside of surface **ds** is determined by the total curl of field **B** inside ds.”

or:

“If you drop a ball near a whirlpool, the stronger the whirlpool is, the faster the ball will travel around it.”

Field Math

One last note on whirlpools:

We previously mentioned that flux is often due to a *distribution* of divergence.

Circulation of a vector field in practice is often due to a *distribution* of curl - that is, the vector field act less like it has one whirlpool and more like it has a distribution of infinitesimally-small whirlpools.

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Field Math

Curl Operator

Curl operator simulation:

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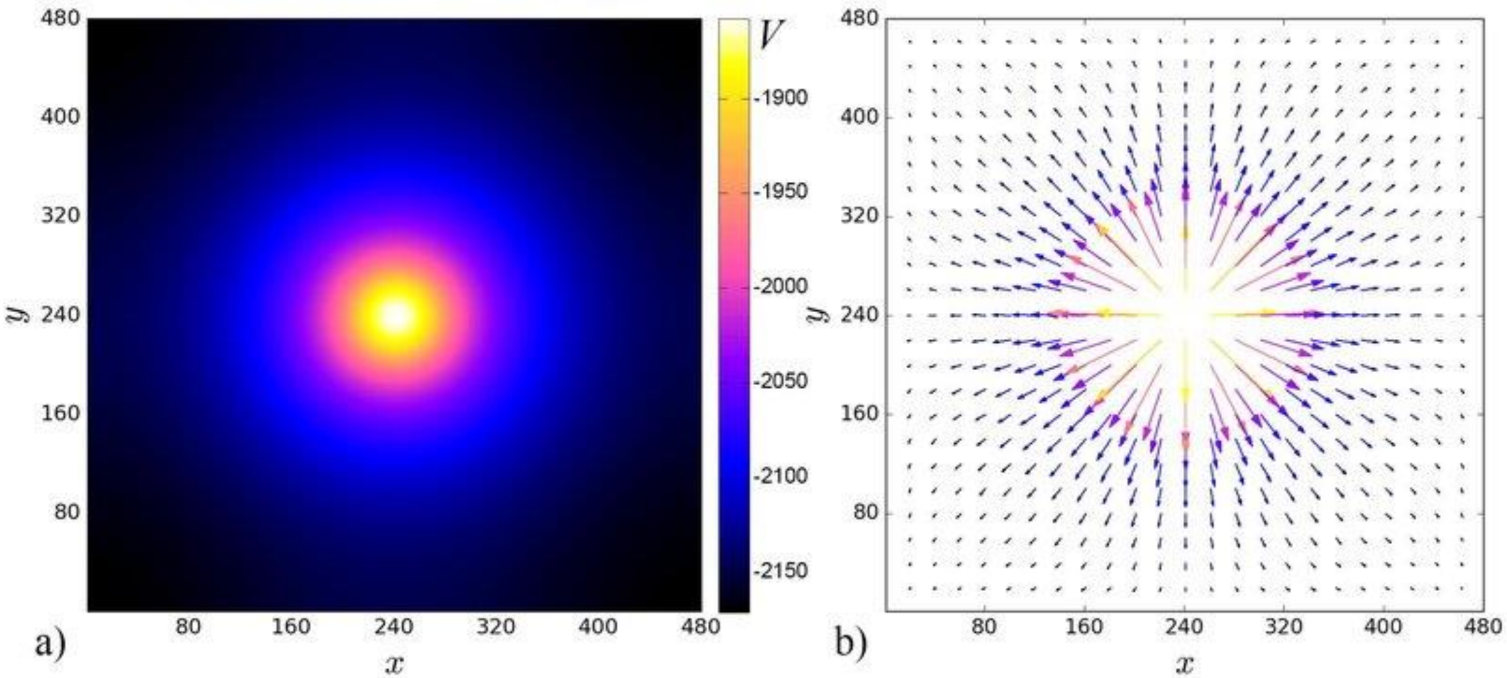
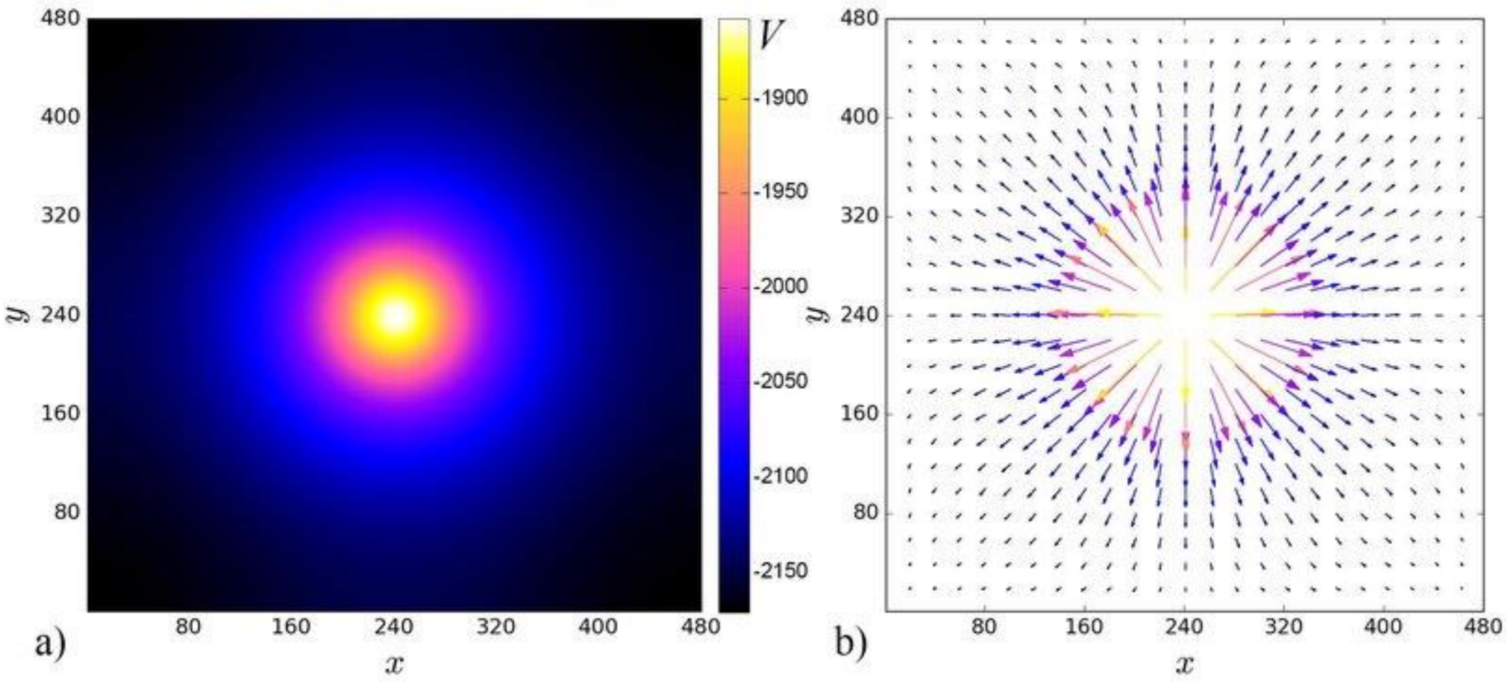
Field Math



