ECSE 2500
Lec 15
March 20

Exam 2: After HW 6, April 3

Coverage: HWs 4–6

Topic: ① Expectation of functions of continuous RVs

② CDF/PDF of functions of continuous RVs

☐ Expected values of functions of continuous RV

As before, we consider a function of RV X, given by g(X), and its expectation is

$$E[g(X)] = \int_{-\infty}^{\infty} g(y) f_X(y)\,dy$$

Applications – Expectation/Variance of X^2

✓ Variance of RV X, e.g.,

$$Var(X) = E[(X - \underbrace{E[X]}_{\text{Constant}})^2]$$

$g(X) = (X - \mu)^2$

As is in discrete RV setting, we can have variance decomposition in continuous RV case,

$$Var(X) = E[(X - E[X])^2]$$

$$= \int_{-\infty}^{+\infty} (x - E[X])^2 f_X(x)\, dx$$

$$= \int_{-\infty}^{+\infty} (x^2 + E^2[X] - 2x E[X]) f_X(x) dx$$

$$= \int_{-\infty}^{+\infty} x^2 f_X(x) dx + E^2[X] \int_{-\infty}^{+\infty} f_X(x) dx$$

$$- 2E[X] \int_{-\infty}^{+\infty} x\, f_X(x) dx$$

$$= E[X^2] + E^2[X] - 2E^2[X]$$

$$= E[X^2] - E^2[X] \Leftarrow \text{Same as discrete RV.}$$

① Start with linear function of X, Y

$$g(\underset{\uparrow \atop RV}{X}, \underset{\uparrow \atop RV}{Y}) = aX + bY$$

i) $E[g(X,Y)] = aE[X] + bE[Y]$

Using the Linearity of Expectation

$$g(X) = \underset{\text{RV}}{X} + \underset{\text{Constant}}{c}$$

ii) $E[g(X)] = E[X] + c$

iii) $g(X) = aX$, given $Var(X)$

$$Var(g(X)) = E[g^2(X)] - E^2[g(X)]$$
$$= E[a^2X^2] - E^2[aX]$$
$$= a^2E[X^2] - a^2E^2[X]$$
$$= a^2\underbrace{(E[X^2] - E^2[X])}_{Var(X)}$$

Be careful, this is not $aVar(X)$

$$= a^2 Var(X)$$

Example: Variance of a uniform continuous RV over [a, b].

Answer: Use the variance decomposition

$$\rightarrow \quad Var(X) = E[X^2] - E^2[X]$$

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{3(b-a)} x^3 \Big|_a^b$$

$$= \frac{1}{3} \cdot \frac{b^3 - a^3}{b-a}$$

$$E[X] = \int_{-\infty}^{+\infty} x \cdot \frac{1}{b-a}\, dx = \frac{a+b}{2}$$

Plug in $\rightarrow$

$$Var(X) = \frac{b^3 - a^3}{3(b-a)} - \frac{(a+b)^2}{4}$$

$$= \frac{1}{12}(b-a)^2$$

Example: Variance of a Gaussian RV

Answer:

$$\text{PDF} \quad f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

$$Var(X) = E[(X - \underbrace{E[X]}_{\mu})^2]$$

$$= \int_{-\infty}^{+\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} dx$$

change of varible
$x - \mu \Leftrightarrow x$

$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}} dx$$

Integration by parts

$$= -x\sigma^2 e^{-\frac{x^2}{2\sigma^2}} \Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} \sigma^2 e^{-\frac{x^2}{2\sigma^2}} dx$$

$$= 0 + \int_{-\infty}^{+\infty} \sigma^2 e^{-\frac{x^2}{2\sigma^2}} dx$$

$$= \sigma^2$$

$\Leftarrow$ Verify that the parameter appears in the PDF is the standard deviation

$\rightarrow$ Homework: Variance of Exponential RV.

□ CDF/PDF of function of continuous RVs

We have a new continuous RV $Y = g(X)$. We know the PDF or CDF of RV X, and we want to find the PDF/CDF of the new RV Y, $F_Y(y)$ or $f_Y(y)$.

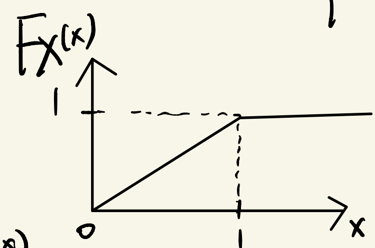
Example: X denotes a uniform RV $[0, 1]$. Let $Y = g(X) = e^X$. What is CDF/PDF of Y?

