2. Linear functions

Outline

Linear and affine functions

Taylor approximation

Regression model

Superposition and linear functions

- $f : \mathbf{R}^n \to \mathbf{R}$ means f is a function mapping n-vectors to numbers
- f satisfies the *superposition property* if

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

  holds for all numbers α, β, and all n-vectors x, y
- be sure to parse this very carefully!
- a function that satisfies superposition is called *linear*

The inner product function

- with a an n-vector, the function

$$f(x) = a^T x = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

  is the *inner product function*

- $f(x)$ is a weighted sum of the entries of x
- the inner product function is linear:

$$\begin{aligned} f(\alpha x + \beta y) &= a^T(\alpha x + \beta y) \\ &= a^T(\alpha x) + a^T(\beta y) \\ &= \alpha(a^T x) + \beta(a^T y) \\ &= \alpha f(x) + \beta f(y) \end{aligned}$$

…and all linear functions are inner products

- suppose $f : \mathbf{R}^n \to \mathbf{R}$ is linear
- then it can be expressed as $f(x) = a^T x$ for some a
- specifically: $a_i = f(e_i)$
- follows from

$$\begin{aligned} f(x) &= f(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n) \\ &= x_1 f(e_1) + x_2 f(e_2) + \cdots + x_n f(e_n) \end{aligned}$$

Affine functions

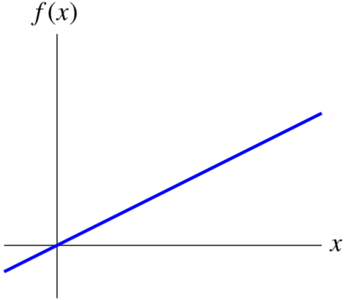
- a function that is linear plus a constant is called *affine*
- general form is $f(x) = a^T x + b$, with a an n-vector and b a scalar
- a function $f : \mathbf{R}^n \to \mathbf{R}$ is affine if and only if

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

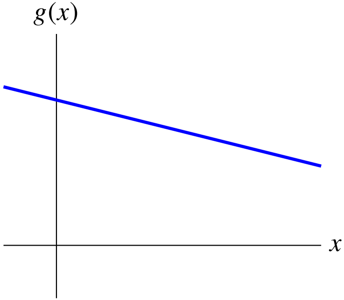
  holds for all α, β with $\alpha + \beta = 1$, and all n-vectors x, y
- sometimes (ignorant) people refer to affine functions as linear

Linear versus affine functions

f is linear



g is affine, not linear



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First-order Taylor approximation

- suppose $f : \mathbf{R}^n \to \mathbf{R}$
- *first-order Taylor approximation* of f, near point z:

$$\hat{f}(x) = f(z) + \frac{\partial f}{\partial x_1}(z)(x_1 - z_1) + \cdots + \frac{\partial f}{\partial x_n}(z)(x_n - z_n)$$

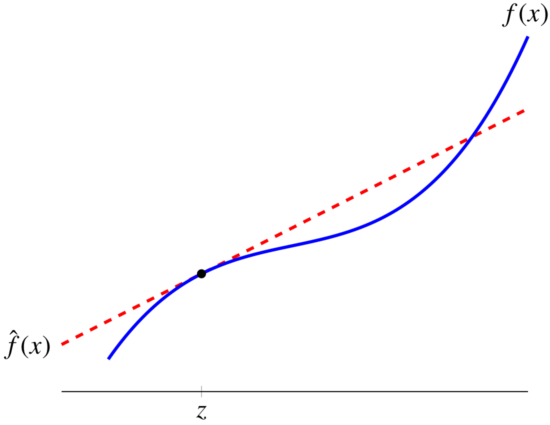
- $\hat{f}(x)$ is *very* close to $f(x)$ when x_i are all near z_i
- $\hat{f}$ is an affine function of x
- can write using inner product as

$$\hat{f}(x) = f(z) + \nabla f(z)^T (x - z)$$

  where n-vector $\nabla f(z)$ is the *gradient* of f at z,

$$\nabla f(z) = \left(\frac{\partial f}{\partial x_1}(z), \ldots, \frac{\partial f}{\partial x_n}(z) \right)$$

Example



Outline

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Regression model

- *regression model* is (the affine function of x)

$$\hat{y} = x^T\beta + v$$

- x is a feature vector; its elements x_i are called *regressors*
- n-vector β is the *weight vector*
- scalar v is the *offset*
- scalar $\hat{y}$ is the *prediction*
  (of some actual outcome or *dependent variable*, denoted y)

Example

- y is selling price of house in $\$1000$ (in some location, over some period)
- regressor is
$$x = (\text{house area}, \#\text{ bedrooms})$$
(house area in 1000 sq.ft.)
- regression model weight vector and offset are
$$\beta = (148.73, -18.85), \qquad v = 54.40$$
- we'll see later how to guess β and v from sales data