

NGR 33-010-082

28

(NASA-CR-149214) COMPUTER SIMULATIONS OF  
PLANETARY ACCRETION DYNAMICS: SENSITIVITY  
TO INITIAL CONDITIONS (Cornell Univ.,  
Ithaca, N.Y.) 57 p HC A04/MF A01 CSCL 03B

N77-12988

Unclas  
G3/91 55837

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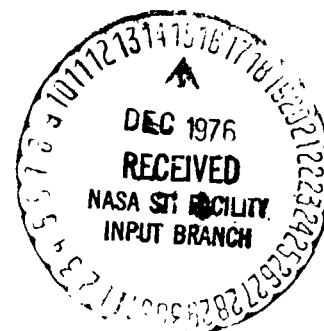
*Center for Radiophysics and Space Research*

ITHACA, N.Y.

CRSR 648

COMPUTER SIMULATIONS OF PLANETARY ACCRETION DYNAMICS:  
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October, 1976

## ABSTRACT

We have tested the implications and limitations of Program ACRETE, a scheme based on Newtonian Physics and accretion with unit sticking efficiency, devised by Dole (1970) to simulate the origin of the planets. The dependence of the results on a variety of radial and vertical density distribution laws, on the ratio of gas to dust in the solar nebula, on the total nebular mass, and on the orbital eccentricity,  $\epsilon$ , of the accreting grains are explored. Only for a small subset of conceivable cases are planetary systems closely like our own generated. Many models have tendencies towards one of two preferred configurations: multiple star systems, or planetary systems in which jovian planets either have substantially smaller masses than in our system or are absent altogether. But for a wide range of cases recognizable planetary systems are generated -- ranging from multiple star systems with accompanying planets, to systems with Jovian planets at several hundred AU, to single stars surrounded only by asteroids. No terrestrial planets were generated more massive than 5 Earth masses. The number of planets per system is for most cases of order 10, and, roughly, inversely proportional to  $\epsilon$ . All systems generated obey a relation of the Titius-Bode variety for relative planetary spacing. The ease with which planetary systems are generated, using such elementary physical assumptions, supports the idea of abundant and morphologically diverse planetary systems throughout the Galaxy.

## I. INTRODUCTION

The ultimate problem in planetary studies is the origin of the solar system. Despite a serious recent attack on the problem by many investigators -- much of which has been published in the pages of Icarus over the last few years -- it seems safe to say that no generally acceptable detailed model of the origin of the solar system exists. Indeed, the rate of change of models of origins, even in the hands of experienced individual investigators is a clear indication of the uncertainty of the subject. Furthermore, almost all of the detailed models have concentrated on the important early stages of solar system history, particularly the solar nebula, and not on the origins of planets per se. For example, the significant paper by Goldreich and Ward (1973) carries the history of the solar nebula up to the generation of planetesimals of about the size of Phobos and Deimos. On the observational side, the most recent work has served to cast doubts on the reliability of claimed identifications of extrasolar planetary systems (Gatewood, 1976), rather than providing a data source against which models of origins can be tested. Under these circumstances any model which purports to generate planetary systems recognizably similar to our own deserves careful scrutiny.

Any acceptable model for the formation of the solar system should be able to account at least for its most obvious characteristics: the distinction between terrestrial and Jovian planets, the spacing of planetary orbits and the distribution of planetary mass with heliocentric distance. Such parameters as the rotation periods of the planets; the orbital configuration of comets, asteroids and the particles

in the rings of Saturn; and the anomalous obliquities of Uranus and Venus are presumably details not essential to an understanding of the formation processes -- although it is possible that they might provide significant clues.

To the best of our knowledge, the only existing model which attempts to generate mature planetary systems as opposed to other stages in the evolution of solar nebulae is that of Dole (1970). In his computer simulation, "accretion nuclei" of specified mass are injected in prograde orbits in the invariable plane of a primitive solar nebula composed of both gas and dust. The physics are simply Newtonian mechanics and perfectly inelastic collisions. When accretion nuclei collide with dust grains the grains adhere with unit efficiency. Growing accretion nuclei beyond a certain mass gravitationally accrete gas as well. When two accretion nuclei collide they stick also and produce a larger planetesimal. The process is permitted to continue until all the dust and some of the gas is gathered into planets. For some choices of input parameters the resulting planetary configurations (Figure 1) are remarkably like those of the solar system.

At first sight it appears quite extraordinary that so simple a physical protocol can lead to so recognizable a set of planetary systems. The program takes no explicit account of chemical segregation with heliocentric distance in the solar nebula, of a clearing out by solar radiation pressure and the solar wind of the inner solar system during the T Tauri stage of the sun, of hydromagnetic effects, turbulent convection; or of planets dynamically unstable because of rapid rotation. If the origin of the planets can indeed be understood with such elementary assumptions

and with plausible input parameters, the model deserves much deeper attention.

Dole was able to produce planetary systems of recognizable characteristics only with a certain choice of input parameters and assumed structure of the solar nebula. The present paper is devoted to a critical examinations of these assumptions and an exploration of the consequences of variations of parameters and assumed solar nebular structure.

For example, the coplanar character of the simulated planetary systems is a direct consequence of the fact that the accretion nuclei are injected with zero inclination in prograde orbits. The formation and dynamical properties of the accretion nuclei are not further justified by Dole; we will discuss them further in the light of more recent research. The increased current skepticism (Gatewood, 1976) on earlier reductions of perturbations in the proper motion of Barnard's Star removes the props from the reduction by Black and Suffolk (1973) according to which the planets of the Barnard Star system would not have been in coplanar orbits.

## II. THE COMPUTER MODEL

The computer simulation program, called ACRETE, was written by J. Rice and generously provided to us by S. Dole. We have varied the program where necessary. In this section we describe the essential features of ACRETE.

1. The solar nebula is taken to have the shape of an "exocone", seen edge-on in Fig. 2. The shape is assumed to arise from an originally spherical cloud of gas and dust with some nonzero net angular momentum in which dust particles with bits highly inclined

to the invariable plane are eventually degraded to orbits of lower inclination through inelastic collisions. Most models of the solar nebula assume either a similar configuration or a cylindrical (disk-shaped) distribution of matter in which the density of gas and dust falls off away from the central plane. In its original application, ACRETE does not take account of the vertical density distribution. A corrected treatment will be discussed in a later section.

2. The mass ratio of gas to dust in the nebula is a constant,  $K = \rho_g / \rho_d$  where  $\rho_g$  and  $\rho_d$  are respectively the radially-dependent density of gas and dust. While it may be reasonable to expect that this ratio will be independent of radial distance in the central plane of the nebula, the mass difference between a dust particle and a gas molecule will insure different scale heights for their respective vertical density distributions. This correction, however, would complicate the computer program greatly, and was not taken into account either by Dole or by us. Dole used the value  $K = 50$ , which will be shown later to be a reasonable number.

3. In Dole's study, the density distribution of dust is  $\rho_d = Ae^{-\alpha r^{1/3}}$  where  $r$  is the distance from the center of mass of the cloud in astronomical units (AU), and  $A$  and  $\alpha$  are adjustable parameters. In addition to experimenting with  $A$  and  $\alpha$ , we have varied the functional form itself. Dole was able to generate aesthetically pleasing (i.e. solar system-like) planetary systems when  $A = 0.0015 M_\odot / \text{AU}^3$  and  $\alpha = 5$ . The justification for these particular choices was one of convenience; Dole was not striving for any generality, since he stated that the object of his exercise was to generate planetary systems similar to our own. We will

examine other choices of  $A$  and  $\alpha$  as well as other choices for the functional form  $\rho_d$ .

4. The dust particles comprising the cloud (other than the accretion nuclei) are all given the same orbital eccentricity  $\epsilon$ , an input parameter (Dole's value is  $\epsilon = 0.25$ ), and are taken to have randomly distributed semimajor axes and inclinations. We will examine the consequences of other choices of  $\epsilon$ , but for simplicity will not assume a distribution function for various values of the orbital eccentricity of dust grains.

5. The accretion nuclei are taken to have some initial mass  $m_0$  which is an input parameter of the program. The nuclei are injected into prograde orbits of zero inclination, with semimajor axes randomly distributed between 0.3 and 50 AU, and with eccentricities given by the distribution function  $e = 1 - (1 - Y)^{0.077}$ , where  $Y$  is random between zero and one. This form is an empirical distribution derived by Dole which reproduces the distribution of planetary eccentricities in the solar system. The small exponent yields small eccentricities and, since any nucleus undergoing accretion would suffer numerous inelastic collisions that would tend to circularize its orbit, it was not felt necessary to change the exponent even though it is an input parameter. Similarly, the bounds of 0.3 and 50 AU for the semimajor axes are also input variables, but changing them changes neither the physics of the problem, nor (to any substantial degree) the results. Occasionally, these limits were moved closer together for convenience when no planets could be formed at the extremities of the cloud.

A nucleus captures all dust particles which cross its orbit (sticking coefficient unity), plus those whose orbits fall in



an unstable region related to its gravitational cross section. The radial extent of this region around the accretion nucleus is given by  $x = r\mu^{1/4}$ , where  $r$  is the distance of the nucleus from the center of the nebula and  $\mu$  is its reduced mass with respect to the Sun:  $\mu = m/(1 + m)$ , where  $m$  is the nuclear mass expressed in solar masses. The expression for  $x$  is an approximation to the solution of the restricted three-body problem. Birn (1973) finds the exponent to be  $1/3$  instead of  $1/4$ , but this was not changed in the program, since the effect of the change can be shown to be small. Also, it is implicitly assumed that the semimajor axes of all orbits precess through all directions in the invariable plane via accumulated gravitational perturbations.

7. Nuclei accrete only dust initially, until their masses (and hence escape velocities) are high enough to permit the retention of gas as well. If we assume that an accreting planetoid of mass  $m$  has uniform density, its escape velocity  $v_e$  is proportional to  $m^{1/3}$ . A gas molecule at temperature  $T$  has a velocity  $\propto T^{1/2}$ , and, if we assume a temperature-distance dependence of  $T(r) = T_0 (r/r_0)^{-1/2}$  (where  $r$  is the radial distance from the central star), then the gas velocity becomes  $v_g \propto r^{-1/4}$ . The functional form chosen for  $T(r)$  is appropriate for an optically thin solar nebula and for some choices of optically thick nebulae. For retention of gas above some critical mass  $m_c$ , we demand  $v_e > v_g$ , or  $C_1 m_c^{1/3} = C_2 r^{-1/4}$ , so that  $m_c = C_3 r^{-3/4}$ , where  $C_1, C_2, C_3$  are constants of proportionality. In practice,  $C_3$  is  $\sim 10^{-5}$  when  $m_c$  is measured in solar masses and  $r$  is taken to be the perihelion distance of the planetoid's orbit. An alternate form of the temperature distribution,  $T(r) \propto r^{-1}$ , has been suggested by Lewis (1974). In this case,  $m_c \propto r^{-3/2}$ .

Once the critical mass is reached, a nucleus will accrete some gas along with the dust. As the mass increases still further, a larger fraction of the gas present near the nucleus will be captured, so that in the limit of a very large mass the net density of captured material will be  $\rho = K\rho_d$ , which corresponds to the capture of all gas near the nucleus. For intermediate masses, the "effective density" of accreted matter is taken to be

$$\rho_e = K\rho_d [1 + (m_c/m)^{1/2} (K-1)]^{-1}, \quad m \geq m_c,$$

which obeys the conditions  $\rho_e = \rho_d$  when  $m = m_c$  and  $\rho_e \rightarrow K\rho_d$  when  $m \rightarrow \infty$ . This function is arbitrary and was selected by Dole primarily for its simplicity and its correct behavior in the limits. A functional form which is more physically exact would require knowledge of the structure of the accreting planet and a detailed dynamical analysis of the solar nebula, both of which are beyond the scope of this treatment. However, the expression is probably at least qualitatively correct and implies that the greater the mass of the planet, the greater the gas/dust ratio of the accreted mass.

8. The nuclei are injected sequentially, with the newest nucleus growing to completion before the next is injected. The growth of the nucleus is calculated iteratively in the program, and "completion" is defined as a fractional mass increase on a given iteration  $< 10^{-4}$ . Ideally, one would like to have all of the nuclei growing simultaneously, since, in the present form of the calculation, the final appearance of a planetary system is weakly dependent on the order in which the accretion nuclei are injected into the nebula. However, while this may change slightly the details of a given planetary system, the overall morphology of a set of planetary systems derived from similar initial conditions remains unchanged.

When the radius of capture of a growing planet intrudes on that of an already-formed planet, the two coalesce into a new body which continues to grow until completion. The new semimajor axis of the orbit of the component planet is taken to be

$$a_3 = [m_1 + m_2] / [(m_1/a_1) + (m_2/a_2)],$$

where  $a_1$  and  $a_2$  are the semimajor axes of the two coalescing bodies, and  $m_1$  and  $m_2$  are their masses. The value  $a_3$  is the maximum allowed from the conservation of energy. The new eccentricity  $e_3$  is calculated from  $a_3$  and the conservation of angular momentum. Clearly, with no information about the position angles of the precollision semimajor axes of the two orbits, the three-body problem admits no unique solution and so (within the confines of the conservation laws) the choice of  $a_3$  and  $e_3$  is somewhat arbitrary. The form given above, however, is both physically realizable and convenient.

9. A nucleus which is injected into a region which has already been swept free of dust by existing planets is a "dud" and cannot grow, since a nucleus cannot initially accumulate gas. Thus, the program ends when all dust between 0.3 and 50 AU has been swept up. A typical run of the program will entail the injection of 100 - 300 nuclei, most of which are duds. The simulations in this paper were run on the IBM 370/168 at Cornell University. The running time necessary for the formation of a single planetary system was on the order of 3 seconds, which (conveniently) is a factor  $\sim 10^{15}$  faster than the process being simulated. The cost was roughly \$1 per solar system, or ten cents per planet.

