Pathologies on the Hilbert scheme of points

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Abstract We prove that the Hilbert scheme of points on a higher dimensional affine space is non-reduced and has components lying entirely in characteristic p for all primes p. In fact, we show that Vakil's Murphy's Law holds up to retraction for this scheme. Our main tool is a generalized version of the Białynicki-Birula decomposition.

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1 Introduction

Vakil defined *singularity type* as an equivalence class of pointed schemes under the relation generated by $(X, x) \sim (Y, y)$ if there is a smooth morphism $(X, x) \rightarrow (Y, y)$. He defined that *Murphy's Law holds* for a moduli space if every singularity type of finite type over $\mathbb{Z}$ appears on this space. In [30] Vakil gave numerous examples when this happens. A notable item missing in his list is the Hilbert scheme of points. In fact little was known about its

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singularities: for example the following classical questions raised by Fogarty and Hartshorne were open.

Question 1.1 ([16, p. 520], [1, Problem 1.25], [9]) Is $\mathrm{Hilb}_{\mathrm{pts}}(\mathbb{A}^n_{\mathbb{Z}})$ reduced for all n? Is $\mathrm{Hilb}_{\mathrm{pts}}(\mathbb{A}^n_{\mathbb{C}})$ reduced for all n?

Question 1.2 ([17, p. 148], [1, Problem 1.2], [22]) Do all finite $\Bbbk$-schemes, for a finite field $\Bbbk$, lift to characteristic zero?

Question 1.1 was completely open, and $\mathrm{Hilb}_4(\mathbb{A}^3_{\mathbb{C}})$ is the only known reduced but singular Hilbert scheme of points [6,28]. It was explicitly asked in [1, Problem 1.6] whether $\mathrm{Hilb}_8(\mathbb{A}^4_{\mathbb{C}})$ is reduced. There was a bit of progress on Question 1.2 in recent years. Classically, Berthelot and Ogus [8, §3] note that the maximal ideal of the algebra $\mathbb{F}_p[x_1, \ldots, x_6]/(x_1^p, x_2^p, \ldots, x_6^p, x_1x_2 + x_3x_4 + x_5x_6)$ admits no divided power structure. Bhatt [7, 3.16] mentioned that this algebra does not lift to $W_2(\mathbb{F}_p) = \mathbb{Z}/p^2$ and Zdanowicz [31, §3.2] gave a short direct proof of this fact. Langer [22] proved that for $p = 2$ this algebra *does* lift to characteristic zero. He also refined Zdanowicz's method to give, for every local Artin ring A with $pA \neq 0$ and residue characteristic p, a finite $\mathbb{F}_p$-scheme nonliftable to A. The constructed scheme depends on A and Langer writes "in principle these schemes could be liftable to characteristic 0 over some more ramified rings but we are unable to check whether this really happens".

In this paper we prove that the answers to Questions 1.1–1.2 above are negative; both pathologies occur as special cases of a Murphy-type Law, which we now describe. We say that *Murphy's Law holds up to retraction* for a space $\mathcal{M}$ if for every singularity type $\mathfrak{S}$ there is a representative (Y, y) of $\mathfrak{S}$, an open subscheme (X, x) of $\mathcal{M}$ and a retraction $(X, x) \to (Y, y)$. Here, a *retraction* $(X, x) \to (Y, y)$ is a morphism of pointed schemes together with a section. The aim of this paper is to prove the following theorem.

Theorem 1.3 *Murphy's Law holds up to retraction for* $\mathrm{Hilb}_{\mathrm{pts}}(\mathbb{A}^{16}_{\mathbb{Z}})$.

On the infinitesimal level, Theorem 1.3 implies that for every singularity type $\mathfrak{S}$, there exists a representative (Y, y) of $\mathfrak{S}$ and a point on $\mathrm{Hilb}_{\mathrm{pts}}(\mathbb{A}^{16}_{\mathbb{Z}})$ with complete local ring $\hat{\mathcal{O}}_{Y,y}[[t_1, \ldots, t_r]]/I$, for some r and I such that $\hat{\mathcal{O}}_{Y,y} \cap I = 0$. In particular, $\hat{\mathcal{O}}_{Y,y}$ is a *subring* of the complete local ring.

Murphy's Law up to retraction holds also for the scheme $\mathrm{Hilb}_{\mathrm{pts}}(\mathbb{P}^{16}_{\mathbb{Z}})$, as $\mathrm{Hilb}_{\mathrm{pts}}(\mathbb{A}^{16}_{\mathbb{Z}})$ is its open subscheme. More generally, the forgetful functor from embedded to abstract deformations of a finite scheme is smooth [2, p. 4], hence the above pathologies appear also for the abstract deformations of finite schemes and for Hilbert schemes of points on every smooth quasi-projective variety of dimension at least sixteen.

The negative answers to Questions 1.1–1.2 with $n = 16$ follow from Theorem 1.3 applied to singularity types $[\mathrm{Spec}(\mathbb{Z}[u]/u^2), (u)]$ and $[\mathrm{Spec}(\mathbb{Z}/p), 0]$

respectively, see Sect. 5. More precisely, for Question 1.2 we obtain finite schemes R which do not lift to any ring A with $pA \neq 0$. In Sect. 5 we give explicit examples of this behavior, for $\Bbbk = \mathbb{Z}/p$, $p = 2, 3, 5$. Using the singularity type $[\mathrm{Spec}(\mathbb{Z}/p^{\nu}), (p)]_{\nu=2,3,\ldots}$ we also obtain finite $(\mathbb{Z}/p)$-schemes R that do lift to $W_{\nu}(\mathbb{Z}/p) = \mathbb{Z}/p^{\nu}$ but do not lift to any A with $p^{\nu}A \neq 0$ and so forth.

The main difficulty with analysing finite schemes is their lack of structure: for example, they admit no non-trivial line bundles. Our proof of Theorem 1.3 proceeds by a series of reductions from objects with more structure, such as projective schemes. The main role is played by the generalized Białynicki-Birula decomposition, which we now recall.

Classically [3], for a smooth variety X with a $\mathbb{G}_m$-action and $X^{\mathbb{G}_m} = \coprod_{i=1}^{r} Y_i$ with Y_i connected, the BB decomposition is a variety $X^{+} = \coprod_{i=1}^{r} X_i^{+}$ with a map $\theta_0 \colon X^{+} \to X$ and a retraction $\pi \colon X^{+} \to X^{\mathbb{G}_m}$ which makes X_i^{+} a locally trivial affine fiber bundle over Y_i.

The generalized Białynicki-Birula decomposition [10,19,20] extends this construction to $\mathbb{G}_m$-schemes X, not necessarily smooth, normal or reduced. We apply it to the scheme $\mathcal{H}_{\mathbb{Z}} := \mathrm{Hilb}_{\mathrm{pts}}(\mathbb{A}_{\mathbb{Z}}^{r})$ with the standard $\mathbb{G}_m$-action on $\mathbb{A}_{\mathbb{Z}}^{r}$. We obtain a locally finite type $\mathbb{Z}$-scheme $\mathcal{H}_{\mathbb{Z}}^{+} = \mathrm{Hilb}_{\mathrm{pts}}^{+}(\mathbb{A}_{\mathbb{Z}}^{r})$ together with a map $\theta_0 \colon \mathcal{H}_{\mathbb{Z}}^{+} \to \mathcal{H}_{\mathbb{Z}}$ and a retraction $\pi \colon \mathcal{H}_{\mathbb{Z}}^{+} \to \mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_m}$ with section i ($\mathcal{H}_{\mathbb{Z}}^{+}$ is not in general a bundle over $\mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_m}$). The $\Bbbk$-points in the image of θ_0 are exactly the $\Bbbk$-schemes supported at the origin of $\mathbb{A}_{\Bbbk}^{r}$, so $\mathrm{im}(\theta_0)$ is nowhere dense. To remedy this, we define $\theta \colon \mathbb{G}_a^{r} \times \mathcal{H}_{\mathbb{Z}}^{+} \to \mathcal{H}_{\mathbb{Z}}$ which maps $(v, [R])$ to the scheme $\theta_0([R])$ translated by v. The structure is summarized in the diagram below, see Sect. 4 for details.

$$
\begin{array}{ccc}
\mathbb{G}_a^{r} \times \mathcal{H}_{\mathbb{Z}}^{+} & \xrightarrow{\;\theta\;} & \mathcal{H}_{\mathbb{Z}} \\
{\scriptstyle 0_{\mathbb{G}_a^{r}} \times \mathrm{id}} \Big\uparrow \Big\downarrow {\scriptstyle \mathrm{pr}_2} & \nearrow {\scriptstyle \theta_0} & \\
\mathcal{H}_{\mathbb{Z}}^{+} & & \\
{\scriptstyle i} \Big\uparrow \Big\downarrow {\scriptstyle \pi} & & \\
\mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_m} & &
\end{array}
\qquad (1.1)
$$

The maps θ_0 and θ are injective on $\Bbbk$-points for all fields $\Bbbk$ and we identify the points of $\mathcal{H}_{\mathbb{Z}}^{+}$ with their images in $\mathcal{H}_{\mathbb{Z}}$. The tangent space to $[R] \in \mathcal{H}_{\Bbbk}(\Bbbk)$ is equal to $\mathrm{Hom}_T(I_R, T/I_R)$, where $T = H^0(\mathbb{A}^{r}, \mathcal{O}_{\mathbb{A}_{\Bbbk}^{r}})$. If $[R]$ is $\mathbb{G}_m$-fixed, then this space is graded and in this case $T_{\mathcal{H}_{\Bbbk}^{+}, [R]} = \mathrm{Hom}_T(I_R, T/I_R)_{\geq 0}$. It follows that $d\theta_{[R]}$ is surjective if and only if

$$
\dim_{\Bbbk} \mathrm{Hom}_T(I_R, T/I_R)_{<0} = \dim T. \qquad (1.2)
$$

If (1.2) holds, we say that R has *trivial negative tangents* (*TNT* for short).
If R has TNT, then θ is an open immersion on its neighbourhood, see [19].
This property is important enough to give it a name: we say that (X, x) *locally
retracts to* (Y, y) if there are open neighbourhoods $x \in U \subset X$ and $y \in V \subset Y$
and a retraction $(U, x) \to (V, y)$. For example, if R has TNT then $(\mathcal{H}_{\mathbb{Z}}, [R])$
locally retracts to $(\mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_m}, [R])$.

Now we discuss the proof of Theorem 1.3. We first present a natural but
unsuccessful line of argument and then refine it to obtain a proof. Fix a singu-
larity type $\mathfrak{S}$. Vakil [30] proved that there is a smooth surface $Z \subset \mathbb{P}^4$ whose
embedded deformations in $\mathbb{P}^4$ are of type $\mathfrak{S}$. For $M \gg 0$ the $\mathbb{G}_m$-equivariant
deformations of the cone $V(I(Z)_{\geq M}) \subset \mathbb{A}^5$ are also of singularity type $\mathfrak{S}$. Let
$\mathfrak{n}$ be the ideal of the origin in $\mathbb{A}^5$. Then the $\mathbb{G}_m$-equivariant deformations of the
zero-dimensional truncation $R_0 := V(I(Z)_{\geq M} + \mathfrak{n}^{M+2})$ are of singularity type
$\mathfrak{S}$, see [14]. In other words, $(\mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_m}, [R_0])$ has type $\mathfrak{S}$. Thus, $(\mathcal{H}_{\mathbb{Z}}^+, [R_0])$ has
singularity type $\mathfrak{S}$ up to retraction. However, there is no reason for θ to be an
open immersion around $[R_0]$ and we cannot say anything about the singularity
type of $[R_0] \in \mathcal{H}_{\mathbb{Z}}$. To obtain a scheme such that that θ is an open immersion
in its neighbourhood, we need a refinement that produces a scheme with TNT.

The refinement is based on the concept of TNT frames. Let $I \subset S = \mathbb{k}[x_1, \ldots, x_n]$ be a homogeneous ideal and $a \geq 2$. A TNT frame *of size a for*
I is an ideal $J \subset T = S[y_1, \ldots, y_n] = \mathbb{k}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ given by

$$J := I \cdot T + (x_1, \ldots, x_n)^{a+1} + (y_1, \ldots, y_n)^2 + (x_1 y_1 + \ldots + x_n y_n).$$
$$(1.3)$$

Informally, the TNT frame is a bifurcated reduction of I to dimension zero.
The quadric $Q = \sum x_i y_i$ is technically useful for the proof that $\mathrm{Spec}(T/J)$
has TNT because the deformations of J inside $V(Q)$ do not admit any neg-
ative tangents under mild assumptions on I, see Corollary 3.8. Since J is
homogeneous with respect to both x_i's and y_i's, the stabilizer of $[J]$ contains
a two dimensional torus $\mathbb{G}_x \times \mathbb{G}_y$. Let $\mathbb{G}_{xy} \subset \mathbb{G}_x \times \mathbb{G}_y$ be the torus acting
diagonally.

