ECSE 2500
Lec 17
March 27

Plan of next two classes

3/30 - Review class of HW 4-6

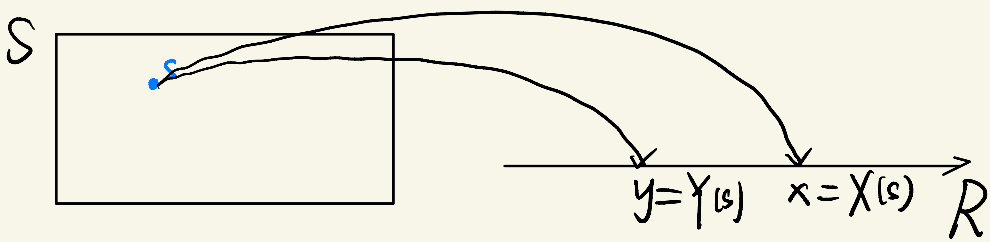
4/3 - Exam 2

Plan of today's class

① CDF / PDF of two random variables

☐ Two random variables

So far we've only looked at one RV at a time.
Now we will start talk about joint / two random variables.
That is, we do an experiment and obtain one outcome of the experiment which creats two real numbes from outcome



$\boxed{\text{Example}}$ Flip coin n times. Define

One random experiment generates two RVs

$X(s) = \#$ of heads over n coin flipping

$Y(s) =$ position of the first head

Clearly, X is a Binomial random variable

Y is a Geometric random variable.

These two RVs follow different distributions, but there is a relationship between them. We will make this relationship clearly in the following note.

$\boxed{\text{Example}}$ We go to bank. Define

X is the length of time spent waiting in line

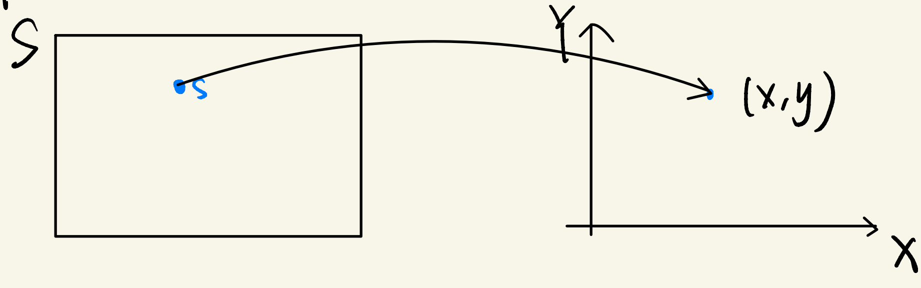
Y is the length of transactions

$\boxed{\text{Example}}$ Choose one person from a class. Define

X = height of the selected person

Y = weight of the selected person

In 2D, we can also think of (X, Y) providing a point in the plane:



For two RVs, we will use almost the same terminology as on RV case: Events, CDF, PDF, PMF, Expectation, variance, etc.

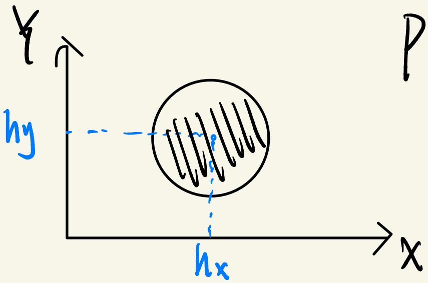
- Joint Event (Extended from Event)