### III. THE ACCRETION PROCESS

Most models of the solar nebula employ a self-gravitating disk or exocone  $< 1$  AU thick and many AU in radius, with a total mass between  $0.1$  and  $1.0 M_{\odot}$ , exclusive of the mass of the Sun itself. Goldreich and Ward (1973) have hypothesized a disk some  $10^{12}$  cm thick. As the nebula cools, the vapor pressures of some of the constituents fall below their partial pressures, and the condensation of small particles ensues. These particles then fall towards the central plane of the disk, accumulating matter as they fall from viscous drag and collisions in the medium. For  $\sim 1$  AU from the center of the disk, this occurs on a time scale  $\sim 10$  years and leads to particles with masses  $\sim 100$ g. This mass is an upper limit, however, being strongly dependent on the number of nucleation sites (i.e. the number of particles descending upon the central plane). ACRETE, in injecting the nuclei sequentially, assumes a number of sites  $\sim 100$ , as stated before. Hills (1973) suggests that there were 100 major accretion sites before mutual collisions led to fragmentation into roughly  $10^3$  nuclei. The precise number, however, is of only marginal importance, for the resultant disk of particles in the central plane is gravitationally unstable and will clump together to form fewer pre-planetary accretion nuclei. This clumping leads to the formation of planetesimals with radii  $r \sim 5$ km and masses  $m \sim 10^{18}$ g on a time scale of only a few thousand years. These planetesimals are largely in prograde orbits of near-zero inclination, and account plausibly for the coplanar nature of the solar system.

In another model, Cameron (1973) states that turbulence in the solar nebula can cause grains to aggregate into bodies of a few tens of cm in radius which can then grow to lunar-sized planetesimals

as they descend to the central plane. As before, the process takes only a few thousand years.

Once the preplanetary accretion nuclei have settled into the central plane of the nebula, their masses are much greater than the masses of the ambient dust particles, so that further growth will be dominated by the gravitational capture mechanism. Weidenschilling (1974) has performed a straightforward analysis of this process and concludes that, from accretion nuclei no larger than  $10^{-3}$  of a terrestrial planetary mass, the solar system could be formed in about  $10^8$  years.

The mass of the injection nuclei which Dole used in ACRETE was  $m_0 = 10^{-15} M_\odot \sim 10^{18} \text{g}$ ; coincidentally the same size as the planetesimals of Goldreich and Ward, but a good deal smaller than Cameron's. We find that varying the seed mass  $m_0$  by many orders of magnitude has absolutely no effect on the final results, since the amount of matter that the particle accretes from the nebula on the first iteration is in most cases vastly greater than its initial mass. Therefore,  $m_0 = 10^{-15} M_\odot$  was used in all subsequent runs. We can postulate a model similar to Goldreich and Ward's in which numerous bodies of mass  $10^{18} \text{g}$  are created by local gravitational instabilities and which subsequently grow via accretion processes like those built into the computer program. Since the program is insensitive to the initial mass of the accretion nuclei, the number of nucleation sites in the solar nebula becomes unimportant. Numerous masses of  $10^{18} \text{g}$  in Keplerian orbits would eventually coalesce into a smaller number of more massive nuclei on which the accretion process would continue as before.

Having established some theoretical basis for the specific model

on which program ACRETE is constructed, we proceed to alter the individual parameters one at a time, to approach physically more realistic models than the one employed by Dole.

#### IV. K: THE GAS TO DUST MASS RATIO

In Figure 3 are displayed three model planetary systems generated with the canonical ACRETE program, but with different values of the ratio of gas to dust. As expected, decreasing the amount of gas in the cloud has no effect on those planets which never reach their critical mass  $m_c$  and hence never accumulate any gas in the first place. This is apparent especially for the cases  $K = 30$  and  $K = 10$  in which the nuclei were injected into the same orbits in the two cases. (The orbits are determined by a random number generator: this is fed a seed number which causes the generation of a random series.)

For  $K = 30$  and  $K = 10$ , the five planets which did not accumulate gas (filled circles) underwent no change in mass, while the gas giants (open circles) are considerably smaller in the latter run. An extension of this result can be seen qualitatively in the  $K = 100$  run, which has two very large gas giants.

What is a reasonable value for  $K$ ? Taking typical values for HI regions (Harwit, 1973), we find that the number density of grains is  $\sim 10^{-10} \text{ cm}^{-3}$ , and that their radii are  $\sim 3 \times 10^{-5} \text{ cm}$ . The mass density of gas in an HI region is  $\sim 2 \times 10^{-22} \text{ g cm}^{-3}$ , so that if we assume unit mass density for each grain, we obtain  $K = 20$ . The value for a nebula of solar composition is  $\sim 100$ , depending on the degree of condensation, but since  $K$  in the program is taken to be

the mass ratio of hydrogen and helium to all other substances (rather than the volatile:refractory ratio), this can be treated as an upper limit. Thus, Dole's value of  $K = 50$  is certainly an acceptable one, since even values as low as  $K = 10$  in ACRETE yield plausible planetary systems.

A more comprehensive model would have included the variation of  $K$  with  $r$ . Because the incidence of condensation should increase with declining temperature,  $K$  should decrease with heliocentric distance. However, we believe that a slowly varying  $K$ , or a bimodal distribution of  $K$  in which the values differ by a factor of no more than about 4, will not alter our results profoundly. We see from the figures that the only perceptible result of a variation of  $K$  by a factor of 5 is a change in the prevalence of gas giants. A distance-dependent  $K$  of the sort described would probably have as its principal consequence, a small inward displacement of the region of the Jovian planets.

## V. THE CENTRAL DENSITY AND THE PARAMETER A

Since an exponentially decreasing density function leads to a total nebular mass which is directly proportional to the density at  $r = 0$ , changing the parameter A in the expression  $\rho_d = A \exp(-\alpha r^\beta)$  is equivalent to scaling the mass of the cloud. More fundamental changes in the functional form itself will be discussed in a later section. This particular form was used by Dole because it has the mathematically desirable properties of being monotonically decreasing with  $r$ , and being integrable over a spherical or cylindrical volume; and because its use in program ACRETE leads to the formation of planetary systems resembling the solar system. To this latter end, Dole used the values  $\alpha = 5$  and  $\beta = 1/3$ , which for the moment we adopt. He employed the value  $0.0015 M_\odot/\text{AU}^3$  for A, or roughly  $10^{-9} \text{ g cm}^{-3}$ , which leads to a total nebular mass of approximately  $0.06 M_\odot$  when the opening angle of the exocone is taken to be  $\pi/2$  so that the "cone" is actually a sphere (see Appendix). (This mass, as the low central density indicates, is exclusive of the mass of the central star).

A mass of  $0.06 M_\odot$  is somewhat low compared to that of most models. Urey (1974), for example, derives a mass of  $0.6 M_\odot$ , although he refers to two other models which call for nebular masses of  $0.2$  and  $0.05 M_\odot$ . ACRETE does not "know" that some of

the dust in the solar nebula is left unaccreted. In the actual formation process, it is possible that accretion onto planetary bodies from the solar nebula is in competition with a T Tauri solar wind tending to sweep away material. Workers concerned with the early



stages of formation of the solar nebula often quote values of mass (exclusive of the mass of the central star) of about  $1M_{\odot}$  (e.g. Cameron, 1976). But we are concerned with the values of solar nebula mass after the generation of accretion nuclei; in the interim a substantial loss of nebular material may have occurred, associated with the T Tauri stage of the central star. If we find that only a small range of nebular density is consistent with familiar solar systems, it follows that such systems are correspondingly uncommon. The relative timing of the generation of accretion nuclei and the T Tauri stage of the central star is an important and as yet unresolved factor in understanding the origin of planetary systems.

The results of decreasing the total mass of the cloud by one-third ( $A = 0.001 M_{\odot}/AU^3$ ) and by two-thirds ( $A = 0.0005 M_{\odot}/AU^3$ ) are shown in Fig. 4. In the latter case, the density at every point in the nebula is so low that only one planet is able to accrete enough dust to exceed its critical mass and begin to accumulate gas. Only two planets have masses greater than one earth mass. For the case  $A = 0.001 M_{\odot}/AU$  the accretion process is somewhat more successful, although the resulting gas giants are small compared to those in the solar system.

For values of  $A > 0.0015 M_{\odot}/AU^3$ , Dole has presented some results (Fig. 5). Even a doubling of the total mass of the cloud leads to a near-catastrophic accumulation of gas by the large planets. Hydrogen thermonuclear reactions occur in the core of a star of mass  $\geq 0.07 M_{\odot}$ , although deuterium burning will have set in long before that. Hence the planetary system generated by  $A = 0.003 M_{\odot}/AU^3$  in Fig. 5 would quite likely be a borderline case of a double star system, while

those generated for  $A = 0.006$  and  $A = 0.015 M_{\odot}/AU^3$  would definitely be so. This is not necessarily a drawback to the model: the statistical studies of Abt and Levy (1976) suggest that virtually all stars are components of multiple systems, two-thirds of which include stellar companions and one-third planetary. Hence the tendency for our accretion model to give rise to stellar or barely substellar companions to the central star mimics a similar tendency in nature, and the probability of the existence of numerous extrasolar planetary systems is correspondingly high.

Thus for the numerical values and functional forms chosen, planetary systems of roughly familiar aspect are produced for nebular masses (exclusive of the central star) between about 0.02 and about  $0.2 M_{\odot}$ . Systems with smaller nebular masses than this will tend to be comprised exclusively of terrestrial planets -- and, eventually, of asteroids only. Systems with larger nebular masses will evolve with the largest secondary components undergoing thermonuclear reactions, and therefore will become double or multiple star systems. In this case there will also be terrestrial and jovian planets produced, some of which will be in orbits gravitationally unstable according to the restricted three body problem. But others will be in one of the three categories of reasonably stable orbits: around the center of mass of the system if the two stellar components have a small separation; around one or the other of the individual stars if the two components have a large separation; and in a figure-8 trajectory around both components, though this is unstable in the long term.

It is necessary to note, as Dole points out, that the generation of multiple star systems pushes program ACRETE somewhat beyond the limits of its intended application (which for Dole was the simulation

of planetary systems similar to our own). When planet formation gives way to star formation, ACRETE breaks down in the sense that the total mass of the companions can exceed the intended mass of the original nebula. This effect can be seen in Fig. 5d, in which the mass of the companions comes to  $0.61 M_{\odot}$ , as compared with a nebular mass of  $0.58 M_{\odot}$  (derived from a central density of  $0.015 M_{\odot}/\text{AU}^3$  and Dole's radial density distribution). We will refer to such a breakdown of ACRETE as a pathological multiple star system. It arises from the breakdown of the approximation  $\rho(r-b_p) = \rho(r+b_a)$  (see section VII) when an accreting body becomes very massive.

#### VI. THE ORBITAL ECCENTRICITY OF THE DUST PARTICLES: $\epsilon$

In ACRETE, the accretion nuclei are assumed to capture all of those dust particles whose orbits cross their own. If a nucleus is injected with a high orbital eccentricity, it will, of course, cross the orbits of more dust particles, hence accumulate more of them and end up correspondingly more massive. Similarly, if the orbital eccentricities of the dust particles are high, then a given dust particle is more likely to cross the path of some nucleus. Thus, more eccentric particle orbits should give rise to more massive planets and, as a corollary, fewer planets in a given planetary system (since fewer nuclei are required to sweep up all of the dust). The results of varying  $\epsilon$ , the eccentricity of the dust particles in the solar nebula, between  $\epsilon = 0.1$  and  $\epsilon = 0.5$  are shown in Figures 6 and 7. The results are as expected: a typical  $\epsilon = 0.5$  run yielded six planets, three of which are quite large, while an  $\epsilon = 0.1$  run yielded fifteen relatively small bodies. Curiously, for values of  $\epsilon \lesssim 0.3$ , the effect is nearly linear (Fig. 8), within the limits of uncertainty caused by the random injection of the nuclei ( $\epsilon = 0.1$

planetary systems will generally have 14, 15, or 16 planets, etc.). The function, of course, must level off to  $N = 1$ , since for the limiting case  $\epsilon \rightarrow 1.0$  the dust particles will have near-parabolic orbits, all of which will cross the orbit of and hence be accreted onto the first nucleus injected, leading to a double star system for all reasonable values of the nebular mass.

We have argued in Section II, however, that frequent collisions in the early nebula would lead to a circularization of the orbits of accretion nuclei. This should apply to the dust particles as well, so that it is interesting that the program works "best" (in the sense of generating planetary systems similar to the solar system) when  $\epsilon = 0.25$ , which is a rather high value. Values of  $\epsilon$  more in accord with what we would expect in the nebula (say  $\epsilon \lesssim 0.1$ ) lead to an inefficient accretion process when inserted into ACRETE (Figs. 7b and 7c).

We have attempted to counteract this effect by trying lower values of  $\epsilon$  and increasing the mass of the cloud (by increasing the central density) to compensate. The results are shown in Figure 9, in which all three systems were generated with the same random number sequence. Figure 9a shows a run of ACRETE with Dole's parameters:  $A = 0.0015 M_{\odot}/\text{AU}^3$  and  $\epsilon = 0.25$ . In Figure 9b,  $A = 0.003 M_{\odot}/\text{AU}^3$  and  $\epsilon = 0.1$ , so that the mass of the cloud is now  $M_c \approx 0.12 \sin \theta_{\max} M_{\odot}$ . This condition was shown earlier to give rise to a system of barely substellar companions when  $\epsilon = 0.25$ , as shown in Figure 5b. Now the accretion process has not run away as dramatically; the largest gas giant is only seven times the mass of Jupiter.

The effect of lowering  $\epsilon$  still further by another factor of five, to  $\epsilon = 0.02$ , and increasing  $A$  again by only 25% (to  $A = 0.00375 M_{\odot}/\text{AU}^3$ ) is illustrated in Figure 9c. The mass of the largest companion has

increased by 50%, to  $0.01 M_{\odot}$ . Clearly, we have pushed A almost to the limit; decreasing  $\epsilon$  to zero and increasing A much further will lead to a binary star system.

