Exam 2

Instructions

1.) Unless otherwise specified, you have one class period to complete the questions below.

2.) Read all directions carefully.

3.) Show your work in enough detail to allow the graders to completely follow your thought process.

4.) Make sure your calculator is set to perform trigonometric functions in radians & not degrees & use at least 2 significant digits.

5.) Make sure to write your answers legibly. You can write on the back of the exam pages or ask for scratch paper

Solutions

1. Boundary Conditions

Consider the boundary between two different materials. The top material has permittivity $\varepsilon_r = 20$ and the bottom material has the same permittivity as free space. Between the top and bottom materials there is a surface charge with density +200 nC / m^2.

a.) (6 pts) Define some electric field in the top region. It may be any field you choose as along as you specify both magnitude and direction and the magnitude is not zero. Then calculate the electric field in the bottom region.

$$\vec{D} = \epsilon \vec{E}$$

Let $\vec{E_1} = 100\hat{x} + 100\hat{y}$ ($\hat{x}$ = boundary tangent, $\hat{y}$ = boundary normal)

$$\vec{E_{1t}} = \vec{E_{2t}} = 100\hat{x} \qquad \vec{D_{1n}} - \vec{D_{2n}} = \rho_s$$

$$(100)20\,\varepsilon_0 - 200 \times 10^{-9}\ C/m^2 = \vec{D_{2n}} = -182\ nC/m^2$$

$$\vec{E_{2n}} = \frac{D_{2n}}{\varepsilon_0} = -20{,}600\ V/m$$

$$\vec{E_2} = 100\hat{x} - 20600\hat{y}\ V/m$$

b.) (6 pts) Now assume that each of the regions consists of a cube 1 meter in length on each side. Calculate the electric field energy density in each region and the total energy stored.

$$|\vec{E_2}| \approx 20{,}600\ V/m$$

$$|\vec{E_1}| = \sqrt{100^2 + 100^2} = 141\ V/m$$

$$W_1 = \frac{1}{2}(20\varepsilon_0)(141\ V/m)^2 \cdot (1\ m)^3 = 3.52 \times 10^{-6}\ J$$

$$W_2 = \frac{1}{2}\varepsilon_0 (20600\ V/m)^2 \cdot (1m)^3 = 3.76 \times 10^{-3}\ J$$

2. Capacitance

Consider a spherical capacitor with a layered structure. Its innermost layer is a grounded spherical conductor of radius 1mm. This is covered by a 1mm thick layer of dielectric with permittivity $2\varepsilon_0$, then a 1mm thick layer of permittivity $10\varepsilon_0$, and finally at the outermost layer, 1mm thick shell of conductor. At any given moment of operation, the capacitor has some charge +Q on the outer conductor and charge -Q on the inner conductor, which we will represent as the charge magnitude Q.

a.) Write an expression for the electric field inside all regions in the interval 0mm ≤ r 5mm as a function of Q. Be sure to specify the direction of the field.

$0mm \le r < 1mm$: $\vec{E} = 0$ (conductor region)

$1mm \le r < 2mm$: $\vec{E} = \dfrac{-Q}{8\pi\varepsilon_0 r^2}\hat{r}$ V/m

$2mm \le r < 3mm$: $\vec{E} = \dfrac{-Q}{40\pi\varepsilon_0 r^2}\hat{r}$ V/m

$3mm \le r < 4mm$: $\vec{E} = 0$ (conductor region)

$r \ge 4mm$: $\vec{E} = 0$ $(Q_{enc} = 0)$

b.) For a given Q, where is there a nonzero surface charge density and what is its value?

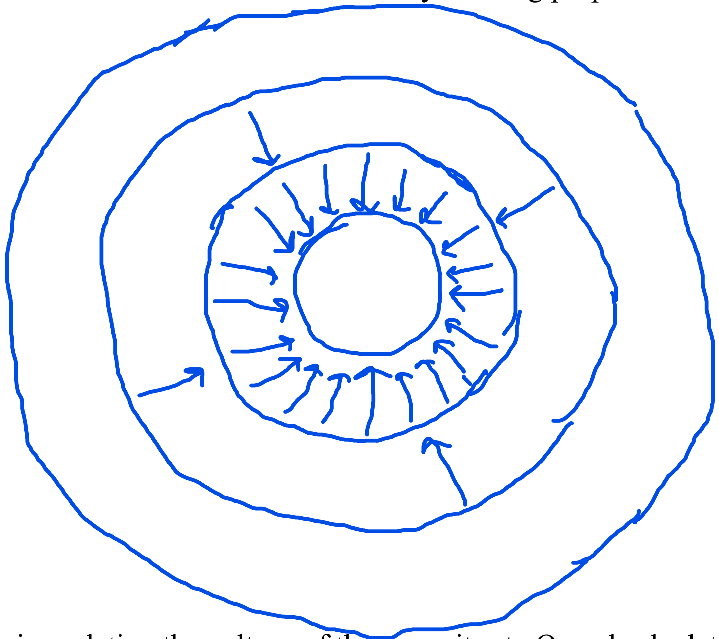
At $r = 1mm$ (outer surface of inner conductor)

$$\rho_s = \frac{-Q}{4\pi(0.001)^2} \quad C/m^2$$

At $r = 3mm$ (inner surface of outer conductor)

$$\rho_s = \frac{Q}{4\pi(0.003)^2}$$

c.) Draw a cross-section of this capacitor and sketch the field inside. Be sure to show the direction of the field and do your best to draw the field line density as being proportional to the field magnitude.



d.) Write an expression relating the voltage of this capacitor to Q, and calculate its capacitance.

