Fuzzy labelling semantics for quantitative argumentation

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Abstract

Abstract argumentation is a well-studied model for evaluating arguments. Recently, evaluating *argument strength* in quantitative argumentation has received increasing attention, in which arguments are evaluated through *acceptability degree*. However, argument strength solely defined on acceptability degree appears not sufficient in practical applications. In this paper, we provide a novel quantitative method called *fuzzy labelling* for *fuzzy argumentation systems*, in which a *triple* of *acceptability*, *rejectability* and *undecidability degrees* is used to evaluate argument strength. Such a setting sheds new light on defining argument strength and provides a deeper understanding of the status of arguments. Specifically, we investigate the *postulates* of fuzzy labelling, which present the rationality requirements for semantics concerning the acceptability, rejectability and undecidability degrees. We then propose a class of fuzzy labelling semantics conforming to the above postulates and investigate the properties. Finally, we demonstrate that fuzzy labelling semantics can be considered both a conservative generalization of classical labelling semantics and a labelling version of fuzzy extension semantics.

Keywords: abstract argumentation, quantitative argumentation, fuzzy labelling semantics, evaluation of strength.

1 Introduction

Abstract argumentation framework (AF) is a well-studied model for reasoning and decision-making in conflict situations [12, 33]. An AF is a directed graph whose nodes represent arguments and arrows represent attacks between arguments. Its applications span diverse domains, including decision-making [3], explainability in AI [28], etc.

To capture the uncertainty in real argumentation, Dung's AF has been extended to various *quantitative argumentation systems* (QuAS) via different quantitative approaches, such as *weighted argumentation systems* [15, 34], *probabilistic argumentation systems* [39, 41, 45], *fuzzy argumentation systems* [29, 42, 51, 57], etc. Generally speaking, each argument or attack in a QuAS is assigned an *initial degree*, usually expressed by a *numerical value* in $[0, 1]$ from a meta-level, so that richer real-world applications can be properly described.

Evaluating arguments is a central topic in the literature and commonly achieved through *semantics* [8]. For example, the well-known *extension semantics* [33] and *labelling semantics* [20] are designed to deal with classical AFs, giving sets of acceptable arguments and labels {accepted, rejected, undecided} over arguments, respectively, while in QuAS, the *gradual semantics* are used for evaluating *argument strength* by assigning each argument a numerical value in $[0, 1]$ as the so-called *acceptability degree*.

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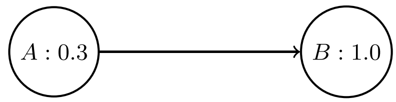


FIGURE 1. Getting vaccinated or not.

The study of gradual semantics has received extensive attention in the literature [7, 11, 14, 25, 46]. Most of the work focuses on the evaluation of the acceptability degree. However, we argue that this approach may not be sufficient in practical applications. Evidence in cognitive science [18, 19] reveals that the strength of positivity and negativity should be separately measured in the evaluative process. Actually, considering both the positive and negative aspects is a common practice in many areas [3, 32, 38, 44, 47, 54]. While the acceptability degree measures the extent to which an argument can be accepted, reflecting its positive strength, it is also important to characterize the extent to which it can be rejected, reflecting its negative strength.

To this end, we propose the concept of *rejectability degree*, measuring the extent to which an argument can be rejected according to the impact of its attackers. The rejectability degree helps to make more informed decisions, especially in cases where minimizing attack is crucial. For instance, politicians may prefer to choose 'safer' arguments (i.e., suffer less attack) to avoid criticism or risks. In addition, we introduce the notion of *undecidability degree*, which measures the extent to which the argument cannot be decided to be accepted or rejected. This notion allows for intuitively capturing the degree of 'uncertainty' or 'don't know', which is widely adopted in various fields, such as Dempster-Shafer theory [30, 50], subjective logic [43], etc. We illustrate the above idea with the example below.

EXAMPLE 1.
Consider the following scenario:

- A: getting vaccinated may cause side effects.
- B: everyone should get vaccinated due to the viral pandemic.

This instance is represented as a QuAS in Figure 1, in which A attacks B, A and B are assigned initial degrees 0.3 and 1, respectively.

We analyze the strength of A and B as follows (shown in Table 1).

1. The acceptability degree of A remains its initial degree 0.3 since A has no attackers, while the acceptability degree of B is $1 - 0.3 = 0.7$, i.e., obtained from weakening its initial degree through attacker A.
2. The rejectability degree of A is 0 since it has no attackers. The rejectability degree of B is 0.3 since its attacker A has acceptability degree 0.3, and it is reasonable to reject B to the same extent.
3. The undecidability degree of A is $1 - 0.3 = 0.7$ and B is $1 - (0.7 + 0.3) = 0$.

Existing evaluation methods suggest that argument B is preferable to A due to its higher acceptability degree ($0.7 > 0.3$). However, in the real world, many people choose not to get vaccinated due to potential side effects, i.e., prefer A to B. Our approach suggests that argument A is preferable to B due to its lower rejectability degree ($0 < 0.3$). So for those people with safety concerns, the rejectability degree appears more critical to avoid risks.

TABLE 1. Extended argument strength for A and B

	acceptability	rejectability	undecidability
A	0.3	0	0.7
B	0.7	0.3	0

In this paper, we propose a more comprehensive method called *fuzzy labelling* for fuzzy argumentation systems. This method describes argument strength as a triple consisting of acceptability, rejectability and undecidability degrees. In essence, fuzzy labelling combines gradual semantics and labelling semantics by assigning a numerical value to each label. This setting offers new insights into argument strength and a deeper understanding of the status of arguments. Furthermore, it is beneficial to identify the status of rejected and undecided in judgment aggregation [23, 24], designing algorithms [21, 27], explaining semantics [49], etc.

After introducing the framework, we propose a class of *fuzzy labelling semantics* using the well-known *postulate-based approach* [53] in two steps. (i) Investigate the *postulates* for fuzzy labelling, which adapt the criteria for classical labelling semantics [22] and incorporate the notion of *tolerable attack* [57]. (ii) Formalize fuzzy labelling semantics that conform to the above postulates. We will explore the properties of our semantics and demonstrate that fuzzy labelling semantics can be viewed as a conservative generalization of classical labelling semantics [8], and can be seen as a labelling version of fuzzy extension semantics in [57].

The remaining part is structured as follows: Section 2 introduces some basic concepts. Section 3 defines postulates and semantics in terms of fuzzy labelling. Section 4 explores the properties of fuzzy labelling semantics and discusses its links with classical labelling semantics. Section 5 examines the relationships between fuzzy labelling semantics and fuzzy extension semantics. Section 6 discusses related work and concludes the paper. All proofs of the formal results are included in the Appendix.

2 Preliminaries

2.1 Fuzzy set theory

DEFINITION 1 ([58]).
A *fuzzy set* is a pair (X, S) in which X is a nonempty set called the *universe* and $S : X \to [0,1]$ is the associated *membership function*. For each $x \in X$, $S(x)$ is called the *grade* of membership of x in X.

For convenience, when the universe X is fixed, a fuzzy set (X, S) is identified by its membership function S, which can be represented by a set of pairs (x, a) with $x \in X$ and $a \in [0,1]$. We stipulate that all pairs $(x, 0)$ are omitted from S.

For instance, the following are fuzzy sets with universe $\{A, B, C\}$:

$$S_1 = \{(A, 0.5)\},\ S_2 = \{(B, 0.8), (C, 0.9)\},\ S_3 = \{(A, 0.8), (B, 0.8), (C, 1)\}.$$

Note that $S_1(A) = 0.5, S_1(B) = S_1(C) = S_2(A) = 0$ and in S_3 every element has a non-zero grade.

A *fuzzy point* is a fuzzy set containing a unique pair (x, a). We may identify a fuzzy point by its pair. For example, S_1 is a fuzzy point and identified by $(A, 0.5)$.

Let S_1 and S_2 be two fuzzy sets. Say S_1 is a *subset* of S_2, denoted by $S_1 \subseteq S_2$, if for any $x \in X$, $S_1(x) \leq S_2(x)$. Conventionally, we write $(x, a) \in S$ if a fuzzy point (x, a) is a subset of S. Moreover, we shall use the following notations:

- the *union* of S_1 and S_2: $S_1 \cup S_2 = \{(x, \max\{S_1(x), S_2(x)\}) \mid x \in X\}$;
- the *intersection* of S_1 and S_2: $S_1 \cap S_2 = \{(x, \min\{S_1(x), S_2(x)\}) \mid x \in X\}$;
- the *complement* of S: $S^c = \{(x, 1 - a) \mid S(x) = a\}$.

In the example, $S_1(x) \leq S_3(x)$ for each element x, thus fuzzy point S_1 is a subset of S_3, written as $(A, 0.5) \in S_3$. Similarly, it is easy to check: (i) $S_2 \subseteq S_3$; (ii) $S_2 \cup S_3 = \{(A, 0.8), (B, 0.8), (C, 1)\}$; (iii) $S_1 \cap S_3 = \{(A, 0.5)\}$; (iv) $S_3^c = \{(A, 0.2), (B, 0.2)\}$.

2.2 Fuzzy argumentation system

In this paper, we concentrate on *fuzzy argumentation system* where arguments and attacks are initially assigned degrees within $[0, 1]$ to characterize the uncertainty in the world.

DEFINITION 2.
A *fuzzy argumentation system* (FAS) over a finite set of arguments *Args* is a pair $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ in which $\mathcal{A} : Args \to [0, 1]$ and $\mathcal{R} : Args \times Args \to [0, 1]$ are total functions.

In Definition 2, $\mathcal{A}$ and $\mathcal{R}$ are fuzzy sets of arguments and attacks. $\mathcal{A}$ can be denoted by pairs $(A, \mathcal{A}(A))$ in which $\mathcal{A}(A)$ is the *initial degree* of A, and $\mathcal{R}$ can be denoted by pairs $\big((A, B), \mathcal{R}(A, B)\big)$ or simply $\big((A, B), \mathcal{R}_{AB}\big)$. Moreover, we denote by $Att(A)$ the set of all attackers of A, i.e., $Att(A) = \{B \in Args \mid \mathcal{R}_{BA} \neq 0\}$. Intuitively, when all the arguments and attacks are assigned initial degree 0 or 1, the framework coincides with Dung's AF.