With nebular masses of the order of  $0.1 M_{\odot}$  the eccentricity of dust orbits of roughly 0.15 seems to produce recognizable planetary systems. Both values seem to be in reasonable conformity with our expectations for the solar nebula. We also note that the mean eccentricity of the asteroids is  $\epsilon \approx 0.15$ . However, because the accretion process as simulated in ACRETE produces familiar solar systems when  $\epsilon = 0.25$  for  $A = 0.0015 M_{\odot}/\text{AU}^3$ , subsequent computer runs will for convenience use that value, as we vary other parameters.

## VII. MODIFICATIONS TO THE DENSITY DISTRIBUTION

The mass density as a function of heliocentric distance is a critical attribute of any model of the solar nebula. The functional form of the density which Dole used in ACRETE is  $\rho_d = A \exp(-\alpha r^{1/3})$ , with A and  $\alpha$  as free parameters. More generally, we may use the form (which we will call form A)  $\rho_d = A \exp(-\alpha r^{\beta})$ , and treat  $\beta$  as a free parameter as well. The mass which a nucleus at radial distance r will accumulate is roughly proportional to  $r^3 \rho$ . For form A, with  $\beta = 1/3$ , this reaches a maximum when the r derivative of  $r^3 \rho_d$  vanishes; i.e., when  $r_m = (9/\alpha)^3$ . Dole used the value  $\alpha = 5$ , which leads to the largest planets near  $r \approx 5.8 \text{ AU}$ , i.e. not very far from Jupiter's orbit. Thus, we can move the position of the largest planets by altering the value of  $\alpha$ . Large values of  $\alpha$  will make the exponential drop off faster, so that distant planets

become smaller as the maximum of  $r^3\rho$  moves inward.

The assumption of a solar nebula with a density maximum at  $\sim 10$  AU is supported by the frequency histogram of separations of double star systems (Kuiper, 1951) which is also peaked near 10 AU. This is in reasonable accord with the implications of program ACRETE, that in many cases the formation of double stars is due to the condensation of a particularly massive jovian planet from a solar nebula. If we are to preserve the total mass of the nebula as  $\alpha$  is increased we must increase the value of A. If we take an exocone of angle  $\theta = \pi/2$  (i.e. a sphere), then for a density distribution with form A and  $\beta = 1/3$ , the total mass of the cloud is (see Appendix)  $M_{\odot} = 483840\pi KA/\alpha^9 M_{\odot}$ . For  $K = 50$ ,  $A = 0.0015 M_{\odot}/\text{AU}^3$ , and  $\alpha = 5$ , this becomes  $M_c = 0.06 M_{\odot}$ . If we keep  $K = 50$  and wish to preserve  $M_c = 0.06 M_{\odot}$ , then the relation between A and  $\alpha$  is  $A = \alpha^9/(1.3 \times 10^7) M_{\odot}/\text{AU}^3$ . The results of varying  $\alpha$  (and A with  $\alpha$ ) are shown in Figure 10. For small values of  $\alpha$ , more planets would be formed at  $r \gg 50$  AU if the program were allowed to inject accretion nuclei out that far. For  $\alpha = 1$ , for example, Jovian planets would be formed near 700 AU.

It is apparent that, although in principle an exponentially decreasing density distribution is reasonable, the particular form  $\exp(-\alpha r^{1/3})$  is quite arbitrary and only serves well for values of  $\alpha$  not very different from 5. This form was used in the first place because of its pleasing tendency to produce familiar end results. In fact, however, it falls off much more rapidly than most other theoretical models, decreasing to 1 percent of the central density at only 0.78 AU (just outside the orbit of Venus). This indicates that, for such a model, planetary formation takes place at a very late stage

in the condensation of the nebula.

Heppenheimer (1974) has managed to find a meeting point between Cameron and Pine's models and a density distribution of form A, by matching the pressure implied by such a distribution to the pressure obtained by Cameron and Pine (their Figure 2). If we assume that the nebular material obeys the ideal gas law Larson's (1969) adiabat gives  $p \propto \rho^{5/3}$ . Hence Heppenheimer fits the relation  $p = p_0 \exp[-(5/3)\alpha r^\beta]$  to the pressure curve of Cameron and Pine and finds  $\alpha = 4.4$  and  $\beta = 0.22$ . The mass of the nebula then becomes

$$M_c = 1.18 \times 10^4 A \sin \theta_{\max} M_\odot$$

In the model of Cameron and Pine, the nebula has roughly the exocone geometry, with a semithickness of approximately 1 AU at about 50 AU from the center. Hence  $\theta_{\max} \approx 0.02$ , leading to  $M_c \approx 236A M_\odot$  for A given in  $M_\odot/\text{AU}^3$ . If, to retain consistency with other models, we demand  $M_c = 0.1 M_\odot$ , we find  $A = 4.2 \times 10^{-4} M_\odot/\text{AU}^3$ .

Running ACRETE with the parameters  $\alpha = 4.4$ ,  $\beta = 0.22$ , and  $A = 4.2 \times 10^{-4}$  leads to a pathological multiple star system. Clearly, this same result will occur if A is increased to  $0.0015 M_\odot/\text{AU}^3$ , in which case all of the parameters would be identical to those used by Dole except for  $\beta$ , which is 0.22 instead of 0.33. The companion star, then, forms near the edge of the nebula; Table 1 indicates that it is the more gradual decrease in the former case that is causing the difficulty.

TABLE 1

DEPENDENCE OF DENSITY ON HELIOCENTRIC DISTANCE FOR TWO MODELS OF FORM A

<u>r</u>	<u><math>\exp(-5r^{1/3})</math></u>	<u><math>\exp(-4.4r^{0.22})</math></u>
0.1 AU	$9.82 \times 10^{-2}$	$7.06 \times 10^{-2}$
1	$6.73 \times 10^{-3}$	$1.23 \times 10^{-2}$
2	$1.84 \times 10^{-3}$	$5.95 \times 10^{-3}$
10	$2.10 \times 10^{-5}$	$6.74 \times 10^{-4}$
20	$1.28 \times 10^{-6}$	$2.02 \times 10^{-4}$
50	$1.00 \times 10^{-8}$	$3.03 \times 10^{-5}$

It is nonetheless striking that a fairly small change in the qualitative behavior of  $\rho$  should lead to such violent changes in the results, especially in light of the fact that the form  $\exp(-4.5r^{1/3})$ , which is even more similar to Heppenheimer's result, gives rise to a comfortably familiar planetary system (see Figure 10d).

A more complicated aspect of the entire problem -- and one which was not taken into account at all in the original version of ACRETE -- is the variation of nebular density with  $z$ , the distance perpendicular to the plane of symmetry. Since in general the protoplanetary orbits will have some nonzero inclination to the central plane, the mass accreted on each orbit will then be determined by the nebular surface density at appropriate heliocentric distance. ACRETE, however, assumes orbits with zero inclination so that the only accreted material is that within the toroid (shown in cross section in Figure 11) defined by the orbital eccentricities of the accretion nuclei and dust particles and the gravitational cross section of the nucleus. The volume of the toroid is approximately  $V = 2\pi r(b_a + b_p)(x_a + x_p)$ , where  $x_a$  and  $x_p$  are the gravitational capture distances of aphelion and



perihelion (see section II), and  $b_a$  and  $b_p$  include the effects of the dust's orbital eccentricity. If the density does not vary out of the plane, and if  $\rho(r - b_p) = \rho(r + b_a)$ , the mass of dust within the toroid is  $M_d = 2\pi r(b_a + b_p)(x_a + x_p)\rho_d(r)$ , a formulation employed by Dole.

From studies of a self-gravitating set of mass points, originally due to Ledoux and to Chandrasekhar, Urey (1974) argues that the vertical variation takes the form

$$\rho(r, z) = \rho(r) \operatorname{sech}^2 [z/H(r)],$$

where  $H(r)$  is some radially-dependent characteristic vertical scale height. In this case, the mass of the dust within the toroid becomes

$$M_d = 2\pi r(b_a + b_p) \left\{ \int_0^{x_a} \operatorname{sech}^2 \frac{z}{H(r)} dz + \int_0^{x_p} \operatorname{sech}^2 \frac{z}{H(r)} dz \right\} \rho_d(r, 0)$$

$$= 2\pi r(b_a + b_p)H(r) \left\{ \tanh [x_a/H(r)] + \tanh [x_p/H(r)] \right\} \rho_d(r, 0)$$

which approaches Dole's form in the limit  $x_{a,p}/H(r) \rightarrow 0$ . Urey finds (his Table III) that  $H(r) = 0.00267r$  AU; for  $r$  measured in AU. For a planet of mass  $m$  with a circular orbit, we have  $x_{a,p}/H = 374(m/M_\odot)^{1/4}$ , so that  $\tanh (x_a/H)$  will differ appreciably from its argument (and therefore depart from Dole's limiting case) whenever  $m \gtrsim 10^{-11}M_\odot$ . Since this is  $< 10^{-3}$  the mass of the Moon, we can conclude that, for every case of interest, Urey's vertical density distribution will lead to results substantially different from Dole's vertically-uniform model.

A  $\text{sech}^2 z$  vertical decrease in density is faster than exponential. Urey's nebula, therefore, is much thinner than Dole's. A characteristic scale height of  $0.00267r$  leads to a semithickness of roughly  $0.1 \text{ AU}$  at  $r = 50 \text{ AU}$ , or about an order of magnitude thinner than the nebular model of Cameron and Pine. Hills (1973) also concludes that the scale height for gas and dust in the solar nebula is very small: roughly  $0.1 r \text{ AU}$  for  $\text{H}_2$  gas and  $10^{-3} r \text{ AU}$  for dust. For models which are so concentrated into the central plane, however, radial density distributions as steep as Dole's (see Table I) lead to very low nebular masses. Consider an exocone of radius  $R$  and opening angle  $\theta_{\max}$  in which the only density variation is radial,  $\rho(r)$ . Then the mass of the cloud is

$$M_{c,Dole} = 4\pi \int_0^R r^2 \rho(r) \sin \theta_{\max} dr$$

Urey's nebula has a vertical density distribution as well, and has cylindrical geometry which we will characterize by some thickness  $h$ . Then the mass becomes

$$M_{c,Urey} = 2\pi \int_0^R \int_{-h}^h r \rho(r) \text{sech}^2(z/\gamma r) dr dz$$

where  $\gamma r = 0.00267r = H(r)$ . If we let  $h \gg \gamma r$  (true for  $h \gtrsim 0.25 \text{ AU}$ ) we can, to good approximation, extend the limits on the  $z$  integral to  $\pm \infty$  and integrate to get

$$M_{c,Urey} = 4\pi \int_0^R \gamma r^2 \rho(r) dr$$

so that  $M_{c,Urey}/M_{c,Dole} = \gamma/\sin \theta_{\max}$ . In Cameron and Pine's nebula,  $\theta_{\max} \approx 0.02$ , so that if we were to apply this opening angle to Dole's exocone, we could get  $M_{c,Urey}/M_{c,Dole} \sim 0.1$ . For a complete sphere, the ratio becomes  $2 \times 10^{-3}$ . Thus, for identical radial density distributions, Urey's model leads to nebular masses much smaller than Dole's.

In principle the masses of the two models could be reconciled by either increasing the central density in Urey's model or inserting a radial density function that is less steep than Dole's. We have simulated a  $\text{sech}^2(z/\gamma r)$  dependence in the vertical direction by modifying the mass contained in a toroidal volume in the fashion derived earlier. Using a radial density function of form A with  $\alpha = 5$ ,  $\beta = 1/3$  (Dole's values) and a central density increased by an order of magnitude from Dole's value (to  $0.015 M_{\odot}/\text{AU}^3$ ), ACRETE consistently generates pathological multiple star systems. Nonpathological systems are generated with any degree of regularity only when the central density becomes  $\rho_0 \lesssim 0.006 M_{\odot}/\text{AU}^3$ . This implies

$$M_{c,\text{Dole}} = 0.23 \sin \theta_{\text{max}} M_{\odot}$$

$$M_{c,\text{Urey}} = 6 \times 10^{-4} M_{\odot}$$

which is unrealistically low for reasonable values of  $\theta$ . The failure of ACRETE to accommodate a vertical density distribution must unfortunately be interpreted as being due to the exclusion of some pertinent physics from the computer model. It seems clear that a vertical distribution will exist in a rotating preplanetary nebula and that the accretion nuclei, like the planets, will travel in orbits with nonzero inclination. The importance of the surface density (as opposed to the vertical and radial volume densities) arises from these conditions, but is overlooked in the computer program in the assumption of perfectly coplanar orbits.

# VIII. THE ARBITRARY NATURE OF $\rho(r)$

A rigorous derivation of the radial density distribution requires a detailed knowledge of the equation of state at all points in the cloud, a treatment which is well beyond the scope of this paper and, apparently, many others. Numerous assumptions, including consideration of the ambient magnetic field and the solar wind enter into the problem, and the final results must be strongly model-dependent.

Any formulation of the density function in the cloud must at present be, to a certain extent, arbitrary, so that it is perhaps the safest course to choose one which is characterized solely by physically reasonable qualitative attributes. This is essentially what Urey, Dole and Cameron and Pine all did, and what we shall proceed to do.

One of the most obvious forms to try, because of its simplicity and wide applicability, is a simple exponential. This is just another manifestation of form A, with  $\beta = 1$ . Since form A contains  $r^\beta$ , we see that, because  $1 > 1/3$ ,  $\rho$  will fall off more quickly than Dole's form. Various radial scale lengths and central densities were tried, with the most success in generating planetary systems obtained with scale lengths of  $\sim 0.4 \pm 0.1$  AU. The results of the density functions

$$\rho(r,z) = 0.004 e^{-3r} \text{sech}^2(z/\gamma r) M_\odot/\text{AU}^3$$

and

$$\rho(r,z) = 0.002 e^{-2r} \operatorname{sech}^2(z/\gamma r) M_{\odot}/\text{AU}^3$$

(for  $r$  in AU) are shown in Figures 12b and 12c. The masses of the nebulae in these cases are  $10^{-3} M_{\odot}$ , and are kept deliberately low because of a propensity for this form of the density distribution to generate pathological multiple star systems. Removing the vertical density dependence does not help; the resultant increase in mass concentrated into the inner region of the nebula only exaggerates the tendency towards pathological results.

