MATH-2400 INTRODUCTION TO DIFFERENTIAL EQUATIONS FALL 2022

Problem Set 6

Due: 11pm, Tuesday, October 25, 2022

Submitted to LMS By:
Joseph Hutchinson
662022852
Section 17

NOTES

1. Practice problems listed below and taken from the textbook are for your own practice, and are not to be turned in.
2. There are two parts of the Problem Set, an objective part consisting of multiple choice questions (with no partial credit available) and a subjective part (with partial credit possible). Please complete all questions.
3. Writing your solutions in LaTeX is preferred but not required.
4. Show all work for problems in the subjective part. Illegible or undecipherable solutions will not be graded.
5. Figures, if any, should be neatly drawn by hand, properly labelled and captioned.
6. Your completed work is to be submitted electronically to LMS as a single pdf file. Be sure that the pages are properly oriented and well lighted. (Please do not e-mail your work to Muhammad or me.)

Practice Problems from the textbook (Not to be turned in)

- Exercises from Chapter 3, pages 72–75: 1(c,d), 2, 4, 7, 8, 12, 13.

Objective part (Choose A, B, C or D; no work need be shown, no partial credit available)

1. (5 points) The displacement $u(t)$ of a mass-spring-damper system is governed by $mu'' + cu' + ku = 0$, where $m = 2$ and $k = 8$. For what value of the damping coefficient c is the system **critically damped**?

   So, where $\lambda = \frac{-c}{2m}$ and $\omega_0^2 = \frac{k}{m}$, then $\lambda^2 - \omega_0^2 = 0$ to be critically damped.
   $\lambda = \frac{-c}{2(2)} = \frac{-c}{4}$
   and $\omega_0^2 = \frac{8}{2} = 4$
   So, $(\frac{-c}{4})^2 - (4)^2 = 0$
   $\frac{c^2}{16} - 16 = 0$
   $c^2 = 16^2$
   $\boldsymbol{c = 16}$

   A $c = 4$

   B $c = 8$

   [C] $\boldsymbol{c = 16}$

   D None of these choices

2. (5 points) The displacement $u(t)$ of a forced mass-spring-damper system is governed by the linear DE $mu'' + cu' + ku = 5\cos(2t)$. For what values of the mass m, damping coefficient c and spring constant k is the system **in resonance**?

For this system with forced oscillation, it can't have resonance when the system is damped with a value of $c > 0$. In order to have resonance, c must $= 0$. So, options B and C are improbable.

For the forcing function $5\cos(2t)$, $\omega = 2$.

Looking at option A:
For the homogeneous solution $u_h(t)$, $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{2}$
Since $\omega \neq \omega_0$, there will be no resonance for A.
Thus, none of the choices A, B, or C could bring the system to resonance.

A $m = 1$, $c = 0$, $k = 2$

B $m = 2$, $c = 1$, $k = 4$

C $m = 2$, $c = 1$, $k = 8$

[D] None of these choices

3. (5 points) The displacement $u(t)$ of a forced mass-spring system is governed by $u''+2u'+3u = 4\cos(t)$. The amplitude R of the forced response is given by:

For this case where the system has a forcing function and there is damping ($c > 0$), the amplitude R of the forced response is:
$R = \frac{F_0}{\sqrt{D}}$ where $\boldsymbol{F_0 = 4}$.

The value of $D = c^2\omega^2 + (k - m\omega^2)^2$
$D = (2)^2(1)^2 + (3 - (1)(1)^2)^2$
$D = 4 + 4$
$\boldsymbol{D = 8}$

So, plug in $D = 8$ and $F_0 = 4$ to the equation for R:
$R = \frac{4}{\sqrt{8}} = \frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$
$\boldsymbol{R = \sqrt{2}}$

A $R = 1$

[B] $\boldsymbol{R = \sqrt{2}}$

C $R = 2$

D None of these choices

Subjective part (Show work, partial credit available)

1. (15 points) A mass weighing 8 lb stretches a spring 4 in. Assume the mass is pulled **downward**, stretching the spring a distance of 6 in, and then set in motion with an upward velocity of 3 ft/s. There is **no damping** in the system and the acceleration due to gravity is $g = 32$ ft/s^2.