In Definition 3, we define *attack intensity* to measure the impact of a set of arguments towards an argument. We choose the 'min' operator to aggregate the degree of the attacker and attack relation, and the 'max' operator to aggregate the impact of several attackers.[1]

DEFINITION 3.
Let $\langle \mathcal{A}, \mathcal{R} \rangle$ be an FAS and $A, B \in Args$. We define that

- the *attack intensity* of (B, b) towards A w.r.t. $\mathcal{R}_{BA}$ is $b * \mathcal{R}_{BA}$, and
- the *attack intensity* of $\mathcal{S}$ towards A is $\max_{B \in Att(A)} \mathcal{S}(B) * \mathcal{R}_{BA}$,

where $*$ is a shorthand s.t. $a * b = \min\{a, b\}$.

To preserve classical semantics and their properties, we incorporate the notion of *tolerable attack* [57]: an attack is considered *tolerable* if the sum of the attacker's attack intensity and the attackee's degree is not greater than 1.

DEFINITION 4.
Let $\langle \mathcal{A}, \mathcal{R} \rangle$ be an FAS, $(A, a), (B, b) \in \mathcal{A}$ and $\big((A, B), \mathcal{R}_{AB}\big) \in \mathcal{R}$. If $a * \mathcal{R}_{AB} + b \leq 1$, then the attack from (A, a) to (B, b) is called *tolerable*, otherwise it is called *sufficient*.

[1] There exist alternative operators, such as the product operator and the Łukasiewicz operator [35, 42]. Using 'min' and 'max' provides a uniform and convenient setting for elaborations in the paper.

EXAMPLE 2 (Cont.).
Consider the FAS $\langle \mathcal{A}, \mathcal{R} \rangle$ depicted in Example 1, in which $\mathcal{A} = \{(A, 0.3), (B, 1)\}$ and $\mathcal{R} = \{\big((A, B), 1\big)\}$. Then the initial degree of A is 0.3 and B is 1. Additionally, $Att(A) = \varnothing$ and $Att(B) = \{A\}$. Moreover, the attack intensity of $(A, 0.3)$ towards B w.r.t. $\big((A, B), 1\big)$ is $0.3 * 1 = \min\{0.3, 1\} = 0.3$. Since $0.3 * 1 + 1 > 1$, the attack from $(A, 0.3)$ to $(B, 1)$ is sufficient.

3 Fuzzy labelling semantics for FAS

3.1 Fuzzy labelling and its postulates

In this section, we extend classical labelling [20] to *fuzzy labelling* for FAS. While classical labelling assigns each argument a label from {accepted, rejected, undecided}, fuzzy labelling assigns each label a degree corresponding to acceptability, rejectability and undecidability, respectively.

DEFINITION 5.
Let $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an FAS over a finite set of arguments *Args*. A *fuzzy labelling* for $\mathcal{F}$ is a total function

$$FLab_{\mathcal{F}} : Args \rightarrow [0,1] \times [0,1] \times [0,1].$$

We denote $FLab_{\mathcal{F}}(A)$ by a triple (A^a, A^r, A^u), where each element is, respectively, called the *acceptability*, *rejectability*, *undecidability degree* of argument A. For convenience, *FLab* can also be written as a triple $(FLab^a_{\mathcal{F}}, FLab^r_{\mathcal{F}}, FLab^u_{\mathcal{F}})$, where each $FLab^{\circ}_{\mathcal{F}}$ is a fuzzy set defined as $\{(A, A^{\circ}) \mid A \in Args\}$ with $\circ \in \{a, r, u\}$.

For simplicity, we will use the shorthand *FLab* instead of $FLab_{\mathcal{F}}$, and use *acceptability* (resp. *rejectability, undecidability*) *arguments* to refer to the elements in $FLab^a$ (resp. $FLab^r$, $FLab^u$).

EXAMPLE 3 (Cont.).
Continue Example 2. Let

$$FLab = (\{(A, 0.3), (B, 0.7)\}, \{(B, 0.3)\}, \{(A, 0.7)\})$$

be a fuzzy labelling for FAS. Then $FLab(A) = (0.3, 0, 0.7)$ and $FLab(B) = (0.7, 0.3, 0)$. That is, the acceptability, rejectability and undecidability degree of A is 0.3, 0 and 0.7, respectively. Similarly, the corresponding degree of B is 0.7, 0.3 and 0, respectively.

We aim to use fuzzy labelling to generalize several widely studied classical semantics, including grounded, preferred, etc. (see [8] for an overview). To achieve this, we provide a set of *postulates*, each representing a rational constraint on acceptability, rejectability or undecidability degree.

In the literature, the initial degree usually represents the maximal degree to which an argument can be accepted [7, 11, 29, 57]. Accordingly, our first postulate, called *Bounded*, states that the acceptability degree of an argument is bounded by its initial degree.

POSTULATE 1 (Bounded, BP). A fuzzy labelling satisfies the *Bounded Postulate* over an FAS $\langle \mathcal{A}, \mathcal{R} \rangle$ iff $\forall A \in Args$, $A^a \leq \mathcal{A}(A)$. □

As shown before, the undecidability degree measures the extent to which an argument cannot be decided to be accepted or rejected. It allows representing the degree of 'uncertainty' or 'don't know' about the argument. This leads to the second postulate, called *Uncertainty*.

POSTULATE 2 (Uncertainty, UP). A fuzzy labelling satisfies the *Uncertainty Postulate* over an FAS $\langle \mathcal{A}, \mathcal{R} \rangle$ iff $\forall A \in Args$, $A^u = 1 - A^a - A^r$. □

In the following, we establish postulates to refine three basic elements of classical semantics: *conflict-free*, *admissible* and *complete*. Note that conflict-free is a precondition for defining semantics. Additionally, following [10], we adopt the term 'admissible semantics' for the convenience of presentation, even though it is not considered as a semantics in Dung's original work.

According to [8], classical conflict-free requires that no conflict should be allowed within the set of accepted arguments. The *Tolerability Postulate* extends the idea, stating that attacks among acceptability arguments should be tolerable.

POSTULATE 3 (Tolerability, TP). A fuzzy labelling satisfies the *Tolerability Postulate* over an FAS $\langle \mathcal{A}, \mathcal{R} \rangle$ iff $\forall A \in Args$,

$$\max_{B \in Att(A)} B^a * \mathcal{R}_{BA} + A^a \leq 1.$$

We stipulate that $\max_{B \in Att(A)} B^a * \mathcal{R}_{BA} = 0$ if $Att(A) = \varnothing$. □

Classical admissible semantics requires that an argument can be accepted (or rejected) only if there is a reason to do so. We introduce two postulates, *Weakened* and *Defense*, to generalize this idea.

Weakened states that an argument can be rejected to some degree only if it receives at least the same attack intensity from its acceptability attackers. This postulate extends the classical idea that an argument can be 'rejected' only if it has an 'accepted' attacker.

POSTULATE 4 (Weakened, WP). A fuzzy labelling satisfies the *Weakened Postulate* over an FAS $\langle \mathcal{A}, \mathcal{R} \rangle$ iff $\forall A \in Args$,

$$A^r \leq \max_{B \in Att(A)} B^a * \mathcal{R}_{BA}.$$

□

Defense states that an argument can be accepted to some degree only if all its sufficient attackers are rejected to at least the same degree. This postulate extends the classical idea that an argument can be 'accepted' only if all its attackers are 'rejected'.

POSTULATE 5 (Defense, DP). A fuzzy labelling satisfies the *Defense Postulate* over an FAS $\langle \mathcal{A}, \mathcal{R} \rangle$ iff $\forall A \in Args$,

$$A^a \leq \min_{B \in Att(A)} \{\max\{B^r, 1 - \mathcal{A}(B) * \mathcal{R}_{BA}\}\}.$$

We stipulate that $\min_{B \in Att(A)} \{\max\{B^r, 1 - \mathcal{A}(B) * \mathcal{R}_{BA}\}\} = 1$ if $Att(A) = \varnothing$. □

Theorem 1 provides an explanation for DP, indicating that a fuzzy labelling satisfies DP if and only if the acceptability degree of any argument is not greater than the rejectability degree of its sufficient attackers. Therefore, DP precisely conveys the intended concept.

THEOREM 1.
A fuzzy labelling *FLab* satisfies DP iff for any argument $A \in Args$ and $B \in Att(A)$, $(B, \mathcal{A}(B))$ sufficiently attacks (A, A^a) implies $A^a \leq B^r$.

Finally, we establish postulates to refine complete semantics. While admissible semantics requires a reason for accepting and rejecting an argument to some degree, complete semantics goes further by requiring that one cannot leave the degree undecided that should be accepted or rejected.

The *Strict Weakened Postulate* is the strict version of WP. It states that the rejectability degree of an argument should equal the attack intensity from its acceptability attackers. This postulate ensures that one cannot leave the degree undecided that should be rejected.

POSTULATE 6 (Strict Weakened, SWP). A fuzzy labelling satisfies the *Strict Weakened Postulate* over an FAS $\langle \mathcal{A}, \mathcal{R} \rangle$ iff $\forall A \in Args$,

$$A^r = \max_{B \in Att(A)} B^a * \mathcal{R}_{BA}.$$

The *Strict Defense Postulate* is the strict version of DP. □

POSTULATE 7 (Strict Defense, SDP). A fuzzy labelling satisfies the *Strict Defense Postulate* over an FAS $\langle \mathcal{A}, \mathcal{R} \rangle$ iff $\forall A \in Args$,

$$A^a = \min\{\min_{B \in Att(A)} \{\max\{B^r, 1 - \mathcal{A}(B) * \mathcal{R}_{BA}\}\}, \mathcal{A}(A)\}.$$

□

Theorem 2 provides an explanation for SDP, demonstrating that if a fuzzy labelling satisfies SDP, then the acceptability degree of an argument is either constrained by its attackers or determined by its initial degree. Consequently, SDP ensures that one cannot leave the degree undecided that should be accepted.