Another obvious form to try is a power law,  $\rho(r) \propto r^{-n}$  where  $n > 0$ . This form has the disadvantage of diverging at zero, although this is clearly not a problem physically, since we are only interested in the nebula at  $r \gtrsim 0.1$  AU. For the form to be integrable, we further require that  $n > 2$  in a strictly cylindrical nebula, and  $n > 3$  for a spherical one (although this is not a rigid restriction since the nebula has a finite diameter). We can deal in another way with the divergence at small  $r$  with the following ad hoc argument: most models of the formation of the solar system suggest that the young sun was in a T Tauri stage during the epoch of planetary formation. The T Tauri solar wind would considerably deplete the interior portion of the nebula of both refractory and gaseous material. Evidence for the size of a depleted region is suggested by recent observations of the T Tauri star RU Lupi by Gahm, et al. (1975), who found concentrations of dust, presumably driven out by the stellar wind, at distances of a few tenths of an AU from the star. We therefore choose to modify our power law distribution so that it reflects some flattening of the mass density function in regions close to the star. This

we refer to as form b:  $\rho(r) = \rho_1(r^n + C)^{-1}$  where C is some dimensionless constant and r is measured in AU.  $\rho(r)$  approaches the simple power law  $r^{-n}$  when  $r \gg C^{1/n}$ . For a given n, we can solve for  $\rho_1$  and C by demanding a particular central density  $\rho_0 = \rho_1/C$  and a particular nebular mass, which for Dole's exocone is

$$M_{c,Dole} = 4\pi K \rho_1 \int_0^R \frac{r^2 dr}{(r^n + C)} \sin \theta_{max} M_{\odot}$$

if  $\rho_1/C$  is the central value of  $\rho_d(r)$  and R is the radius, in the symmetry plane, of the nebula. From the arguments given earlier, when one includes the  $\text{sech}^2 z$  vertical distribution the mass becomes approximately  $M_{c,Urey} = M_{c,Dole} \gamma / \sin \theta_{max}$ , where  $\gamma = 0.00267$ .  $M_c$  is evaluated for various values of n in the Appendix.

Note that for any density distribution of form B, the function  $r^3 \rho$  reaches a maximum at  $r_m = (3C/n-3)^{1/n}$ . As stated earlier,  $r_m$  represents the distance at which the largest planets in the system will tend to form.

If we choose to simulate Urey's  $r^{-3}$  density function with the T Tauri modification, we must decide on R, since, with an infinite upper limit,  $r^2(r^3+C)^{-1}$  is not integrable. Taking  $R = 70$  AU and, as before,  $K = 50$ , the mass becomes

$$M_{c,Dole} = 67\pi \rho_1 \ln(1+70 C^{-1/3}) \sin \theta_{max} M_{\odot}$$

Letting  $\theta_{max} = \pi/2$ , the conditions  $M_{c,Dole} = 0.06 M_{\odot}$  and  $\rho_0 = \rho_1/C = 0.0015 M_{\odot}/\text{AU}^3$  yield the approximate solution

$$\rho_d(r) = \frac{5 \times 10^{-5}}{r^3 + 0.032} M_{\odot}/\text{AU}^3$$

The distribution remains fairly flat out to  $r \sim 0.032^{1/3} = 0.3$  AU.

A planetary system generated by ACRETE with this density distribution

is shown in Figure 13a. Two of the planets are very large, but are still sub-stellar (the largest is ten times the mass of Jupiter). Note, however, that for  $n = 3$ ,  $r^3\rho$  has no maximum at a finite  $r_m$ ; in models of this type, the largest planets will always be formed at the outer edge of the nebula.

If we insert the  $\text{sech}^2 z$  vertical distribution, a tendency towards pathological multiple star systems develops. As before, we can combat this only by lowering the total mass of the cloud by a factor that inhibits the formation of gas giants. The resultant planetary systems, generated when  $M_{c,Urey} \sim 10^{-3} M_\odot$ , resemble the one in Figure 10a.

The choice of  $n$  is arbitrary and is open to considerable experimentation. When  $n = 6$ , for example, the mass integral yields the result

$$M_{c,Dole} = (2\pi^2/3)(K\rho_1 C^{-1/2}) \sin \theta_{\max}$$

In this case, the integral converges as  $R \rightarrow \infty$ . Taking  $\theta_{\max} = \pi/2$ ,  $M_{c,Dole} = 0.06 M_\odot$  and  $\rho_0 = \rho_1/C = 0.0015 M_\odot/\text{AU}^3$  as in the  $n = 3$  case, the density becomes

$$\rho_d(r) = \frac{2 \times 10^{-5}}{r^6 + 0.014} M_\odot/\text{AU}^3$$

Now  $r^3\rho$  reaches a maximum at  $r_m = (0.052/3)^{1/6} \sim 0.5 \text{ AU}$ , out to which distance  $\rho$  is flat. The result of this distribution is shown in Figure 13b. Gas giants can only form close to the sun, followed by terrestrial planets and asteroids, moving outward. That such a planetary system can form at all is highly questionable; the inner terrestrial planets might not have stable orbits, and both

the T Tauri wind and Jeans escape would make the accretion of large amounts of gas by a planet so close to the sun unlikely.

Introducing the vertical density function causes the same problems as before. A low cloud mass permits the formation of terrestrial planets close to the sun, but otherwise gas giants turn into stellar companions.

Finally, we note that Larson's adiabat suggests a power law. If  $T(r) \propto \rho^{2/3}$  and we use the simple temperature law mentioned earlier,  $T(r) \propto r^{-1/2}$ , we obtain  $\rho(r) \propto r^{-3/4}$ . If  $T(r) \propto r^{-1}$ ,  $\rho(r) \propto r^{-3/2}$ . Such extremely shallow distributions, however, generate only pathological systems.

## IX. PLANETARY DISTANCES

The geometric spacing of the planets in the solar system is one of its most striking properties, represented by a number of formal schemes the most famous of which is the so-called Titius-Bode "law" -- in which the semi-major axes of planetary orbits in AU are written  $r = 0.4 + 0.3 \times 2^n$ . The value  $-\infty$  must be assigned to  $n$  in order to explain Mercury; thereafter, integer values are adopted beginning with zero. It is then necessary to identify Ceres as a planet. Even so the values for  $n = 7$  (Neptune) and  $n = 8$  (Pluto) are in unsatisfactory agreement. Thus Bode's law can be described as a fit to eight numbers by an equation with five or six free parameters or arbitrary indexing conventions -- not a very impressive "law".

Dermott (1968), however, has proposed a simple, quasi-geometric form which describes adequately the spacing of satellites around some



of the major planets and meets with moderate success when applied to the solar system as a whole. If  $P_c$  is taken to be a constant of proportionality then the periods of the planets can be expressed approximately as  $P_n = P_c j^{n/2}$ , where  $j$  is a small integer ( $j = 6$  for the solar system) and  $n$  is a given planet's "orbital" integer, generally about the same as its serial position outward from the sun. Differences between  $n$  and the serial position arise because Dermott allows that two planets can share the same value of  $n$  and furthermore that all values of  $n$  in a sequence need not be used. For the solar system, both Earth and Venus are in the  $n = 2$  orbital, and both Neptune and Pluto share the value  $n = 8$ . The advantage of this "law" over Bode's is that the relationship between  $P_n$  and  $n$  can be graphed as a straight line semilogarithmically; in light of the amount of freedom in the choice  $P_c$ ,  $j$ , and the  $n$ 's, however, it is probably no less arbitrary.

A measure of the adequacy of the relation, used in part by Dermott, can be made by comparing the  $n$ 's to the  $m$ 's in the equation  $P_m = P_c j^{m/2}$ : here, the  $P_m$  values represent the actual periods of the planets, and  $m$  takes on any values (not necessarily integer) to ensure that that is the case. Taking Dermott's quantity  $\Delta n = m - n$ , we define the quantity  $\sigma = [N_p^{-1} \sum (\Delta n)^2]^{1/2}$ : the rms derivation from the law per planet when  $N_p$  is the number of planets in the system.

In order to compare our simulated planetary systems against Dermott's law, we reformulate the latter as  $a_n = C_a j^{n/3}$ , using Kepler's third law to utilize the orbital semimajor axes  $a_n$  rather than the period (all appropriate constants are now absorbed into  $C_a$ ). Values of  $j$ ,  $C_a$ ,  $N_p$  and  $\sigma$  for the solar system and for some of the systems generated in this paper are shown in Table 2. Note that allowing a half-integer

value of  $j$  (6.5) for the solar system results in a significant improvement in  $\sigma$  over the integral case; this serves to illustrate the somewhat arbitrary nature of the procedure. It must also be borne in mind that even for randomly distributed values of  $m$  and  $n$ , the rms value of  $\Delta n$  is  $1/2\sqrt{3} = 0.289$ . Since  $\sigma$  is nearly half this value in even the best case, it is apparent that our model planetary systems follow a Bode-type law about as well as the solar system.

The agreement of Bode-type laws with our model solar systems even in such bizarre cases as, say, 12a or 13a is of some interest. It cannot be due to multiple resonances in the  $n$ -body problem as proposed by Molchanov (1968) because the appropriate physics is not contained in the computer simulations [See also other criticisms and Molchanov's reply: Backus (1969), Henon (1969), Molchanov (1969), Molchanov (1969), Gingerich (1969), Dermott (1969)].

Instead, what seems clearly to be happening is a kind of collisional natural selection. The solar system begins with gas, accretion nuclei and dust grains, and a variety of orbital eccentricities and heliocentric distances. But because of the high sticking efficiency in nucleus-grain and nucleus-nucleus collisions, those accreting planets with interacting orbits merge. In all cases the final configuration shows planets nicely separated one from another. Because larger quantities of mass are required to generate the jovian planets, they are required to sweep up larger volumes of dust and therefore have larger mutual separations than do the terrestrial planets. Because there were then more objects on more eccentric orbits in a time before the completion of this collisional natural selection, the rate of planetary collision very early in the history of the solar system may have been

considerable--quite apart from the infall of matter in debris rings in the vicinity of forming planets.

TABLE 2

FITS OF REAL AND MODEL SOLAR SYSTEMS TO A MODIFIED DERMOTT RELATION

<u>System</u>	<u>N<sub>p</sub></u>	<u>C<sub>a</sub> (AU)</u>	<u>j</u>	<u>σ</u>	<u>Comments</u>
Solar System	10	0.263	6	0.217	Includes Ceres
Solar System	10	0.230	6.5	0.169	22% improvement in σ
Figure 6(a)	6	0.185	8	0.195	Form A
Figure 12(a)	11	0.236	4	0.174	sech <sup>2</sup> z vertical distribution
Figure 13(a)	8	0.235	7	0.134	$\rho(r) \propto 1/(r^3+C)$

## X. CONCLUSIONS

The original results of Dole's program are so provocative that it is natural to question whether the remarkable planetary systems generated by ACRETE are the result of careful tailoring of the assumed radial and vertical density distributions in the solar nebula and the accompanying free parameters; or whether they are properties of any reasonable set of assumptions about the solar nebula. We have confirmed that for a perhaps plausible radial density law (form A) and the arbitrary choices  $\alpha = 5$  and  $\beta = 0.33$  recognizable planetary systems of solar system type are generated for nebular masses between 0.02 and about 0.2  $M_{\odot}$  and dust grain orbital eccentricities not extremely high or extremely low. However, in Section VII we found

that an apparently small change from  $\beta = 0.33$  to  $\beta = 0.22$  leads to a striking change in the end product. It is, of course, possible that a correct reconstruction of the underlying physics of the solar nebula will yield values of  $\beta$  near 0.33 and density distributions like form A. But in the absence of such a justification we can only conclude either a) that ACRETE is missing some of the essential physics of solar system cosmogony, or b) that planetary systems of our type are only one example in a rich array of alternative varieties of planetary systems. Likewise, more fundamental changes in the nebular morphology (e.g. from an exponential to a power law density distribution function) generate planetary systems some of which, although they do not closely resemble our own, are not fundamentally objectionable (Figs. 12 and 13).

Abt and Levy (1976) have found that the frequency of secondary masses for binaries with periods less than a century varies as the one-third power of the secondary mass. If this function can be extrapolated, it implies that about 20 percent of stars of solar mass have a largest companion of mass  $\sim 10^{-2} M_{\odot}$  and about 10 percent of stars of solar mass have a largest companion of mass  $\sim 10^{-3} M_{\odot}$ . In at least a crude way this result is consistent with our findings: where the solar nebular mass is between about  $1 M_{\odot}$  and  $10^{-1} M_{\odot}$  binary stars form; while for smaller solar nebular masses during accretion, jovian planets form. (This is a model-dependent consistency, however, because Abt and Levy believe that the short period binaries are fission systems from a single protostar).

Perhaps the most striking result of this exercise is that in all cases planetary systems are generated which satisfy a Titius-Bode sort of law; and in all cases the number of planets generated

is between several and about 20 -- that is  $\sim 10$ . This property may be understood in a very qualitative way: an accreting planet can be expected to perturb the orbits of dust particles in the solar nebula up to a few AU distant, depending on the planetary mass, and to sweep up material in such a zone. The number of such zones of a few AU in width in a nebula 50 AU in radius is  $\sim 10$ ; hence the number of planets. The remarkable result on the number of planets then is attributable to the size of the solar nebula which is given as an input parameter. However, there is at least a hint in our results (see Fig. 13a) that more massive and extensive solar nebulae lead to very large jovian planets or very small stars; but that, even in such a case, the number of dust lanes swept up is still  $\sim 10$ . The width of the lanes, i.e., the spacing of the planets, arises in part from the dependence of the accreting planets' gravitational capture lengths on their orbital radii,  $x \propto r$  (see Section II).

The computer simulations described in this paper and in Dole's take no explicit account of chemical fractionation, the T Tauri stage of early stellar evolution, nebular opacity, frozen-in magnetic fields and a number of other factors. The accretion process is imagined to be purely dynamical, and in that respect is similar to the work of Weidenschilling (1974). Furthermore, for purposes of computational convenience even the dynamics is simplified; as, for example, when all dust grains are taken to have the same eccentricity. We have found that both major and minor changes in the model, of equal apparent plausibility as the initial conditions assumed by Dole, lead to dramatic changes in the resulting planetary systems. Of course, it must be borne

In mind that with at least six free parameters to describe the model nebula, we have investigated only a small fraction of our "parameter space." In fact, there exist a multitude of such spaces, each defined by distinct, plausible density distributions, of which only two--forms A and B--were considered in this work. Considering these, we have covered a relatively minute number of cases indeed.

