Rensselaer Polytechnic Institute
Department of Electrical, Computer, and Systems Engineering
ECSE 2500: Engineering Probability, Spring 2023
Homework #7 Solutions

1. (a) Each value of X in $[5,7]$ has a $\frac{1}{3}$ probability of occurring since the distribution is uniform. If we let E be the number of eggs available, the distribution of E given X is a binomial with probability of success $\frac{1}{2}$. The number of eggs is the carton Y will then be $\max(6,E)$. That is, once we have 6 eggs, we don't collect any more. So the joint PMF table looks like:

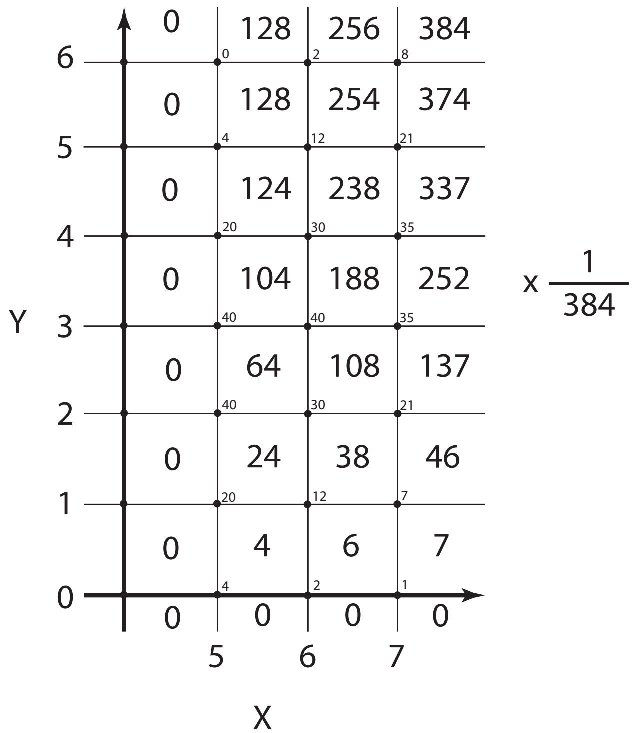
		X		
		5	6	7
	0	$\frac{1}{3}\cdot\left(\frac{1}{2}\right)^5$	$\frac{1}{3}\cdot\left(\frac{1}{2}\right)^6$	$\frac{1}{3}\cdot\left(\frac{1}{2}\right)^7$
	1	$\frac{1}{3}\cdot 5\cdot\left(\frac{1}{2}\right)^5$	$\frac{1}{3}\cdot 6\cdot\left(\frac{1}{2}\right)^6$	$\frac{1}{3}\cdot 7\cdot\left(\frac{1}{2}\right)^7$
	2	$\frac{1}{3}\cdot\binom{5}{2}\cdot\left(\frac{1}{2}\right)^5$	$\frac{1}{3}\cdot\binom{6}{2}\cdot\left(\frac{1}{2}\right)^6$	$\frac{1}{3}\cdot\binom{7}{2}\cdot\left(\frac{1}{2}\right)^7$
Y	3	$\frac{1}{3}\cdot\binom{5}{3}\cdot\left(\frac{1}{2}\right)^5$	$\frac{1}{3}\cdot\binom{6}{3}\cdot\left(\frac{1}{2}\right)^6$	$\frac{1}{3}\cdot\binom{7}{3}\cdot\left(\frac{1}{2}\right)^7$
	4	$\frac{1}{3}\cdot\binom{5}{4}\cdot\left(\frac{1}{2}\right)^5$	$\frac{1}{3}\cdot\binom{6}{4}\cdot\left(\frac{1}{2}\right)^6$	$\frac{1}{3}\cdot\binom{7}{4}\cdot\left(\frac{1}{2}\right)^7$
	5	$\frac{1}{3}\cdot\left(\frac{1}{2}\right)^5$	$\frac{1}{3}\cdot\binom{6}{5}\cdot\left(\frac{1}{2}\right)^6$	$\frac{1}{3}\cdot\binom{7}{5}\cdot\left(\frac{1}{2}\right)^7$
	6	0	$\frac{1}{3}\cdot\left(\frac{1}{2}\right)^6$	$\frac{1}{3}\cdot\binom{7}{6}\cdot\left(\frac{1}{2}\right)^7+\left(\frac{1}{2}\right)^7$

or getting everything into a common denominator,

		X		
		5	6	7
	0	$\frac{4}{384}$	$\frac{2}{384}$	$\frac{1}{384}$
	1	$\frac{20}{384}$	$\frac{12}{384}$	$\frac{7}{384}$
	2	$\frac{40}{384}$	$\frac{30}{384}$	$\frac{21}{384}$
Y	3	$\frac{40}{384}$	$\frac{40}{384}$	$\frac{35}{384}$
	4	$\frac{20}{384}$	$\frac{30}{384}$	$\frac{35}{384}$
	5	$\frac{4}{384}$	$\frac{12}{384}$	$\frac{21}{384}$
	6	0	$\frac{2}{384}$	$\frac{8}{384}$