THEOREM 2.
If a fuzzy labelling *FLab* satisfies SDP, then for any $A \in Args$,

$$A^a = \min\{\min_{B \in S} B^r, 1 - \max_{B \notin S} \mathcal{A}(B) * \mathcal{R}_{BA}, \mathcal{A}(A)\}$$

where $S = \{B \in Args \mid (B, \mathcal{A}(B))$ sufficiently attacks $(A, A^a)\}$. We stipulate that $\min_{B \in S} B^r = 1$ if $S = \varnothing$ and $\max_{B \notin S} \mathcal{A}(B) * \mathcal{R}_{BA} = 0$ if $Args \setminus S = \varnothing$.

Theorem 3 shows the links between the above postulates.

THEOREM 3.
The postulates have the following properties:

1. *BP* + *UP* + *DP* implies *TP*;
2. *SWP* implies *WP*;
3. *SDP* implies *BP*;
4. *SDP* implies *DP*.

3.2 Fuzzy labelling semantics

In this section, we apply fuzzy labelling to extend classical semantics. In essence, fuzzy labelling semantics can be seen as a quantitative generalization of classical labelling semantics. Consider a fixed FAS $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ and a fuzzy labelling *FLab*.

We start by introducing *conflict-free fuzzy labelling*, which requires that attacks within the set of acceptability arguments should be tolerable. The corresponding postulates are BP, UP and TP.

TABLE 2. Postulates for conflict-free/admissible/complete fuzzy labellings

Fuzzy Labelling	Postulates
conflict-free	BP, UP, TP
admissible	BP, UP, WP, DP
complete	BP, UP, SWP, SDP

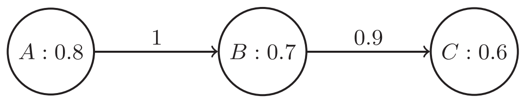


FIGURE 2. The FAS $\mathcal{F}$ in Example 4

DEFINITION 6 (Conflict-free Fuzzy Labelling).
A fuzzy labelling is *conflict-free* iff it satisfies BP, UP and TP.

We now define *admissible fuzzy labelling*, which requires that an argument can be accepted and rejected to some degree only if there is a reason to do so. The corresponding postulates are BP, UP, WP and DP.

DEFINITION 7 (Admissible Fuzzy Labelling).
A fuzzy labelling is *admissible* iff it satisfies BP, UP, WP and DP.

While admissible fuzzy labelling requires a reason for accepting and rejecting an argument to some degree, *complete fuzzy labelling* goes further by requiring that one cannot leave the degree undecided that should be accepted or rejected. The corresponding postulates are BP, UP, SWP and SDP.

DEFINITION 8 (Complete Fuzzy Labelling).
A fuzzy labelling is *complete* iff it satisfies BP, UP, SWP and SDP.

The postulates for conflict-free, admissible and complete fuzzy labellings are summarized in Table 2.

EXAMPLE 4.
Consider a fuzzy argumentation system (Figure 2) over $Args = \{A, B, C\}$

$$\mathcal{F} = \langle \{(A, 0.8), (B, 0.7), (C, 0.6)\}, \{\big((A, B), 1\big), \big((B, C), 0.9\big)\}\rangle.$$

Consider two fuzzy labellings $FLab_1$ and $FLab_2$ as presented in Table 3. First, BP and UP are easy to check by definition. Then from

$$A^{a_1} * \mathcal{R}_{AB} + B^{a_1} = \min\{0.5, 1\} + 0.4 \leq 1, \qquad \text{(By TP)}$$

$$B^{a_1} * \mathcal{R}_{BC} + C^{a_1} = \min\{0.4, 0.9\} + 0.6 \leq 1, \qquad \text{(By TP)}$$

TABLE 3. Two fuzzy labellings $FLab_1$ and $FLab_2$

	acceptability	rejectability	undecidability
	$A^{a_1} = 0.5$	$A^{r_1} = 0$	$A^{u_1} = 0.5$
$FLab_1$	$B^{a_1} = 0.4$	$B^{r_1} = 0.5$	$B^{u_1} = 0.1$
	$C^{a_1} = 0.6$	$C^{r_1} = 0.4$	$C^{u_1} = 0$
	$A^{a_2} = 0.8$	$A^{r_2} = 0$	$A^{u_2} = 0.2$
$FLab_2$	$B^{a_2} = 0.2$	$B^{r_2} = 0.8$	$B^{u_2} = 0$
	$C^{a_2} = 0.6$	$C^{r_2} = 0.2$	$C^{u_2} = 0.2$

we deduce that $FLab_1$ satisfies TP. Therefore, $FLab_1$ is conflict-free. However, since $(B, 0.7)$ sufficiently attacks $(C, 0.6)$ but $C^{a_1} > B^{r_1}$, it follows that $FLab_1$ violates DP by Theorem 1 and thus it is not admissible.

Next, we demonstrate that $FLab_2$ satisfies SDP and SWP:

$$
\begin{aligned}
A^{a_2} &= \min\{1, \mathcal{A}(A)\} = \{1, 0.8\} = 0.8 && \text{(By SDP)}\\
B^{a_2} &= \min\left\{\max\{A^{r_2}, 1 - \mathcal{A}(A) * \mathcal{R}_{AB}\}, \mathcal{A}(B)\right\} && \text{(By SDP)}\\
&= \min\{\max\{0, 1 - 0.8 * 1\}, 0.7\}\\
&= 0.2\\
C^{a_2} &= \min\left\{\max\{B^{r_2}, 1 - \mathcal{A}(B) * \mathcal{R}_{BC}\}, \mathcal{A}(C)\right\} && \text{(By SDP)}\\
&= \min\{\max\{0.8, 1 - 0.7 * 0.9\}, 0.6\}\\
&= 0.6\\
A^{r_2} &= 0 && \text{(By SWP)}\\
B^{r_2} &= A^{a_2} * \mathcal{R}_{AB} = \min\{0.8, 1\} = 0.8 && \text{(By SWP)}\\
C^{r_2} &= B^{a_2} * \mathcal{R}_{BC} = \min\{0.2, 0.9\} = 0.2 && \text{(By SWP)}
\end{aligned}
$$

Therefore, $FLab_2$ is admissible and also complete.

In the following, we refine several widely studied classical semantics, including grounded, preferred, semi-stable and stable, by imposing maximality or minimality constraints on complete semantics.

Grounded semantics is characterized by the minimal set of accepted arguments. *Grounded fuzzy labelling* refines this notion by requiring that the set of acceptability arguments be minimal (w.r.t. fuzzy set inclusion, similarly in subsequent discussions) among all complete fuzzy labellings.

DEFINITION 9 (Grounded Fuzzy Labelling).
FLab is a *grounded fuzzy labelling* iff it is a complete fuzzy labelling where $FLab^a$ is minimal among all complete fuzzy labellings.

Preferred semantics is characterized by the maximal set of accepted arguments. To refine it, *preferred fuzzy labelling* requires the set of acceptability arguments to be maximal among all complete fuzzy labellings.

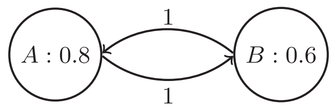


FIGURE 3. The FAS $\mathcal{F}$ in Example 5

DEFINITION 10 (Preferred Fuzzy Labelling).
FLab is a *preferred fuzzy labelling* iff it is a complete fuzzy labelling where $FLab^a$ is maximal among all complete fuzzy labellings.

Semi-stable semantics impose the minimality constraint on the set of undecided arguments. *Semi-stable fuzzy labelling* refines it by requiring the set of undecidability arguments to be minimal among all complete fuzzy labellings.

DEFINITION 11 (Semi-stable Fuzzy Labelling).
FLab is a *semi-stable fuzzy labelling* iff it is a complete fuzzy labelling where $FLab^u$ is minimal among all complete fuzzy labellings.

Stable semantics requires that no argument can be left undecided. *Stable fuzzy labelling* refines this by requiring the set of undecidability arguments to be empty.

DEFINITION 12 (Stable Fuzzy Labelling).
FLab is a *stable fuzzy labelling* iff it is a complete fuzzy labelling with $FLab^u = \varnothing$.

EXAMPLE 5.
Consider a fuzzy argumentation system (Figure 3) with a cycle

$$\mathcal{F} = \langle\{(A, 0.8), (B, 0.6)\}, \{\big((A, B), 1\big), \big((B, A), 1\big)\}\rangle.$$

Consider three fuzzy labellings $FLab_1$, $FLab_2$ and $FLab_3$, where
$FLab_1 = (\{(A, 0.8), (B, 0.2)\}, \{(A, 0.2), (B, 0.8)\}, \varnothing)$;
$FLab_2 = (\{(A, 0.4), (B, 0.6)\}, \{(A, 0.6), (B, 0.4)\}, \varnothing)$;
$FLab_3 = (\{(A, 0.4), (B, 0.2)\}, \{(A, 0.2), (B, 0.4)\}, \{(A, 0.4), (B, 0.4)\})$.
First, it is straightforward to check that all three fuzzy labellings are complete by definition. Since $FLab_1^a$ and $FLab_2^a$ are maximal among all complete fuzzy labellings, it follows that $FLab_1$ and $FLab_2$ are preferred. Analogously, $FLab_3^a$ is minimal among all complete fuzzy labellings, thereby making $FLab_3$ grounded. Since $FLab_1^u = FLab_2^u = \varnothing$, both $FLab_1$ and $FLab_2$ are semi-stable and also stable.

4 Links to classical labelling semantics

Fuzzy labelling semantics can be viewed as a conservative generalization of classical labelling semantics. To see this, we first give several key properties of fuzzy labelling and then illustrate that most classical labelling semantics can be preserved.

TABLE 4. Complete semantics under various restrictions

Complete semantics	Resulting Semantics
no restrictions	complete fuzzy labelling
empty $FLab^u$	stable fuzzy labelling
maximal $FLab^a$	preferred fuzzy labelling
maximal $FLab^r$	preferred fuzzy labelling
maximal $FLab^u$	grounded fuzzy labelling
minimal $FLab^a$	grounded fuzzy labelling
minimal $FLab^r$	grounded fuzzy labelling
minimal $FLab^u$	semi-stable fuzzy labelling

4.1 Properties of fuzzy labelling semantics

It is well known that classical labelling semantics assigns each argument a label from {*in, out, undec*} to represent {accepted, rejected, undecided} respectively, which characterizes extension semantics by sets of *in*-labelled arguments.

