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Data Assimilation

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The Ensemble Kalman Filter

With 63 Figures

 Springer

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To Tina and Endre

Preface

The aim of this book is to introduce the formulation and solution of the data assimilation problem. The focus is mainly on methods where the model is allowed to contain errors and where the error statistics evolve through time. So-called strong constraint methods and simple methods where the error statistics are constant in time are only briefly explained, and then as special cases of more general weak constraint formulations.

There is a special focus on the Ensemble Kalman Filter and similar methods. These are methods which have become very popular, both due to their simple implementation and interpretation and their properties with nonlinear models.

The book has been written during several years of work on the development of data assimilation methods and the teaching of data assimilation methods to graduate students. It would not have been completed without the continuous interaction with students and colleagues, and I particularly want to acknowledge the support from Laurent Bertino, Kari Brusdal, François Counillon, Mette Eknes, Vibeke Haugen, Knut Arild Lisæter, Lars Jørgen Natvik, and Jan Arild Skjervheim, with whom I have worked closely for several years. Laurent Bertino and François Counillon also provided much of the material for the chapter on the TOPAZ ocean data assimilation system. Contributions from Laurent Bertino, Theresa Lloyd, Gordon Wilmot, Martin Miles, Jennifer Trittschuh-Vallès, Brice Vallès and Hans Wackernagel, on proof-reading parts of the final version of the book are also much appreciated.

It is hoped that the book will provide a comprehensive presentation of the data assimilation problem and that it will serve as a reference and textbook for students and researchers working with development and application of data assimilation methods.

Contents

List of symbols	xv
1 Introduction	1
2 Statistical definitions	5
2.1 Probability density function	5
2.2 Statistical moments	8
2.2.1 Expected value	8
2.2.2 Variance	8
2.2.3 Covariance	9
2.3 Working with samples from a distribution	9
2.3.1 Sample mean	9
2.3.2 Sample variance	10
2.3.3 Sample covariance	10
2.4 Statistics of random fields	10
2.4.1 Sample mean	10
2.4.2 Sample variance	10
2.4.3 Sample covariance	11
2.4.4 Correlation	11
2.5 Bias	11
2.6 Central limit theorem	12
3 Analysis scheme	13
3.1 Scalar case	13
3.1.1 State-space formulation	13
3.1.2 Bayesian formulation	15
3.2 Extension to spatial dimensions	16
3.2.1 Basic formulation	16
3.2.2 Euler–Lagrange equation	17
3.2.3 Representer solution	19
3.2.4 Representer matrix	20

3.2.5	Error estimate	20
3.2.6	Uniqueness of the solution	22
3.2.7	Minimization of the penalty function.....	23
3.2.8	Prior and posterior value of the penalty function	24
3.3	Discrete form	24
4	Sequential data assimilation	27
4.1	Linear Dynamics	27
4.1.1	Kalman filter for a scalar case.....	28
4.1.2	Kalman filter for a vector state.....	29
4.1.3	Kalman filter with a linear advection equation	29
4.2	Nonlinear dynamics.....	32
4.2.1	Extended Kalman filter for the scalar case.....	32
4.2.2	Extended Kalman filter in matrix form.....	33
4.2.3	Example using the extended Kalman filter.....	35
4.2.4	Extended Kalman filter for the mean	36
4.2.5	Discussion	37
4.3	Ensemble Kalman filter	38
4.3.1	Representation of error statistics	38
4.3.2	Prediction of error statistics	39
4.3.3	Analysis scheme.....	41
4.3.4	Discussion	43
4.3.5	Example with a QG model	44
5	Variational inverse problems	47
5.1	Simple illustration	47
5.2	Linear inverse problem	50
5.2.1	Model and observations	51
5.2.2	Measurement functional.....	51
5.2.3	Comment on the measurement equation	51
5.2.4	Statistical hypothesis	52
5.2.5	Weak constraint variational formulation	52
5.2.6	Extremum of the penalty function	53
5.2.7	Euler–Lagrange equations	54
5.2.8	Strong constraint approximation	55
5.2.9	Solution by representer expansions.....	56
5.3	Representer method with an Ekman model	57
5.3.1	Inverse problem	58
5.3.2	Variational formulation	58
5.3.3	Euler–Lagrange equations	59
5.3.4	Representer solution	60
5.3.5	Example experiment	61
5.3.6	Assimilation of real measurements	64
5.4	Comments on the representer method	67