The concept of TNT frames is geared towards the following result.

Proposition 1.4 *Let* char $\mathbb{k} \neq 2$ *and let* $I \subset S$ *be a homogeneous ideal with*
$I_2 = 0$. *Assume that* depth$(S_+, S/I) \geq 3$ *and* dim $S \geq 3$. *Then*

(1) $\mathrm{Spec}(T/J)$ *has TNT, hence* $(\mathcal{H}_{\mathbb{Z}}, [J])$ *locally retracts to* $(\mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_{xy}}, [J])$,
(2) $(\mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_{xy}}, [J])$ *locally retracts to* $(\mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_x \times \mathbb{G}_y}, [J])$,
(3) $(\mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_x \times \mathbb{G}_y}, [J])$ *locally retracts to* $(\mathrm{Hilb}^{\mathbb{G}_x}(\mathbb{A}^n), [I + (x_1, \ldots, x_n)^{a+1}])$.

For $a \gg 0$ the $\mathbb{G}_x$-equivariant deformations of $I + (x_1, \ldots, x_n)^{a+1} \subset S$ and
$I \subset S$ are canonically isomorphic [14]. For such a number a, the composi-
tion of the local retractions from Proposition 1.4 gives a local retraction of

$(\mathcal{H}_{\mathbb{Z}}, [J])$ to $(\mathrm{Hilb}^{\mathbb{G}_x}(\mathbb{A}^n), [I])$. The analogue of Proposition 1.4 holds also in char $\Bbbk = 2$, for a slightly modified J, see Sect. 3.3.

We return to the proof of Theorem 1.3. Fix a singularity type $\mathfrak{S}$, a surface $Z \subset \mathbb{P}^4$ and its truncation $V(I(Z)_{\geq M}) \subset \mathbb{A}^5$ as above. The ideal $I(Z)_{\geq M}$ does not satisfy the depth assumption of Proposition 1.4, so fix a linear embedding $\mathbb{A}^5 \hookrightarrow \mathbb{A}^8$ and consider the extended ideal $I := I(Z)_{\geq M} \cdot S$ in $S = H^0(\mathbb{A}^8, \mathcal{O}_{\mathbb{A}^8})$. Let J be a TNT frame for I with $a \gg 0$. Proposition 1.4 implies that the (not necessarily equivariant!) deformations of $[J]$ in $\mathrm{Spec}(T) = \mathbb{A}^{16}$ locally retract to the $\mathbb{G}_x$-equivariant deformations of $[I]$ in $\mathbb{A}^8$. These in turn retract to the equivariant deformations of $[I(Z)_{\geq M}]$ in $\mathbb{A}^5$. Thus, $(\mathrm{Hilb}_{\mathrm{pts}}(\mathbb{A}_{\mathbb{Z}}^{16}), [J])$ locally retracts to a scheme of singularity type $\mathfrak{S}$ and the proof is concluded.

In the course of the proof we give two side-results. First, in Corollary 3.10 we present a class of ideals, generalizing TNT frames above, which have TNT. By [19, Theorem 1.2], each of those ideals lies on an elementary component of the Hilbert scheme; thus we obtain a new, very large class of elementary components. Second, in Corollary 3.17 we prove that thickenings of maximal linear subspaces of the quadric $V(Q)$ are rigid.

Theorem 1.3 together with related combinatorics, in particular TNT frames which subtly balance prescribed homogeneous deformations and the TNT condition, is the main novelty of this paper. The Białynicki-Birula decomposition of $\mathrm{Hilb}_{\mathrm{pts}}(\mathbb{A}_{\Bbbk}^n)$, in the equicharacteristic setting, was introduced in [19]. The ambient dimension $n = 16$ in Theorem 1.3 is chosen to make the argument transparent and is probably not optimal. Hazarding a guess, we would say that it can be reduced to $n = 6$ or even $n = 4$. In any case, the case $n = 3$ is special because of the superpotential description [5,11]; it would be interesting to know the answers to Questions 1.1–1.2 in this case.

The above results exhibit pathologies of the space of based rank n algebras, see [27, §4]. Hilbert schemes of points also appear prominently in the study of secant and cactus varieties and in algebraic complexity [21]. The non-reducedness of $\mathrm{Hilb}_{\mathrm{pts}}(\mathbb{A}_{\mathbb{C}}^n)$ strongly suggests that the equations for these varieties obtained in [4] are only set-theoretic, not ideal-theoretic. To make this suggestion rigorous, one would need to know that the Gorenstein locus of $\mathrm{Hilb}_{\mathrm{pts}}(\mathbb{A}_{\mathbb{C}}^n)$ is non-reduced, but this remains open. Another interesting open question is whether there are generically non-reduced components of the Hilbert scheme of points.
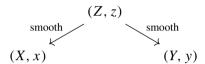
The outline is as follows: in Sect. 3 we prove the necessary prerequisite results on the tangent spaces and maps. In Sects. 3.1–3.2 we deal with char $\Bbbk \neq$ 2, while in Sect. 3.3 we present the modified construction for characteristic two. Section 4 contains main ideas of the paper: we discuss Białynicki-Birula decompositions, prove a generalized version of Proposition 1.4 and finally prove Theorem 1.3. In Sect. 5 we discuss consequences of specific singularity

types and give explicit examples of non-reduced points on the Hilbert scheme and components lying in positive characteristic.
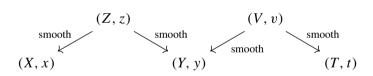
2 Pointed schemes and smooth equivalence

A *pointed scheme* (X, x) is a scheme X of finite type over $\mathbb{Z}$ together with a point x of the underlying topological space of X. A *morphism of pointed schemes* $f \colon (X, x) \to (Y, y)$ is a morphism of schemes $f' \colon X \to Y$ such that $f'(x) = y$. We say that f is *smooth* if the underlying morphism $f' \colon X \to Y$ is smooth. We say that pointed schemes (X, x) and (Y, y) are *smoothly equivalent* if there exists a pointed scheme (Z, z) and smooth maps

$$
\begin{array}{ccc}
 & (Z, z) & \\
{\scriptstyle\text{smooth}}\swarrow & & \searrow{\scriptstyle\text{smooth}} \\
(X, x) & & (Y, y)
\end{array}
$$

Lemma 2.1 *Being smoothly equivalent is an equivalence relation of pointed schemes.*

Proof The relation is clearly reflexive and symmetric. To prove transitivity, suppose that the pairs (X, x), (Y, y) and (Y, y), (T, t) are smoothly equivalent. By definition there exist pointed schemes (Z, z) and (V, v) together with smooth maps forming the diagram

$$
\begin{array}{ccccc}
 & (Z, z) & & (V, v) & \\
{\scriptstyle\text{smooth}}\nearrow & & \searrow{\scriptstyle\text{smooth}} \quad {\scriptstyle\text{smooth}}\swarrow & & \searrow{\scriptstyle\text{smooth}} \\
(X, x) & & (Y, y) & & (T, t)
\end{array}
$$

Let $W := Z \times_Y V$ and let $\kappa(-)$ denote the residue field. The algebra $\kappa(z) \otimes_{\kappa(y)} \kappa(v)$ is a tensor product of nonzero algebras over a field, hence it is nonzero. Therefore the scheme $\mathrm{Spec}(\kappa(z) \otimes_{\kappa(y)} \kappa(v)) \simeq \mathrm{Spec}(\kappa(z)) \times_{\mathrm{Spec}(\kappa(y))} \mathrm{Spec}(\kappa(v))$ is nonempty. Choose a point p of this scheme. The natural maps $\mathrm{Spec}(\kappa(z)) \to Z$ and $\mathrm{Spec}(\kappa(v)) \to V$ induce a map $\mathrm{Spec}(\kappa(p)) \to W$. Let $w \in W$ be the image of p. The pointed scheme (W, w) comes with smooth maps to (X, x) and (T, t) and proves that those schemes are smoothly equivalent. $\qquad\square$

Lemma 2.1 shows in particular that the above definition of smooth equivalence agrees with Vakil's one given in the introduction. The equivalence classes of smooth equivalence are called *singularity types*. The singularity type of (X, x)

is denoted by $[(X, x)]$ or simply $[X]$ if X has only one point. For example, the singularity type $[\mathrm{Spec}(\mathbb{F}_p)]$ consists of pointed schemes (X, x) over $\mathrm{Spec}(\mathbb{F}_p)$ and such that $x \in X$ is smooth. A *retraction* $(X, x) \to (Y, y)$ is a pair (f, s) where $f\colon (X, x) \to (Y, y)$ and $s\colon (Y, y) \to (X, x)$ are morphisms of pointed schemes such that $f \circ s = \mathrm{id}_Y$. For each such retraction, the residue fields of x and y are isomorphic. The retractions we encounter in this article come from diagrams similar to (1.1).

3 Tangent spaces

In this section we prove tangent-map-surjectivity lemmas, such as the TNT condition, needed for the proof of a generalized version of Proposition 1.4. Specifically, the aim is to prove Proposition 3.10, Corollary 3.16 (for characteristic two, respectively Proposition 3.23, Corollary 3.26), which are applied in Sect. 4. These results follow from the chain of quite technical partial results, obtained using linear algebra and representation theory. We encourage the reader to consult Sect. 4 for motivation before diving into details.

Throughout, let $\mathbb{k}$ be field. Let $I \subset S = \mathbb{k}[\mathbf{x}]$ be a homogeneous ideal. Let $T = S[\mathbf{y}] = \mathbb{k}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Fix $a \geq 2$, $b \geq 1$ and let $\mathfrak{m}_x := (x_1, \ldots, x_n) \subset T$, $\mathfrak{m}_y := (y_1, \ldots, y_n) \subset T$, $Q := \sum x_i y_i$ and

$$J := I \cdot T + \mathfrak{m}_x^{a+1} + \mathfrak{m}_y^{b+1} + (Q).$$

The ideal J is $\mathbb{N}^2$-graded by $(\deg \mathbf{x}, \deg \mathbf{y})$. Throughout this section the word *homogeneous* for elements of T refers to this bi-grading. For elements of S, the word *homogeneous* refers to the usual grading by the total degree (when viewing S as subring of T, these agree). The ideal J is generated by elements of degrees $(*, 0)$, $(0, b + 1)$, and $(1, 1)$. The space $\mathrm{Hom}_T(J, T/J)$ is graded by

$$
\begin{aligned}
&\mathrm{Hom}_T(J, T/J)_{(\alpha, \beta)} \\
&\quad := \left\{ \varphi\colon J \to T/J \mid \varphi\left(J_{(\gamma, \delta)}\right) \subset (T/J)_{(\gamma + \alpha, \delta + \beta)} \text{ for all } \gamma, \delta \right\}. \quad (3.1)
\end{aligned}
$$

This section is devoted to showing that certain homogeneous pieces of $\mathrm{Hom}_T(J, T/J)$ vanish or are as small as possible. The following Table 1 provides for each such piece a reference to the corresponding result. Stars denote pieces which are not of interest. The table is subtly asymmetric with respect to $\mathbf{x}$ and $\mathbf{y}$.

Now we introduce two tricks which recur in our computation of the homogeneous components of the space (3.1). Consider the canonical epimorphism $p\colon T/(I \cdot T + (Q)) \to T/J$ and note that there exists a unique homogeneous linear section of the map p:

Table 1 Reference for computations of pieces of $\mathrm{Hom}_T(J, T/J)_{(\alpha,\beta)}$, characteristic not equal to two. The pieces marked with Corollary 3.5 or Lemma 3.9 vanish, others do not

$\mathrm{Hom}_T(J, T/J)_{(\alpha,\beta)}$	$\beta \leq -2$	$\beta = -1$	$\beta = 0$	$\beta \geq 1$
$\alpha \leq -2$	Corollary 3.5	Corollary 3.5	Corollary 3.5	Corollary 3.5
$\alpha = -1$	Corollary 3.5	Lemma 3.9	Corollary 3.8	★
$\alpha = 0$	Corollary 3.5	Corollary 3.8	★	★
$\alpha = 1$	Corollary 3.5	Corollary 3.15	★	★
$\alpha \geq 2$	Corollary 3.5	★	★	★

$$s: \frac{T}{J} \to \frac{T}{I \cdot T + (Q)}$$

and that s is zero in degrees $(*, \geq a + 1)$, $(\geq b + 1, *)$ and an isomorphism in degrees $(\leq a, \leq b)$. Our first trick is the following lemma.