Nevertheless, in all cases--even when pathological binary star systems are generated--planetary systems are formed. One interesting result is that while terrestrial planets can be formed without jovian planets (in very low mass solar nebulae) the converse never occurs. We continue to be impressed that so simple a dynamical model generates recognizable if not familiar planetary systems with  $\sim$  ten planets per system and a Bode's law spacing for a wide variety of initial conditions. The results suggest that planetary systems are widely prevalent in the Milky Way Galaxy, but that substantial morphological differences between extrasolar planetary systems and our own can be expected.

# APPENDIX: EVALUATION OF THE MASS OF THE NEBULA

The total mass density throughout the cloud (both gas and dust) is

$$\rho(r) = \rho_d(r) + \rho_g(r) = \rho_d + K\rho_d \approx K\rho_d$$

since  $K \sim 50 \gg 1$ . Thus, for the geometry of an exocone of opening angle  $\theta_{\max}$  and radius  $R$ , the mass of the cloud is

$$M_c = 4\pi K \int_0^R \rho_d(r) r^2 dr \sin \theta_{\max}$$

For a cylindrical geometry with Urey's  $\text{sech}^2 z$  vertical distribution this becomes (See Section VII)

$$M_c = 4\pi K \gamma \int_0^R \rho_d(r) r^2 dr$$

If the density distribution takes form A,  $\rho_d = Ae^{-\alpha r^\beta}$ , the mass in either case is

$$M_A = 4 KA \xi \int_0^R r^2 e^{-\alpha r^\beta} dr$$

where  $\xi = \sin \theta_{\max}$  for an exocone, and  $\xi = \gamma = 0.00267$  for Urey's cylinder. Typically,  $R \sim 50$  and  $\alpha \sim 5$ . If  $\beta = 1$ , the centroid of the distribution is at  $r_c = 1/5 \ll 50$ , so that, to excellent approximation, we can extend the upper limit on the the integral to  $\infty$ . This approximation is still quite good for  $\beta < 1$ , so that we can make the substitution  $u = r^\beta$  and perform the semi-infinite integral to find

$$M_A = \frac{4 KA \xi}{\beta} \alpha^{-3/\beta} \Gamma\left(\frac{3}{\beta}\right)$$

If the distribution takes form B,  $\rho_d = \rho_1(r^n + C)^{-1}$ , the integral becomes

$$\begin{aligned} M_B &= 4\pi K \rho_1 \xi \int_0^R \frac{r^2 dr}{r^n + C} \\ &= \frac{4}{3} \pi K \xi \frac{\rho_1}{\sqrt{C}} \tan^{-1} \frac{R}{\sqrt{C}} \quad n = 6 \end{aligned}$$

$$= \frac{2\pi^2}{3} K\xi \frac{\rho_1}{\sqrt{C}} \quad \text{as } R \rightarrow \infty$$

The approximation  $R \rightarrow \infty$  is good provided only that the weaker condition  $R/C^{1/2} \gg 1$  holds, and, since  $R \sim 50$  and  $C \sim 10^{-2}$ , this is generally true.

If  $n = 3$ ,  $M_B$  does not converge as  $R \rightarrow \infty$ . The result of the integration is

$$M_B = \frac{4\pi}{3} K\xi\rho_1 \cdot \ln(1 + RC^{-1/3}) \quad n = 3$$

Fortunately, the divergence is logarithmically slow, so that the choice of  $R$  is not critical. Since ACRETE is usually run so as to inject nuclei out as far as  $r = 50$  AU, we calculate  $M_B$  with  $R = 70$ .

#### ACKNOWLEDGEMENTS

We are grateful to S. Dole for comments and for generously providing us with a copy of the computer program ACRETE, and to B. J. Levin, S. J. Weidenschilling, and J. Veverka for helpful comments. This research was supported in part by NASA grant NGR 33-010-082, and in part by Grant NGR 33-010-220, Planetology Program Office, NASA Headquarters. One of us (R.I.) was supported by the National Astronomy and Ionosphere Center, which is operated by Cornell University under contract to the National Science Foundation (NSF C-600).



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## FIGURE CAPTIONS

- Fig. 1 Planetary systems generated by Dole (1970) using program ACRETE. Solid circles represent terrestrial planets, while open circles indicate Jovian planets that have accreted gas as well as dust. The radius of each circle is scaled solely by the cube root of the planet's mass, given in the figures in units of Earth masses. The positions of jovian planets are given by the centers of the circles. Those cases in which planets to be abutting or overlapping are, of course, only artifacts of the schematic diagram. All planets produced are well separated, as the relative positions of their centers indicate. The fourth system displayed is our own; the others are generated by ACRETE.
- Fig. 2 The exocone. In the original model, the density of gas and dust depended only on the radial distance from the central star.
- Fig. 3 Planetary systems generated by ACRETE for different values of the gas:dust mass ratio in the solar nebula. Dole employed the value  $K = 50$ .
- Fig. 4 The effect of decreasing the central density of the nebula, which is equivalent to scaling the mass of the cloud. Densities are measured in units of  $M_{\odot}/AU^3$ .
- Fig. 5 The effect of increasing the central density of the cloud from Dole's value of  $0.0015 M_{\odot}/AU^3$ . In the pathological system (d), the sum of the masses of the bodies slightly exceeds the original mass of the nebula.
- Fig. 6 Increasing the eccentricity,  $\epsilon$ , of the orbits of the dust particles in the nebula results in a more efficient accretion process and a higher incidence of Jovian planets. Results which resemble the solar system are obtained when  $\epsilon \approx 0.25$  as shown in (c).
- Fig. 7 Decreasing  $\epsilon$  leads to numerous terrestrial bodies and relatively few gas giants. Small values of  $\epsilon$  may have prevailed in the actual solar nebula.
- Fig. 8 Average number of planets  $N$  generated by ACRETE as a function of  $\epsilon$ , the orbital eccentricity of dust particles in the solar nebula. For  $\epsilon \leq 0.3$ , the effect is nearly linear.
- Fig. 9 Compensation for the inefficiency of the accretion process associated with small values of  $\epsilon$  by increasing the mass of the nebula, proportional to  $A$ . Although the results illustrated in (b) compare favorably with the system shown in Fig. 5(b), the central density  $A$  cannot be increased much beyond the value  $0.00375 M_{\odot}/AU^3$ , as shown in (c), without generating binary star systems.

- Fig. 10 Systems generated by ACRE1. by varying the steepness with which the density profile falls off.  $\alpha$  is the parameter in the expression  $\rho_d = 0.0015 \exp(-\alpha r^{1/3})$ . In cases (a), (b) and (c), more planets with orbital semimajor axes  $r > 50$  would be formed if injected accretion nuclei were allowed out that far.
- Fig. 11 Cross-section of the toroidal volume swept out by an accretion nucleus with orbital semimajor axis  $r$ .  $x_a$  and  $x_p$  are related to the gravitational cross-section of the nucleus.  $b_a$  and  $b_p$  are also related to this cross-section as well as to the orbital eccentricities of both the nucleus and the dust particles in the nebula.
- Fig. 12 The effect of including Urey's  $\text{sech}^2 z$  vertical density profile. Figure (a) was generated by the insertion of  $0.0015 \exp(-5r^{1/3})$ . Figures (b) and (c) were obtained by using a simple exponential radial density profile, with scale lengths of  $1/2$  AU and  $1/3$  AU, respectively.
- Fig. 13 Planetary systems obtained with radial density distributions different from an exponential. Figure (a) was derived for the form  $\rho_d = \rho_1(r^3 + C)^{-1}$ , Figure (b) for  $\rho_d = \rho_1(r^6 + C)^{-1}$ , where  $\rho_1$  and  $C$  are constants. Figure (c) shows the solar system.

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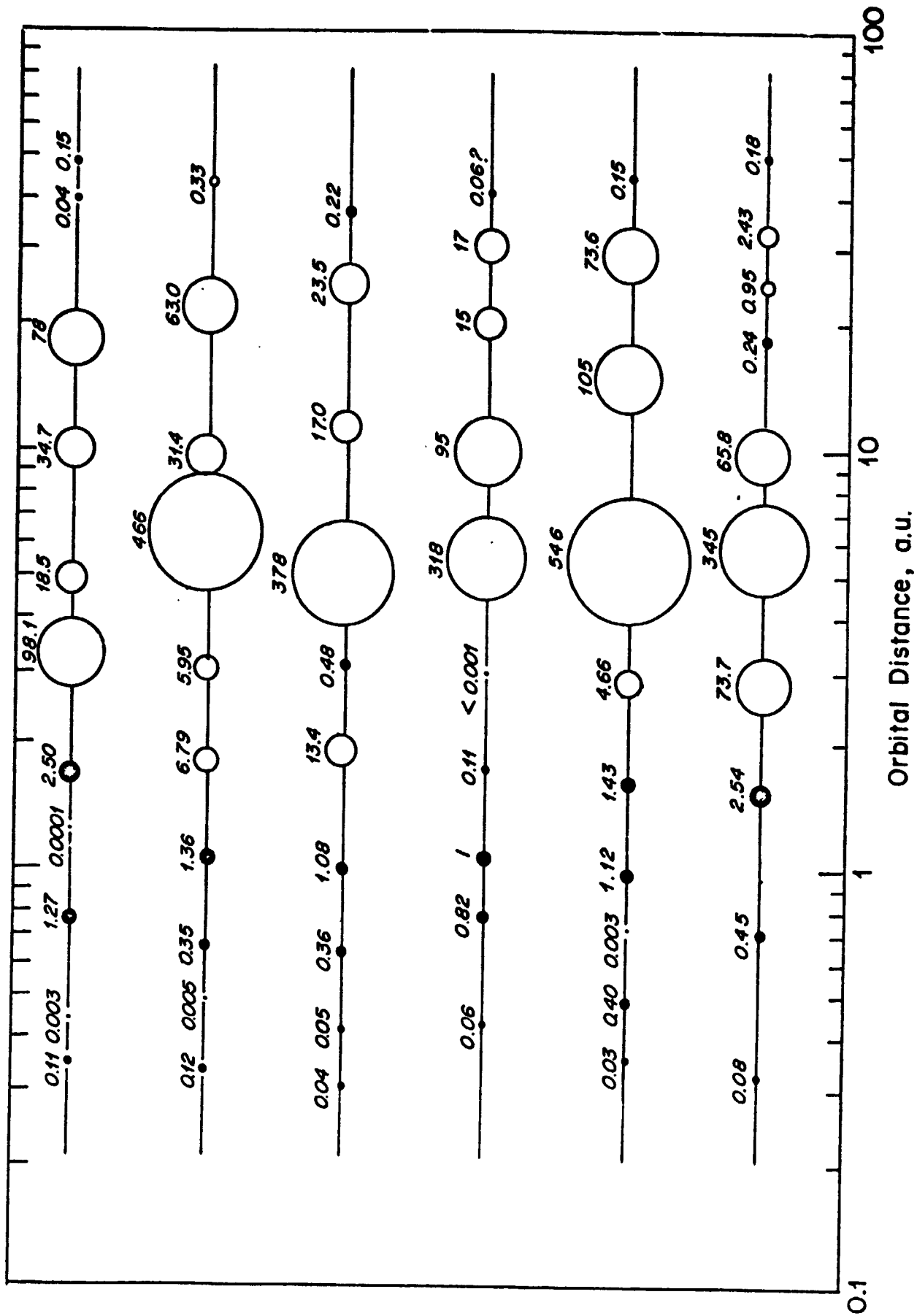