What is the relationship between the potential field (left) and the electric field (right)?

Field Math



What is the relationship between the potential field (left) and the electric field (right)?
The electric field is defined by the gradient of the potential field.

Gradient operator

- The gradient operator

  GRADIENT measures CHANGE in a SCALAR FIELD

  - the result is a VECTOR pointing in the direction of increase

  For a Cartesian system:

$$\nabla f = \frac{\partial f}{\partial x} \cdot \hat{a}_x + \frac{\partial f}{\partial y} \cdot \hat{a}_y + \frac{\partial f}{\partial z} \cdot \hat{a}_z$$

- Main property

  You will find that $\nabla \times \nabla f = 0$ ALWAYS

Field Math

What is gradient? It's this.

Gfycat

Field Math

Gradient Operator

Gradient simulation:

http://em8e.eecs.umich.edu/jws/ch3/mod3_2/mod3_2_webstart.jnlp

Field Math

Gradient Operator

Compute the gradient of the following functions.

a. $f = 8a^2 \cos\phi + 2rz$ (cylindrical)

b. $f = a \cos 2\theta / r$ (spherical)

a. $\nabla f = \frac{\partial f}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{a}_\phi + \frac{\partial f}{\partial z}\hat{a}_z = \boxed{2z\,\hat{a}_r + \frac{1}{r}(-8a^2 \sin\phi)\hat{a}_\phi + 2r\,\hat{a}_z}$

$(f = 8a^2 \cos\phi + 2rz)$

b. $f = \frac{a\cos 2\theta}{r}$; $\nabla f = \frac{\partial f}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{a}_\phi$

$\nabla f = \frac{-a\cos 2\theta}{r^2}\hat{a}_r + \frac{1}{r}\frac{a}{r}(-2\sin 2\theta)\hat{a}_\theta$

Field Math

- Curl
  - Measures the circulation of a vector field

$$curl B \text{ or } \nabla \times B$$

Result is a <u>VECTOR</u>

- Gradient
  - Measures the change in a scalar field

$$grad(f) \text{ or } \nabla f$$

Result is a <u>VECTOR</u>

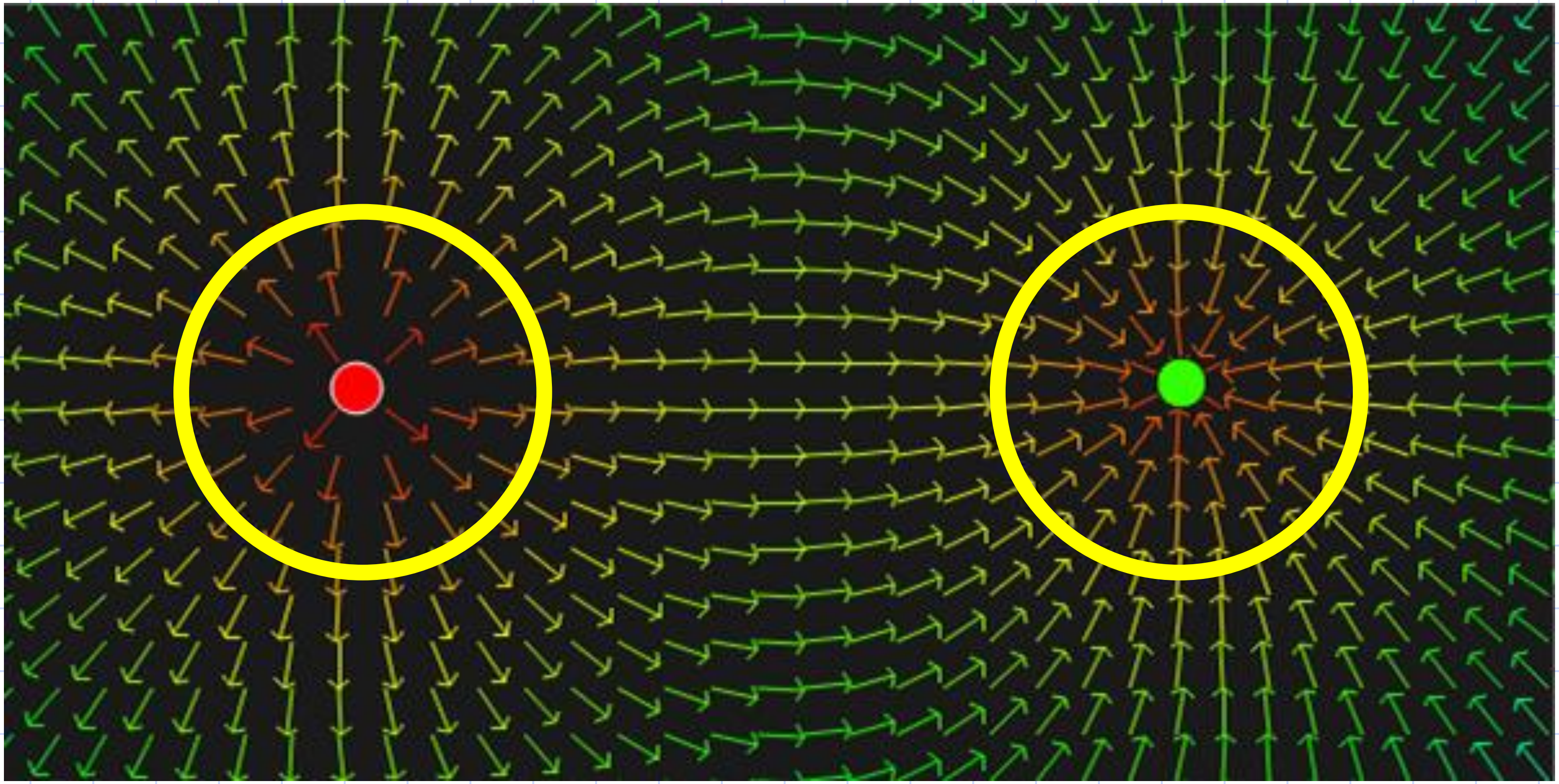
- Divergence
  - Measures the flux of a vector field through a surface

