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above that it is necessary to investigate the character of the emulsion used and to determine the value of the constant ϵ. The work of Redman, quoted above, is of particular interest, as it shows that for a certain blue sensitive emulsion time effects can be ignored over a large range. For work in the blue region of the spectrum, provided high speed is not required, an emulsion such as this should, of course, be employed in preference to one for which the value of ϵ is appreciable. It would be a great boon to workers in spectrophotometry if a high-speed panchromatic emulsion possessing a similarly negligible curvature of the reciprocity failure curve could be discovered.

In this paper temperature variations have been ignored. The recent work of Webb * has shown that the shape of the reciprocity failure curves is effected by temperature, and that for very low temperatures of the order of $-75°$ C. the curves are practically linear (so that $\epsilon = 0$ for this temperature). It does not appear from Webb's results that for ordinary working temperatures serious error will be introduced by neglecting the variation of ϵ with temperature. The recent work, however, does emphasise the desirability of calibrating plates at approximately the same temperature as that at which they were exposed.

Royal Observatory, Greenwich :
1936 *September* 15.

THE INTERNAL CONSTITUTION OF THE PLANETS.

D. S. Kothari, Ph.D.

(Communicated by Professor E. A. Milne)

Introduction.—The present paper is concerned with the following problem : Is it possible from purely theoretical considerations (1) to deduce a relation connecting the radius and mass of a planet and (2) to predict the maximum radius for a *cold* † (stellar or planetary) body ? The maximum value for the radius, as estimated from observational data, is about one-tenth of the solar radius, a value about equal to the diameter of Jupiter.‡ The problem that we take up in this paper is the same as considered by Majumdar and the author in a recent paper § (referred to hereafter as KM I), but the method here employed—which is the same as that given in Article 1 of a

* *J. Opt. Soc. Amer.*, 25, 4, 1935.

† We are using here the word *cold* in the technical sense. Matter will be referred to as *cold* if its temperature and density are such that any *free* electrons present constitute a *degenerate* gas in the sense of Fermi-Dirac statistics.

‡ Attention to this very significant fact has been drawn by Professor Russell (as reported in *Observatory*, 58, 260, 1935). The present paper owes its origin to the above remarks of Professor Russell.

§ Kothari and Majumdar, *Nat. Acad. Sci. U.P. India*, 6, 57, 1936 ; also *Nature*, 137, 157, 1936.

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previous paper * on "White Dwarfs and Electro-static Correction"—is, I believe, simpler and more straightforward, and the final result (expressible by a simple formula) is in better accord with observation.

Our starting-point in KM I as well as the present paper is the usual theory of the white dwarf stars, but with one significant modification in that the degree of ionisation in degenerate matter is not treated as fixed, *i.e.* μ the mean molecular weight *per* free electron is not treated as independent of the mass of the star, but is regarded, as it must be, as a variable, and its dependence on the mass M of the configuration is taken account of. Thus we introduce no new hypothesis, but merely free the usual theory from a serious limitation and then work out its consequences. The theory in this form not only explains—as the usual theory does—the structure of the white dwarf stars, but also gives an *insight into the structure of the planets and predicts a maximum radius for a cold body.*

In KM I the dependence of μ on the mass of the configuration is evaluated by applying the theory of pressure-ionisation. In the case of non-degenerate matter the degree of ionisation is determined essentially by temperature and to some extent by pressure in accordance with Saha's formula. For degenerate matter the temperature concept is relegated to the background and the degree of ionisation is determined essentially by the pressure (or density), and for this reason it is spoken of as pressure-ionisation. The theory of pressure-ionisation has been discussed in previous papers † and also in KM I.

In this paper we make use of the *virial* theorem, and by including the electrostatic potential energy with the gravitational and total kinetic energy of the configuration automatically take account of the dependence of the degree of ionisation in degenerate matter on the mass of the configuration. In section 1 the relevant results of the usual white dwarf theory are summarised. In section 2 we derive the fundamental formula of this paper and discuss its applications in section 3.

1. In the case of the white dwarf stars the researches beginning with Fowler and later developed extensively by Milne and others have shown that the essential features of their internal constitution can be explained by an application of Fermi-Dirac statistics. A white dwarf star is characterised by a low luminosity, a high effective temperature and a very large mean density. In a preliminary theoretical treatment it is usual to assume the luminosity of a white dwarf to be zero, *i.e.* to regard it as a *black dwarf*—a term originally due to Fowler. This procedure is not as unreasonable as it may appear at first sight, for, as has been shown elsewhere,‡ the small luminosity of a white dwarf has little effect on the calculated values of its radius and mean density.

