Homework 1 Solutions

1. Coaxial Cables

a) The Ulaby textbook's companion website (https://em8e.eecs.umich.edu/) has many resources that we'll make use of during the course this semester. One of these resources is a set of applications that allow you to interact with fundamental concepts in electricity and magnetism. Navigate to the site and click on the Modules tab.

b) Once on the Interactive Modules page, click on "Chapter 2: Transmission Lines", then "Coaxial Cable". Coaxial cables are one of the main types of transmission line, which you'll find in many different applications. It may take some time to load (and you may need to install Java).

c) Use the drop-down menu next to "Select" to select "Impedance vs. Radius a".

d) Let's consider the case of a coaxial cable with an inner diameter (*2a*) of 3mm, an outer diameter (*2b*) of 9mm, and a dielectric constant (ε_r) of 2.1. What is the characteristic impedance of this coaxial cable to the nearest ohm? Additionally, we will be considering a lossless coaxial cable here, so ignore imaginary components of the characteristic impedance Z_0 (and thus any frequency-dependence of Z_0).

Z_0 = 45 Ohms [1 pt]

e) What is the velocity of the signal on this coaxial cable in m/s?

u = 1/sqrt(c*l) = 1/sqrt(106.3pF/m*219.5nH/m) = 2.07×10^8 m/s [2 pts: 1 for eq; 1 for calculation]

f) If the wavelength of the signal being transmitted on this coaxial cable is 2m, what are the approximate phase constant β in rad/m and the frequency *f* in Hz?

β = 2*π/ λ = 2*π/2m = 3.14 rad/m; [2 pts: 1 pt for eq; 1 pt for calculation]

f = u / λ = 2.07×10^8 [m/s] / 2 [m] = 1.035×10^8 Hz = 103.5 MHz [2 pts: 1 pt for eq; 1 pt for calculation]

NOTE: we should accept correct answers in Hz and rad/s since I accidentally asked for ω in Hz: 103.5 MHz = 650.3×10^6 rad/s.

g) Now consider the case of a lossy transmission line, which the app calculates by default. How does Z_0 change as frequency increases? For a given frequency, what happens to Z_0 as the conductivity of the conductor (σ_c) decreases (as the line resistance increases)?

As frequency increases, the imaginary part of Z_0 gets smaller. As the conductivity decreases for a given frequency, the imaginary part of Z_0 increases. [2 pts: 1 for each question]

2. Phasor Notation

Note: Remember when you convert from the *z-t* domain to phasor notation, that you first have to convert to a cosine base, and that when converting from phasor notation back to the *z-t* domain, you obtain a cosine.

a) Convert the following voltage from the time domain into phasor notation: $V(t) = 5j\sin(\omega t)$

Step 1 is to convert from a sin base to a cosine base by shifting the sine wave by +π/2:

$$V(t) = 5jsin(\omega t) = 5jcos\left(\omega t - \frac{\pi}{2}\right) \text{ [1 pt]}$$

Next, since V(t) is complex, we need to convert the *j* in front of the cosine to a phase:

$$V(t) = 5jcos\left(\omega t - \frac{\pi}{2}\right) = 5cos\left(\omega t - \frac{\pi}{2}\right) e^{j\frac{\pi}{2}} \text{ [1 pt]}$$

Finally, we can convert this expression into a phasor form:

$$\tilde{V} = 5e^{-\frac{j\pi}{2}} e^{\frac{j\pi}{2}} = 5 = 5 \text{ [2 pts: 1 for approach; 1 for calculation]}$$

b) Convert the following current phasor into the time domain: $\tilde{I} = 0.10e^{-j(\beta z + \frac{\pi}{4})}$.

The formula for converting a phasor back into the *t*-*z* domain is:

$$V(z,t) = Re\{\tilde{V}e^{j\omega t}\} = 0.10 \cos\left(\omega t - \beta z - \frac{\pi}{4}\right) \text{ [2 pts: 1 for correct approach; 1 for calculation]}$$

c) Is the current wave in b) a traveling wave or standing wave? If it's a traveling wave, which direction is it traveling?