Lemma 3.1 (lifting homogeneous homomorphisms) *Let N be a graded T-module with a minimal presentation*

$$\bigoplus_j T(-d_{1j}, -d_{2j}) \xrightarrow{\delta_1} \bigoplus_i T(-c_{1i}, -c_{2i}) \xrightarrow{\delta_0} N \longrightarrow 0$$

and ψ be a homomorphism $\psi: N \to T/J$ of degree (α, β). Suppose that $d_{1j} + \alpha \leq a$ and $d_{2j} + \beta \leq b$ for all j. Then ψ lifts to a homomorphism $N \to T/(I \cdot T + (Q))$ of degree (α, β).

Proof Let $K := \frac{J}{IT+(Q)}$. Take a lifting of ψ to a homogeneous chain complex map

$$
\begin{array}{ccccccc}
\bigoplus T(-d_{1j}, -d_{2j}) & \xrightarrow{\delta_1} & \bigoplus_i T(-c_{1i}, -c_{2i}) & \xrightarrow{\delta_0} & N & \longrightarrow & 0 \\
\downarrow{\rho} & & \downarrow & & \downarrow{\psi} & & \\
0 \longrightarrow K & \longrightarrow & \frac{T}{IT+(Q)} & \longrightarrow & \frac{T}{J} & \longrightarrow & 0
\end{array}
$$

The map ρ maps the generator of $T(-d_{1j}, -d_{2j})$ into $K_{d_{1j}+\alpha, d_{2j}+\beta}$. But $d_{1j} + \alpha \leq a$ and $d_{2j} + \beta \leq b$, hence $K_{d_{1j}+\alpha, d_{2j}+\beta} = 0$. Therefore $\rho = 0$, which completes the claim. $\qquad\square$

Our second trick is as follows: suppose that $K \subset T$ is an ideal and M is a T-module with $\mathrm{depth}(K, M) \geq 2$. Then $\mathrm{Ext}^i(T/K, M) = 0$ for $i = 0, 1$, see [12, Proposition 18.4], so $M = \mathrm{Hom}_T(T, M) \to \mathrm{Hom}_T(K, M)$ is an isomorphism. To make the trick applicable to $K = I \cdot T + (Q)$, we give a lower bound of the depth of $T/(I \cdot T + (Q))$, as follows.

Lemma 3.2 *Let $d = \mathrm{depth}(S_+, S/I)$. Suppose that $d \geq 2$ and $f_1, \ldots, f_{d-1} \in S_+$ is a regular sequence on S/I consisting of homogeneous elements. Then Q is a non-zero divisor on both $T/(I \cdot T)$ and $T/(I \cdot T + f_1 \cdot T)$. Moreover, $f_1, \ldots, f_{d-1}$ is a regular sequence on $T/(I \cdot T + (Q))$.*

Proof By base change, we may assume that $\Bbbk$ is algebraically closed.

The quotient module $M = S/(I + (f_1, \ldots, f_{d-1}))$ has S_+-depth at least one, hence there exists a quadric $f_d = \sum_{i=1}^n x_i l_i \in S_2$ which is a regular element for M. Thus, the sequence

$$(y_1 - l_1, y_2 - l_2, \ldots, y_n - l_n, f_1, \ldots, f_{d-1}, Q)$$

is regular for the T-module $T/(I \cdot T)$. This sequence consists of elements homogeneous with respect to the total degree, hence each of its permutations is also a regular sequence [24, Theorem 16.3]. In particular, the sequences $(f_1, Q, f_2, \ldots, f_{d-1}, y_1 - l_1, \ldots, y_n - l_n)$ and $(Q, f_1, \ldots, f_{d-1}, y_1 - l_1, \ldots, y_n - l_n)$ are regular for $T/(I \cdot T)$. The first one implies that Q is regular on $T/(I \cdot T + (f_1))$ and the second that Q is regular on $T/(I \cdot T)$ and that $f_1, \ldots, f_{d-1}$ is regular on $T/(I \cdot T + (Q))$. $\square$

Now we begin direct computations of specific degrees of the tangent space at J.

3.1 Negative tangents

In this section we verify that $\mathrm{Spec}(T/J)$ has TNT.

Lemma 3.3 (*I-ignoring lemma*) *Suppose that $\mathrm{depth}(S_+, S/I) \geq 2$. Let α, β be integers, with $\alpha \leq -1$. Let $\varphi \in \mathrm{Hom}(J, T/J)_{(\alpha, \beta)}$ be a homomorphism such that $\varphi(\mathfrak{m}_x^{a+1}) = 0$. Then $\varphi(I) = 0$.*

Proof Choose any homogeneous $i \in I$ and let a' be its degree. If $a' \geq a + 1$, then $i \in \mathfrak{m}_x^{a+1}$ and so $\varphi(i) = 0$. Otherwise, we have $\mathfrak{m}_x^{a+1-a'} i \subset \mathfrak{m}_x^{a+1}$, so

$$0 = \varphi\left(\mathfrak{m}_x^{a+1-a'} i\right) = \mathfrak{m}_x^{a+1-a'} \varphi(i).$$

But $\deg(S_{a+1-a'}\varphi(i)) = (a + 1 + \alpha, \beta)$ and $a + 1 + \alpha < a + 1$. Therefore,

$$\left(\frac{T}{I \cdot T + (Q)}\right)_{(a+1+\alpha, \beta)} \rightarrow \left(\frac{T}{J}\right)_{(a+1+\alpha, \beta)}$$

is an isomorphism. Hence, $\varphi(i)$ uniquely lifts to an element F of $T/(I \cdot T + (Q))$ such that $\mathfrak{m}_x^{a+1-a'} F = 0$. By Lemma 3.2, we have $\mathrm{depth}(\mathfrak{m}_x, T/(I \cdot T + (Q))) > 0$, hence $F = 0$, so $\varphi(i) = 0$. As $i \in I$ was chosen arbitrarily, we have $\varphi(I) = 0$. $\square$

Lemma 3.4 (depth lemma) *Suppose that* $\operatorname{depth}(S_+, S/I) \geq 3$ *and* $n \geq 3$. *Fix* (α, β) *so that* $\alpha < 0$ *or* $\beta < 0$. *Let* $\mathfrak{p}_x$, $\mathfrak{p}_y$ *be homogeneous ideals with radicals* $\mathfrak{m}_x$, $\mathfrak{m}_y$ *respectively. Then*

$$\operatorname{Hom}\left(\mathfrak{p}_x, \frac{T}{I \cdot T + (Q)}\right)_{(\alpha, \beta)} = 0 \quad and \quad \operatorname{Hom}\left(\mathfrak{p}_y, \frac{T}{I \cdot T + (Q)}\right)_{(\alpha, \beta)} = 0$$

Proof Let $M = T/(I \cdot T + (Q))$ viewed as a T-module. By Lemma 3.2, $\operatorname{depth}(\mathfrak{p}_x, M) \geq 2$. Hence $\operatorname{Ext}^i(T/\mathfrak{p}_x, M) = 0$ for $i = 0, 1$, so

$$\operatorname{Hom}_T(T, M) \to \operatorname{Hom}_T(\mathfrak{p}_x, M)$$

is an isomorphism. But $\operatorname{Hom}_T(T, M) = M$ has no non-zero elements of degree (α, β) with $\alpha < 0$ or $\beta < 0$. The same argument applies with $\mathfrak{p}_y$ instead of $\mathfrak{p}_x$; the depth assumption is satisfied as the sequence (y_1, y_2) is regular. $\square$

Corollary 3.5 *Suppose that* $\operatorname{depth}(S_+, S/I) \geq 3$ *and* $n \geq 3$. *Then* $(T_{[J]}\mathcal{H})_{(\alpha, \beta)} = 0$ *for* $\alpha \leq -2$ *or* $\beta \leq -2$.

Proof Consider the case $\alpha \leq -2$. Take an element $\varphi \in \operatorname{Hom}(J, T/J)_{(\alpha, \beta)}$. By degree reasons, φ sends $\mathfrak{m}_y^{b+1}$ and Q to zero. Consider $\varphi' = \varphi_{|\mathfrak{m}_x^{a+1}} : \mathfrak{m}_x^{a+1} \to T/J$. This map sends generators of $\mathfrak{m}_x^{a+1}$ to elements of $(T/J)_{(\leq a-1, \leq b-1)}$ and the syzygies of $\mathfrak{m}_x^{a+1}$ are linear, of degree $(1, 0)$. Hence, by Lemma 3.1, the map φ' lifts to a map

$$\varphi'' : \mathfrak{m}_x^{a+1} \to \frac{T}{I \cdot T + (Q)},$$

of degree (α, β) with $\alpha \leq -2$. Such a map is zero by Lemma 3.4 applied to $\mathfrak{p}_x = \mathfrak{m}_x^{a+1}$. Therefore, $\varphi(\mathfrak{m}_x^{a+1}) = 0$. From Lemma 3.3 it follows that $\varphi(I) = 0$, hence $\varphi = 0$. The case $\beta \leq -2$ is symmetric, minus the use of Lemma 3.3. $\square$

Now we will analyse homomorphisms of degree (α, β) with $\alpha + \beta = -1$. These do exist (e.g. the tangents corresponding to the $\mathbb{G}_a^{2n}$-action by translations), so the depth considerations as above are not directly applicable.

Lemma 3.6 *Suppose that* $\operatorname{depth}(S_+, S/I) \geq 2$. *Then*

$$\operatorname{Hom}_T\left(\frac{\mathfrak{m}_x^{a+1}}{\mathfrak{m}_x^{a+1} \cap (Q)}, \frac{T}{J}\right)_{(-1,0)} = 0.$$

Proof Let

$$\varphi \in \operatorname{Hom}_T\left(\frac{\mathfrak{m}_x^{a+1}}{\mathfrak{m}_x^{a+1} \cap (Q)}, \frac{T}{J}\right)_{(-1,0)}$$

Take the unique lift of φ to a linear map $\varphi' : S_{a+1} \to (S/I)_a$. Let $m \in S_a$ be a monomial. Then $\deg(mQ) = (a+1, 1)$ and

$$0 = \varphi(mQ) = \sum y_i \varphi(mx_i),$$

hence there exists a form $n_m \in (S/I)_{a-1}$ such that

$$\sum y_i \varphi'(mx_i) \equiv Q n_m = \sum y_i x_i n_m \quad \mod IT + \mathfrak{m}_x^{a+1},$$

so $\varphi(mx_i) \equiv x_i n_m \mod IT + \mathfrak{m}_x^{a+1}$. We define a linear map $\psi : S_a \to (S/I)_{a-1}$ by $\psi(m) := n_m$. Suppose that $m_1, m_2 \in S_a$ are monomials such that $x_i m_1 = x_j m_2$ for some i, j. Then

$$x_i \psi(m_1) - x_j \psi(m_2) = x_i n_{m_1} - x_j n_{m_2} \equiv \varphi(x_i m_1) - \varphi(x_j m_2)$$
$$= \varphi(x_i m_1 - x_j m_2) = \varphi(0) = 0 \quad \mod IT + \mathfrak{m}_x^{a+1}.$$

But $\deg(x_i \psi(m_1) - x_j \psi(m_2)) = a$, hence $x_i \psi(m_1) - x_j \psi(m_2) \in IT$ and so ψ extends to a S-module homomorphism $\psi' : S_{\geq a} \to S/I$ of degree -1. But $\mathrm{depth}(S_+, S/I) \geq 2$, so $\mathrm{Ext}^i(S/S_{\geq a}, S/I) = 0$ for $i = 1, 2$, hence $\mathrm{Hom}(S, S/I) \to \mathrm{Hom}(S_{\geq a}, S/I)$ is an isomorphism. In particular, there are no homomorphisms of negative degrees, so $\psi' = 0$. Accordingly, $\psi = 0$, hence $\varphi = 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 3.7 *Let $n \geq 2$. Then*

$$\mathrm{Hom}_T \left(\frac{\mathfrak{m}_y^{b+1}}{\mathfrak{m}_y^{b+1} \cap (Q)}, \frac{T}{J} \right)_{(0,-1)} = 0.$$

Proof The proof of Lemma 3.6 can be repeated with **x** interchanged with **y** and $I = 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Let $\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_n}$ be the derivations with respect to variables of T. By Leibniz's rule, each such derivation ∂ induces a T-linear map $J \to T/J$. This map is an element of the tangent space $T_{[J]}\mathcal{H}$. By slight abuse, we denote it by ∂. Geometrically, these elements arise from the action of $\mathbb{G}_a^{2n}$ on $\mathbb{A}^{2n} = \mathrm{Spec}(T)$ by translation.

