Fields and Waves I

Lecture 9

Fields & Math

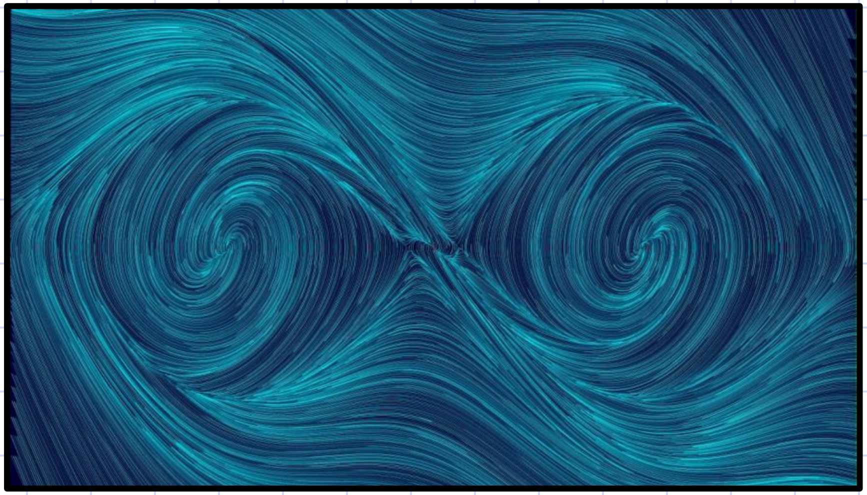
James D. Rees
Electrical, Computer, and Systems Engineering Department
Rensselaer Polytechnic Institute, Troy, NY

These slides were prepared through the work of the following people:

- Kenneth A. Connor – ECSE Department, Rensselaer Polytechnic Institute, Troy, NY (Emeritus)
- J. Darryl Michael – GE Global Research Center, Niskayuna, NY
- Thomas P. Crowley – National Institute of Standards and Technology, Boulder, CO
- Sheppard J. Salon – ECSE Department, Rensselaer Polytechnic Institute, Troy, NY (Emeritus)
- Lale Ergene – ITU Informatics Institute, Istanbul, Turkey
- Jeffrey Braunstein – ECE Department, University at Albany
- James Lu - ECSE Department, Rensselaer Polytechnic Institute, Troy, NY
- James Dylan Rees - ECSE Department, Rensselaer Polytechnic Institute, Troy, NY

Materials from other sources are referenced where they are used.
Those listed as Ulaby are figures from Ulaby's textbook.

Overview



- Exam 1
- Review
- Vector Notation
- Coordinate Systems
- Line, Area and Volume Integrals
- Gradient, Divergence and Curl

Field Math

- In our treatment of transmission lines so far, we have been able to derive voltage and current everywhere on the line using an extension of circuit theory.
- But to understand the more general picture, we need Maxwell's Equations. **Now we enter the "fields" portion of the class.**

$\nabla . E = \dfrac{\rho}{\epsilon_0}$	(1)	Gauss' law
$\nabla . B = 0$	(2)	Magnetic monopoles
$\nabla \times E = -\dfrac{\partial B}{\partial t}$	(3)	Faraday's law
$\nabla \times H = J + \dfrac{\partial D}{\partial t}$	(4)	Ampere-Maxwell law

Field Math

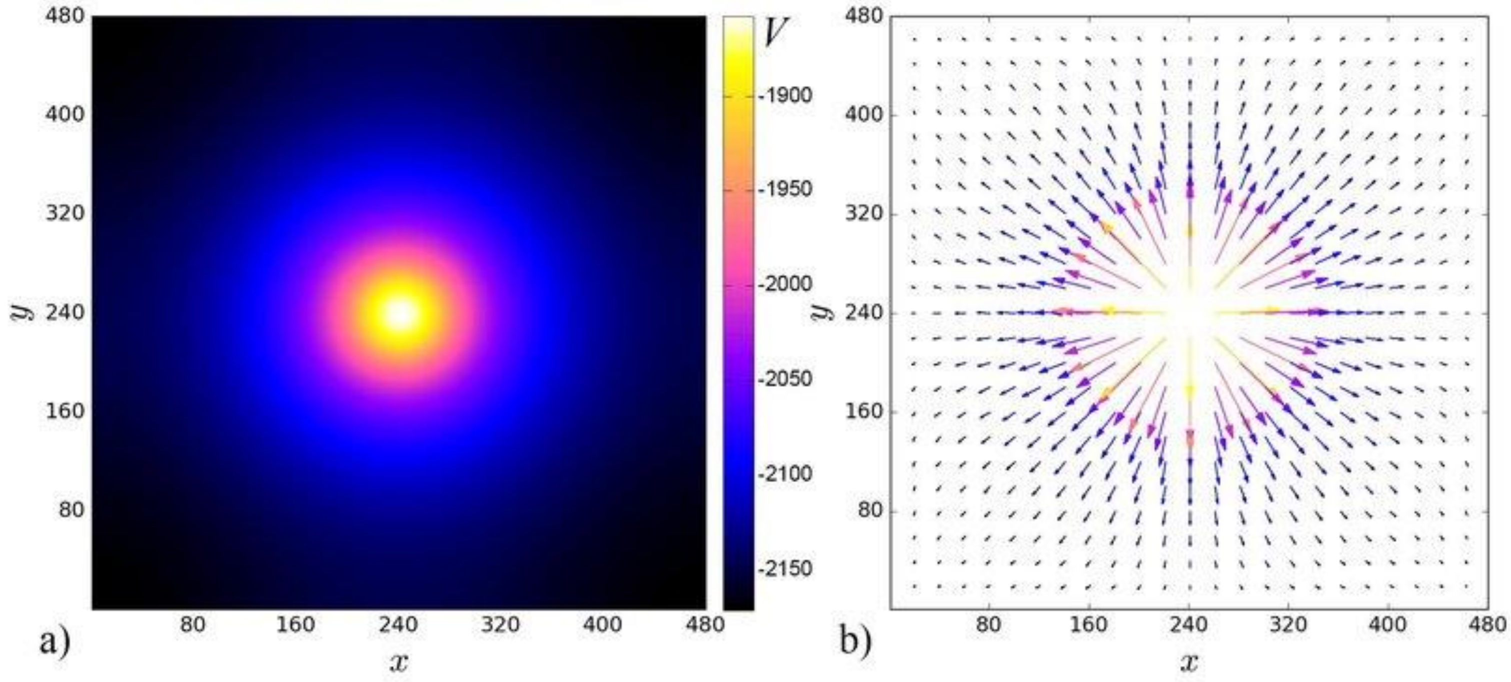
...but what is a field, anyway?

Field Math

...but what is a field, anyway?

- Has a value at every point in space
- Could be a scalar field, where every point has scalar quantity (i.e. electric potential)
- Could be a vector field, where every point has a vector (i.e. electric field, magnetic field)
- Scalar and vector fields are often related through gradients

Field Math



Electric potential field (left) and corresponding electric field (vector field, right) for a charged droplet ([Lauricella]())

Field Math

Which of these are represented mathematically by fields? Why or why not?

- Wind velocity
- Energy density
- Temperature
- Drag force due to wind

Field Math

- 3 Types of coordinate systems we will use for fields:
  - Rectangular
  - Cylindrical
  - Spherical
- Examples of when to use them:
  - Conductive sheets (rectangular)
  - Wires and cables (cylindrical)
  - Spheres (spherical)
- We can make our lives easier by choosing our coordinate systems for symmetry reasons. (for instance, both cables and the cylindrical coordinate system have radial symmetry)

Field Math

Rectangular / Cartesian Coordinates Definition

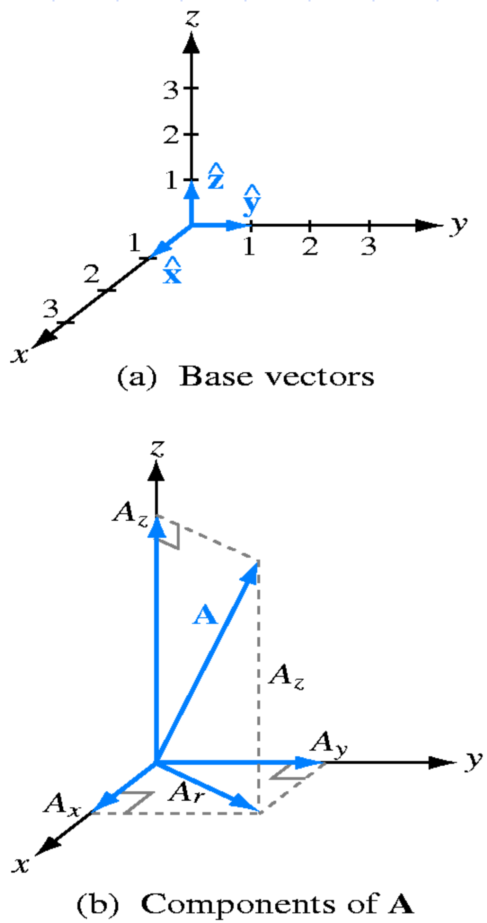
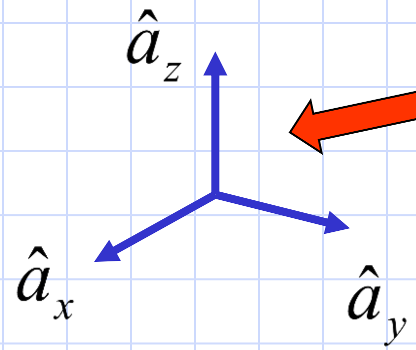


Figure 3-2

$$A = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$$



Unit Vector Representation for Rectangular Coordinate System

The Unit Vectors imply :

$\hat{a}_x$ → Points in the direction of increasing **x**

$\hat{a}_y$ → Points in the direction of increasing **y**

$\hat{a}_z$ → Points in the direction of increasing **z**

Field Math

Rectangular / Cartesian Coordinates Dot Product

- Definition

$$A \bullet B = \left|\vec{A}\right|\left|\vec{B}\right|\cos(\theta_{AB})$$

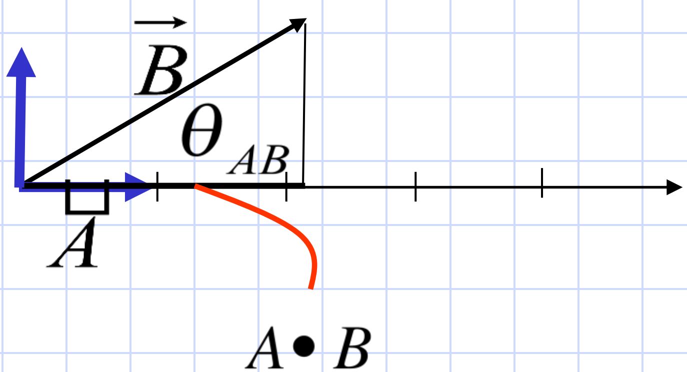
Dot Product (scalar)

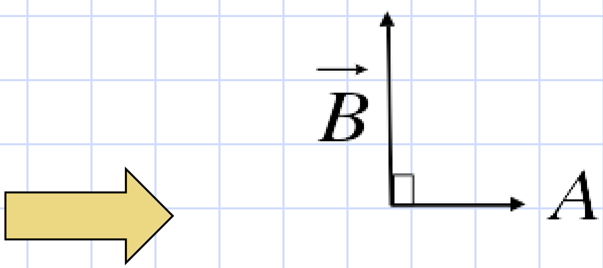
$$A \bullet B = A_x B_x + A_y B_y + A_z B_z$$

$$\left|\vec{A}\right| = \sqrt{\vec{A}.\vec{A}} = \left(A_x^2 + A_y^2 + A_z^2\right)^{\frac{1}{2}}$$

Magnitude of vector

- Meaning of dot product



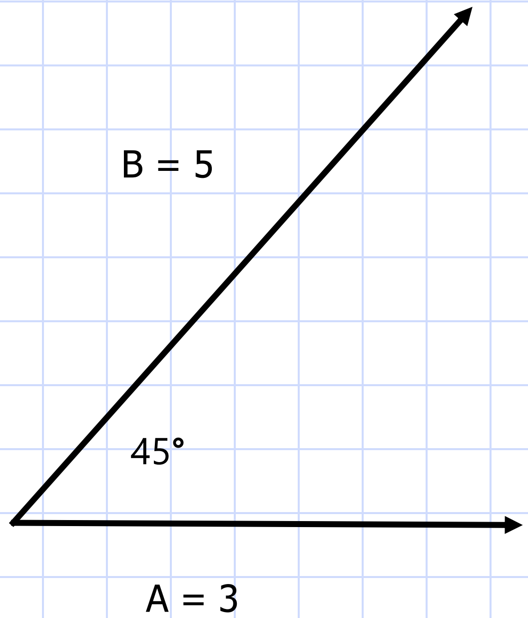


$$A \bullet B = 0$$

Field Math

Rectangular / Cartesian Coordinates Dot Product

- What's the dot product of **A** and **B**?