The wave in b) is a traveling wave which is traveling in the +z direction. [2pts: 1 for each question]

3. Traveling Waves and Standing Waves Part 1

The voltage and current on a transmission line are given by

$$v(z,t) = 9 \cos(2\pi \times 10^7 t + 1.5\pi z)$$

$$i(z,t) = 0.09 \cos(2\pi \times 10^7 t + 1.5\pi z)$$

a) Are the voltage and current waves traveling waves or standing waves or neither? If they are traveling waves, what is their frequency *f* in Hz and which direction are they traveling (+z or -z)?

Traveling waves take the form $\cos(\omega t \pm \beta z + \varphi)$ [3 pts: 1 for traveling wave; 1 for frequency; 1 for correct direction]

Both waves are traveling waves with a frequency $f = 1 \times 10^7$ Hz and in the -z direction (β is positive).

b) What are the period *T* and wavelength λ of the voltage and current waves?

$T = 1/f = 1/(1 \times 10^7 \text{ Hz}) = 100\text{ns}$ [1.5 pts: 1 for equation; 0.5 for calculation]
$\lambda = 2*\pi/\beta = 2*\pi/(1.5\pi \text{ rad/s}) = 1.33\text{m}$ [1.5: 1 for equation; 0.5 for calculation]

c) What is the characteristic impedance of the transmission line?

$Z_0 = V_0/I_0 = 9\text{V} / 0.09\text{A} = 100\Omega$ [2 pts: 1 for equation; 1 for calculation]

d) What is the velocity of the signal on the transmission line?

$u = f * \lambda = 1 \times 10^7 \text{ Hz} * 1.33\text{m} = 1.33*10^7 \text{ m/s}$ [2 pts: 1 for equation; 1 for calculation]

e) What are the capacitance and inductance per unit length of the transmission line?

$u = 1/\text{sqrt}(l*c)$ and $Z_0 = \text{sqrt}(l/c)$. Using both equations: $c = 1/(u*Z_0)$ and $l = Z_0/u$
$c = 1/(1.33*10^7 \text{ m/s} * 100\Omega) = 752$ pF/m [2 pts: 1 for equation; 1 for calculation]
$l = 100\Omega/(1.33*10^7 \text{ m/s}) = 7.52$ μH/m [2 pts: 1 for equation; 1 for calculation]

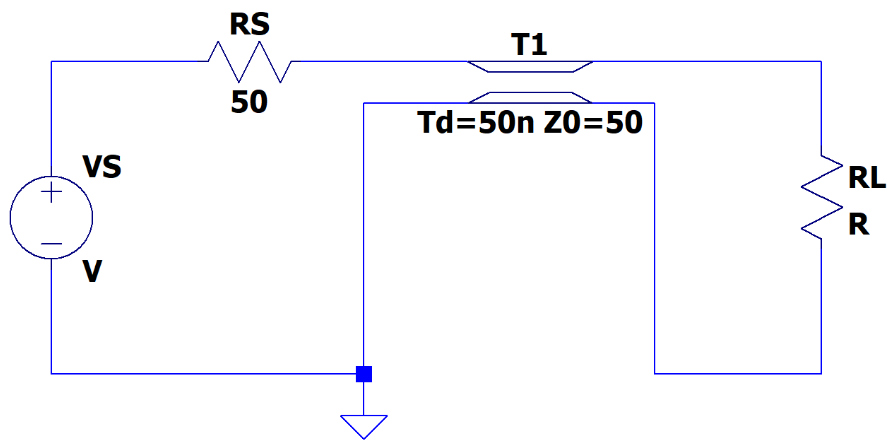
f) Express both the voltage and current waves in phasor form.