Example: In the case of using GPS, it will generate two RVs (X, Y) representing location information. What is

$$P(\text{location is less than 5 km from origin})?$$

$$\{(X,Y) \mid (X-h_x)^2+(Y-h_y)^2 \le 25\}$$

(h_x, h_y) location of origin

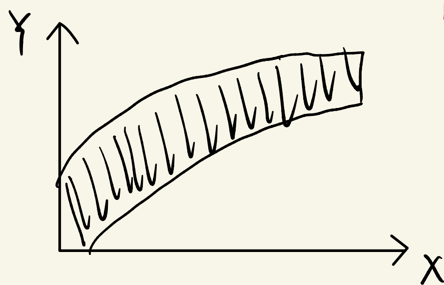


Example: In the height/weight example, what is

$$P(\text{Body mass index of selected person is normal})?$$

$$BMI = \frac{\text{Weight}}{(\text{height})^2}$$

$$\{(X,Y) \mid \frac{X}{Y^2} \in \underbrace{[n_1, n_2]}_{\text{normal BMI range}}\}$$

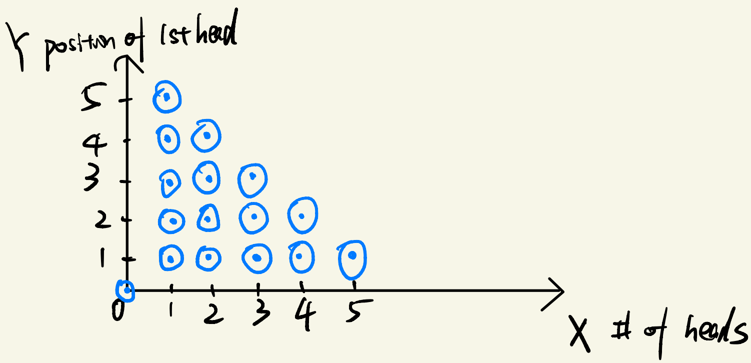


Example (Discrete)

In coin flip case, what is

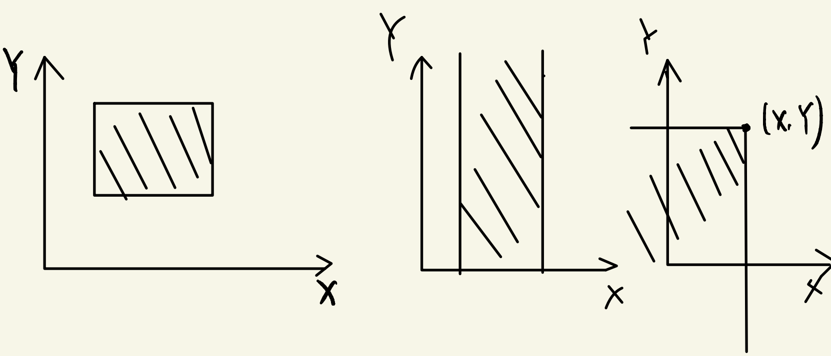
$$P(\underbrace{\text{less than 3 heads and first time head after 2 flips}}_{\{(X,Y)\,|\,X \leq 3,\ Y \geq 2\}},\ n=5)?$$

⊙ denotes the outcome with non-zero probability



In particular, we often interested in the "product" from events of following form