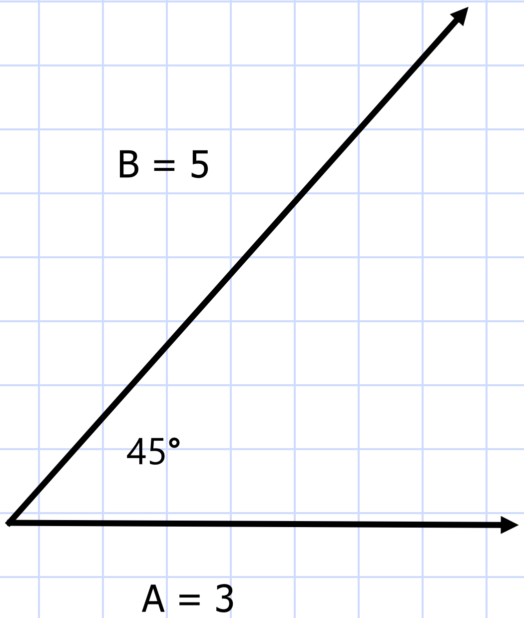


Note: we can denote vectors either by putting an arrow over their letters or by writing their letters in bold.

Field Math

Rectangular / Cartesian Coordinates Dot Product

- What's the dot product of **A** and **B**?



5 * 3 * cos(45) = 15*sqrt(2)/2

Field Math

Rectangular / Cartesian Coordinates Cross Product

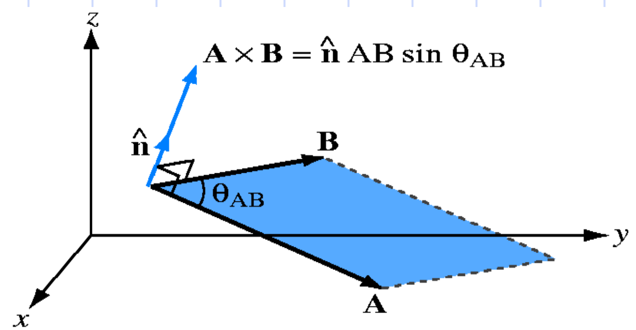
- Definition

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

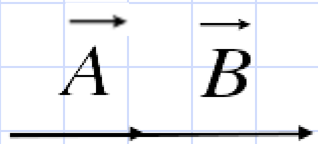
Cross Product

(VECTOR)

- Meaning of the cross product



(a) Cross product

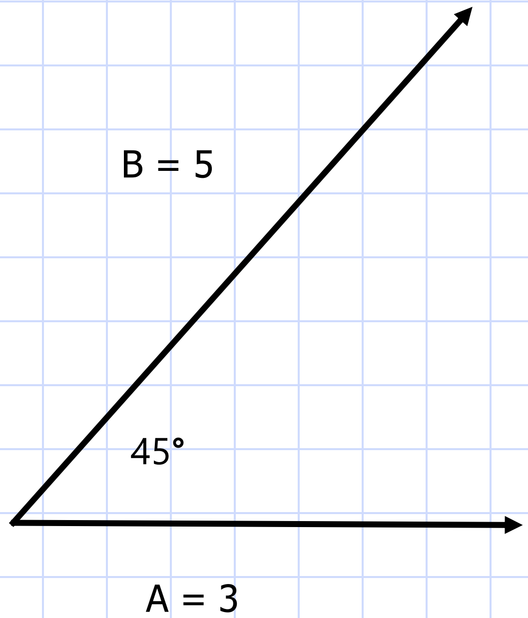


$$A \times B = 0$$

Field Math

Rectangular / Cartesian Coordinates Dot Product

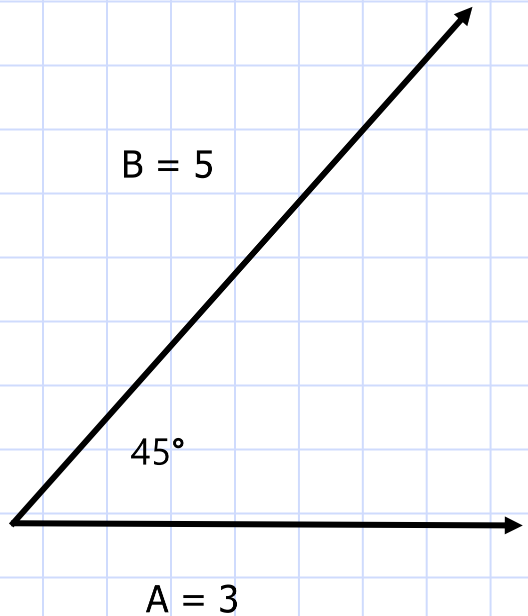
- What's the **cross product** of **A** and **B**?



Field Math

Rectangular / Cartesian Coordinates Dot Product

- What's the **cross product** of **A** and **B**?



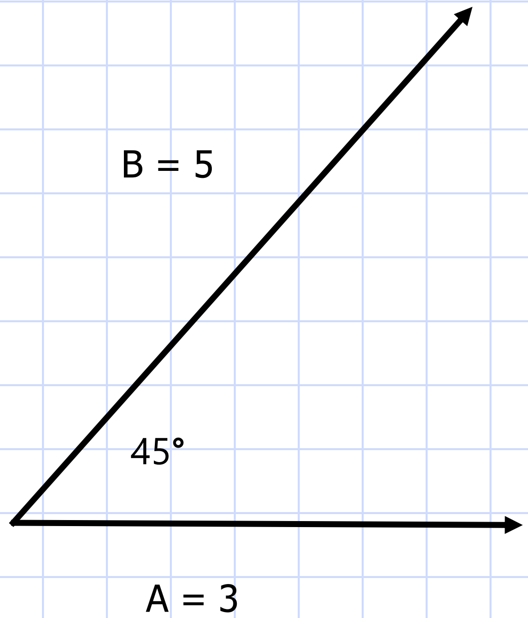
Magnitude is 5 * 3 * sin(45) = 15*sqrt(2)/2

But the answer is a vector. What direction does it point?

Field Math

Rectangular / Cartesian Coordinates Dot Product

- What's the **cross product** of **A** and **B**?



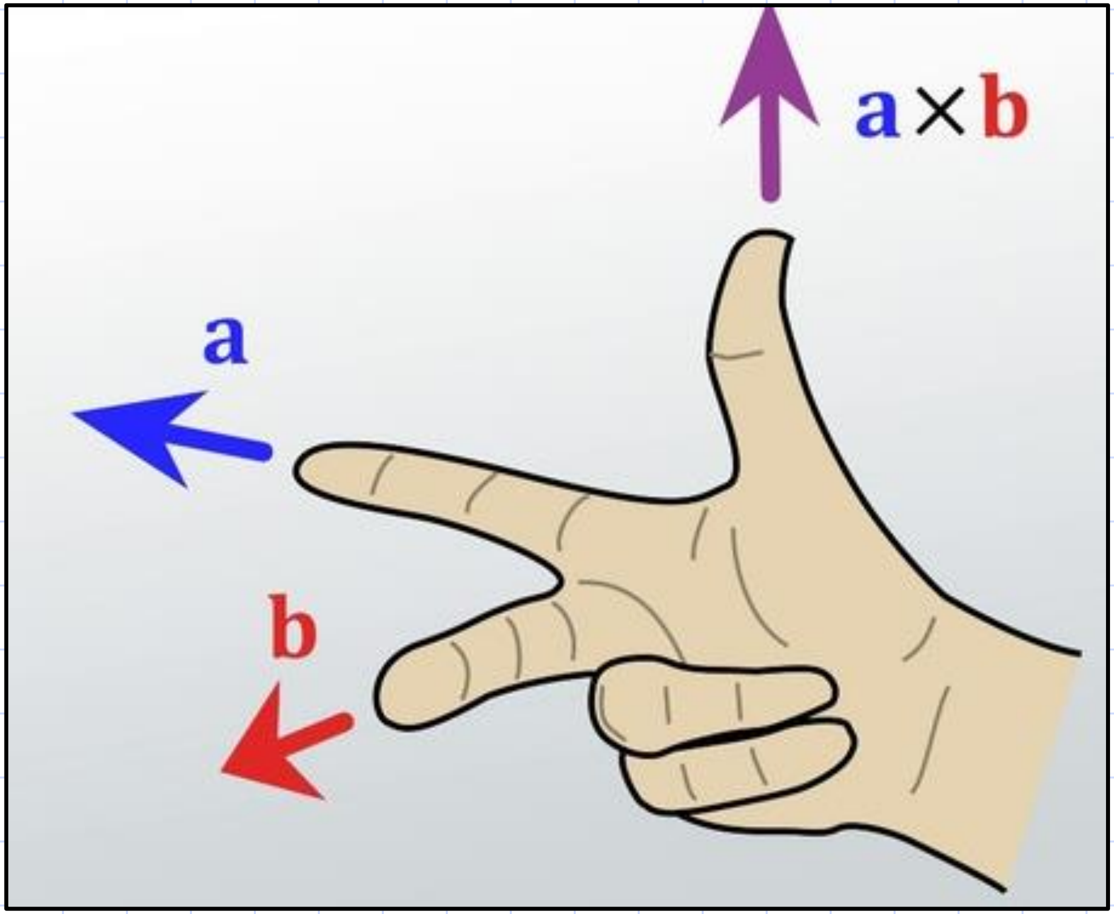
Magnitude is 5 * 3 * sin(45) = 15*sqrt(2)/2

But the answer is a vector. What direction does it point?

Out of screen, toward you.

