# 5. Linear independence

#### **Outline**

Linear independence

Basis

Orthonormal vectors

Gram-Schmidt algorithm

#### Linear dependence

▶ set of *n*-vectors  $\{a_1, \ldots, a_k\}$  (with  $k \ge 1$ ) is *linearly dependent* if

$$\beta_1 a_1 + \cdots + \beta_k a_k = 0$$

holds for some  $\beta_1, \ldots, \beta_k$ , that are not all zero

- equivalent to: at least one  $a_i$  is a linear combination of the others
- we say ' $a_1, \ldots, a_k$  are linearly dependent'
- $\{a_1\}$  is linearly dependent only if  $a_1 = 0$
- $\{a_1, a_2\}$  is linearly dependent only if one  $a_i$  is a multiple of the other
- for more than two vectors, there is no simple to state condition

# **Example**

the vectors

$$a_1 = \begin{bmatrix} 0.2 \\ -7 \\ 8.6 \end{bmatrix}, \qquad a_2 = \begin{bmatrix} -0.1 \\ 2 \\ -1 \end{bmatrix}, \qquad a_3 = \begin{bmatrix} 0 \\ -1 \\ 2.2 \end{bmatrix}$$

are linearly dependent, since  $a_1 + 2a_2 - 3a_3 = 0$ 

can express any of them as linear combination of the other two, e.g.,

$$a_2 = (-1/2)a_1 + (3/2)a_3$$

# Linear independence

▶ set of n-vectors  $\{a_1, \ldots, a_k\}$  (with  $k \ge 1$ ) is *linearly independent* if it is not linearly dependent, *i.e.*,

$$\beta_1 a_1 + \cdots + \beta_k a_k = 0$$

holds only when  $\beta_1 = \cdots = \beta_k = 0$ 

- we say ' $a_1, \ldots, a_k$  are linearly independent'
- ightharpoonup equivalent to: no  $a_i$  is a linear combination of the others

• example: the unit *n*-vectors  $e_1, \ldots, e_n$  are linearly independent

# Linear combinations of linearly independent vectors

▶ suppose x is linear combination of linearly independent vectors  $a_1, \ldots, a_k$ :

$$x = \beta_1 a_1 + \cdots + \beta_k a_k$$

▶ the coefficients  $\beta_1, \ldots, \beta_k$  are *unique*, *i.e.*, if

$$x = \gamma_1 a_1 + \cdots + \gamma_k a_k$$

then  $\beta_i = \gamma_i$  for  $i = 1, \dots, k$ 

- this means that (in principle) we can deduce the coefficients from x
- to see why, note that

$$(\beta_1 - \gamma_1)a_1 + \dots + (\beta_k - \gamma_k)a_k = 0$$

and so (by linear independence)  $\beta_1 - \gamma_1 = \cdots = \beta_k - \gamma_k = 0$ 

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#### Independence-dimension inequality

- ▶ a linearly independent set of n-vectors can have at most n elements
- ▶ put another way: any set of n + 1 or more n-vectors is linearly dependent

#### **Basis**

- ▶ a set of n linearly independent n-vectors  $a_1, \ldots, a_n$  is called a *basis*
- ▶ any *n*-vector *b* can be expressed as a linear combination of them:

$$b = \beta_1 a_1 + \dots + \beta_n a_n$$

for some  $\beta_1, \ldots, \beta_n$ 

- and these coefficients are unique
- formula above is called *expansion of b in the*  $a_1, \ldots, a_n$  *basis*
- ightharpoonup example:  $e_1, \ldots, e_n$  is a basis, expansion of b is

$$b = b_1 e_1 + \dots + b_n e_n$$

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#### **Orthonormal vectors**

- ▶ set of *n*-vectors  $a_1, \ldots, a_k$  are (mutually) orthogonal if  $a_i \perp a_j$  for  $i \neq j$
- they are *normalized* if  $||a_i|| = 1$  for i = 1, ..., k
- they are orthonormal if both hold
- can be expressed using inner products as

$$a_i^T a_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

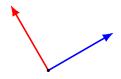
- orthonormal sets of vectors are linearly independent
- ▶ by independence-dimension inequality, must have  $k \le n$
- when  $k = n, a_1, \dots, a_n$  are an *orthonormal basis*