$$div A \text{ or } \nabla \bullet A$$

Result is a <u>SCALAR</u>

Field Math

Maxwell's Equations



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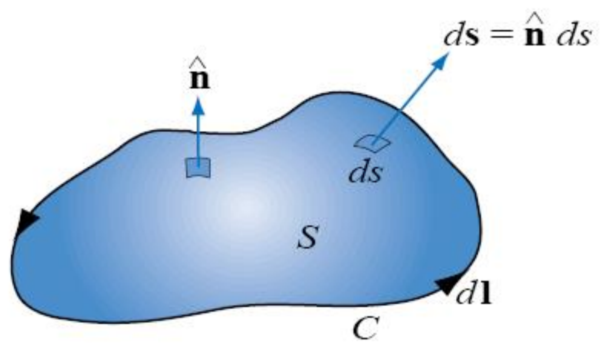

Figure 3-23

Field Math

Curl Operator

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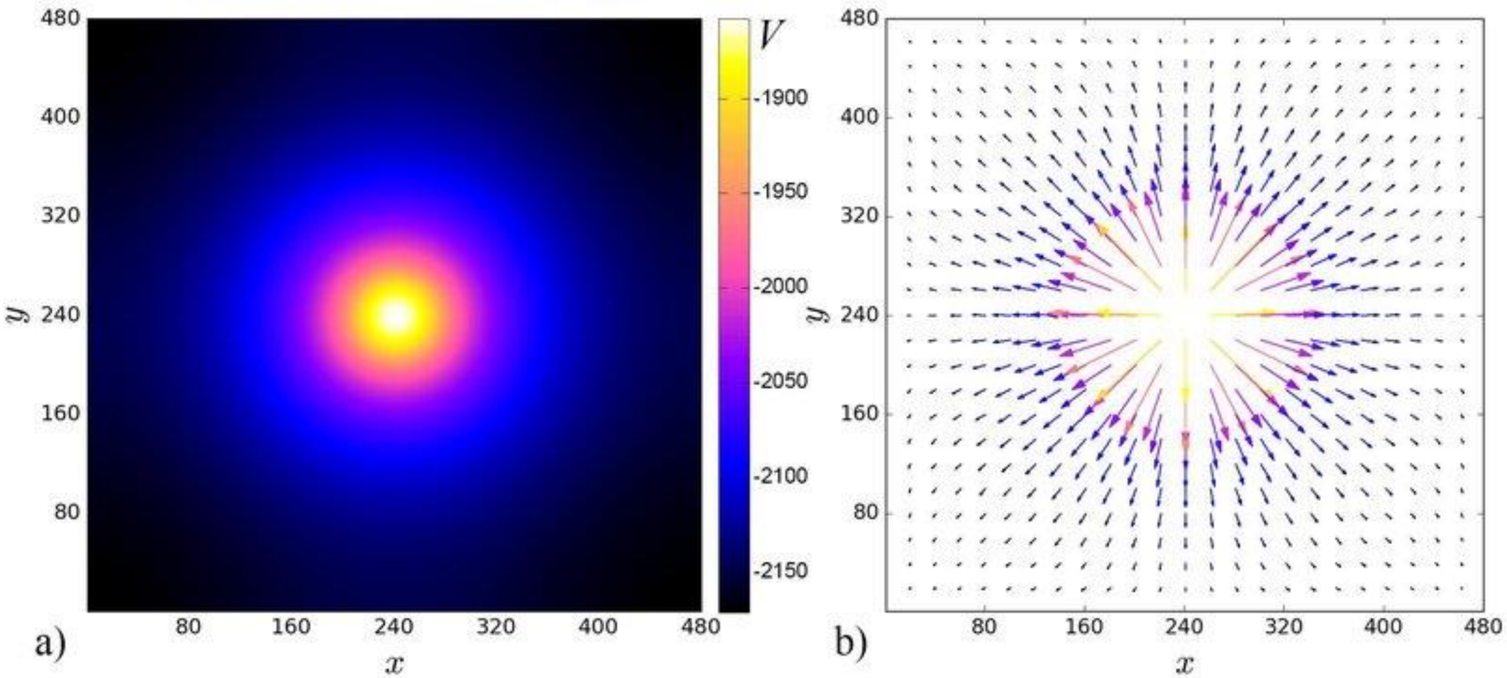
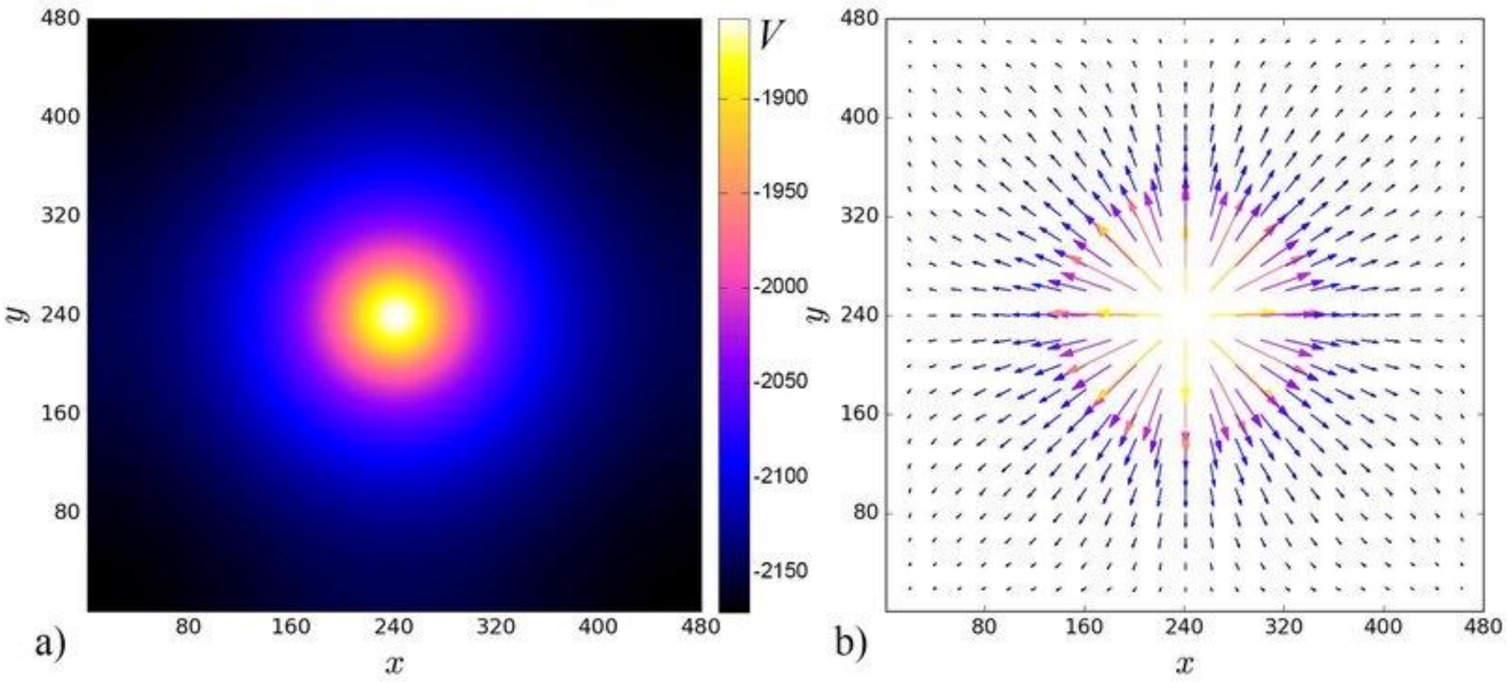
Field Math



