Homework 3

Released: February 1st

Due: 11:59pm February 8th

1. Lossy Transmission Lines

You are given a length of transmission line with the following properties: $u_p = 0.66c$ and $Z_0 = 50\Omega$.

a) You measure the amplitude of the voltage at two positions on the transmission line and obtain the following measurements:

Distance (m)	Voltage (V)
0 (source)	5.00
900	4.97

What is the value of α, the attenuation factor for this line?

$$V(z) = V(0)\exp^{-\alpha z}$$

$$4.97V = 5V\exp^{-\alpha 900m} \rightarrow \alpha = -\frac{\ln\left(\frac{4.97}{5}\right)}{900} \approx 6.7 \times 10^{-6}\, m^{-1}$$

[2 pts: 1 pt for valid approach; 1 pt for no math error]

b) Assuming this is a low-loss transmission line, what is the value of *r*, the resistance per unit length of the line?

$$\text{For a low-loss line, } \alpha = \frac{r}{2Z_0} \rightarrow r = 2\alpha Z_0 = 2 \cdot 6.7 \times 10^{-6} m^{-1} \cdot 50\Omega = 670\,\frac{\mu\Omega}{m}$$

[2 pts: 1 pt for valid approach; 1 pt for no math error]

c) Knowing both u_p and Z_0 (assumed to be purely real in this case), you can calculate *c* (capacitance per unit length) and *l* (inductance per unit length) for the transmission line. What value of *g* (conductance per unit length) would you need to make this low-loss transmission line dispersionless?

$$u_p = \frac{1}{\sqrt{lc}};\, Z_0 = \sqrt{\frac{l}{c}}$$

$$u_p Z_0 = \frac{1}{c} \rightarrow c = \frac{1}{u_p Z_0} = \frac{1}{0.66 \cdot 3 \times 10^8 \left[\frac{m}{s}\right] \cdot 50\Omega} = 101\,\frac{pF}{m}$$

$$\frac{u_p}{Z_0} = \frac{1}{l} \rightarrow l = \frac{Z_0}{u_p} = \frac{50\Omega}{0.66 \cdot 3 \times 10^8 \left[\frac{m}{s}\right]} = 253\,\frac{nH}{m}$$

$$\text{Using the Heaviside condition: } \frac{r}{l} = \frac{g}{c}$$

$$g = \frac{cr}{l} = \frac{101\frac{pF}{m} \cdot 670\frac{\mu\Omega}{m}}{253\frac{nH}{m}} = 2.67 \times 10^{-7}\frac{S}{m} = 267\frac{nS}{m}$$

[3 pts: 1 pt for correct *c* and *l*; 1 pt for correct approach to finding *g*; 1 pt for not making a math error]

d) What is the new value for α on this dispersionless transmission line?

$$\alpha = r\sqrt{\frac{c}{l}} = 670\frac{\mu\Omega}{m}\sqrt{\frac{101\frac{pF}{m}}{253\frac{nH}{m}}} = 1.34 \times 10^{-5}\ m^{-1}$$

[2 pts: 1 pt for correct approach; 1 pt for not making a math error]

e) If you were to measure the voltage at z = 900m on this new, dispersionless transmission line, what would it be? Has the addition of conductance noticeably affected the loss of this transmission line?

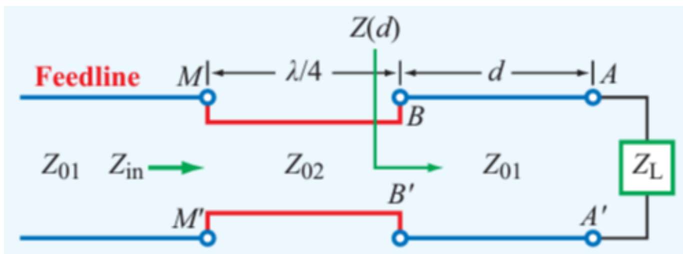
$$V(z) = V(0)\exp^{-\alpha z}$$
$$V(900) = 5V\exp^{-1.34\times10^{-5}\cdot900} = 4.94V$$

Yes, the addition of the conductance increased the voltage loss on the line by an additional 100% (30 mV).

