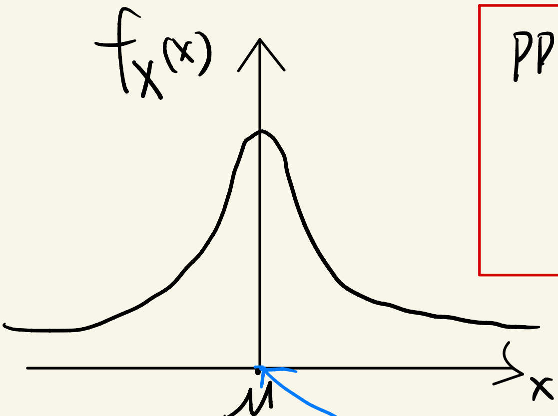
ECSE 4500
Lec 13
March 13

Topic: $\boxed{\text{Gaussian}}$ Random Variable

① What is Gaussian RV?
② What is its PDF/CDF?
③ How to use PDF/CDF compute some probabilities related Gaussian RV?

☐ Gaussian RV: Probably the most important random variable is Gaussian, e.g.,



PPF of Gaussian

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

σ: standard deviation
σ^2: variance
μ: mean of Gaussian

Q: Why this Gaussian is an important RV?

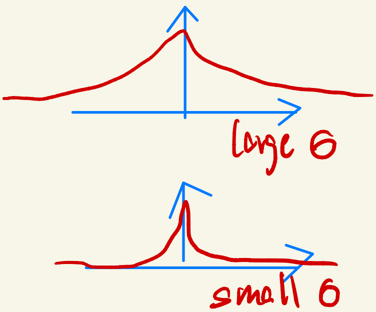
① In many applications, we see Gaussian RV can approximate the outcome of random experiements

- Test (Medical)
- Stock market indices
- Noise in Communication systems

② Central limit theory

Regarding Gaussian RV,

$\boxed{\text{Intuition:}}$ High probability that X is near mean μ, and falls symmetrically as we get further from mean μ, controlled by standard deviation σ.



Q: Whether Gaussian PDF is a valid PDF function ?

1) Non-negative ✓

2) Integrate into 1 ? ✓

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

Change of variable

$x \Leftrightarrow x-\mu$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}\,dx$$

Change of variable

$x' = \frac{x}{\sigma}$, $dx' = \frac{1}{\sigma}dx$

$$= \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx}_{\text{Define as } k}$$

If we can calculate $k^2 = a$

we can also obtain k by $\sqrt{a}$.

Consider

$$k^2 = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} dx\, dy$$

Change of variable

$r^2 = x^2 + y^2$

$r\,dr\,d\theta = dx\,dy$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-\frac{r^2}{2}} r\,dr\,d\theta$$

$$= \int_0^{\infty} r e^{-\frac{r^2}{2}} dr$$

$$= -e^{-\frac{r^2}{2}} \Big|_{r=0}^{r=\infty} = 0 - (-1) = 1$$

$$\Rightarrow \quad k = \sqrt{k^2} = 1$$

We have verified that Gaussian PDF we introduced at the beginning is a valid PDF!

Preview: Central limit theorem

If $X_1, X_2, \cdots, X_N$ are N independent, identically distributed random variables (iid), no matter what the distribution is, their "average" as $N \to \infty$ turns out to be a Gaussian distribution.

$\square$ What is the CDF of Gaussian RV?

big   small

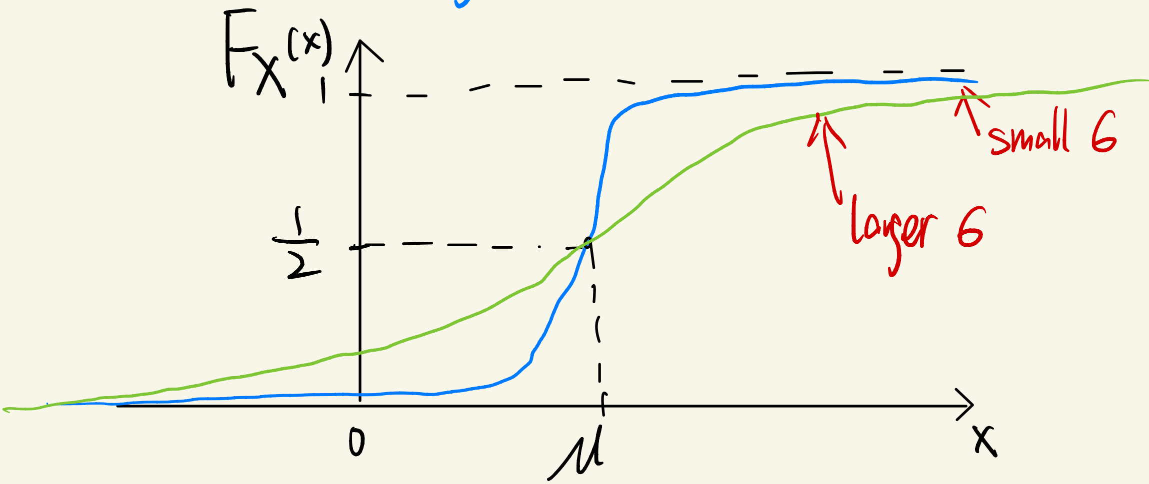
$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(y)\, dy$$