Corollary 3.8 *Suppose that $\mathrm{depth}(S_+, S/I) \geq 2$ and $n \geq 2$. Then*

$$(T_{[J]}\mathcal{H})_{(-1,0)} = \langle \partial_{x_1}, \ldots, \partial_{x_n} \rangle \quad \text{and} \quad (T_{[J]}\mathcal{H})_{(0,-1)} = \langle \partial_{y_1}, \ldots, \partial_{y_n} \rangle$$

Proof Choose any homomorphism $\varphi : J \to T/J$ of degree $(-1, 0)$. Then $\varphi(Q) \in \langle y_1, \ldots, y_n \rangle$. But $\partial_{x_i}(Q) = y_i$, hence there is a unique linear combination D of $\{\partial_{x_i}\}$ such that $(\varphi - D)(Q) = 0$. Replacing φ by $\varphi - D$, we may

assume $\varphi(Q) = 0$. By Lemma 3.6, we have $\varphi(\mathfrak{m}_x^{a+1}) = 0$. By Lemma 3.3, we have $\varphi(I) = 0$. Finally, $\varphi(\mathfrak{m}_y^{b+1}) = 0$ by degree reasons, so $\varphi = 0$ and the claim follows for $(T_{[J]}\mathcal{H})_{(-1,0)}$. The argument for degree $(0, -1)$ is symmetric.   □

Out of all tangents of negative degrees, there is a single degree left to consider: $(-1, -1)$.

Lemma 3.9 [$(-1, -1)$ tangents] *Suppose that* $\mathfrak{m}_x^a \not\subset I$. *Then* $(T_{[J]}\mathcal{H})_{(-1,-1)} = 0$.

Proof Let $\varphi\colon J \to T/J$ a homomorphism of degree $(-1, -1)$. Then $\varphi(\mathfrak{m}_x^{a+1}) = \varphi(\mathfrak{m}_y^{b+1}) = \varphi(I) = 0$ by degree reasons. Moreover, $\varphi(Q) \in \Bbbk$. If $\varphi(Q) \neq 0$, then

$$\mathfrak{m}^a = \mathfrak{m}^a \varphi(Q) = \varphi(\mathfrak{m}^a Q) \subset \varphi(\mathfrak{m}_x^{a+1}) = 0,$$

so $\mathfrak{m}^a \subset I$, a contradiction. Hence $\varphi(Q) = 0$, so $\varphi = 0$.   □

Proposition 3.10 (TNT for J) *Suppose that* depth$(S_+, S/I) \geq 3$ *and* $n \geq 3$. *Then the scheme* Spec(T/J) *has TNT.*

Proof This follows from Corollary 3.5, Corollary 3.8, and Lemma 3.9.   □

3.2 Degree zero tangents in characteristic $\neq 2$

Proposition 3.10 is the essential part to obtain the local retraction from Point 1 in Proposition 1.4. To obtain the retraction from Point 2, we need to compute the tangents of $[\mathrm{Spec}(T/J)]$ that have degree $(\alpha, -\alpha)$ for $\alpha > 0$. All homomorphisms coming from such tangents kill I by degree reasons. Hence, the material in this section is independent of the choice of I. To emphasise this further, we introduce the linear space $V = \langle x_1, \ldots, x_n \rangle$ and identify $y_1, \ldots, y_n$ with the dual space V^*. Then

$$T_{(\alpha,\beta)} \simeq S^\alpha V \otimes S^\beta V^*.$$

for all α, β and, in particular, Q becomes the trace element in $V \otimes V^*$, hence is SL(V)-invariant. For $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$ we let $\mathbf{x}^\mathbf{a}, \mathbf{y}^\mathbf{b}$ denote the monomials $x_1^{\mathbf{a}_1} \ldots x_n^{\mathbf{a}_n}$, $y_1^{\mathbf{b}_1} \ldots y_n^{\mathbf{b}_n}$ respectively. We will denote the space $(S^\alpha V^*)^*$ by $\Gamma^\alpha V$ and denote by $\mathbf{x}^{[\mathbf{a}]} \in \Gamma^\alpha V$ the functional which is dual to $\mathbf{y}^\mathbf{a}$. There is an contraction action $(-) \lrcorner (-)\colon V^* \otimes \Gamma^\alpha V \to \Gamma^{\alpha-1} V$ given by

$$(\ell \lrcorner \varphi)(f) = \varphi(\ell \cdot f) \tag{3.3}$$

for all $\ell \in V^*$, $\varphi \in \Gamma^\alpha V$ and $f \in S^{\alpha-1} V^*$. For char $\Bbbk > \alpha$ there is a unique $\mathrm{GL}(V)$-equivariant isomorphism $\Gamma^\alpha V \simeq S^\alpha V$ and it sends $\mathbf{x}^{[\mathbf{a}]}$ to $\mathbf{x}^{\mathbf{a}}/\mathbf{a}!$. Under this isomorphism, contraction corresponds to differentiation $\ell \circ \varphi \mapsto \partial_\ell(\varphi)$. See [12, A2.4] or [4, §3] for details.

Now we introduce the group G giving trivial degree zero tangents; this is the degree-zero counterpart of $\mathbb{G}_a^{2n}$. Let $G \subset \mathrm{GL}_{2n,\mathbb{Z}}$ be a subgroup given in the basis $x_1, \ldots, x_n, y_1, \ldots, y_n$ by

$$G = \left\{ \begin{pmatrix} I_n & A \\ 0 & I_n \end{pmatrix} \mid A \in \mathbb{M}_{n \times n} \right\}. \tag{3.4}$$

The group G is smooth and acts naturally on $\mathcal{H}_\mathbb{Z}$ and $\mathcal{H}_\mathbb{Z}^{\mathbb{G}_{xy}}$. The Lie algebra $\mathfrak{g}$ of $G_\Bbbk$ maps y_i's to combinations of x_j's. Hence, the tangent to the orbit map $G_\Bbbk \ni g \mapsto g \cdot [J] \in \mathcal{H}_\Bbbk^{\mathbb{G}_m}$ is

$$\mathfrak{g} \to \mathrm{Hom}_T(J, T/J)_{(1,-1)}. \tag{3.5}$$

Let G' be the stabilizer of Q in G. In the description (3.4), it consists of anti-symmetric matrices A. Let $\mathfrak{g}'$ be the Lie algebra of G'. The action of $\mathfrak{g}'$ annihilates Q and we obtain a tangent map

$$\mathfrak{g}' \to \mathrm{Hom}_T(J/Q, T/J)_{(1,-1)}. \tag{3.6}$$

Now we proceed to computations. Throughout this subsection, $\otimes$ denotes $\otimes_\Bbbk$. The S-module $T_{*,b+1}$ is free. Let $M := (T/Q)_{*,b+1}$. This is an S-module with presentation

$$0 \to S(-1, -b-1) \otimes S^b V^* \to S(0, -b-1) \otimes S^{b+1} V^* \to M, \tag{3.7}$$

where the twists correspond to the fact that generators of M have $\mathbf{x}$-degree $(0, b+1)$ and its syzygies have degree $(1, b+1)$ with respect to natural bigrading. The presentation map is just the multiplication by Q. Explicitly, it is given by

$$f \otimes g \mapsto \sum_{i=1}^n x_i f \otimes y_i g. \tag{3.8}$$

The module M plays a key role in the computation of $(1, -1)$ tangents of $[\mathrm{Spec}(T/J)]$. There is a natural injection $M \subset \mathfrak{m}_y^{b+1}$, hence we obtain restriction maps

$$\mathrm{Hom}_T\left(\tfrac{J}{Q}, \tfrac{T}{\mathfrak{m}_y^{b+1}+(Q)}\right)_{(1,-1)} \longrightarrow \mathrm{Hom}_T\left(\tfrac{\mathfrak{m}_y^{b+1}}{\mathfrak{m}_y^{b+1}\cap(Q)}, \tfrac{T}{\mathfrak{m}_y^{b+1}+(Q)}\right)_{(1,-1)} \longrightarrow \mathrm{Hom}_S\left(M, \tfrac{T}{Q}\right)_{(1,-1)}$$
$$\downarrow \qquad\qquad\qquad\qquad \downarrow \qquad\qquad\qquad\qquad \downarrow r$$
$$\mathrm{Hom}_T\left(\tfrac{J}{Q}, \tfrac{T}{J}\right)_{(1,-1)} \longrightarrow \mathrm{Hom}_T\left(\tfrac{\mathfrak{m}_y^{b+1}}{\mathfrak{m}_y^{b+1}\cap(Q)}, \tfrac{T}{J}\right)_{(1,-1)} \xrightarrow{r'} \mathrm{Hom}_S\left(M, \tfrac{T}{J}\right)_{(1,-1)}.$$

$$(3.9)$$

Lemma 3.11 *Suppose $I_2 = 0$. Then all maps in the diagram (3.9) are bijective.*

Proof Left-side horizontal maps are bijective, because the homomorphisms kill other generators of J/Q by degree reasons. The right-side horizontal maps are bijective since homomorphisms of degree $(1, -1)$ into $T/\mathfrak{m}_y^{b+1}$ annihilate $T_{(*,\geq b+2)}$. Finally, $I_2 = 0$ implies that the surjection $T/J \to T/Q$ is bijective in degrees $(\leq 1, \leq 2)$. Then the rightmost downward arrow is bijective by applying Lemma 3.1 to (3.7). $\qquad\square$

We concentrate on analysing $\mathrm{Hom}_S(M, T/Q)_{(1,-1)}$. For brevity, let $K := \mathrm{Hom}_S\left(M, \tfrac{T}{Q}\right)_{(1,-1)}$. From the presentation (3.7) we obtain an exact sequence

$$0 \to K \to \mathrm{Hom}_S\left(S(0, -b-1) \otimes S^{b+1} V^*, \frac{T}{Q}\right)_{(1,-1)}$$
$$\to \mathrm{Hom}_S\left(S(-1, -b-1) \otimes S^{b+1} V^*, \frac{T}{Q}\right)_{(1,-1)},$$

which simplifies to $0 \to K \to \Gamma^{b+1} V \otimes (T/Q)_{(1,b)} \to \Gamma^b V \otimes (T/Q)_{(2,b)}$. Hence, we obtain a commutative diagram with exact columns and bottom row.

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\Gamma^{b+1} V \otimes S^{b-1} V^* & \xrightarrow{\Phi_0} & \Gamma^b V \otimes V \otimes S^{b-1} V^* \\
\downarrow{\scriptstyle \mathrm{id}\otimes(-\cdot Q)} & & \downarrow{\scriptstyle \mathrm{id}\otimes(-\cdot Q)} \\
\Gamma^{b+1} V \otimes V \otimes S^b V^* & \xrightarrow{\Phi} & \Gamma^b V \otimes S^2 V \otimes S^b V^* \\
\downarrow & & \downarrow \\
0 \to K \to \Gamma^{b+1} V \otimes \tfrac{V\otimes S^b V^*}{Q\cdot S^{b-1} V^*} & \xrightarrow{\overline{\Phi}} & \Gamma^b V \otimes \tfrac{S^2 V\otimes S^b V^*}{Q\cdot V\otimes S^{b-1} V^*} \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

$$(3.10)$$

Let us write down Φ and Φ_0 explicitly. The map Φ comes from applying $\operatorname{Hom}_S(-, T)_{(1,-1)}$ to the map (3.8). Therefore, it is given in coordinates by

$$\Phi(\varphi \otimes f \otimes g) = \sum_{i=1}^{n} (y_i \lrcorner \varphi) \otimes (x_i f) \otimes g, \qquad (3.11)$$

where $\lrcorner$ denotes contraction, as defined in (3.3).

Proposition 3.12 *Suppose that $b = 1$ and char $\Bbbk \neq 2$. Then the map*

$$\mathfrak{g}' \to \operatorname{Hom}_S(M, T/Q)_{(1,-1)} \qquad (3.12)$$

obtained by composing (3.6) and (3.9) is bijective.

Proof For an element of $\mathfrak{g}'$, the image of y_i is read off the image of $y_i^{b+1} = y_i^2$ in the corresponding homomorphism, so the map (3.12) is injective and it is enough to check that $\dim \operatorname{Hom}_S(M, T/Q)_{(1,-1)} = \dim \mathfrak{g}' = \dim \Lambda^2 V$. We do this directly.

