HW1

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1 Problem 1

1.1 1. DNF

$\phi = (x \land y) \lor (x \land z) \lor (y \land z)$

1.2 2. CNF

$\psi = (x \lor y) \land (x \lor z) \land (y \lor z)$

1.3 3. Z3 script

```
(declare-const x Bool)
(declare-const y Bool)
(declare-const z Bool)

;; (x \land y) \lor (x \land z) \lor (y \land z)
(declare-fun phi (Bool Bool Bool) Bool)
(assert (= (phi x y z)
           (or (and x y) (or (and x z) (and y z)))))

;; (x \lor y) \land (x \lor z) \land (y \lor z)
(declare-fun psi (Bool Bool Bool) Bool)
(assert (= (psi x y z)
           (and (or x y) (and (or x z) (or y z)))))

;; constraint: check sat for not equal, so "equal" is not falsifiable
```

```
(assert (not (= (phi x y z) (psi x y z))))

(check-sat)
```

1.4 4. Z3py script

```
from z3 import *

x,y,z = Bools('x y z')

# (x \land y) \lor (x \land z) \lor (y \land z)
phi = Or (And(x,y), And(x,z), And(y,z))

# (x \lor y) \land (x \lor z) \land (y \lor z)
psi = And (Or(x,y), Or(x,z), Or(y,z))

s = Solver()

# constraint: check sat for not equal, so "equal" is not falsifiable
s.add(Not (phi == psi))
print (s.check())
```

2 Problem 2

2.1 (a) $((p \to q) \to q) \to ((q \to p) \to p)$

1.	$(p \to q) \to q$	assumption
2.	$(\neg p)$	assumption
3.	p	assumption
4.	$\perp$	$\neg$E 2, 3
5.	q	$\perp$E 4
6.	$(p \to q)$	$\to$I 3–5
7.	q	$\to$E 1, 6
8.	$\neg p \to q$	$\to$I 2–7
9.	$q \to p$	assumption
10.	$\perp$	$\neg$E 8, 9
11.	p	$\perp$E 10
12.	$((q \to p) \to p)$	$\to$I 9–11
13.	$((p \to q) \to q) \to ((q \to p) \to p)$	$\to$I 1–12

2.2 (c) $((p \to q) \wedge (q \to p)) \to ((p \vee q) \to (p \wedge q))$

1.	$(p \to q) \wedge (q \to p)$	assumption
2.	$(p \to q)$	$\wedge$E$_1$ 1
3.	$(q \to p)$	$\wedge$E$_2$ 1
4.	$(p \vee q)$	assumption
5.	p	assumption
6.	q	$\to$E 2
7.	$p \wedge q$	$\wedge$I 5, 6
8.	q	assumption
9.	p	$\to$E 3
10.	$p \wedge q$	$\wedge$I 8, 9
11.	$(p \wedge q)$	$\wedge$I 4, 5–7, 8–10
12.	$((p \vee q) \to (p \wedge q))$	$\to$I 4–11
13.	$((p \to q) \wedge (q \to p)) \to ((p \vee q) \to (p \wedge q))$	$\to$I 1–12

2.3 (d) $(p \to q) \to ((\neg p \to q) \to q)$

1.	$(p \to q)$	assumption
2.	$\neg p \to q$	assumption
3.	$\neg p \vee p$	LEM
4.	$\neg p$	assumption
5.	q	$\to$E 2,4
6.	p	assumption
7.	q	$\to$E 1,6
8.	q	$\to$E 3,4–5,6–7
9.	$(\neg p \to q) \to q$	$\to$I 2–8
10.	$(p \to q) \to ((\neg p \to q) \to q)$	$\to$I 1–9

3 Problem 3

3.1 rule $\neg$I

1.	$\phi \to \bot$	given
2.	ϕ	assumption
3.	$\bot$	$\to$E 1 2
4.	$\phi \to \bot$	$\to$I 2–3

3.2 rule $\neg$E

1.	ϕ	given
2.	$\phi \to \bot$	given
3.	$\bot$	$\to$E 1,2

3.3 rule $\neg\neg$E

1.	$(\phi \to \bot) \to \bot$	given

This won't work. We won't be able to conclude ϕ, as I cannot find a way to reach conclusion with only ϕ.

3.4 rule $\neg\neg$I

1.	ϕ	given
2.	$\phi \to \bot$	assumption
3.	$\bot$	$\to$E 1,2
4.	$(\phi \to \bot) \to \bot$	$\to$I 2,3

4 Problem 4

4.1 Base Case

Given wff ϕ, assume its height is 1. By the definition of height, there is no path in its parse tree, so it is an atom. Then $rank(\phi) = rank(p) = 1$ is satisfied.