   (a) Determine an initial-value problem for the downward displacement $u(t)$ in **units of ft**.

   No damping in the system, so $\boldsymbol{c = 0}\ \mathbf{\frac{N{\cdot}s}{ft}}$

   Use Newton's Form of the equation, with constants m and k (c has been proven uneccessary for this problem):
   $mu'' + ku = 0$

   The spring constant $k = \frac{\text{weight}}{\text{length displaced}} = \frac{8 \text{ lbs}}{4/12 \text{ feet}}$
   $\boldsymbol{k = 24 \frac{\textbf{lbs}}{\textbf{ft}}}$

   The value of m is the mass of the object. We know its weight $= 8$ lbs under earth's gravitational acceleration $g = 32$ ft/s^2. So:
   $m = \frac{\text{weight (mg)}}{g} = \frac{8 \text{ lbs}}{32 \text{ ft/s}^2} = \frac{1}{4}$ "slugs" of mass
   $\boldsymbol{m = \frac{1}{4}}$ **slugs**

   Plug in constants m and k to arrive at our equation to solve:
   $(\frac{1}{4})u'' + (24)u = 0$

   At $t = 0$ seconds, the mass is displaced 6 inches $= 0.5$ feet downwards, and is released with an upward starting velocity of 3 ft/s. We define downwards as the positive direction. So, the initial conditions are:
   $\boldsymbol{u(0) = u_0 = 0.5}$ **ft**
   $\boldsymbol{u'(0) = v_0 = -3}$ **ft/s**

   (b) Solve the IVP and express the solution in the polar form $u(t) = R\cos(\omega_0 t - \phi)$.

   $\frac{1}{4}u'' + 24u = 0$
   Let $u = e^{rt}$, and simplify to get:
   $\frac{1}{4}r^2 + 24 = 0$
   $r^2 = -24 * 4$
   $r = \pm\sqrt{-96} = \pm\sqrt{-1 * 16 * 6}$
   $r = \pm 4\sqrt{6}\, i$

   The value of $\boldsymbol{\omega_0 = 4\sqrt{6}}$. These two complex roots lead to a solution of the form:
   $u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$
   $u(t) = C_1 \cos(4\sqrt{6}\, t) + C_2 \sin(4\sqrt{6}\, t)$

   Plug in the initial condition $u(0) = 0.5$ft:
   $0.5 = C_1 \cos(0) + C_2 \sin(0)$
   $0.5 = C_1(1)$
   $\boldsymbol{C_1 = 0.5}$

   Find $u'(t)$ in order to apply the second initial condition.
   $u(t) = C_1 \cos(4\sqrt{6}\, t) + C_2 \sin(4\sqrt{6}\, t)$
   $u'(t) = -4\sqrt{6} C_1 \sin(4\sqrt{6}\, t) + 4\sqrt{6} C_2 \cos(4\sqrt{6}\, t)$

Apply the IC $u'(0) = -3$ ft/s :
$-3 = -4\sqrt{6}C_1 \sin(0) + 4\sqrt{6}C_2 \cos(0)$
$-3 = -4\sqrt{6}(0.5)(0) + 4\sqrt{6}C_2(1)$
$4\sqrt{6}C_2 = -3$
$C_2 = -\frac{3}{4\sqrt{6}}$
$\boldsymbol{C_2 = -\frac{\sqrt{6}}{8}}$

Plug in constants $\boldsymbol{C_1 = 0.5}$ and $\boldsymbol{C_2 = -\frac{\sqrt{6}}{8}}$ into the solution:
$u(t) = 0.5\cos(4\sqrt{6}\,t) - \frac{\sqrt{6}}{8}\sin(4\sqrt{6}\,t)$