What is the relationship between the potential field (left) and the electric field (right)?

Field Math



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Gfycat

Field Math

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$$(f = 8a^2\cos\phi + 2rz)$$

$$b.\quad f = \frac{a\cos 2\theta}{r}\;; \qquad \nabla f = \frac{\partial f}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{a}_\phi$$

$$\nabla f = \frac{-a\cos 2\theta}{r^2}\hat{a}_r + \frac{1}{r}\frac{a}{r}\left(-2\sin 2\theta\right)\hat{a}_\theta$$

Field Math

- Curl
  - Measures the circulation of a vector field

$$curl B \text{ or } \nabla \times B$$

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  - Measures the change in a scalar field

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Result is a <u>VECTOR</u>

- Divergence
  - Measures the flux of a vector field through a surface

$$div A \text{ or } \nabla \bullet A$$

Result is a <u>SCALAR</u>

Electrostatics

Today we will begin applying the mathematical tools from last lecture to solve Maxwell's equations. But given the complexity of this task, we will start with a simplified case: electrostatics.

- Charges do not move
- No current
- As a consequence, there is no magnetic field; only electric field

Electrostatics

Full Version of Maxwell's Equations

Integral Form | Differential Form

$$\oint \vec{B} \cdot d\vec{S} = 0 \qquad \nabla \cdot \vec{B} = 0$$

$$\oint \vec{D} \cdot dS = \oint \rho dv = Q_{encl} \qquad \nabla \cdot \vec{D} = \rho$$

$$\oint \vec{H} \cdot d\vec{l} = \int \vec{J} \cdot d\vec{S} + \int \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S} \qquad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{S} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Electrostatics

Electrostatic Version of Maxwell's Equations

Integral Form | Differential Form

$$\oint \vec{D} \cdot d\vec{S} = \int \rho dV \qquad \nabla \cdot \vec{D} = \rho$$

$$\oint \vec{E} \cdot d\vec{l} = 0 \qquad \nabla \times \vec{E} = 0$$

Electrostatics

Integral Form

Differential Form

$$\boxed{\oint \vec{D} \cdot d\vec{S} = \int \rho dV \qquad \nabla \cdot \vec{D} = \rho}$$

$$\oint \vec{E} \cdot d\vec{l} = 0 \qquad \nabla \times \vec{E} = 0$$

These equations represent Gauss's Law (just written in a different form)

Electrostatics

Integral Form Differential Form

$$\oint \vec{D} \cdot d\vec{S} = \int \rho dV \qquad \nabla \cdot \vec{D} = \rho$$

$$\boxed{\oint \vec{E} \cdot d\vec{l} = 0 \qquad \nabla \times \vec{E} = 0}$$

These equations tell you that in electrostatics, the E-field has no curl (and there are no net voltages around loops).

Electrostatics

There are 2 different ways of writing the electric field.