Since $b = 1$, the map $\Phi \colon \Gamma^2 V \otimes V \otimes V^* \to V \otimes S^2 V \otimes V^*$ has source and target of the same dimension. We will prove that it is bijective. It is enough to prove surjectivity. By (3.11), we have $\Phi = \Psi \otimes \operatorname{id}_{V^*}$ for the map $\Psi \colon \Gamma^2 V \otimes V \to V \otimes S^2 V$ given by

$$\Psi(\varphi \otimes f) = \sum_{i=1}^{n} (y_i \lrcorner \varphi) \otimes (x_i f).$$

It is enough to prove that Ψ is surjective. For pairwise distinct i, j, k, we have

$$\Psi\left((x_i \cdot x_j) \otimes x_k + (x_i \cdot x_k) \otimes x_j - (x_j \cdot x_k) \otimes x_i\right) = 2 x_i \otimes x_j \cdot x_k.$$

The same holds for not necessarily distinct i, j, k under the convention that $x_i \cdot x_i = 2 x_i^{[2]}$ in $\Gamma^2 V$. Thus, the map Ψ is surjective, hence the map Φ is bijective. In particular, Φ and Φ_0 are injective. From the snake lemma applied to (3.10), we have $K \simeq \operatorname{coker} \Phi_0$, in particular $\dim K = \dim \Lambda^2 V$ as claimed. $\square$

Remark 3.13 Proposition 3.12 fails for char $\Bbbk = 2$; in particular Ψ and Φ are not injective.

The restriction to $b = 1$, while sufficient for our purposes, is not very satisfactory. For $b > 1$ we have the following result in large enough characteristics. We will not use it in the proof of Theorem 1.3, so we only sketch a proof.

Proposition 3.14 *Let $b \geq 1$ be arbitrary and char $\Bbbk = 0$. Then the map $\mathfrak{g}' \to \operatorname{Hom}_S(M, T/Q)_{(1,-1)}$ is bijective.*

Proof (Sketch of proof) By Proposition 3.12 we may assume $b > 1$ (we only need this for notational reasons in Schur functors). The maps in Diagram (3.10) split into maps between simple $GL(V)$-modules, which are indexed by Young diagrams [15, §6].

The cokernel of Φ is isomorphic to $\mathbb{S}_{b,2} V \otimes S^b V^*$. Similarly, the cokernel of $\mathrm{id} \otimes (- \cdot Q)$ is isomorphic to $\Gamma^{b+1} V \otimes \mathbb{S}_{b+1,b,\ldots,b} V$. Hence, the cokernel of $\overline{\Phi}$ is obtained from $\Gamma^b V \otimes V^{\otimes 2} \otimes S^b V^* \simeq S^b V \otimes V^{\otimes 2} \otimes S^b V^*$ by applying two partial symmetrizations (corresponding to dividing by images of $\mathrm{id} \otimes (- \cdot Q)$ and Φ), which together imply that $\mathrm{coker}\, \overline{\Phi} = \mathbb{S}_{2b,b+2,b,\ldots,b} V$. Now, a dimension count on the bottom row shows that $\dim K = \dim \Lambda^2 V$ as claimed. $\qquad\square$

Corollary 3.15 *Suppose that $I_2 = 0$ and char $\Bbbk \neq 2$. Suppose further that either $b = 1$ or char $\Bbbk = 0$. Then the map (3.5) is bijective.*

Proof Let

$$\mathfrak{lt} = \left\{ \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \mid A \text{ is lower-triangular} \right\} \text{ and } \mathfrak{g}' = \left\{ \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \mid A \text{ is anti-symmetric} \right\}$$

Take an element $\varphi \in \mathrm{Hom}(J, T/J)_{(1,-1)}$. By degree reasons, $\varphi(\mathfrak{m}_x^{a+1}) = \varphi(I) = 0$ and $\varphi(Q) \in T_{2,0} = S_2$. There is a unique element $D \in \mathfrak{lt}$ such that $D(Q) = \varphi(Q)$. Replacing φ by $\varphi - D$, we can assume $\varphi(Q) = 0$. Therefore, φ comes from an element of

$$\mathrm{Hom}_T \left(\frac{\mathfrak{m}_y^{b+1}}{\mathfrak{m}_y^{b+1} \cap (Q)}, \frac{T}{J} \right)_{(1,-1)}.$$

By Lemma 3.11 and Proposition 3.12 or Proposition 3.14, there exists a unique element of $\mathfrak{g}'$ mapping to φ, which concludes the proof. $\qquad\square$

Corollary 3.16 *Let $n \geq 3$ and char $\Bbbk \neq 2$. Suppose that $\mathrm{depth}(S_+, S/I) \geq 3$ and $I_2 = 0$. Suppose further that either $b = 1$ or char $\Bbbk = 0$. The map*

$$\mathfrak{g} \to \bigoplus_{\alpha \geq 1} \mathrm{Hom}_T(J, T/J)_{(\alpha, -\alpha)}$$

is bijective.

Proof This follows from Corollary 3.15 and Corollary 3.5. $\qquad\square$

The above computations of degree-zero tangents can be translated to a geometric statement, which is of some independent interest at least as a motivation.

Corollary 3.17 *Let char $\Bbbk \neq 2$ and $n \geq 3$. Assume either $b = 1$ or char $\Bbbk = 0$. Let*

$$L_y^b = V((y_1, \ldots, y_n)^{b+1}) \subset V(Q) \subset \mathbb{P}^{2n-1}$$

be a thickening of a projective subspace on a quadric. Then L_y^b is a rigid scheme: we have $T_{L_y^b}^1 = 0$, where T^1 is the Schlessinger's functor [17, §3].

Proof Let $B = T/((Q) + (y_1, \ldots, y_n)^{b+1})$ be the coordinate ring of L_y^b. Since $n \geq 3$, we have $\mathrm{depth}(\mathfrak{m}_y, B) \geq 2$, so $T_{L_y^b}^1 \simeq (T_B^1)_0$ naturally, see e.g. [17, Theorem 5.4]. Corollary 3.15 proves that $(T_B^1)_0 = 0$, directly from the construction of T_B^1. □

3.3 Tangents in characteristic 2

Assumption 3.18 In this subsection $\Bbbk$ is a field of characteristic two.

In this case, Proposition 3.12 fails and we need to replace $\mathfrak{m}_y^2$ in the definition of J by another ideal. There are many possible replacements; in any case the symmetry has to be broken. We choose $\mathfrak{p} \subset \Bbbk[\mathbf{y}]$ defined by

$$\mathfrak{p} := (y_1^2, y_2^2, \ldots, y_n^2) + (y_1) \cdot (y_1, y_2, y_3, \ldots, y_n)$$
$$= (y_1^2, y_2^2, \ldots, y_n^2) + (y_1) \cdot \mathfrak{m}_y,$$

mainly for the (relatively) simple computations. Let $I \subset S = \Bbbk[x_1, \ldots, x_n]$ be a homogeneous ideal and $a \geq 2$. A *tweaked frame* of size a for I is an ideal $J \subset T = \Bbbk[\mathbf{x}, \mathbf{y}]$ defined by

$$J := I \cdot T + \mathfrak{m}_x^{a+1} + \mathfrak{p} + (Q). \tag{3.13}$$

As in Sect. 3.1, we calculate some graded pieces of $T_{[J]}\mathcal{H} = \mathrm{Hom}_T(J, T/J)$. Most of the arguments will directly pass to this setup. There is some additional work needed in degrees $(*, -1)$. Anyway, for clarity we provide statements and sketches of proofs of all steps.

An important additional piece is the following easy result about the syzygies of $(y_1^2, y_2^2, \ldots, y_n^2, Q)$.

Lemma 3.19 *Let $\alpha \in \{0, 1\}$ and $n \geq 3$. Suppose that $F_1, \ldots, F_n \in T_{\alpha, 1}$ are forms satisfying $\sum_{i=1}^n x_i^2 F_i \in (Q)$. Then $F_i \in (Q)$ for $i = 1, \ldots, n$. In particular, if $\alpha = 0$, then $F_i = 0$ for all i.*

Proof Let $\sum x_i^2 F_i = Q \cdot G$, then $\deg(G) = (\alpha + 1, 0)$. Differentiate with respect to y_j to obtain

$$\sum x_i^2 \frac{\partial F_i}{\partial y_j} = x_j G. \tag{3.14}$$

In follows that $G \in (x_1^2, \ldots, x_n^2) + (x_j)$. Intersecting over all x_j, we get $G = \sum_{i=1}^n \lambda_i x_i^2$ for some $\lambda_i \in \mathbb{k}$. The forms $\frac{\partial F_i}{\partial y_j}$ have degree $(\alpha, 0) \leq (1, 0)$, hence $\frac{\partial F_i}{\partial y_j} = \lambda_i x_j$. Consequently, we have $F_i = \sum_{j=1}^n \frac{\partial F_i}{\partial y_j} y_j = \lambda_i \sum_j x_j y_j \in (Q)$.
                                                                                    $\square$

Lemma 3.20 *Suppose that* $\mathrm{depth}(S_+, S/I) \geq 3$ *and* $n \geq 3$. *Then* $(T_{[\mathcal{J}]}\mathcal{H})_{(\alpha, \beta)}$ $= 0$ *for* $\alpha \leq -2$ *or* $\beta \leq -2$.

Proof The non-Koszul syzygies of $\mathfrak{p}$ are linear, hence the proof of Corollary 3.5 applies without changes. $\square$

Lemma 3.21 *Suppose that* $n \geq 3$ *and* $I_2 = 0$. *Then*

$$\mathrm{Hom}_T \left(\frac{\mathfrak{p}}{\mathfrak{p} \cap (Q)}, \frac{T}{\mathcal{J}} \right)_{(0, -1)} = 0. \qquad (3.15)$$

Proof Take a homomorphism φ as in the left-hand side of (3.15) and its unique lift of φ to a linear map $\varphi' \colon T_{0,2} \to T_{0,1}$. First, from $Q^2 = \sum_{i=1}^n x_i^2 y_i^2$ we get

$$\sum_{i=1}^n x_i^2 \varphi'(y_i^2) \in (Q),$$

so $\varphi'(y_i^2) = 0$ for $i = 1, \ldots, n$, by Lemma 3.19. Next, $y_1 Q = \sum_{i=1}^n x_i(y_1 y_i)$ so we have $0 = \sum_{i=1}^n x_i \varphi(y_1 y_i)$ and

$$\sum_{i=1}^n x_i \varphi'(y_1 y_i) = \lambda \cdot Q,$$

for some $\lambda \in \mathbb{k}$. Comparing coefficients near x's, we get $\varphi'(y_1 y_i) = \lambda y_i$. In particular $0 = \varphi'(y_1^2) = \lambda y_1$, hence $\lambda = 0$, thus $\varphi' = 0$. $\square$

Corollary 3.22 *Suppose that* $\mathrm{depth}(S_+, S/I) \geq 2$, $I_2 = 0$ *and* $n \geq 3$. *Then*

$$(T_{[\mathcal{J}]}\mathcal{H})_{(-1,0)} = \langle \partial_{x_1}, \ldots, \partial_{x_n} \rangle \quad and \quad (T_{[\mathcal{J}]}\mathcal{H})_{(0,-1)} = \langle \partial_{y_1}, \ldots, \partial_{y_n} \rangle$$

Proof Homomorphisms of degree $(-1, 0)$ kill $\mathfrak{p}$ and this case reduces to the one considered in Corollary 3.8. Consider a homomorphism $\varphi \in \mathrm{Hom}_T(\mathcal{J}, T/\mathcal{J})_{(0,-1)}$. Then $\varphi(Q) \in \langle x_1, \ldots, x_n \rangle$. But $\partial_{y_i}(Q) = x_i$, hence there is a unique linear combination D of $\{\partial_{y_i}\}$ such that $(\varphi - D)(Q) = 0$. Replacing φ by $\varphi - D$, we may assume $\varphi(Q) = 0$. By Lemma 3.21 we have $\varphi(\mathfrak{p}) = 0$. Moreover, $\varphi(I + \mathfrak{m}_x^{a+1}) = 0$ for degree reasons. This shows that $\varphi = 0$. $\square$

The following Proposition 3.23 summarizes the above discussion. We stress once more, that we assume char $\Bbbk = 2$, see Assumption 3.18.

Proposition 3.23 *Suppose that* $\mathrm{depth}(S_+, S/I) \geq 3$ *and* $n \geq 3$. *Then the scheme* $\mathrm{Spec}(T/\mathcal{J})$ *has TNT.*

Proof This follows from Lemma 3.20, Corollary 3.22, and a direct analogue of Lemma 3.9. □

Now we analyse degree zero tangents of $\mathcal{J}$. In (3.4) we defined the group $G \subset \mathrm{GL}_{2n,\mathbb{Z}}$ by

$$G = \left\{ \begin{pmatrix} I_n & A \\ 0 & I_n \end{pmatrix} \mid A \in \mathbb{M}_{n \times n} \right\}.$$

We also introduce its Lie algebra $\mathfrak{g}$ and the tangent map

$$\mathfrak{g} \to \mathrm{Hom}_T(\mathcal{J}, T/\mathcal{J})_{(1,-1)}. \tag{3.16}$$

As before, we will prove that it is bijective.