Grading criteria: 10 points in total
-1 point: 1-2 errors
-2 point: 3-4 errors
-3 point: 5-6 errors
-4 point: 7-8 errors
-5 point: 9-10 errors
-6 point: 11-12 errors
-7 point: 13-14 errors
-8 point: 15-17 errors
-9 point: 18-20 errors
-10 point: Incorrect or no answer

(b) The joint CDF looks like a stairstep function that's easiest to represent as a grid. Remember that the joint CDF is the sum of all probability to the left and below a given point (indicated by small numbers at the dots); we accumulate probability every time we cross a line.



Grading criteria: 10 points in total

-1 point: 1-2 errors

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-6 point: 11-12 errors

-7 point: 13-14 errors

-8 point: 15-17 errors

-9 point: 18-20 errors

-10 point: Incorrect or no answer

(c) From the joint PMF in part (a), it's easy to compute the marginal PMF of Y just by summing up the rows; we obtain

	$p_y(Y)$
0	$\frac{7}{384}$
1	$\frac{39}{384}$
2	$\frac{91}{384}$
3	$\frac{115}{384}$
4	$\frac{85}{384}$
5	$\frac{37}{384}$
6	$\frac{10}{384}$

As expected these probabilities sum to 1, since the marginal is a valid PMF.

Grading criteria: 5 points in total
-1 point: small error
-2 point: two errors
-3 point: 3-4 errors
-4 point: 5-6 errors
-2 point: 5-6 errors
-2 point: did not compute final answer
-10 point: Incorrect or no answer

2. (a) Let's integrate this function and see what we get:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,dx\,dy = k\int_0^1\int_0^1 x(1-x)y\,dx\,dy$$
$$= k\left(\frac{1}{2}x^2 - \frac{1}{3}x^3\Big]_{x=0}^{x=1}\right)\left(\frac{1}{2}y^2\Big]_{y=0}^{y=1}\right)$$
$$= k(\frac{1}{6})(\frac{1}{2})$$
$$= \frac{k}{12}$$

Since for a valid PDF we need this integral to equal 1, this means that $k = 12$.

> Grading criteria: 5 points in total
> -0 point: Incorrect, but correct based on your part (a)
> -1 point: Small error
> -1.5 point: Computation error
> -2.5 point: The integration is incorrect
> -4 point: Incorrect, some efforts
> -5 point: Incorrect or no answer

(b) The joint CDF in the "interesting" range $x \in [0,1], y \in [0,1]$ is computed as

$$F_{XY}(x,y)\,dx\,dy = \int_0^x\int_0^y f_{XY}(x,y)\,dx\,dy$$
$$= 12\left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\left(\frac{1}{2}y^2\right)$$
$$= (3x^2 - 2x^3)y^2$$

> Grading criteria: 5 points in total
> -0 point: small error
> -2 point: Equation is correct, final answer is incorrect
> -3 point: Partially correct
> -4 point: Incorrect, some efforts
> -5 point: Incorrect or no answer
> -2 point: Incomplete process
> -1 point: Missing conclusion (what is the value of c?)

(c) To compute the marginals, we integrate out the variable we don't care about. For $x \in [0,1]$ we have

$$f_X(x) = \int_0^1 12x(1-x)y\,dy$$
$$= (12x(1-x))\left(\frac{1}{2}y^2\Big]_{y=0}^{y=1}\right)$$
$$= (12x(1-x))(\frac{1}{2})$$
$$= 6x(1-x) \quad x \in [0,1], 0 \text{ otherwise}$$

Grading criteria: 5 points in total
-1 point: Small error
-2.5 point: The integration is incorrect
-1.5 point: Did not specify the pdf at all values of x
-4 point: Incorrect, some efforts
-5 point: Incorrect or no answer

(d) Similarly for $y \in [0,1]$ we have

$$\begin{aligned} f_Y(y) &= \int_0^1 12x(1-x)y\,dx \\ &= y\left(6x^2 - 4x^3\right]_{x=0}^{x=1}\Big) \\ &= y(2) \\ &= 2y \quad y \in [0,1], 0 \text{ otherwise} \end{aligned}$$

Grading criteria: 5 points in total
-1 point: Small error
-2.5 point: The integration is incorrect
-1.5 point: Did not specify the pdf at all values of x
-4 point: Incorrect, some efforts
-5 point: Incorrect or no answer