Particularly, classical complete labellings can be uniquely determined by either the set of accepted arguments or the set of rejected arguments. Based on the result, preferred or grounded semantics can be equivalently expressed through complete semantics (see Table 1 in [22]). A similar result also holds in fuzzy labelling semantics. In the following, Lemma 1 says that a complete fuzzy labelling is uniquely determined by the set of acceptability arguments or the set of rejectability arguments. This eventually leads to a semantics correspondence shown in Table 4, analogous to that of [22].

LEMMA 1.
Let $FLab_1$ and $FLab_2$ be two complete fuzzy labellings of an FAS. Then $FLab_1^a \subseteq FLab_2^a$ iff $FLab_1^r \subseteq FLab_2^r$.

From Lemma 1, it is easy to derive Propositions 1 and 2, which provide alternative characterizations of preferred and grounded semantics.

PROPOSITION 1.
Let $FLab$ be a fuzzy labelling of an FAS. Then $FLab$ is grounded iff it is complete with minimal $FLab^a$, minimal $FLab^r$ or maximal $FLab^u$ among all complete fuzzy labellings.

PROPOSITION 2.
Let $FLab$ be a fuzzy labelling of an FAS. Then $FLab$ is preferred iff it is complete with maximal $FLab^a$ or maximal $FLab^r$ among all complete fuzzy labellings.

In Table 4, we summarize the aforementioned semantics in terms of restrictions on complete fuzzy labellings. Note that all combinations of minimality or maximality conditions have been covered.

Theorem 4 states implications between fuzzy labelling semantics (shown in Figure 4), which are similar to those of classical labelling semantics [8].

THEOREM 4 (Semantics inclusions).
Let $FLab$ be a fuzzy labelling of an FAS. Then:

(1) if $FLab$ is admissible, then it is conflict-free;
(2) if $FLab$ is complete, then it is admissible;

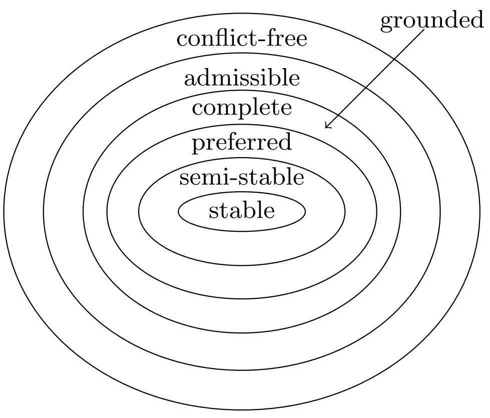


FIGURE 4. Semantics inclusion relationship

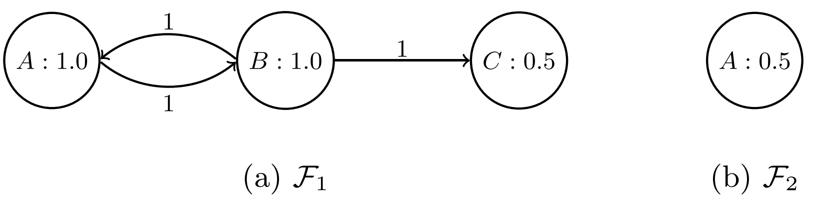


FIGURE 5. Counterexamples for semantics inclusion

(3) if *FLab* is grounded/preferred, then it is complete;
(4) if *FLab* is semi-stable, then it is preferred;
(5) if *FLab* is stable, then it is semi-stable.

The converse of each implication in Theorem 4 does not hold. Counterexamples for (1)–(3) are evident, e.g., see Example 4 and 5. For (4), consider $\mathcal{F}_1$ in Figure 5(a), wherein $FLab_1 = (\{(A,1),(C,0.5)\},\{(B,1)\},\{(C,0.5)\})$ is preferred but not semi-stable. Because $FLab_2 = (\{(B,1)\},\{(A,1),(C,1)\},\varnothing)$ is complete and $FLab_2^u \subsetneq FLab_1^u$. For (5), consider $\mathcal{F}_2$ depicted in Figure 5(b). $FLab_3 = (\{(A,0.5)\},\varnothing,\{(A,0.5)\})$ is clearly semi-stable but not stable. Additionally, Figure 5(b) also illustrates that stable fuzzy labellings may not exist, similar to classical stable labellings.

The uniqueness of classical grounded semantics is an interesting property [33]. Proposition 3 claims the uniqueness of grounded fuzzy labelling.

PROPOSITION 3.
Every FAS has a unique grounded fuzzy labelling.

The existence of classical preferred semantics was originally studied in [33]. Similarly, Proposition 4 claims the existence of preferred fuzzy labellings.

PROPOSITION 4.
Every FAS has at least one preferred fuzzy labelling.

In the following, we consider the coherence problem, which studies the conditions under which different semantics coincide [10]. Based on Lemma 2, we present a sufficient condition for the coherence of preferred, semi-stable and stable fuzzy labellings.

LEMMA 2.
For any argument with the initial degree of 1, its undecidability degree is 0 in any preferred fuzzy labelling.

THEOREM 5.
For an FAS where all arguments are assigned the initial degree of 1, preferred, semi-stable and stable fuzzy labellings coincide.

From Theorem 5, a sufficient condition for the existence of stable fuzzy labellings can be obtained.

COROLLARY 1.
For an FAS where all arguments are assigned the initial degree of 1, stable fuzzy labellings always exist.

4.2 Classical labelling as fuzzy labelling

In this section, we show that most classical labelling semantics can be preserved in fuzzy labelling semantics. Let us introduce the notions of *argumentation framework* [33] and *classical labelling semantics* [22].

DEFINITION 13 ([33]).
An *argumentation framework* is a pair $AF = (Args, Att)$, where $Args$ is a set of *arguments* and $Att \subseteq Args \times Args$ is a set of *attacks*. A *attacks* B iff $(A, B) \in Att$, and A is called the *attacker* of B. The set of attackers of A is denoted by $Att(A) = \{B \mid (B, A) \in Att\}$.

DEFINITION 14 ([22]).
Let $AF = (Args, Att)$ be an argumentation framework. An *argument labelling* is a total function $Lab : Args \to \{in, out, undec\}$. Given a Lab, we write $in(Lab)$ for $\{A \in Args | Lab(A) = in\}$, $out(Lab)$ for $\{A \in Args | Lab(A) = out\}$, and $undec(Lab)$ for $\{A \in Args | Lab(A) = undec\}$.

- Lab is a *conflict-free* labelling iff for each argument $A \in in(Lab)$, there exists no argument $B \in Att(A)$ s.t. $Lab(B) = in$.
- Lab is an *admissible* labelling iff for each argument $A \in Args$, it holds that
  1. if A is labelled *in*, then all its attackers are labelled *out*;
  2. if A is labelled *out*, then it has at least one attacker that is labelled *in*.
- Lab is a *complete* labelling iff for each argument $A \in Args$, it holds that
  1. if A is labelled *in*, then all its attackers are labelled *out*;
  2. if A is labelled *out*, then it has at least one attacker that is labelled *in*;
  3. if A is labelled *undec*, then not all its attackers are labelled *out* and it does not have an attacker that is labelled *in*.
- The *grounded* labelling is a complete labelling Lab where $in(Lab)$ is minimal (w.r.t. set inclusion) among all complete labellings.
- A *preferred* labelling is a complete labelling Lab where $in(Lab)$ is maximal (w.r.t. set inclusion) among all complete labellings.

- A *semi-stable* labelling is a complete labelling Lab where $undec(Lab)$ is minimal (w.r.t. set inclusion) among all complete labellings.
- A *stable* labelling is a complete labelling Lab where $undec(Lab) = \varnothing$.

Now we provide an intuitive way to transform AF to FAS as follows:

DEFINITION 15.
Given an argumentation framework $AF = (Args, Att)$, the *corresponding FAS* $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ is defined over $Args$ such that

- for any $A \in Args$, $\mathcal{A}(A) = 1$;
- if $(A, B) \in Att$, then $\mathcal{R}(A, B) = 1$;
- if $(A, B) \notin Att$, then $\mathcal{R}(A, B) = 0$.

Given a classical labelling Lab, the *corresponding fuzzy labelling FLab* is defined as follows: for any $A \in Args$,

- if $Lab(A) = in$, then $FLab(A) = (1, 0, 0)$;
- if $Lab(A) = out$, then $FLab(A) = (0, 1, 0)$;
- if $Lab(A) = undec$, then $FLab(A) = (0, 0, 1)$.

The theorem below states that a number of classical labelling semantics can be directly preserved in fuzzy labelling semantics.

THEOREM 6.
Let Lab be an argument labelling of an AF. For each semantics $\mathcal{S}$ ∈{admissible, complete, grounded, stable}, if Lab is an $\mathcal{S}$ labelling of the AF, then the corresponding fuzzy labelling is also an $\mathcal{S}$ fuzzy labelling of the corresponding FAS.

For preferred and semi-stable semantics, the following result holds.

PROPOSITION 5.
Preferred and semi-stable labellings can be preserved for frameworks without odd cycles.

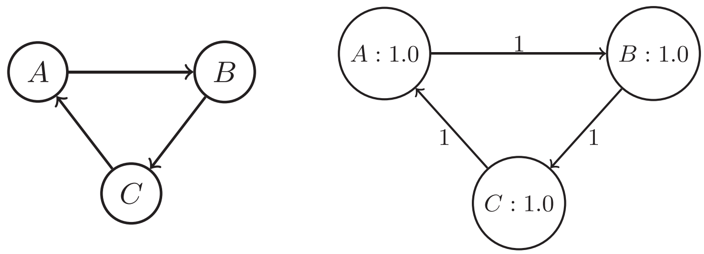
It should be noted that preferred and semi-stable semantics may not be preserved within frameworks containing odd cycles. For instance, the preferred and semi-stable labelling for the AF in Figure 6(a) is $(\varnothing, \varnothing, \{A, B, C\})$. However, for the corresponding FAS in Figure 6(b), the preferred and semi-stable fuzzy labelling is $(\{(A, 0.5), (B, 0.5), (C, 0.5)\}, \{(A, 0.5), (B, 0.5), (C, 0.5)\}, \varnothing)$.