$E_X \cap E_Y$



- Joint CDF of two random variables

$$F_{X,Y}(x, y) = P(X \le x, Y \le y)$$

big letter   small letter

As before. CDF has the following properties

$$F_{X,Y}(\infty, \infty) = P(X \le \infty, Y \le \infty)$$
$$= 1$$

$$F_{X,Y}(-\infty, y) = P(X \le -\infty, Y \le y)$$
$$= 0$$

$$F_{X,Y}(x, -\infty) = P(X \le x, Y \le -\infty)$$
$$= 0$$

- Joint PMF for two discrete RVs

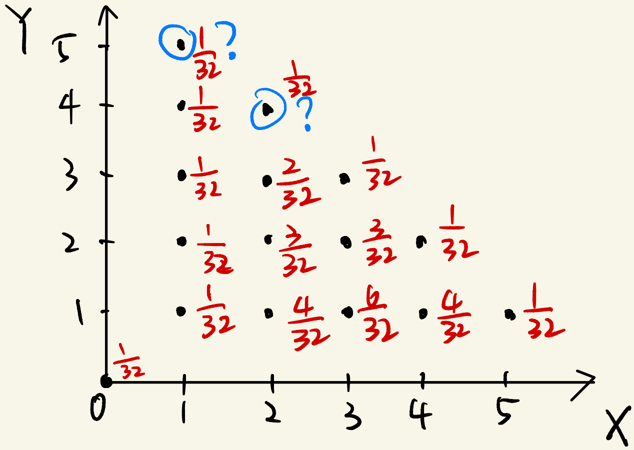
$$P_{X,Y}(x, y) = P(X = x, Y = y)$$

big case   small case

In coin flip case, we have the following joint PMF function (assum a fair coin)

Joint PMF

$P_{X,Y}(x,y)$



$$P(X=1, Y=5) = P(TTTTH) = \frac{1}{2^5} = \frac{1}{32}$$

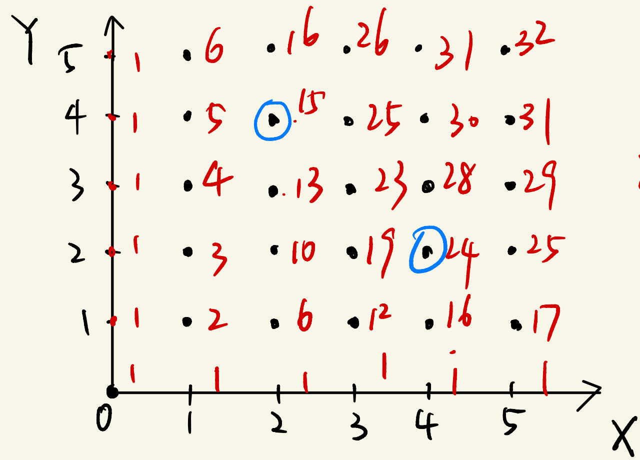
$$P(X=2, Y=4) = P(TTTHH) = \frac{1}{32}$$

Then the joint CDF is what we get by summing up everything below and to the left of a given dot

Joint CDF

$F_{X,Y}(x,y)$

$= P(X \le x, Y \le y)$



$\times \frac{1}{32}$

$$F_{X,Y}(2,4) = P(X \le 2, Y \le 4)$$

$$= \sum_{\substack{x=0,1,2 \\ y=0,1,2,3,4}} P_{X,Y}(x,y) = \frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{2}{32}+\frac{3}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{4}{32}=\frac{15}{32}$$

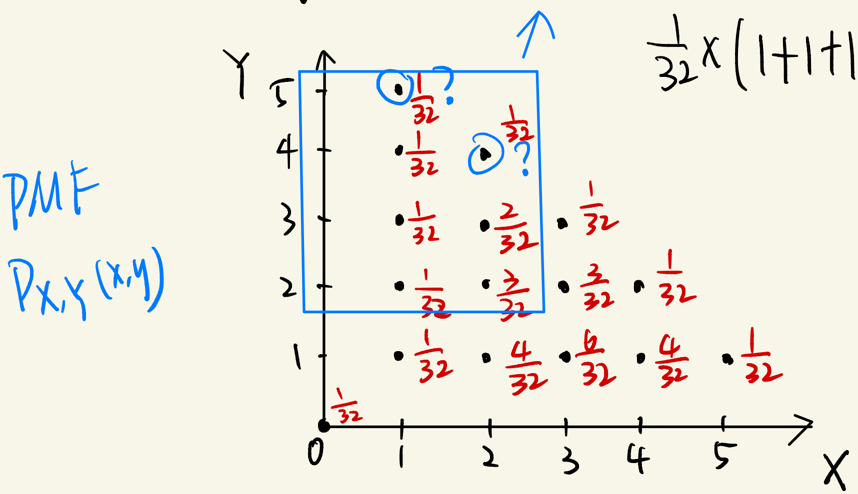
Q: Why we need joint PMF and CDF?