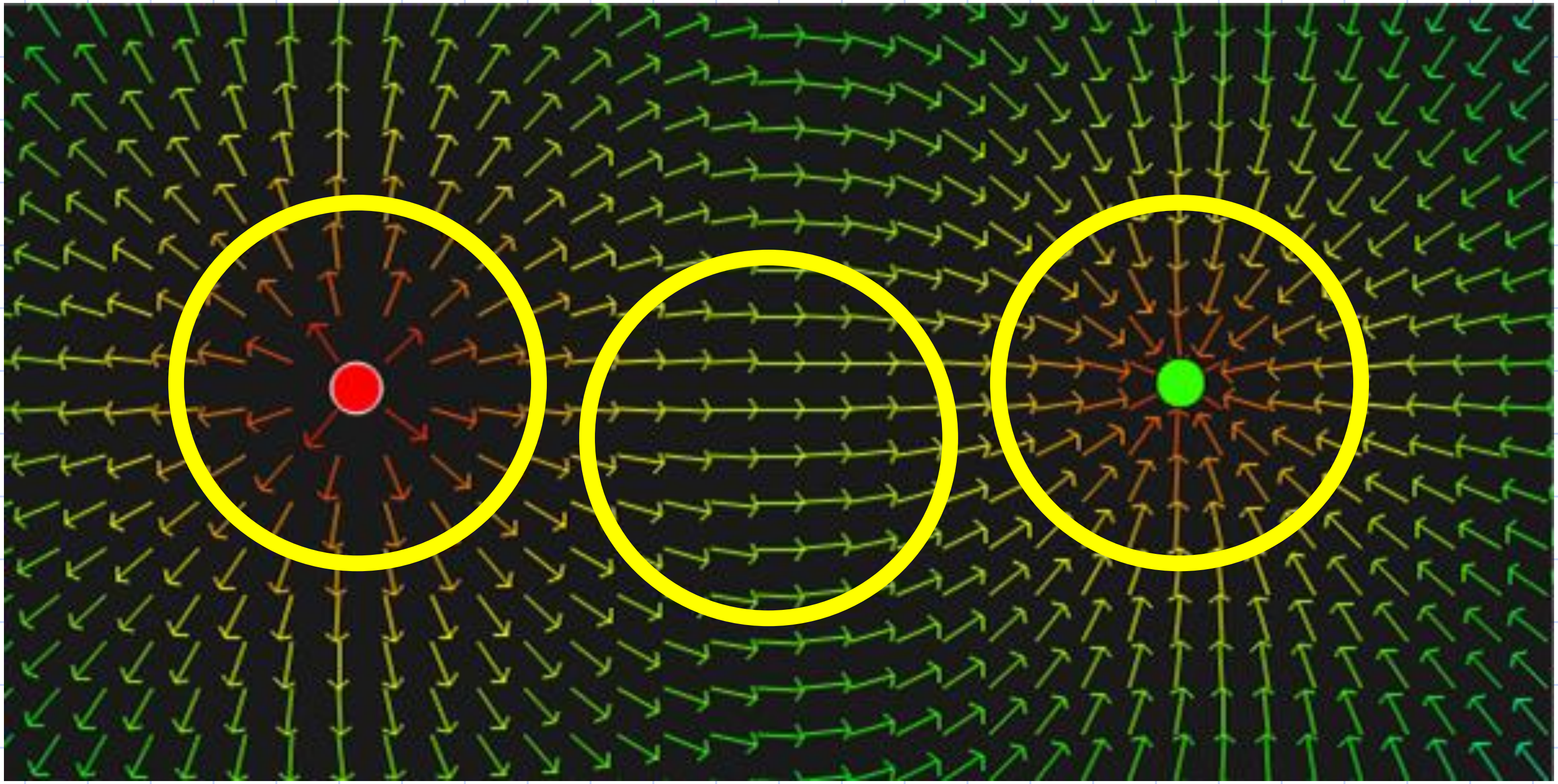
$\vec{E}$ (often called just called "electric field" or "E-field") has units of volts per meter (V/m). Thus it has a direct relationship with voltage.

$\vec{D}$ (often called "electric displacement field"), has units of coulombs per square meter (C/m^2) and thus has a direct relationship with charge or charge density.

The two quantities are related by: $\vec{D} = \epsilon \vec{E}$

ϵ represents *permittivity*, a property of the medium that the electric field is propagating through.

Electrostatics



Consider these three circles containing different amounts of charge.

Electrostatics

$$\nabla \cdot \vec{D} = \rho$$

We had stated that the points that contain charge have a non zero divergence, whereas the points with no charge have a divergence of zero. *(This is the differential form of Gauss's Law.)*

Electrostatics

$$\oint \vec{D} \cdot d\vec{S} = \int \rho dV$$

This could take the form of point charges or line, surface, or volume charge density

The integral form of Gauss's Law states that the flux through the circles is determined by the amount of charge inside. This can be rewritten as:

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q_{enc}}{\epsilon}$$

for empty space, $\epsilon = \epsilon_0$ (8.85e-12 farads per meter)

Electrostatics

Show that the electric field of a point charge satisfies Gauss' Law by integrating $\oint \vec{E} \cdot d\vec{S}$ over the surface of a sphere of radius *a*.

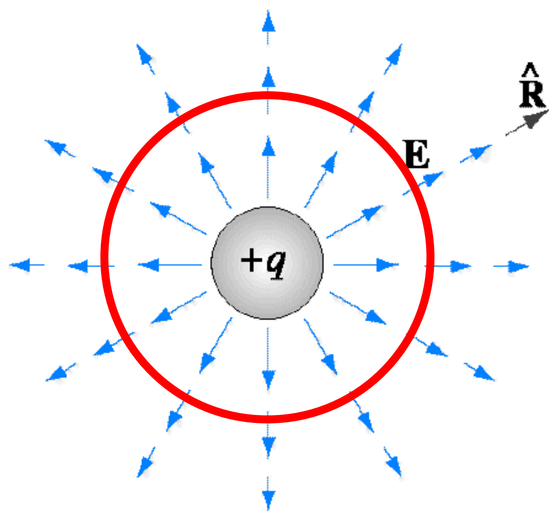


Figure 1-5

Electrostatics

For point charge $\vec{E} = \dfrac{Q}{4\pi\varepsilon_0 r^2}\,\hat{a}_r$

$\vec{ds}$

$\vec{ds} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{a}_r$

$$\oint \vec{E}\cdot\vec{ds} = \int_0^{2\pi}\int_0^{\pi} \frac{Q}{4\pi\varepsilon_0 \cancel{r^2}}\,\cancel{r^2}\,\sin\theta \, d\theta \, d\phi$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(-\cos\theta\Big]_0^{\pi}\right) 2\pi = \frac{Q}{\varepsilon_0}$$

Gauss' law $\oint \vec{D}\cdot\vec{ds} = Q_{enc} \Rightarrow \varepsilon_0 \oint \vec{E}\cdot\vec{ds} = Q$ ✓

NOTE: E is constant on surface of sphere

$\therefore \oint \vec{E}\cdot\vec{ds} = E_r \int ds_r = E_r \, 4\pi r^2 = Q/\varepsilon_0$