This solution can be split into its horizontal and vertical components, where its maximum polar amplitude is "R". R is the hypotenuse of a triangle with legs length C_1 and C_2. By the pythagorean theorem:
$R = \sqrt{{C_1}^2 + {C_2}^2}$
$R = \sqrt{(0.5)^2 + (-\frac{\sqrt{6}}{8})^2}$
$R = \sqrt{\frac{1}{4} + \frac{3}{32}} = \sqrt{\frac{11}{32}}$
$\boldsymbol{R = \sqrt{\frac{11}{32}}}$

The polar form of the solution will look like:
$u(t) = R\cos(\omega_0 t - \phi)$
Where ϕ is the base angle of the triangle formed by C_1, C_2, and R. Solve for it via a trig relationship:
$\phi = \tan^{-1}(\frac{C_2}{C_1})$
$\phi = \tan^{-1}(\frac{-\sqrt{6}/8}{1/2}) = \tan^{-1}(-\sqrt{6}/4)$
$\boldsymbol{\phi = \tan^{-1}(-\sqrt{6}/4)}$

Thus, by plugging in all values, the polar form of the solution $\boldsymbol{u(t)}$ is:
$\boldsymbol{u(t) = \sqrt{\frac{11}{32}}\cos(\ 4\sqrt{6}t - \tan^{-1}(-\sqrt{6}/4)\ )}$

(c) Determine the frequency, period and amplitude of the oscillation. **Sketch the solution**.

The amplitude of the oscillation is simply the value of R:
Amplitude $\boldsymbol{= \sqrt{\frac{11}{32}}}$ **feet**

The period of the oscillation is T, where $T = \frac{2\pi}{\omega_0}$.
$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{4\sqrt{6}} = \frac{\pi}{2\sqrt{6}} = \pi\frac{\sqrt{6}}{12}$
$\boldsymbol{T = \pi\frac{\sqrt{6}}{12}}$ **seconds**

The frequency of the oscillation is f, where $f = \frac{1}{T}$.
$f = 1/T = 1/(\pi\frac{\sqrt{6}}{12})$
$\boldsymbol{f = \frac{12}{\pi\sqrt{6}}}$ **Hz**

Sketch of the solution:

2. (15 points) A force of 4 N stretches a spring 10 cm. A mass of 2 kg is hung from the spring, and the mass is also **attached to a viscous damper** that exerts a force of 16 N when the velocity of the mass is 2 m/s. The mass is set into motion **from its equilibrium position** by an initial downward velocity of 20 cm/s.

(a) Determine an initial-value problem for the **upward** displacement $u(t)$ in units of meters.

The coefficient of damping, c, is equal to the ratio of Force it applies at some Velocity. Here, the damper exerts $F = 16 \text{ N}$ when $v = 2 \text{ m/s}$. So:
$c = \frac{F}{v} = \frac{16 \text{ N}}{2 \text{ m/s}} = 8$
$\boldsymbol{c = 8 \ \frac{\textbf{N}\cdot\textbf{s}}{\textbf{m}}}$

The spring constant $k = \frac{\text{force}}{\text{length displaced}} = \frac{4 \text{ Newtons}}{0.1 \text{ meters}}$
$\boldsymbol{k = 40 \ \frac{\textbf{N}}{\textbf{m}}}$

The value of m is the mass of the object. We know its mass is 2 kilograms:
$\boldsymbol{m = 2 \ \ \textbf{kg}}$

At $t = 0$ seconds, the mass is displaced 0 meters from its equilibrium position. It is released with a downward starting velocity of 0.2 m/s. We define upwards as the positive direction.
So, the initial conditions are:
$\boldsymbol{u(0) = u_0 = 0 \ \textbf{m}}$
$\boldsymbol{u'(0) = v_0 = -0.2 \ \textbf{m/s}}$

Plug these values into the general equation for a damped oscillating spring system, to get:
$mu'' + cu' + ku = 0$
$\boldsymbol{2u'' + 8u' + 40u = 0}$

(b) Solve the IVP and sketch the solution.