Lemma 3.24 (special liftings) *Let* $n \geq 4$. *Let* $\varphi \colon \mathcal{J} \to T/\mathcal{J}$ *be a homomorphism of degree* $(1, -1)$. *Then* $\varphi(y_i^2) = 0$ *for all* i. *Moreover, there exists a special lifting of* $\varphi_{|\mathfrak{p}}$, *that is, a degree* $(1, -1)$ *linear map*

$$\varphi' \colon \left\langle y_i^2, y_1 y_i \mid i = 1, \ldots, n \right\rangle \to T_{1,1}$$

such that

(1) $\varphi'(g) \mod (Q) = \varphi(g)$ *for every generator of* $\mathfrak{p}$,
(2) $\varphi'(y_i^2) = 0$ *and* $\varphi'(y_1 y_i) \in (y_1, y_i)$ *for all* i.

Proof Take a homomorphism $\varphi \colon \mathcal{J}/Q \to T/\mathcal{J}$ of degree $(1, -1)$ and any lift to a linear map $\varphi' \colon T_{0,2} \to T_{1,1}$. The syzygy $Q \cdot Q = \sum x_i^2 y_i^2$ implies $\sum_{i=1}^n x_i^2 \varphi'(y_i^2) \in (Q)$. By Lemma 3.19 we have $\varphi'(y_i^2) = 0$ for all i, after possible changing the lifting.

The syzygy $y_i \cdot (y_1 y_i) = y_1 \cdot y_i^2$ implies that $y_i \varphi'(y_1 y_i) \in \mathfrak{p} + (Q)$. Take $p \in \mathfrak{p}$ and $l \in T_{0,1}$ such that

$$y_i \varphi'(y_1 y_i) = p + l Q. \tag{3.17}$$

Let $l = \sum_{j=1}^n l_j y_j$ for $l_j \in \Bbbk$. Replacing p with $p + l_1 y_1 Q$, we may assume $l_1 = 0$. Pick $j \neq i$ and any $j' \neq 1, i, j$ (we use $n \geq 4$). The monomial $x_{j'} y_{j'} y_j$ appears in lQ with coefficient l_j, but does not appear elsewhere in (3.17), so $l_j = 0$. Then $l = l_i y_i$. Then $p \in (y_i) \cap \mathfrak{p} = y_i(y_1, y_i)$, so $p = y_i p'$ for some $p' \in (y_1, y_i)$. Clearly, $\varphi'(y_1 y_i) - p' \in (Q) \subset \mathcal{J}$, hence we may replace $\varphi'(y_1 y_i)$ by p'. The lifting thus obtained satisfies the conditions. □

Proposition 3.25 *Let $n \geq 4$ and $I_2 = 0$. The map (3.16) is bijective.*

Proof Take a lift φ' as in Lemma 3.24 and the unique lift $\hat{Q} \in T_{2,0}$ of $\varphi(Q)$. Recall from Lemma 3.24 that $\varphi'(y_i^2) = 0$ and $\varphi(y_i^2) = 0$ for all i. Write

$$\varphi'(y_1 y_i) = y_i L_i + y_1 M_i \quad \text{for} \quad i = 1, \ldots, n, \tag{3.18}$$

where $L_1 = M_1 = 0$. Since $y_1 Q = \sum_{i=1}^{n} x_i (y_1 y_i)$, we have $y_1 \varphi(Q) = \sum_{i=1}^{n} x_i \varphi(y_1 y_i)$. Since $I_2 = 0$, we have

$$y_1 \hat{Q} - \sum_{i=1}^{n} x_i \varphi'(y_1 y_i) \in \mu \cdot Q. \tag{3.19}$$

for some $\mu \in T_{1,0}$. Putting (3.18) into (3.19), we get

$$y_1 \hat{Q} - \sum_{i=1}^{n} x_i (y_i L_i + y_1 M_i) = \mu Q. \tag{3.20}$$

Comparing coefficients of y_i in (3.20), for $i > 1$, we obtain $L_2 = L_3 = \ldots = L_n = -\mu$. Moreover, $\varphi'(y_1 \cdot y_1) = \varphi'(y_1^2) = 0$ by the first paragraph, so $L_1 = 0$ and $M_1 = 0$. From the Eq. (3.20), we compute $\hat{Q} = \sum_{i \geq 2} x_i M_i + \mu x_1$.

Define $D \in \mathfrak{g}$ by setting $D(y_1) = \mu$ and $D(y_i) = M_i$ for $i \geq 2$. Then $D(y_i^2) = 0 = \varphi'(y_i^2)$ and moreover $\varphi'(y_1 y_i) = \mu y_i + y_1 M_i = D(y_1 y_i)$ for all $i \geq 2$. Therefore φ is the image of $D \in \mathfrak{g}$. □

Corollary 3.26 *Let $n \geq 4$ and $I_2 = 0$. The map*

$$\mathfrak{g} \to \bigoplus_{\alpha \geq 1} \mathrm{Hom}_T(\mathcal{J}, T/\mathcal{J})_{(\alpha, -\alpha)}.$$

is bijective.

Proof This follows from Proposition 3.25 and Lemma 3.20. □

4 Białynicki-Birula decompositions and retractions

In this section we formally define Białynicki-Birula decompositions and apply the results from the previous section to obtain the local retractions and prove Proposition 1.4 and Theorem 1.3. In total, these proofs apply three BB decompositions; we consider three different linear $\mathbb{G}_m$-actions, which correspond to the three retractions from Proposition 1.4.

Let $\mathbb{G}_m := \mathrm{Spec}(\mathbb{Z}[t^{\pm 1}])$ and $\overline{\mathbb{G}}_m := \mathrm{Spec}(\mathbb{Z}[t^{-1}]) \simeq \mathbb{A}^1_{\mathbb{Z}}$ be its compactification at infinity. Every $\mathbb{G}_m$-action on $\mathbb{A}^r_{\mathbb{Z}}$ induces a $\mathbb{G}_m$-action on $\mathcal{H}_{\mathbb{Z}}$.

The *Białynicki-Birula decomposition* of the $\mathbb{G}_m$-scheme $\mathcal{H}_{\mathbb{Z}}$ is a functor from $\mathbb{Z}$-schemes to sets given by

$$\mathcal{H}_{\mathbb{Z}}^+(B) := \left\{ \varphi \colon \overline{\mathbb{G}}_m \times B \to \mathcal{H}_{\mathbb{Z}} \mid \varphi \text{ is } \mathbb{G}_m\text{-equivariant} \right\}.$$

This functor is represented by a scheme $\mathcal{H}_{\mathbb{Z}}^+$, whose connected components are quasi-projective over $\mathbb{Z}$. This scheme comes with naturals maps:

- *Forgetting about the limit* by restricting φ to $\varphi_{|1 \times B} \colon B \to \mathcal{H}_{\mathbb{Z}}$. This induces a map $\theta_0 \colon \mathcal{H}_{\mathbb{Z}}^+ \to \mathcal{H}_{\mathbb{Z}}$.
- *Restricting to the limit* by restricting φ to $\varphi_{|\infty \times B} \colon B \to \mathcal{H}_{\mathbb{Z}}$. The family $\varphi_{|\infty \times B}$ is equivariant, hence the image lies in $\mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_m}$. We obtain a map

$$\pi \colon \mathcal{H}_{\mathbb{Z}}^+ \to \mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_m}.$$

- *Embedding of fixed points.* The trivial $\mathbb{G}_m$-action on $\mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_m}$ extends to $\overline{\mathbb{G}}_m \times \mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_m} \to \mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_m}$ and hence induces a map $i \colon \mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_m} \to \mathcal{H}_{\mathbb{Z}}^+$. We have $\pi \circ i = \mathrm{id}$ and $\theta_0 \circ i \colon \mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_m} \to \mathcal{H}_{\mathbb{Z}}$ is the embedding of fixed points. In particular, π is a retraction.

The existence of Białynicki-Birula decompositions for Hilbert schemes of points and totally divergent $\mathbb{G}_m$-action is proven in [19, Proposition 3.1]; in that paper $\Bbbk$ denotes a field, but the proof holds equally well for $\Bbbk = \mathbb{Z}$: indeed the proof of [19, Proposition 3.1] goes through without changes for $\Bbbk = \mathbb{Z}$ and its main nontrivial ingredient is the existence of the multigraded Hilbert scheme which holds over any commutative ring $\Bbbk$ [18]. Alternatively and more explicitly, one can take the standard affine $\mathbb{G}_m$-stable covering $\{U_\lambda\}_\lambda$ of $\mathcal{H}_{\mathbb{Z}}^+$, see [26, §18.1] and then glue $\mathcal{H}_{\mathbb{Z}}^+$ from U_λ^+, as in [20, Proposition 5.3]; neither of these two steps depends on the base ring.

Remark 4.1 The image of θ_0 is frequently nowhere dense. This happens in particular for the dilation action $\mathbb{G}_m \times \mathbb{A}^r \to \mathbb{A}^r$, given by $t \cdot (x_1, \ldots, x_r) = (tx_1, \ldots, tx_r)$, see [19, Proposition 3.2]. Hence, we cannot in general hope to prove that θ_0 is an open immersion. To remedy this, we choose a smooth algebraic group $\mathbb{Z}$-scheme G acting on $\mathbb{A}^r$ and extend the map θ_0 to

$$\theta \colon G \times \mathcal{H}_{\mathbb{Z}}^+ \to \mathcal{H}_{\mathbb{Z}},$$

which maps (g, x) to $g \cdot \theta_0(x)$. Below, G will be either $\mathbb{G}_a^r$ acting by translation or the unipotent group G defined in (3.4) and recalled below.

Recall the group G of linear transformations given in the basis $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ by

$$G = \left\{ \begin{pmatrix} I_n & A \\ 0 & I_n \end{pmatrix} \mid A \in \mathbb{M}_{n \times n} \right\}.$$

and its Lie algebra $\mathfrak{g}$.

We would now like to apply Białynicki-Birula decompositions to prove Proposition 1.4 for frames (for char $\mathbb{k} \neq 2$) and tweaked frames (for char $\mathbb{k} = 2$). To avoid dichotomy in proofs and for clarity, we abstract the necessary properties into a standalone definition.

Definition 4.2 (*Frame-like ideals*) Let $I \subset S$ be a homogeneous ideal and $J \subset T = \mathbb{k}[\mathbf{x}, \mathbf{y}]$ be a $\mathbb{N}^2$-homogeneous ideal of the form

$$J = IT + \mathfrak{m}_x^{a+1} + \mathfrak{p} + (Q),$$

where $\mathfrak{p} \subset (y_1, \ldots, y_n)^2$. We say that J is *frame-like* if the following conditions hold

(a) the scheme $\mathrm{Spec}(T/J)$ has TNT,
(b) the map $\mathfrak{g} \to \bigoplus_{\alpha \geq 1} \mathrm{Hom}_T(J, T/J)_{(\alpha, -\alpha)}$ is bijective,
(c) there exists a b such that $\mathfrak{p} \supset T_{2,b+1}$ and $I_b = 0$.

Lemma 4.3 *Let char $\mathbb{k} \neq 2$ and $n = \dim S \geq 3$. Let $I \subset S$ be an ideal with $\mathrm{depth}(S_+, S/I) \geq 3$, $I_2 = 0$. Let J be a frame of size a for I. Then J is frame-like (for all $b \geq 1$).*

Proof This follows from Proposition 3.10 and Corollary 3.16.   □

Lemma 4.4 *Let char $\mathbb{k} = 2$ and $n = \dim S \geq 4$. Let $I \subset S$ be an ideal with $\mathrm{depth}(S_+, S/I) \geq 3$, $I_n = 0$. Let $\mathcal{J}$ be a tweaked frame of size a for I. Then $\mathcal{J}$ is frame-like (for all $b \geq n$).*

Proof This follows from Proposition 3.23, Corollary 3.26 and from $(y_1^2, \ldots, y_n^2) \supset \mathfrak{m}_y^{n+1}$.   □

Let us fix a frame-like ideal J. The ideal J is $\mathbb{N}^2$-graded by $(\deg \mathbf{x}, \deg \mathbf{y})$ and hence its stabilizer contains a two-dimensional torus $\mathbb{G}_m^2$. We consider three of its one-dimensional sub-tori: $\mathbb{G}_x, \mathbb{G}_y, \mathbb{G}_{xy}$. They act on T by respectively

$$t \cdot (\mathbf{x}, \mathbf{y}) = \begin{cases} (t\mathbf{x}, \mathbf{y}) & \text{for } t \in \mathbb{G}_x \\ (\mathbf{x}, t\mathbf{y}) & \text{for } t \in \mathbb{G}_y \\ (t\mathbf{x}, t\mathbf{y}) & \text{for } t \in \mathbb{G}_{xy} \end{cases}$$

