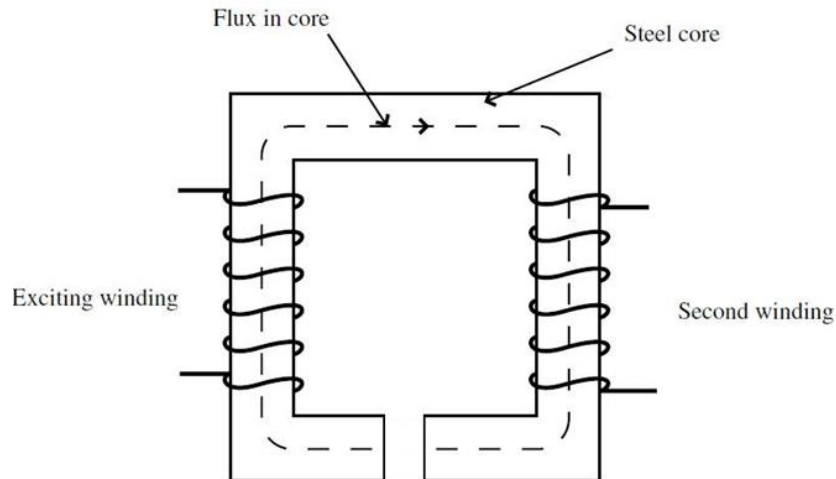


Homework 7

1. Transformer on an Iron Core with an Air Gap [12 Points]

The transformer below consists of two coils on an iron core with an air gap. The iron core portion has a perimeter of 475cm and the air gap is 25cm wide, giving a total perimeter through the iron core region and air gap of 500cm. Additionally, the iron core has a square cross-sectional profile with side lengths of 25cm. The permeability of the iron core region is $3000\mu_0$ and the permeability of the air gap region is μ_0 . The exciting winding has 1000 turns and the second winding has 3000 turns.



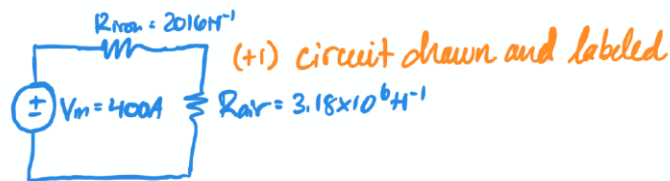
- a) Draw a magnetic circuit to represent the structure when the current through the coil is 400mA DC and calculate and label the following:
- the MMF
 - the reluctance of the iron core and air gap region
 - the total magnetic flux through the core. (For this part of the problem, ignore the second winding.)

$$i) \text{ m.m.f. } = N \cdot I = 1000 \cdot 400 \text{ A} = 400 \text{ A} \quad (+1)$$

$$ii) R = \frac{l}{\mu A} \rightarrow R_{\text{iron}} = \frac{0.475 \text{ m}}{3000 \mu_0 \cdot (0.25 \text{ m})^2} = 2016 \text{ H}^{-1} \quad (+1)$$

$$R_{\text{air}} = \frac{0.25 \text{ m}}{\mu_0 \cdot (0.25 \text{ m})^2} = 3.18 \times 10^6 \text{ H}^{-1} \quad (+1)$$

$$iii) \psi_m = \frac{V_m}{R_{iron} + R_{air}} = \frac{400A}{2016 H^{-1} + 3.18 \times 10^6 H^{-1}} = 1.26 \times 10^{-4} Wb \quad (+1)$$



- b) Calculate the magnitude of the H field in the air gap.

$$H = \frac{B}{\mu_0} = \frac{\psi_m / A}{\mu_0} = \frac{\psi_m}{\mu_0 A} = \frac{1.26 \times 10^{-4}}{4\pi \times 10^{-7} \frac{H}{m} \cdot (0.25m)^2} = 1004 A/m \quad (+1)$$

- c) Calculate the magnetic force felt by the two pieces of the core on either side of the air gap. Is this an attractive or repulsive force?

$$F_m = \frac{B_{air}^2}{2\mu_0} = \frac{(\mu_0 \cdot 1004 A/m)^2}{2\mu_0} = 25.9 N \quad (+1)$$

attractive force, as F points in the direction of the region with higher energy density, which is in the air gap. (+1) attractive / repulsive

$$\vec{F} = -\vec{\nabla} \left(\frac{B^2}{2\mu_0} \right)$$

- d) Suppose that the 400mA DC current is replaced with a 400mA 100Hz AC current. What is the resulting emf across the secondary coil?

