ECSE 2500
Lec 3
Jan. 13th

Topic: Axioms of probability
Rules

Last lecture, we have introduced several key elements of random experiments such as sample space, events and event class.

Ex. Flip coin twice, record the results of two flips

Sample space $S := \{HH, HT, TH, TT\}$

Event class (all possible subsets of S)

$$F := \left\{ \begin{array}{llll} \phi, & \{HH\}, & \{HH, HT\}, & \{HH, HT, TH\}, & \{HH, HT, TH, TT\} \\ & \{HT\}, & \{HH, TH\}, & \{HH, HT, TT\} \\ & \{TH\} & \{HH, TT\}, & \{HH, TH, TT\} \\ & \{TT\} & \{HT, TH\}, & \{HT, TH, TT\} \\ & & \{HT, TT\} \\ & & \{TH, TT\} \end{array} \right\}$$

$$1 + 4 + 6 + 4 + 1 = 16$$
$$= 2^4$$

Now we want to assign a probality to each event A in the event class F, denoted $P(A)$.

$$\underbrace{P(\cdot)}_{\text{Function}} : \text{Event } A \rightarrow \text{Scalar value } [0,1]$$

Function: A mapping between input $\rightarrow$ output

In order to be self-consistent, the probability function $P(\cdot)$ must satisfy 3 Axioms / Rules:

Foundation of discussing probability

1. $P(A) \geq 0$ (No negative probability)
2. $P(S) = 1$ (There is a fixed "amount" or "mass" of probability $= 1$)
3. If $A \cap B = \phi$ then

$$P(A \cup B) = P(A) + P(B)$$

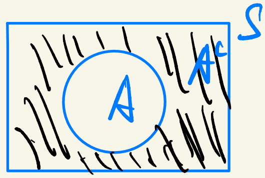
Remark: ③ If events are disjoint (no overlap in common), probability of union is the sum of individual events.

Techinically, we can generalize to 3 or more events. If $A_1, A_2 \cdots A_\infty$ is a set of events with $A_i \cap A_j = \phi, \forall i, j$, then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

→ From these basic principles, we can derive several other important properties of probability.

1. $P(A^c) = 1 - P(A)$

$\uparrow$
Complement of A



Since we have $P(A \cup A^c) = P(S) \overset{\text{Axiom 2}}{=} 1$

$\overset{\text{Axiom 3}}{=} P(A) + P(A^c)$

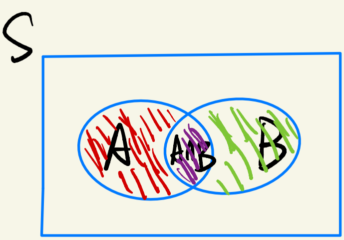
$\Rightarrow P(A^c) = 1 - P(A)$. Q.E.D.

2. $P(A) \leq 1$

since $P(A) = 1 - \underbrace{P(A^c)}_{\geq 0} \overset{\text{Axiom 1}}{\leq} 1$

3. $P(\phi) = 0$, since $P(\underset{A^c}{\phi}) = 1 - P(\underset{A}{S}) \overset{\text{Axiom 2}}{=} 0$

4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



Next page

$$A \cup B = \underbrace{(A-B)}_{\text{III}} \cup \underbrace{(A \cap B)}_{\text{III}} \cup \underbrace{(B-A)}_{\text{III}}$$

Disjoint sets

Since $P(A \cup B) \overset{\text{Axiom 3}}{=} P(A-B) + P(A \cap B) + P(B-A)$

$$= \underbrace{P(A-B) + P(A \cap B)}_{} + \underbrace{P(B-A) + P(A \cap B)}_{} - P(A \cap B)$$

$$\overset{\text{Axiom 3}}{=} P(A) + P(B) - P(A \cap B). \quad \text{Q.E.D.}$$

5. If $A \subset B$, $P(A) \leq P(B)$

Since $P(B) \overset{\text{Axiom 3}}{=} \underbrace{P(A) + P(B-A)}_{\text{Disjoint sets}}$

$$\overset{\text{Axiom 1}}{\geq} P(A)$$

☐ Discrete, Finite Sample Spaces

If we have a finite number of outcomes, then any event is just a subset of outcomes, and the probabilities is the sum of the individual probabilities.

$\boxed{\text{Exp.}}$ Roll a 6-sided die and observe the number

$$S := \{1, 2, 3, 4, 5, 6\}$$

Event $A := \{\text{Number is even}\} = \{2, 4, 6\}$

$$P(\text{Number is even}) = P(\{2,4,6\})$$

$$= P(\{2\} \cup \{4\} \cup \{6\})$$

$$\overset{\text{Axiom 3}}{=} P(\{2\}) + P(\{4\}) + P(\{6\})$$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

This also works for discrete sample space with an infinite number of outcomes.

$\boxed{\text{Exp.}}$ Roll a fair 6-sided die. until the current top number matches the previous top number, and record the number of rolls as X.

In this experiment, $S := \{2, 3, 4, \ldots\}$

However, we can still compute the probability of rolling a certain number of times.

$$P(X=1) = 0$$

$$P(X=2) = \frac{1}{6}$$ How ?

$$\frac{\text{6 outcomes that have same number twice}}{\text{36 total outcomes}}$$

$P(X=3)$ is a little more complicated

We need Roll 3 = Roll 2, but also

$$\underline{\text{Roll } 2 \neq \text{Roll } 1}$$

$$P(X=3) = \left(1 - \frac{\text{6 outcomes}}{\text{36 outcomes}}\right) \times \left(\frac{\text{6 outcomes}}{\text{36 outcomes}}\right)$$

$$= \frac{5}{6} \times \frac{1}{6} = \frac{5}{36} \approx 0.13$$

In general, we can show that

$$P(X=n) = P(\text{roll } 2 \neq \text{roll } 1)$$
$$\cdot P(\text{roll } 3 \neq \text{roll } 2)$$
$$\cdot P(\text{roll } 4 \neq \text{roll } 3)$$
$$\cdots$$
$$\cdot P(\text{roll } n = \text{roll } n-1)$$

$$= \left(\frac{5}{6}\right)^{n-2} \left(\frac{1}{6}\right)$$ holds $n=2,3$ as well