# **Examples of orthonormal bases**

- standard unit *n*-vectors  $e_1, \ldots, e_n$
- the 3-vectors

$$\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

the 2-vectors shown below



# **Orthonormal expansion**

• if  $a_1, \ldots, a_n$  is an orthonormal basis, we have for any *n*-vector x

$$x = (a_1^T x)a_1 + \dots + (a_n^T x)a_n$$

- called orthonormal expansion of x (in the orthonormal basis)
- ▶ to verify formula, take inner product of both sides with a<sub>i</sub>

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#### Gram-Schmidt (orthogonalization) algorithm

- an algorithm to check if  $a_1, \ldots, a_k$  are linearly independent
- we'll see later it has many other uses

# **Gram-Schmidt algorithm**

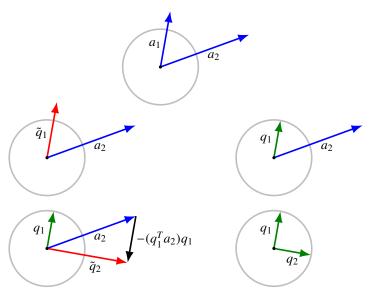
**given** 
$$n$$
-vectors  $a_1, \ldots, a_k$ 

for 
$$i = 1, \ldots, k$$

- 1. Orthogonalization:  $\tilde{q}_i = a_i (q_1^T a_i)q_1 \cdots (q_{i-1}^T a_i)q_{i-1}$
- 2. Test for linear dependence: if  $\tilde{q}_i = 0$ , quit
- 3. Normalization:  $q_i = \tilde{q}_i / \|\tilde{q}_i\|$

- ▶ if G–S does not stop early (in step 2),  $a_1, \ldots, a_k$  are linearly independent
- ▶ if G–S stops early in iteration i = j, then  $a_j$  is a linear combination of  $a_1, \ldots, a_{j-1}$  (so  $a_1, \ldots, a_k$  are linearly dependent)

# **Example**



Boyd & Vandenberghe

#### **Analysis**

let's show by induction that  $q_1, \ldots, q_i$  are orthonormal

- ▶ assume it's true for i − 1
- orthogonalization step ensures that

$$\tilde{q}_i \perp q_1, \ldots, \tilde{q}_i \perp q_{i-1}$$

ightharpoonup to see this, take inner product of both sides with  $q_j, j < i$ 

$$q_j^T \tilde{q}_i = q_j^T a_i - (q_1^T a_i)(q_j^T q_1) - \dots - (q_{i-1}^T a_i)(q_j^T q_{i-1})$$
  
=  $q_j^T a_i - q_j^T a_i = 0$ 

- ightharpoonup so  $q_i \perp q_1, \ldots, q_i \perp q_{i-1}$
- ▶ normalization step ensures that  $||q_i|| = 1$

#### **Analysis**

assuming G-S has not terminated before iteration i

•  $a_i$  is a linear combination of  $q_1, \ldots, q_i$ :

$$a_i = \|\tilde{q}_i\|q_i + (q_1^T a_i)q_1 + \dots + (q_{i-1}^T a_i)q_{i-1}$$

•  $q_i$  is a linear combination of  $a_1, \ldots, a_i$ : by induction on i,

$$q_i = (1/||\tilde{q}_i||) \left( a_i - (q_1^T a_i) q_1 - \dots - (q_{i-1}^T a_i) q_{i-1} \right)$$

and (by induction assumption) each  $q_1,\ldots,q_{i-1}$  is a linear combination of  $a_1,\ldots,a_{i-1}$ 

# **Early termination**

#### suppose G-S terminates in step j

•  $a_i$  is linear combination of  $q_1, \ldots, q_{i-1}$ 

$$a_j = (q_1^T a_j)q_1 + \dots + (q_{j-1}^T a_j)q_{j-1}$$

- ▶ and each of  $q_1, ..., q_{j-1}$  is linear combination of  $a_1, ..., a_{j-1}$
- ▶ so  $a_j$  is a linear combination of  $a_1, \ldots, a_{j-1}$

# Complexity of Gram-Schmidt algorithm

▶ step 1 of iteration i requires i - 1 inner products,

$$q_1^T a_i, \dots, q_{i-1}^T a_i$$

which costs (i-1)(2n-1) flops

- ▶ 2n(i-1) flops to compute  $\tilde{q}_i$
- ▶ 3*n* flops to compute  $\|\tilde{q}_i\|$  and  $q_i$
- total is

$$\sum_{i=1}^{k} ((4n-1)(i-1) + 3n) = (4n-1)\frac{k(k-1)}{2} + 3nk \approx 2nk^2$$

using 
$$\sum_{i=1}^{k} (i-1) = k(k-1)/2$$