If we have a star of mass M which has ceased to radiate and is composed of ionised matter degenerate in the sense of Fermi-Dirac statistics, then, as has been shown by several authors, the radius R of the star is connected with

* Kothari, *Phil. Mag.*, **12**, 665, 1931.

† Kothari and Majumdar, *Astr. Nachr.*, **244**, 65, 1931 ; Swirles, *Proc. Roy. Soc.*, **141**, 561, 1933.

‡ Kothari, *M.N.*, **93**, 70, 1932.

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its mass by the relation * (neglecting relativity effects, this being justified so long as the mass is not much larger than the Sun)

$$R = \frac{5(\omega^0{}_{3/2})^{1/3}K}{2^{7/3}\pi^{2/3}G\mu^{5/3}} \frac{1}{M^{1/3}}$$

$$= 2\cdot 79 \times 10^9 \frac{1}{\mu^{5/3}} \left(\frac{\odot}{M}\right)^{1/3} \text{cm.}, \tag{1}$$

where $\odot$ is the mass of the Sun, μ is the mean molecular weight *per* free electron and $K/\mu^{5/3}$ denotes the "degenerate gas constant." † If ρ denotes the density of the material, then $\rho/\mu m_H$—m_H being the mass of the hydrogen atom—gives the number of *free electrons* per unit volume. It may be noted here that the usual definition of μ is average weight per particle, but as in degenerate gas the "heavy particle" is to be consistently ignored, we define μ as the molecular weight *per free electron*. We shall speak of it as the "electron molecular weight" to distinguish it from the usual definition.‡

If ρ_m denotes the mean density of the star, then from (1) we have

$$\rho_m = \frac{96\pi}{125(\omega^\circ{}_{3/2})} \left(\frac{G}{K}\right)^3 \mu^5 M^2 \tag{2}$$

$$= 2\cdot 14 \times 10^4 \mu^5 \left(\frac{M}{\odot}\right)^2 \text{gm./cm.}^3.$$

Further, if ρ_c denotes the central density, then ρ_c and ρ_m are related by the relation (given in Milne's notation)

$$\frac{\rho_m}{\rho_c} = 3(\omega^\circ{}_{3/2})/(\sigma_{3/2})^{3/2} = 1/5\cdot 99. \tag{3}$$

For any *fully ionised* element $\mu = A/Z$, where A is its atomic weight and Z its atomic number, and therefore μ is nearly 2 (*e.g.* for iron $A=56$, $Z=26$, $\mu=2\cdot 15$) except for ionised hydrogen, for which $\mu=1$. The value A/Z represents the minimum value for μ; in the case of partial ionisation of the element μ will be greater. In the case of hydrogen of abundance such that

* Milne, *M.N.*, **92**, 610, 1932. The notation used in (1) above is slightly different from that of Milne's paper in that our μ denotes the molecular weight on the chemical scale whereas Milne's μ is in grams, and therefore his μ equals our μm_H, m_H being the mass of the hydrogen atom. The "degenerate gas constant" $K/\mu^{5/3}$ used here is Milne's K. This is done to show explicitly the dependence on μ (see Kothari, *M.N.*, **93**, 70, 1932).

† The "degenerate gas constant" is defined by the relation :

$$\text{pressure of degenerate gas} = \frac{8\pi h^2}{15m}\left(\frac{3n}{8\pi}\right)^{5/3} = \frac{8\pi h^2}{15m}\left(\frac{3}{8\pi m_H}\right)^{5/3}\frac{\rho^{5/3}}{\mu^{5/3}} = \frac{K\rho^{5/3}}{\mu^{5/3}},$$

n in above is the number of *free* electrons per unit volume.

‡ This distinction between the two definitions for μ is often not stated explicitly and this may lead to some confusion. Thus Chandrasekhar in *M.N.*, **95**, 208, 1935, equation (3), implies by μ the molecular weight per free electron, but refers to it simply as molecular weight. Similarly μ in Fairclough's paper (*M.N.*, **95**, 585, 1935) implies the electron molecular weight.