$\tilde{V}(z) = 9e^{j1.5\pi z}$ [2 pts: 1 for approach; 1 for calculation]

$\tilde{I}(z) = 0.09e^{j1.5\pi z}$ [2 pts: 1 for approach; 1 for calculation]

4. Traveling Waves and Standing Waves Part 2

Consider the following system consisting of a voltage source V_s, source resistance $R_s = 50\Omega$, transmission line with characteristic impedance $Z_0 = 50\Omega$, and load resistance R_L, as shown below. At time t = 0, the voltage source V_s is switched on and sends a voltage wave of the form $\tilde{V} = 2V^+e^{-j\beta z}$ towards the transmission line.



a) If $R_L = 50\Omega$, write the expression for $\widetilde{V_L}$, the voltage across the load resistor R_L. How much of the incident wave $\tilde{V}$ is reflected between the end of the transmission line and R_L?

Due to the voltage divider between R_S and Z_0, only half of the source voltage V_s appears at the input to the transmission line, so that the voltage at the input to the t-line is

$\tilde{V}_{in} = V^+e^{-j\beta z}$. [1 pt for correct input voltage]

Since the load resistance R_L is perfectly matched to the characteristic impedance of the t-line, Γ=0 and V^- (the reflected wave component of the voltage on the t-line) is zero. Therefore, no voltage wave is reflected and the entire voltage wave traveling down the t-line is transmitted to the load resistance R_L, thus:

$\tilde{V}_L = V^+e^{-j\beta z}$ [1 pt for correct reflection coefficient; 1 pt for correct voltage]

b) For the rest of the problem, suppose instead that $R_L = 0$. What is the reflection coefficient Γ? How much of the incident wave is reflected when the load is a short circuit?

$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{0\Omega - 50\Omega}{0\Omega + 50\Omega} = -1$ [2pts: 1 pt for equation; 1 pt for calculation]

100% of the incident wave is reflected and none of it is transmitted to the load R_L [1 pt]

c) Write the expression for the total voltage on the transmission line, including both incident and reflected waves. Recall that the general solution for the Telegrapher's Equations in phasor form is $\tilde{V} = V^+e^{-j\beta z} + V^-e^{+j\beta z}$ and that $V^- = \Gamma V^+$. Leave your answer in phasor form and in terms of V^+.

The general solution to the Telegrapher's Equations for voltage is

$$\tilde{V} = V^+e^{-j\beta z} + V^-e^{+j\beta z}$$

From b), we calculated that $V^- = -V^+$, so we can plug that into the solution above to obtain [1 pt]

$\tilde{V} = V^+(e^{-j\beta z} - e^{+j\beta z})$ [1 pt]

d) Convert your answer to the *z-t* domain. Your answer should be in the form of a sum of two cosines.

Converting from phasor form back to the *z-t* domain:

$V(z,t) = Re\{\tilde{V}e^{j\omega t}\} = V^+ [\cos(\omega t - \beta z) - \cos(\omega t + \beta z)]$ [2 pts: 1 pt for approach; 1 pt for calculation]

e) Using the trigonometric identity $\cos(\theta) - \cos(\phi) = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$, convert your answer into a product of sine functions. What kind of wave is this?

Using the above identity, we can convert the answer to d) into the following expression:

$V(z,t) = V^+ [\cos(\omega t - \beta z) - \cos(\omega t + \beta z)]$

$= -2V^+ [\sin(\omega t)\sin(-\beta z)] = 2V^+ [\sin(\omega t)\sin(\beta z)]$ [2 pts: 1 pt for correct approach; 1 pt for calculation]

This is a standing wave! [1 pt]

f) What is the reflection coefficient when the load resistance R_L is an open circuit ($R_L=\infty$)? Would you expect a standing wave to form in this case? Why or why not?

The reflection coefficient in the case of an open circuit is

$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{\infty\Omega - 50\Omega}{\infty\Omega + 50\Omega} = +1$ [2 pts: 1 pt for correct approach; 1 pt for calculation]

This leads to a case in which the reflected wave amplitude is $V^- = +V^+$, which also forms a standing wave, but this time the standing wave is a product of cosines.

$\tilde{V} = 2V^+ [\cos(\omega t)\cos(\beta z)]$ [2 pts: 1 pt for correct answer; 1 pt for correct justification]