Figure 1

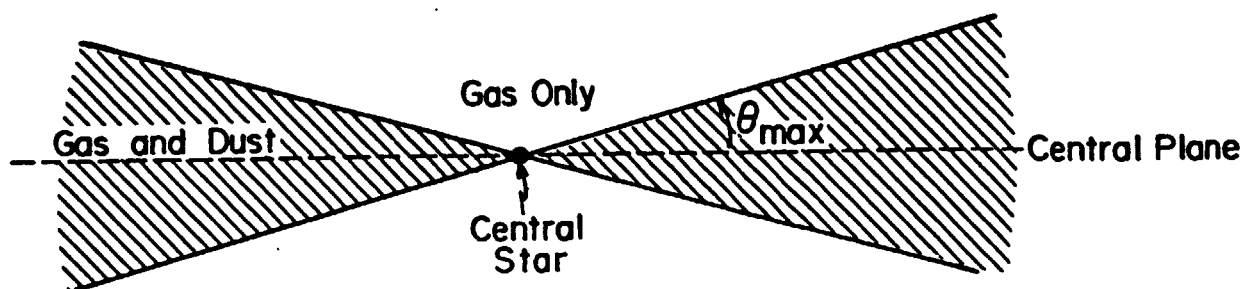


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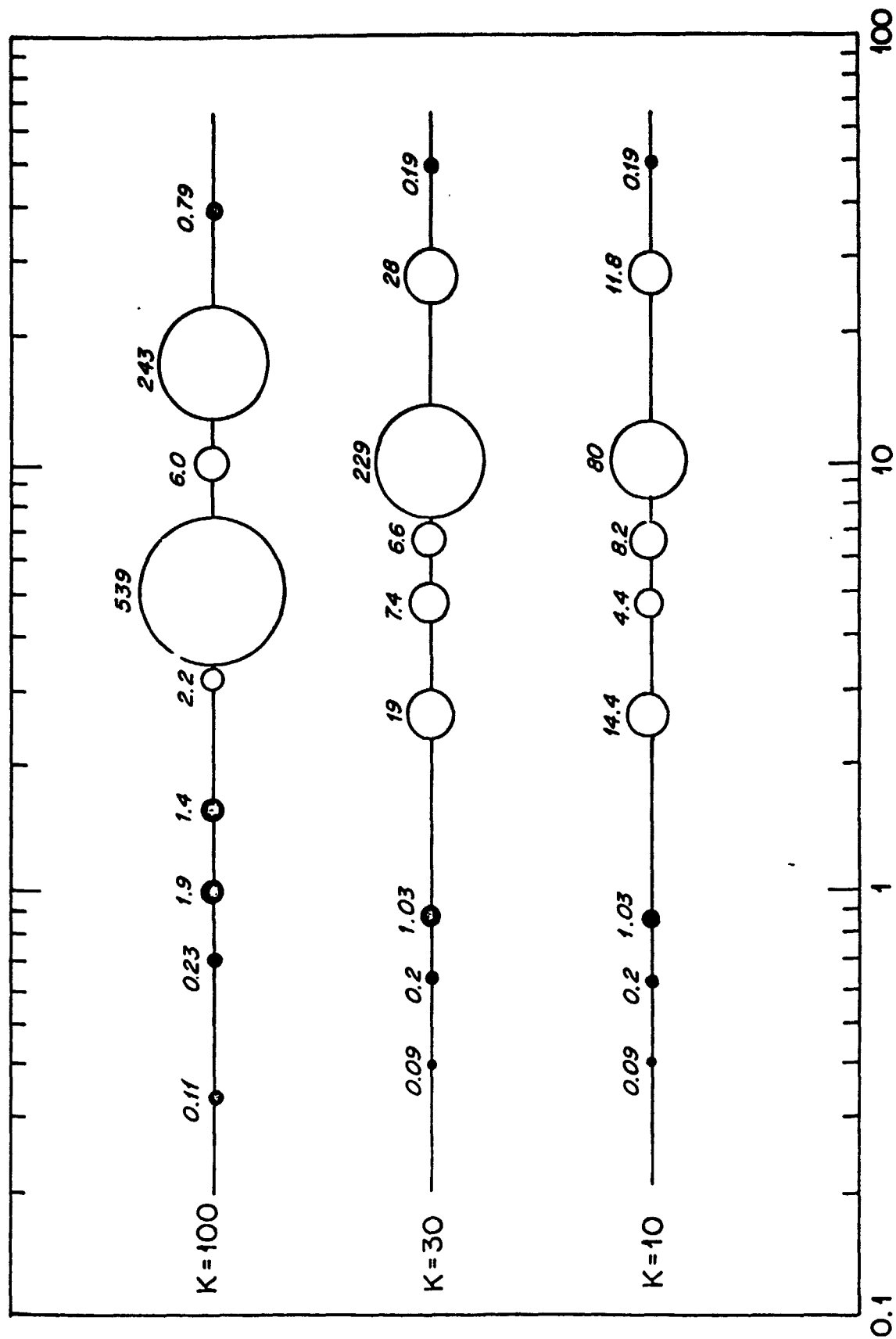


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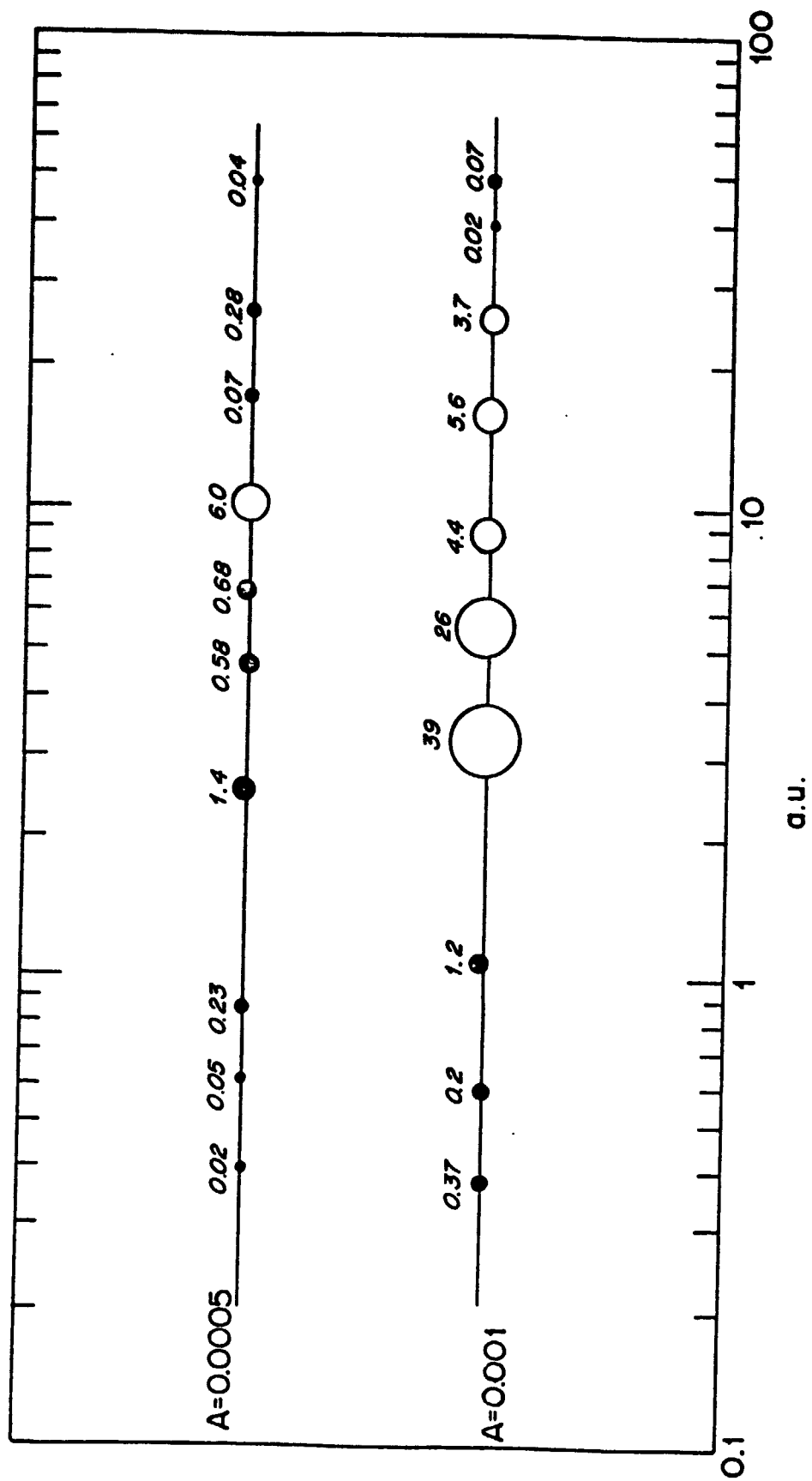


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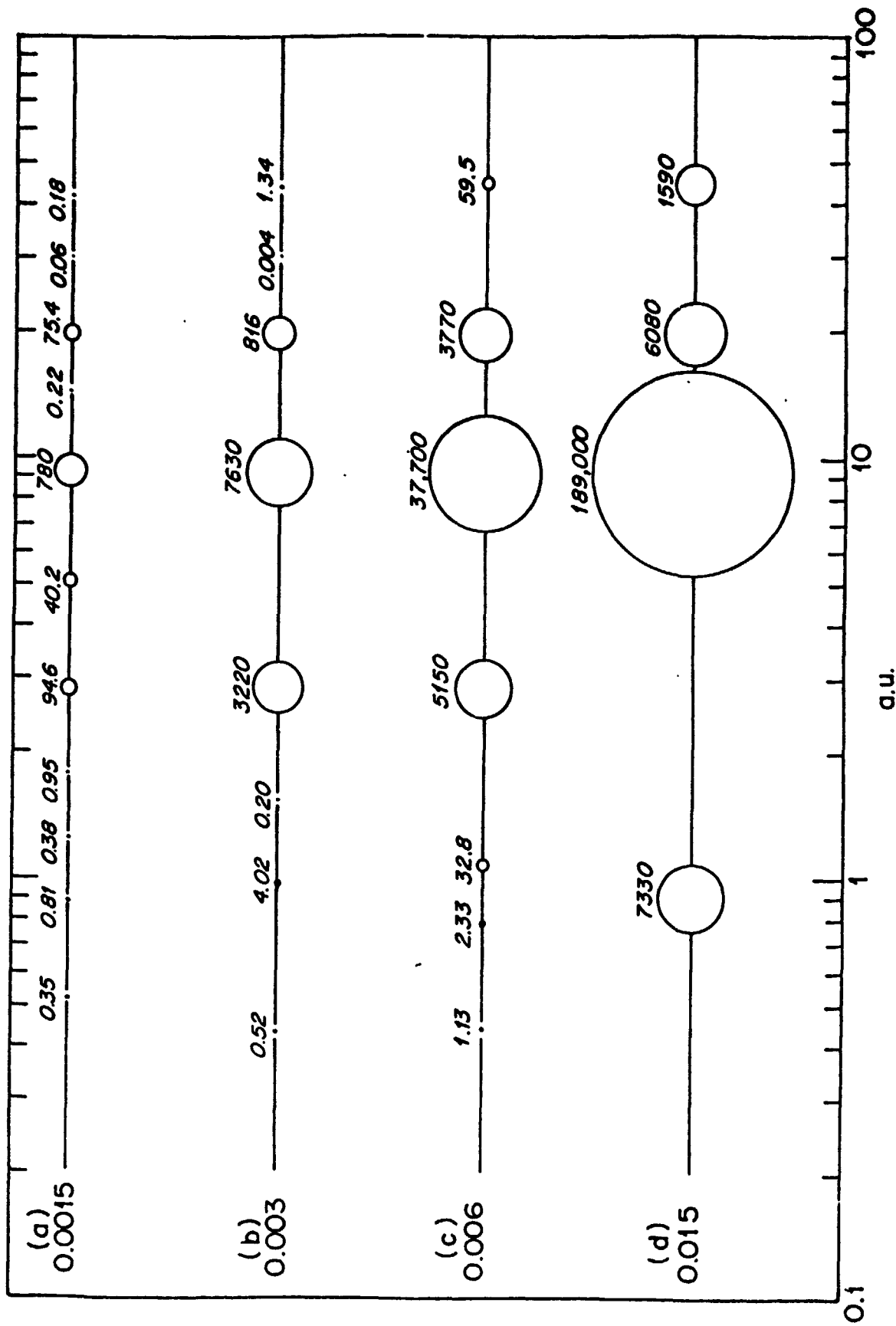


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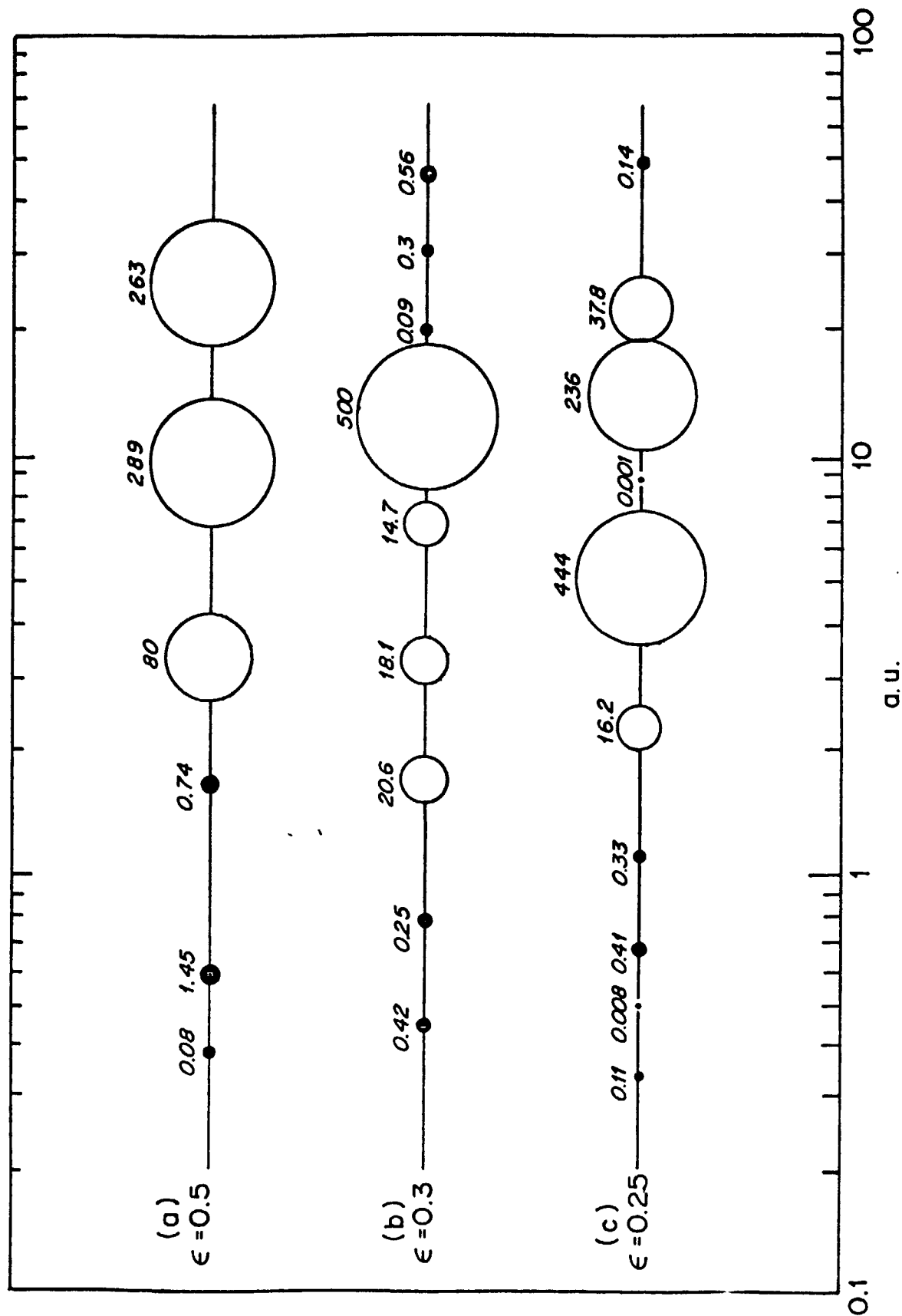