6	Nonlinear variational inverse problems	71
6.1	Extension to nonlinear dynamics	71
6.1.1	Generalized inverse for the Lorenz equations	72
6.1.2	Strong constraint assumption	73
6.1.3	Solution of the weak constraint problem	76
6.1.4	Minimization by the gradient descent method	77
6.1.5	Minimization by genetic algorithms	78
6.2	Example with the Lorenz equations	82
6.2.1	Estimating the model error covariance	82
6.2.2	Time correlation of the model error covariance	83
6.2.3	Inversion experiments	84
6.2.4	Discussion	92
7	Probabilistic formulation	95
7.1	Joint parameter and state estimation	95
7.2	Model equations and measurements	96
7.3	Bayesian formulation	97
7.3.1	Discrete formulation	98
7.3.2	Sequential processing of measurements	99
7.4	Summary	101
8	Generalized Inverse	103
8.1	Generalized inverse formulation	103
8.1.1	Prior density for the poorly known parameters	103
8.1.2	Prior density for the initial conditions	104
8.1.3	Prior density for the boundary conditions	104
8.1.4	Prior density for the measurements	105
8.1.5	Prior density for the model errors	105
8.1.6	Conditional joint density	107
8.2	Solution methods for the generalized inverse problem	108
8.2.1	Generalized inverse for a scalar model	108
8.2.2	Euler–Lagrange equations	109
8.2.3	Iteration in α	111
8.2.4	Strong constraint problem	111
8.3	Parameter estimation in the Ekman flow model	113
8.4	Summary	117
9	Ensemble methods	119
9.1	Introductory remarks	119
9.2	Linear ensemble analysis update	121
9.3	Ensemble representation of error statistics	122
9.4	Ensemble representation for measurements	124
9.5	Ensemble Smoother (ES)	124
9.6	Ensemble Kalman Smoother (EnKS)	126
9.7	Ensemble Kalman Filter (EnKF)	129

9.7.1	EnKF with linear noise free model	129
9.7.2	EnKS using EnKF as a prior	130
9.8	Example with the Lorenz equations	131
9.8.1	Description of experiments	131
9.8.2	Assimilation Experiment	132
9.9	Discussion	137
10	Statistical optimization	139
10.1	Definition of the minimization problem	139
10.1.1	Parameters	140
10.1.2	Model	140
10.1.3	Measurements	140
10.1.4	Cost function	141
10.2	Bayesian formalism	141
10.3	Solution by ensemble methods	142
10.3.1	Variance minimizing solution	144
10.3.2	EnKS solution	144
10.4	Examples	145
10.5	Discussion	154
11	Sampling strategies for the EnKF	157
11.1	Introduction	157
11.2	Simulation of realizations	158
11.2.1	Inverse Fourier transform	159
11.2.2	Definition of Fourier spectrum	159
11.2.3	Specification of covariance and variance	160
11.3	Simulating correlated fields	162
11.4	Improved sampling scheme	163
11.5	Experiments	167
11.5.1	Overview of experiments	167
11.5.2	Impact from ensemble size	170
11.5.3	Impact of improved sampling for the initial ensemble	171
11.5.4	Improved sampling of measurement perturbations	171
11.5.5	Evolution of ensemble singular spectra	173
11.5.6	Summary	174
12	Model errors	175
12.1	Simulation of model errors	175
12.1.1	Determination of ρ	175
12.1.2	Physical model	176
12.1.3	Variance growth due to the stochastic forcing	176
12.1.4	Updating model noise using measurements	180
12.2	Scalar model	180
12.3	Variational inverse problem	181
12.3.1	Prior statistics	181