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there are x atoms of hydrogen to one atom of the other element (following a usual practice we shall take the other element to be iron in numerical work) the value of μ for complete ionisation of the material will be given by

$$\mu = \frac{A+x}{Z+x} \sim 1 + \frac{Z}{Z+x}. \tag{4}$$

In the case of Sirius B, which has a mass of 0·85 $\odot$, the observed radius is about 22×10^8 cm. If we assume the stellar matter to be (on an average) composed of iron—and we shall speak of it as assumption F—then assuming complete ionisation ($\mu = 2 \cdot 15$) we get from (1) $R = 8 \cdot 2 \times 10^8$ cm. This value is smaller than the observed one by a factor of about 3. No doubt in formula (1) we assumed the luminosity to be zero, but, as already remarked, even if we took account of the luminosity, the calculated value of R will not be appreciably modified.* As will be seen later, in the case of the planets also the calculated radius on assumption F comes out smaller than the observed radius, and the discrepancy again is of a factor of about 3. However, the observed value of R can be made to agree with the calculated value by introducing the hypothesis of hydrogen abundance. Thus, to take the case of Sirius B, the observed R agrees with the calculated value if we take $\mu = 1 \cdot 2$, *i.e.* assume the stellar material to contain a large proportion of hydrogen (about 70 per cent. by weight).†

Equation (1), therefore, is in agreement with the observed mass-radius relation for white dwarfs, provided we make the two assumptions that (i) the stellar matter contains a good proportion of hydrogen, and (ii) the material is completely ionised. Assumption (i) is in harmony with recent researches on the constitution of stars in general; one cannot, of course, justify it on purely theoretical grounds. Assumption (ii) is, however, on a different footing. We should expect that in a complete theory of white dwarfs this result, instead of being assumed, should follow naturally from the theory. The theory developed in this paper is an attempt in this direction.

We have so far discussed the case of *cold* bodies of mass nearly that of the Sun.‡ We now ask the question as to what will happen for masses $M << \odot$.

* For the case of the model white dwarf $M = \frac{1}{2}\odot$, $L = 10^{31}$ erg/sec., $L/M = 10^{-2}$ erg/sec. grm. considered in *M.N.*, **93**, 70, 1932, R is found to be $1 \cdot 03 \left(\frac{2 \cdot 1}{\mu}\right)^{5/3} 10^9$ cm. If L be assumed zero, then R will have the value $1 \cdot 02 \left(\frac{2 \cdot 1}{\mu}\right)^{5/3} 10^9$ cm. The effect of luminosity in this case is to increase R by about 3 per cent. For Sirius B, $L = 10^{31}$ erg/sec., and the effect of luminosity on the calculated radius would in all probability be not much more than that for the model white dwarf.

† The hypothesis of hydrogen abundance has been applied to the white dwarf stars by several investigators, *e.g.* Mitra, *Zeits. für Astrophys.*, **4**, 329, 1932, and Strömgren, *Zeits. für Astrophys.*, **7**, 222, 1933. For o_2 Eridani B the proportion of hydrogen required is much less than for Sirius B.

‡ The white dwarf stars for which data exist so far have all masses near about that of the Sun—in no case is it less than about $\odot/5$.

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The relation (1) shows that if we assume μ to be constant (independent of M), R continues to increase as M decreases. But this inference stands in contradiction to observation, for in the case of planets which are *cold* bodies of mass much less than the Sun (their cores correspond, one may say, to ideal black dwarfs) a larger mass is associated, not with a smaller, but with a larger radius. Further, for any planet the observed radius is larger than the radius calculated from (1) for $\mu = 2\cdot15$ by a factor of the order of 100.* Thus, if we restrict ourselves to the assumption of μ being independent of M, then for $M << \odot$, equation (1) leads to absurd results. However, there is no *a priori* justification for this assumption, and as it leads to wrong results it must be discarded. In the next section we develop the theory so as to be free from the above restriction and work out its consequences.

2. Let us consider a collection of particles moving under their own mutual forces. If the force between any two particles varies inversely as the square of the distance between them, then it is a well-known result † (easily established classically and also wave-mechanically) that

$$2T + W = 0, \tag{5}$$

where T denotes the total kinetic energy of all the particles in the assembly and W is the total potential energy.

We shall apply this result to a *cold* spherical aggregate of mass M composed of matter of atomic weight A and atomic number Z. We proceed to estimate T and W for such a mass.