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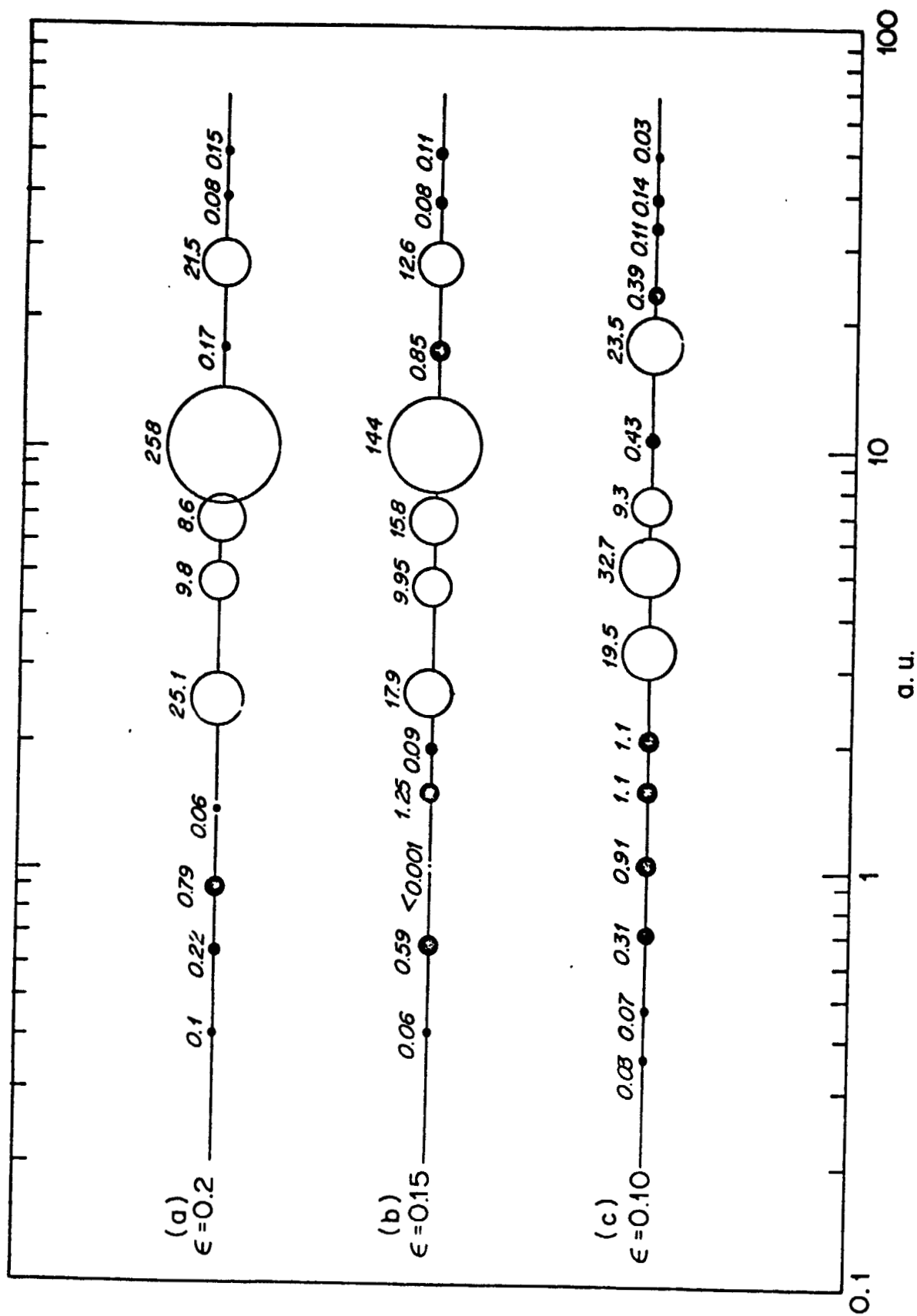
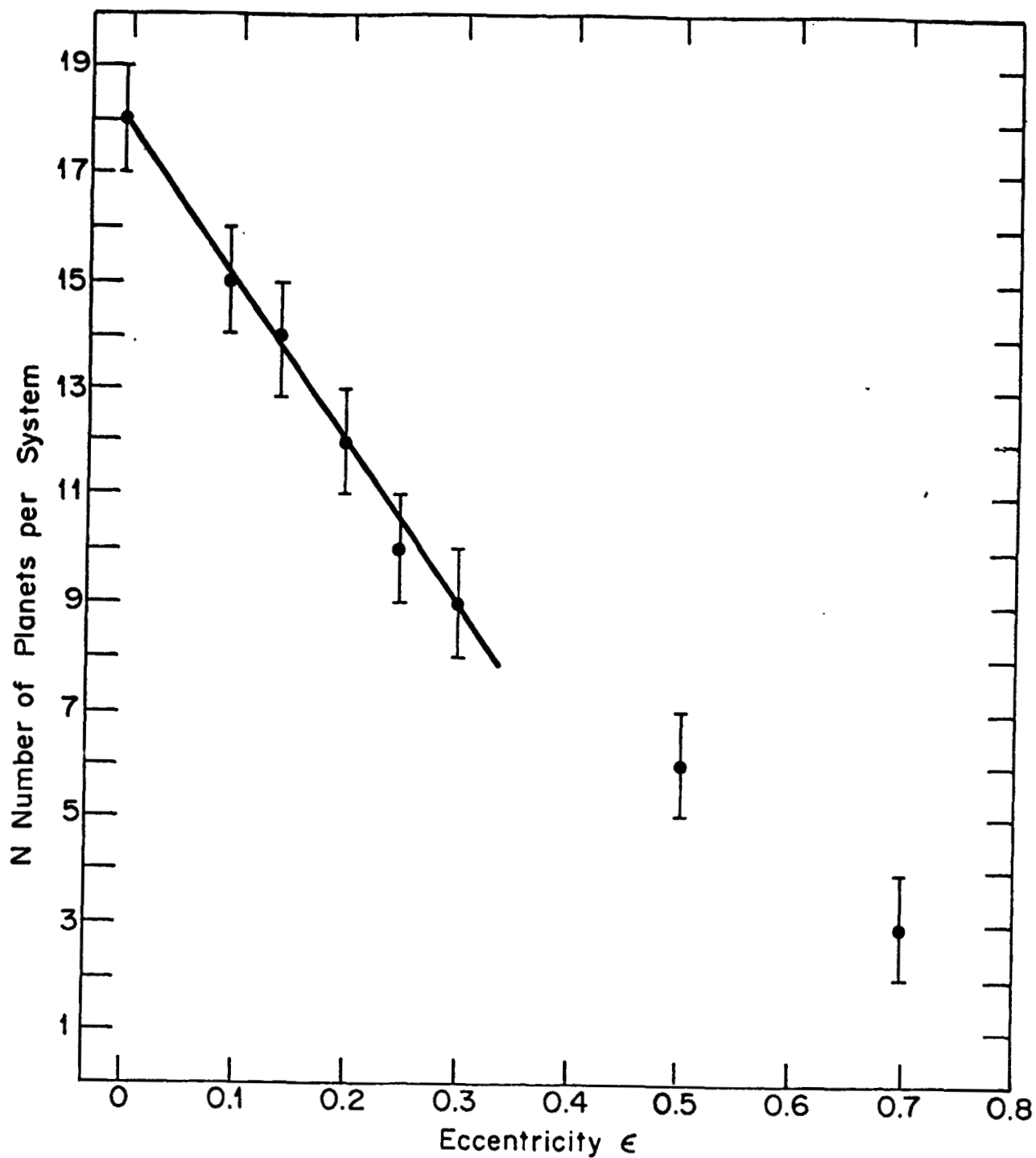