12.3.2	Penalty function	182
12.3.3	Euler–Lagrange equations	182
12.3.4	Iteration of parameter	182
12.3.5	Solution by representer expansions	183
12.3.6	Variance growth due to model errors	184
12.4	Formulation as a stochastic model	185
12.5	Examples	185
12.5.1	Case A0	186
12.5.2	Case A1	186
12.5.3	Case B	189
12.5.4	Case C	192
12.5.5	Discussion	193
13	Square Root Analysis schemes	195
13.1	Square root algorithm for the EnKF analysis	195
13.1.1	Updating the ensemble mean	196
13.1.2	Updating the ensemble perturbations	196
13.1.3	Randomization of the analysis update	197
13.1.4	Final update equation in the square root algorithms	200
13.2	Experiments	201
13.2.1	Overview of experiments	201
13.2.2	Impact of the square root analysis algorithm	203
14	Rank issues	207
14.1	Pseudo inverse of \mathbf{C}	207
14.1.1	Pseudo inverse	208
14.1.2	Interpretation	209
14.1.3	Analysis schemes using the pseudo inverse of \mathbf{C}	209
14.1.4	Example	209
14.2	Efficient subspace pseudo inversion	212
14.2.1	Derivation of the subspace pseudo inverse	212
14.2.2	Analysis schemes based on the subspace pseudo inverse	216
14.2.3	An interpretation of the subspace pseudo inversion	217
14.3	Subspace inversion using a low-rank $\mathbf{C}_{\epsilon\epsilon}$	218
14.3.1	Derivation of the pseudo inverse	218
14.3.2	Analysis schemes using a low-rank $\mathbf{C}_{\epsilon\epsilon}$	219
14.4	Implementation of the analysis schemes	220
14.5	Rank issues related to the use of a low-rank $\mathbf{C}_{\epsilon\epsilon}$	221
14.6	Experiments with $m \gg N$	224
14.7	Summary	229

15	An ocean prediction system	231
15.1	Introduction	231
15.2	System configuration and EnKF implementation	232
15.3	Nested regional models	235
15.4	Summary	236
16	Estimation in an oil reservoir simulator	239
16.1	Introduction	239
16.2	Experiment	241
16.2.1	Parameterization	242
16.2.2	State vector	243
16.3	Results	245
16.4	Summary	248
A	Other EnKF issues	249
A.1	Local analysis	249
A.2	Nonlinear measurements in the EnKF	251
A.3	Assimilation of non-synoptic measurements	253
A.4	Time difference data	254
A.5	Ensemble Optimal Interpolation (EnOI)	255
A.6	Chronology of ensemble assimilation developments	255
A.6.1	Applications of the EnKF	255
A.6.2	Other ensemble based filters	264
A.6.3	Ensemble smoothers	264
A.6.4	Ensemble methods for parameter estimation	264
A.6.5	Nonlinear filters and smoothers	265
	References	267
	Index	277

List of symbols

a	De-correlation lengths in variogram models (11.2–11.4); Error in initial condition for scalar models (5.5), (5.22), (8.22) and (12.22)
$A(z)$	Vertical diffusion coefficient in Ekman model, Sect. 5.3
$A_0(z)$	First guess of vertical diffusion coefficient in Ekman model, Sect. 5.3
\mathbf{a}	Error in initial condition for vector models (5.70), (6.8–6.10), (7.2)
\mathbf{A}_i	Ensemble matrix at time t_i , Chap. 9
\mathbf{A}	Ensemble matrix, Chap. 9
\mathbf{b}	Vector of coefficients solved for in the analysis scheme (3.38)
$\mathbf{b}(\mathbf{x}, t)$	Error in boundary condition, Chap. 7
\mathbf{b}_0	Stochastic error in upper condition of Ekman model, Sect. 5.3
\mathbf{b}_H	Stochastic error in lower condition of Ekman model, Sect. 5.3
c	Constant in Fourier spectrum (11.10) and (11.14); constant multiplier used when simulating model errors (12.20)
c_i	Multiplier used when simulating model errors (12.54)
c_d	Wind-drag coefficient in Ekman model, Sect. 5.3
c_{d_0}	First guess value of wind-drag coefficient in Ekman model, Sect. 5.3
c_{rep}	Constant multiplier used when modelling model errors in representer method, Chap. 12
$C_{\psi\psi}$	Covariance of a scalar state variable ψ
$C_{c_d c_d}$	Covariance of error in wind-drag c_{d_0}
$C_{AA}(z_1, z_2)$	Covariance of error in vertical diffusion $A_0(z)$
$C_{\psi\psi}(\mathbf{x}_1, \mathbf{x}_2)$	Covariance of scalar field $\psi(\mathbf{x})$ (2.25)