Electrostatics

Coulomb's Law

$\vec{E}$,of Q_1 is $= \dfrac{Q_1}{4\pi\varepsilon_0 R^2} \cdot \hat{a}_R$

Unit vector pointing away from Q_1

Then,

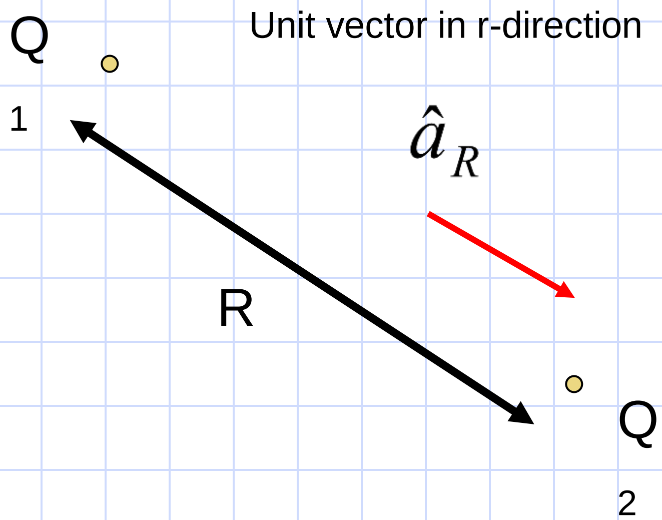
$$\vec{F}_{12} = Q_2 \cdot \vec{E}$$

- we work with <u>E-Field</u> because Maxwell's equations written in those terms

Electrostatics

Coulomb's Law

$\vec{F}$ (force), between point charges



$$\vec{F}_{12} = \frac{Q_1 \cdot Q_2}{4\pi\varepsilon_0 R^2} \cdot \hat{a}_R$$

Force on Charge 2 by Charge 1

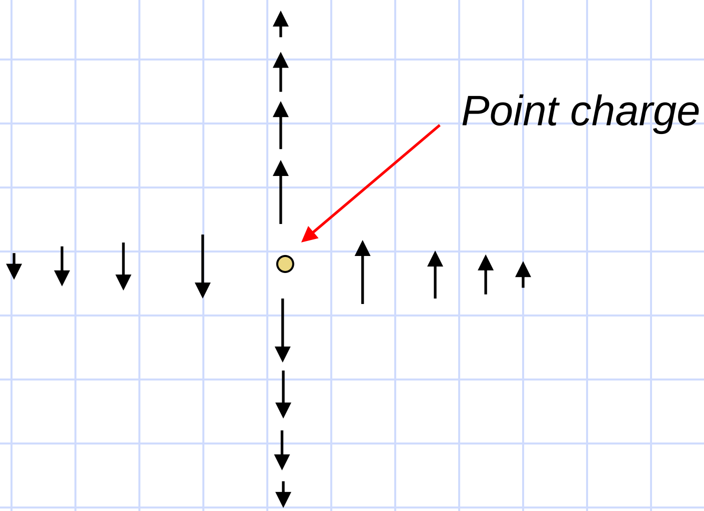
Electrostatics

$\vec{E}$, is a VECTOR Field *How do we represent it?*

- Field points in the direction that a +q test charge would move

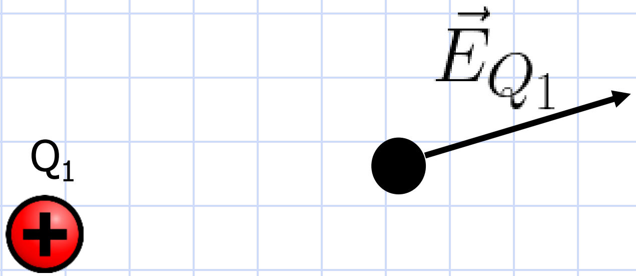
Represent using Arrows : Direction and Length

Proportional to Magnitude or strength of E-Field



Electrostatics

To compute E-fields from multiple charges, apply superposition of fields.



Electrostatics

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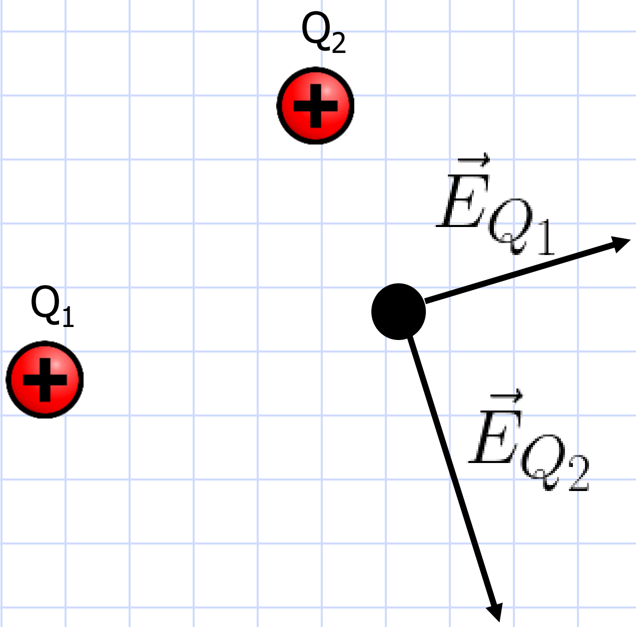

Electrostatics

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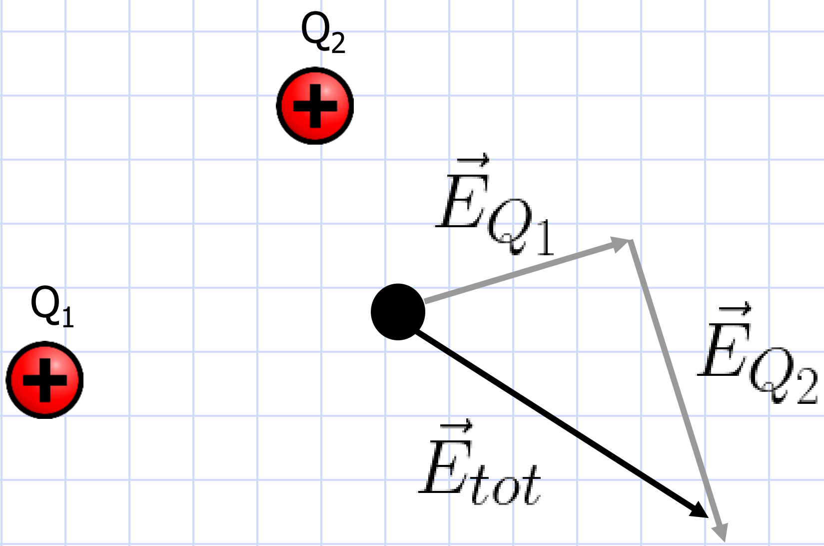
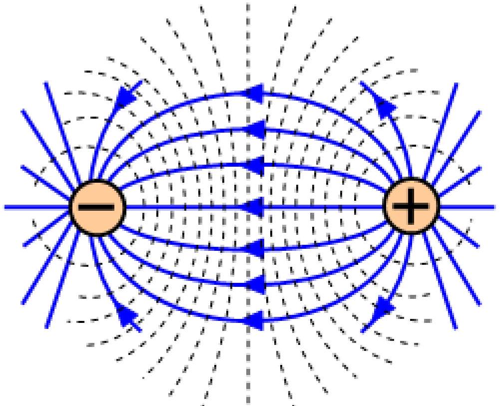