Furthermore, the AF in Figure 6(a) has no stable labellings, while the above fuzzy labelling is stable for the FAS in Figure 6(b). From this example, we can see that our semantics provides a different view on frameworks with odd cycles.

5 Relation to fuzzy extension semantics

We demonstrate that fuzzy labelling semantics can be viewed as a labelling version of fuzzy extension semantics in [57].

DEFINITION 16 ([57]).
Let $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an FAS and $E \subseteq \mathcal{A}$ be a fuzzy set.



(a) An odd-cycle AF (b) An odd-cycle FAS

FIGURE 6. Odd-cycle AF and the corresponding FAS

- ***Weaken:*** If (A,a) sufficiently attack (B,b) w.r.t. $\big((A,B),\mathcal{R}_{AB}\big)$, we say that (A,a) *weakens* (B,b) to (B,b'), where $b' = 1 - a * \mathcal{R}_{AB}$.
- ***Weakening defend:*** A fuzzy set $S \subseteq \mathcal{A}$ *weakening defends* (C,c) if, for any $(B,\mathcal{A}(B))$ sufficiently attacks (C,c), there exists $(A,a) \in S$ s.t. (A,a) weakens $(B,\mathcal{A}(B))$ to (B,b'), which tolerably attacks (C,c).
- A fuzzy set E is *conflict-free* if all attacks in E are tolerable.
- A conflict-free fuzzy set E is an *admissible* fuzzy extension if E weakening defends each element in E.
- An admissible fuzzy extension E is *complete* if it contains all the elements in $\mathcal{A}$ that E weakening defends.
- The *grounded* fuzzy extension is the least complete fuzzy extension.
- A *preferred* fuzzy extension is a maximal complete fuzzy extension.

Now we establish the correspondence between fuzzy labelling semantics and fuzzy extension semantics using transforming functions. Roughly speaking, the set of acceptability arguments can be viewed as a fuzzy extension through functions *FLab2Ext* and *Ext2FLab*.

DEFINITION 17.
Given an FAS and a conflict-free fuzzy labelling *FLab*, the corresponding fuzzy extension is defined as $FLab2Ext(FLab) = FLab^a$.

DEFINITION 18.
Given an FAS and a conflict-free fuzzy set E, the corresponding fuzzy labelling is defined as $Ext2FLab(E) = \{E, E^+, (E \oplus E^+)^c\}$, where

$$E^+ = \{(A,a) \mid A \in Args \text{ and } a = \max_{B \in Args} E(B) * \mathcal{R}_{BA}\}$$

$$(E \oplus E^+)^c = \{(A,a) \mid A \in Args \text{ and } a = 1 - E(A) - E^+(A)\}.$$

The following theorem examines the relationships between fuzzy labelling semantics and fuzzy extension semantics.

THEOREM 7.
Let $FLab$ be a fuzzy labelling and E be a fuzzy extension. For each semantics $\mathcal{S} \in$ {admissible, complete, grounded, preferred},

1. if E is an $\mathcal{S}$ fuzzy extension, then $Ext2FLab(E)$ is an $\mathcal{S}$ fuzzy labelling;
2. if $FLab$ is an $\mathcal{S}$ fuzzy labelling, then $FLab2Ext(FLab)$ is an $\mathcal{S}$ fuzzy extension.

There is a one-to-one correspondence between complete (resp. grounded, preferred) fuzzy labellings and complete (resp. grounded, preferred) fuzzy extensions, as stated below.

THEOREM 8.
Let $FLab$ be a fuzzy labelling of an FAS. Then $FLab$ is a complete (resp. grounded, preferred) fuzzy labelling iff there is a complete (resp. grounded, preferred) fuzzy extension E s.t. $FLab = Ext2FLab(E)$.

According to Theorems 7 and 8, fuzzy extension semantics can be regarded as a special version of fuzzy labelling semantics. This correspondence is similar to that of classical labelling semantics and classical extension semantics. However, the observation does not hold for stable semantics, as shown by the coherence of stable and preferred fuzzy extensions in [57].

Defining semi-stable semantics through fuzzy extension is not straightforward due to the limitation of fuzzy set operations. Fortunately, this limitation does not apply to fuzzy labelling as it employs a numerical expression. In this sense, fuzzy labelling may be extended to other semantics such as *stage*, *eager* and *naive*, which appear difficult to describe through fuzzy extension.

6 Discussion and conclusion

Gabbay and Rodrigues [37] measured the degrees of '*in*', '*out*' and '*undec*' in Dung's AF. The basic idea is to first compute the labellings under a given semantics, categorising arguments into '*in*', '*out*' and '*undec*'. Each argument is then assigned a numerical value as the degree of being '*in*', '*out*' or '*undec*', using the equational semantics described in [35].

Dondio [31] evaluated arguments by integrating the labelling approach with the so-called *sensitivity index* derived from subgraph analysis. The sensitivity index of an argument and a label $l \in \{in, out, undec\}$ is the portion of subgraphs where the argument is labelled l. Arguments are ranked first by the label order ($in > undec > out$), and then according to the sensitivity index when arguments with the same label.

In [56], we introduced the so-called *bilateral gradual semantics* to evaluate argument strength through both the acceptability and rejectability degrees in weighted argumentation. This approach builds upon the gradual semantics framework initially proposed by Amgoud et al. [6], incorporating the notion of rejectability degree non-reciprocally.

Many researchers have focused on the semantics that calculate the acceptability degree in QuAS. Da Costa Pereira et al. introduced trust-based semantics for FAS in [29]. Gabbay and Rodrigues introduced iterative-based semantics for numerical AF in [36]. Amgoud et al. proposed weighted max/card/h-categorizer-based semantics for WAS in [6, 7]. Baroni et al. proposed QuAD semantics for Quantitative Bipolar AF in [9], which was later extended to DF-QuAD semantics by Rago et al. in [48].

Extension or labelling semantics for QuAS have also been deeply investigated. Dunne et al. [34] obtained the extension semantics of WAS by disregarding the attacks whose total weight is less than a given budget. In [42], Janssen et al. proposed extension semantics for FAS that allow minor internal attacks. Wu et al. proposed fuzzy extension semantics over fuzzy set for FAS in [57]. Bistarelli et al. [17] redefined extension semantics for WAS by considering *weighted defence*, and a labelling version was subsequently proposed in [16].

The study of *probabilistic semantics* in probabilistic argumentation has attracted increasing interest. Li et al. [45] proposed to compute the probability of sub-frameworks and then calculate the classical extension semantics of these sub-frameworks. Hunter [39] proposed the *epistemic extensions* that assign each argument a probability that satisfies *rationality* conditions. Thimm [52] generalized classical extension semantics by associating extensions with probabilities. Later in [40], Hunter and Thimm combined the idea in [39, 52] by assigning each argument a probability to represent the degree of belief that the argument is acceptable.

However, all these evaluation methods differ from our fuzzy labelling semantics that evaluate arguments through a triple consisting of acceptability, rejectability and undecidability degrees. It is worth noting that such an evaluation methodology has been widely employed in many areas. For instance, in Dempster-Shafer theory, each assertion is associated with three non-negative degrees (p, q, r) s.t. $p + q + r = 1$. Here, p is the degree 'for' the assertion, q is the degree 'against' the assertion and r is the degree of 'don't know' [30, 50]. Similarly in [43], an agent's opinion is associated with a triple (b, d, i), where b for the degree of 'belief', d for the degree of 'disbelief' and i for the degree of 'ignorance' in the field of subjective logic. In [38], Haenni evaluated arguments (not in Dung-style argumentation) using the *degrees of belief/disbelief/ignorance*, and discussed the desirable properties of this method, particularly the *non-additivity* for classifying disbelief and ignorance.

In conclusion, this paper proposed fuzzy labelling approach for fuzzy argumentation systems, which describes argument strength as a triple consisting of acceptability, rejectability and undecidability degrees. Such a setting sheds new light on defining argument strength and provides a deeper understanding of the status of arguments. For the purpose of evaluating arguments, we provided a class of fuzzy labelling semantics that generalize the classical semantics, such as complete, semi-stable, etc. Finally, we demonstrated that fuzzy labelling semantics can be considered both a conservative generalization of classical labelling semantics and a labelling version of fuzzy extension semantics.

In future work, we shall continue to explore fuzzy labelling semantics and its applications. For instance, we can extend it to systems like probabilistic argumentation [1, 45] or other more general systems [5, 26], investigate related computational complexity issues and develop algorithms for implementation, etc. Additionally, it would be interesting to apply fuzzy labelling semantics for explainability in AI [28], decision-making [3] and dealing with preferences [2, 4, 13], as our methodology considers a common phenomenon in real-world cognition.

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A Appendix

This appendix contains the proofs of the results presented in the paper.

A.1 Proof of Theorem 1 in Section 3.1

PROOF. Consider an FAS $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ over a set of arguments *Args*.