In estimating the total kinetic energy T we include all electrons,‡ bound as well as free. As is now well known from the investigations of Fermi, Thomas and others, the *bound* electrons in an atom can be treated as forming a degenerate gas, and as the matter is *cold* the free electrons (if they be present) will also be degenerate.§ We can assign to each atomic nucleus with its Z electrons (in general some of these electrons will be *bound* and the others *free*) an element of volume or spherical cell of radius a which can be expressed in terms of the density ρ by the relation

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* It is possible that one may feel it not quite reasonable to apply (1) to planets and to criticise it on the account that the predicted radius is wrong. It should be remarked that we nowhere assert whether (1) ought or ought not to apply to planets —we merely examine whether it does or does not. We find as a matter of fact that it does not apply when μ is taken independent of M, and this merely serves as an impetus for working out a theory which will be free from the above limitation regarding μ. The theory so developed, surprisingly enough, predicts all the essential features of planetary structure.

† This is called the *virial* theorem. For a proof from wave mechanics, see Frenkel, *Wave Mechanics*, vol. ii, p. 30. The general result for force varying as the nth power of the distance is $2T = (n+1)W$. The sign of the force (attraction or repulsion) is immaterial and it may vary from particle to particle. The state of zero potential energy is taken to be that when the particles constituting the assembly are all dispersed to infinity.

‡ For *cold* matter the total kinetic energy is almost entirely due to electrons, the nuclei making negligible contribution.

§ If the matter was not *cold* the free electrons would be non-degenerate and their kinetic energy determined by the temperature.

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$$\frac{a^3}{\gamma}\frac{\rho}{Am_H}=1, \tag{6}$$

where m_H is the mass of the hydrogen atom and γ is a factor of the order unity. It will lead to no serious error in estimating the kinetic energy if we assume the Z electrons as uniformly distributed in the cell.* The usual formula for the kinetic energy of a degenerate electron gas is

$$E_0=\frac{3}{10}\frac{h^2}{m}\left(\frac{3n}{8\pi}\right)^{2/3}N, \tag{7}$$

where n is the electron concentration, *i.e.* number of electrons per unit volume, and N is the total number of electrons. Substituting Z for N and $\frac{Z\gamma}{a^3}$ for n, we obtain for the kinetic energy per cell T' the expression

$$T'=\frac{3}{10}\frac{h^2}{m}\left(\frac{3}{8\pi}\frac{Z\gamma}{a^3}\right)^{2/3}Z. \tag{8}$$

Substituting for $\frac{\gamma}{a^3}$ from (6) we have

$$T'=\frac{3}{10}\frac{h^2}{m}\left(\frac{3}{8\pi}\frac{Z\rho}{Am_H}\right)^{2/3}Z. \tag{9}$$

In estimating to a first approximation the value of T we may neglect the variation of density from the boundary towards the centre of the configuration and take for ρ in (9) the mean density ρ_m of the configuration,†

$$\rho_m=M\Big/\frac{4\pi}{3}R^3, \tag{10}$$

where R is the radius of the configuration. Multiplying T' by the number of cells in mass M which is M/Am_H, we obtain for the total kinetic energy T

$$\begin{aligned} T&=\frac{3}{10}\frac{h^2}{m}\left(\frac{3}{8\pi}\frac{Z\rho_m}{Am_H}\right)^{2/3}\frac{M}{Am_H}Z \\ &=\beta\left(\frac{Z}{A}\right)^{5/3}\frac{1}{R^2}M^{5/3}, \end{aligned} \tag{11}$$

where

$$\begin{aligned} \beta&=\frac{3}{5}\frac{1}{2^{1/3}}\left(\frac{3}{8\pi}\right)^{4/3}\frac{h^2}{m}\frac{1}{m_H{}^{5/3}} \\ &=5{\cdot}71\ 10^{12}. \end{aligned} \tag{12}$$

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* See, for example, Feinberg, *Physikalische Zeit. der sowjetunion*, **8**, 409, 1935.

† The density distribution will be very nearly polytropic with index 3/2. For this polytrope the mean density is 1/6th the central density, and except close to the boundary there is no wild departure between the actual and the mean density. If necessary the variation in density could be taken account of by a simple numerical integration (see Kothari, *Phil. Mag.*, *loc. cit.*, § 5). However, it will not affect essentially the results presented here.