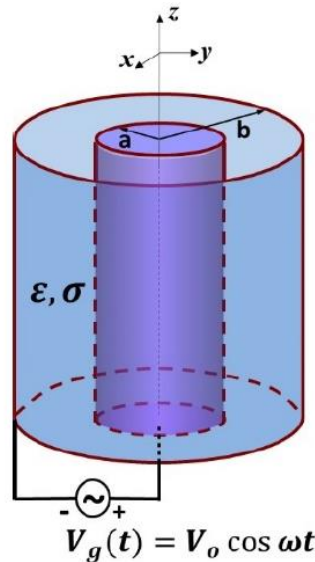
$$\begin{aligned} emf &= -\frac{\partial \Phi}{\partial t} = -\frac{\partial}{\partial t} \{ N_{secondary} \psi_m(t) \} \quad (+1) \text{ equation} \\ &= -3000 \cdot 1.26 \times 10^{-4} Wb \frac{\partial}{\partial t} \{ \sin(2\pi f t) \} \\ &= -0.378 (2\pi \cdot 100 \text{ Hz}) \cos(200\pi t) \\ &= -237.5 \cos(200\pi t) V \quad (+1) \end{aligned}$$

- e) Is it preferable to use a hard or a soft ferromagnetic material for the core of this transformer? Why?

It is preferable to use a soft magnetic material for this core because it'll suffer fewer losses due to hysteresis effects than a hard core would. (+1) soft magnetic material (+1) justification

2. Displacement Current and the Quasi-static Approximation [16 Points]

For a coax-cable capacitor, the volume between the cylindrical copper conductors is filled with a lossy dielectric with permittivity ϵ_r and conductivity σ . The radius of the inner conductor is a and the effective inner radius for the outer conductor is b . The cable length l is much shorter than the wavelength, but $l \gg b$. The voltage applied across the coax-cable capacitor is $V(t) = V_0 \cos(\omega t)$ V.



- a) Determine the displacement field \vec{D} between the conductors, the displacement current I_d and the conduction current I_c passing through the capacitor. What is the phase angle between I_c and I_d ? Which current leads (or which one lags)? Hint: you can use either time domain or phasor domain; you need to integrate the current passing through a cylindrical surface.

displacement field $\vec{D} = \epsilon \vec{E}$
 Via Gauss's Law: Note: can also be solved using Poisson's Equation

$$\oint \vec{D} \cdot d\vec{s} = Q_{inner}$$

$$\epsilon \cdot E \cdot l \cdot 2\pi r = Q_{inner}$$

$$\rightarrow \vec{E} = \frac{Q_{inner}}{2\pi \epsilon l r} \hat{r} \quad (+1) \text{ E-field expression}$$

$$V(r) - V(b) = - \int_b^r \frac{Q_{inner}}{2\pi \epsilon l r} dr = - \frac{Q_{inner}}{2\pi \epsilon l} \ln(r/b) \quad (+1) \text{ V-expression}$$

$$V(r=a) = V_0 = - \frac{Q_{inner}}{2\pi \epsilon l} \ln(a/b), \text{ so}$$

$$Q_{inner} = - \frac{2\pi \epsilon l V_0}{\ln(a/b)} \cos(\omega t) \quad (+1) \text{ solve for } Q_{inner}$$