A: We can use this to compute the probability of any event.

For example, in our previous example, we want

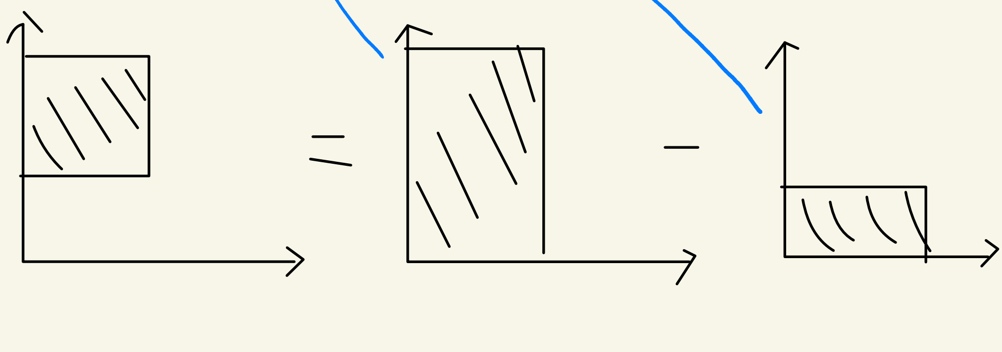
$$P(X<3 \text{ and } Y \geq 2) =$$

$$\frac{1}{32} \times (1+1+1+1+1+2+3) = \boxed{\frac{10}{32}}$$



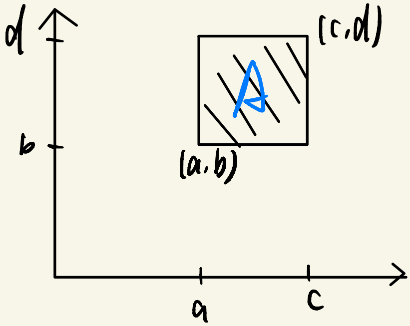
We can also compute this probability using joint CDF.

$$F_{X,Y}(2,5) - F_{X,Y}(2,1) = \frac{16}{32} - \frac{6}{32} = \boxed{\frac{10}{32}}$$

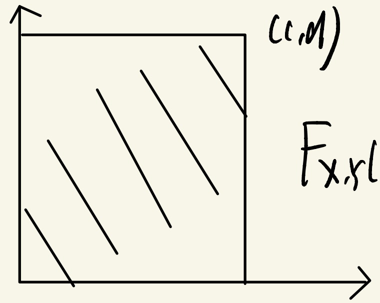


In general, we can compute the probability of a product-form event using CDF as follows

$P(A) =$

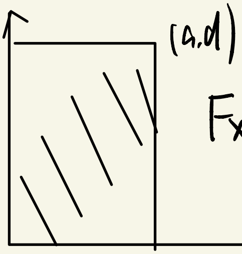


$=$



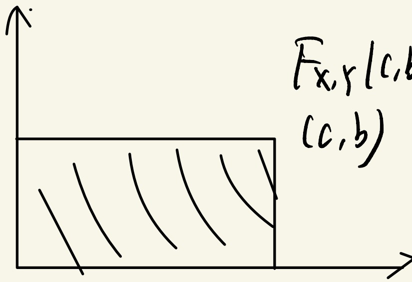
$F_{X,Y}(c,d)$

$-$



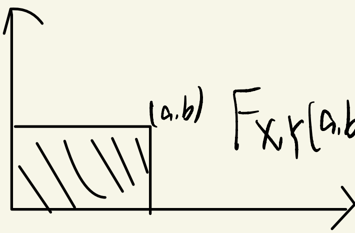
$F_{X,Y}(a,d)$

$-$



$F_{X,Y}(c,b)$

$+$



$F_{X,Y}(a,b)$

$$= F_{X,Y}(c,d) - F_{X,Y}(a,d) - F_{X,Y}(c,b) + F_{X,Y}(a,b)$$