ECSE 2500
Feb. 9th
Lec 9

2/13 Monday	2/16 Thursday	2/21 Monday → Tuesday
Review of problems related HW1-3/Exam	Cancelled	Exam 1
TA Guo		One-page sheet

Today's topic: Expected value and moments of RV.

Last time, we talked about discrete random variables; we will perform an expectation of that random variable.

Q: What is the key component of an RV?

A: Probability Mass Function (PMF): tells us the probability of a random variable taking a particular value.

Need a table to store PMF

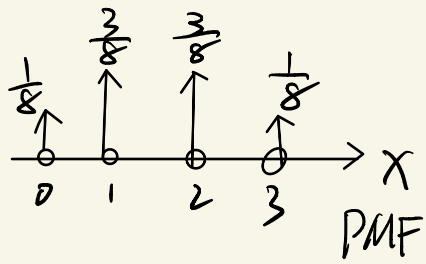
s	$X(s)$	$P(X = X(s))$

○ The Expected Value of a (discrete) random variable (aka. Expectation or Mean) is defined as the sum of all possible values the RV X can take weighted by their probabilities:

$$\boxed{E[X] = \sum_{x_k \in S_X} x_k \, \underbrace{P_X(x_k)}_{P(X = x_k)}}$$

Example: $X = \#$ of heads in 3 flips of fair coin

$P_X(x_k) =$ Probability $X = x_k$



$E[X]$

$= 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8}$

$= \frac{3}{2} = 1.5$

Makes sense intuitively since PMF is symmetric around this value 1.5.

Note: $E[X]$ is not the same as the most likely value X takes; And most of the times, it is not actually an outcome of this experiment (e.g., we cannot get 1.5 heads)

In fact, the intuition is that $E[X]$ is the limit of sample mean: If do a large number of Experiments and record outcomes, we have

$$\text{Sample mean} = \sum_{X_k=0}^{3} \frac{\#\text{ of trials that we see } X_k \text{ heads}}{\text{Total }\#\text{ of trial}} X_k$$

As # of trials $\to \infty$, Sample mean $\to E[X]$.

$\boxed{\text{Example}}$ 1) X is a Bernoulli RV, $P_X(0) = 1-P$
$P_X(1) = P$.

$$E[X] = 0 \times (1-P) + 1 \times P = P$$

e.g., Say I get one dollar every time a trial succeeds, My expected payoff is $P \cdot$ one dollar.

2) X is a Geometric RV.

$$P_X(k) = (1-p)^{k-1} \cdot p \quad , k = 1, 2, \cdots \infty$$

$$E[X] = \sum_{k=1}^{\infty} k \, P_X(k)$$

$$= p \sum_{k=1}^{\infty} k (1-p)^{k-1}$$

$$= p \cdot \left(\frac{1}{1-(1-p)} \right)^2 = \frac{1}{p}$$

How to simplify?

$$\sum_{k=1}^{\infty} k a^{k-1} \qquad a \in (0,1)$$

$$= 1 + 2a + 3a^2 + 4a^3 + \cdots$$

$$= 1 + a + a^2 + a^3 + \cdots \qquad \frac{1}{1-a}$$

$$a + a^2 + a^3 + \cdots \qquad \frac{a}{1-a}$$

$$+ a^2 + a^3 + \cdots \qquad \frac{a^2}{1-a}$$

$$= (1 + a + a^2 + \cdots) \frac{1}{1-a} = \frac{1}{(1-a)^2}$$

3) X is Poisson RV with parameter α

$$P_X(k) = \frac{\alpha^k}{k!} e^{-\alpha}$$

$$E[X] = \sum_{k=0}^{\infty} k \cdot \frac{\alpha^k}{k!} e^{-\alpha}$$

$$= e^{-\alpha} \sum_{k=0}^{\infty} \frac{\alpha^k}{(k-1)!}$$

$$= e^{-\alpha} \cdot \alpha \sum_{k=0}^{\infty} \frac{\alpha^{k-1}}{(k-1)!}$$

$$= \alpha \cdot e^{-\alpha} \cdot e^{\alpha}$$

$$= \alpha$$

Taylor series for e^{α}

$e^{\alpha} = 1 + \alpha + \frac{1}{2!}\alpha^2 + \frac{1}{3!}\alpha^3 + \cdots$

Which means that the expected value of a Poisson RV is the parameter α.