See next page

e.) How much voltage must be applied to this capacitor before dielectric breakdown occurs, and where will it occur? Assume that the dielectric strength of all dielectric materials is 20 MV/m. (That's megavolts, not millivolts.)

Field will be strongest at the inner radius of the inner dielectric, so breakdown occurs here. ($r = 0.001$ m)

$$\vec{E}(0.001) = \frac{-Q}{8\pi\varepsilon_0 (0.001)^2} \qquad Q = CV = V(0.417 \times 10^{-12})$$

$$\frac{-(0.417 \times 10^{-12})V}{8\pi\varepsilon_0 (0.001)^2} = 20 \times 10^6 \qquad V = 10,700 \text{ V}$$

d.) $V = -\int \vec{E} \cdot \vec{dl}$

Calculating voltage from the inner conductor to the outer,

$$V = -\int_{0.001}^{0.002} \frac{-Q}{8\pi \varepsilon_0 r^2} \, dr - \int_{0.002}^{0.003} \frac{-Q}{40\pi \varepsilon_0 r^2} \, dr$$

$$V = \left| \frac{-Q}{8\pi \varepsilon_0 r} \right|_{0.001}^{0.002} + \left| \frac{-Q}{40\pi \varepsilon_0 r} \right|_{0.002}^{0.003}$$

$$V = -\frac{Q}{4\pi \varepsilon_0} \left[\frac{1}{2} \left(\frac{1}{0.002} - \frac{1}{0.001} \right) + \frac{1}{10} \left(\frac{1}{0.003} - \frac{1}{0.002} \right) \right]$$

$$V = \frac{Q}{4\pi \varepsilon_0} \left[266.67 \right]$$

$$C = \frac{Q}{V} = \frac{\cancel{Q}}{[266.67] \frac{\cancel{Q}}{4\pi \varepsilon_0}} = \frac{4\pi \varepsilon_0}{266.67} = 0.417 \, pF$$

Electric Charge

A cylinder is 1m tall and has a radius of 0.5m. The surface of the cylinder has a charge density of -z nC/m^2. *(the z axis points along the center of the cylinder and the bottom of the cylinder is at z=0, so the charge density starts as zero at the bottom of the cylinder and becomes increasingly negative toward the top.)*

a.) What is the total charge Q on the surface of the cylinder?

in nanocoulombs

$$Q = \underbrace{\int_0^{2\pi}}_{\phi} \underbrace{\int_0^{1}}_{z} (-z)\, dS_R = \int_0^{2\pi}\int_0^{1} (-z)\, r\, d\phi\, dz \qquad r = 0.5m$$

$$= (2\pi)(0.5)\int_0^1 (-z)\, dz = (2\pi)(0.5)\left|\frac{-z^2}{2}\right|$$

$$Q = 2\pi \cdot 0.5 \cdot 0.5\ nC = -1.57\ nC$$

b.) If a +100nC charge is located at a height of 50m above the cylinder, what is the magnitude and direction of the force it will experience? *(For this calculation you may approximate the cylinder as being a point charge.)*

$$\vec{F} = \frac{q_1 q_2}{4\pi\varepsilon r^2}\hat{r} = \frac{(100nC)(-1.57nC)}{4\pi\varepsilon_0 (50m)^2}\hat{r}$$

$$\vec{F} = 5.64\times 10^{-10}\ N$$ downward toward the cylinder

(4) a.) Laplace's Equation (because there is no free charge)

b.)
$$V_1 = \frac{1}{4}(0+0+0+0) = 0$$
$$V_2 = \frac{1}{4} = (0+0+100+0) = 25$$
$$V_3 = \frac{1}{4}(0+25+0+0) = 6.25$$
$$V_4 = \frac{1}{4}(0+0+0+100) = 25$$
$$V_5 = 100$$
$$V_6 = \frac{1}{4}(6.25+100+0+0) = 26.56$$
$$V_7 = \frac{1}{4}(25+0+0+0) = 6.25$$
$$V_8 = \frac{1}{4}(100+6.25+0+0) = 26.56$$
$$V_9 = \frac{1}{4}(26.56+26.56+0+0) = 13.28$$

c.) Suppose that the material inside the box has conductivity 5 x 10^{-5} S/m. Calculate the magnitude and direction of the density of current flowing between point V9 and the bottom grounded plate of the box.

Let the width of each box be 1m.

At point V9, $\vec{E}$ is 13.28 V/1m = 13.28 V/m.

$$\vec{J} = \sigma \vec{E} = (5 \times 10^{-5})(13.28)$$

$$\vec{J} = 664 \, \mu A/m^2 \text{ pointing down}$$

5. a.) We choose $f = 1\ MHz$.

$$Z_L = 100 + j\omega L = 100 + j(2\pi)(10^6)(5\times10^{-6}) = 100 + j31.42$$

to normalize: $$z_L = \frac{Z_L}{Z_o} = \frac{100 + j31.42}{50} = 2 + j0.628$$

b.) $y_L = 0.45 - j0.14$

c.) On the Smith Chart, y_{in} where the stub is added is labeled as y'_L.

y_L is located at 0.472λ or $0.472 - 0.5 = -0.028\lambda$

y'_L is located at 0.154λ.

The distance between them is

$0.154\lambda - (-0.028\lambda) = 0.182\lambda$.

$$\lambda = \frac{v}{f} = \frac{0.6 \times 3\times10^8}{10^6} = 180\,m$$

$0.182 \cdot 180\,m = 32.76\,m$ (distance of stub from load)

The Complete Smith Chart

Black Magic Design