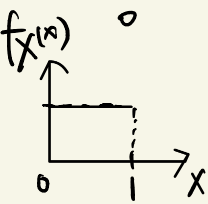
Answer: Based on definition,

$$F_Y(y) = P(Y \le y)$$

$$= P(e^X \le y)$$

$$= P(X \le \ln y)$$





Using the CDF of a uniform RV

$$= \begin{cases} \ln y, & \text{if } \ln y \in [0, 1] \\ 1, & \ln y > 1 \\ 0, & \ln y < 0 \end{cases}$$

$$= \begin{cases} \ln y, & \text{if } y \in [e^0, e^1] \\ 1, & \text{if } y \geq e^1 \\ 0, & \text{if } y \leq e^0 \end{cases}$$

Based on the CDF of Y, we can obtain

PDF $f_Y(y) = \frac{\partial}{\partial y} F_Y(y)$

$$= \begin{cases} \frac{1}{y}, & \text{if } y \in [e^0, e^1] \\ 0, & \text{if } y > e^1 \text{ or } < e^0 \end{cases}$$

To generalize this steps, what we did is like:

Special case 1 If $Y = g(X)$ and g is invertible (1 to 1 mapping), and both X and Y are positive, then

CDF of Y will be $F_Y(y) = P(Y \leq y) = P(X \leq g^{-1}(y))$

Using invertible of $g(x)$

$= F_X(g^{-1}(y))$ ← Composition of $F_X \circ g^{-1}$

PDF of Y $f_Y(y) = \dfrac{d F_Y(y)}{dy}$

(Using Chain rule) $= f_X(g^{-1}(y)) \cdot \dfrac{d}{dy} g^{-1}(y)$

Special case 2

A linear function / transformation

Let RV X have CDF $F_X(x)$ and PDF $f_X(x)$.

Let RV $Y = aX + b$, a, b are constants.

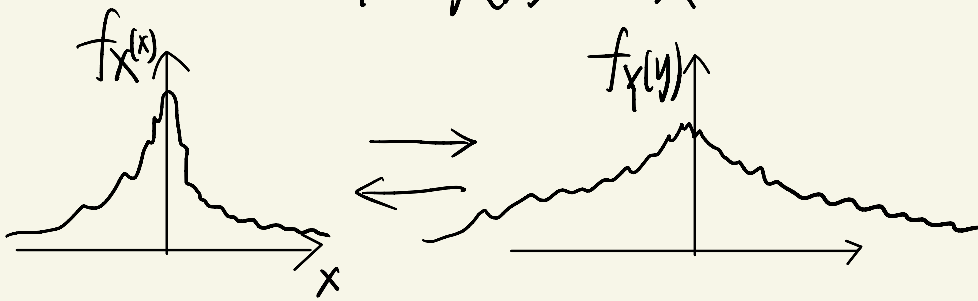
What will be $F_Y(y)$ and $f_Y(y)$?

$$
\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= P(aX + b \le y) \\
&= P(aX \le y - b) \quad \text{Use } F_X(x) \\
&= \begin{cases} P\left(X \le \frac{y-b}{a}\right), & \text{if } a > 0 \\ P\left(X \ge \frac{y-b}{a}\right), & \text{if } a < 0 \end{cases} \stackrel{\downarrow}{=} \begin{cases} F_X\left(\frac{y-b}{a}\right), & a > 0 \\ 1 - F_X\left(\frac{y-b}{a}\right), & a < 0 \end{cases}
\end{aligned}
$$

$$f_Y(y) = \frac{\partial F_Y(y)}{\partial y} = \begin{cases} f_X\left(\frac{y-b}{a}\right)\cdot\frac{1}{a}, & a>0 \\ -f_X\left(\frac{y-b}{a}\right)\cdot\frac{1}{a}, & a<0 \end{cases}$$

$$= \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

Visualization : $Y = g(X) = 2X$



Example: X is a Gaussian RV with mean μ and variance σ^2, and $Y = aX+b$.

Use Special case 2

What is PDF/CDF of Y?

PDF of Y: $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$

$$= \frac{1}{|a|}\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\left(\frac{y-b}{a}-\mu\right)^2}{2\sigma^2}}$$

PDF of Gaussian with mean $b+a\mu$, variance $a^2\sigma^2$.

$$= \frac{1}{\sqrt{2\pi}\,\sigma|a|} e^{-\left(\frac{(y-(b+a\mu))^2}{2\sigma^2|a|^2}\right)}$$

Linear transformation of Gaussian RV is still Gaussian!