$C_{\psi\psi}$	Covariance of a discrete ψ (sometimes short for $C_{\psi\psi}(\mathbf{x}_1, \mathbf{x}_2)$)
$C_{\psi\psi}(\mathbf{x}_1, \mathbf{x}_2)$	Covariance of a vector of scalar state variables $\psi(\mathbf{x})$
$C_{\epsilon\epsilon}$	Used for variance of ϵ in Chap. 3
C_{aa}	Scalar initial error covariance
C_{qq}	Model error covariance
C	Matrix to be inverted in the ensemble analysis schemes, Chap. 9
$C_{\epsilon\epsilon}$	Covariance of measurement errors ϵ
$C_{\epsilon\epsilon}^e$	Low-rank representation of measurement error covariance
C_{aa}	Initial error covariance
C_{qq}	Model error covariance
d	Measurement
\mathbf{d}	Vector of measurements
D	Perturbed measurements, Chap. 9
D_j	Perturbed measurements at data time j , Chap. 9
E	Measurement perturbations, Chap. 9
$f(\mathbf{x})$	Arbitrary function, e.g. in (3.55)
$f(\psi)$	Probability density, e.g. $f(\psi)$ or $f(\boldsymbol{\psi})$ where ψ is a vector or a vector of fields
F	Distribution function (2.1)
$g(\mathbf{x})$	Arbitrary function, e.g. in (3.55) and (3.59)
G	Model operator for a scalar state; linear (4.1) or nonlinear (4.14)
\mathbf{G}	Model operator for a vector state; linear (4.11) or nonlinear (4.21), (9.1) and (7.1)
$h()$	Arbitrary function used at different occasions
H	Depth of bottom boundary in Ekman model, Sect. 5.3
\mathbf{h}	Innovation vector (3.51); spatial distance vector (11.1)
$i(j)$	Time index corresponding to measurement j , Fig. 7.1
I	Identity matrix
J	Number of measurement times, Fig. 7.1
\mathbf{k}	Vertical unit vector (0, 0, 1) in Ekman model Sect. 5.3; wave number $\mathbf{k} = (\kappa, \lambda)$, Chap. 11
k_h	Permeability, Chap. 16
K	Kalman gain matrix (3.85)
m_j	Total number of measurements at measurement time j , Fig. 7.1
m	Sometimes used as abbreviation for m_j
$m(\psi)$	Nonlinear measurement functional in the Appendix

M	Total number of measurements over the assimilation interval
\mathbf{M}	Measurement matrix for a discrete state vector (3.76); measurement matrix operator (10.20)
n	Dimension of state vector $n = n_\psi + n_\alpha$, Chap. 9 and 10
n_α	Number of parameters, Chaps. 9 and 10
n_ψ	Dimension of model state, Chaps. 7–10
n_x	Gridsize in x -direction, Chap. 11
n_y	Gridsize in y -direction, Chap. 11
N	Sample or ensemble size
p	Error of first guess of a scalar or scalar field, Chap. 3; matrix rank, Chap. 14; probability (6.24)
p_A	Error of first guess vertical diffusion coefficient in Ekman model, Sect. 5.3
p_{cd}	Error of first guess wind drag coefficient in Ekman model, Sect. 5.3
P	Reservoir pressure, Chap. 16
q	Stochastic error of scalar model used in Kalman filter formulations
$q(i)$	Discrete model error at time t_i , (6.16)
\mathbf{q}	Stochastic error of vector model used in Kalman filter formulations
\mathbf{Q}	Ensemble of model noise, Sect. 11.4
r	De-correlation length in Fourier space (11.10)
r_1	De-correlation length in principal direction in Fourier space (11.11)
r_2	De-correlation length orthogonal to principal direction in Fourier space (11.11)
r_x	De-correlation length in principal direction in physical space (11.23)
r_y	De-correlation length orthogonal to principal direction in physical space (11.23)
$\mathbf{r}(\mathbf{x}, t)$	Vector of representer functions (3.39) and (5.48)
\mathbf{r}	Matrix of representer (3.80)
\mathbf{R}	Representer matrix (3.63)
R_s	Gas in a fluid state at reservoir conditions, Chap. 16
R_v	Oil in a gas state at reservoir conditions, Chap. 16
S_w	Water saturation, Chap. 16
S_g	Gas saturation, Chap. 16