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The potential energy W of the configuration is the sum of the electrostatic energy W_S and the gravitational energy W_G. It is no simple matter to calculate W_S accurately, but an approximate value for it can easily be obtained. The total electrostatic energy of the configuration will be obtained by multiplying the electrostatic energy of a single cell by the number of cells. Assuming that the Z electrons are uniformly distributed in the cell of radius a, the cell having at its centre a nucleus of charge $+Ze$, the potential $U(r)$ at any point inside the cell and distant r from its centre will be, after the cell has been built up to radius r,

$$U(r) = (Ze - \tfrac{4}{3}\pi r^3 ne)\frac{1}{r}, \tag{13}$$

where n is the electron concentration.

Therefore the potential energy W_S' of the cell will be given by

$$\begin{aligned} -W_S' &= \int_0^a \frac{(Ze - \frac{4}{3}\pi r^3 ne)}{r} \cdot 4\pi r^2 ne\,dr \\ &= \frac{9}{10}\frac{Z^2e^2}{a} \end{aligned} \tag{14}$$

and hence

$$-W_S = -W_S'\frac{M}{Am_H} = \frac{9}{10}\frac{Z^2e^2}{a}\frac{M}{Am_H}. \tag{15}$$

Substituting for a from (6) and taking γ equal to unity * we have

$$\begin{aligned} -W_S &= \tfrac{9}{10}Z^2e^2 \cdot \frac{\rho_m^{1/3}}{(Am_H)^{4/3}}M \\ &= \alpha_1\frac{M^{4/3}}{R}\frac{Z^2}{A^{4/3}}, \end{aligned} \tag{16}$$

where

$$\alpha_1 = \tfrac{9}{10}\left(\frac{3}{4\pi}\right)^{1/3}\frac{e^2}{m_H^{4/3}} = 6{\cdot}47\ 10^{12}. \tag{17}$$

In calculating both T and W_S we have neglected the effect of the variation of density inside the configuration and have treated it as if the density were uniform and equal to the mean density ρ_m. The gravitational energy of a sphere of mass M and uniform density is easily shown to be

$$-W_G = \tfrac{3}{5}G\frac{M^2}{R} = \alpha_2\frac{M^2}{R}, \tag{18}$$

where

$$\alpha_2 = \tfrac{3}{5}G = 4{\cdot}00\ 10^{-8}. \tag{19}$$

Substituting the values for T, W_S and W_G in (5), which expresses the virial theorem, we have

$$2\beta\left(\frac{Z}{A}\right)^{5/3}\frac{M^{5/3}}{R^2} = \alpha_1\frac{Z^2}{A^{4/3}}\frac{M^{4/3}}{R} + \alpha_2\frac{M^2}{R}. \tag{20}$$

* The exact value of γ depends on a number of factors—particularly the lattice arrangement of the atoms—but in any case it will be of the order unity.

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From this we obtain for R

$$R = \frac{2\beta\left(\frac{Z}{A}\right)^{5/3} M^{5/3}}{\alpha_1 \frac{Z^2}{A^{4/3}} M^{4/3} + \alpha_2 M^2} \tag{21}$$

$$= \frac{\frac{2\beta}{\alpha_2}\left(\frac{Z}{A}\right)^{5/3} \frac{1}{\odot^{1/3}}}{1 + \frac{\alpha_1}{\alpha_2}\frac{Z^2}{A^{4/3}}\frac{1}{\odot^{2/3}}\left(\frac{\odot}{M}\right)^{2/3}}\left(\frac{\odot}{M}\right)^{1/3}, \tag{22}$$

or

$$R = \frac{\frac{2\beta}{\alpha_1}\frac{1}{(ZA)^{1/3}}\odot^{1/3}}{1 + \frac{\alpha_2}{\alpha_1}\frac{A^{4/3}}{Z^2}\odot^{2/3}\left(\frac{M}{\odot}\right)^{2/3}}\left(\frac{M}{\odot}\right)^{1/3}. \tag{23}$$

We have obtained a relation connecting the radius and mass of the configuration. Equation (23) is the fundamental equation of the present paper. *It shows that as* M *increases from zero upwards,* R *also increases from zero upwards, attains a maximum value and then begins to decrease, and finally vanishes as* M $\rightarrow \infty$.