Figure 7



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Figure 8

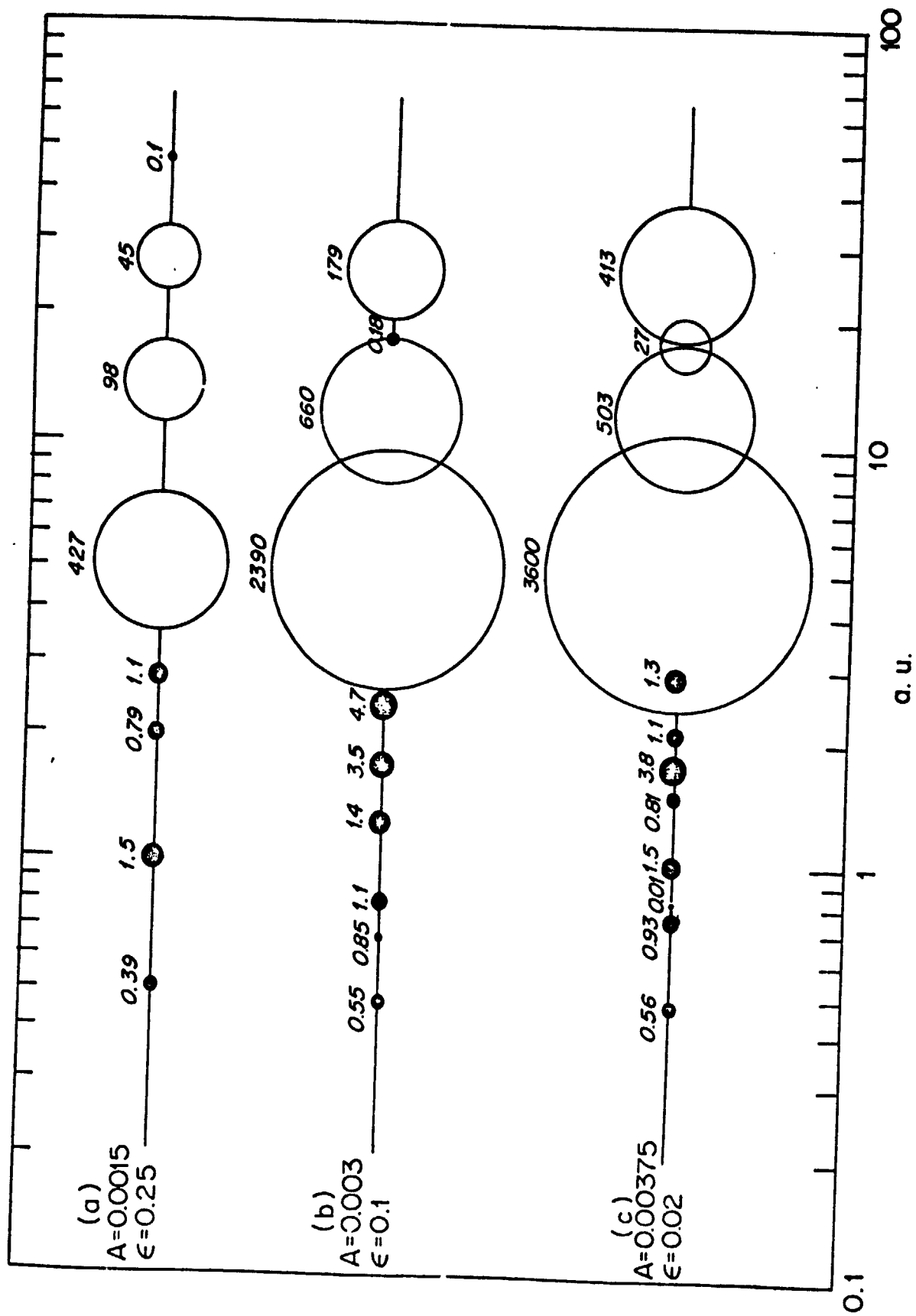


Figure 9

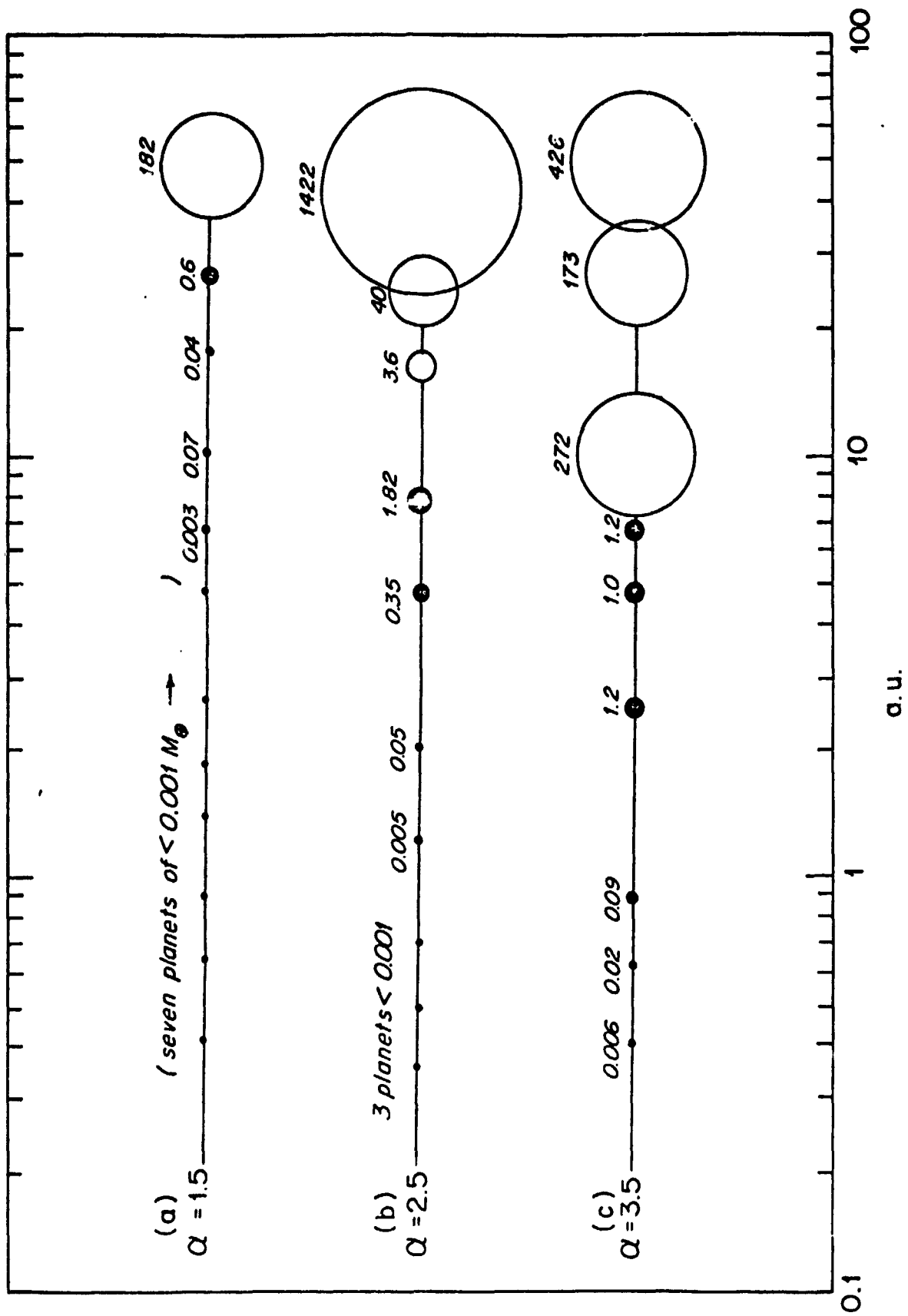


Figure 10 a b c

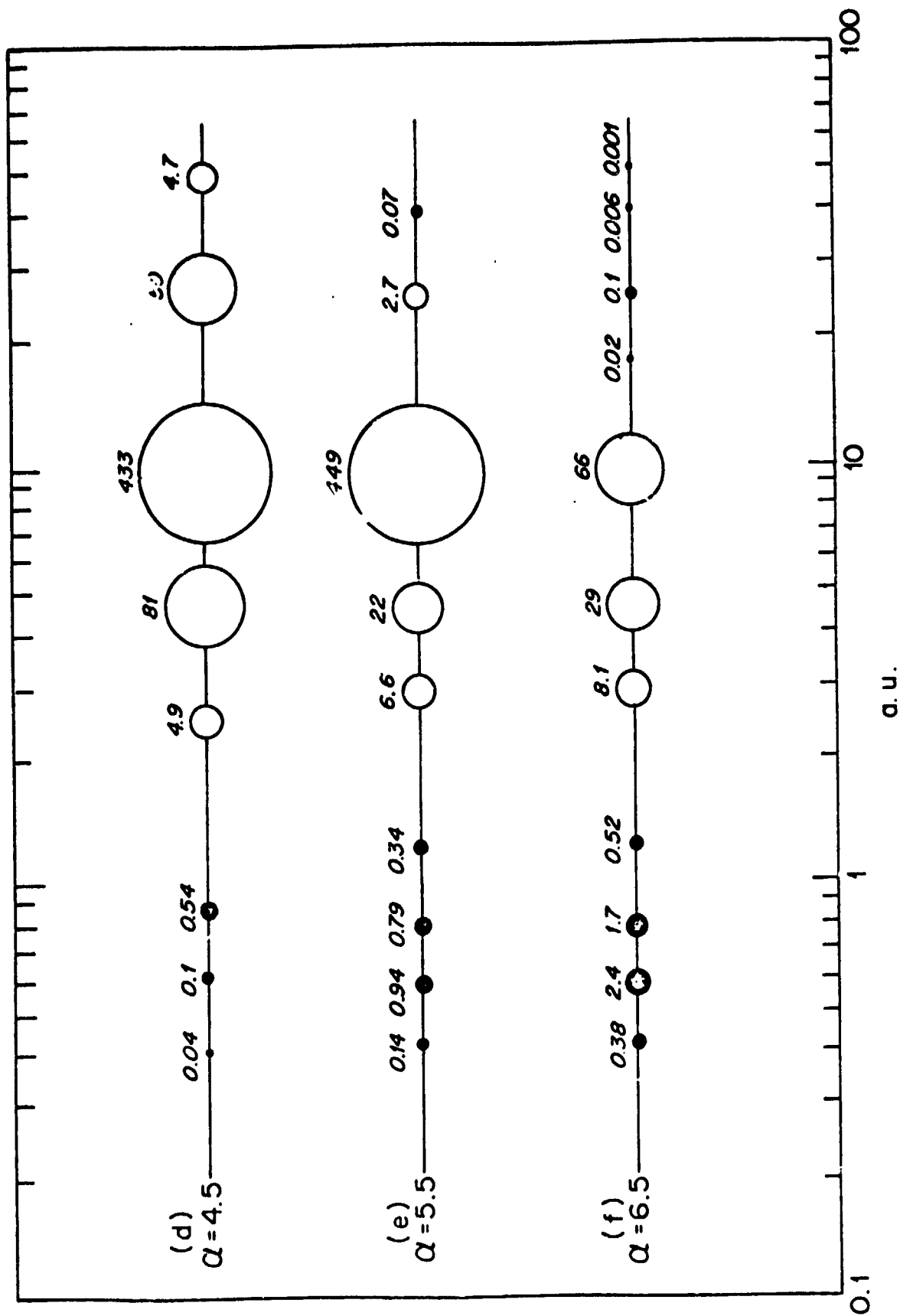


Figure 10 d e f

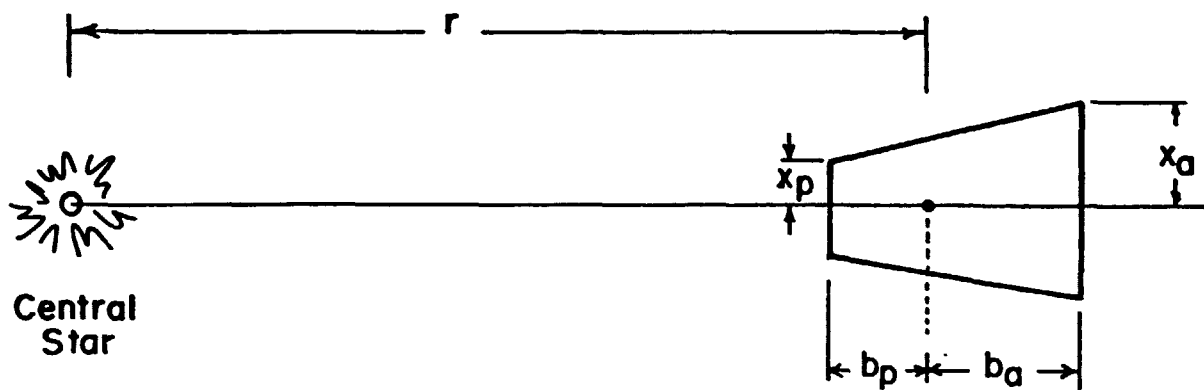


Figure 11



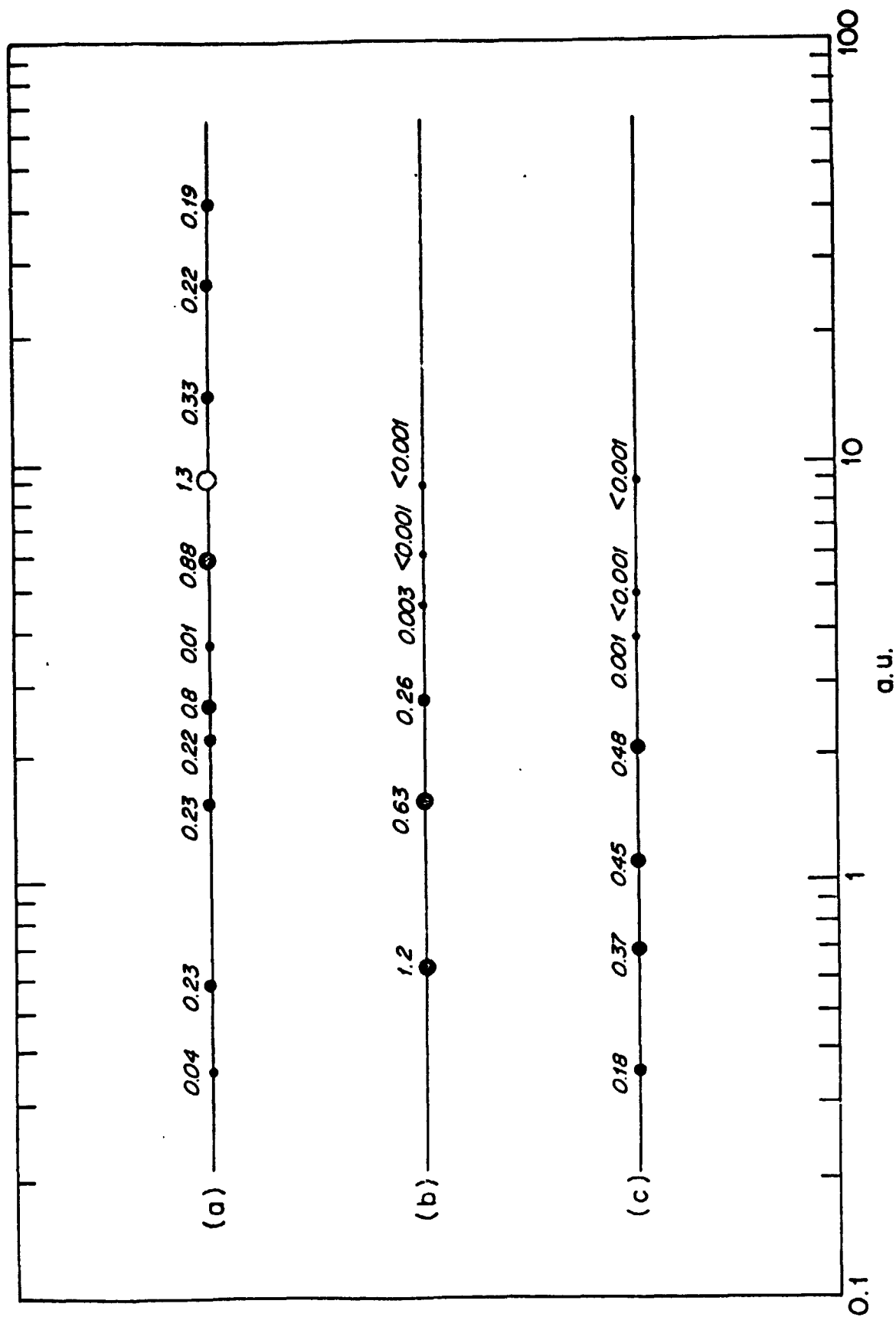


Figure 12

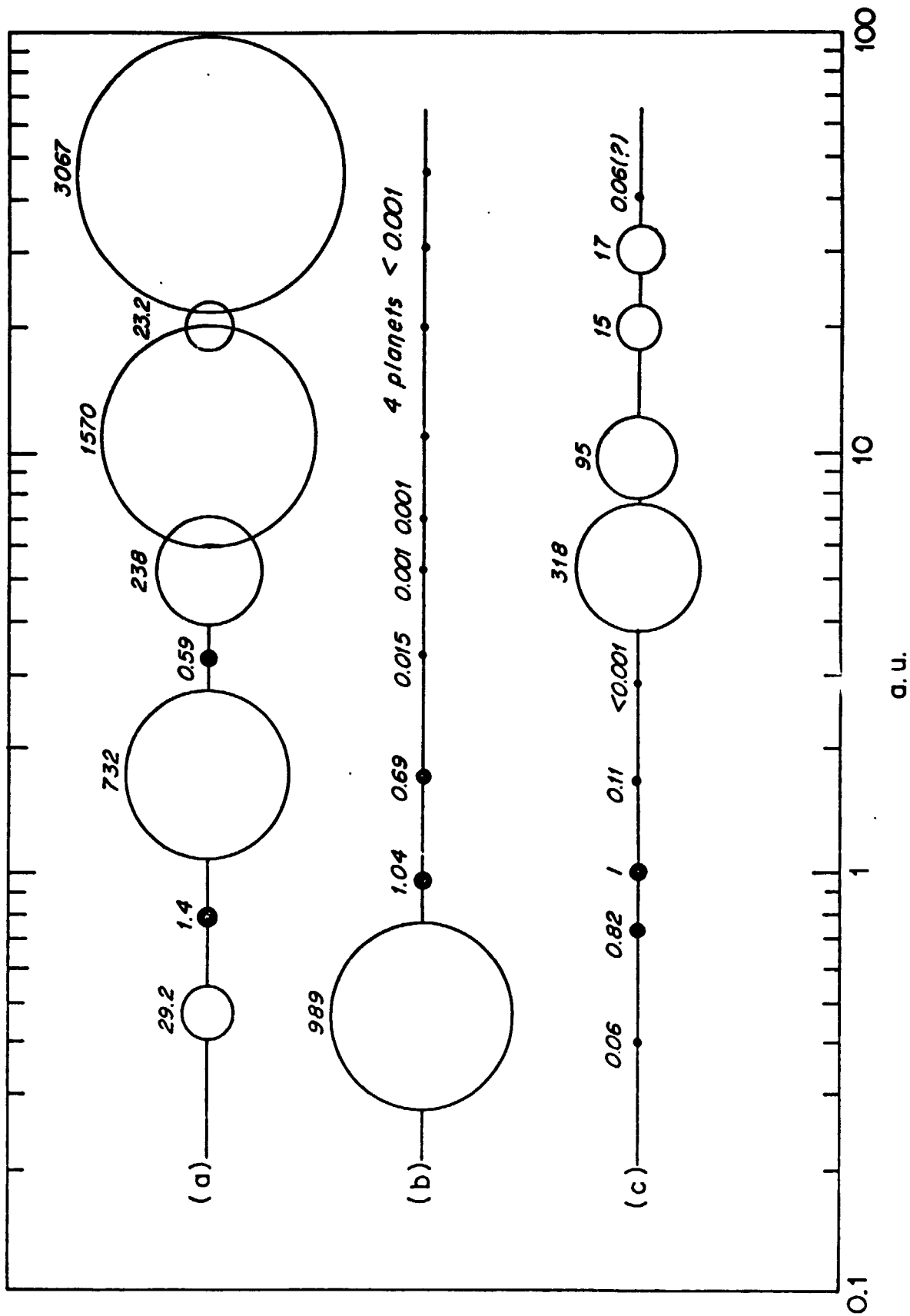


Figure 13