Field Math

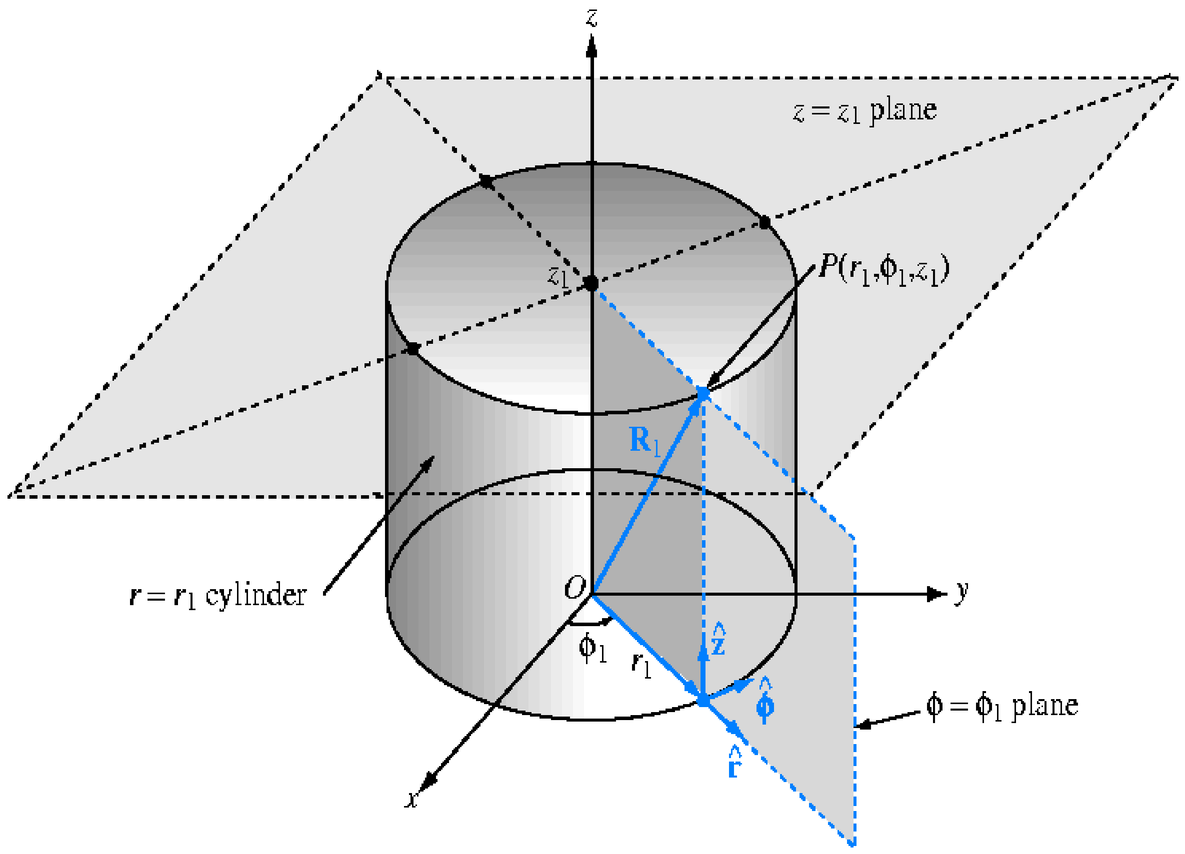
The Right Hand Rule



Study.com

Field Math

Cylindrical Coordinates Definition

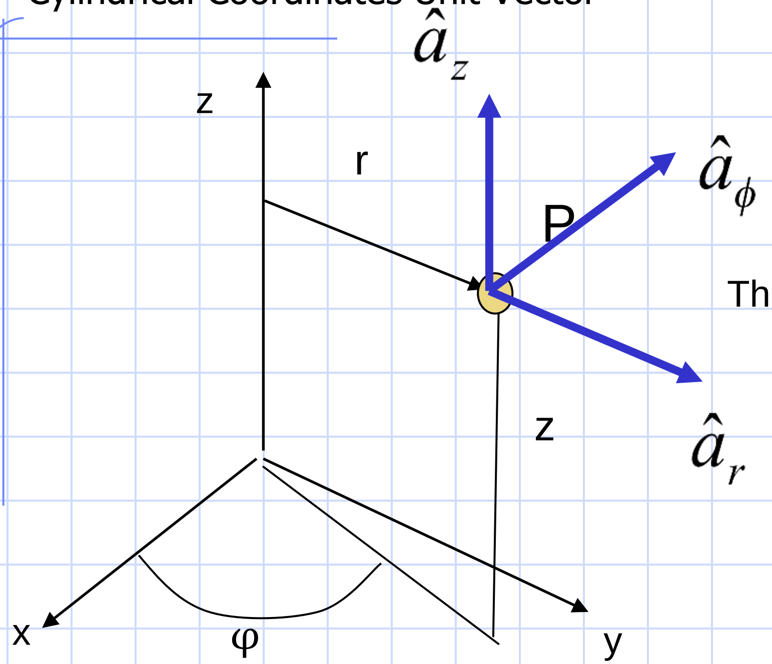


Ulaby

$$\overrightarrow{OP} = r_1\hat{\mathbf{r}} + z_1\hat{\mathbf{z}} \quad \text{for} \quad P(r_1, \phi_1, z_1)$$

Field Math

Cylindrical Coordinates Unit Vector



The Unit Vectors imply :

$\hat{a}_r$ ➡ Points in the direction of increasing **r**

$\hat{a}_\phi$ ➡ Points in the direction of increasing **Φ**

$\hat{a}_z$ ➡ Points in the direction of increasing **z**

In cylindrical coordinates, both $\hat{a}_r$ and $\hat{a}_\phi$ are functions of φ

Field Math

Cylindrical Coordinates Dot Product

Cylindrical representation uses: r, φ, z

UNIT VECTORS:

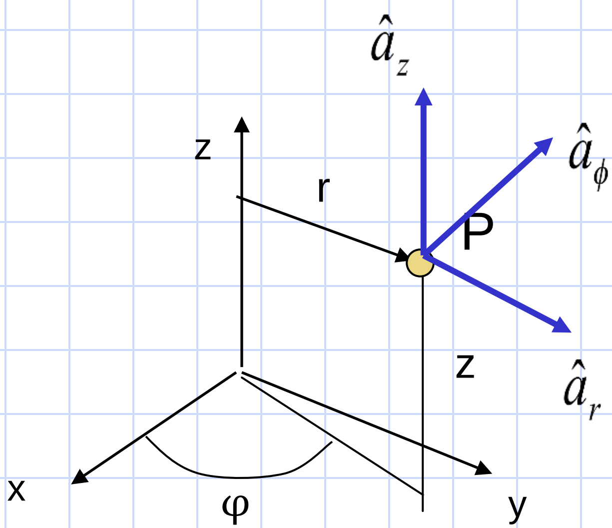
$$\left(\hat{a}_r \quad \hat{a}_\phi \quad \hat{a}_z\right)$$

$$A = A_r\hat{a}_r + A_\phi\hat{a}_\phi + A_z\hat{a}_z$$

$$A \bullet B = A_rB_r + A_\phi B_\phi + A_zB_z$$

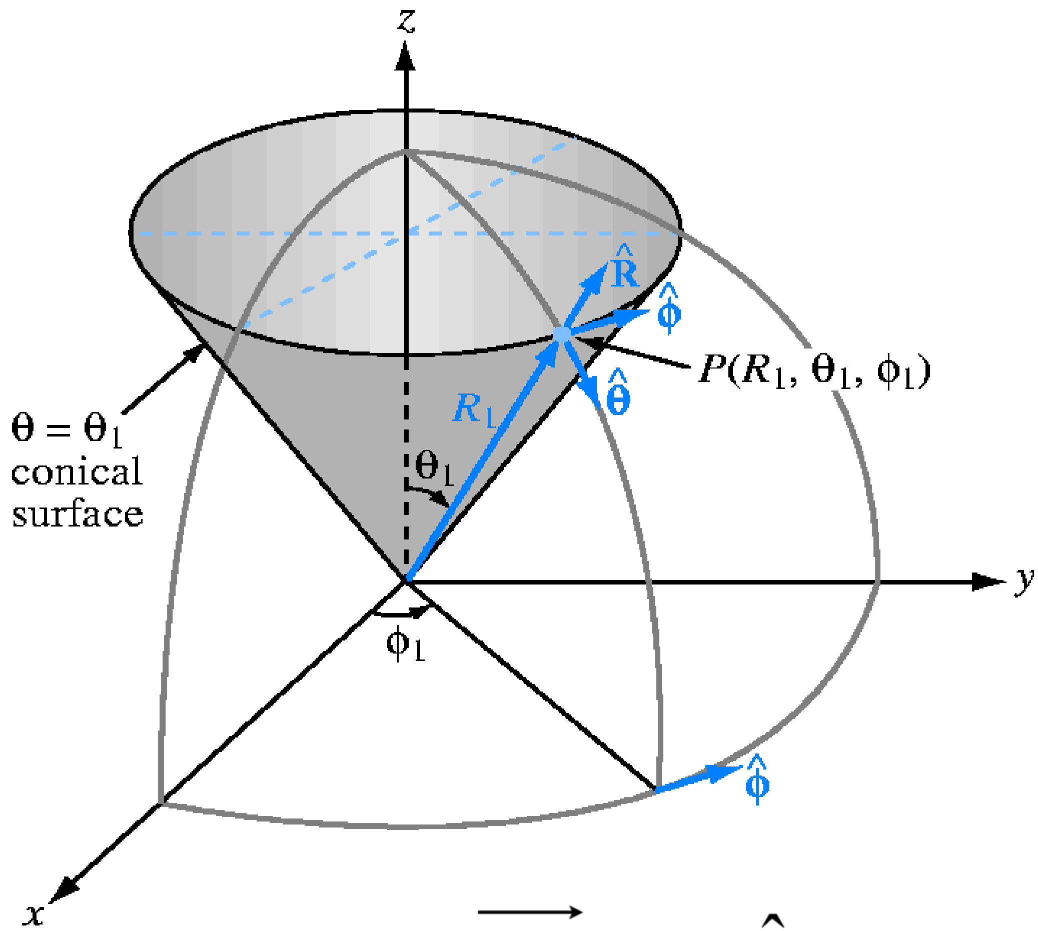
Dot Product

(SCALAR)



Field Math

Spherical Coordinates Definition



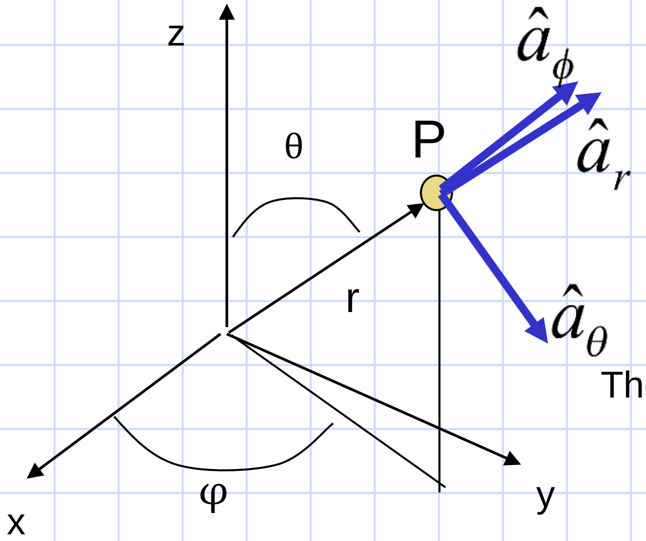
Ulaby

Figure 3-13

$$\overrightarrow{OP} = R_1 \hat{\mathbf{R}} \quad \text{for} \quad P(R_1, \theta_1, \phi_1)$$

Field Math

Spherical Coordinates Unit Vector



The Unit Vectors imply :

$\hat{a}_r$ ➡ Points in the direction of increasing **r**

$\hat{a}_\theta$ ➡ Points in the direction of increasing **θ**

$\hat{a}_\phi$ ➡ Points in the direction of increasing **φ**

In spherical coordinates, $\hat{a}_r$ $\hat{a}_\theta$ and $\hat{a}_\phi$ are functions of φ and θ

Field Math

Spherical Coordinate Dot Product

UNIT VECTORS:

$$\left(\hat{a}_r \quad \hat{a}_\vartheta \quad \hat{a}_\phi\right)$$

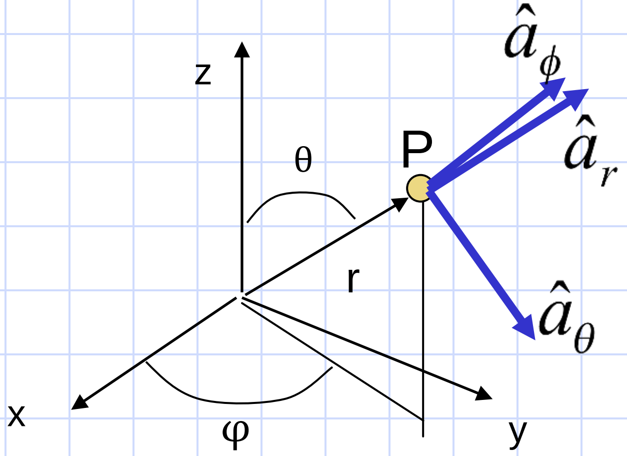
Spherical representation uses: r, θ, φ

$$A = A_r\hat{a}_r + A_\vartheta\hat{a}_\theta + A_\phi\hat{a}_\phi$$

$$A \bullet B = A_rB_r + A_\theta B_\theta + A_\phi B_\phi$$

Dot Product

(SCALAR)



Field Math

Vector Representation

RECTANGULAR Coordinate Systems	CYLINDRICAL Coordinate Systems	SPHERICAL Coordinate Systems
$\left(\hat{a}_x \quad \hat{a}_y \quad \hat{a}_z\right)$	$\left(\hat{a}_r \quad \hat{a}_\phi \quad \hat{a}_z\right)$	$\left(\hat{a}_r \quad \hat{a}_\vartheta \quad \hat{a}_\phi\right)$

NOTE THE ORDER!