Differentiating R with respect to M we find that—

(i) the radius R has its maximum value for $M = M_0$, where M_0 is given by

$$M_0 = \left(\frac{\alpha_1}{\alpha_2}\right)^{3/2}\frac{Z^3}{A^2} \tag{24}$$

or, substituting the numerical values of α_1 and α_2,

$$M_0 \sim 1{\cdot}04\ 10^{-3}\frac{Z^3}{A^2}\odot, \tag{25}$$

and (ii) this maximum value of the radius is

$$R_{\max} = \frac{\beta}{\sqrt{\alpha_1\alpha_2}}\frac{Z^{2/3}}{A} \sim 1{\cdot}12\ 10^{10}\frac{Z^{2/3}}{A}\text{cm.} \tag{26}$$

Expressing equation (21) in terms of $R_{\max}$ and M_0 we have

$$\frac{R}{R_{\max}} = \frac{2(M/M_0)^{1/3}}{1 + (M/M_0)^{2/3}}. \tag{26'}$$

When $M >> M_0$, equation (23) approximates to

$$R \sim \frac{2\beta}{\alpha_2}\left(\frac{Z}{A}\right)^{5/3}\frac{1}{M^{1/3}} = 2{\cdot}28\ 10^{9}\left(\frac{Z}{A}\right)^{5/3}\left(\frac{\odot}{M}\right)^{1/3}\text{cm.} \tag{27}$$

and for $M << M_0$ it approximates to

$$R \sim \frac{2\beta}{\alpha_1}\frac{1}{(ZA)^{1/3}}M^{1/3} = 2{\cdot}22\ 10^{11}\left(\frac{1}{ZA}\right)^{1/3}\left(\frac{M}{\odot}\right)^{1/3}\text{cm.} \tag{28}$$

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We notice at once that equation (27) is just the same as equation (1) of the usual white dwarf theory, except that μ in (1) is now replaced by Z/A and the numerical factor in (1) is a little larger than in (27). The small difference in the numerical factors is due to the fact that (1) is deduced for a polytropic distribution of density, whereas in deriving (27) the density has been assumed uniform. The fact that our theory predicts for μ in (1) the value Z/A shows that for configurations of mass $M >> M_0$—and the known white dwarfs satisfy this condition—the stellar material is completely ionised. *In the usual white dwarf theory this is taken as an assumption; here it follows naturally from the theory.* The second case of $M << M_0$ corresponds to planetary masses. A larger mass in this case is associated with a larger radius.

For $M > M_0$ gravitational energy predominates over the electrostatic energy, whereas for $M < M_0$ the opposite is the case.*

3. Before we can compute numerical results and compare them with observation we must make some assumption about the chemical composition of the material, *i.e.* assume some reasonable values for A and Z. The numerical results given here are worked out on two alternative assumptions :

(i) *Assumption F.*—The material is assumed to be on an average iron with $Z = 26$ and $A = 55 \cdot 8$.

(ii) *Assumption H.*—The material is assumed to be hydrogen with $Z = 1$ and $A = 1$.

It is found that for a given mass M the calculated radius on assumption F is smaller than the observed value, whereas on assumption H the calculated radius is larger.

Fig. 1 exhibits the results of our calculations. The full-line curves which bound the shaded region on the upper and lower sides represent the theoretical mass-radius relation given by equation (23). The upper curve corresponds to the case when the material is assumed to be all hydrogen, the lower curve is for iron. The observed (M, R) values for the planets and the white dwarf stars are indicated in the figure. It is to be particularly noted that all the observed (M, R) values, except for the white dwarf Procyon B, fall within the region between the *H* and *Fe* curves. This is indeed a satisfactory result. Roughly speaking, assumption H gives for the lighter planets a radius about 4 times the observed value, and on assumption F the calculated radius is about 3 times too small. For the heavier planets, Jupiter and Saturn, assumption H gives a value about $1\frac{1}{2}$ times too large and assumption F a value 3 times too small compared to the observed value. It appears therefore possible that by taking a suitable proportion of hydrogen and iron, *i.e.* by a suitable choice regarding the chemical composition of the material, the theoretical value could be made to fit with the observed value,

* In my 1931 paper on the white dwarfs and the electrostatic correction (*loc. cit.*) I observed that the electrostatic correction becomes predominant for any white dwarf, if such exists, with $M < \odot/10$. However, I had entirely overlooked the applicability of these results to the case of planets, it never having occurred to me that a planet was an (ideal) "black dwarf."