CDF of continuous

$$\overset{\text{plug pdf}}{=} \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

Unfortunately, we cannot compute the integral in closed form. We have to compute it using some numerical integration methods.

But intuitively, it should look like this



By using numerical integration, people have already stored the values of Gaussian CDF into a printed table in terms of

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$$

→ CDF of Gaussian RV with mean 0 and variance 1.

True CDF for Gaussian with mean μ and variance σ^2

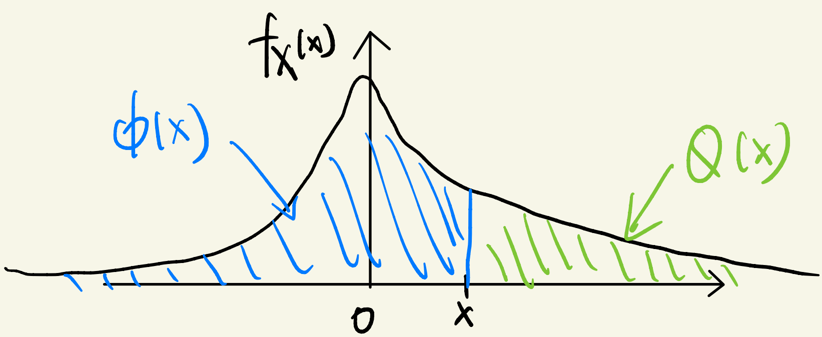
$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

The book reports the numerical values $Q(x)$,

$$Q(x) = 1 - \phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{y^2}{2}} dy$$

TABLE 4.2 Comparison of $Q(x)$ and approximation given by Eq. (4.54).

x	$Q(x)$	Approximation	x	$Q(x)$	Approximation
0	5.00E-01	5.00E-01	2.7	3.47E-03	3.46E-03
0.1	4.60E-01	4.58E-01	2.8	2.56E-03	2.55E-03
0.2	4.21E-01	4.17E-01	2.9	1.87E-03	1.86E-03
0.3	3.82E-01	3.78E-01	3.0	1.35E-03	1.35E-03
0.4	3.45E-01	3.41E-01	3.1	9.68E-04	9.66E-04
0.5	3.09E-01	3.05E-01	3.2	6.87E-04	6.86E-04
0.6	2.74E-01	2.71E-01	3.3	4.83E-04	4.83E-04
0.7	2.42E-01	2.39E-01	3.4	3.37E-04	3.36E-04
0.8	2.12E-01	2.09E-01	3.5	2.33E-04	2.32E-04
0.9	1.84E-01	1.82E-01	3.6	1.59E-04	1.59E-04
1.0	1.59E-01	1.57E-01	3.7	1.08E-04	1.08E-04
1.1	1.36E-01	1.34E-01	3.8	7.24E-05	7.23E-05
1.2	1.15E-01	1.14E-01	3.9	4.81E-05	4.81E-05
1.3	9.68E-02	9.60E-02	4.0	3.17E-05	3.16E-05
1.4	8.08E-02	8.01E-02	4.5	3.40E-06	3.40E-06
1.5	6.68E-02	6.63E-02	5.0	2.87E-07	2.87E-07
1.6	5.48E-02	5.44E-02	5.5	1.90E-08	1.90E-08
1.7	4.46E-02	4.43E-02	6.0	9.87E-10	9.86E-10
1.8	3.59E-02	3.57E-02	6.5	4.02E-11	4.02E-11
1.9	2.87E-02	2.86E-02	7.0	1.28E-12	1.28E-12
2.0	2.28E-02	2.26E-02	7.5	3.19E-14	3.19E-14
2.1	1.79E-02	1.78E-02	8.0	6.22E-16	6.22E-16
2.2	1.39E-02	1.39E-02	8.5	9.48E-18	9.48E-18
2.3	1.07E-02	1.07E-02	9.0	1.13E-19	1.13E-19
2.4	8.20E-03	8.17E-03	9.5	1.05E-21	1.05E-21
2.5	6.21E-03	6.19E-03	10.0	7.62E-24	7.62E-24
2.6	4.66E-03	4.65E-03			



$Q(x)$ is related to Matlab erf function

(But not exactly the same, e.g.,

$$Q(x) = \frac{1}{2}\left(1 - \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right)$$)

□ How to use these stored CDF values?