We identify $\Bbbk$-points of $\mathcal{H}_{\mathbb{Z}}$ with finite subschemes of affine space and with their ideals, with the conversion that $I \subset S$ and $J \subset T$, so for example $(\mathcal{H}_{\mathbb{Z}}, [J])$ is the pointed scheme $(\text{Hilb}(\mathbb{A}_{\mathbb{Z}}^{2n}), [J])$. Finally, we consider the following Diagram (4.1) of Hilbert schemes, where subscripts indicate the points of interest (see below for explanations).

$$
\begin{array}{c}
\mathbb{G}_a^{2n} \times (\mathcal{H}_{\mathbb{Z}}^+, [J]) \xrightarrow{\theta_{xy}} (\mathcal{H}_{\mathbb{Z}}, [J]) \\
\quad\quad {}_{i_{xy}}\!\nwarrow\uparrow\ \downarrow{}^{\pi_{xy}} \\
G \times (\mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_{xy}}, [J])^{+,\mathbb{G}_y} \xrightarrow{\theta_x} (\mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_{xy}}, [J]) \\
\quad\quad {}_{i_x}\!\nwarrow\uparrow\ \downarrow{}^{\pi_x} \\
(\mathcal{H}\mathrm{flag}_{\mathbb{Z}}^{\mathbb{G}_x \times \mathbb{G}_y}, [IT + \mathfrak{m}_x^{a+1} \subset J]) \xrightarrow{\mathrm{pr}_2} (\mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_x \times \mathbb{G}_y}, [J]) \\
\quad {}_{\mathrm{pr}_1}\!\downarrow \\
(\mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_x}, [I + \mathfrak{m}_x^{a+1}])
\end{array}
\tag{4.1}
$$

The scheme $(\mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_{xy}}, [J])^{+,\mathbb{G}_y}$ is the Białynicki-Birula decomposition of $(\mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_{xy}}, [J])$ with respect to the natural $\mathbb{G}_y$-action. The scheme $\mathcal{H}\mathrm{flag}_{\mathbb{Z}}$ is the flag Hilbert scheme, which parameterizes deformations of pairs of finite subschemes $R_1 \subset R_2$ in affine space (the flag Hilbert scheme is constructed as a closed subscheme of $\mathcal{H}_{\mathbb{Z}} \times \mathcal{H}_{\mathbb{Z}}$).

The map θ_{xy} was denoted by θ in the introduction. It is the forgetful map $\mathbb{G}_a^{2n} \times \mathcal{H}_{\mathbb{Z}}^+$ to $\mathbb{G}_a^{2n} \times \mathcal{H}_{\mathbb{Z}}$ composed with the translation action $\mathbb{G}_a^{2n} \times \mathcal{H}_{\mathbb{Z}} \to \mathcal{H}_{\mathbb{Z}}$. Similarly, the map θ_x is the forgetful map followed by the G-action on $\mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_{xy}}$.

The proof of Proposition 1.4 is a journey on Diagram (4.1), from its upper-right corner to the lower-left one. Specifically, each of the three parts of the proposition asserts the existence of a local retraction for one "hook" on this diagram. First two retractions will be obtained from the BB decompositions corresponding to $(\theta_{xy}, \pi_{xy}, i_{xy})$ and (θ_x, π_x, i_x) respectively. The last one is easily deduced from $(\mathrm{pr}_1, \mathrm{pr}_2)$. The conditions (a), (b) of the definition of frame-like ideal imply that $d\theta_{xy}$ and $d\theta_x$ are bijective in relevant points.

Proposition 4.5 *There exists an open $\mathbb{G}_{xy}$-stable neighbourhood $[J] \in U_{xy} \subset \mathcal{H}_{\mathbb{Z}}^+$ such that $(\theta_{xy})_{|\mathbb{G}_a^{2n} \times U_{xy}} \colon \mathbb{G}_a^{2n} \times U_{xy} \to \mathcal{H}_{\mathbb{Z}}$ is an open immersion.*

Proof The map θ_{xy} at $(0, [J])$ induces an injection on obstruction spaces [19, Thm 4.2] and a bijection on tangent spaces by Condition 4.2(a). Hence it is étale at this point. The torus $\mathbb{G}_{xy}$ normalizes $\mathbb{G}_a^{2n}$ in the automorphism scheme of $\mathcal{H}_{\mathbb{Z}}$, hence we obtain a semidirect subgroup $H := \mathbb{G}_{xy} \ltimes \mathbb{G}_a^{2n} \subset \mathrm{Aut}(\mathcal{H}_{\mathbb{Z}})$. This subgroup acts on $\mathbb{G}_a^{2n} \times \mathcal{H}_{\mathbb{Z}}^+$ by $(t, v_1) \cdot (v_2, x) := (t(v_1 + v_2)t^{-1}, tx)$ and θ_{xy} is H-equivariant. Hence, the étale locus of θ_{xy} has the form $\mathbb{G}_a^{2n} \times U$ for some $\mathbb{G}_m$-stable open $U \subset (\mathcal{H}_{\mathbb{Z}}, [J])^{+,\mathbb{G}_{xy}}$. Now, θ_{xy} is universally injective, hence $(\theta_{xy})_{|U}$ is an open immersion, see [29, Tag 025G]. Take $U_{xy} := U$. □

To repeat the argument above for θ_x we need to check universal injectivity for the G-action.

Lemma 4.6 *Let* $[J] \in (\mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_{xy}})^{+,\mathbb{G}_y}$ *be a point with limit point* $[J_0] \in \mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_x \times \mathbb{G}_y}$. *Assume that* $\mathfrak{g} \to \operatorname{Hom}_T(J_0, T/J_0)$ *is injective. Then the stabilizer of* $[J]$ *in* G *is trivial.*

Proof Suppose that there exists a point of the stabilizer of $[J]$ and identify it with a matrix A as in (3.4). For every $t \in \overline{\mathbb{G}}_m$ the element of G corresponding to the matrix $t^{-1}A$ stabilizes $t \cdot [J]$. Therefore, the tangent vector $A \in \mathfrak{g}$ maps to zero in $\operatorname{Hom}_T(J_0, T/J_0)$. Since the map is injective, $A = 0$. □

Proposition 4.7 *There exists an open* $\mathbb{G}_y$-*stable neighbourhood* $[J] \in U_y \subset \mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_{xy}}$ *such that* $(\theta_x)_{|G \times U_y} \colon G \times U_y \to \mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_{xy}}$ *is an open immersion.*

Proof Let $\mathfrak{g}$ be the Lie algebra of G. The tangent map $d(\theta_x)$ at $[J]$ is

$$\mathfrak{g} \oplus \bigoplus_{\alpha \geq 0} \operatorname{Hom}(J, T/J)_{(-\alpha, \alpha)} \to \bigoplus_{\alpha \in \mathbb{Z}} \operatorname{Hom}(J, T/J)_{(-\alpha, \alpha)},$$

so it is bijective by Condition 4.2(b). As in the proof of Proposition 4.5, we find a $\mathbb{G}_y$-stable U_y such that $(\theta_x)_{|G \times U_y}$ is étale. Then by Lemma 4.6 the map $(\theta_x)_{|G \times U_y}$ is universally injective, hence an open immersion [29, Tag 025G]. □

Proposition 4.8 *The map* pr_2 *is an isomorphism on the connected component of* $[IT + \mathfrak{m}_x^{a+1} \subset J] \in \mathcal{H}\mathrm{flag}_{\mathbb{Z}}^{\mathbb{G}_x \times \mathbb{G}_y}$.

Proof This is an easy consequence of the bi-homogeneity of J. The map $(T/(IT + \mathfrak{m}_x^{a+1}))_{(*,0)} \to (T/J)_{(*,0)}$ is bijective. Consider a base ring A and an equivariant deformation $T_A/\widetilde{J}$ of T/J over A. Let $g_1, \ldots, g_r$ be homogeneous generators of I and $\widetilde{g}_1, \ldots, \widetilde{g}_r$ be their unique lifts in $\widetilde{J}$. Let $\widetilde{I} := (\widetilde{g}_1, \ldots, \widetilde{g}_r) + \mathfrak{m}_x^{a+1}$, then $(T_A/\widetilde{I})_{(*,0)} \to (T_A/\widetilde{J})_{(*,0)}$ is bijective as well. Take any syzygy $s \in T^r$ between $g_1, \ldots, g_r$. Since $T_A/\widetilde{J}$ is A-flat, there is a syzygy $\widetilde{s} \in T_A^r$ lifting s. This means that $\widetilde{s}(\widetilde{g}_1, \ldots, \widetilde{g}_r) \in \widetilde{J}$. We have $\deg(s) = \deg(\widetilde{s}) = (*, 0)$, so $\widetilde{s}(\widetilde{g}_1, \ldots, \widetilde{g}_r) \in \widetilde{I}$. Hence, $[\widetilde{I} \subset \widetilde{J}]$ is the required deformation of $[IT + \mathfrak{m}_x^{a+1} \subset J]$. Uniqueness is evident. □

Proposition 4.9 *The projection* pr_1 *is a retraction of the connected component of* $[IT + \mathfrak{m}_x^{a+1} \subset J] \in \mathcal{H}\mathrm{flag}_{\mathbb{Z}}^{\mathbb{G}_x \times \mathbb{G}_y}$ *to the connected component of* $[I + \mathfrak{m}_x^{a+1}] \in \mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_x}$.

Proof The element Q is a non-zero-divisor on $T/I \cdot T$ by Lemma 3.2. Moreover, $T_{2,b} \subset \mathfrak{p}$ by Condition 4.2(c), thus every of $I + (x_1, \ldots, x_n)^{a+1} \subset S$ induces a deformation of J by keeping Q and $\mathfrak{p}$ fixed. This gives the required section of pr_1. □

Now, we prove the following abstract version of Proposition 1.4 from the introduction.

Proposition 4.10 *Let as before, J be a frame-like ideal*

(1) *the scheme $(\mathcal{H}_{\mathbb{Z}}, [J])$ locally retracts to $(\mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_{xy}}, [J])$,*

(2) *the scheme $(\mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_{xy}}, [J])$ locally retracts to $(\mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_x \times \mathbb{G}_y}, [J])$,*

(3) *$(\mathcal{H}_{\mathbb{Z}}^{\mathbb{G}_x \times \mathbb{G}_y}, [J])$ locally retracts to $(\mathrm{Hilb}^{\mathbb{G}_x}(\mathbb{A}^n), [I + (x_1, \ldots, x_n)^{a+1}])$.*

Proof The existence of the local retractions (1), (2), (3) is proven respectively in Proposition 4.5, Proposition 4.7, Propositions 4.8–4.9. □

Proof of Theorem 1.3 Fix a singularity type $\mathfrak{S}$ and let $n = 8$. By [30, Proposition 4.4] there exists a field $\Bbbk$ and a smooth general type surface Z over $\Bbbk$ and an embedding $Z \hookrightarrow \mathbb{P}_{\mathbb{Z}}^4$ such that $[Z] \in \mathrm{Hilb}(\mathbb{P}_{\mathbb{Z}}^4)$ has singularity type $\mathfrak{S}$. Let $S_0 := \bigoplus_i H^0(\mathbb{P}^4, \mathcal{O}(i)) = \Bbbk[x_1, \ldots, x_5]$. Let $M \gg 0$, in particular $M > n$, and let $I_0 := I(Z)_{\geq M} \subset S_0$, so that $[I_0] \in \mathrm{Hilb}^{\mathbb{G}_m}(\mathbb{A}^5)$ has singularity type $\mathfrak{S}$, see [18, Lemma 4.1]. Let $S = S_0[x_6, x_7, x_8]$ be a polynomial ring over S_0 and let $I = I_0 \cdot S$. Let

$$\mathrm{Tan}_I := \mathrm{Hom}_S(I, S/I). \tag{4.2}$$

Fix an action of $\mathbb{G}_{\mathrm{res}} = \mathbb{G}_m$ on S acting with weight one on coordinates x_6, x_7, x_8 and fixing S_0. Since I is generated by elements of S_0, we have $(\mathrm{Tan}_I) = (\mathrm{Tan}_I)_{\geq 0}$ with respect to the grading induced by $\mathbb{G}_{\mathrm{res}}$. Let $\mathcal{H} = \mathrm{Hilb}^{\mathbb{G}_m}(\mathbb{A}_{\mathbb{Z}}^8)$ and let $\theta \colon \mathcal{H}^+ \to \mathcal{H}$ be its Białynicki-Birula decomposition with respect to the $\mathbb{G}_{\mathrm{res}}$-action. Since Tan_I is non-negatively graded, the map θ is an open immersion near $[I]$. Hence, on an neighbourhood U of $[I] \in \mathcal{H}$ there is a retraction

$$U \to \mathcal{H}^{\mathbb{G}_{\mathrm{res}}}. \tag{4.3}$$

But a neighbourhood of $[I] \in \mathcal{H}^{\mathbb{G}_{\mathrm{res}}}$ is canonically isomorphic to a neighbourhood of $[I_0] \in \mathrm{Hilb}^{\mathbb{G}_m}(\mathbb{A}^5)$. Hence, $(\mathrm{Hilb}^{\mathbb{G}_m}(\mathbb{A}^5), [I_0])$ is a local retract of $(\mathrm{Hilb}^{\mathbb{G}_m}(\mathbb{A}^8), [I])$. We note that

(1) $I_n = 0$,

(2) $\mathrm{depth}(S_+, S/I) \geq 3$, because x_6, x_7, x_8 form a regular sequence.