$\Rightarrow$: Let *FLab* be a fuzzy labelling that satisfies DP. Then for any $A \in Args$,

$$A^a \leq \min_{B \in Att(A)} \{\max\{B^r, 1 - \mathcal{A}(B) * \mathcal{R}_{BA}\}\}.$$

For any $B \in Att(A)$, if $(B, \mathcal{A}(B))$ sufficiently attacks (A, A^a), then $1 - \mathcal{A}(B) * \mathcal{R}_{BA} < A^a$. It follows that

$$A^a \leq \max\{B^r, 1 - \mathcal{A}(B) * \mathcal{R}_{BA}\} = B^r.$$

$\Leftarrow$: Let *FLab* be a fuzzy labelling such that for any $A \in Args$ and $B \in Att(A)$, if $(B, \mathcal{A}(B))$ sufficiently attacks (A, A^a), then $A^a \leq B^r$. It follows that

$$A^a \leq \min_{B \in S} \{\max\{B^r, 1 - \mathcal{A}(B) * \mathcal{R}_{BA}\}\}$$

where $S = \{B \in Args \mid (B, \mathcal{A}(B)) \text{ sufficiently attacks } (A, A^a)\}$. For any argument $B \in Att(A) \setminus S$, i.e., where $(B, \mathcal{A}(B))$ tolerably attacks (A, A^a), we have

$$\begin{aligned} A^a &\leq \min_{B \in Att(A) \setminus S} \{1 - \mathcal{A}(B) * \mathcal{R}_{BA}\} \\ &\leq \min_{B \in Att(A) \setminus S} \{\max\{B^r, 1 - \mathcal{A}(B) * \mathcal{R}_{BA}\}\}. \end{aligned}$$

Therefore, $A^a \leq \min_{B \in Att(A)} \{\max\{B^r, 1 - \mathcal{A}(B) * \mathcal{R}_{BA}\}\}$, *FLab* satisfies DP. □

A.2 Proof of Theorem 2 in Section 3.1

PROOF. Let $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an FAS over a set of arguments *Args* and *FLab* be a fuzzy labelling of $\mathcal{F}$ that satisfies SDP. Then for any argument $A \in Args$,

$$A^a = \min\{\min_{B \in Att(A)} \{\max\{B^r, 1 - \mathcal{A}(B) * \mathcal{R}_{BA}\}\}, \mathcal{A}(A)\}.$$

Let $S = \{B \in Args \mid (B, \mathcal{A}(B)) \text{ sufficiently attacks } (A, A^a)\}$. By Theorem 1, $A^a \leq \min_{B \in S} B^r$. Moreover, it holds that $A^a \leq 1 - \max_{B \notin S} \mathcal{A}(B) * \mathcal{R}_{BA}$. Thus

$$A^a \leq \min\{\min_{B \in S} B^r, 1 - \max_{B \notin S} \mathcal{A}(B) * \mathcal{R}_{BA}, \mathcal{A}(A)\}. \tag{A.1}$$

Now we have two cases concerning SDP:

Case 1:

$$\min_{B\in S}\{\max\{B^r, 1-\mathcal{A}(B)*\mathcal{R}_{BA}\}\} \geq \min_{B\in Att(A)\setminus S}\{\max\{B^r, 1-\mathcal{A}(B)*\mathcal{R}_{BA}\}\}. \tag{A.2}$$

Then

$$\begin{aligned}
A^a &= \min\left\{\min_{B\in Att(A)}\{\max\{B^r, 1-\mathcal{A}(B)*\mathcal{R}_{BA}\}\}, \mathcal{A}(A)\right\} && \text{(By SDP)}\\
&= \min\left\{\min_{B\in Att(A)\setminus S}\{\max\{B^r, 1-\mathcal{A}(B)*\mathcal{R}_{BA}\}\}, \mathcal{A}(A)\right\} && \text{(By Inequality 2)}\\
&= \min\left\{\min_{B\notin S}\{\max\{B^r, 1-\mathcal{A}(B)*\mathcal{R}_{BA}\}\}, \mathcal{A}(A)\right\} && \text{(Trivial)}\\
&\geq \min\left\{1-\max_{B\notin S}\mathcal{A}(B)*\mathcal{R}_{BA}, \mathcal{A}(A)\right\} && \text{(By max)}\\
&\geq \min\left\{\min_{B\in S}B^r, 1-\max_{B\notin S}\mathcal{A}(B)*\mathcal{R}_{BA}, \mathcal{A}(A)\right\} && \text{(By min)}\\
&\geq A^a. && \text{(By Inequality 1)}
\end{aligned}$$

Case 2:

$$\min_{B\in S}\left\{\max\{B^r, 1-\mathcal{A}(B)*\mathcal{R}_{BA}\}\} \leq \min_{B\in Att(A)\setminus S}\{\max\{B^r, 1-\mathcal{A}(B)*\mathcal{R}_{BA}\}\right\}. \tag{A.3}$$

Then

$$\begin{aligned}
A^a &= \min\left\{\min_{B\in Att(A)}\{\max\{B^r, 1-\mathcal{A}(B)*\mathcal{R}_{BA}\}\}, \mathcal{A}(A)\right\} && \text{(By SDP)}\\
&= \min\left\{\min_{B\in S}\{\max\{B^r, 1-\mathcal{A}(B)*\mathcal{R}_{BA}\}\}, \mathcal{A}(A)\right\} && \text{(By Inequality 3)}\\
&\geq \min\left\{\min_{B\in S}B^r, \mathcal{A}(A)\right\} && \text{(By max)}\\
&\geq \min\left\{\min_{B\in S}B^r, 1-\max_{B\notin S}\mathcal{A}(B)*\mathcal{R}_{BA}, \mathcal{A}(A)\right\} && \text{(By min)}\\
&\geq A^a. && \text{(By Inequality 1)}
\end{aligned}$$

Combining the two cases, we conclude

$$A^a = \min\left\{\min_{B\in S}B^r, 1-\max_{B\notin S}\mathcal{A}(B)*\mathcal{R}_{BA}, \mathcal{A}(A)\right\}.$$

□

A.3 Proof of Theorem 3 in Section 3.1

PROOF. 1. Let $\mathcal{F} = \langle\mathcal{A}, \mathcal{R}\rangle$ be an FAS over a set of arguments *Args* and *FLab* be a fuzzy labelling satisfying BP, UP and DP. If *FLab* does not satisfy TP, then there exists an argument $A \in Args$ s.t.

$A^a > 1 - C^a * \mathcal{R}_{CA}$ for some $C \in Att(A)$. It follows that

$$
\begin{aligned}
A^a &\leq \min_{B \in Att(A)} \left\{ \max\{B^r, 1 - \mathcal{A}(B) * \mathcal{R}_{BA}\} \right\} && \text{(By DP)} \\
&\leq \max \left\{ C^r, 1 - \mathcal{A}(C) * \mathcal{R}_{CA} \right\} && \text{(By min)} \\
&\leq \max \left\{ C^r, 1 - C^a * \mathcal{R}_{CA} \right\} && \text{(By BP)} \\
&= 1 - C^a * \mathcal{R}_{CA} && \text{(By UP)} \\
&< A^a. && \text{(By hypothesis)}
\end{aligned}
$$

A contradiction. Proofs for 2–4 are straightforward. □

A.4 Proof of Lemma 1 in Section 4.1

PROOF. Let $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an FAS over a set of arguments *Args*, and $FLab_1, FLab_2$ be two complete fuzzy labellings of $\mathcal{F}$.

⇒: Suppose $FLab_1^a \subseteq FLab_2^a$. Then for any $A \in Args$, $FLab_1^a(A) \leq FLab_2^a(A)$. By SWP, $FLab_1^r(A) \leq FLab_2^r(A)$ holds for any $A \in Args$. Therefore, $FLab_1^r \subseteq FLab_2^r$.

⇐: Suppose $FLab_1^r \subseteq FLab_2^r$. Then for any $A \in Args$, $FLab_1^r(A) \leq FLab_2^r(A)$. By SDP, $FLab_1^a(A) \leq FLab_2^a(A)$ holds for any $A \in Args$. Therefore, $FLab_1^a \subseteq FLab_2^a$. □

A.5 Proof of Proposition 1 in Section 4.1

PROOF. Let *FLab* be a fuzzy labelling of an FAS. The cases for minimal $FLab^a$ and minimal $FLab^r$ are straightforward by Definition 9 and Lemma 1. Now we prove *FLab* is a grounded fuzzy labelling iff it is complete with maximal $FLab^u$ among all complete fuzzy labellings. Assume *FLab* is complete with maximal $FLab^u$ but not grounded. Then there exists a complete fuzzy labelling $FLab_*$ s.t. $FLab_*^a \subsetneq FLab^a$. It follows from Lemma 1 that $FLab_*^r \subsetneq FLab^r$. Therefore, $FLab^u \subsetneq FLab_*^u$, contradicting that $FLab^u$ is maximal. Consequently, *FLab* is grounded. □

A.6 Proof of Proposition 2 in Section 4.1

PROOF. The proof is straightforward by Lemma 1. □

A.7 Proof of Theorem 4 in Section 4.1

PROOF. 1. Follows from Theorem 3.

2. Follows from Theorem 3.

3. Refer to Definitions 8, 9 and 10.

4. Let *FLab* be a semi-stable fuzzy labelling of an FAS. If *FLab* is not preferred, then there exists a preferred fuzzy labelling $FLab_*$ such that $FLab^a \subsetneq FLab_*^a$. By Lemma 1, it follows that $FLab^r \subsetneq FLab_*^r$. Hence, $FLab_*^u \subsetneq FLab^u$, contradicting the fact that *FLab* is semi-stable. Consequently, every semi-stable fuzzy labelling is preferred.