$$\begin{aligned}
 \vec{D} &= \epsilon \vec{E} = \frac{q}{2\pi\ell r} \frac{2\pi\ell V_0 \cos(\omega t)}{\ln(a/b)} = \frac{\epsilon V_0 \cos(\omega t)}{\ln(a/b)} \frac{1}{r} \hat{r} \\
 \cdot I_d &= \frac{\partial}{\partial t} \iint \vec{D} \cdot d\vec{s} = \frac{\partial}{\partial t} \left\{ \frac{\epsilon \tilde{V}_0}{\ln(a/b)} \int_0^l \int_0^{2\pi} \frac{1}{a} a d\phi dz \right\} \quad (+1) \text{ D-field} \\
 &= \frac{2\pi\ell\epsilon}{\ln(a/b)} \frac{\partial \tilde{V}_0}{\partial t} = j \frac{2\pi\omega\ell\epsilon V_0 \sin(\omega t)}{\ln(a/b)} \quad (+1) \\
 I_c &= \iint \vec{J} \cdot d\vec{s} = \sigma \int_0^l \int_0^{2\pi} \frac{V_0 \cos(\omega t)}{\ln(a/b)} \frac{1}{r} r d\phi dz \quad (+1) \text{ equation} \\
 &= \frac{2\pi\sigma\ell V_0 \cos(\omega t)}{\ln(a/b)} \quad (+1)
 \end{aligned}$$

- b) Evaluate I_d , I_c and the ratio of their amplitudes when $\epsilon_r = 27$, $\sigma = 2 \times 10^{-8} \text{ S/m}$, $a = 0.45 \text{ mm}$, $b = 1.57 \text{ mm}$, $l = 100 \text{ m}$, $V_0 = 10 \text{ V}$ and $f = 1 \text{ MHz}$.

$$\begin{aligned}
 |I_d| &= \frac{2\pi\omega\ell\epsilon V_0 \sin(\omega t)}{\ln(a/b)} = 7.54 \sin(2\pi \times 10^6 t) [\text{A}] \\
 &= 7.54 \cos(2\pi \times 10^6 t + \frac{\pi}{2}) [\text{A}] \quad (+1) \\
 |I_c| &= \frac{2\pi\sigma\ell V_0 \cos(\omega t)}{\ln(a/b)} = 0.0001 \cos(\omega t) \text{ A} \\
 &= 100 \cos(\omega t) \mu\text{A} \quad (+1) \\
 \frac{|I_d|}{|I_c|} &\approx \frac{7.54 \text{ A}}{100 \times 10^{-6} \text{ A}} = \frac{7.54 \times 10^4}{(+1)} \rightarrow \text{Displacement Current Dominates}
 \end{aligned}$$

- c) Using your answer from part b, determine at which frequency the amplitudes of I_d and I_c are equal.

$$\begin{aligned}
 \text{When does } I_d &= I_c? \\
 \frac{2\pi\omega\ell\epsilon V_0}{\ln(a/b)} &= \frac{2\pi\sigma\ell V_0}{\ln(a/b)} \rightarrow \omega\epsilon = \sigma \rightarrow f = \frac{\sigma}{2\pi\epsilon} \\
 f &= \frac{2 \times 10^{-8} \text{ S/m}}{2\pi \cdot 27 \cdot 8.85 \times 10^{-12} \text{ F/m}} = 13.3 \text{ Hz} \quad (+1)
 \end{aligned}$$

- d) Using your knowledge from circuit theory, calculate I_d and I_c , then compare them with your results from part a. *Hint*: you will need the expressions for capacitance and conductance per unit length of a coaxial cable to calculate these quantities.

$$\begin{aligned}
 I_d &= C \frac{dV}{dt} = \frac{2\pi\epsilon l}{\ln(b/a)} \cdot V_0 \frac{d}{dt}(\sin(\omega t)) \\
 &\quad (+1) \text{ equation} \\
 &= \frac{2\pi\omega\epsilon l V_0}{\ln(b/a)} \sin(\omega t) (+1)
 \end{aligned}$$

* This differs by a "-" sign due to the fact that we defined the direction of displacement current flow to go from low voltage to high voltage ($-\hat{r}$).