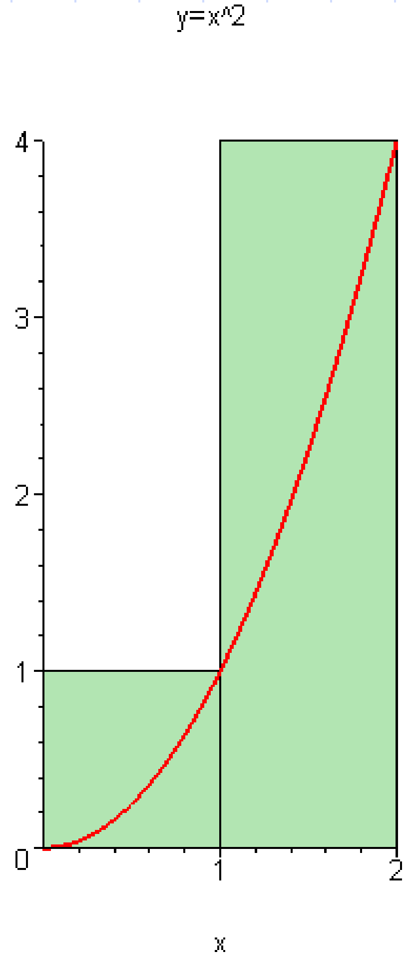
r,φ, z

r,θ ,φ

Note: We do not emphasize transformations between coordinate systems

Field Math

Differential Calculus



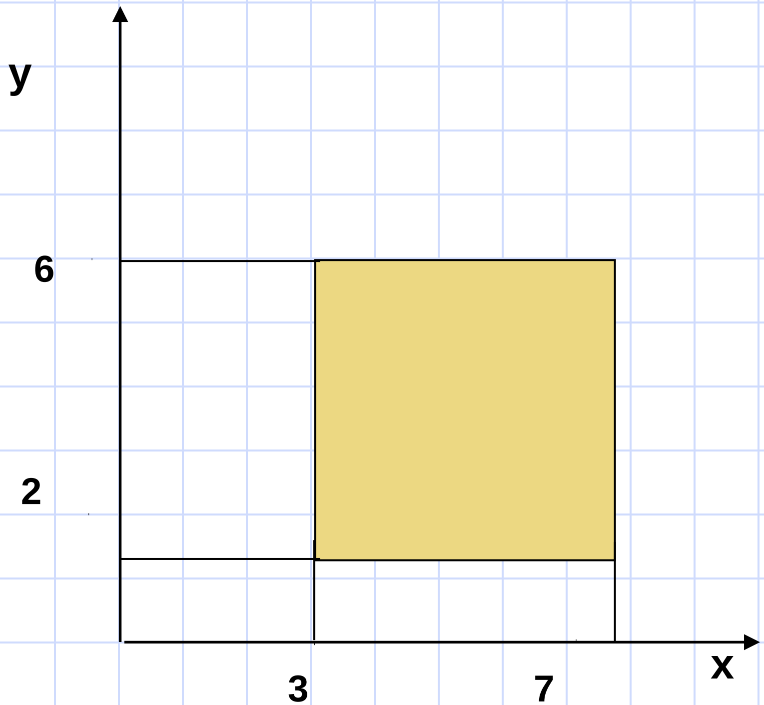
Wikipedia

- Integration = adding up differential elements (infinitesimally small pieces.)
- These differential elements have some some specific geometry depending on dimensions/coordinates
- At left, we integrate infinitesimal slices of y(x)

$$\int f(x)dx$$

Field Math

Differential Calculus



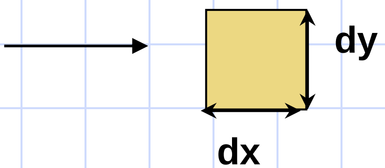
- For 2D Cartesian integrals, we have a square differential element.

$$Area = \int_3^7 \int_2^6 dx.dy = 16$$

- Or if we wanted to integrate some function over this area:

$$\int_3^7 \int_2^6 f(x,y)dxdy$$

• integration over 2 “delta” distances



Field Math

Differential Calculus

- For integration in three dimensions, we use a triple integral with a differential volume element.
- You can visualize this differential volume element by visualizing the volume that is traced out if you allow a point to "wiggle" a small amount in all three directions specified in the coordinate system.

Field Math

Differential Calculus

- In Cartesian coordinates, the differential element is a cube with size lengths dx, dy and dz and area dxdydz.

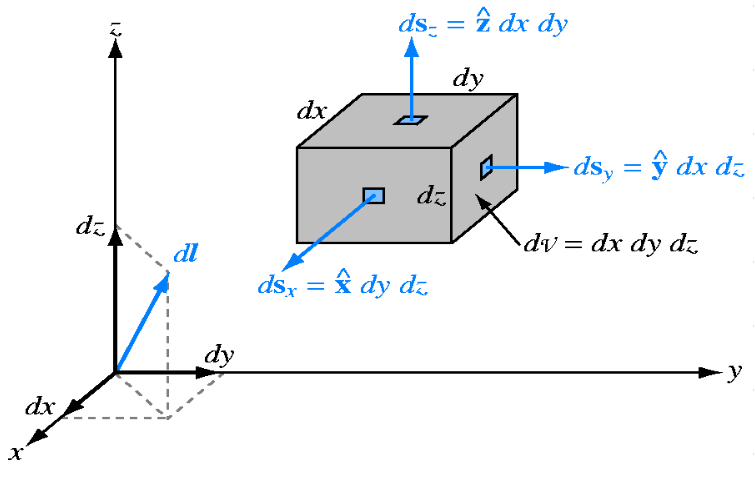


Figure 3-8

$$\int\int\int f(x,y,z)dxdydz$$

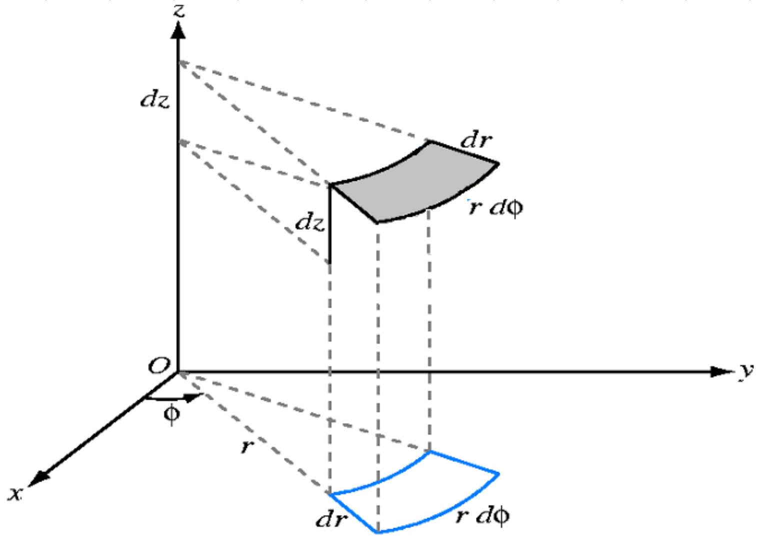
(Recall that we can change the order of integration if we want.)

$$\int\int\int f(x,y,z)dzdxdy$$

Field Math

Differential Calculus

- In cylindrical coordinates, the differential element will be a wedge.

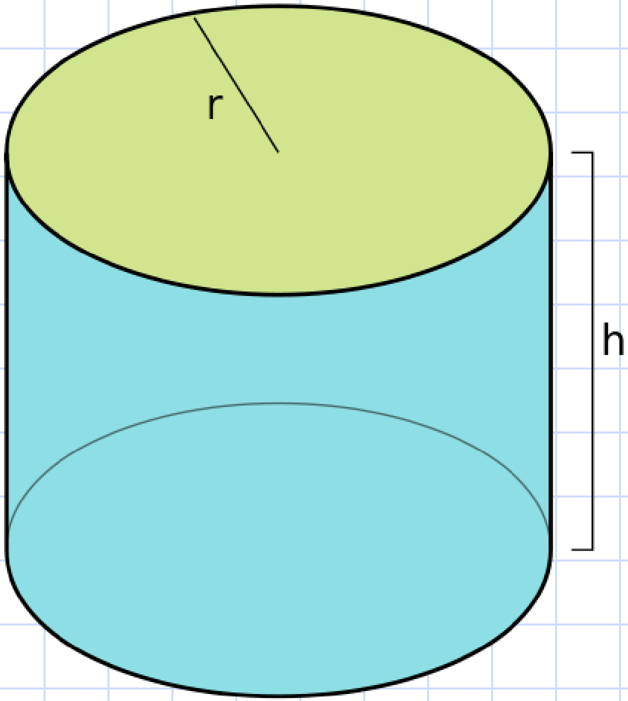


$$\int\int\int f(r,\phi,z)\,r\,dr\,d\phi\,dz$$

Field Math

Differential Calculus

- You can find the volume of this cylinder as follows:

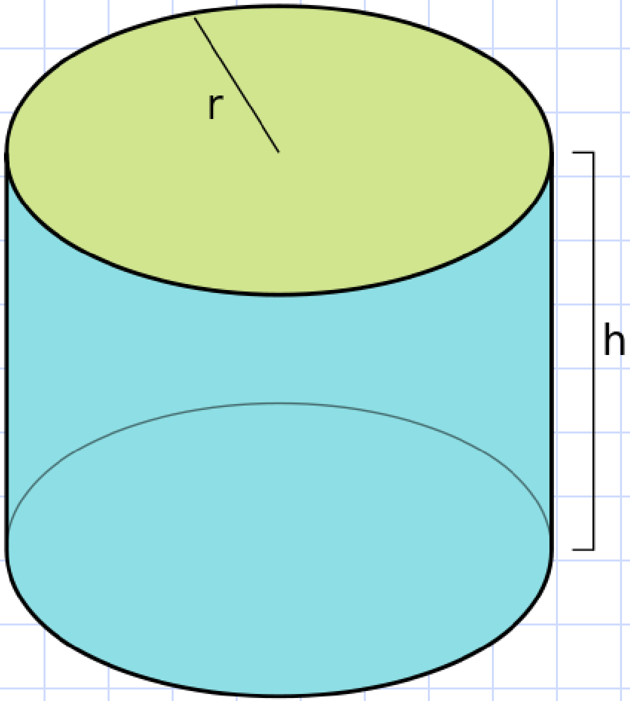


$$\int_0^h \int_0^{2\pi} \int_0^r r\,dr\,d\phi\,dz$$

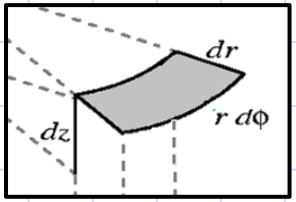
Field Math

Differential Calculus

- You can find the volume of this cylinder as follows:



$$\int_0^h \int_0^{2\pi} \int_0^r \boxed{r\,dr\,d\phi\,dz}$$



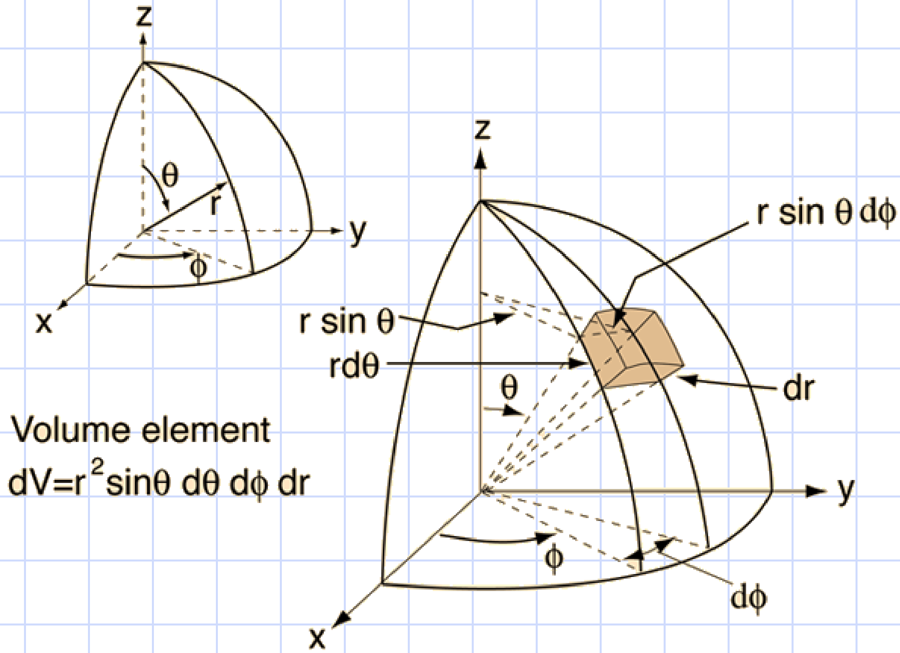
the volume of this

(the volume of the differential wedge looks like that of a rectangular prism. we don't need to worry about curvature due to its infinitesimal size!)