We fix $a \geq \mathrm{reg}(I) + 1$, where I is the regularity of ideal I. Then by [14, Proposition 3.1] the component of H containing $[I + \mathfrak{m}_x^{a+1}]$ is isomorphic to the component of H containing $[I]$ (here the choice of large enough a is crucial).

If char $\Bbbk \neq 2$, then let J be a TNT frame for I of size a. If char $\Bbbk = 2$, then let J be a tweaked frame for I. By Lemma 4.3 or Lemma 4.4, the ideal J is n-frame-like.

By Proposition 4.10 we obtain a retraction from an open neighbourhood of $[J]$ in $(\mathcal{H}_{\mathbb{Z}}, [J])$ to a neighbourhood of $[I + \mathfrak{m}_x^{a+1}]$ in $H := \mathrm{Hilb}^{\mathbb{G}_m}(\mathbb{A}^8)$, which is isomorphic to a neighbourhood of $[I]$. Composing with (4.3), we obtain the desired retraction. □

Remark 4.11 Erman [14] proved that $\coprod_n \mathrm{Hilb}^{\mathbb{G}_m}(\mathbb{A}^n)$ satisfies Murphy's Law by a different reduction from Z to $I(Z)$ using a sufficiently positive embedding of Z. His method is not applicable here, since the obtained $I(Z)$ is generated by quadrics and we require $I_2 = 0$.

5 Corollaries of Theorem 1.3 and examples

Corollary 5.1 (Answer to Question 1.1) *There are non-reduced points on the schemes $\mathcal{H}_{\mathbb{Z}}$ and on $\mathcal{H}_{\mathbb{K}}$, where $\mathbb{K}$ is any field.*

Proof For every pointed scheme (Y, y) in the singularity type of $[\mathrm{Spec}(\mathbb{Z}[u]/u^2), V(u)]$ the ring $\mathcal{O}_{Y,y}$ is $\mathbb{Z}$-flat but not reduced, and hence contains an element v such that $\mathbb{Z}[v]/v^2 \subset \mathcal{O}_{Y,y}$. As in Theorem 1.3, suppose that (Y, y) is such a scheme with a retraction $(X, x) \to (Y, y)$ from an open subscheme X of $\mathcal{H}_{\mathbb{Z}}$. Then the pullback $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ has a section, hence is an injective homomorphism. In particular, $\mathcal{O}_{X,x}$ is non-reduced as well. This proves the claim for $\mathcal{H}_{\mathbb{Z}}$. The injective homomorphisms

$$\mathbb{Z}[v]/v^2 \hookrightarrow \mathcal{O}_{Y,y} \hookrightarrow \mathcal{O}_{X,x}$$

stay injective under $(-) \otimes_{\mathbb{Z}} \mathbb{Q}$, hence the claim follows for $\mathcal{H}_{\mathbb{Q}}$. To prove the claim for $\mathcal{H}_{\mathbb{F}_p}$, we argue as above for the singularity $[\mathrm{Spec}(\mathbb{F}_p[u]/(u^2))]$. The claim for arbitrary field $\mathbb{K}$ now follows from base change. □

Corollary 5.2 (Answer to Question 1.2) *The scheme $\mathcal{H}_{\mathbb{Z}} \to \mathrm{Spec}(\mathbb{Z})$ has components lying entirely in the fiber over $\mathrm{Spec}(\mathbb{Z}/p)$ for all primes p.*

Proof For every pointed scheme (Y, y) in the singularity type of $[\mathrm{Spec}(\mathbb{Z}/p)]$ we have $p\mathcal{O}_{Y,y} = 0$. As in Theorem 1.3, suppose that (Y, y) is such a scheme with a retraction $(X, x) \to (Y, y)$ from an open subscheme X of $\mathcal{H}_{\mathbb{Z}}$. Then the pullback map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ implies that $p\mathcal{O}_{X,x} = 0$, hence each component containing x lies entirely in characteristic p. □

We can extend Corollary 5.2 by considering higher infinitesimal neighbourhoods.

Corollary 5.3 *For every prime p and every $v \geq 0$ there exists a finite field $\mathbb{k}$ and a finite irreducible $\mathbb{F}_p$-scheme R with residue field $\mathbb{k}$ which lifts to $\mathbb{Z}/p^v$ but not to any ring A with $p^v A \neq 0$. In particular R does not lift to $\mathbb{Z}/p^{v+1}$.*

Proof Using Theorem 1.3, choose a member (Y, y) in the singularity type of $\mathfrak{S} = [\mathrm{Spec}(\mathbb{Z}/p^{\nu})]$ such that there exists a retraction $(X, x) \to (Y, y)$ from an open subscheme X of $\mathcal{H}_{\mathbb{Z}}$. Let $\mathbb{k} := \kappa(x) = \kappa(y)$ and $[R] := x$. Since (Y, y) in the type of $\mathfrak{S}$ in the smooth equivalence relation, we have $p^{\nu}\mathcal{O}_{Y,y} = 0$ and a morphism $\varphi\colon \mathrm{Spec}(\mathbb{Z}/p^{\nu}) \to Y$. Composing φ with the section of $X \to Y$, we get a lifting of R to $\mathrm{Spec}(\mathbb{Z}/p^{\nu})$. The scheme R embeds into $\mathbb{A}^{16}_{\mathbb{k}}$, hence its lifting over A would embed into $\mathbb{A}^{16}_A$ and give a morphism $\mathrm{Spec}(A) \to \mathcal{H}_{\mathbb{Z}}$, which (after perhaps localizing A) restricts to $\mathrm{Spec}(A) \to X$. Hence we obtain $\mathrm{Spec}(A) \to Y$ and so $p^{\nu}A = 0$. □

So far our arguments built upon Vakil's construction, which in turn depends on Mnëv-Sturmfels universality for incidence schemes [23] and on results about abelian covers [30, §4]. The Mnëv-Sturmfels construction requires $\mathbb{P}^2_{\mathbb{k}}$ to have enough $\mathbb{k}$-points, hence usually it does not work over $\mathbb{k} = \mathbb{F}_p$ (this is the reason why in Corollary 5.3 we do not obtain algebras with residue field $\mathbb{F}_p$). The theory of abelian covers, while in principle constructive, is not very prone to become explicit either.

In this final part we explicitly construct appropriate points of the Hilbert scheme by hand, bypassing Vakil's work, for several singularity types. First, we note that one can obtain explicit examples of non-reduced points on $\mathcal{H}_{\mathbb{Z}}$ by taking a TNT frame for the truncation of the cone over a curve from Mumford's famous example [17, §13] or the examples of Martin-Deschamps and Perrin [25]. We give an explicit example by framing a degree 3, genus -2 reducible curve from [25, Prop 0.6].

Example 5.4 Let $\mathbb{k}$ be of characteristic zero. Let $S = \mathbb{k}[x_1, \ldots, x_7, y_1, \ldots, y_7]$ be a polynomial ring and $K = (x_1^2, \; x_1x_2, \; x_2^2(x_3+x_4), \; x_1x_4^3+x_2(x_3+x_4)x_3^2)$. The regularity of K is four. Let $I = K \cap (x_1, \ldots, x_4)^4$ and let J be a TNT frame for I with $a = 5$. Then $[J] \in \mathcal{H}_{\mathbb{k}}$ is non-reduced.

Below, we give explicit components of $\mathrm{Hilb}_{\mathrm{pts}}(\mathbb{A}^6_{\mathbb{Z}})$ lying in characteristic p for small p; in fact we give $\mathbb{F}_p$-points of these components. The proof is obtained by replacing the construction of Theorem 1.3 by some explicit computations. Let $\mathbb{k} = \mathbb{F}_p$ and let $R \subset \mathbb{A}^n_{\mathbb{k}}$ be a finite scheme given by a homogeneous ideal. The examples below employ the following line of argument:

(1) check that $\dim(\mathrm{GL}_n \cdot [R]) = \dim_{\mathbb{k}} T_{\mathcal{H}^{\mathbb{G}_m}_{\mathbb{k}}, [R]}$,

(2) conclude that $[R] \in \mathcal{H}^{\mathbb{G}_m}_{\mathbb{k}}$ is smooth,

(3) verify that R does not lift to $W_2(\mathbb{k})$.

(4) use 1-3 to conclude that the component of $\mathcal{H}^{\mathbb{G}_m}_{\mathbb{Z}}$ containing $[R]$ lies entirely over $\mathbb{k}$. This is a known argument, see e.g. [13, Lemma 5.7]. If this holds, also a neighbourhood of $[R] \in \mathcal{H}^+_{\mathbb{Z}}$ lies over $\mathbb{k}$.

(5) check that R has TNT and conclude that $\theta\colon \mathcal{H}^+_{\mathbb{Z}} \to \mathcal{H}_{\mathbb{Z}}$ is an open immersion in a neighbourhood of $[R]$.

The heart of all examples is the observation that the ideal

$$K = (x_1 x_2 + x_3 x_4 + x_5 x_6) + (x_2^p, x_4^p, x_6^p)$$

satisfies Properties (1)-(3). It remains to reduce K to dimension zero so as not to lose these properties and additionally gain TNT. We present one such reduction below.

Example 5.5 Let $q = p^e$ be a prime power, let $\Bbbk = \mathbb{F}_p$ as before and consider the ideal $I' = (x_1 x_2 + x_3 x_4 + x_5 x_6) + (x_2^q, x_4^q, x_6^q) \subset S = \Bbbk[x_1, \ldots, x_6]$. Let I be its saturation and let $J = I + (x_1, x_3, x_5)^{q+1}$. Below all unjustified claims are checked with `Macaulay2` for $q = 3, 4, 5$. Hence, we obtain examples in characteristics ≤ 5. First, the stabilizer of J is 10-dimensional, given by

$$\begin{pmatrix} \lambda D^{-1} & 0 \\ 0 & D^T \end{pmatrix}, \tag{5.1}$$

where $\lambda \in \Bbbk^*$ and $D \in \mathrm{GL}_3$. Hence the GL_6-orbit is 26-dimensional.

To prove that J does not lift to $W_2(\Bbbk)$, we argue similarly as in [31, Proposition 6.1.1]. Let $\widetilde{S} := W_2(\Bbbk)[X_1, \ldots, X_6]$. Suppose that R lifts to $W_2(\Bbbk)$. Then there exists an ideal $\widetilde{J} \subset \widetilde{S}$ such that $\widetilde{S}/\widetilde{J}$ is an embedded deformation of S/J over $W_2(\Bbbk)$. In particular, the syzygy

$$(x_1 x_2 + x_3 x_4 + x_5 x_6)^3 = (x_1)^3 \cdot x_2^3 + (x_3)^3 \cdot x_4^3 + (x_5)^3 \cdot x_6^3$$

lifts to a syzygy between generators of $\widetilde{J}$, which means that

$$(X_1 X_2 + X_3 X_4 + X_5 X_6)^3 - X_1^3 X_2^3 - X_3^3 X_4^3 - X_5^3 X_6^3 \in p\widetilde{J}. \tag{5.2}$$

We have $p\widetilde{S} \simeq S$ as $\widetilde{S}$-modules and $p\widetilde{J} \simeq J$ in this isomorphism. Equation (5.2) translates into

$$2x_1 x_2 x_3 x_4 x_5 x_6 + (x_1 x_2)^2 (x_3 x_4 + x_5 x_6) + (x_3 x_4)^2 (x_1 x_2 + x_5 x_6)$$
$$+ (x_5 x_6)^2 (x_1 x_2 + x_3 x_4) \in J, \tag{5.3}$$

But $(x_1 x_2)^2 (x_3 x_4 + x_5 x_6) \equiv (x_1 x_2)^3 \equiv 0 \mod J$, hence (5.3) is equivalent to $2x_1 x_2 x_3 x_4 x_5 x_6 \in J$, which is false. Hence, we obtain a contradiction (the argument for $q = 4$ should be ramified here). Finally, we check directly that for $\mathrm{Tan} := \mathrm{Hom}_S(J, S/J)$ we have $\dim_\Bbbk(\mathrm{Tan})_0 = 26$, which verifies Property 1 and that $\dim_\Bbbk(\mathrm{Tan})_{<0} = 6$, which verifies Property 5.

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