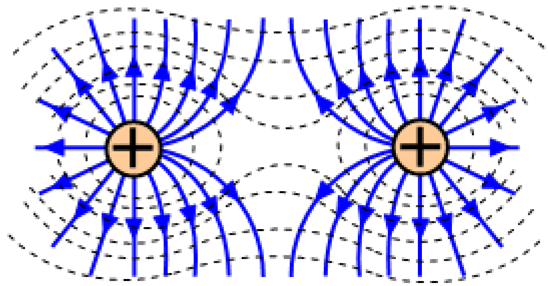
Electrostatics

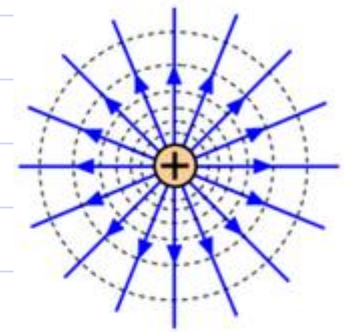
Do Lecture 10 Exercise 1 in groups of up to 4.

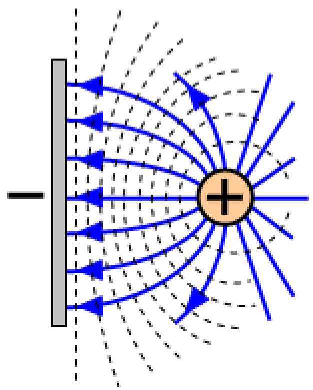
Electrostatics

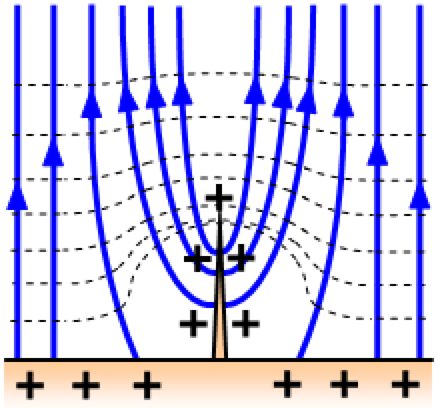
Some examples of E-Fields



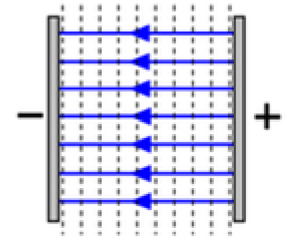






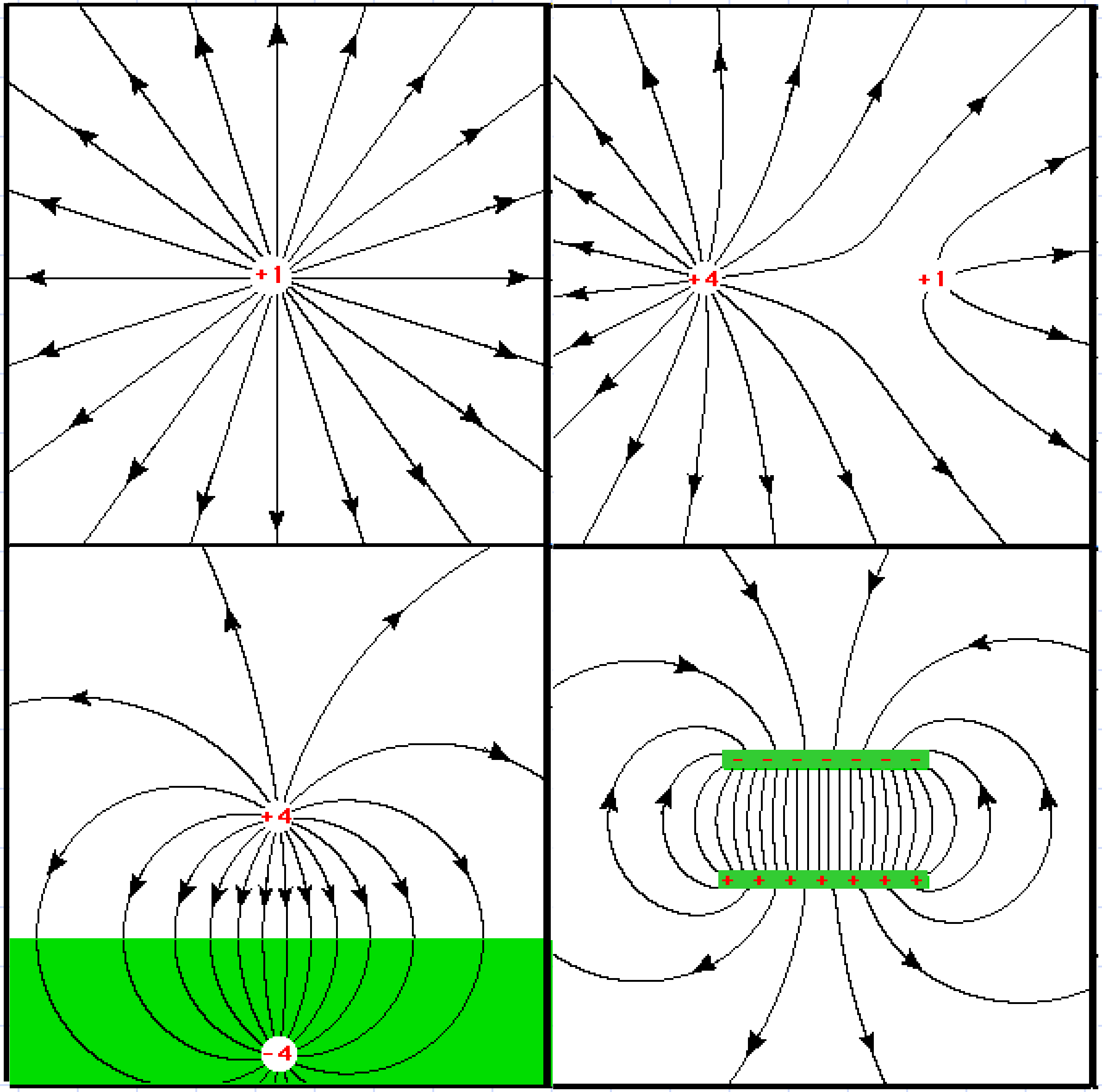


(resourcefulphysics.org)