Field Math

Differential Calculus

- In spherical coordinates, two of the wedge dimensions are proportional to r.



$$\int\int\int r^2 sin\theta d\theta d\phi dr$$

Field Math

Differential Calculus

- Representation of differential lengths **dl** in the 3 coordinate systems

rectangular $$dl = dx \bullet \hat{a}_x + dy \bullet \hat{a}_y + dz \bullet \hat{a}_z$$

cylindrical $$dl = dr \bullet \hat{a}_r + r \bullet d\phi \bullet \hat{a}_\phi + dz \bullet \hat{a}_z$$

spherical $$dl = dr \bullet \hat{a}_r + rd\theta \bullet \hat{a}_\theta + r\sin\theta d\phi \bullet \hat{a}_\phi$$

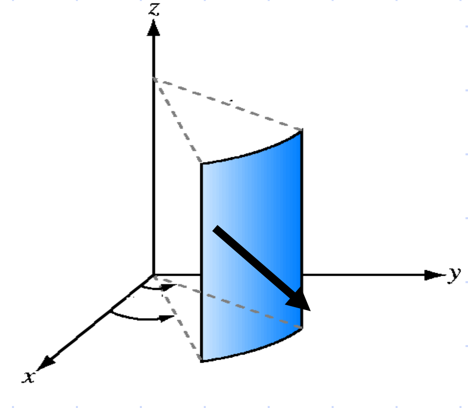
Field Math

Differential Surfaces and Volumes

Example of surface differentials

$$ds = dx.dy.\hat{a}_z \qquad \text{or} \qquad ds = rd\phi.dz.\hat{a}_r$$

Representation of differential surface element:



Vector is <u>NORMAL</u> to surface

$$ds = dx.dy.\hat{a}_z$$

Differential volume (a scalar)

$$dv = dx.dy.dz$$

Field Math

Differential Surfaces and Volumes

- When doing surface and volume integrals we must consider:
  - What is the right system of coordinates ?
  - What is kept constant?
  - What integral limits do we use?
  - What differential element do we use?

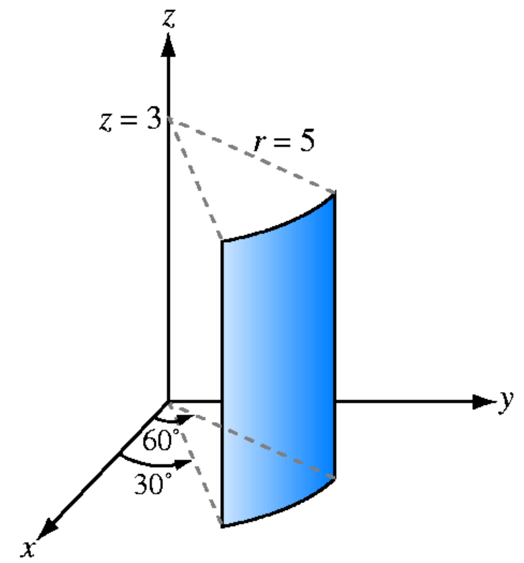


Figure 3-12

Field Math

Differential Surfaces and Volumes

- To visualize a **ds** element:
  - Take the direction that the **ds**'s normal vector points in and hold it constant.
  - Sweep the other two coordinates through all possible values.
  - What shape is traced out by this process? This shape is made of of all the ds elements for the vector you specified.

Field Math

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation $\mathbf{A} =$	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A $\quad \lvert\mathbf{A}\rvert =$	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\quad \overrightarrow{OP_1} =$	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P(R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product $\quad \mathbf{A}\cdot\mathbf{B} =$	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product $\quad \mathbf{A}\times\mathbf{B} =$	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length $\quad d\mathbf{l} =$	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume $\quad dV =$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

Field Math

Differential Surfaces and Volumes

- Do Lecture 9 Exercise 1 in groups of up to 4.

Field Math

Differential Surfaces and Volumes

How do you describe the shapes of all the ds surfaces in the following coordinate systems?

- Cartesian coordinates:

https://mathinsight.org/cartesian_coordinates

- Cylindrical coordinates:

https://mathinsight.org/cylindrical_coordinates

- Spherical coordinates:

https://mathinsight.org/spherical_coordinates

Field Math

- The electric charge density inside a sphere is given by $4cos^2(\theta)$. How do we set up the integral to find total charge Q contained in a sphere of radius 2cm?

Field Math

- The electric charge density inside a sphere is given by $4\cos^2(\theta)$. How do we set up the integral to find total charge Q contained in a sphere of radius 2cm?

$$4\int_0^{0.02}\int_0^{\pi}\int_0^{2\pi}\cos^2(\theta)r^2\sin(\theta)\,dr\,d\theta\,d\phi$$

Field Math

Maxwell's Equations

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

Source: spectrumscientifics.wordpress.com

Gauss' Law: Electric charge gives rise to an electric field.

Field Math

Divergence Operator

- Points of electric charge are "sources" or "sinks" of electric field lines.
- In mathematical terms, we say that the *divergence* of a point is proportional to how much charge it has.

Notation: $$div A = \nabla \bullet A$$ *NOT a DOT product but has similar features*

Result is a SCALAR, composed of derivatives

$$\nabla \bullet \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$ in Cartesian coordinates

Field Math

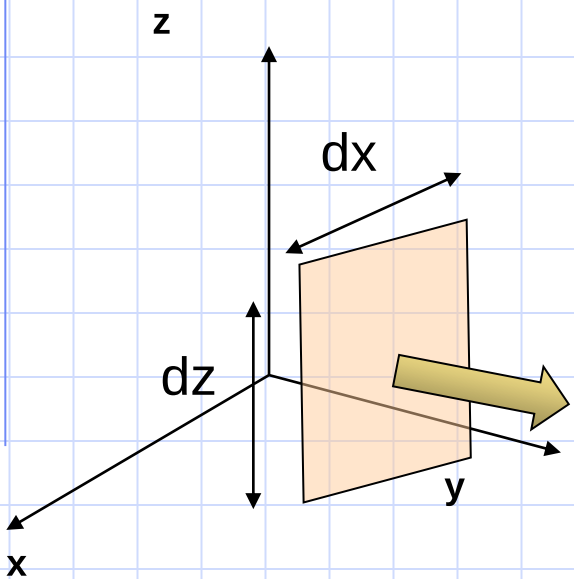
Maxwell's Equations

What does that field actually look like?

https://davidawehr.com/projects/electric_field.html

Field Math

Surface Integrals

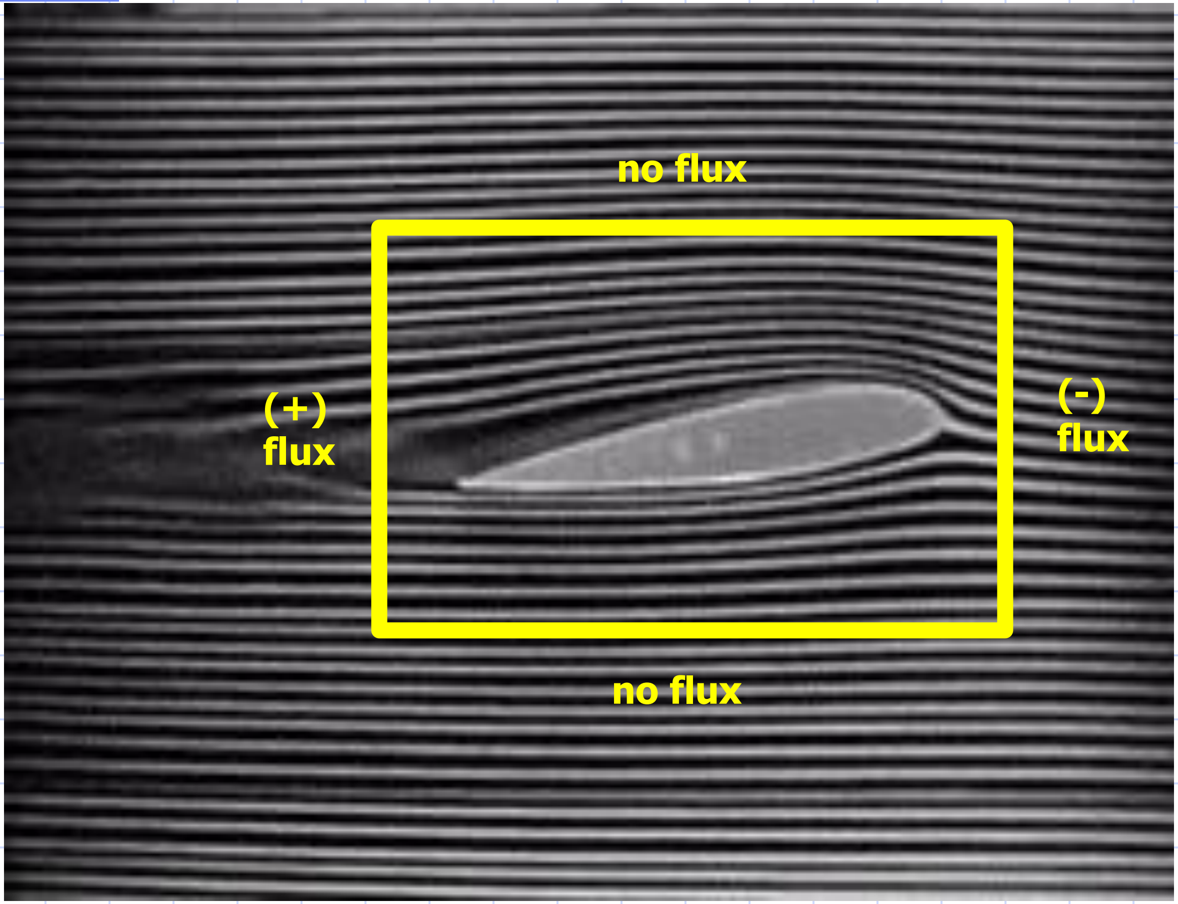


- An important property in field calculations is **flux:** how *much* field is flowing through a surface.
- Whenever we have a vector field, we can calculate a flux by taking the following integral:

$$\boxed{\text{Flux} = \iint_S \vec{\mathbf{F}} \cdot \hat{\mathbf{n}} \underbrace{dS}_{\text{surface area element}}}$$

- Note that the total flux will depend on both the area of the surface through which the field is flowing and the magnitude of the field.