S_o	Oil saturation, Chap. 16
$\mathbf{s}(\mathbf{x}, t)$	Vector of adjoints of representer functions (5.49)
\mathbf{S}_j	Measurement of ensemble perturbations at data time j , Chap. 9
\mathbf{S}	Measurement of ensemble perturbations, Chap. 9
t	Time variable
T	Final time of assimilation period for some examples
u	Dependent variable (5.99)
$\mathbf{u}(z)$	Horizontal velocity vector in Ekman model, Sect. 5.3
$\mathbf{u}_0(z)$	Initial condition for velocity vector in Ekman model, Sect. 5.3
\mathbf{U}	Left singular vectors from the singular value decomposition, Sect. 11.4 and (14.68)
\mathbf{U}_0	Left singular vectors from the singular value decomposition (14.19)
\mathbf{U}_1	Left singular vectors from the singular value decomposition (14.52)
\mathbf{v}	Dummy vector (5.101)
\mathbf{V}	Right singular vectors from the singular value decomposition, Sect. 11.4 and (14.68)
\mathbf{V}_0	Right singular vectors from the singular value decomposition (14.19)
\mathbf{V}_1	Right singular vectors from the singular value decomposition (14.52)
w_k	Random realization with mean equal to zero and variance equal to one (11.33)
W_{aa}	Inverse of scalar initial error covariance
\mathbf{W}	Matrix (14.63) and (14.64)
$\mathbf{W}_{\psi\psi}(\mathbf{x}_1, \mathbf{x}_2)$	Functional inverse of $\mathbf{C}_{\psi\psi}(\mathbf{x}_1, \mathbf{x}_2)$, e.g. (3.27)
$\mathbf{W}_{aa}(\mathbf{x}_1, \mathbf{x}_2)$	Functional inverse of initial error covariance
\mathbf{W}_{aa}	Inverse of initial error covariance
$\mathbf{W}_{\eta\eta}$	Smoothing weight (6.19)
$\mathbf{W}_{\epsilon\epsilon}$	Matrix inverse of the covariance $\mathbf{C}_{\epsilon\epsilon}$
\mathbf{x}	Independent spatial variable
x_n	x -position in grid $x_n = n\Delta x$, Chap 11
\mathbf{X}_0	Matrix (14.26) and (14.51)
\mathbf{X}_1	Matrix (14.30) and (14.55)
\mathbf{X}_2	Matrix (14.34) and (14.59)
x, y, z	Dependent variables in Lorenz equations (6.5–6.6)

\mathbf{x}	Dependent variable $\mathbf{x}^T = (x, y, z)$ in Lorenz equations
\mathbf{x}_0	Initial condition $\mathbf{x}_0^T = (x_0, y_0, z_0)$ in Lorenz equations
y_m	y -position in grid $y_m = m\Delta y$, Chap 11
\mathbf{Y}	Matrix (14.65)
\mathbf{Z}	Matrix of eigenvectors from eigenvalue decomposition
\mathbf{Z}_1	Matrix of eigenvectors from eigenvalue decomposition (14.27)
\mathbf{Z}_p	Matrix of p first eigenvectors from eigenvalue decomposition (14.15)
\mathcal{B}	Penalty function in measurement space, e.g. $\mathcal{B}[\mathbf{b}]$ in (3.66)
\mathcal{D}	Model domain
$\partial\mathcal{D}$	Boundary of model domain
\mathcal{H}	Hamiltonian, used in hybrid Monte Carlo algorithm (6.25)
\mathcal{H}	Hessian operator (second derivative of model operator)
\mathcal{J}	Penalty function, e.g. $\mathcal{J}[\psi]$
\mathcal{M}	Scalar measurement functional (3.24)
\mathcal{M}	Vector of measurement functionals
\mathcal{N}	Normal distribution
\mathcal{P}	Matrix to be inverted in representer method (3.50)
α_1, α_2	Coefficients used in Chap. 3
α_{ij}	Coefficient used in (16.1)
$\boldsymbol{\alpha}(\mathbf{x})$	Poorly known model parameters to be estimated, Chap. 7
$\boldsymbol{\alpha}'(\mathbf{x})$	Errors in model parameters, Chaps. 7 and 10
β	Coefficient in Lorenz equations (6.7); constant (also β_{ini} and β_{mes}), Sect. 11.4
$\delta\psi$	Variation of ψ
ϵ	Real measurement error, Chap. 3
$\epsilon_{\mathcal{M}}$	Representation errors in measurement operator, Sect. 5.2.3
ϵ_d	Actual measurement errors, Sect. 5.2.3
ϵ	Random or real measurement errors, Sect. 5.2.3
$\boldsymbol{\eta}$	Smoothing operators used in gradient method (6.19)
γ	Constant used in smoothing norm analysis (6.32); step length (6.22)
$\gamma(\mathbf{h})$	Variogram (11.1)
$\kappa_2(\mathbf{A})$	Condition number, Chap. 11
κ_l	Wave number in x direction, Chap. 11
λ_p	Wave number in y direction, Chap. 11
λ	Eigenvalue (13.8); scalar adjoint variable (5.37)