[3 pts: 1 pt for correct approach; 1 pt for not making a math error; 1 pt for explanation]

2. Impedance Matching with Smith Charts Part I

Given the circuit below, you are to use a transmission line of length *d* with characteristic impedance $Z_{01} = 50\Omega$ and a quarter-wave transformer of characteristic impedance Z_{02} to match the feedline impedance (which is also $Z_{01} = 50\Omega$) to the load impedance $Z_L = (40 + j35)\Omega$.



(Ulaby 2-33)

Finding the Length L of the Transmission Line Needed to Make $Z_{in,0}$ Purely Real

a) Calculate the normalized impedance *z*.

$$z = \frac{Z_L}{Z_{01}} = \frac{40}{50} + j\frac{35}{50} = (0.8 + j0.7)\Omega$$

[2 pts: 1 pt for correct equation; 1 pt for no math error]

b) Using the Smith chart on the next page, plot *z* and the VSWR circle.

See Smith chart on next page.

[2 pts: 1 pt for correct z on Smith chart; 1 pt for correct VSWR circle]

c) Using the Smith chart, determine the magnitude and phase of Γ and the SWR. Explain how you found each of these values.

$|\Gamma| = 0.38$ and is the radius of the VSWR circle. You can find $|\Gamma|$ by going to the bottom of the chart, finding $|\Gamma| = 0$ (marked as "center" on the 3rd line from the top) then traveling a distance equal to the VSWR circle radius to the left and reading off the value for "Rfl. Coeff, E or I".

[4 pts: 1 pt for correct |Γ|; 1 pt for correct ∠Γ; 1 pt for correct VSWR circle; 1 pt for correct process for finding these values]

d) What length of transmission line (in terms of λ) is needed to transform *z* into a purely real impedance and what is the resulting impedance?

See Smith chart: moving clockwise (towards the generator = making the t-line longer) along the VSWR circle from the location of z on the Smith chart to the real Γ axis, we must move a total of about 85°. That corresponds to a line length of $d = \frac{85°}{180°} \cdot \frac{\lambda}{4} = 0.118\lambda$.

The real impedance reached on the VSWR circle is $z = 2.2Z_{01} = 110\Omega$.

[3 pts: 1 pt for correct approach; 1 pt for correct *d*; 1 pt for correct real *z*]

e) The VSWR circle you drew in part b) is a curve of constant $|\Gamma_L|$. Why must we remain on a curve of constant $|\Gamma_L|$ on the Smith chart when we are transforming impedances using transmission lines? (Hint: what parameters determine the magnitude of Γ_L?)

The magnitude of Γ_L is fixed by the characteristic impedance of the line and the load impedance, so this cannot be changed without changing one of the two (which we are not doing). We are only changing the length of the t-line.

[1 pt for valid answer]

f) When you determined the length of the transmission line needed in part d), you traveled along the VSWR circle, on which the magnitude of Γ is constant, but the phase of Γ is changing. What does a change in the phase of Γ on a Smith chart physically correspond to? (Hint: Γ on the Smith chart can be thought of as the phase-shifted reflection coefficient Γ_d).

The phase of Γ_d (the phase-shifted reflection coefficient) depends on β (the phase constant, which is not changing) and *d* (the distance on the t-line towards the generator, as measured from the load): $\Gamma_d = \Gamma_L e^{-j2\beta d}$. Thus, a change in the phase corresponds to moving a distance down a transmission line away from the load OR lengthening an existing transmission line towards the generator, as measured from the load.

[1 pt for valid answer]

g) What do the impedances on the VSWR circle on the Smith chart represent? (Hint: recall the equation for wave impedance on the transmission line $Z(d) = \frac{1+\Gamma_L e^{-j2\beta d}}{1-\Gamma_L e^{-j2\beta d}}$, where θ_Γ is the phase of Γ_L at the load and *d* is the distance measured from the load in the direction of the generator).

The impedances on the VSWR circle on the Smith chart represent the normalized wave impedance on the transmission line at a distance *d* from the load. If we were to terminate a t-line connected to the load at a length *d*, this Smith chart impedance would be Z_{in}, as seen "looking into" the t-line.

[1 pt for valid answer]

Finding the Characteristic Impedance Z_{02} Needed to Match Z_{01} to Z_L

h) Determine the characteristic impedance Z_{02} of the quarter-wave transformer required to match the generator resistance to the transmission line you determined the length of in part d).