5. Refer to Definitions 11 and 12. □

A.8 Proof of Proposition 3 in Section 4.1

PROOF. Let $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an FAS over a set of arguments *Args*. By Definition 10 and Theorem 2 in [57], the grounded fuzzy extension is the least complete fuzzy extension (see Definition 16). So every FAS has a unique grounded fuzzy extension. Moreover, by Theorem 8, there exists a one-to-one correspondence between the grounded fuzzy labelling and the grounded fuzzy extension. Hence every FAS has exactly one grounded fuzzy labelling. □

A.9 Proof of Proposition 4 in Section 4.1

PROOF. Consider an FAS $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ over a set of arguments *Args*. For any complete fuzzy labelling $FLab_1$, if $FLab_1^a$ is not maximal among all complete fuzzy labellings, then $FLab_1^r$ is also not maximal from Lemma 1. So there exists a complete fuzzy labelling $FLab_2$ s.t. $FLab_1^a \subsetneq FLab_2^a$ and $FLab_1^r \subsetneq FLab_2^r$. By iterating this process, we construct a sequence of complete fuzzy labellings $\{FLab_1, FLab_2, ..., FLab_n, ...\}$ until a preferred fuzzy labelling is found. The sequence is monotonically increasing, with $FLab_i^a \subsetneq FLab_{i+1}^a$ and $FLab_i^r \subsetneq FLab_{i+1}^r$. If this process could be repeated infinitely, then the Monotone Convergence Theorem ensures the existence of a fuzzy labelling $FLab_\#$ s.t. for any $A \in Args$, $FLab_\#^a(A) = \lim_{n\to\infty} FLab_n^a(A)$ and $FLab_\#^r(A) = \lim_{n\to\infty} FLab_n^r(A)$. Now we prove that $FLab_\#$ is complete. For any $A \in Args$ and $n \in \mathbb{N}$, we have

$$FLab_n^a(A) = \min\left\{\min_{B\in Att(A)}\{\max\{FLab_n^r(B), 1 - \mathcal{A}(B) * \mathcal{R}_{BA}\}\}, \mathcal{A}(A)\right\}$$

$$FLab_n^r(A) = \max_{B\in Att(A)} FLab_n^a(B) * \mathcal{R}_{BA}$$

$$FLab_n^u(A) = 1 - FLab_n^a(A) - FLab_n^r(A)$$

Taking the limit on both sides, we obtain

$$FLab_\#^a(A) = \min\left\{\min_{B\in Att(A)}\{\max\{FLab_\#^r(B), 1 - \mathcal{A}(B) * \mathcal{R}_{BA}\}\}, \mathcal{A}(A)\right\}$$

$$FLab_\#^r(A) = \max_{B\in Att(A)} FLab_\#^a(B) * \mathcal{R}_{BA}$$

$$FLab_\#^u(A) = 1 - FLab_\#^a(A) - FLab_\#^r(A)$$

Therefore, $FLab_\#$ satisfies BP, UP, SDP and SWP, making it complete. As a result, $FLab_\#$ is a complete fuzzy labelling with maximal $FLab_\#^a$ among all complete fuzzy labellings, i.e., it is preferred. □

A.10 Proof of Lemma 2 in Section 4.1

PROOF. Consider an FAS $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ over a set of arguments *Args*. By Theorem 2 in [57], for a preferred fuzzy extension E, every element not in E is sufficiently attacked by an element in E. Furthermore, Theorem 8 establishes a one-to-one correspondence between preferred fuzzy extensions and preferred fuzzy labellings. So for any preferred fuzzy labelling *FLab*, any element not in $FLab^a$ is sufficiently attacked by an element in $FLab^a$. Let A be an argument with $\mathcal{A}(A) = 1$. Then for any $c > A^a$, (A, c) is sufficiently attacked by some $(B, b) \in FLab^a$, giving $b * \mathcal{R}_{BA} + c > 1$. It follows that $b * \mathcal{R}_{BA} + A^a \geq 1$. By SWP, $A^o \geq b * \mathcal{R}_{BA}$, and then $A^a + A^o = 1$. Consequently, for each argument with the initial degree of 1, its undecidability degree is 0 in any preferred fuzzy labelling. □

A.11 Proof of Theorem 5 in Section 4.1

PROOF. By Lemma 2, the initial degree of an argument is 1 implies its undecidability degree is 0 in any preferred fuzzy labelling. Therefore, if all arguments are assigned initial degree 1, then the set of undecidability arguments is empty in any preferred fuzzy labelling, rendering it stable. According to the semantics inclusion theorem, preferred, semi-stable and stable fuzzy labellings coincide. □

A.12 Proof of Corollary 1 in Section 4.1

PROOF. Consider an FAS where all arguments are assigned initial degree 1. By Theorem 5, in this case preferred and stable fuzzy labellings coincide. Furthermore, Proposition 4 states that preferred fuzzy labellings always exist for any FAS. Consequently, stable fuzzy labellings always exist in such an FAS. □

A.13 Proof of Theorem 6 in Section 4.2

PROOF. Consider an argumentation framework $AF = (Args, Att)$ and an argument labelling Lab. Let $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ be the corresponding FAS of AF and $FLab$ be the corresponding fuzzy labelling of Lab.

1. Under admissible semantics, we prove $FLab$ is an admissible fuzzy labelling of $\mathcal{F}$. We verify the following conditions for any $A \in Args$:

$$\begin{aligned}
A^a &\le \min\left\{\min_{B\in Att(A)}\{\max\{B^r, 1-\mathcal{A}(B) * \mathcal{R}_{BA}\}\}, \mathcal{A}(A)\right\} && \text{(By BP,DP)}\\
&= \min\left\{\min_{B\in Att(A)}\{\max\{B^r, 0\}\}, 1\right\}\\
&= \min_{B\in Att(A)} B^r\\
A^r &\le \max_{B\in Att(A)} B^a * \mathcal{R}_{BA} = \max_{B\in Att(A)} B^a && \text{(By WP)}\\
A^u &= 1 - A^a - A^r && \text{(By UP)}
\end{aligned}$$

There are three cases:

- $Lab(A) = in$. Then $Lab(B) = out$ for any $B \in Att(A)$. So for any $B \in Att(A)$, $B^a = 0$ and $B^r = 1$. Hence

$$\begin{aligned}
A^a &= 1 \le \min_{B\in Att(A)} B^r\\
A^r &= 0 \le \max_{B\in Att(A)} B^a\\
A^u &= 0 = 1 - A^a - A^r.
\end{aligned}$$

- $Lab(A) = out$. There exists at least one argument $B \in Att(A)$ and $Lab(B) = in$. So $\exists B \in Att(A)$ s.t. $B^a = 1$ and $B^r = 0$. Hence

$$A^a = 0 \le \min_{B \in Att(A)} B^r$$

$$A^r = 1 \le \max_{B \in Att(A)} B^a$$

$$A^u = 0 = 1 - A^a - A^r.$$

- $Lab(A) = undec$. Trivial.

Consequently, *FLab* is an admissible fuzzy labelling.

2. Under complete semantics, we prove *FLab* is a complete fuzzy labelling of $\mathcal{F}$. We verify the following conditions for any $A \in Args$:

$$\begin{aligned} A^a &= \min\left\{\min_{B \in Att(A)}\{\max\{B^r, 1 - \mathcal{A}(B) * \mathcal{R}_{BA}\}\}, \mathcal{A}(A)\right\} && \text{(By SDP)} \\ &= \min\left\{\min_{B \in Att(A)}\{\max\{B^r, 0\}\}, 1\right\} \\ &= \min_{B \in Att(A)} B^r \\ A^r &= \max_{B \in Att(A)} B^a * \mathcal{R}_{BA} = \max_{B \in Att(A)} B^a && \text{(By SWP)} \\ A^u &= 1 - A^a - A^r && \text{(By UP)} \end{aligned}$$

There are three cases:

- $Lab(A) = in$. Then $Lab(B) = out$ for any $B \in Att(A)$. So for any argument $B \in Att(A)$, $B^a = 0$ and $B^r = 1$. Hence

$$A^a = 1 = \min_{B \in Att(A)} B^r$$

$$A^r = 0 = \max_{B \in Att(A)} B^a$$

$$A^u = 0 = 1 - A^a - A^r.$$

- $Lab(A) = out$. There exists at least one argument $B \in Att(A)$ and $Lab(B) = in$. So $\exists B \in Att(A)$ s.t. $B^a = 1$ and $B^r = 0$. Hence

$$A^a = 0 = \min_{B \in Att(A)} B^r$$

$$A^r = 1 = \max_{B \in Att(A)} B^a$$

$$A^u = 0 = 1 - A^a - A^r.$$

- $Lab(A) = undec$. There exists no argument $B \in Att(A)$ s.t. $Lab(B) = in$, and there exists at least one argument $C \in Att(A)$ s.t. $Lab(C) = undec$. Therefore, for any $B \in Att(A)$, $B^a = 0$ and

$\exists C \in Att(A)$ s.t. $C^r = 0$. Hence

$$A^a = 0 = \min_{B \in Att(A)} B^r$$

$$A^r = 0 = \max_{B \in Att(A)} B^a$$

$$A^u = 1 = 1 - A^a - A^r.$$

Consequently, *FLab* is a complete fuzzy labelling.

3. Under grounded semantics, we prove by the definitions of grounded extension and grounded fuzzy extension. For simplicity, we use S to represent both classical and fuzzy sets such that $A \in S$ iff $S(A) = 1$.

As shown in [33], the grounded extension of AF is defined as the least fixed point of F_{AF}, where F_{AF} is a function from the set of all the subsets of $Args$ to itself such that for any $S \subseteq Args$,

$$F_{AF}(S) = \{A \in Args | (B, A) \in Att \Rightarrow \exists C \in S \text{ s.t.} (C, B) \in Att\}.$$

The grounded extension is obtained by

$$E = \bigcup_{i=1,\ldots,\infty} F^i_{AF}(\varnothing)$$

where

$$F^1_{AF}(\varnothing) = F_{AF}(\varnothing) \text{ and for } i > 1, F^i_{AF} = F_{AF}(F^{i-1}_{AF}(\varnothing)).$$

Similarly, as shown in [57], the grounded fuzzy extension (see Definition 16) of $\mathcal{F}$ is defined as the least fixed point of $F_{\mathcal{F}}$, where $F_{\mathcal{F}}$ is a function from the set of all the subsets of $\mathcal{A}$ to itself, such that for any fuzzy set $S \subseteq \mathcal{A}$,

$$F_{\mathcal{F}}(S) = \{(A, a) | S \text{ weakening defends } (A, a)\}.$$

The grounded fuzzy extension is obtained by

$$E = \bigcup_{i=1,\ldots,\infty} F^i_{\mathcal{F}}(\varnothing)$$

where

$$F^1_{\mathcal{F}}(\varnothing) = F_{\mathcal{F}}(\varnothing) \text{ and for } i > 1, F^i_{\mathcal{F}} = F_{\mathcal{F}}(F^{i-1}_{\mathcal{F}}(\varnothing)).$$

We now prove for any $A \in Args$ and $S \subseteq Args$, if $A \in F_{AF}(S)$, then $(A, 1) \in F_{\mathcal{F}}(S)$. Suppose $A \in F_{AF}(S)$. Then for any $B \in Att(A)$, $\exists C \in S$ s.t. $(C, B) \in Att$. Therefore, for any $(B, 1)$ sufficiently attacks $(A, 1)$, $\exists (C, 1) \in S$ weakens $(B, 1)$ to $(B, 0)$, which tolerably attacks $(A, 1)$, and then $(A, 1) \in F_{\mathcal{F}}(S)$. Accordingly, for any $i \in \{1, 2, ...\}$, if $A \in F^i_{AF}(\varnothing)$, then $(A, 1) \in F^i_{\mathcal{F}}(\varnothing)$. Therefore if A is in the grounded extension, then $(A, 1)$ is in the grounded fuzzy extension.