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and we could derive information about the average chemical composition of the planetary material.* However, we shall not attempt here the calculations for a mixture of elements, as in this case the evaluation of the electrostatic energy proves to be a difficult matter. All the same the theory does bring out the essential points that for white dwarfs a smaller mass means a

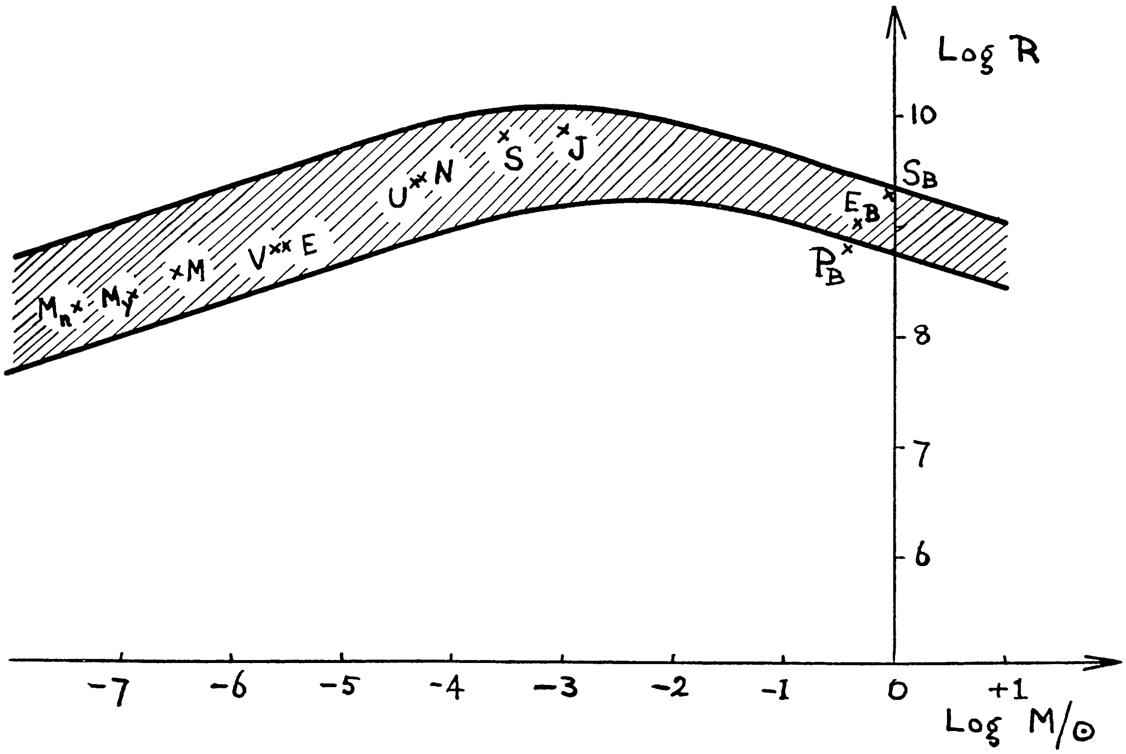

FIG. 1.—*The full-line curves bounding the shaded region are theoretical. The upper curve is for Hydrogen and the lower for Iron.*

M_n = *Moon.*	E = *Earth.*	J = *Jupiter.*
M_y = *Mercury.*	U = *Uranus.*	S_B = *Sirius B.*
M = *Mars.*	N = *Neptune.*	E_B = o_2 *Eridani B.*
V = *Venus.*	S = *Saturn.*	P_B = *Procyon B.*

larger radius, whereas for the planetary bodies a smaller mass corresponds to a smaller radius; *and that there exists a maximum radius for a "cold" body.* The values of R (maximum) and M_0 are at once found from (26) and (25), and are given in the following table. The values under the heading "Observed" have been roughly estimated from the run of the observed (M, R) values.

The observed values lie between the two theoretical values. The value

* As the planets have in all probability been formed out of the Sun, a reasonable assumption about their chemical composition would be to take it to be the same as assumed for the Sun in theories dealing with its internal constitution. The non-hydrogen elements would be represented by the Russell mixture and the proportion of hydrogen would be about one-third by weight (Strömgren, *Zeits. für Astrophysik*, 4, 118, 1932; 7, 322, 1933). Preliminary calculations seem to indicate that the theoretical (M, R) value in this case would be in fairly good accord with observation.

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