$2u'' + 8u' + 40u = 0$, with ICs: $u(0) = 0 \text{ m}$, and $u'(0) = -0.2 \text{ m/s}$

Let $u = e^{rt}$, and simplify to get:
$2r^2 + 8r + 40 = 0$

With $a = 2$, $b = 8$, and $c = 40$, use the quadratic formula to find the roots, r:
$r = \frac{-b \pm \sqrt{(b)^2 - (4ac)}}{2a} = \frac{-8 \pm \sqrt{64 - (320)}}{4}$
$r = -2 \pm \frac{\sqrt{-256}}{4} = -2 \pm \frac{\sqrt{-1 * (16)^2}}{4}$
$\boldsymbol{r = -2 \pm 4i}$

So $\lambda = -2$, and $\omega = 4$. The roots are of complex form, so this system is **underdamped**. The "rectangular" form of the solution will be:
$u(t) = e^{\lambda t}(\ C_1 \cos(\omega t) + C_2 \sin(\omega t)\ )$
$u(t) = e^{-2t}(\ C_1 \cos(4t) + C_2 \sin(4t)\ )$

Plug in the IC $u(0) = 0 \text{ m}$:
$0 = e^0(\ C_1 \cos(0) + C_2 \sin(0)\ )$
$\boldsymbol{C_1 = 0}$

Next, find $u'(t)$ in order to apply the second IC:
$u(t) = e^{-2t}(\ (0)\cos(4t) + C_2 \sin(4t)\ )$
$u(t) = C_2\, e^{-2t} \sin(4t)$
$u'(t) = -2C_2\, e^{-2t} \sin(4t) + 4C_2\, e^{-2t} \cos(4t)$

Apply the IC $u'(0) = -0.2$:
$-0.2 = -2C_2\, e^0 \sin(0) + 4C_2\, e^0 \cos(0)$
$-0.2 = 4C_2$
$C_2 = \frac{-0.2}{4}$
$\boldsymbol{C_2 = -0.05}$

Plug in $\boldsymbol{C_1 = 0}$ and $\boldsymbol{C_2 = -0.05}$ to find the solution of the IVP:
$\boldsymbol{u(t) = -0.05e^{-2t} \sin(4t)}$

Since $C_1 = 0$, the "triangle" formed by C_1 and C_2 is merely a straight line on the C_2 axis. So, $\phi = 0$ and the hypotenuse $R = \sqrt{{C_2}^2 + 0^2} = C_2 = -0.05$. So, this solution is already in polar form!

The pseudo-period of the solution's graph will be $T = \frac{2\pi}{\omega}$, where $\omega = 4$:
$T = \frac{2\pi}{4}$
Pseudo-period $\boldsymbol{T = \frac{\pi}{2}}$

Sketch of the solution's behavior:

3. (15 points) The displacement $u(t)$ of a forced mass-spring-damper system satisfies the DE

$$u'' + 2u' + 3u = \cos(\omega t)$$

(a) The forced response of the system has the form $u_p(t) = A\cos(\omega t) + B\sin(\omega t)$. Determine formulas for A and B. (Note: your formulas will involve the frequency ω of the forcing.)

From the provided DE, $\boldsymbol{m = 1}$, $\boldsymbol{c = 2}$, and $\boldsymbol{k = 3}$.
The natural frequency of the system ω_0 is given by: $\omega_0 = \sqrt{\frac{k}{m}}$
$\boldsymbol{\omega_0 = \sqrt{3}}$

Since the system has damping, there *cannot be resonance* between the natural and forced responses. So, no need to determine $\boldsymbol{u_h(t)}$ in order to check for resonance.