http://shinliang.blogspot.com/2009/04/21-coulombs-law.html

Electrostatics

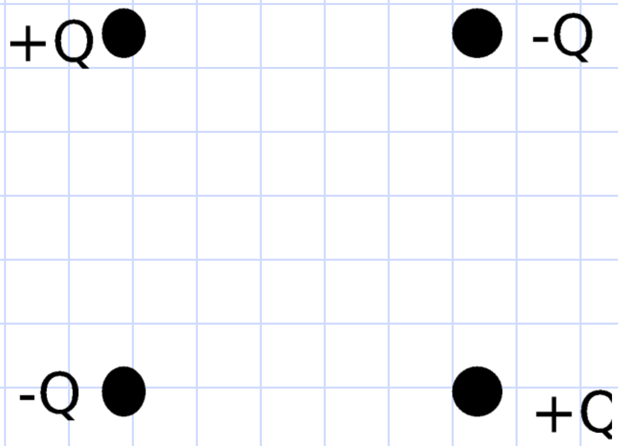


Note, in the upper right figure, that four times as many field lines leave the +4 positive charge as leave the +1 charge. All of the field lines end at infinity, as they do with a single positive charge.

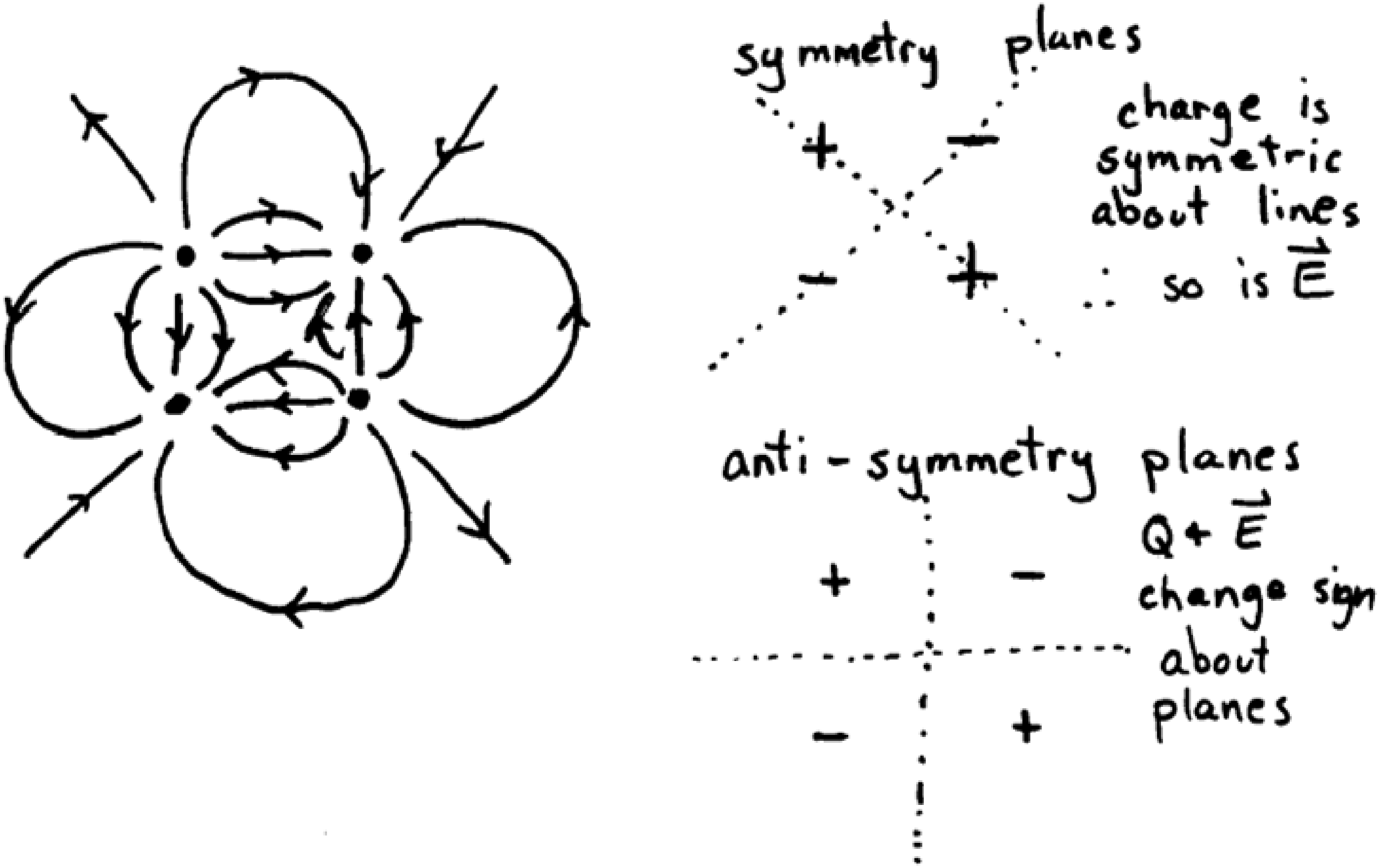
http://people.seas.harvard.edu/~jones/cscie129/nu_lectures/lecture6/field_vis/e_vis.html

Electrostatics

Sketch the electric field lines for the electric quadrupole shown. Sketch the planes for which you expect the field to be symmetric. *After completing your sketch,* verify your result with the applet at https://davidawehr.com/projects/electric_field.html



Electrostatics



Gauss's Law

$$\Phi_E = \frac{Q}{\varepsilon_0}$$

$$\Phi_E = \oiint \vec{E} . d\vec{A}$$