(e) Yes, X and Y are independent since we can see that

$$\begin{aligned} f_{XY}(x,y) &= 12x(1-x)y \\ &= (6x(1-x))(2y) \\ &= f_X(x)\, f_Y(y) \end{aligned}$$

Grading criteria: 5 points in total
-0 point: Incorrect, but correct based on your previous parts
-2.5 point: X and Y are independent
-2 point: No conclusion
-4 point: Correct conclusion without reason
-3.5 point: Correct conclusion with incorrect reason
-4 point: Incorrect, some efforts
-5 point: Incorrect or no answer

(f)

$$
\begin{aligned}
P(Y < \sqrt{X}) &= \int_0^1 \int_0^{\sqrt{x}} 12x(1-x)y \, dy \, dx \\
&= \int_0^1 (12x(1-x)) \left(\frac{1}{2} y^2 \Big]_{y=0}^{y=\sqrt{x}} \right) dx \\
&= \int_0^1 (12x(1-x)) \left(\frac{1}{2} x \right) dx \\
&= \int_0^1 6x^2(1-x) \, dx \\
&= 2x^3 - \frac{6}{4} x^4 \Big]_{x=0}^{x=1} \\
&= \frac{1}{2}
\end{aligned}
$$

Grading criteria: 5 points in total
-0 point: Incorrect, but correct based on your previous parts
-2 point: Computation error
-4 point: Incorrect, some efforts
-5 point: Incorrect or no answer

3. (a) We're told that X and Y are jointly Gaussian with the PDF

$$
f_{XY}(x, y) = c \exp\left(-\frac{3}{64} (12x^2 - 80x + 3y^2 + 24y - 4xy) \right)
$$

where c is an unknown constant. We are also told that the correlation coefficient $\rho = \frac{1}{3}$. We need to pattern-match this against the 2D Gaussian PDF of the form

$$
f_{XY}(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(\frac{-1}{2(1-\rho^2)} \left[\left(\frac{x-\mu_x}{\sigma_x} \right)^2 - 2\rho \left(\frac{x-\mu_x}{\sigma_x} \right) \left(\frac{y-\mu_y}{\sigma_y} \right) + \left(\frac{y-\mu_y}{\sigma_y} \right)^2 \right] \right)
$$

We don't care about the constant in front; let's just expand the exponent, plugging in $\rho = \frac{1}{3}$:

$$
\begin{gathered}
\frac{-1}{2(1-\rho^2)\sigma_x^2} x^2 - \frac{1}{2(1-\rho^2)\sigma_y^2} y^2 + \left(\frac{\mu_x}{(1-\rho^2)\sigma_x^2} - \frac{\rho\mu_y}{(1-\rho^2)\sigma_x\sigma_y} \right) x + \left(\frac{\mu_y}{(1-\rho^2)\sigma_y^2} - \frac{\rho\mu_x}{(1-\rho^2)\sigma_x\sigma_y} \right) y \\
+ \frac{\rho}{(1-\rho^2)\sigma_x\sigma_y} xy + \text{constant}
\end{gathered}
$$

$$
= \frac{-9}{16\sigma_x^2} x^2 - \frac{9}{16\sigma_y^2} y^2 + \left(\frac{9\mu_x}{8\sigma_x^2} - \frac{3\mu_y}{8\sigma_x\sigma_y} \right) x + \left(\frac{9\mu_y}{8\sigma_y^2} - \frac{3\mu_x}{8\sigma_x\sigma_y} \right) y + \frac{3}{8\sigma_x\sigma_y} xy + \text{constant}
$$

Matching up the coefficients on the x^2 and y^2 terms gives us

$$
\frac{-9}{16\sigma_x^2} = \frac{-9}{16} \qquad\qquad \frac{-9}{16\sigma_y^2} = \frac{-9}{64}
$$

which tells us that $\sigma_x = 1$ and $\sigma_y = 2$. As a sanity check we can see that the xy term also agrees.

Grading criteria: 10 points in total
-2 point: Small error
-2 point: Mess up variance and standard deviation
-5 point: Incorrect, partial efforts for equation simplification
-4 point: Missing final answer
-7 point: Correct answer with no process shown
-8 point: Incorrect, some efforts
-9 point: Incorrect answer with no process
-10 point: Incorrect or no answer

(b) Now that we know σ_x and σ_y we can look at the x and y terms:

$$\frac{9\mu_x}{8} - \frac{3\mu_y}{16} = \frac{15}{4} \quad \rightarrow \quad 18\mu_x - 3\mu_y = 60$$
$$\frac{9\mu_y}{32} - \frac{3\mu_x}{16} = \frac{-9}{8} \quad \rightarrow \quad -6\mu_x + 9\mu_y = -36$$

Solving this linear system gives $\mu_x = 3, \mu_y = -2$.