In order to match Z_{01} to Z_L, we need to use a quarter-wave transformer to convert $Z_{in} = 2.2Z_{01}$ into Z_{01}

$$Z_{01} = \frac{Z_{02}^2}{Z_{in}} \rightarrow Z_{02} = \sqrt{Z_{01}Z_{in}} = \sqrt{50\Omega \cdot 110\Omega} = 74.16\Omega$$

[2 pts: 1 pt for valid approach; 1 pt for no math error]

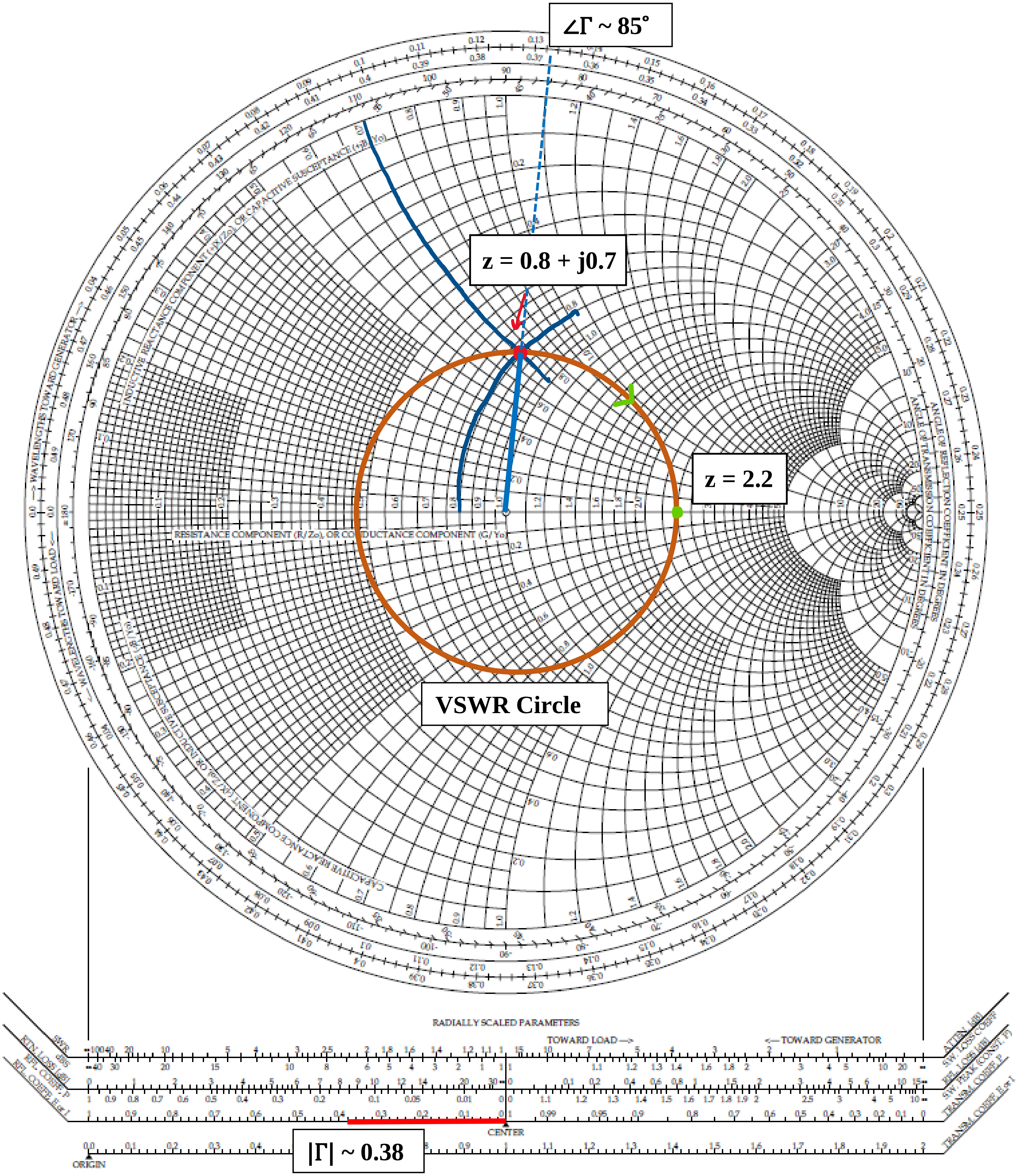
i) If the frequency of the signal is *f* = 10MHz and the velocity factor of both transmission lines is 0.66, how long is your transmission line from part d? How long is your quarter wave transformer from part g?