Surface integrals and flux

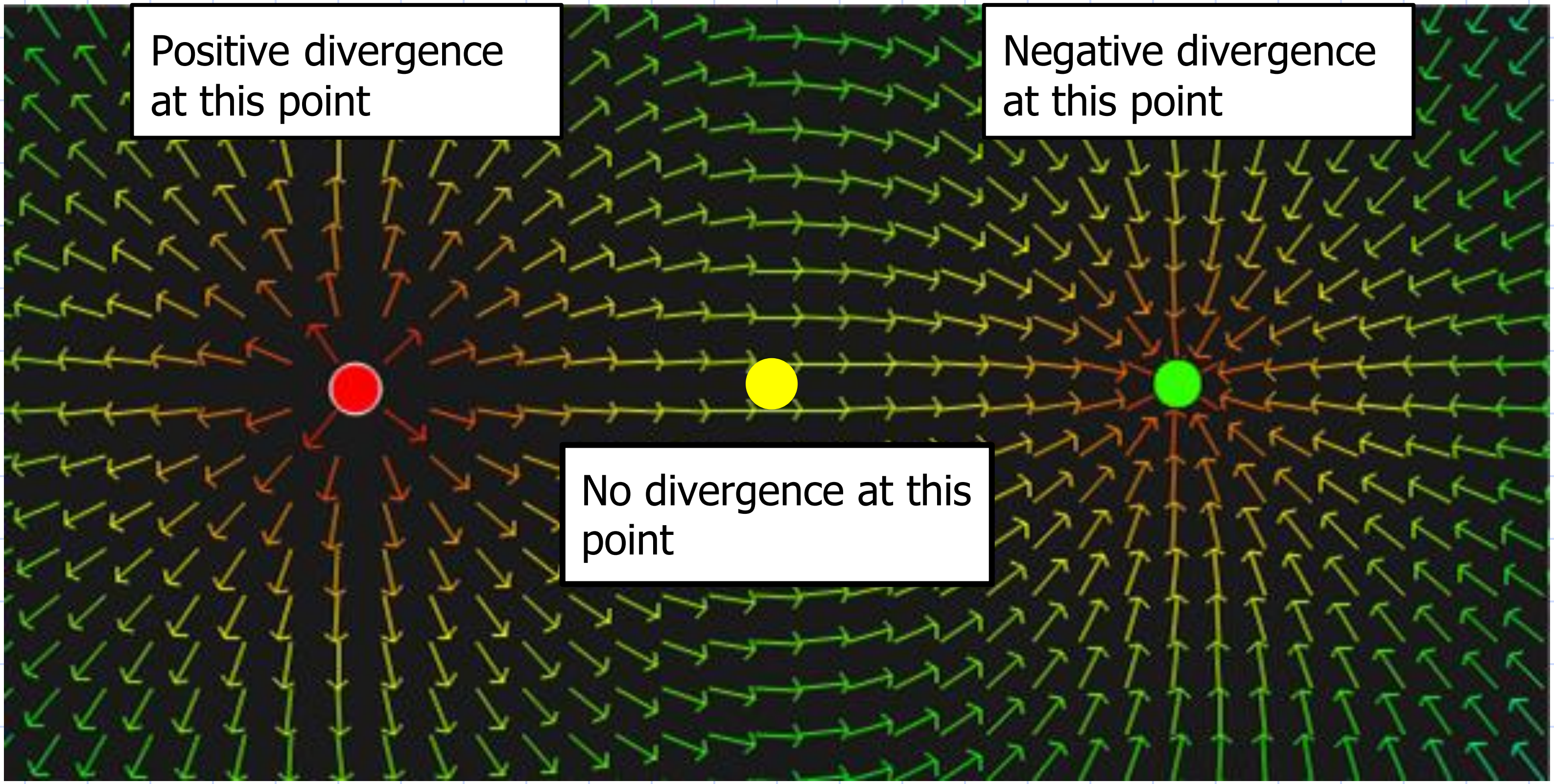


Gfycat

What is the total flux in the image above?

Field Math

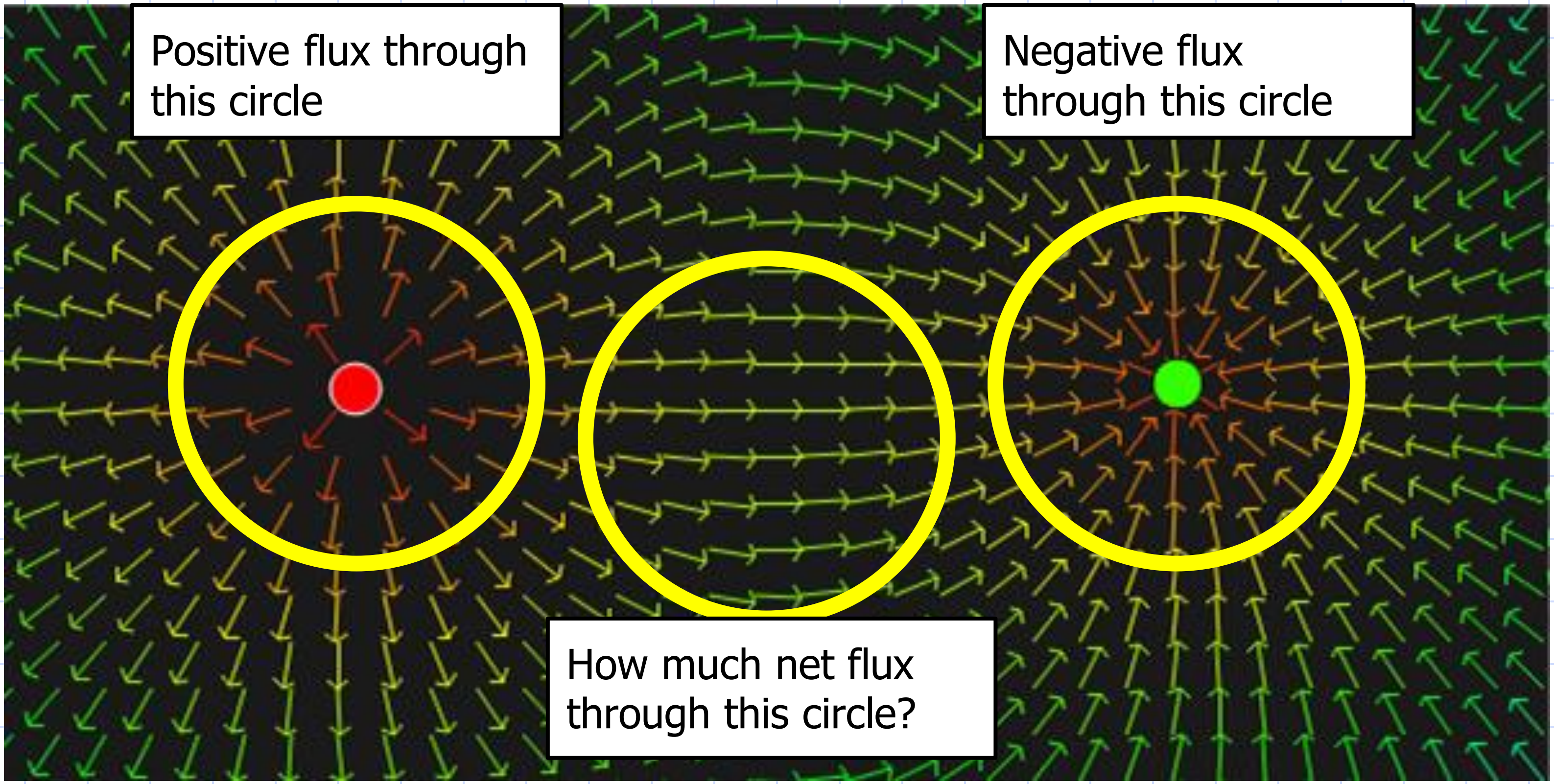
Maxwell's Equations



Field Math

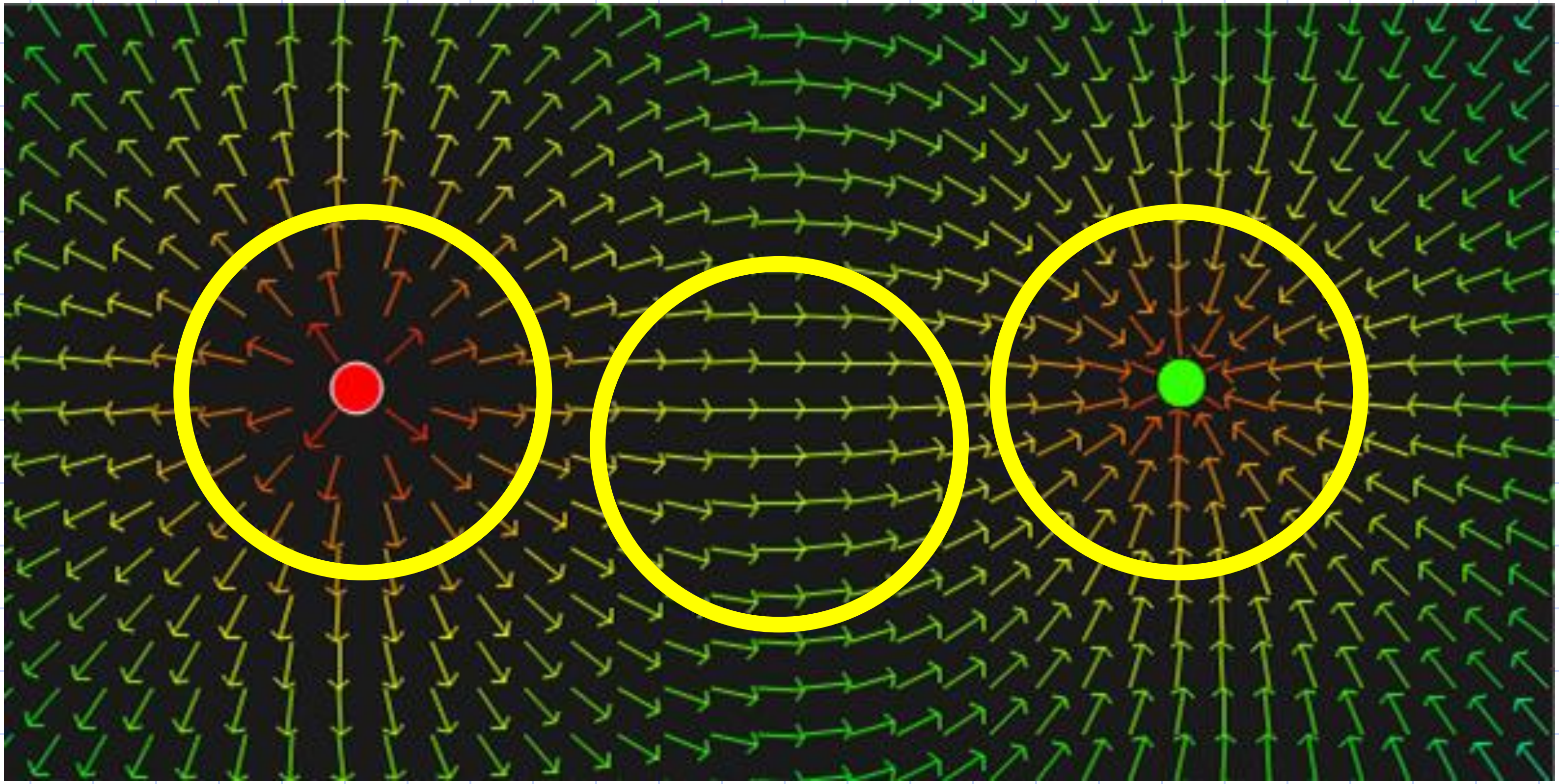
Maxwell's Equations

Flux = magnitude of flow through a surface



Field Math

Maxwell's Equations



You can see that the flux through any of these three circles has a relationship to the amount of charge inside. (To consider the effect of all the charge inside each circle, we must do the integral of the divergence operator inside the circle.)