(Gaussian mean 0, variance 1)

if X is a Gaussian RV with mean μ and variance σ^2.

$$F_X(x) = P(X \le x) = P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right)$$

mean μ, var σ^2 mean 0, Var 1

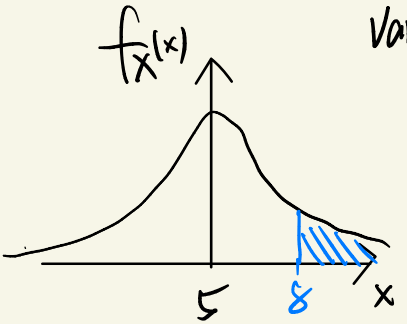
$$= \Phi\left(\frac{x-\mu}{\sigma}\right) = 1 - Q\left(\frac{x-\mu}{\sigma}\right)$$

If we are interested in $F_X(x)$,

we check $Q(x)$ at $\frac{x-\mu}{\sigma}$, then

we calculate $1 - Q\left(\frac{x-\mu}{\sigma}\right)$.

Example. X is Gaussian with mean 5 and variance 4. What is $P(X>8)$?

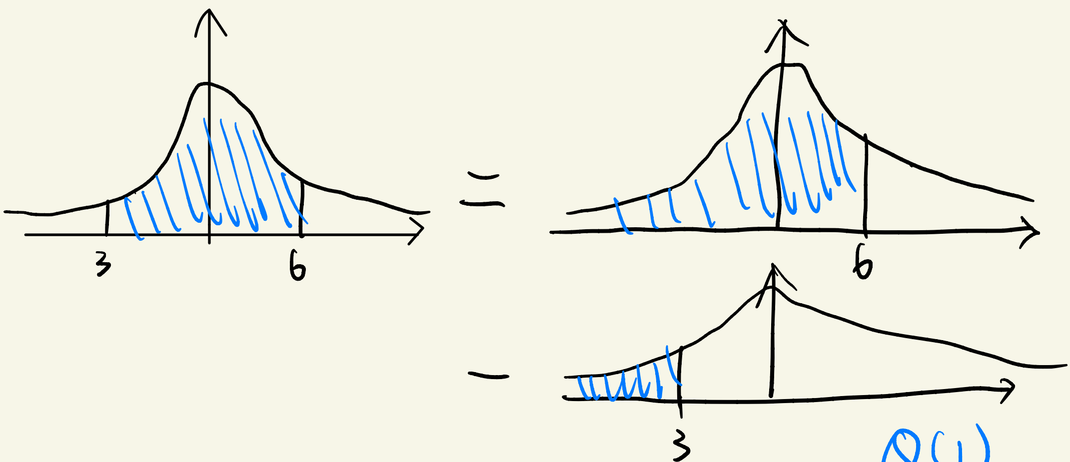


$$P(X>8)$$
$$= 1 - P(X \le 8)$$
$$= 1 - \left(1 - Q\left(\frac{8-\mu}{\sigma}\right)\right)$$

$\mu=5,\ \sigma=2$

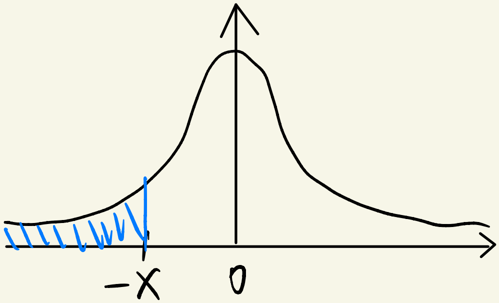
$$= Q\left(\frac{8-5}{2}\right) = Q(1.5)$$
$$= 6.68 \times 10^{-2} = 0.0668$$

What is $P(3 < X < 6)$?

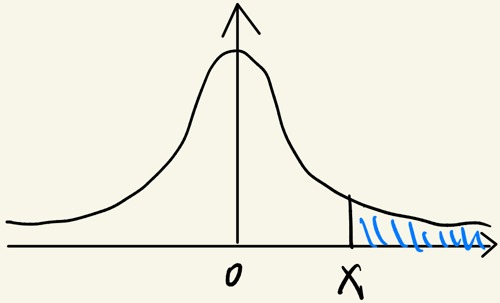


$$= \Phi\left(\frac{6-5}{2}\right) - \Phi\left(\frac{3-5}{2}\right) = 1 - Q\left(\tfrac{1}{2}\right) - \overbrace{\left(1 - Q(-1)\right)}^{Q(1)} =$$

Check table



$\phi(-x)$



$Q(x) = 1 - \phi(x)$