3. Plane Wave in a Lossy Medium [13 Points]

Today's computer microprocessors (and many other electronic and photonic devices) are built on a silicon (Si) substrate. For pure Si, $\epsilon_r \approx 12$. Assume the Si substrate resistivity is $\rho = 10 \Omega \cdot \text{cm}$.

- a) Determine the frequency range in which the Si substrate can be treated as a good insulator (dielectric), and the frequency range in which the Si substrate can be treated as a good conductor.

$$\rho = 10 \Omega \cdot \text{cm} \times \frac{1 \text{ m}}{100 \text{ cm}} = 0.1 \Omega \cdot \text{m}$$

for a good insulator: $\frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega \epsilon'} \ll 0.01$ (+1) condition

$$\text{so } f \gg \frac{\sigma}{2\pi \epsilon'} = \frac{1}{0.01 \cdot 2\pi \epsilon' \rho} \approx 1.5 \times 10^{12} \text{ Hz}$$

• good insulator when $f \gg 1.5 \text{ THz}$ | (+1)

• for a good conductor: $\frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega \epsilon'} \gg 100$ (+1) condition

$$\text{so } f \ll \frac{1}{100 \cdot 2\pi \cdot \epsilon' \cdot \rho} = 1.5 \times 10^5 \text{ Hz} \rightarrow f \ll 150 \text{ GHz}$$
 | (+1)

- b) If a plane wave is traveling in the Si substrate at 100 MHz, find the attenuation constant and phase constant (α and β), wavelength, and intrinsic impedance (η).

at $f = 100 \text{ MHz}$, Si acts like a good conductor, so: (+1) correct regime

$$\alpha = \sqrt{\pi f \mu \sigma} = 20\pi \text{ Np/m}$$
 | (+1)
$$\beta = \alpha = 20\pi \text{ rad/m}$$
 | (+1)
$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{20\pi} = 0.1 \text{ m}$$
 | (+1)
$$\eta = (1+j) \frac{\alpha}{\sigma} = \frac{20\pi}{10} \sqrt{2} e^{j\frac{\pi}{4}} = 2\sqrt{2}\pi e^{j\frac{\pi}{4}} \Omega$$
 | (+1)

- c) Using your results from b, if the wave is traveling in the y-direction with an \mathbf{E} -field amplitude of 10 V/m measured at $y = 0$, find \mathbf{E} and \mathbf{H} for the wave in the phasor domain. $\vec{E} = E\hat{x}$.

$$|\vec{E}| = 10 \text{ V/m}$$

$$\tilde{E} = 10 \text{ V/m} e^{-20\pi y} e^{-j20\pi y} \hat{x} \quad (+1) \text{ correct } \tilde{E}$$

$$\tilde{H} = \frac{\hat{k}}{\eta} \times \tilde{E} = (+\hat{y}) \times \frac{(10 e^{-20\pi y} e^{-j20\pi y} \hat{x})}{2\sqrt{2}\pi e^{j\pi/4}}$$

$$\tilde{H} = -\frac{5}{\sqrt{2}\pi} e^{-20\pi y} e^{-j(20\pi y - \pi/4)} \hat{z}$$

(+1) correct H magnitude

(+1) correct H direction

(+1) correct H phase

4. Wave Polarization [12 Points]

Determine the polarization of the following waves (i.e., linear, circular or elliptical) and their propagation direction. If a wave is linearly polarized, also determine the inclination angle. For non-linear polarization, determine the rotation direction (LH or RH). Draw a polarization diagram for each case with a few data points to show your work.

a) $\tilde{E}(z) = (3\hat{x} - j3\hat{y})e^{j25\pi z}$ [V/m]

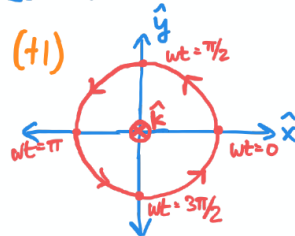
$$\tilde{E}(z) = (3\hat{x} - j3\hat{y})e^{j25\pi z} \text{ [V/m]}$$