Field Math

Divergence Operator

Divergence Theorem:

$$\oint \vec{A} \cdot \vec{ds} = \int (\nabla \cdot \vec{A}) \cdot dV$$

Volume integral on right is volume enclosed by surface on the left

In plain English:

"The amount and magnitude of field sources or sinks inside a volume will determine the flux through the volume's surface."

Field Math

Divergence Operator

"Global" quantities

$\int A \bullet ds$ Measures Flux through any surface

$\oint A \bullet ds$ Measures Flux through closed surfaces

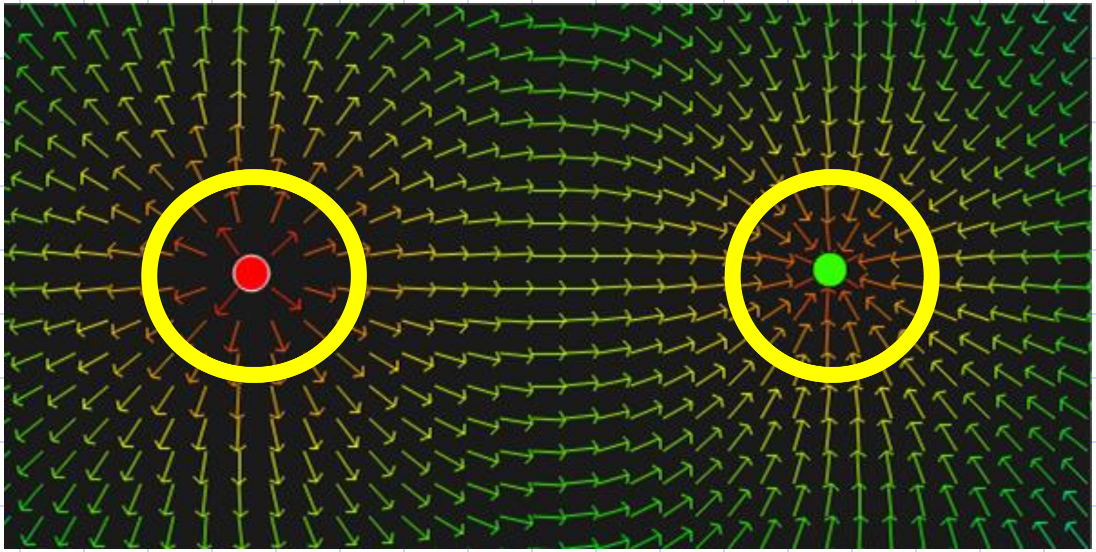
is related to

$\nabla \bullet A$, is a "local" measure of flux property

Divergence = degree to which a given point is a source or a sink

Field Math

Maxwell's Equations



$$\oint \vec{A} \cdot \vec{ds} = \int (\nabla \cdot \vec{A}) \cdot dV$$

One last important point:

The source of flux could be a *single point* with a nonzero divergence, or the divergence could come from an *area*. This parallels the fact that electric charges can be treated as points, but it is often more accurate to speak of an *area of charge distribution*.

Field Math

Divergence Operator

Calculate $\nabla \bullet \mathbf{A}$ for each of the vectors below.

a. $\mathbf{A} = x^2 y\,\mathbf{a}_x + c^2 x\,\mathbf{a}_z$

b. $\mathbf{A} = c/r^2\,\mathbf{a}_r + e^{-j\beta r}\sin\theta/r\,\mathbf{a}_\varphi$

$$a.\ \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \underbrace{\cancel{\frac{\partial A_y}{\partial y}}}_{0} + \frac{\partial A_z}{\partial z} = \frac{\partial}{\partial x}(x^2 y) + \underbrace{\cancel{\frac{\partial}{\partial z}(c^2 x)}}_{0} = \boxed{2xy}$$

$$b.\ \nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \underbrace{\cancel{\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(A_\theta \sin\theta\right)}}_{0} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\cancel{r^2}\frac{c}{\cancel{r^2}}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\frac{e^{-j\beta r}\sin\theta}{r}\right) = \boxed{0}$$

Field Math

Divergence Operator

Divergence operator simulation:

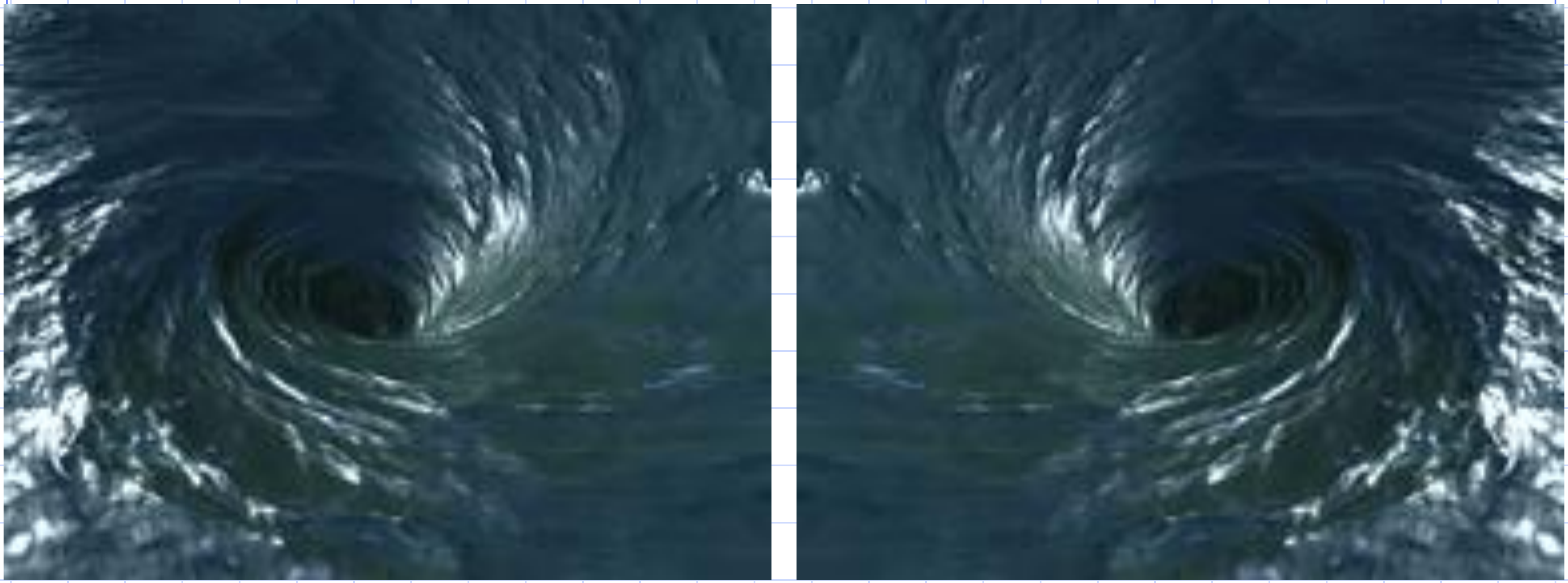
http://em8e.eecs.umich.edu/jws/ch3/mod3_3/mod3_3_webstart.jnlp

Field Math

What's the difference between these two whirlpools?

Field Math

What's the difference between these two whirlpools?



Circulation in opposite directions.
Their velocity fields have opposing curl.

Field Math

Curl Operator

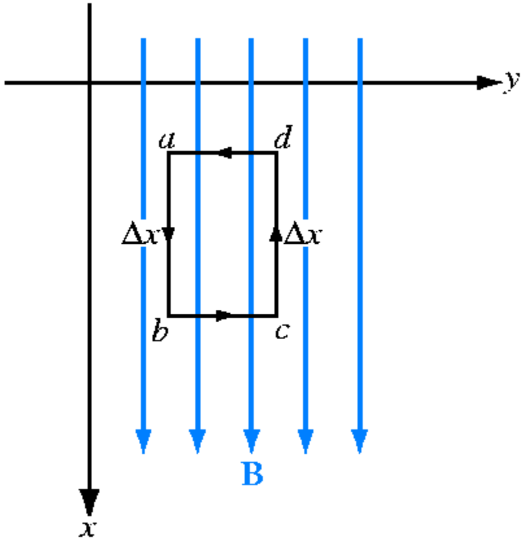
Curl can be calculated at a point using the following expression for Cartesian coordinates (Ulaby pg. 166).
Similar expressions exist for the other coordinate systems.

$$\nabla \times \mathbf{B} = \hat{\mathbf{x}} \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) + \hat{\mathbf{y}} \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) + \hat{\mathbf{z}} \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} . \qquad (3.105)$$

Field Math

Curl Operator

NOTATION: $\oint$ Implies a CLOSED LOOP Integral



(a) Uniform field

$$\oint \vec{B} \cdot d\vec{l}$$ measures circulation (related to Curl)

Example of a uniform field B in the x direction

$$\text{circulation} = \oint \vec{B} \cdot d\vec{l} =$$

$$\int_a^b B\hat{x} \bullet \hat{x}dx + \int_b^c B\hat{x} \bullet \hat{y}dy + \int_c^d B\hat{x} \bullet \hat{x}dx + \int_d^a B\hat{x} \bullet \hat{y}dy = 0$$

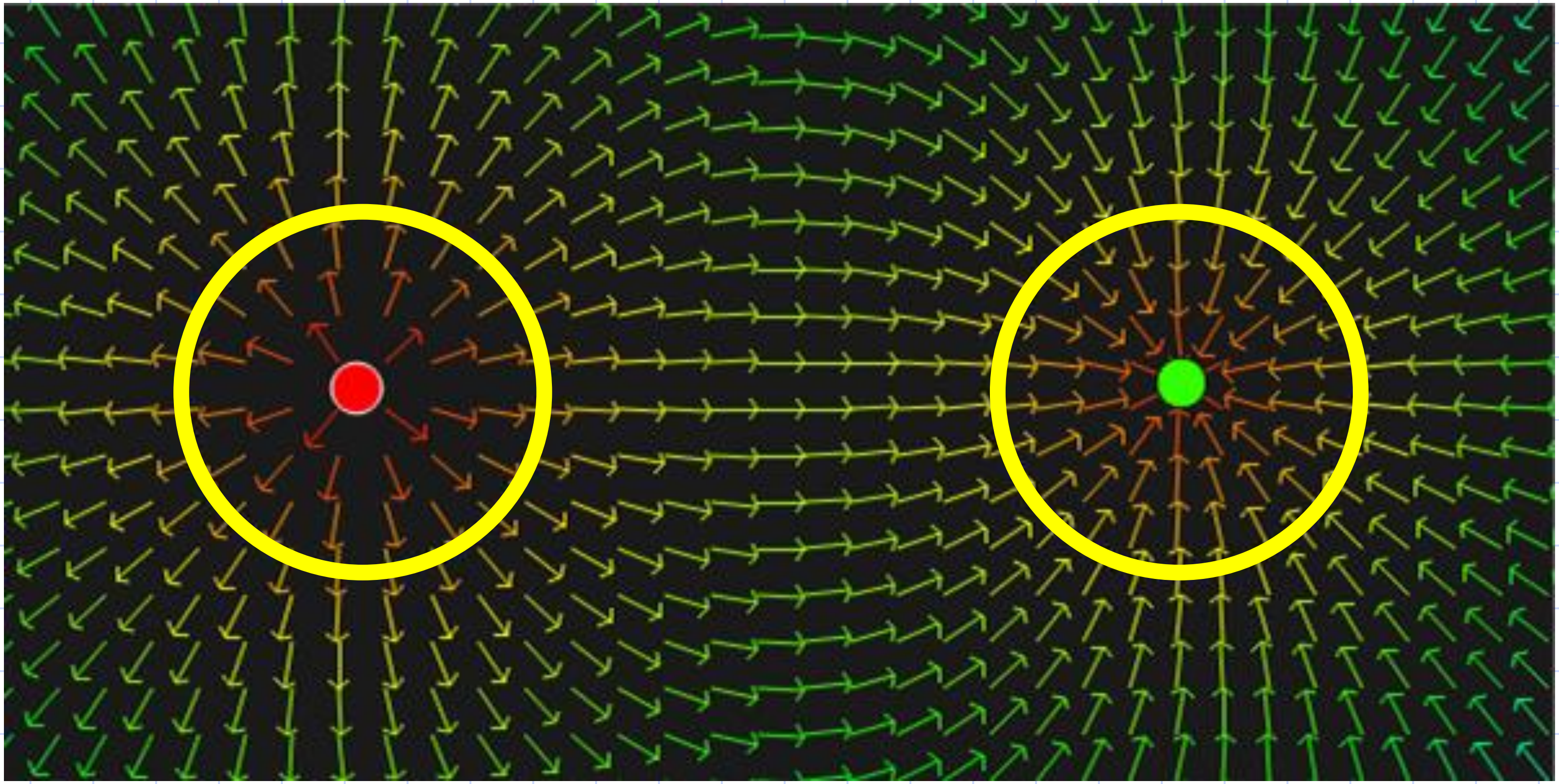
Field Math

The eye of the whirlpool is a place where the curl is nonzero. Whereas the point charges were *sources* or *sinks* of field lines, the whirlpool eye is a point that causes *circulation* of a vector field (in this case velocity) around it.