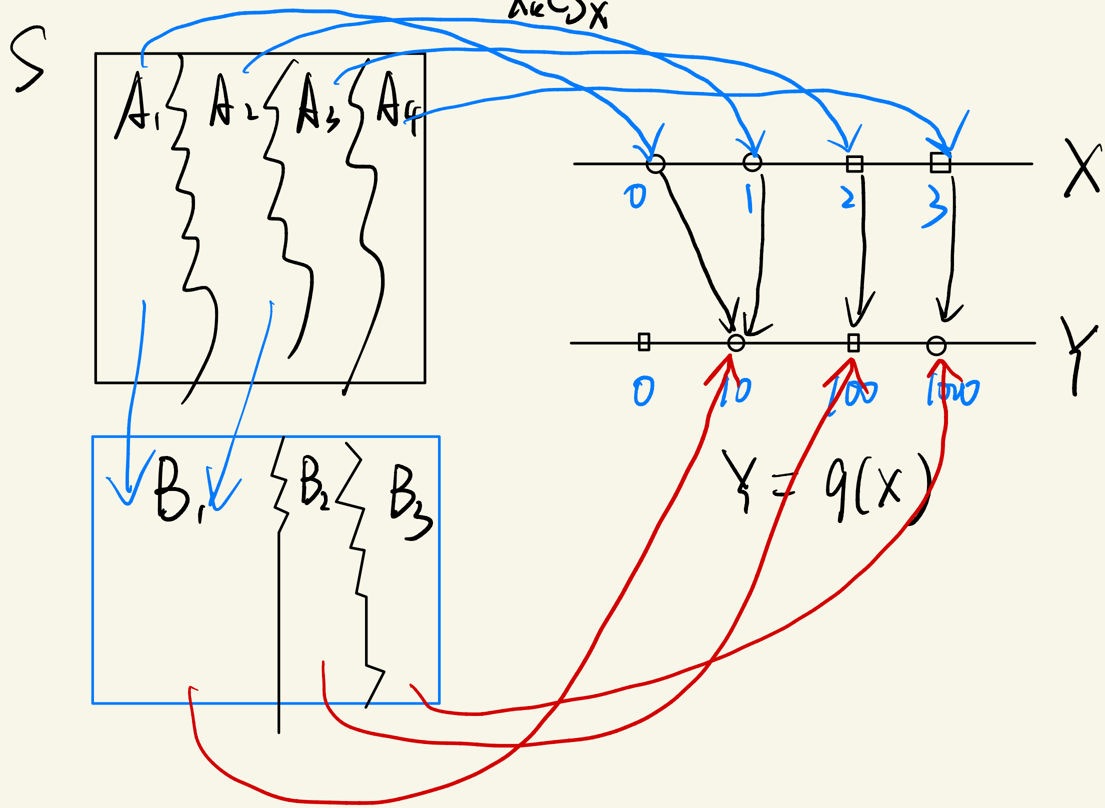
□ Expected Value of A function of RV X

$$E[\,g(X)\,].$$

$R \to R$ $S \to R$

Definition:

$$E[g(X)] = \sum_{x_k \in S_X} g(x_k) \cdot P_X[x_k]$$



Based on the alternative view of partition sample space,

$$E[g(X)] = E[Y]$$

$$= \sum_{y_k \in S_Y} y_k P_Y(y_k)$$

$$= \sum_{y_k \in S_Y} y_k \left(\sum_{x_j \in S_X \mid g(x_j) = y_k} P_X(x_j) \right)$$

$$= \sum_{x_j \in S_X} g(x_j) P_X(x_j) = E[g(X)]$$

Some important properties

$$E[g(X) + h(X)] = E[g(X)] + E[h(X)]$$

$$E[aX] = aE[X]$$

$$E[X + c] = E[X] + c$$

$$E[c] = c$$

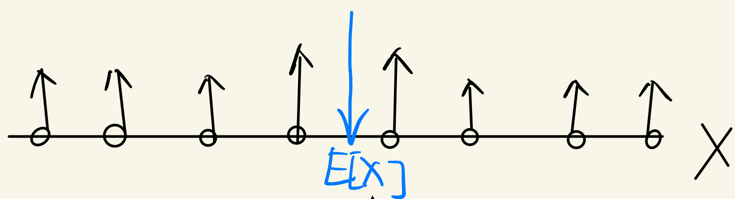
□ Variance — A special case of Expected value of a function of random variable

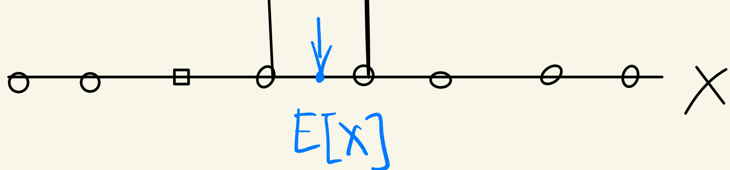
Define a function

$$g(X) := (X - E[X])^2$$

e.g. How far do we expect X to be from $E[X]$?

Give us some sense of the spread of PMF.





Variance

$$Var(X) = E[g(X)] = E[(X - E[X])^2]$$

$$= E[X^2 - 2XE[X] + E^2[X]]$$

$$= E[X^2] - 2E[X]E[X] + (E[X])^2$$

$$= E[X^2] - (E[X])^2 \leftarrow \text{Important}$$

We often use $Var(X) = E[(X - E[X])^2]$

and $\sigma_X = \sqrt{Var(X)}$, a.k.a. standard deviation

Bigger $\sigma_X \Rightarrow$ Values of RV widely spread around the mean of RV.

$\boxed{\text{Moment of } X}$

$$E[X^k] \leftarrow k\text{th moment}$$

$\boxed{\text{Central moment of } X}$

$$E[(X - E[X])^k]$$

$\leftarrow$ kth central moment

Example: 1) X is Binomial RV, what is $E[X]$?

$$P_X(k) = C_k^n p^k (1-p)^{n-k}, \quad k = 0, 1, 2 \cdots n$$

$$E[X] = \sum_{k=0}^{n} k \, C_k^n p^k (1-p)^{n-k}$$ Looks a pain

We can view Binomial RV X

$$\underbrace{X = X_1 + X_2 + \cdots + X_n}_{n \text{ independent Bernoulli RV}}$$

$$E[X] = \sum_{k=1}^{n} E[X_k] = np$$

2) Variance of a Bernoulli RV with parameter p.

$$E[X] = p.$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$= 0^2 \cdot (1-p) + 1^2 \cdot p - p^2 = p - p^2$$

$\boxed{\text{Import property}}$ If X and Y are two independent RVs, their variance

$$Var(X \pm Y) = Var(X) + Var(Y)$$