Grading criteria: 10 points in total
-2 point: Small error
-4 point: Missing final answer
-5 point: Incorrect, partial efforts (some equations)
-7 point: Correct answer with no process shown
-7 point: Incomplete answer
-8 point: Incorrect, some efforts
-9 point: Incorrect answer with no process
-10 point: Incorrect or no answer

4. (a) The correlation of X and Y is defined as $E(XY)$.

$$\begin{aligned}
E(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{XY}(x,y)\,dx\,dy \\
&= \frac{6}{19}\int_0^2\int_0^1 xy(x^2+y^3)\,dy\,dx \quad = \frac{6}{19}\int_0^2 \frac{1}{2}x^3y^2 + \frac{1}{5}xy^5\Big]_{y=0}^{y=1}\,dx \\
&= \frac{6}{19}\int_0^2 \frac{1}{2}x^3 + \frac{1}{5}x\,dx \\
&= \frac{6}{19}\left(\frac{1}{8}x^4 + \frac{1}{10}x^2\Big]_{x=0}^{x=2}\right) \\
&= \frac{6}{19}\cdot\frac{12}{5} \qquad = \frac{72}{95}
\end{aligned}$$

Grading criteria: 5 points in total
-1 point: Small error
-2.5 point: Incorrect, partial efforts
-4 point: Incorrect, some efforts
-5 point: Incorrect or no answer

(b) The covariance of X and Y is defined as $E(XY) - E(X)E(Y)$, which means we first have to compute the marginal means $E(X)$ and $E(Y)$.

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{XY}(x,y)\, dx\, dy \\
&= \frac{6}{19}\int_0^2\int_0^1 x(x^2+y^3)\, dy\, dx \qquad = \frac{6}{19}\int_0^2 x^3y + \frac{1}{4}xy^4\Big]_{y=0}^{y=1} dx \\
&= \frac{6}{19}\int_0^2 x^3 + \frac{1}{4}x\, dx \\
&= \frac{6}{19}\left(\frac{1}{4}x^4 + \frac{1}{8}x^2\Big]_{x=0}^{x=2}\right) \\
&= \frac{6}{19}\cdot\frac{9}{2} \qquad = \frac{27}{19}
\end{aligned}
$$

$$
\begin{aligned}
E(Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{XY}(x,y)\, dx\, dy \\
&= \frac{6}{19}\int_0^2\int_0^1 y(x^2+y^3)\, dy\, dx \qquad = \frac{6}{19}\int_0^2 \frac{1}{2}x^2y^2 + \frac{1}{5}y^5\Big]_{y=0}^{y=1} dx \\
&= \frac{6}{19}\int_0^2 \frac{1}{2}x^2 + \frac{1}{5}\, dx \\
&= \frac{6}{19}\left(\frac{1}{6}x^3 + \frac{1}{5}x\Big]_{x=0}^{x=2}\right) \\
&= \frac{6}{19}\cdot\frac{26}{15} \qquad = \frac{52}{95}
\end{aligned}
$$

Thus

$$\mathrm{Cov}(X,Y) = \frac{72}{95} - \frac{27}{19}\cdot\frac{52}{95} = \frac{-36}{1805}$$

Grading criteria: 10 points in total
-0 point: Incorrect, but correct based on your part (a)
-2 point: Computation error
-3 point: E(X) is incorrect
-3 point: E(Y) is incorrect
-4 point: Errors in expectations
-4 point: Incorrect or missing calculation of covariance
-5 point: Incorrect, partial efforts for integration
-7 point: Incomplete answer
-8 point: Incorrect, some efforts
-9 point: Only provide a general equation
-10 point: Incorrect or no answer

(c) Now we use the properties of expected value and the numbers we already computed:

$$
\begin{aligned}
E(95(X(1+Y)+2Y(1-X))) &= 95E(X+2Y-XY) \\
&= 95(E(X)+2E(Y)-E(XY)) \\
&= 95\left(\frac{27}{19}+\frac{104}{95}-\frac{72}{95}\right) \qquad = 167
\end{aligned}
$$

Grading criteria: 5 points in total

-0 point: Incorrect, but correct based on your part (a)

-1 point: Small error

-2.5 point: Incorrect, partial efforts (equations)

-4 point: Incorrect, some efforts

-4 point: Incomplete answer

-5 point: Incorrect or no answer

(d) No, X and Y are not uncorrelated since $E(XY) \neq 0$.

Grading criteria: 5 points in total

-0 point: Incorrect, but correct based on your part (a)

-3 point: Correct with no justification

-3 point: Correct answer with wrong justification

-5 point: Incorrect or no answer