Using the *Method of Undetermined Coefficients*, we'll solve for $u_p(t)$. "Guess" that:
$u_p(t) = A\cos(\omega t) + B\sin(\omega t)$
Find ${u_p}'(t)$:
${u_p}'(t) = -\omega A\sin(\omega t) + \omega B\cos(\omega t)$
Find ${u_p}''(t)$:
${u_p}''(t) = -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t)$

Let $s = sin(\omega t)$ and $c = cos(\omega t)$.
Now, plug each of these into the original DE, in order to find $L[u_p]$:
$L[u_p] = u'' + 2u' + 3u$
$L[u_p] = (-\omega^2 Ac - \omega^2 Bs) + 2(-\omega As + \omega Bc) + 3(Ac + Bs)$
Set equal to the forcing function $cos(\omega t)$:
$(-\omega^2 Ac - \omega^2 Bs) + 2(-\omega As + \omega Bc) + 3(Ac + Bs) = c$
$c[-\omega^2 A + 2\omega B + 3A] + s[-\omega^2 B - 2\omega A + 3B] = c$

Since the coefficient on the right hand side for cosine (c) is 1, we can form an equation:
$-\omega^2 A + 2\omega B + 3A = 1$
$A(3 - \omega^2) + 2\omega B = 1$ (eq. 1)

The coefficient for sine (s) on the right hand side is 0, so form another equation:
$-\omega^2 B - 2\omega A + 3B = 0$
$-2\omega A + B(3 - \omega^2) = 0$ (eq. 2)

Multiply (eq. 1) by $(3 - \omega^2)$:
$A(3 - \omega^2)^2 + B(6\omega - 2\omega^3) = (3 - \omega^2)$ (eq. 1)

Multiply (eq. 2) by (-2ω):
$4\omega^2 A + B(-6\omega + 2\omega^3) = 0$ (eq. 2)

Add the two equations and get:
$A[(3 - \omega^2)^2 + 4\omega^2] = (3 - \omega^2)$
$A = \frac{(3-\omega^2)}{(3-\omega^2)^2 + 4\omega^2}$
$\boldsymbol{A = \frac{(3-\omega^2)}{D}}$ **where** $\boldsymbol{D = (3 - \omega^2)^2 + 4\omega^2}$

Solve for B. Plug A into the original (non-modified) (eq. 2) to get:
$-2\omega(\frac{(3-\omega^2)}{D}) + B(3 - \omega^2) = 0$
$B(3 - \omega^2) = 2\omega(\frac{(3-\omega^2)}{D})$
$\boldsymbol{B = \frac{2\omega}{D}}$ **where** $\boldsymbol{D = (3 - \omega^2)^2 + 4\omega^2}$

(b) The amplitude of the forced response R is given by $R = \sqrt{A^2 + B^2}$, where A and B are given by the formulas from part (a). Determine the frequency ω that maximizes the amplitude of the forced response. (Hint: consult an in-class example.)

Plug in A and B from above:
$R = \sqrt{(\frac{3-\omega^2}{D})^2 + (\frac{2\omega}{D})^2}$ where $D = (3-\omega^2)^2 + 4\omega^2$
$R = \sqrt{\frac{1}{(D)^2}[(3-\omega^2)^2 + (4\omega^2)]}$
$R = \sqrt{\frac{1}{(D)^2}(D)} = \sqrt{\frac{1}{D}} = D^{-1/2}$

The amplitude of the forced response, $R(\omega)$, will be maximized when its derivative $R'(\omega) = 0$.
Since $R = D^{-1/2}$ and only depends on D, we're looking for when $D' = 0$:
$D = (3-\omega^2)^2 + 4\omega^2$
$D = \omega^4 - 2\omega^2 + 9$
$D' = 4\omega^3 - 4\omega = \omega(4\omega^2 - 4)$
Set equal to 0:
$\omega(4\omega^2 - 4) = 0$
$\omega(2\omega + 2)(2\omega - 2) = 0$
$\omega = 0$ or $\omega = \pm 1$
Ignore $\omega = 0$ and $\omega = -1$, because a frequency value must be positive and nonzero for this system to function in a sensical manner.
$\boldsymbol{\omega = 1}$ **Hz**

- A value of $\boldsymbol{\omega = 1}$ Hz will maximize the amplitude of the forced response.
- This value makes intuitive sense, because it is close to the natural frequency $\boldsymbol{w_0 = \sqrt{3} \approx 1.732}$ Hz of the system.
- If there were no damping coeffiecient $\boldsymbol{c}$, the ideal value would be $\boldsymbol{\omega = \omega_0}$.