$$\lambda = \frac{u_p}{\omega} = \frac{0.66 \cdot 3 \times 10^8 \left[\frac{m}{s}\right]}{2\pi \cdot 10 \times 10^6 Hz} = 3.15m$$

Length of first line: d = 0.118λ = 0.372m

Length of quarter-wave transformer = λ/4 = 0.788m

[2 pts: 0.5 pt (each) for valid approach; 0.5 pt (each) for no math error]


∠Γ ~ 85°
z = 0.8 + j0.7
z = 2.2
VSWR Circle
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
RADIALLY SCALED PARAMETERS
TOWARD LOAD —>
<— TOWARD GENERATOR
CENTER
ORIGIN
|Γ| ~ 0.38

3. Impedance Matching with Smith Charts Part II

In this question, you will verify you results from Problem 2 using the equations for the impedance of lossless transmission lines, instead of the Smith Chart.

a) Using the equation $\Gamma_L = \frac{z-1}{z+1}$, where z is the normalized impedance from 2a, calculate the magnitude and phase of the reflection coefficient at the load. Do they match the values you determined using the Smith chart in 2c?

$$\Gamma_L = \frac{0.8 + j0.7 - 1}{0.8 + j0.7 + 1} = \frac{-0.2 + j0.7}{1.8 + j0.7} \cdot \frac{1.8 - j0.7}{1.8 + j0.7} = \frac{0.13 + j1.4}{3.73} = 0.0349 + j0.3753$$

$$|\Gamma_\mathrm{L}| = 0.3769 \sim 0.38$$

$$\angle\Gamma_L = \tan^{-1}\left(\frac{0.3753}{0.0349}\right) = 84.7° \sim 85°$$

These values match those found on the Smith chart in 2c.

[4 pts: 1 pt for correct formulas; 1 pt for | Γ |; 1 pt for ∠Γ; 1 pt for matching 2c values]

b) Insert the transmission line length L you determined in 2d into the equation for input impedance: $Z_{in}(L) = Z_0 \frac{z + j\tan(\beta L)}{1 + jz\tan(\beta L)}$, where $Z_0 = 50\Omega$ as in problem 2. Does Z_{in} agree with the impedance you found on the Smith chart in 2d?

$$Z_{in} = Z_0 \frac{z + j\tan(\beta d)}{1 + jz\tan(\beta d)} = Z_0 \frac{\left(0.8 + j0.7 + j\tan\left(\frac{2\pi}{\lambda} \cdot 0.118\lambda\right)\right)}{1 + j(0.8 + j0.7)\tan\left(\frac{2\pi}{\lambda} \cdot 0.118\lambda\right)}$$

$$= Z_0 \frac{0.8 + j1.6157}{0.359 + j0.7326} \cdot \frac{(0.359 - j0.7326)}{(0.359 - j0.7326)} = Z_0\left(\frac{1.4709 - j0.006}{0.6656}\right)$$

$$= Z_0(2.21 - j0.009) \approx 2.21 Z_0$$

This result is very close to the $2.20Z_0$ determined using the Smith Chart.

[3 pts: 1 pt for correct approach; 1 pt for no math. errors; 1 pt for result that matches 2d]

c) Assuming that your quarter-wave transformer was appropriately designed in 2h, what impedance does the generator circuit "see" looking to the right from R_g, into the transmission line?

If the quarter-wave transformer is appropriately designed, the generator circuit should see $Z_{in} = \frac{Z_{0T}^2}{Z_L} = \frac{(74.16\Omega)^2}{110\Omega} = 50\Omega = R_g$.