λ	Vector adjoint variable
Λ	Diagonal matrix of eigenvalues from eigenvalue decomposition
Λ_1	Diagonal matrix of eigenvalues from eigenvalue decomposition (14.26)
Λ_p	Diagonal matrix of eigenvalues from eigenvalue decomposition (14.14)
μ	Sample mean (2.20)
$\mu(\mathbf{x})$	Sample mean (2.23)
ω	Frequency variable (6.33)
ω_i	Unit variance noise process (8.10)
Ω	Error covariance of ω_i noise process (8.12)
ϕ	Scalar variable, Chap. 2
$\phi(\mathbf{x})$	Porosity in Chap. 16
$\phi_{l,p}$	Uniform random number (11.10)
Φ	Random scalar variable, Chap. 2
π	3.1415927
π	Momentum variable used in hybrid Monte Carlo algorithm (6.25)
ψ	Scalar state variable (has covariance $C_{\psi\psi}$)
$\psi(\mathbf{x})$	Scalar state variable field (has error covariance $\mathbf{C}_{\psi\psi}(\mathbf{x}_1, \mathbf{x}_2)$)
$\hat{\psi}(\mathbf{k})$	Fourier transform of $\psi(\mathbf{x})$, Chap. 11
$\boldsymbol{\psi}$	Vector state variable, e.g. from a discretized $\psi(\mathbf{x})$ (has error covariance $\mathbf{C}_{\psi\psi}$)
$\boldsymbol{\psi}(\mathbf{x})$	Vector of scalar state variables (has error covariance $\mathbf{C}_{\psi\psi}(\mathbf{x}_1, \mathbf{x}_2)$)
Ψ	Random scalar variable, Chap. 2
Ψ_0	Best guess initial condition for dynamical scalar models, may be function of \mathbf{x}
$\boldsymbol{\psi}(\mathbf{x})$	Vector of fields, sometimes written just $\boldsymbol{\psi}$
$\boldsymbol{\psi}_0$	Estimate of initial condition $\boldsymbol{\Psi}_0$, Chap. 7
$\boldsymbol{\psi}_b$	Estimate of boundary condition $\boldsymbol{\Psi}_b$, Chap. 7
$\boldsymbol{\Psi}$	Combined state vector, Chap. 10
$\boldsymbol{\Psi}_0$	Best guess initial condition
$\boldsymbol{\Psi}_b$	Best guess boundary condition, Chap. 7
ρ	Correlation parameter (11.33); coefficient in Lorenz equations (6.6)
Σ	Matrix of singular values from the singular value decomposition, Sect. 11.4 and (14.68)

Σ_0	Matrix of singular values from the singular value decomposition (14.19)
Σ_1	Matrix of singular values from the singular value decomposition (14.52)
σ	Standard deviation defined in Chap. 2; used as coefficient in Lorenz equations (6.5); singular values, Sect. 11.4, Chap. 13 and Chap. 14
τ	De-correlation time (12.1)
θ	Pseudo temperature variable used in simulated annealing algorithm; rotation of principal direction (11.11)
Θ	Random rotation used in SQRT analysis scheme, Chap. 13
ξ	Random number used in Metropolis algorithm
ξ	Coordinate running over boundary of model domain
$\mathbf{1}_N$	$N \times N$ matrix with all elements equal to 1
$\delta()$	Dirac delta function (3.24)
δ_{ψ_i}	Vector used to extract a component of the state vector, Chap. 7
$E[]$	Expectation operator
$\mathcal{O}()$	Order of magnitude function
\Re	Space of real numbers, e.g. $\Re^{n \times m}$ for a real $n \times m$ matrix