What is the curl inside the yellow circle?

Field Math

Maxwell's Equations



What is the curl of these two circles?

Field Math

Curl Operator

- The curl operator

NOTATION: $\nabla \times B$ Result of this operation is a VECTOR

This is <u>NOT</u> a CROSS-PRODUCT

- Stokes's theorem:

$$\oint \vec{B} \cdot \vec{dl} = \int\int \nabla \times \vec{B} \cdot ds$$

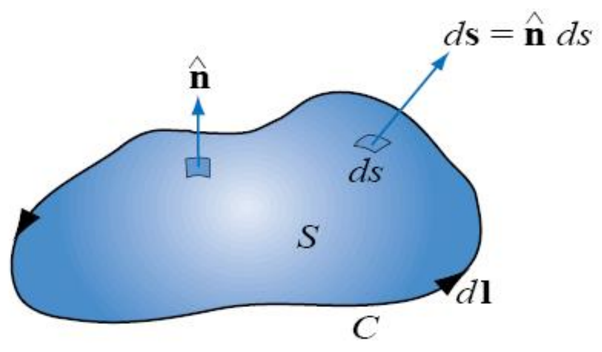


Figure 3-23

Surface integral on right is surface enclosed by line on the left

Field Math

Curl Operator

$$\oint \vec{B} \cdot \vec{dl} = \int\int \nabla \times \vec{B} \cdot ds$$

In plain English:

“The amount of **B** pointing in a loop around the outside of surface **ds** is determined by the total curl of field **B** inside ds.”

or:

“If you drop a ball near a whirlpool, the stronger the whirlpool is, the faster the ball will travel around it.”

Field Math

One last note on whirlpools:

We previously mentioned that flux is often due to a *distribution* of divergence.

Circulation of a vector field in practice is often due to a *distribution* of curl - that is, the vector field act less like it has one whirlpool and more like it has a distribution of infinitesimally-small whirlpools.

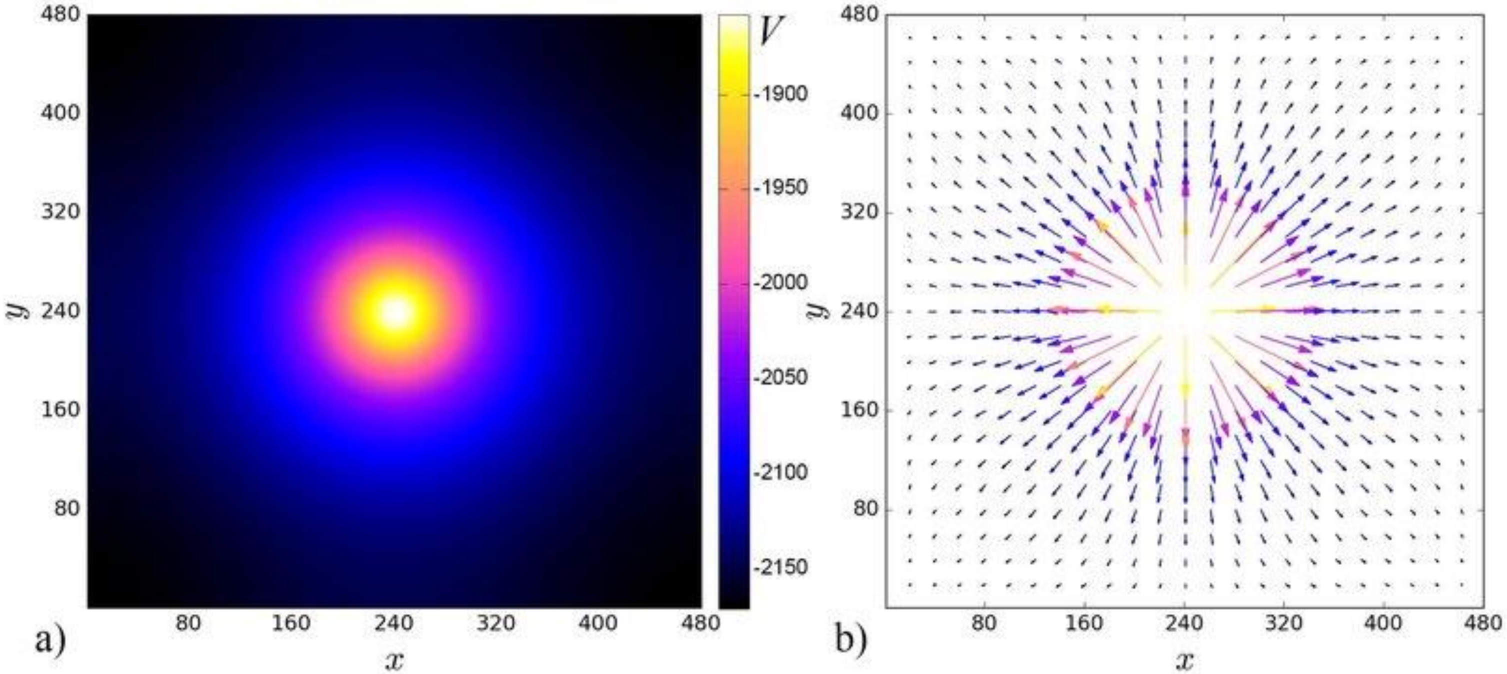
Field Math

Curl Operator

Curl operator simulation:

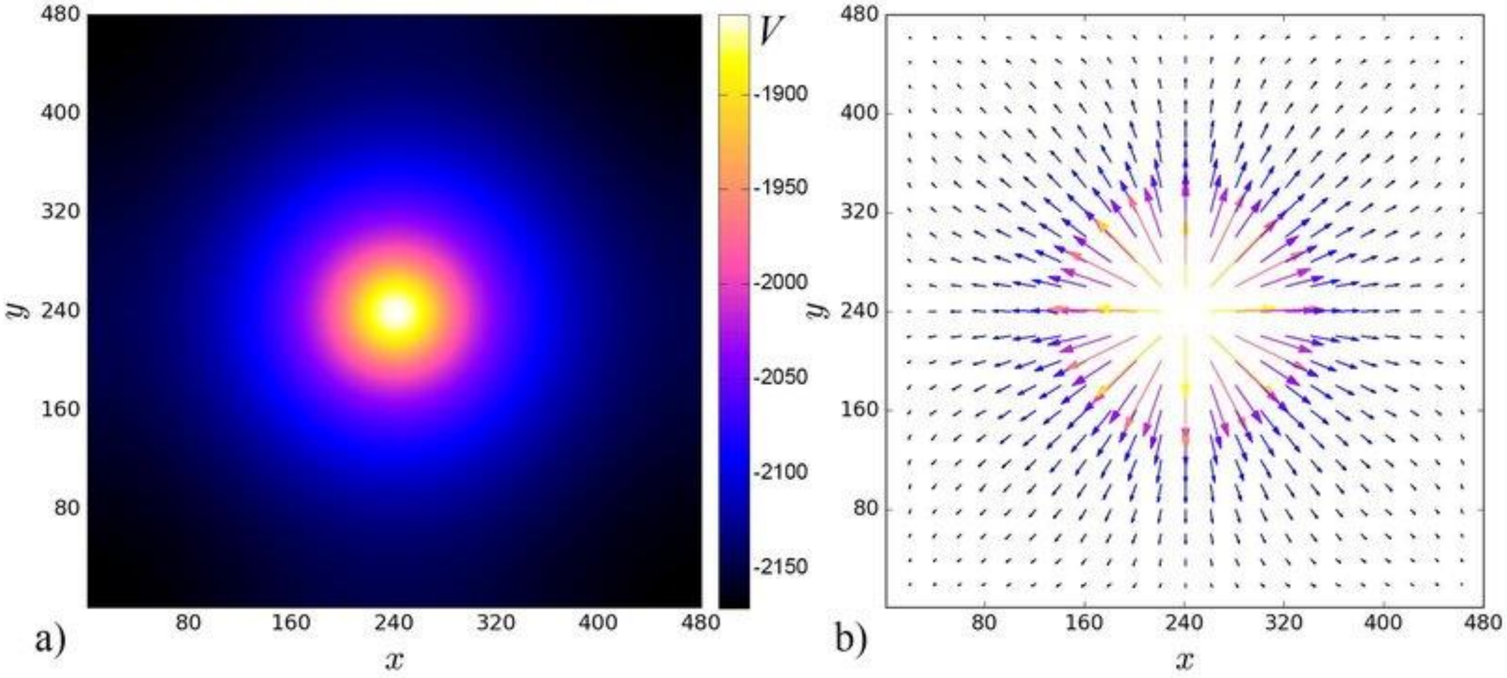
http://em8e.eecs.umich.edu/jws/ch3/mod3_4/mod3_4_webstart.jnlp

Field Math



What is the relationship between the potential field (left) and the electric field (right)?

Field Math



What is the relationship between the potential field (left) and the electric field (right)?
The electric field is defined by the gradient of the potential field.

Gradient operator

- The gradient operator

GRADIENT measures CHANGE in a SCALAR FIELD

• the result is a VECTOR pointing in the direction of increase

For a Cartesian system:

$$\nabla f = \frac{\partial f}{\partial x} \cdot \hat{a}_x + \frac{\partial f}{\partial y} \cdot \hat{a}_y + \frac{\partial f}{\partial z} \cdot \hat{a}_z$$

- Main property

You will find that $\nabla \times \nabla f = 0$ ALWAYS

Field Math

What is gradient? It’s this.

Gfycat

Field Math

Gradient Operator

Gradient simulation:

http://em8e.eecs.umich.edu/jws/ch3/mod3_2/mod3_2_webstart.jnlp

Field Math

Gradient Operator

Compute the gradient of the following functions.

a. $f = 8a^2 \cos\phi + 2rz$ (cylindrical)

b. $f = a \cos 2\theta / r$ (spherical)

$$\text{a. } \nabla f = \frac{\partial f}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{a}_\phi + \frac{\partial f}{\partial z}\hat{a}_z = \boxed{2z\,\hat{a}_r + \frac{1}{r}\left(-8a^2\sin\phi\right)\hat{a}_\phi + 2r\,\hat{a}_z}$$

$$(f = 8a^2\cos\phi + 2rz)$$

$$\text{b. } f = \frac{a\cos 2\theta}{r}\,;\quad \nabla f = \frac{\partial f}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{a}_\varphi$$

$$\nabla f = \frac{-a\cos 2\theta}{r^2}\hat{a}_r + \frac{1}{r}\frac{a}{r}\left(-2\sin 2\theta\right)\hat{a}_\theta$$

Field Math

- Curl
  - Measures the circulation of a vector field

$$curl B \text{ or } \nabla \times B$$

Result is a <u>VECTOR</u>

- Gradient
  - Measures the change in a scalar field

$$grad(f) \text{ or } \nabla f$$

Result is a <u>VECTOR</u>

- Divergence
  - Measures the flux of a vector field through a surface

$$div A \text{ or } \nabla \bullet A$$

Result is a <u>SCALAR</u>