[2 pts: 1 pt for correct approach (calculation or reasoning); 1 pt for verifying a matched impedance]

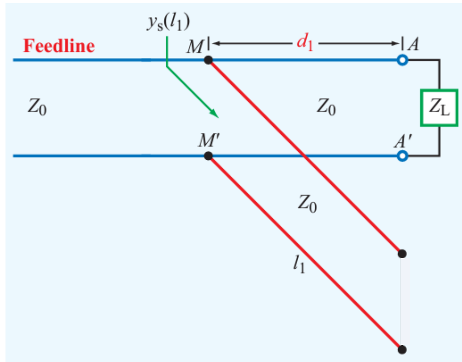
d) Even though you've matched Z_L to R_g to achieve maximum power transfer to the load via a quarter-wave transformer (no reflections at the input to the transmission line in steady-state), the reflection coefficient at the load Γ_L (and at the generator Γ_g) is non-zero, meaning that reflections still occur on the λ/4 line. How is it possible that maximum power is still being transferred despite $\Gamma_L \neq 0$?

Even though there are reflections, the t-line is lossless and after an infinite number of reflections between the load and the source on the t-line, the total power that is transferred to the load sums to the time-average maximum of $P_{av}^{L} = \frac{|V_0^+|^2}{2Z_0}$.

[1 pt for valid answer]

4. Impedance Matching with Stubs (also using Smith Charts)

You are to match a transmission line with a characteristic impedance of Z_0 =100Ω to a load impedance Z_L = (50 - j25)Ω using an open-circuited stub, as shown below:



a) Find the normalized impedance of the load. Plot it on the Smith chart.

$$z = \frac{50}{100} - j\frac{25}{100} = 0.5 - j0.25$$

[2 pts: 1 pt for correct z; 1 pt for correct location on Smith chart]

b) Using the Smith chart, find and plot the normalized admittance y.

From the Smith chart: $y = 1.6 + j0.8$

[2 pts: 1 pt for correct approach on Smith chart; 1 pt for correct y]

c) What length of transmission line d_1 is required to transform the real part of y to be 1? What is y at this point? What is the significance of Re{y} = 1?

$$d_1 = \frac{36° - (-68°)}{180} \cdot \frac{\lambda}{4} = 0.144\lambda$$

From the Smith chart: y = 1 - j0.78

During this step, we match the real part of Y to $1/Z_0$, which only leaves a reactive component to match, which we can do with the open-circuited stub.

An alternate solution is to rotate further to the next time the SWR circle intersects the matching circle, which occurs at y = 1 + j0.78. This is a rotation of 328° and corresponds to a line of length d_1 = 0.456λ.

[4 pts: 1 pt for correct approach on Smith chart; 1 pt for correctly-calculated d_1 value; 1 pt for correct y; 1 pt for correct explanation]

d) Plot the location of the open-circuit admittance and the susceptance that we will need for matching the load on the Smith chart. What length of stub l_1 do we need in order to provide this susceptance for matching?

$$l_1 = \frac{180° - 104°}{180°} \cdot \frac{\lambda}{4} = 0.106\lambda$$

For the 2nd answer above, the length that is needed is located 284° counter-clockwise from Y = 0, which results in a stub of length l_1 = 0.394λ.

[2 pts: 1 pt for correct approach on Smith chart; 1 pt for correct l_1]

e) Now we need to verify that our procedure on the Smith chart has actually matched the load impedance to the transmission line characteristic impedance of 100Ω. The input impedance seen by the transmission line is $Z_{in} = \frac{Z_{stub}^{oc} \cdot Z_{d1}}{Z_{stub}^{oc} + Z_{d1}}$, where Z_{stub}^{oc} is the input impedance of the open-circuited stub and Z_{d1} is the input impedance of the transmission line of length d_1.