· direction of propagation : $-z$ (+1)

· polarization:

$$\tilde{E}(z) = (3\hat{x} + 3\hat{y}e^{-j\frac{\pi}{2}})e^{j25\pi z}$$

→ E_x out of phase with E_y by $-\frac{\pi}{2}$,
so the wave is circularly polarized (+1)



$$\vec{E}(0,t) = 3\cos(\omega t)\hat{x} + 3\cos(\omega t - \frac{\pi}{2})\hat{y}$$

$$\omega t = 0: 3\hat{x} - 0\hat{y}$$

$$\omega t = \frac{\pi}{2}: 0\hat{x} + 3\hat{y}$$

$$\omega t = \pi: -3\hat{x} + 0\hat{y}$$

$$\omega t = \frac{3\pi}{2}: 0\hat{x} - 3\hat{y}$$

→ left-hand circular (+1)

b) $\vec{E}(z,t) = 2\cos(10^6\pi t - 0.5z + 45^\circ)\hat{x} + \sin(10^6\pi t - 0.5z - 45^\circ)\hat{y}$ [V/m]

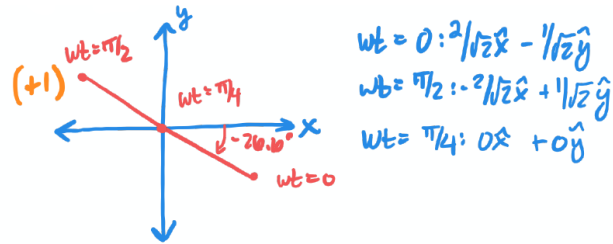
$$\vec{E}(z,t) = 2\cos(10^6\pi t - 0.5z + 45^\circ)\hat{x} + \sin(10^6\pi t - 0.5z - 45^\circ)\hat{y} \text{ [V/m]}$$

· direction of propagation: $+z$ (+1)

$$\text{polarization: } 2\cos(10^6\pi t - 0.5z + \frac{\pi}{4})\hat{x} - \cos(10^6\pi t - 0.5z + \frac{\pi}{4})\hat{y}$$

→ E_x is $\pm\pi$ out of phase with E_y , so
it is linearly polarized (+1)

→ Angle of inclination: $\tan^{-1}(1/-2) = -\underline{20.6^\circ}$ (+1)



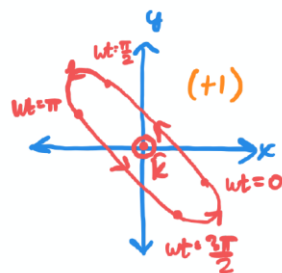
c) $\tilde{E}(z) = (3e^{j\frac{\pi}{6}}\hat{x} - 3e^{j\frac{\pi}{3}}\hat{y})e^{-j3\pi z}$ [V/m]

$$\tilde{E}(z) = (3e^{j\frac{\pi}{6}}\hat{x} - 3e^{j\frac{\pi}{3}}\hat{y})e^{-j3\pi z}$$

· direction of propagation: $+z$ (+1)

· polarization: E_x is $\frac{\pi}{6}$ out of phase with E_y
 \rightarrow elliptical polarization (+1)

$\omega t = 0: 3\cos(\pi/6)\hat{x} - 3\cos(\pi/3)\hat{y} = 2.6\hat{x} - 1.5\hat{y}$
 $\omega t = \pi/2: 3\cos(4\pi/6)\hat{x} - 3\cos(5\pi/6)\hat{y} = -1.5\hat{x} + 2.6\hat{y}$
 $\omega t = \pi: 3\cos(7\pi/6)\hat{x} - 3\cos(4\pi/3)\hat{y} = -2.6\hat{x} + 1.5\hat{y}$
 $\omega t = 3\pi/2: 3\cos(10\pi/6)\hat{x} - 3\cos(11\pi/6)\hat{y} = 1.5\hat{x} - 2.6\hat{y}$



· right-hand elliptical
 (+1)