It has been proved in [22] that the grounded extension and the grounded labelling are in one-to-one correspondence. Moreover, Theorem 8 shows that the grounded fuzzy extension and the grounded fuzzy labelling are in one-to-one correspondence. Therefore, if A is labelled as *in* in the grounded labelling, then $A^a = 1$ in the grounded fuzzy labelling. Furthermore, from the proof of point 2, the corresponding fuzzy labelling of the grounded labelling is also complete. Consequently, the corresponding fuzzy labelling *FLab* is a complete fuzzy labelling with minimal $FLab^a$, i.e., it is grounded.

4. Under stable semantics, we prove that *FLab* is a stable fuzzy labelling. *Lab* is stable implies that it is a complete labelling with $undec(Lab) = \varnothing$. By the proof of point 2, *FLab* is a complete fuzzy labelling with $FLab^u = \varnothing$. Consequently, *FLab* is a stable fuzzy labelling. □

A.14 Proof of Proposition 5 in Section 4.2

PROOF. From Theorem 33 in [33] and Figure 12 in [8], it is established that stable, preferred and semi-stable semantics coincide in frameworks without odd cycles. Consequently, since stable labellings can be preserved, the same applies to preferred and semi-stable labellings. □

A.15 Proof of Theorem 7 in Section 5

PROOF. Let $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an FAS over a set of arguments *Args*.

1. Let E be a conflict-free fuzzy set. We denote the corresponding fuzzy labelling as

$$FLab = Ext2FLab(E) = \{E, E^+, (E \oplus E^+)^c\}.$$

Suppose E is admissible, we prove *FLab* is an admissible fuzzy labelling. E is admissible implies that E weakening defends every element in E. Namely for any $A \in Args$, if $(B, \mathcal{A}(B))$ sufficiently attacks $(A, E(A))$, then $\exists (C, c) \in E$ s.t. (C, c) weakens $(B, \mathcal{A}(B))$ to $(B, 1 - E(A))$, i.e., $\exists (C, c) \in E$ s.t. $c * \mathcal{R}_{CB} = E(A)$. Therefore

$$E(A) = c * \mathcal{R}_{CB} \leq \max_{X \in Att(B)} E(X) * \mathcal{R}_{XB} = E^+(B).$$

Consequently, for any $B \in Att(A)$, if $(B, \mathcal{A}(B))$ sufficiently attacks $(A, E(A))$, then $E(A) \leq E^+(B)$, i.e., $A^a \leq B^r$. Thus *FLab* satisfies DP by Theorem 1. Easy to see the rest conditions hold as well.

Suppose E is complete, we prove *FLab* is a complete fuzzy labelling. By Definition 18, BP, UP and SWP are easy to check. It is sufficient to show *FLab* satisfies SDP. Assume for any $A \in Args$,

$$A_d = \min\left\{ \min_{X \in Att(A)} \{\max\{E^+(X), 1 - \mathcal{A}(X) * \mathcal{R}_{XA}\}\}, \mathcal{A}(A) \right\}.$$

E is complete implies *FLab* is admissible. Then for any $A \in Args$, $E(A) \leq A_d$ by DP and BP. Next we prove $E(A) \geq A_d$. By the definition of complete fuzzy extension, it remains to show (A, A_d) is weakening defended by E. For any $B \in Att(A)$, if $(B, \mathcal{A}(B))$ sufficiently attacks (A, A_d), then $A_d > 1 - \mathcal{A}(B) * \mathcal{R}_{BA}$. Since $A_d \leq \max\{E^+(B), 1 - \mathcal{A}(B) * \mathcal{R}_{BA}\}$, we have $A_d \leq E^+(B)$. By Definition 18, there exists $(C, c) \in E$ s.t. $A_d = c * \mathcal{R}_{CB}$. This implies that $\exists (C, c) \in E$ weakens $(B, \mathcal{A}(B))$ to $(B, 1 - A_d)$, which tolerably attacks (A, A_d). Hence (A, A_d) is weakening defended by E, it follows that $E(A) \geq A_d$. Consequently, *FLab* satisfies SDP.

Suppose E is a grounded fuzzy extension. Clearly it is the least complete fuzzy extension. From point 1 of Theorem 8, there is a one-to-one relationship between complete fuzzy labellings and complete fuzzy extensions. Therefore, *FLab* is a complete fuzzy labelling with the least $FLab^a$ among all complete fuzzy labellings, i.e., it is grounded.

The case of preferred semantics is analogous to grounded semantics.

2. Let *FLab* be a conflict-free fuzzy labelling. We denote the corresponding fuzzy extension as

$$E = FLab2Ext(FLab) = FLab^a.$$

Since *FLab* satisfies TP, for any $A \in Args$,

$$\max_{B \in Att(A)} FLab^a(B) * \mathcal{R}_{BA} + FLab^a(A) \leq 1,$$

ensuring that all attacks within $FLab^a$ are tolerable. Therefore E is conflict-free.

Suppose *FLab* is admissible, we prove E is an admissible fuzzy extension. Since *FLab* is admissible, easy to see E is conflict-free. It is sufficient to prove that any $(A, E(A)) \in E$ is weakening defended by E. By Theorem 1, for any $B \in Att(A)$, if $(B, \mathcal{A}(B))$ sufficiently attacks $(A, E(A))$, then $E(A) = A^a \leq B^r$. Thus $\exists(C, c) \in E$ s.t. $c * \mathcal{R}_{CB} = E(A)$ by WP. So for any $(A, E(A)) \in E$, if $(B, \mathcal{A}(B))$ sufficiently attacks $(A, E(A))$, then $\exists(C, c) \in E$ weakens $(B, \mathcal{A}(B))$ to $(B, 1 - E(A))$, which tolerably attacks $(A, E(A))$. Consequently, E is an admissible fuzzy extension.

Suppose *FLab* is complete, we prove E is a complete fuzzy extension. Since *FLab* is complete, easy to see E is admissible. It is sufficient to prove that E contains all the elements in $\mathcal{A}$ that it weakening defends. Let $(A, a) \in \mathcal{A}$ be weakening defended by E. So if $(B, \mathcal{A}(B))$ sufficiently attacks (A, a), then $\exists(C, c) \in E$ s.t. (C, c) weakens $(B, \mathcal{A}(B))$ to $(B, 1 - a)$. It follows that $c * \mathcal{R}_{CB} = a$, which gives

$$a = c * \mathcal{R}_{CB} \leq \max_{X \in Att(B)} X^a * \mathcal{R}_{XB} = B^r.$$

Consequently, for any $B \in Att(A)$, if $(B, \mathcal{A}(B))$ sufficiently attacks (A, a), then $a \leq B^r$. Let $S = \{B \in Args \mid (B, \mathcal{A}(B))$ sufficiently attacks $(A, a)\}$. It follows that

$$\begin{aligned} a &\leq \min\left\{\min_{B \in S} B^r, \mathcal{A}(A)\right\} \\ &\leq \min\left\{\min_{B \in S}\{\max\{B^r, 1 - \mathcal{A}(B) * \mathcal{R}_{BA}\}\}, \mathcal{A}(A)\right\}. \end{aligned}$$

Moreover, it is easy to see

$$\begin{aligned} a &\leq \min\left\{\min_{B \notin S}\{1 - \mathcal{A}(B) * \mathcal{R}_{BA}\}, \mathcal{A}(A)\right\} \\ &\leq \min\left\{\min_{B \notin S}\{\max\{B^r, 1 - \mathcal{A}(B) * \mathcal{R}_{BA}\}\}, \mathcal{A}(A)\right\}. \end{aligned}$$

By SDP, we have

$$a \leq \min\left\{\min_{B \in Att(A)}\{\max\{B^r, 1 - \mathcal{A}(B) * \mathcal{R}_{BA}\}\}, \mathcal{A}(A)\right\} = A^a.$$

Therefore, $(A, a) \in FLab^a = E$. Consequently, E is a complete fuzzy extension.

Suppose *FLab* is the grounded fuzzy labelling. Clearly, it is a complete fuzzy labelling with the least $FLab^a$ among all complete fuzzy labellings. From point 1 of Theorem 8, there is a one-to-one relationship between complete fuzzy labellings and complete fuzzy extensions. Therefore, E is the least complete fuzzy extension, i.e., it is grounded.

The case of preferred semantics is analogous to grounded semantics. □

A.16 Proof of Theorem 8 in Section 5

PROOF. 1. Under complete semantics, it is sufficient to prove that *Ext2FLab* and *FLab2Ext* are the inverse of each other, when restricted to complete fuzzy extensions and complete fuzzy labellings. Note that for each complete fuzzy extension E, $Ext2FLab\big(Ext2FLab(E)\big) = E$. Now suppose *FLab* is a complete fuzzy labelling. We have $FLab2Ext(FLab) = FLab^a$ and $Ext2FLab\big(FLab^a\big) = \big(FLab^a, (FLab^a)^+, \big(FLab^a \oplus (FLab^a)^+\big)^c$. We show $FLab^r = (FLab^a)^+$ as follows:

- For any $(A, a) \in FLab^r$, by SWP, there exists $(B, b) \in FLab^a$ satisfying $b * \mathcal{R}_{BA} = a$. Clearly, $(A, a) \in (FLab^a)^+$.
- For any $(A, a) \in (FLab^a)^+$, there exists $(B, b) \in FLab^a$ s.t. $b * \mathcal{R}_{BA} = a$. Clearly, $(A, a) \in FLab^r$ by SWP.

Consequently, $Ext2FLab\big(FLab2Ext(FLab)\big) = FLab$.
Points 2 and 3 are straightforward from point 1 and Theorem 7. □

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