4.2 Inductive Step

Given ϕ is a wff and its height is greater than 1. Thus it has at least 1 connective in it. Assume $height(\phi_n) = rank(\phi_n)$.

4.2.1 Suppose ϕ_{n+1} is the form of $\neg\phi_n$

Here the $\neg$ connective would add 1 to $height(\phi_n)$ as it introduce another path from $\neg$ to ϕ_n.
Thus $height(\neg\phi_n) = height(\phi_n) + 1 = rank(\phi_n) + 1 = rank(\neg\phi_n)$
Thus $height(\phi_{n+1}) = rank(\phi_{n+1})$

4.2.2 Suppose ϕ_{n+1} is the form of $\phi_{n_1} o \phi_{n_2}$

Here the o connective would add 1 to the higher one between ϕ_{n_1} and ϕ_{n_2}.
Thus $height(\phi_{n+1}) = 1 + max(height(\phi_{n_1}), height(\phi_{n_2})) = 1 + max(rank(\phi_{n_1}), rank(\phi_{n_2}))$
$= rank(\phi_{n+1})$
Thus $height(\phi_{n+1}) = rank(\phi_{n+1})$

5 Problem 5

5.1 (a)

5.1.1 $\neg, \wedge$

$A \vee B \equiv \neg(\neg A \wedge \neg B)$
Prove by truth table

A	B	¬ (¬ A ∧ ¬ B)	A ∨ B
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

$A \rightarrow B \equiv \neg(A \wedge \neg B)$
Prove by truth table

A	B	¬ (A ∧ ¬ B)	A → B
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

$\perp \equiv A \wedge \neg A$ By contradiction
$\top \equiv A \vee \neg A$ By LEM

5.1.2 $\neg, \rightarrow$

$A \vee B \equiv \neg A \rightarrow B$
Prove by truth table

A	B	(¬ A → B)	A ∨ B
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

$A \wedge B \equiv \neg(A \rightarrow \neg B)$
Prove by truth table

A	B	¬ (A → ¬ B)	A ∧ B
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

$\bot \equiv \neg(A \to A)$
Prove by truth table

A	$\neg$ (A $\to$ A)
T	F
F	F

$\top \equiv A \to A$
Prove by truth table

A	A $\to$ A
T	T
F	T

5.1.3 $\to, \bot$

$\neg A \equiv (A \to \bot)$ (proved in problem 3.1)

$A \vee B \equiv (A \to \bot) \to B$ (proved in problem 5.a.2: $A \vee B \equiv \neg A \to B$)

$A \wedge B \equiv (A \to (B \to \bot)) \to \bot$ (proved in problem 5.a.2: $A \wedge B \equiv \neg(A \to \neg B)$)

$\top \equiv \bot \to \bot$
Prove by truth table

$\bot$	$\bot \to \bot$
F	T

5.2 (b)

Suppose C contains all but neither $\neg$ nor $\bot$.
Assume there is a wff ϕ with connectives in C and assume all atoms in ϕ to be T. Then the conclusion of such wff will always be T, so C cannot simulate those wff's with F for conclusion.

5.3 (c)

6 Problem 6

Assume $\phi_1 \to \phi_2 \to \phi_3 \to ... \to \phi_n \to \psi$ is a tautology.
By the definition of completeness, which is when $\phi_1, \phi_2, \phi_3, ..., \phi_n$ are T, then ψ is T, and here tautology makes sure any value for $\phi_1, \phi_2, \phi_3, ..., \phi_n$ will imply ψ to be T, which is tighter than the assumption of completeness, $\phi_1, \phi_2, \phi_3, ..., \phi_n \vdash \psi$ has a proof.

Assume $\phi_1, \phi_2, \phi_3, ..., \phi_n \vdash \psi$ has a proof. By the definition of soundness, if $\phi_1, \phi_2, \phi_3, ..., \phi_n \vdash \psi$ has a proof, $\phi_1, \phi_2, \phi_3, ..., \phi_n \vDash \psi$ meaning if all premise are T, the conclusion is also T. This make sure that in $\phi_1 \to \phi_2 \to \phi_3 \to ... \to \phi_n \to \psi$ if ϕ_n is T, ψ is T. And for $\phi_1 \to \phi_2 \to \phi_3 \to ... \to \phi_n \to \psi$ to be F, the only case is that when ϕ_n is T and ψ becomes F, which is eliminated by soundness. Thus $\phi_1 \to \phi_2 \to \phi_3 \to ... \to \phi_n \to \psi$ is a tautology.

Thus $\phi_1, \phi_2, \phi_3, ..., \phi_n \vdash \psi$ has a proof iff $\phi_1 \to \phi_2 \to \phi_3 \to ... \to \phi_n \to \psi$ is a tautology.