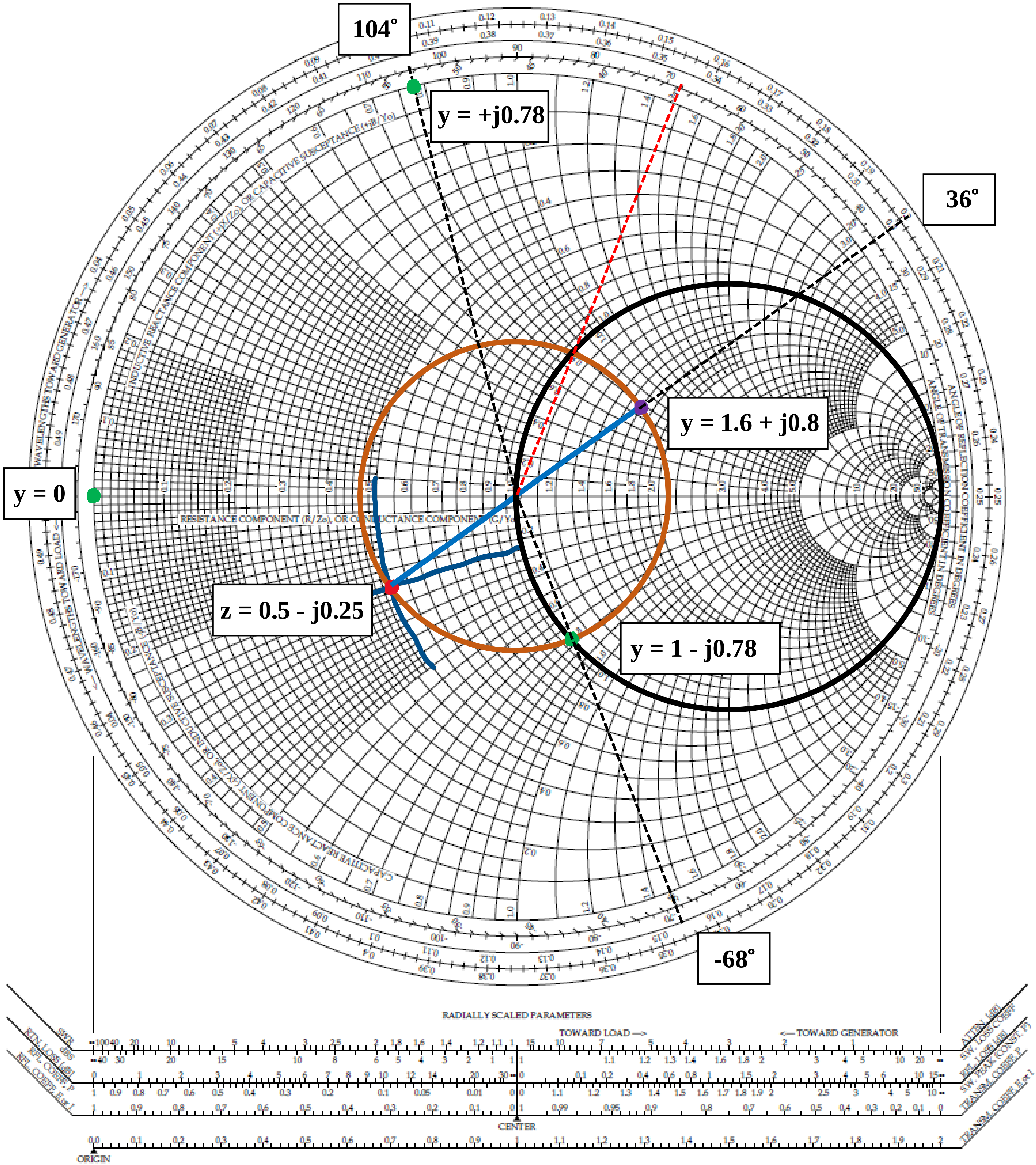
$$Z_{stub}^{oc} = -jZ_0 \cot \beta l_1 = -jZ_0 \cot(2\pi \cdot 0.106) = -j1.273 Z_0$$

$$Z_{d1} = Z_0 \left(\frac{z + j \tan \beta d_1}{1 + jz \tan \beta d_1} \right) = Z_0 \left(\frac{(0.5 - j0.25) + j \tan(2\pi \cdot 0.144)}{1 + j(0.5 - j0.25) \tan(2\pi \cdot 0.144)} \right) = Z_0(0.611 + j0.481)$$

$$Z_{in} = \frac{Z_{stub}^{oc} \cdot Z_{d1}}{Z_{stub}^{oc} + Z_{d1}} = Z_0(0.989 + j0.009)$$

This is close to the expected answer of $Z_{in} = Z_0$.

[3 pts: 1 pt (each) for correct approach to calculating the relevant Z value; 1 pt for not making a math mistake / recognizing that $Z_{in} = Z_0$ is the correct result]


104°
y = +j0.78
36°
y = 0
y = 1.6 + j0.8
z = 0.5 - j0.25
y = 1 - j0.78
-68°
RADIALLY SCALED PARAMETERS
TOWARD LOAD —>
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